Chapter 2 Notes

Note Title 12/16/2017

2.1 Linear Differential Equations; Method of Integrating Factors

Given $\frac{dy}{dt} + p(t)y = g(t)$ [1]

$y(t)$ is a solution to [1] $\Longleftrightarrow$

$y(t)$ is a solution to:

$u(t)\frac{dy}{dt} + u(t)p(t)y = u(t)g(t)$, $u(t) \neq 0$ [2]

Pf: Clearly $y(t)$ a solution to [1] will be a solution to [2]. And if $y(t)$ is a solution to [2], then dividing by $u(t)$, since $u(t) \neq 0$, $y(t)$ will be a solution to [1].

Note for $$y = \frac{1}{\mu(t)}\left(\int_{t_0}^{t} \mu(s)g(s)\,ds + c\right), \quad (33)$$ p. 29,

from the Fundamental Theorem of Calculus,

$$\frac{d}{dt}\int_{t_0}^{t} \mu(s)g(s)\,ds = \mu(t)g(t)$$

$$\therefore \frac{dy}{dt} = -\frac{1}{\mu(t)^2}\cdot\frac{d\mu}{dt}\cdot\left[\int_{t_0}^{t}\mu(s)g(s)\,ds + c\right] + \frac{1}{\mu(t)}\left[\mu(t)g(t)\right]$$

$$= -\frac{1}{\mu(t)}\frac{d\mu}{dt}\cdot y(t) + g(t)$$

$$\therefore \mu(t)\frac{dy}{dt} + \frac{d\mu}{dt}\cdot y(t) = \mu(t)\,g(t) \qquad [3]$$

But by definition, $\mu(t) = \exp\int_{t_0}^{t} p(s)\,ds$

$$\therefore \frac{d\mu(t)}{dt} = p(t)\exp\int_{t_0}^{t} p(s)\,ds = p(t)\,\mu(t)$$

since $\frac{d}{dt}\int_{t_0}^{t} p(s)\,ds = p(t)$

$\therefore$ [3] becomes,

$$\mu(t)\frac{dy}{dt} + \mu(t)p(t)\,y = \mu(t)\,g(t)$$

Dividing by $\mu(t)$ since $\mu(t) \neq 0$,

$$\frac{dy}{dt} + p(t)\,y = g(t) \qquad [4]$$

$\therefore$ (33) on p.29 is a solution to [4]

Extending the financial model to one involving deposits/withdrawals is not good. It assumes such deposits/withdrawals are made continuously, rather than one time lump transactions. The resulting solution should be piecewise continuous, not fully continuous, demonstrating jumps or drops at the time of deposit or withdrawal.

2.4 Differences Between Linear and Nonlinear Differential Equations

The statement, at the bottom of page 52,

An important geometrical consequence of the uniqueness parts of Theorems 2.4.1 and 2.4.2 is that the graphs of two solutions cannot intersect each other. Otherwise, there would

Should really state, "... is that the graphs of two general solutions cannot intersect each other." Then, "two solutions" becomes more meaningful.

The statement on page 55, under "Interval of Existance",

point where it fails to exist and to be differentiable. The one exceptional solution shows that solutions may sometimes remain continuous even at points of discontinuity of the coefficients.

refers to the existance of the solution $y = t^2$ to $ty' + 2y = 4t^2$, $y(1) = 1$, as described on page 29, of section 2.1, Example 4.

On p. 61, the text states,

> in the ty-plane. Thus, although other solutions may be asymptotic to the equilibrium solution as $t \to \infty$, they cannot intersect it at any finite time. Consequently, a solution that starts in the interval $0 < y < K$ remains in this interval for all time, and similarly for a solution that starts in $K < y < \infty$.

Any intial condition (t_0, y_0), where $y_0 \neq K$, means there is only one solution, and it is asymptotic to $y = K$. $\therefore$ Every solution for which $y_0 \neq K$ means the solution can't intersect the line $y = K$.

On p. 62, $$\ln|y| - \ln\left|1 - \frac{y}{K}\right| = rt + c, \qquad (9)$$

Suppose $y_0 > K > 0$. $\therefore |y| = y$, $\left|1 - \frac{y}{K}\right| = \frac{y}{K} - 1$

$$\therefore \ln(y) - \ln\left(\frac{y}{K} - 1\right) = rt + c$$

$$\frac{y}{\frac{y}{K} - 1} = C' e^{rt}, \quad C' = e^{c} \text{ a constant,}$$

or $$\frac{y}{1 - \frac{y}{K}} = C e^{rt}, \quad C = -C', \text{ a constant}$$

$\therefore$ same form as (10), & $\therefore$ (11)

On pp. 65-66, $\frac{dy}{dt} = -r\left(1 - \frac{y}{T}\right)\left(1 - \frac{y}{K}\right)y,$ (17)

$$y' = -r\frac{(y-T)(y-K)y}{TK}$$

$$= \frac{-r}{TK}\left[y^3 - (T+K)y^2 + TKy \right]$$

$$\therefore\ y'' = \frac{-r}{TK}\left[3y^2 - 2(T+K)y + TK \right]$$

$$y'' = 0 \iff 3y^2 - 2(T+K)y + TK = 0$$

$$\iff y = \frac{2(T+K) \pm \sqrt{4(T+K)^2 - 12TK}}{6}$$

$$\iff y = \frac{(T+K) \pm \sqrt{(T+K)^2 - 3TK}}{3}$$

$$\iff y = \frac{(T+K) \pm \sqrt{T^2 - TK + K^2}}{3}$$

The "proof" of Theorem 2.6.1 on p. 72 assumes $Q(x,y)$ is a C^2 function. This is not immediately apparent from choosing $Q(x,y) = \int_{x_0}^{x} M(s,y)\,ds$

$Q_x = M(x,y)$ and $\therefore$ $Q_{xy} = M_y$, continuous under assumptions of Th. 2.6.1.

But need differentiation under integral for

$$Q_y = \frac{\partial}{\partial y}\int_{x_0}^{x} M(s,y)\,ds = \int_{x_0}^{x} M_y(s,y)\,ds$$

$\therefore$ $Q_{yx} = M_y(x,y)$, again continuous by Th. 2.6.1

$\therefore$ $Q_{xy} = Q_{yx} = M_y(x,y)$

In Coddington's "An Introduction to Ordinary Differential Equations", pp. 195-196, Fubini's Theorem is used, requiring $M_y - N_x$ be continuous:

$$\int_{x_0}^{x}\left[\int_{y_0}^{y} M_y(s,t)\,dt\right]ds - \int_{y_0}^{y}\left[\int_{x_0}^{x} N_x(s,t)\,ds\right]dt = \int_{x_0}^{x}\int_{y_0}^{y}(M_y - N_x)\,ds\,dt$$

Rather than restricting argument to $t_0 = 0,\ y(0) = 0$, the rectangle of continuity can be confined to

$$R:\ |x - x_0| \le a,\ |y - y_0| \le b,\ (a, b > 0)$$

and

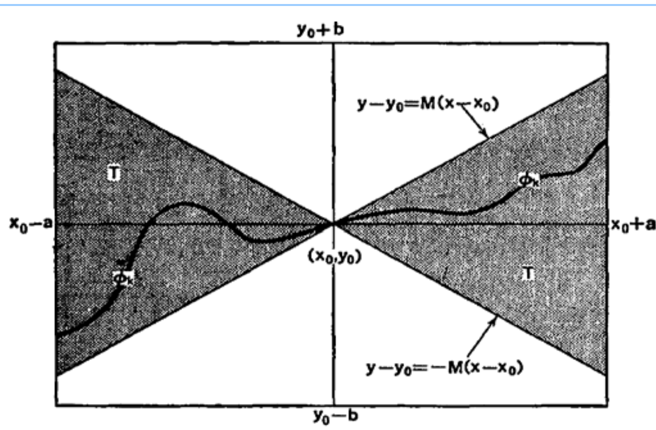
$$\phi(t) = y_0 + \int_{t_0}^{t} f(s, \phi(s))\, ds$$

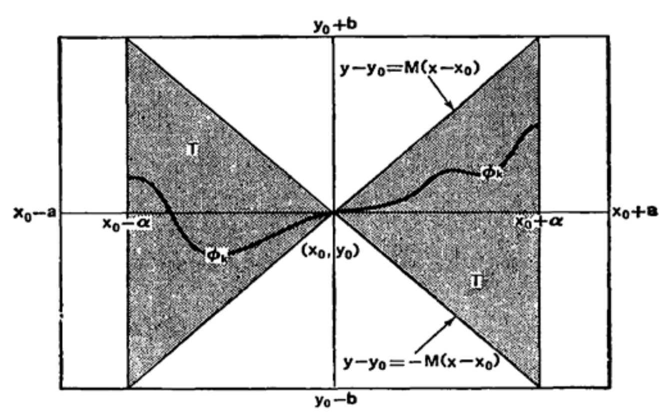
$$\therefore\ \phi(t_0) = y_0 \quad \text{and} \quad \phi'(t) = f(t, \phi(t))$$

$$\therefore\ \phi_{n+1}(t) = y_0 + \int_{t_0}^{t} f(s, \phi_n(s))\, ds$$

$$\therefore\ \phi_{n+1}(t_0) = y_0 \quad \text{and} \quad \phi'_{n+1}(t) = f(t, \phi_n(t)) = \phi_n(t)$$

From Coddington, the bow-tie regions look as shown:





2.9 First-Order Difference Equations

Example 1, page 93

It is assumed 12% interest rate reduces to

1% monthly interest rate. $\therefore \rho = 1.01$

$\therefore$ $y_n = \rho^n \left(y_0 - \frac{b}{1-\rho} \right) + \frac{b}{1-\rho}$ becomes

$$y_n = (1.01)^n \left(10{,}000 - \frac{b}{-0.01} \right) + \frac{b}{-0.01}$$

$$= (1.01)^n (10{,}000 + 100b) - 100b$$

$$\therefore \; y_n - 10{,}000(1.01)^n = [(1.01)^n - 1]\, 100b$$

$$b = \frac{y_n - 10{,}000(1.01)^n}{100[(1.01)^n - 1]}$$

where y_n = balance at end of nth month

from $y_{n+1} = \rho y_n + b = (1.01) y_n + b$,

b is paid at end of nth month (b is negative)

Nonlinear Equations, page 94

$\frac{dy}{dt} = ry\left(1 - \frac{y}{K}\right)$ becomes $\frac{y_{n+1} - y_n}{h} = r y_n \left(1 - \frac{y_n}{K}\right)$

$$\therefore \; y_{n+1} - y_n = hry_n\left(1 - \frac{y_n}{K}\right)$$

$$\therefore \; y_{n+1} = y_n + hry_n\left(1 - \frac{y_n}{K}\right)$$

$$= y_n\left(1 + hr - \frac{hry_n}{K}\right)$$

$$= (1+hr)\,y_n\left[1 - \frac{hr}{K(1+hr)}\,y_n\right]$$

$\therefore$ let $\rho = (1+hr)$, $\quad \kappa = \dfrac{K(1+hr)}{hr}$

$$\therefore \; y_{n+1} = \rho\, y_n\left(1 - \frac{y_n}{\kappa}\right)$$

Nonlinear Equilibrium Solution $u_n = \frac{\rho - 1}{\rho}$, p. 95

Using $v_n = u_n - \frac{\rho-1}{\rho}$, and assuming v_n small (so u_n is close to $\frac{\rho-1}{\rho}$), converts analyzing u_n to analyzing the sequence on v_n. If one converges, so does the other.

From v_n, look at $v_{n+1} = u_{n+1} - \frac{\rho-1}{\rho}$, and see if v_{n+1} is even smaller and converges.

From (21), $u_{n+1} = \rho u_n(1 - u_n)$,

$$\rho\left(v_n + \frac{\rho-1}{\rho}\right)\left(1 - v_n - \frac{\rho-1}{\rho}\right)$$

$$= \rho\left[v_n - v_n^2 - v_n \cdot \frac{\rho-1}{\rho} + \frac{\rho-1}{\rho} - v_n \cdot \frac{\rho-1}{\rho} - \frac{(\rho-1)^2}{\rho^2}\right]$$

$$= \rho v_n - 2v_n(\rho-1) + (\rho-1) - \frac{(\rho-1)^2}{\rho}$$ ignoring v_n^2 term

$$= -\rho v_n + 2v_n + (\rho-1) - \frac{(\rho-1)^2}{\rho}$$

$$= v_n(2-\rho) + \rho - 1 - \left(\frac{\rho^2 - 2\rho + 1}{\rho}\right)$$

$$= v_n(2-\rho) + \rho - 1 - \rho + 2 - \frac{1}{\rho}$$

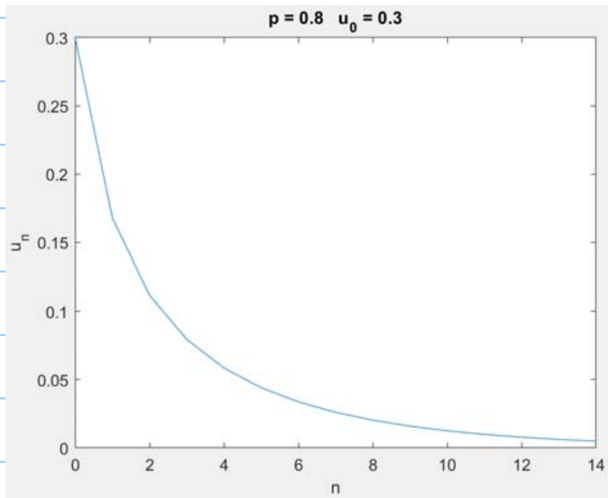
$$= v_n(2-\rho) + 1 - \frac{1}{\rho} = v_n(2-\rho) + \frac{\rho-1}{\rho}$$

$$\therefore v_{n+1} + \frac{\rho-1}{\rho} = u_{n+1} = v_n(2-\rho) + \frac{\rho-1}{\rho}$$

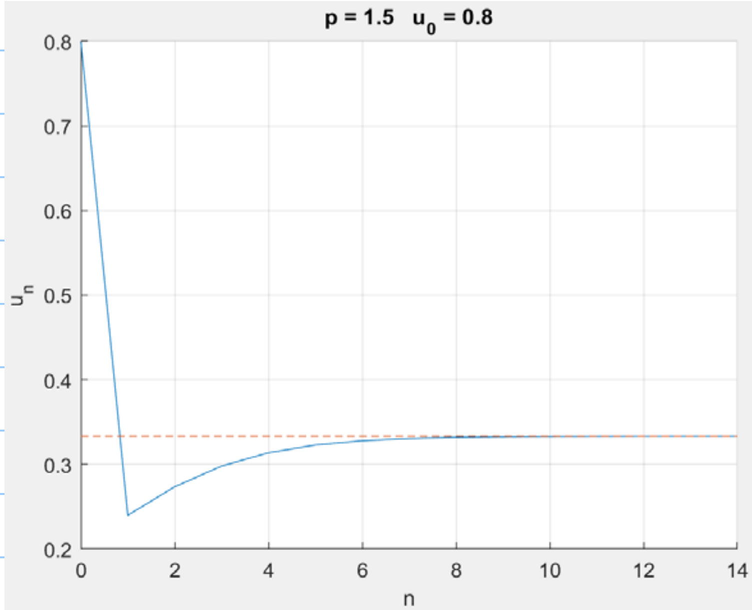
$$\therefore \underline{v_{n+1} = v_n(2-\rho)}$$

MATLAB code

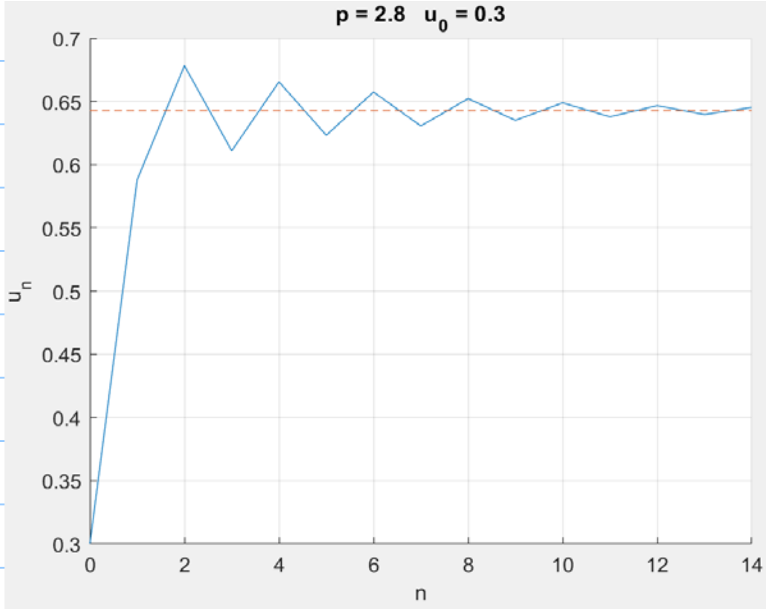
```
clear,clc;
u = zeros([1 15]);
n = zeros([1 15]);
p = 0.8;
u(1) = 0.3;
n(1) = 0;
for i = 2:15
    n(i) = i - 1;
    u(i) = p*u(i-1)*(1 - u(i-1));
end
plot(n,u)
xlabel 'n', ylabel 'u_n'
title 'p = 0.8    u_0 = 0.3'
```



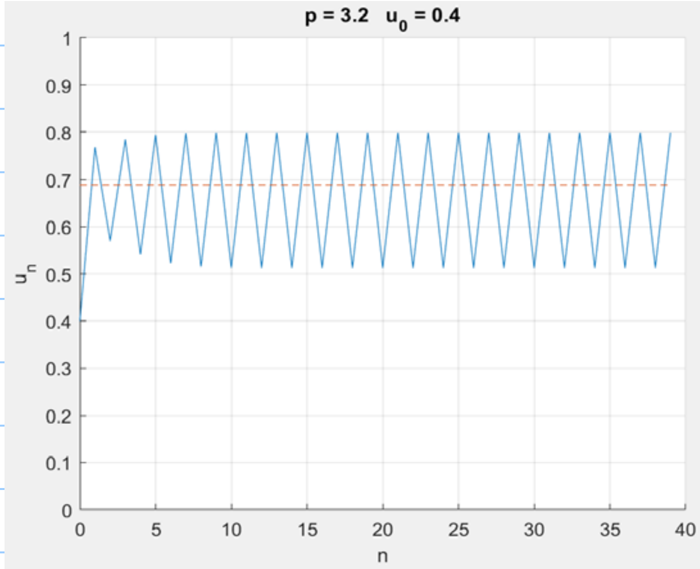
```
clear,clc;
u = zeros([1 15]);
n = 0:1:14;
p = 1.5;
eq = (p-1)/p;
u(1) = 0.8;
for i = 2:15
    u(i) = p*u(i-1)*(1 - u(i-1));
end
hold on
plot(n,u)
plot(n,0*u + eq, '--')
grid on
xlabel 'n', ylabel 'u_n'
title 'p = 1.5   u_0 = 0.8'
```




```
clear,clc;
u = zeros([1 15]);
n = 0:1:14;
p = 2.8;
eq = (p-1)/p;
u(1) = 0.3;
for i = 2:15
    u(i) = p*u(i-1)*(1 - u(i-1));
end
hold on
plot(n,u)
plot(n,0*u + eq, '--')
grid on
xlabel 'n', ylabel 'u_n'
title 'p = 2.8   u_0 = 0.3'
```




```
clear,clc;
steps = 40;
u = zeros([1 steps]);
n = 0:1:steps-1;
p = 3.2;
eq = (p-1)/p;
u(1) = 0.4;
for i = 2:steps
    u(i) = p*u(i-1)*(1 - u(i-1));
end
hold on
plot(n,u)
plot(n,0*u + eq, '--')
grid on
axis([0 steps 0 1])
xlabel 'n', ylabel 'u_n'
title 'p = 3.2   u_0 = 0.4'
```



Chapter 3 Notes

Note Title 5/23/2018

3.1 Homogeneous Differential Equations with Constant Coefficients

On p. 106,

$$c_1 e^{r_1 t_0} + c_2 e^{r_2 t_0} = y_0. \quad (22)$$

$$c_1 r_1 e^{r_1 t_0} + c_2 r_2 e^{r_2 t_0} = y_0'. \quad (23)$$

can be written as $\begin{bmatrix} e^{r_1 t_0} & e^{r_2 t_0} \\ r_1 e^{r_1 t_0} & r_2 e^{r_2 t_0} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_0' \end{bmatrix}$

$$\therefore \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} r_2 e^{r_2 t_0} & -e^{r_2 t_0} \\ -r_1 e^{r_1 t_0} & e^{r_1 t_0} \end{bmatrix} \begin{bmatrix} y_0 \\ y_0' \end{bmatrix}$$

$\Delta = (r_2 - r_1) e^{r_1 t_0 + r_2 t_0}$, the determinant

$$\therefore c_1 = \frac{y_0 r_2 e^{r_2 t_0} - y_0' e^{r_2 t_0}}{(r_2 - r_1) e^{r_1 t_0 + r_2 t_0}} = \frac{y_0 r_2 - y_0'}{(r_2 - r_1) e^{r_1 t_0}}$$

$$c_2 = \frac{-y_0 r_1 e^{r_1 t_0} + y_0' e^{r_1 t_0}}{(r_2 - r_1) e^{r_1 t_0 + r_2 t_0}} = \frac{y_0' - y_0 r_1}{(r_2 - r_1) e^{r_2 t_0}}$$

Or, $c_1 = \dfrac{y_0' - y_0 r_2}{r_1 - r_2} e^{-r_1 t_0}$ $\quad c_2 = \dfrac{y_0 r_1 - y_0'}{r_1 - r_2} e^{-r_2 t_0}$

which is equation (24)

3.2 Solutions of Linear Homogeneous Equations; the Wronskian

Further clarification of the statement on p.113 is needed:

> On the other hand, if $W = 0$, then the denominators appearing in equations (10) and (11) are zero. In this case equations (8) have no solution unless y_0 and y_0' have values that also make the numerators in equations (10) and (11) equal to zero. Thus, if $W = 0$, there are many initial conditions that cannot be satisfied no matter how c_1 and c_2 are chosen.

From:

$$\begin{aligned} c_1 y_1(t_0) + c_2 y_2(t_0) &= y_0, \\ c_1 y_1'(t_0) + c_2 y_2'(t_0) &= y_0'. \end{aligned} \qquad (8)$$

Multiplying top by $y_1'(t_0)$, bottom by $y_1(t_0)$,

$$c_1 y_1(t_0) y_1'(t_0) + c_2 y_2(t_0) y_1'(t_0) = y_0 y_1'(t_0)$$

$$c_1 y_1'(t_0) y_1(t_0) + c_2 y_2'(t_0) y_1(t_0) = y_0' y_1(t_0)$$

Subtracting, $c_2(y_2 y_1' - y_2' y_1) = y_0 y_1'(t_0) - y_0' y_1(t_0)$

and $\therefore$ $0 = y_0 y_1'(t_0) - y_0' y_1(t_0)$ [1]

since $W = y_1 y_2' - y_2 y_1' = 0$

But depending upon y_0 and y_0', [1] may or may not be true, irrespective of c_1, c_2.

Similarly, multiplying (8) top by y_2', bottom by y_2, you get $0 = y_0 y_2'(t_0) - y_0' y_2(t_0)$ [2], which may or may not be true. If [1], [2] are true, then the numerators in (10) are zero:

$$c_1 = \frac{y_0 y_2'(t_0) - y_0' y_2(t_0)}{y_1(t_0)\, y_2'(t_0) - y_1'(t_0)\, y_2(t_0)}, \quad c_2 = \frac{-y_0 y_1'(t_0) + y_0' y_1(t_0)}{y_1(t_0)\, y_2'(t_0) - y_1'(t_0)\, y_2(t_0)}, \qquad (10)$$

and then c_1, c_2 can be anything, so there are infinitely many solutions.

$\therefore$ If $W \neq 0$, the linear equations can be solved. If $W = 0$, it is not always possible to choose c_1 and c_2 to get a solution

$y = c_1 y_1 + c_2 y_2$ (i.e., [1], [2] not true).

This is how Theorem 3.2.3 is proved:

$a \Longleftrightarrow b$: if $b \Rightarrow a$
if not $b \Rightarrow$ not a

=

For Theorem 3.2.4, proof also follows the form: if $b \Rightarrow a$
if not $b \Rightarrow$ not a

For the part if not $b \Rightarrow$ not a, remember it is given y_1, y_2 are solutions to $y'' + py' + qy = 0$

The "not b" part is assuming not $W \neq 0$; i.e., $W = 0$ for all t. ∴ Choose y_0 and y_0' s.t. no c_1, c_2 can be found to make $c_1 y_1 + c_2 y_2$ a solution, given y_0, y_0'. But a solution does exist given y_0 and y_0' by The Existence Theorem 3.2.1. ∴ y_1 and y_2 can't span the space of solutions, which means "not a", where "a" says they do span all solutions.

Note: Theorem 3.2.4 should be stated for an interval I, where p, q are continuous and $t_0 \in I$.

Theorem 3.2.4

Suppose that y_1 and y_2 are two solutions of the second-order linear differential equation (2),

$$L[y] = y'' + p(t)y' + q(t)y = 0.$$

Then the two-parameter family of solutions

$$y = c_1 y_1(t) + c_2 y_2(t)$$

with arbitrary coefficients c_1 and c_2 includes every solution of equation (2) if and only if there is a point t_0 where the Wronskian of y_1 and y_2 is not zero.

Note that no intial conditions are stated in premise.

That is, y_1 and y_2 satisfy a homogeneous equation without initial conditions given. Theorem 3.2.4 says you only need 2 independent solutions s.t. $W[y_1, y_2] \neq 0$ at some t_0 in order to find all solutions.

3.3 Complex Roots of the Characteristic Equation

Let $u(t) = e^{at}\cos(bt)$, $v(t) = e^{at}\sin(bt)$

$\therefore \; W[u,v](t) = b e^{2at}$

```
clear, clc
syms a b t
u = exp(a*t)*cos(b*t);
v = exp(a*t)*sin(b*t);
A = [u v; diff(u) diff(v)]
simplify(det(A))  % Compute Wronskian
```

A =

$$\begin{pmatrix} e^{a\,t}\cos(b\,t) & e^{a\,t}\sin(b\,t) \\ a\,e^{a\,t}\cos(b\,t) - b\,e^{a\,t}\sin(b\,t) & a\,e^{a\,t}\sin(b\,t) + b\,e^{a\,t}\cos(b\,t) \end{pmatrix}$$

ans = $b\,e^{2\,a\,t}$

$\therefore$ For $b \neq 0$, $e^{at}\cos(bt)$ and $e^{at}\sin(bt)$ always form a fundamental set of solutions.

On page 131 of text, from

$$y = v(t)y_1(t);$$

$$y' = v'(t)y_1(t) + v(t)y_1'(t)$$

$$y'' = v''(t)y_1(t) + 2v'(t)y_1'(t) + v(t)y_1''(t).$$

and $$y'' + p(t)y' + q(t)y = 0. \tag{29}$$

you get,

$$(v''y_1 + 2v'y_1' + vy_1'') + p(v'y_1 + vy_1') + qvy_1 = 0$$

$$\therefore\ y_1 v'' + (2y_1' + py_1)v' + \underbrace{(y_1'' + py_1' + qy_1)}_{=0}v = 0$$

$$\therefore\ y_1 v'' + (2y_1' + py_1)v' = 0$$

Assuming $y_1 \neq 0$,

$$\frac{v''}{v'} = \frac{-(2y_1' + py_1)}{y_1}$$

$$\therefore\ \ln(v') = -\int \frac{(2y_1' + py_1)}{y_1} + K$$

$$v' = c_1 \exp\left[-\int \frac{(2y_1' + py_1)}{y_1}\right], \quad c_1 = e^K$$

$$V = C_1 \int \exp\left[-\int \frac{(2y_1' + py_1)}{y_1}\right] + C_2$$

Or,
$$V = C_1 \int e^{-\int \frac{(2y_1' + py_1)}{y_1}} + C_2$$

In Example 3, p. 131, $2t^2y'' + 3ty' - y = 0, \quad t > 0,$

becomes $y'' + \frac{3}{2t} - \frac{1}{2t^2} y = 0$ and $y_1 = \frac{1}{t}$

$$\therefore \frac{2y' + py_1}{y_1} = \frac{2\left(-\frac{1}{t^2}\right) + \left(\frac{3}{2t}\right)\left(\frac{1}{t}\right)}{\frac{1}{t}} = \frac{-\frac{4}{2t^2} + \frac{3}{2t^2}}{\frac{1}{t}} = -\frac{1}{2t}$$

$$\therefore -\int -\frac{1}{2t} = \frac{1}{2} \ln(t) = \ln t^{\frac{1}{2}}$$

$$\therefore \exp\left[-\int \frac{(2y_1' + py_1)}{y_1}\right] = \exp\left[\ln t^{\frac{1}{2}}\right] = t^{\frac{1}{2}}$$

$$\therefore V = C_1 \int t^{\frac{1}{2}} + C_2 = \frac{2}{3} C_1 t^{3/2} + C_2$$

which is the same as $v(t) = \frac{2}{3}ct^{3/2} + k.$

3.5 Nonhomogeneous Equations; Method of Undetermined Coefficients

On page 141 of text, it states,

the right-hand side of equation (33). Note that if $a\alpha^2 + b\alpha + c$ is zero, then $e^{\alpha t}$ is a solution of the homogeneous equation.

For $ay'' + by' + cy = e^{\alpha t} P_n(t)$,

the characteristic equation is: $ar^2 + br + c = 0$

$\therefore$ Since α is a root to this equation, $e^{\alpha t}$ is a solution to $ay'' + by' + cy = 0$

Also on p. 141,

If both $a\alpha^2 + b\alpha + c$ and $2a\alpha + b$ are zero (and this implies that both $e^{\alpha t}$ and $te^{\alpha t}$ are solutions of the homogeneous equation), then the correct form for $u(t)$ is $t^2(A_0 t^n + \cdots + A_n)$.

For $ar^2 + br + c = 0$, $r = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

If $2a\alpha + b = 0$, then $\alpha = \dfrac{-b}{2a}$ is a root,

so $b^2 - 4ac$ must be 0, which means α is a repeated root. $\therefore$ $e^{\alpha t}$ and $te^{\alpha t}$ are solutions to $ay'' + by' + cy = 0$ from section 3.4.

Summary:

TABLE 3.5.1 The Particular Solution of $ay'' + by' + cy = g_i(t)$

$g_i(t)$	$Y_i(t)$
$P_n(t) = a_0t^n + a_1t^{n-1} + \cdots + a_n$	$t^s(A_0t^n + A_1t^{n-1} + \cdots + A_n)$
$P_n(t)e^{\alpha t}$	$t^s(A_0t^n + A_1t^{n-1} + \cdots + A_n)e^{\alpha t}$
$P_n(t)e^{\alpha t}\begin{cases}\sin\beta t\\ \cos\beta t\end{cases}$	$t^s\big((A_0t^n + A_1t^{n-1} + \cdots + A_n)e^{\alpha t}\cos(\beta t) + (B_0t^n + B_1t^{n-1} + \cdots + B_n)e^{\alpha t}\sin(\beta t)\big)$

Notes: Here, s is the smallest nonnegative integer ($s = 0$, 1, or 2) that will ensure that no term in $Y_i(t)$ is a solution of the corresponding homogeneous equation. Equivalently, for the three cases, s is the number of times 0 is a root of the characteristic equation, α is a root of the characteristic equation, and $\alpha + i\beta$ is a root of the characteristic equation, respectively.

"Notes" perhaps better stated as:

s is the number of times:

(1) $g_i(t) = P_n(t)$: 0 is a root of $ar^2+br+c=0$

(2) $g_i(t) = e^{\alpha t}P_n(t)$: α is a root of $ar^2+br+c=0$

(3) $g_i(t) = e^{(\alpha \pm \beta t)}P_n(t)$: $\alpha+\beta i$ is a root of $ar^2+br+c=0$

3.6 Variation of Parameters

$$u_1'(t)\,y_1(t) + u_2'(t)\,y_2(t) = 0. \tag{21}$$

$$u_1'(t)\,y_1'(t) + u_2'(t)\,y_2'(t) = g(t). \tag{25}$$

These can be written as,

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ g \end{bmatrix}$$

$$\therefore \quad u_1' = \frac{\begin{bmatrix} 0 & y_2 \\ g & y_2' \end{bmatrix}}{W[y_1, y_2]} = -\frac{g(t)\,y_2(t)}{W[y_1, y_2](t)}$$

$$u_2' = \frac{\begin{bmatrix} y_1 & 0 \\ y_1' & g \end{bmatrix}}{W[y_1, y_2]} = \frac{g(t)\,y_1(t)}{W[y_1, y_2](t)}$$

Note that in the derivation of $mu'' + \gamma u' + Ku = F(t)$, $u(t)$ denotes "from the equilibrium". $\therefore$ If a weight hangs from a spring, gravity is involved, and $mg - KL = 0$, so this term disappears. If no gravity is involved, there is no mg and no KL since $L = 0$. For example, a block and spring on a horizontal table. Here, $u(t)$ denotes position from $x = 0$. With gravity, $u(t)$ denotes position from $x = L$.

=

On page 153, equation (27) comes from a Taylor expansion:

$$\left(1 - \frac{\gamma^2}{4Km}\right)^{\frac{1}{2}} \approx 1 - \frac{\gamma^2}{8Km}$$

i.e., if $f(x) = (1-x)^{\frac{1}{2}}$, $f'(x) = -\frac{1}{2}(1-x)^{-\frac{1}{2}}$, $f''(x) = -\frac{1}{4}(1-x)^{-\frac{3}{2}}$

$$f(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \ldots$$

$$= 1 - \frac{1}{2}x - \frac{1}{8}x^2 + \ldots$$

Here, $x = \frac{\gamma^2}{4Km}$, so if $0 < x << 1$, $x^2 <<< 1$,

so $\frac{1}{8}x^2$ can be ignored.

Also, p. 153, second paragraph at end states:

The motion is called a damped oscillation or a damped vibration. The amplitude factor R depends on m, γ, k, and the initial conditions.

R comes from determining A and B from initial conditions. But initial conditions involve taking derivative of $\sin(\mu t)$ and $\mu = \frac{1}{2m}(4Km - \gamma^2)^{\frac{1}{2}}$

So, R depends on m, γ, K and initial conditions.

3.8 Forced Periodic Vibrations

Recall the general solution to damped free vibration:

$mu'' + ku = 0$ is $A\cos(\omega_0 t) + B\sin(\omega_0 t)$

where $\omega_0 = \sqrt{\frac{k}{m}}$

For damped vibrations, $mu'' + \gamma u' + ku = 0$, there is always an exponential term (overdamped, critically damped, or underdamped).

For forced vibrations of sort $F(t) = F_0\cos(\omega t)$ or $F_0\sin(\omega t)$, the particular solution $A\cos(\omega t) + B\sin(\omega t)$ has the frequency of the forcing function (using undetermined coefficients).

For damped forced vibrations, $mu'' + \gamma u' + ku = F(t)$, where $F(t) = F_0\cos(\omega t)$, the particular solution has form $A\cos(\omega t) + B\sin(\omega t)$, or

$R\cos(\omega t - \delta) = R\cos(\omega t)\cos(\delta) + R\sin(\omega t)\sin(\delta)$.

Using $u(t) = R\cos(\omega t - \delta)$, assuming $\omega \neq \omega_0$,

and substituting into $mu'' + \gamma u' + Ku = F_0 \cos(\omega t)$,

$u' = -R\omega \sin(\omega t - \delta) \quad u'' = -R\omega^2 \cos(\omega t - \delta)$,

$$-Rm\omega^2 \cos(\omega t - \delta) - R\gamma\omega \sin(\omega t - \delta) + RK\cos(\omega t - \delta) =$$

$$R(K - m\omega^2)\cos(\omega t - \delta) - R\gamma\omega \sin(\omega t - \delta) =$$

$$R(K - m\omega^2)\left[\cos(\omega t)\cos(\delta) + \sin(\omega t)\sin(\delta)\right.$$

$$\left. - R\gamma\omega\left[\sin(\omega t)\cos(\delta) - \cos(\omega t)\sin(\delta)\right]\right. =$$

$$\left[R(K - m\omega^2)\cos(\delta) + R\gamma\omega\sin(\delta)\right]\cos(\omega t)$$

$$+ \left[R(K - m\omega^2)\sin(\delta) - R\gamma\omega\cos(\delta)\right]\sin(\omega t) = F_0\cos(\omega t)$$

$$\therefore \quad R\left[(K - m\omega^2)\cos(\delta) + \gamma\omega\sin(\delta)\right] = F_0 \qquad [1]$$

$$R\left[(K - m\omega^2)\sin(\delta) - \gamma\omega\cos(\delta)\right] = 0 \qquad [2]$$

Letting $\omega_0^2 = \frac{K}{m}$ or $K = m\omega_0^2$, the above equations

become $\begin{bmatrix} m(\omega_0^2 - \omega^2) & \gamma\omega \\ -\gamma\omega & m(\omega_0^2 - \omega^2) \end{bmatrix} \begin{bmatrix} \cos(\delta) \\ \sin(\delta) \end{bmatrix} = \begin{bmatrix} F_0/R \\ 0 \end{bmatrix}$

Let $d = m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2 \qquad [3]$

(the determinant of the left matrix, and always ≥ 0)

$\therefore$ From Cramer's Rule,

$$\cos(\delta) = \frac{(F_0/R)\, m(\omega_0^2 - \omega^2)}{d}, \quad \sin(\delta) = \frac{(F_0/R)\gamma\omega}{d} \quad [4]$$

$$\therefore \cos^2(\delta) + \sin^2(\delta) = 1 = \frac{(F_0/R)^2 \left[m^2(\omega_0^2 - \omega^2) + \gamma^2\omega^2\right]}{d^2},$$

Or, using [3], $1 = \frac{(F_0/R)^2 d}{d^2}$, $\therefore \frac{F_0}{R} = \sqrt{d}$

Let $\Delta = \sqrt{d} = \sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}$

$$\therefore \underline{R = \frac{F_0}{\Delta}}$$

Using this to substitute into [4], and $d = \Delta^2$,

$$\underline{\cos(\delta) = \frac{m(\omega_0^2 - \omega^2)}{\Delta}}, \quad \underline{\sin(\delta) = \frac{\gamma\omega}{\Delta}}$$

Recall $\omega_0 = \sqrt{\frac{K}{m}}$, from $mu'' + Ku = 0$,

$u = A\cos(\omega_0 t) + B\sin(\omega_0 t)$, so ω_0 is the frequency of an undamped unforced system.

From $R = \dfrac{F_0}{\Delta}$, $\dfrac{R}{F_0} = \dfrac{1}{\sqrt{m^2(\omega_0^2-\omega^2)^2 + \gamma^2\omega^2}}$

$$= \frac{1}{\sqrt{m^2\omega_0^4\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)^2 + m^2\omega_0^4\frac{\gamma^2\omega^2}{m^2\omega_0^4}}}$$

now use $\omega_0^2 = \dfrac{K}{m}$

$$= \frac{1}{K\sqrt{\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \frac{\gamma^2}{m^2\omega_0^2}\left(\frac{\omega}{\omega_0}\right)^2}}$$

$$\therefore \frac{RK}{F_0} = \frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_0}\right)^2\right]^2 + \frac{\gamma^2}{Km}\left(\frac{\omega}{\omega_0}\right)^2}} = \frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_0}\right)^2\right]^2 + \Gamma\left(\frac{\omega}{\omega_0}\right)^2}}$$

where $\Gamma = \dfrac{\gamma^2}{Km}$

$$\therefore R = \frac{F_0}{K}\left[\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \frac{\gamma^2}{Km}\left(\frac{\omega}{\omega_0}\right)^2\right]^{-\frac{1}{2}} \qquad [R]$$

$$\therefore \frac{dR}{d\omega} = -\frac{F_0}{2K}\left[\quad\right]^{-\frac{3}{2}}\left[2\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)\left(\frac{-2\omega}{\omega_0^2}\right) + \frac{2\gamma^2\omega}{Km\omega_0^2}\right]$$

Setting to zero, $\dfrac{\gamma^2\omega}{Km\omega_0^2} = \dfrac{2\omega}{\omega_0^2}\left(1-\dfrac{\omega^2}{\omega_0^2}\right)$

Or, $\dfrac{\gamma^2}{Km} = \dfrac{2(\omega_0^2-\omega^2)}{\omega_0^2}$, $\omega^2 = \omega_0^2 - \dfrac{\omega_0^2\gamma^2}{2Km}$

$$\therefore\ \omega_{max}^2 = \omega_0^2\left(1 - \frac{\gamma^2}{2mK}\right),\ \text{or}\ \frac{\omega_{max}^2}{\omega_0^2} = 1 - \frac{\gamma^2}{2mK}$$

Substituting into [R],

$$R_{max} = \frac{F_0}{K}\left[\left(\frac{\gamma^2}{2mK}\right)^2 + \frac{\gamma^2}{mK}\left(1 - \frac{\gamma^2}{2mK}\right)\right]^{-\frac{1}{2}} \qquad [R_{max}]$$

$$= \frac{F_0}{K}\left[\frac{\gamma^4}{4m^2K^2} + \frac{\gamma^2}{mK} - \frac{\gamma^4}{2m^2K^2}\right]^{-\frac{1}{2}}$$

$$= \frac{F_0}{K}\left[\frac{\gamma^2}{mK} - \frac{\gamma^4}{4m^2K^2}\right]^{-\frac{1}{2}} = \frac{F_0}{\gamma K}\left[\frac{1}{mK} - \frac{\gamma^2}{4m^2K^2}\right]^{-\frac{1}{2}}$$

Now using $K = m\omega_0^2$

$$= \frac{F_0}{\gamma K}\left[\frac{1}{m^2\omega_0^2} - \frac{\gamma^2}{4m^3K\omega_0^2}\right]^{-\frac{1}{2}}$$

$$= \frac{F_0 m\omega_0}{\gamma K}\left[1 - \frac{\gamma^2}{4mK}\right]^{-\frac{1}{2}} = \frac{F_0\omega_0}{\gamma\omega_0^2}\left(1 - \frac{\gamma^2}{4mK}\right)^{-\frac{1}{2}}$$

$$\therefore\ R_{max} = \frac{F_0}{\gamma\omega_0\sqrt{1 - \gamma^2/4mK}}$$

To test if ω_{max} is a relative max or minimum, rather than doing the second derivative test, consider evaluating R at $\frac{\omega_{max}^2}{\omega_0^2} + \epsilon$, $\epsilon \neq 0$

Using [R],

$$R_\epsilon = \frac{F_0}{K}\left[\left(\frac{\gamma^2}{2mK} - \epsilon\right)^2 + \frac{\gamma^2}{mK}\left(1 - \frac{\gamma^2}{2mK} + \epsilon\right)\right]^{-\frac{1}{2}}$$

$$= \frac{F_0}{K}\left[\left(\frac{\gamma^2}{2mK}\right)^2 + \frac{\gamma^2}{mK}\left(1 - \frac{\gamma^2}{2mK}\right) - \frac{\gamma^2\epsilon}{mK} + \epsilon^2 + \frac{\gamma^2}{mK}\epsilon\right]^{-\frac{1}{2}}$$

$$= \frac{F_0}{K}\left[\left(\frac{\gamma^2}{2mK}\right)^2 + \frac{\gamma^2}{mK}\left(1 - \frac{\gamma^2}{2mK}\right) + \epsilon^2\right]^{-\frac{1}{2}} \quad [R_\epsilon]$$

Comparing $[R_{max}]$ to $[R_\epsilon]$,

$$\left[\left(\frac{\gamma^2}{2mK}\right)^2 + \frac{\gamma^2}{mK}\left(1 - \frac{\gamma^2}{2mK}\right) + \epsilon^2\right] > \left[\left(\frac{\gamma^2}{2mK}\right)^2 + \frac{\gamma^2}{mK}\left(1 - \frac{\gamma^2}{2mK}\right)\right]$$

$$\therefore R_\epsilon = \frac{F_0/K}{\sqrt{\left(\frac{\gamma^2}{2mK}\right)^2 + \frac{\gamma^2}{mK}\left(1 - \frac{\gamma^2}{2mK}\right) + \epsilon^2}} < \frac{F_0/K}{\sqrt{\left(\frac{\gamma^2}{2mK}\right)^2 + \frac{\gamma^2}{mK}\left(1 - \frac{\gamma^2}{2mK}\right)}} = R_{max}$$

$\therefore$ For any small $\epsilon < 0$ or $\epsilon > 0$, $R_\epsilon < R_{max}$, so ω_{max} represents a point of relative maximum for R.

Note, for $|x| < 1$, $(1+x)^k = 1 + kx + \frac{k(k-1)}{2}x^2 + \ldots$ for any real number k (Taylor expansion).

$\therefore R_{max} = \frac{F_0}{\gamma \omega_0}\left(1 - \frac{\gamma^2}{4mk}\right)^{-\frac{1}{2}} \approx \frac{F_0}{\gamma \omega_0}\left(1 + \frac{\gamma^2}{8mk}\right)$

as long as $\frac{\gamma^2}{4mk} < 1$. For small γ, this is true.

If $\gamma = 0$, then from $R = \frac{F_0}{\Delta}$ and $\Delta = m(\omega_0^2 - \omega^2)$

$\therefore R = \frac{F_0}{m(\omega_0^2 - \omega^2)}$ and clearly $R \to \infty$ as $\omega \to \omega_0$

If $\gamma \neq 0$, then from $\frac{\omega_{max}^2}{\omega_0^2} = 1 - \frac{\gamma^2}{2mk}$, obtained from setting $\frac{dR}{d\omega} = 0$, the max γ can be is

$1 - \frac{\gamma^2}{2mk} \geq 0$, or $\gamma^2 \leq 2mk$, $\gamma_{max} = \sqrt{2mk}$

$\therefore 0 \leq \gamma \leq \sqrt{2mk}$, and for $\gamma_{max} = \sqrt{2mk}$,

$\omega_{max} = 0$. $\therefore R = \frac{F_0}{k}\left[\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \frac{\gamma^2}{km}\left(\frac{\omega}{\omega_0}\right)^2\right]^{-\frac{1}{2}}$

means $R_{max} = \frac{F_0}{k}$. But really, $\omega = 0$

means $mu'' + \gamma u' + ku = F_0 \cos(\omega t) = F_0$

and $u(t) = \frac{F_0}{k}$ is the particular solution.

Assuming $\omega \neq 0$, setting $\gamma = \sqrt{2mk}$,

$$R = \frac{F_0}{k}\left[\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \frac{2mk}{km}\left(\frac{\omega}{\omega_0}\right)^2\right]^{-\frac{1}{2}}$$

$$= \frac{F_0}{k}\left[1 - 2\left(\frac{\omega}{\omega_0}\right)^2 + \left(\frac{\omega}{\omega_0}\right)^4 + 2\left(\frac{\omega}{\omega_0}\right)^2\right]^{-\frac{1}{2}}$$

$$= \frac{F_0/k}{\sqrt{1 + \left(\frac{\omega}{\omega_0}\right)^4}}$$

$\therefore$ From $\omega = 0$ to increasing $\omega > 0$,

R is a monotone decreasing function of ω.

Summary (assuming $\omega \neq 0$)

For $0 \le \gamma < \sqrt{2mk}$, R can $\to \infty$ as $\gamma\omega_0 \to 0$

and for small γ, R is max when $\omega \approx \omega_0$ $\left(\omega_{max} = \omega_0\left(1 - \frac{\gamma^2}{2mk}\right)^{\frac{1}{2}}\right)$

For $2mk \le \gamma^2 < 4mk$ (critical damping),

let $\gamma_\epsilon^2 = 2mk + \epsilon$. Then, from above,

$$R = \frac{F_0}{k}\left[1 + \left(\frac{\omega}{\omega_0}\right)^4 + \frac{\epsilon}{km}\left(\frac{\omega}{\omega_0}\right)^2\right]^{-\frac{1}{2}} \quad \text{and}$$

R is a monotone decreasing function of ω, max $\frac{F_0}{k}$.

$\therefore \dfrac{\gamma^2}{mK} > 4$: over damping, $u_c = Ae^{r_1 t} + Be^{r_2 t}$
no resonance

$\dfrac{\gamma^2}{mK} = 4$: critical damping, $u_c = (A + Bt)e^{rt}$
no resonance

$2 \leq \dfrac{\gamma^2}{mK} < 4$: under damping, $u_c = e^{-\frac{\gamma}{2m}}(A\cos\mu t + B\sin\mu t)$
no resonance
$R(\omega)$ a monotone decreasing function

$0 < \dfrac{\gamma^2}{mK} < 2$: under damping, $u_c = e^{-\frac{\gamma}{2m}}(A\cos\mu t + B\sin\mu t)$
resonance at

$$\omega_{max} = \omega_0\left(1 - \frac{\gamma^2}{2mK}\right)^{-\frac{1}{2}}$$

Example 2 (p. 163) Note: $\omega_0 = \frac{K}{m} = 1$

$\dfrac{\omega}{\omega_0} = 0.3$: $\Delta = \sqrt{m(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2} = \sqrt{(1-.09)^2 + \frac{.09}{64}}$

$= 0.9108$

$\therefore R = \dfrac{F_0}{\Delta} = \dfrac{3}{0.9108} = \underline{3.2939}$

$\delta = \text{Arctan}\left(\dfrac{\gamma\omega}{m(\omega_0^2 - \omega^2)}\right) = \text{Arctan}\left(\dfrac{\frac{1}{8}(.3)}{1-.09}\right)$

$= \underline{0.041185}$

$\dfrac{\omega}{\omega_0} = 1$: $\Delta = \sqrt{\frac{1}{64}} = \frac{1}{8}$

$$\therefore R = \frac{F_0}{\Delta} = \frac{3}{1/8} = \underline{24}$$

$$\delta = \text{Arcsin}\left(\frac{\gamma w}{\Delta}\right) = \text{Arcsin}\left(\frac{1/8}{1/8}\right) = \underline{\frac{\pi}{2}}$$

$$\frac{w}{w_0} = 2 : \quad \Delta = \sqrt{(1-4)^2 + \frac{4}{64}} = 3.010$$

$$\therefore R = \frac{F_0}{\Delta} = \frac{3}{3.010} = \underline{0.9965}$$

$$\delta = \text{Arctan}\left(\frac{\frac{1}{8}(2)}{1-4}\right) = -0.08314$$

$$\equiv \pi - 0.08314 = \underline{3.0585}$$

Note $\frac{\gamma^2}{mK} = \frac{1}{64}$, so $0 < \frac{\gamma^2}{mK} < 2$: underdamping

and resonance can exist

Solution :

$$u(t) = e^{-t/16}\left(A\cos\left(\frac{\sqrt{255}}{16}t\right) + B\sin\left(\frac{\sqrt{255}}{16}t\right)\right) + R\cos(wt - \delta)$$

$$u(0) = 2, \quad u'(0) = 0$$

$$\frac{w}{w_0} = 0.3 : \quad u(0) = A + 3.2939\cos(0.041185)$$

$$= A + 3.2911 = 2, \quad A = \underline{-1.2911}$$

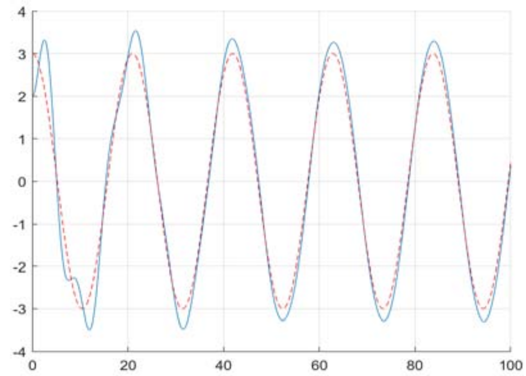
$$u'(0) = -\frac{1}{16}(A) + \left(B\frac{\sqrt{255}}{16}\right) - Rw\sin(-\delta)$$

$$= 0.08069 + B(0.9980) + (3.2939)(0.3)(0.04117)$$

$= (0.9980)B + 0.12138 = 0, \quad \underline{B = -0.1216}$

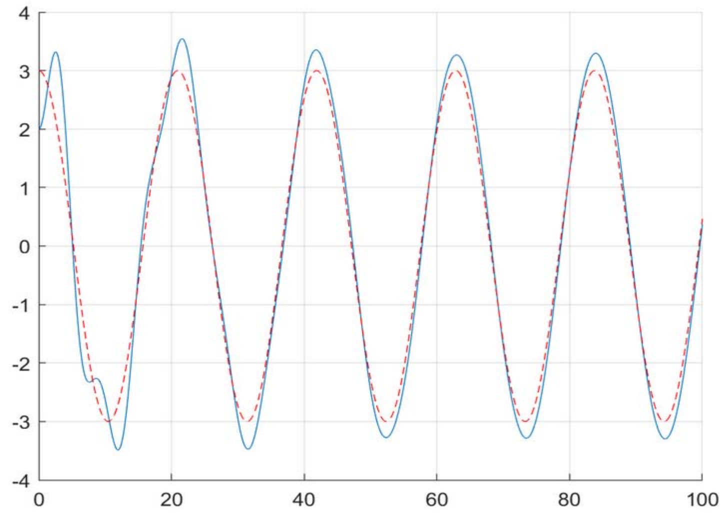
From MATLAB

```
clear,clc;
t = 0:0.01:100;
sq = sqrt(255);
sq_16 = sq/16;
A = -1.2911; B = -0.1216;
uc = exp(-t/16).*(A*cos(sq_16*t) + B*sin(sq_16*t));
R = 3.2939; d = 0.041185; w = 0.3;
u = uc + R*cos(w*t - d);
hold on
plot(t,u)
plot(t, 3*cos(w*t), 'r--')
grid on
```



Or, more simply:

```
clear,clc;
w = 0.3;
syms u(t)
Du = diff(u,t);
eqn = diff(u,t,2) + (1/8)*Du + u == 3*cos(w*t);
cond = [u(0) == 2, Du(0) == 0];
uSol(t) = dsolve(eqn, cond);
t = 0:0.1:100;
hold on
plot(t,uSol(t))
plot(t, 3*cos(w*t), 'r--')
grid on
```



Forced Vibrations Without Damping

From p.161, (10) is

$$U(t) = R\cos(\omega t - \delta). \tag{10}$$

$$R = \frac{F_0}{\Delta}, \quad \cos\delta = \frac{m(\omega_0^2 - \omega^2)}{\Delta}, \quad \text{and} \quad \sin\delta = \frac{\gamma\omega}{\Delta}, \tag{11}$$

$$\Delta = \sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2} \quad \text{and} \quad \omega_0^2 = \frac{k}{m}. \tag{12}$$

Setting $\gamma = 0$, $\Delta = m(\omega_0^2 - \omega^2)$, and $\delta = 0$

$\therefore U(t) = \dfrac{F_0}{m(\omega_0^2-\omega^2)} \cos(\omega t)$, used in (18):

$\omega_0 \neq \omega$

$$u = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) + \frac{F_0}{m(\omega_0^2 - \omega^2)}\cos(\omega t). \tag{18}$$

When $\omega = \omega_0$, similar to problem #9, Section 3.5,

$u'' + \omega_0^2 u = \dfrac{F_0}{m}\cos(\omega_0 t)$, $\omega_0^2 = \dfrac{K}{m}$. For the particular

solution, let $u(t) = At\cos(\omega_0 t) + Bt\sin(\omega_0 t)$

MATLAB:

```
clear,clc
syms t A B w0 m
c2 = 1; c1 = 0; c0 = w0^2;  %coeffs of diff eq
y = (A*t)*cos(w0*t) + (B*t)*sin(w0*t);    %attempt
u = c2*diff(y,t,2) + c1*diff(y,t,1) + c0*y;
collect(u, [cos(w0*t), sin(w0*t)])
```

ans = $(2\,B\,w_0)\cos(t\,w_0) + (-2\,A\,w_0)\sin(t\,w_0)$

$\therefore \quad 2B\omega_0 = \frac{F_0}{m}, \quad B = \frac{F_0}{2m\omega_0}$

$-2A\omega_0 = 0, \quad A = 0$

$\therefore \quad u_p(t) = \frac{F_0}{2m\omega_0} t \sin(\omega_0 t)$

$\therefore \quad u = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{F_0}{2m\omega_0} t \sin(\omega_0 t).$

Chapter 4 Notes

Note Title 12/17/2018

4.1 General Theory of nth Order Linear Differential Equations

On page 171, definition of linear dependence is more accurately stated as,

The functions $f_1, f_2, \ldots, f_n$ are said to be linear dependent on an interval I if, for all $t \in I$, there exists a single set of constants $K_1, K_2, \ldots, K_n$, not all zero, such that $K_1 f_1(t) + K_2 f_2(t) + \ldots + K_n f_n(t) = 0$

4.4 The Method of Variation of Parameters

From $$Y' = (u_1 y_1' + u_2 y_2' + \cdots + u_n y_n') + (u_1' y_1 + u_2' y_2 + \cdots + u_n' y_n), \qquad (4)$$

$$u_1' y_1 + \dots + u_n' y_n = 0$$

From $y^{(2)}$, $u_1' y_1' + \dots + u_n' y_n' = 0$

From $y^{(n-1)}$, $u_1' y_1^{(n-2)} + \dots + u_n' y_n^{(n-2)} = 0$

$\therefore$ So far,
$$\underbrace{\begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-2)} & & \cdots & y_n^{(n-2)} \end{bmatrix}}_{(n-1)\times n} \underbrace{\begin{bmatrix} u_1' \\ u_2' \\ \vdots \\ u_n' \end{bmatrix}}_{n\times 1} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{(n-1)\times 1}$$

Also,
$$Y = u_1 y_1 + \dots + u_n y_n$$
$$Y' = u_1 y_1' + \dots + u_n y_n'$$
using above conditions
$$Y^{(2)} = u_1 y_1^{(2)} + \dots + u_n y_n^{(2)}$$
$$Y^{(n-1)} = u_1 y_1^{(n-1)} + \dots + u_n y_n^{(n-1)}$$
$$\therefore Y^{(n)} = u_1 y_1^{(n)} + \dots + u_n y_n^{(n)} + u_1' y_1^{(n-1)} + \dots + u_n' y_n^{(n-1)}$$

Substitute these into:

$$y^{(n)} + p_1(t) y^{(n-1)} + \cdots + p_{n-1}(t) y' + p_n(t) y = g(t) \qquad (1)$$

$$\therefore\ p_n y = p_n u_1 y_1 + \dots + p_n u_n y_n$$

$$p_{n-1} y' = p_{n-1} u_1 y_1' + \dots + p_{n-1} u_n y_n'$$

$$p_{n-2} y^{(2)} = p_{n-2} u_1 y_1^{(2)} + \dots + p_{n-2} u_n y_n^{(2)}$$

$$p_1 y^{(n-1)} = p_1 u_1 y_1^{(n-1)} + \dots + p_1 u_n y_n^{(n-1)}$$

$$y^{(n)} = u_1 y_1^{(n)} + \dots + u_n y_n^{(n)} + u_1' y_1^{(n-1)} + \dots + u_n' y_n^{(n-1)}$$

Adding by columns,

$$y^{(n)} + p_1 y^{(n-1)} + \dots + p_{n-2} y^{(2)} + p_{n-1} y' + p_n y =$$

$$u_1 \big(\underbrace{y_1^{(n)} + p_1 y_1^{(n-1)} + \dots + p_{n-2} y_1^{(2)} + p_{n-1} y_1' + p_n y_1}_{=0} \big) +$$

$$\vdots$$

$$u_n \big(\underbrace{y_n^{(n)} + p_1 y_n^{(n-1)} + \dots + p_{n-2} y_n^{(2)} + p_{n-1} y_n' + p_n y_n}_{=0} \big)$$

$$\therefore\ y^{(n)} + p_1 y^{(n-1)} + \dots + p_{n-1} y' + p_n y =$$

$$u_1' y_1^{(n-1)} + \dots + u_n' y_n^{(n-1)} = g(t)$$

$\therefore$ This latter equation can be used to expand the matrix to an $n \times n$ matrix:

$$\underbrace{\begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & & & \\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_n^{(n-2)} \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{bmatrix}}_{n \times n} \underbrace{\begin{bmatrix} u_1' \\ u_2' \\ \vdots \\ \vdots \\ u_n' \end{bmatrix}}_{n \times 1} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ g(t) \end{bmatrix}}_{n \times 1}$$

$$\therefore W[y_1, \ldots, y_n] \begin{bmatrix} u_1' \\ \vdots \\ u_n' \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ g \end{bmatrix}$$

Chapter 5 Notes

Note Title 2/13/2019

5.1 Review of Power Series

Lemma 1 Let $s_n = \sum_{i=0}^{n} a_i$. If $\lim_{n\to\infty} s_n = L$, then $\lim_{n\to\infty} a_n = 0$

Proof: Let $\epsilon > 0$. $\exists N'$ s.t. if $m > N'$ then $|S_m - L| < \frac{\epsilon}{2}$. $\therefore |S_{m+1} - L| < \frac{\epsilon}{2}$ as $m+1 > N'$

$$\therefore |a_{m+1}| = |S_{m+1} - S_m| = |S_{m+1} - L + L - S_m|$$

$$\leq |S_{m+1} - L| + |S_m - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

$\therefore$ Choose $N = N' + 1$. $\therefore$ if $n > N$, then $n > N'+1$, $(n-1) > N'$, so

$$|a_n - 0| = |a_{(n-1)+1} - 0| = |a_{(n-1)+1}|$$

$$= |S_{(n-1)+1} - S_{(n-1)}| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

$\therefore \lim_{n\to\infty} a_n = 0$

Property 4, p. 190

4. If the power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converges at $x = x_1$, it converges absolutely for $|x-x_0| < |x_1-x_0|$; and if it diverges at $x = x_1$, it diverges for $|x-x_0| > |x_1-x_0|$.

Pf: (1) Assume $\sum_{n=0}^{\infty} a_n(x_1-x_0)^n$ converges, and

assume $x_1 \neq x_0$

By Lemma 1 above, $\lim_{n\to\infty} a_n(x_1-x_0)^n = 0$

$\therefore$ For $\epsilon = 1$, $\exists N > 0$ s.t. if $n > N$, then

$|a_n(x_1-x_0)^n| < 1$ [1]

$\therefore$ For all x s.t. $|x-x_0| < |x_1-x_0|$, $\left|\frac{(x-x_0)}{(x_1-x_0)}\right| < 1$

and $\therefore$ $\left|\frac{(x-x_0)}{(x_1-x_0)}\right|^n < 1$

$\therefore$ For all $n > N$, and x s.t. $|x-x_0| < |x_1-x_0|$,

$|a_n(x-x_0)^n| = |a_n(x_1-x_0)^n| \left|\frac{(x-x_0)^n}{(x_1-x_0)^n}\right| < \left|\frac{(x-x_0)}{(x_1-x_0)}\right|^n$

using [1]

$\therefore \sum_{n=N+1}^{\infty} |a_n(x-x_0)^n|$ is dominated by $\sum_{n=N+1}^{\infty} \left|\frac{x-x_0}{x_1-x_0}\right|^n$

and the latter series is a convergent geometric series.

$\therefore \sum_{n=N}^{\infty} |a_n(x-x_0)|$ converges, and so

$$\sum_{n=0}^{\infty} |a_n(x-x_0)| \text{ converges.}$$

(2) Suppose $|x'-x_0| > |x_1-x_0|$ and $\sum_{n=0}^{\infty} |a_n(x'-x_0)^n|$ converges. Then by (1), since $|x_1-x_0| < |x'-x_0|$, $\sum_{n=0}^{\infty} |a_n(x_1-x_0)^n|$ must converge, a contradiction.

Property 7, p. 191

7. The two series can be formally multiplied, and

$$f(x)g(x) = \left(\sum_{n=0}^{\infty} a_n(x-x_0)^n\right)\left(\sum_{n=0}^{\infty} b_n(x-x_0)^n\right) = \sum_{n=0}^{\infty} c_n(x-x_0)^n,$$

where $c_n = a_0b_n + a_1b_{n-1} + \cdots + a_nb_0$. The resulting series converges at least for $|x-x_0| < \rho$.

The term $c_n(x-x_0)^n$ is the sum of all the factors of $f(x)$ and $g(x)$ whose exponents add up to n.

$\therefore (a_0)[b_n(x-x_0)^n] + [a_1(x-x_0)^1][b_{n-1}(x-x_0)^{n-1}] + \ldots + [a_n(x-x_0)^n](b_0)$

$$= \left[a_0 b_n + a_1 b_{n-1} + \ldots + a_n b_0 \right] (x - x_0)^n$$

$$\therefore c_n = \sum_{k=0}^{n} a_k b_{n-k} = a_0 b_n + a_1 b_{n-1} + \ldots + a_n b_0$$

Further, if $b_0 \neq 0$, then $g(x_0) \neq 0$, and the series for $f(x)$ can be formally divided by the series for $g(x)$, and

$$\frac{f(x)}{g(x)} = \sum_{n=0}^{\infty} d_n (x - x_0)^n.$$

In most cases the coefficients d_n can be most easily obtained by equating coefficients in the equivalent relation

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n = \left[\sum_{n=0}^{\infty} d_n (x - x_0)^n \right] \left[\sum_{n=0}^{\infty} b_n (x - x_0)^n \right]$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} d_k b_{n-k} \right) (x - x_0)^n.$$

$\therefore a_0 = d_0 b_0 \qquad a_1 = d_0 b_1 + d_1 b_0 \qquad a_2 = d_0 b_2 + d_1 b_1 + d_2 b_0$

$$\therefore \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} b_0 & 0 & 0 & \cdots & 0 \\ b_1 & b_0 & 0 & \cdots & 0 \\ \vdots & & & & \\ b_n & b_{n-1} & & \cdots & b_0 \end{bmatrix} \begin{bmatrix} d_0 \\ d_1 \\ \vdots \\ d_n \end{bmatrix}, \text{ or } A = BD$$

B is diagonal and $\therefore$ invertible if $b_0 \neq 0$

$\therefore$ can solve for D.

As an example, consider $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$

$$\therefore \sin(x) = \tan(x)\cos(x)$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$\therefore \underset{\overset{\uparrow}{x^3}}{A_3} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1/6 \end{bmatrix} \quad \underset{\overset{\uparrow}{x^3}}{B_3} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{1}{2} & 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 0 & 1 \end{bmatrix} \quad D_3 = \begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$\therefore A_3 = B_3 D_3 \,, \quad D_3 = B_3^{-1} A_3$$

note $b_0 = 1 \neq 0$

Using MATLAB,

```
clear, clc
A = [0, 1, 0, -1/6]'
B = [1, 0, 0, 0;
     0, 1, 0, 0;
     -1/2, 0, 1, 0;
     0, -1/2, 0, 1]
D = rats(inv(B)*A)
```

```
A = 4×1
         0
    1.0000
         0
   -0.1667

B = 4×4
    1.0000         0         0         0
         0    1.0000         0         0
   -0.5000         0    1.0000         0
         0   -0.5000         0    1.0000

D = 4×14 char array
    '      0       '
    '      1       '
    '      0       '
    '     1/3      '
```

$$\therefore d_0 = 0 \quad d_1 = 1 \quad d_2 = 0 \quad d_3 = \frac{1}{3}$$

$$\therefore \tan(x) = \sum_{n=0}^{\infty} d_n x^n = x + \frac{1}{3}x^3 + \dots$$

for $\tan(x)$: $= x + \frac{2x^3}{3!} + \frac{16x^5}{5!} + \frac{272x^7}{7!} + \dots = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \dots$

$\therefore$ If $f(x)$ and $g(x)$ are analytic about a point x_0, and $b_0 \neq 0$ for $g(x) = \sum_{n=0}^{\infty} b_n (x - x_0)^n$, then $g(x)$ can be factored out of $f(x)$:

$$f(x) = g(x) \left[\frac{f(x)}{g(x)} \right]$$

So if $y(x) = K_1 \sum_{n=0}^{\infty} b_n x^n + K_2 \sum_{n=0}^{\infty} a_n x^n$

then $$y(x) = \sum_{n=0}^{\infty} b_n x^n \left[K_1 + K_2 \frac{\sum_{n=0}^{\infty} a_n x^n}{\sum_{n=0}^{\infty} b_n x^n} \right]$$

$$= \sum_{n=0}^{\infty} b_n x^n \left[K_1 + K_2 \sum_{n=0}^{\infty} d_n x^n \right]$$

Example 1, p.198

$$y'' + y = 0, \quad -\infty < x < \infty. \tag{4}$$

$$= a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}. \tag{11}$$

We identify two series solutions of equation (4):

$$y_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \quad \text{and} \quad y_2(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}.$$

Note that $y(x)$ defined by (11) satisfies (4) and no stipulations are put on a_0 or a_1.

$\therefore$ $y(x)$ with $a_0 = 1$ and $a_1 = 0$ in (11) satisfies (4), as does $y(x)$ with $a_0 = 0$ and $a_1 = 1$.

This justifies saying "two series solutions of (4)".

Ratio test for $y_1(x)$:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1} x^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{(-1)^n x^{2n}}\right| = \left|\frac{x^2}{(2n+2)(2n+1)}\right|$$

$$\therefore \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^2}{(2n+2)(2n+1)}\right| = 0, \text{ for all } x.$$

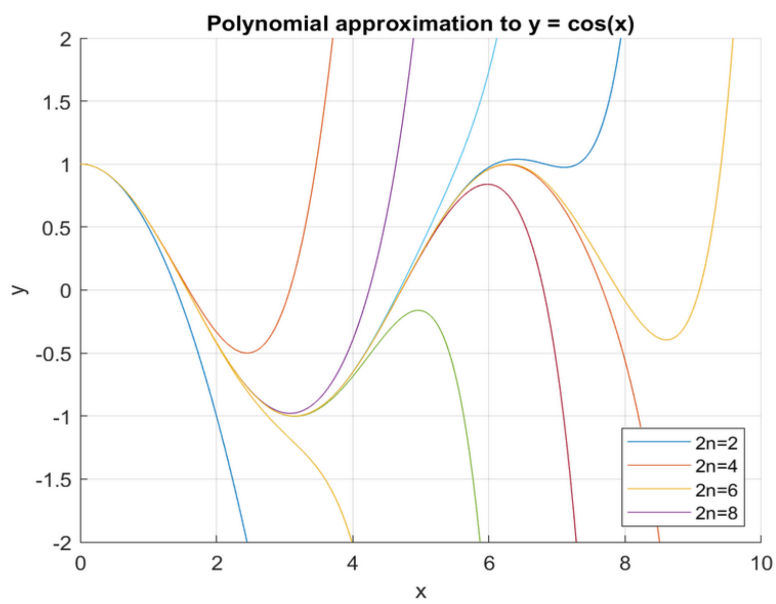
Ratio test for $y_2(x)$:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1}x^{2n+3}}{(2n+3)!}\cdot\frac{(2n+1)!}{(-1)^n x^{2n+1}}\right| = \left|\frac{x^2}{(2n+3)(2n+2)}\right|$$

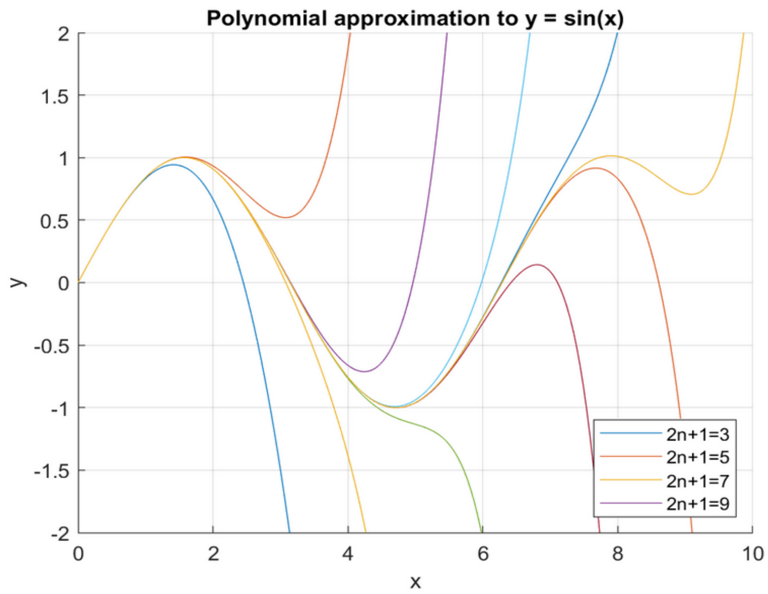
As above, $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0$, for all x.

Using MATLAB to plot:

```
clear, clc
x = 0:0.01:10;
hold on
ylim([-2 2])    % set vert axis range
grid on
S = 1;
for n = 1:10
    y = S + (-1)^n*(x.^(2*n))/factorial(2*n);
    S = y;
    plot(x,y)
end
xlabel 'x', ylabel 'y'
title('Polynomial approximation to y = cos(x)')
legend('2n=2','2n=4','2n=6','2n=8','Location','southeast')
```



```
clear, clc
x = 0:0.01:10;
hold on
ylim([-2 2])    % set vert axis range
grid on
S = x;
for n = 1:10
    y = S + (-1)^n*(x.^(2*n+1))/factorial(2*n+1);
    S = y;
    plot(x,y)
end
xlabel 'x', ylabel 'y'
title('Polynomial approximation to y = sin(x)')
legend('2n+1=3','2n+1=5','2n+1=7','2n+1=9','Location','southeast')
```



5.5 Series Solutions Near a Regular Singular Point, Part I

On p. 220,

$$y = x^r(a_0 + a_1 x + \cdots + a_n x^n + \cdots) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{r+n}, \qquad (7)$$

it is reasonable to assume $a_0 \neq 0$.

If $a_0 = 0$, let a_k be the first coefficient that is nonzero.

$$\therefore\ y = x^r\left(a_k x^k + a_{k+1} x^{k+1} + \ldots\right)$$

$$= a_k x^{r+k} + a_{k+1} x^{r+k+1} + \ldots$$

$$= x^{r+k}\left(a_k + a_{k+1} x + a_{k+2} x^2 + \ldots\right)$$

$$= x^s\left(b_0 + b_1 x + b_2 x^2 + \ldots\right)$$

i.e., it is the same form letting $y = x^s \sum_{n=0}^{\infty} b_n x^n$ with $b_0 \neq 0$.

5.6 Series Solutions Near a Regular Singular Point, Part II

On p. 227, for Equal Roots, it states,

of the indicial equation were equal. We consider r to be a continuous variable and determine a_n as a function of r by solving the recurrence relation (8). For this choice of $a_n(r)$ for $n \geq 1$, the terms in equation (6) involving $x^{r+1}, x^{r+2}, x^{r+3}, \ldots$ all have coefficients equal to zero.

$$L[\phi] = a_0 F(r) x^r + \sum_{n=1}^{\infty} \left(F(r+n) a_n + \sum_{k=0}^{n-1} a_k((r+k)p_{n-k} + q_{n-k}) \right) x^{r+n} = 0, \quad (6)$$

$$F(r+n) a_n + \sum_{k=0}^{n-1} a_k((r+k)p_{n-k} + q_{n-k}) = 0, \quad n \geq 1. \quad (8)$$

That is, by solving (8) for a_n, you are requiring all coefficients of $x^{r+1}, x^{r+2}, \ldots$ to be 0.

$$\therefore \quad a_n = \frac{-\sum_{k=0}^{n-1} a_k[(r+k)p_{n-k} + q_{n-k}]}{F(r+n)}, \quad n \geq 1$$

Note $F(r+n)$ very unlikely to be 0 since $F(r)$ only zero at $r = r_1$, so $n+r$ would have to equal r_1. At this point, just trying to be guided in finding another solution form.

This is done by assuming r is a continuous variable. If find a form and need to compute $a_n(r)$ at $r = r_1$, note $F(r_1 + n) \neq 0$ for $n \geq 1$ since $F(r)$ only zero at $r = r_1$.

Roots r1 and r2 differing by an integer N

Reference is made to a_N. Note this is not a_n. The capital N refers to $N = r_1 - r_2$, where $r_1 \geq r_2$. So if $N = 1$, then $a_N = a_1$. This mainly occurs in:

$$a = \lim_{r \to r_2} (r - r_2) a_N(r). \qquad (20)$$

$\therefore$ Substitue the value of N for n in:

$$a_n = \frac{-\sum_{K=0}^{n-1} a_K[(r+K)p_{n-K} + q_{n-K}]}{F(r+n)}$$

Proof of:

$$c_n(r_2) = \frac{d}{dr}[(r - r_2)a_n(r)]\Big|_{r=r_2}, \quad n = 1, 2, \ldots, \qquad (19)$$

$$a = \lim_{r \to r_2} (r - r_2) a_N(r). \qquad (20)$$

Let $\phi(x,r) = a_0 x^r + x^r \sum_{k=1}^{\infty} a_k(r) x^k$, $a_0 \neq 0$

and define the following:

$$a_0(r) = a_0 \neq 0$$

$$F(r) = r(r-1) + p_0 r + q_0$$

[1] $F(r+n)\,a_n(r) = -D_n(r)$, a function of r where

$$D_n(r) = \sum_{k=0}^{n-1} \left[(r+k)p_{n-k} + q_{n-k}\right] a_k(r), \quad n \geq 1$$

and

$$xp(x) = \sum_{n=0}^{\infty} p_n x^n, \quad x^2 q(x) = \sum_{n=0}^{\infty} q_n x^n, \tag{2}$$

$$L[y] = x^2 y'' + x(xp(x))y' + \left(x^2 q(x)\right) y = 0, \tag{1}$$

Thus, the $a_n(r)$, a function of r, is determined recursively, and from the development in the text on p. 225, since

$$F(r+n)\,a_n(r) + D_n(r) = 0 \text{ for } n \geq 1,$$

$$L[\phi] = a_0 F(r) x^r, \text{ so that}$$

$L[\phi](r_1) = a_0 F(r_1) x^{r_1} = 0$ since r_1 is a root (the larger root) of $F(r)$.

$\therefore\ y_1(x) = \phi(x, r_1) = x^{r_1} \sum_{k=0}^{\infty} a_k(r_1) x^k$

Now let r_2, the second root of $F(r)$, be s.t.

$r_1 - r_2 = N$, so $r_2 = r_1 - N$ and $r_1 = r_2 + N$,

N a positive integer.

If $a_0 \neq 0$ is given, $a_1(r_2)$, $a_2(r_2), \ldots, a_{N-1}(r_2)$

all exist as finite numbers from [1] since

$F(r_2 + n) \neq 0$ for $1 \leq n \leq N-1$, as $F()$ only

has 2 roots, and $r_2 + n < r_1$ for $1 \leq n \leq N-1$.

At $n = N$, $F(r_2 + n) = F(r_1) = 0$

$\therefore$ [1] becomes $0 \cdot a_N(r_2) = -D_N(r_2)$

$F(r)$ can be expressed as $F(r) = (r - r_1)(r - r_2)$

since r_1 and r_2 are its roots, and so

$$F(r+N) = (r + N - r_1)(r + N - r_2)$$

$$= [r - (r_1 - N)](r + N - r_2)$$

$$= (r - r_2)(r + N - r_2)$$

(a) If $\Delta_N(r_2) = 0$, then $\Delta_N(r)$ has $(r - r_2)$ as a factor : $-\Delta_N(r) = (r - r_2)\Delta_N^*(r)$, and so

[1] becomes : $(r - r_2)(r + N - r_2)\, a_N(r) = (r - r_2)\Delta_N^*(r)$

and so, $(N)\, a_N(r_2) = \Delta_N^*(r_2)$,

so $a_N(r_2)$ can be calculated, as can

$a_{N+1}(r_2)$, $a_{N+2}(r_2)$, ... since

$F(r_2 + n) \neq 0$ for $n \geq N+1$

$$\therefore \quad \boxed{y_2(x) = x^{r_2} \sum_{n=0}^{\infty} a_n(r_2)\, x^n, \quad a_0(r_2) \neq 0}$$

where the $a_n(r_2)$ are determined as above.

Thus, to check for this special case, just iterate up to $\Delta_N(r_2)$ to see if $\Delta_N(r_2) = 0$

If so, compute $\left. -\dfrac{\Delta_N(r)}{N(r - r_2)} \right|_{r = r_2}$ and

assign $a_N(r_2)$ that value, and use this $a_N(r_2)$ to compute $a_{N+1}(r_2)$, ...

$a_2(r_2)$ can be set to any nonzero value,

e.g., $a_2(r_2)=1$.

(b) If $D_N(r_2) \neq 0$, then there is no solution for $a_N(r_2)$ since $F(r_2+N)\cdot a_N(r_2) = -D_N(r_2)$ becomes $0\cdot a_N(r_2) = -D_N(r_2) \neq 0$

However, by choosing $a_0(r) = r - r_2$, then since $D_n(r)$ is a linear combination of all prior coefficients $a_n(r)$: $a_0(r), a_1(r), a_2(r), \dots, a_{n-1}(r)$, Then from [1], each prior $a_k(r)$ has $(r-r_2)$ as a factor and $\therefore$ so does $D_n(r)$.

$\therefore$ This time set $\phi(x,r) = x^r \sum_{n=0}^{\infty} a_n(r) x^n$, where $a_0(r) = r - r_2$ and the rest of a_n are defined as in [1]. Because of the definition of each a_n, $F(r+n)a_n(r) + D_n(r) = 0$, so

$$\sum_{n=1}^{\infty} \left[F(r+n)a_n(r) + D_n(r)\right] x^{r+n} = 0, \text{ and so}$$

$L[\phi](x,r) = (r-r_2)F(r)x^r$ as $a_0 = (r-r_2)$

$\therefore \phi(x, r_2) = 0$, so $y_2(x) = \phi(x, r_2)$ is a solution to $L[y]=0$.

Note $a_0(r_2) = (r_2 - r_2) = 0$ and $\therefore$ so does

$a_1(r_2) = a_2(r_2) = \dots = a_{N-1}(r_2) = 0$.

For $a_N(r_2)$, $F(r_2+N) \cdot a_N(r_2) = -\Delta_N(r_2)$

From above, $F(r+N) = (r-r_2)(r+N-r_2)$

$\therefore (r-r_2)(r+N-r_2)a_N(r) = -\Delta_N(r)$

But now $\Delta_N(r)$ has a factor of $(r-r_2)$ since it is a linear combination of prior $a_k(r)$

$\therefore$ After factoring out $(r-r_2)$,

$$a_N(r_2) = \left.\frac{-\Delta_N(r)}{N(r-r_2)}\right|_{r=r_2}, \text{ a finite number}$$

or, $$\lim_{r \to r_2} a_N(r) = \left.\frac{-\Delta_N(r)}{N(r-r_2)}\right|_{r=r_2}$$

since $a_n(r)$ and $\Delta_n(r)$ are continuous.

Note in the text, since $a_n(r)$ refers to the a_n derived from:

$$F(r+n)a_n + \sum_{k=0}^{n-1} a_k((r+k)p_{n-k} + q_{n-k}) = 0, \quad n \geq 1. \tag{8}$$

The $a_n(r)$ derived above was from setting $a_0(r) = r - r_2$. $\therefore$ If you multiply the a_n in the text by $(r-r_2)$, you get the equivalent to the $a_n(r)$ used above.

$\therefore \lim_{r \to r_2} a_N(r)$ above is the same as the text form: $\lim_{r \to r_2} (r-r_2) a_N(r)$

Call this limit or calculation "a" as in the text.

$$\therefore \quad y_2(x) = \phi(x, r_2) = x^{r_2} \sum_{n=N}^{\infty} a_n(r_2) x^n$$

$$= x^{r_2} \sum_{n=0}^{\infty} a_{n+N}(r_2) x^{n+N} = x^{r_2+N} \sum_{n=0}^{\infty} a_{n+N}(r_2) x^n$$

$$= x^{r_1} \sum_{n=0}^{\infty} a_{n+N}(r_2) x^n$$

Note that the $a_{n+N}(r_2)$ are determined recursively, beginning with $a_N(r_2)$, and the recursive formula, [1], is the same as that used for $y_1(x)$. Thus, this $y_2(x)$, obtained by setting $a_0 = r - r_2$, is just $a_N(r_2)\, y_1(x)$, $y_1(x)$ obtained by setting $a_0 = 1$.

To better see this, refer to the recursion formula:

$$a_n = -\frac{\sum_{k=0}^{n-1} [(r+k)p_{n-k} + q_{n-k}]a_k}{F(r+n)}, \quad n > N, \quad a_0 = r - r_2$$

But as noted above, $a_0(r_2) = a_1(r_2) = \dots = a_{N-1}(r_2) = 0$

$\therefore$ The above recursion can be written starting with $k = N$:

$$a_n = -\frac{\sum_{k=N}^{n-1} [(r+k)p_{n-k} + q_{n-k}]a_k}{F(r+n)}, \quad n > N$$

now shift index

$$a_n = \frac{-\sum_{k=0}^{n-1-N} \left[(r+k+N)\, p_{n-(k+N)} + q_{n-(k+N)} \right] a_{k+N}}{F(r+n)}, \quad n > N$$

Now let $m = n - N$ and set $b_m = a_n$, $n > N$.

$\therefore$ when $n = k + N$, $m = k$ so $b_k = a_{k+N}$

$$\therefore a_n = b_m = \frac{-\sum_{k=0}^{m-1} \left[(r+k+N)\, p_{m-k} + q_{m-k} \right] b_k}{F(r+m+N)}, \quad m \geq 1$$

and $b_0 = a_N(r_2)$

setting $r = r_2$ and noting $r_2 + N = r_1$, we get:

$$a_n(r_2) = b_m(r_2) = \frac{-\sum_{k=0}^{m-1} \left[(r_1+k)\, p_{m-k} + q_{m-k} \right] b_k}{F(r_1+m)}, \quad m \geq 1$$

These coefficients are just those of $y_1(x)$ with $b_0 = a_N(r_2)$ instead of $b_0 = 1$.

$\therefore$ The $y_2(x)$ above is just $a_N(r_2)\, y_1(x)$

$\therefore$ Just as in the case of $r_1 = r_2$, we will look at $\frac{\partial \phi}{\partial r}$, the $\phi(x)$ obtained using $a_0 = r - r_2$.

$\therefore$ Using $L[\phi](x,r) = (r-r_2)F(r)x^r$,

$$\frac{\partial}{\partial r} L[\phi](x,r) = \frac{\partial}{\partial r}\left[(r-r_2)F(r)\,x^r\right]$$

$$= F(r)x^r + (r-r_2)F'(r)x^r + (r-r_2)F(r)\ln(x)\,x^r$$

$$\therefore \frac{\partial}{\partial r} L[\phi](x,r)\Big|_{r=r_2} = 0 \quad \text{since } F(r_2) = 0.$$

But $\frac{\partial}{\partial r} L[\phi](x,r) = L\left[\frac{\partial \phi}{\partial r}\right](x,r)$ because of equality of mixed partial derivatives (since all derivatives are continuous).

$\therefore \frac{\partial \phi}{\partial r}(x,r_2)$ is a solution to $L[y]=0$.

From $\phi(x,r) = x^r \sum_{n=0}^{\infty} a_n(r)x^n$, where $a_0(r) = r-r_2$, and $a_n(r)$ are defined by [1],

$$\frac{\partial \phi}{\partial r}(x,r) = \ln(x)\,x^r \sum_{n=0}^{\infty} a_n(r)x^n + x^r \sum_{n=0}^{\infty} a_n'(r)x^n$$

$$\therefore \frac{\partial \phi}{\partial r}(x,r_2) = \ln(x)\,x^{r_2} \sum_{n=0}^{\infty} a_n(r_2)x^n + x^{r_2} \sum_{n=0}^{\infty} a_n'(r_2)x^n$$

As above, $x^{r_2}\sum_{n=0}^{\infty} a_n(r_2)x^n$, with $a_0(r) = r - r_2$, is the same as $a\,y_1(x)$, where $a = a_N(r_2)$, where $a_n(r)$ were determined using [1] above, and $a_0(r) = r - r_2$.

$\therefore$ $$\boxed{y_2(x) = a\ln(x)\,y_1(x) + x^{r_2}\sum_{n=0}^{\infty} a_n'(r_2)\,x^n}$$

where $a_0(r) = r - r_2$

The $a_n(r)$ in the text was not derived using $a_0(r) = r - r_2$. $\therefore$ This factor must be multiplied by the text's notation of $a_n(r)$ to be equivalent to the above $a_n(r)$.

$\therefore$ The $\frac{d}{dr}a_n(r)$ above is equivalent to the text $\frac{d}{dr}(r-r_2)a_n(r)$

$=$

The case of $r_1 - r_2 = N$, a positive integer

After solving for $y_1(x)$, when using the substitution method for finding $y_2(x)$, $y_2(x) = a y_1(x) \ln(x) + x^{r_2} \sum_{n=0}^{\infty} c_n x^n$, one encounters situations in which a coefficient, c_i, is undetermined, and so an arbitrary assignment of c_i is made, so all subsequent coefficients can be determined based on the recurrence relation.

Let $y_2(x)$ be one solution from one assignment of c_i, and let $y_3(x)$ be another solution from a different assignment of c_i. $\therefore L[y_2] = L[y_3] = 0$

Let $y_2(x) = a y_1 \ln(x) + x^{r_2} \sum_{n=0}^{\infty} c_n x^n$

$y_3(x) = a y_1 \ln(x) + x^{r_2} \sum_{n=0}^{\infty} d_n x^n$

Note $c_0 = d_0$, $c_1 = d_1$, ..., $c_{i-1} = d_{i-1}$, but $c_i \neq d_i$

$\therefore$ Since $L[y_2 - y_3] = 0$,

$$L\left[x^{r_2} \sum_{n=0}^{\infty} (c_n - d_n) x^n\right] = 0$$

Let $e_n = c_n - d_n$, $n \geq 0$.

From above, $e_0 = e_1 = e_2 = \ldots = e_{i-1} = 0$.

Note also for $y_1(x) = x^{r_1} \sum_{n=0}^{\infty} a_n x^n$, $L[y_1] = 0$, $a_0 \neq 0$

$\therefore$ From $L[(y_2 - y_3) - s y_1] = 0$, $s \neq 0$, a constant

$$x^{r_2} \sum_{n=0}^{\infty} e_n x^n = s x^{r_1} \sum_{n=0}^{\infty} a_n x^n$$

$$\therefore \sum_{n=0}^{\infty} e_n x^n = s x^{r_1 - r_2} \sum_{n=0}^{\infty} a_n x^n = s x^N \sum_{n=0}^{\infty} a_n x^n$$

$$\therefore \sum_{n=0}^{\infty} e_n x^n = \sum_{n=i}^{\infty} e_n x^n = s \sum_{n=0}^{\infty} a_n x^{n+N}$$

Since $a_0 \neq 0$, $e_i x^i = s a_0 x^N$, which means $i = N$.

Since e_i is the first non-zero coefficient of $y_2(x) - y_3(x)$, if $i > N$, then $s a_0 = 0$, a contradiction since $a_0 \neq 0$. If $i < N$, then $e_i = 0$, also a contradiction. $\therefore i = N$.

Thus, the coefficient of arbitrary assignment, if it occurs, is at c_N for $y_2(x)$.

$$\therefore \sum_{n=N}^{\infty} e_n x^n = s \sum_{n=0}^{\infty} a_n x^{n+N} = s x^N \sum_{n=0}^{\infty} a_n x^n$$

Multiplying both sides by x^{r_2},

$$x^{r_2} \sum_{n=N}^{\infty} e_n x^n = s\, x^{r_2+N} \sum_{n=0}^{\infty} a_n x^n = s x^{r_1} \sum_{n=0}^{\infty} a_n x^n = s\, y_1(x)$$

Note that $y_2(x) - y_3(x) = x^{r_2} \sum_{n=N}^{\infty} e_n x^n$

$\therefore$ $y_2(x) - y_3(x) = s\, y_1(x)$, or $y_3 = y_2 - s\, y_1(x)$

$\therefore$ Assigning a different value to the undetermined coeffient only introduces adding or subtracting a multiple of $y_1(x)$.

5.7 Bessel's Equation

Page 232, differention of:

$$f(x) = (x-\alpha_1)^{\beta_1}(x-\alpha_2)^{\beta_2}(x-\alpha_3)^{\beta_3}\cdots(x-\alpha_n)^{\beta_n},$$

is logarithmic differentiation:

$$\log f(x) = \beta_1 \log(x-\alpha_1) + \ldots + \beta_n \log(x-\alpha_n)$$

$\therefore$ Taking derivatives of both sides,

$$\frac{1}{f(x)} \cdot f'(x) = \frac{\beta_1}{x-\alpha_1} + \ldots + \frac{\beta_n}{x-\alpha_n}$$

$$\therefore f'(x) = \left[\frac{\beta_1}{x-\alpha_1} + \ldots + \frac{\beta_n}{x-\alpha_n}\right] f(x)$$

From the text: $a_{2m}(r) = \dfrac{(-1)^m a_0}{(r+2)^2\cdots(r+2m)^2},$

for each m, can choose a_0 to be positive or negative so can take $\log[(-1)^m a_0]$. This is a constant so derivative is 0.

$$\therefore \log[a_{2m}(r)] = \log[(-1)^m a_0] - 2[\log(r+2) + \ldots + \log(r+2m)]$$

$\therefore$ For any m, $\dfrac{a_{2m}'(r)}{a_{2m}(r)} = -2\left[\dfrac{1}{r+2} + \ldots + \dfrac{1}{r+2m}\right]$

Page 234-235

Bessel Equation of Order $\frac{1}{2}$

$$L[y] = x^2 y'' + xy' + \left(x^2 - \frac{1}{4}\right) y = 0. \qquad (16)$$

Note $x p(x) = 1$, so $p_0 = 1$, $p_1 = 0$

$x^2 q(x) = -\frac{1}{4} + x^2$, so $q_0 = -\frac{1}{4}$, $q_1 = 0$, $q_2 = 1$

$\therefore$ from $$F(r+n)a_n + \sum_{k=0}^{n-1} a_k((r+k)\,p_{n-k} + q_{n-k}) = 0, \quad n \geq 1. \qquad (8)$$

$$F(r+1)\,a_1 = -\sum_{k=0}^{0} a_0 [(r+0)p_1 + q_1]$$

$$\therefore F(r+1) = (r+1)r + (r+1)p_0 + q_0$$

$$= r^2 + r + r + 1 - \frac{1}{4} = (r+1)^2 - \frac{1}{4}$$

$\therefore F(-\frac{1}{2}+1) = 0$ But $p_1 = q_1 = 0$

$\therefore 0 \cdot a_1 = 0$

$\therefore a_2, a_3, \ldots$ can be assigned, so don't need log term, and $y_2(x) = x^{-\frac{1}{2}} \sum_{n=0}^{\infty} a_n x^n$

This is the special situation in Coddington, p. 163 at bottom.

From equation (17) on p. 235,

$$L[\phi](x,r) = \left(r^2 - \tfrac{1}{4}\right)a_0 x^r + \left[(r+1)^2 - \tfrac{1}{4}\right]a_1 x^{r+1}$$

$$+ \sum_{n=2}^{\infty}\left[\left[(r+n)^2 - \tfrac{1}{4}\right]a_n + a_{n-2}\right]x^{r+n} = 0$$

For $r = -\frac{1}{2}$,

$$L[\phi](x,-\tfrac{1}{2}) = \sum_{n=2}^{\infty}\left[\left[\left(n-\tfrac{1}{2}\right)^2 - \tfrac{1}{4}\right]a_n + a_{n-2}\right]x^{r+n} = 0$$

$$= \sum_{n=2}^{\infty}\left[\left[n(n-1)\right]a_n + a_{n-2}\right]x^{r+n} = 0$$

$$\therefore \quad a_n = \frac{-a_{n-2}}{n(n-1)}, \quad n \geq 2, \quad a_0, a_1 \text{ independent}$$

$$\therefore \quad a_2 = \frac{-a_0}{2\cdot 1} \qquad\qquad a_3 = \frac{-a_1}{3\cdot 2}$$

$$a_4 = \frac{-a_2}{4\cdot 3} = \frac{a_0}{4\cdot3\cdot2\cdot1} \qquad\qquad a_5 = \frac{-a_3}{5\cdot 4} = \frac{a_1}{5\cdot4\cdot3\cdot2\cdot1}$$

$$\therefore \quad a_{2n} = \frac{(-1)^n a_0}{(2n)!} \qquad\qquad a_{2n+1} = \frac{(-1)^n a_1}{(2n+1)!}, \quad n = 1,2,3,\ldots$$

Bessel Equation of Order One

The Bessel function of the first kind of order one, denoted by J_1, is obtained by choosing $a_0 = 1/2$. Hence

$$J_1(x) = \frac{x}{2}\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m}}{2^{2m}(m+1)!m!}. \tag{27}$$

The series converges absolutely for all x, so the function J_1 is analytic everywhere.

$$\therefore a_m = \frac{(-1)^m x^{2m+1}}{2\cdot 2^{2m}(m+1)!m!}$$

$$\left|\frac{a_{m+1}}{a_m}\right| = \left|\frac{(-1)^{m+1} x^{2m+3}}{2\cdot 2^{2m+2}(m+2)!(m+1)!}\cdot\frac{2\cdot 2^{2m}(m+1)!m!}{(-1)^m x^{2m+1}}\right|$$

$$= \left|\frac{x^2}{2^2(m+2)(m+1)}\right|$$

$$\therefore \text{as } \lim_{m\to\infty}\left|\frac{a_{m+1}}{a_m}\right| = 0 \text{ for all } x$$

For $y_2(x)$: $$y_2(x) = aJ_1(x)\ln x + x^{-1}\left(1+\sum_{n=1}^{\infty}c_n x^n\right), \quad x > 0. \tag{28}$$

$$\therefore y_2(x) = aJ_1\ln(x) + \sum_{n=0}^{\infty}c_n x^{n-1}, \quad x>0, \quad c_0 = 1$$

$$\therefore y_2' = aJ_1'\ln(x) + aJ_1\left(\frac{1}{x}\right) + \sum_{n=0}^{\infty}(n-1)c_n x^{n-2}$$

$$y_2'' = aJ_1''\ln(x) + 2aJ_1'\left(\frac{1}{x}\right) + aJ_1\left(-\frac{1}{x^2}\right) + \sum_{n=0}^{\infty}(n-2)(n-1)c_n x^{n-3}$$

Substitute in: $$L[y] = x^2y'' + xy' + (x^2-1)y = 0. \tag{23}$$

$$\therefore\ x^2 y_2'' = a x^2 J_1'' \ln(x) + 2 a x J_1' - a J_1 + \sum_{n=0}^{\infty} (n-2)(n-1) c_n x^{n-1}$$

cancel cancel

$$x y_2' = a x J_1' \ln(x) + a J_1 + \sum_{n=0}^{\infty} (n-1) c_n x^{n-1}$$

$$(x^2-1) y_2 = a(x^2-1) J_1 \ln(x) + \sum_{n=0}^{\infty} c_n x^{n+1} - \sum_{n=0}^{\infty} c_n x^{n-1}$$

$$\therefore\ L[y_2] = 2 a x J_1' + \sum_{n=0}^{\infty} \underbrace{\left[(n-2)(n-1) + (n-1) - 1\right]}_{(n-1)^2 - 1 = n(n-2)} c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^{n+1}$$

$$\therefore\ \sum_{n=0}^{\infty} n(n-2) c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^{n+1} = -2 a x J_1', \quad c_0 = 1$$

$$\therefore\ -c_1 + 0\cdot c_2 x + \sum_{n=3}^{\infty} n(n-2) c_n x^{n-1} + c_0 x + \sum_{n=1}^{\infty} c_n x^{n+1} = -2 a x J_1'$$

$$\therefore\ -c_1 + (0\cdot c_2 + c_0) x + \sum_{n=2}^{\infty} (n+1)(n-1) c_{n+1} x^n + \sum_{n=2}^{\infty} c_{n-1} x^n = -2 a x J_1'$$

$$\therefore\ -c_1 + c_0 x + \sum_{n=2}^{\infty} \left[(n^2-1) c_{n+1} + c_{n-1}\right] x^n = -2 a x J_1', \quad c_0 = 1$$

From $$J_1(x) = \frac{x}{2} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}(m+1)!\,m!}. \qquad (27)$$

$$J_1(x) = \frac{1}{2} \sum_{m=0}^{\infty} \frac{(-1)\, x^{2m+1}}{2^{2m}(m+1)!\,m!}$$

$$\therefore\ J_1' = \frac{1}{2} \sum_{m=0}^{\infty} \frac{(-1)^m (2m+1) x^{2m}}{2^{2m}(m+1)!\,m!}$$

$$\therefore\ -2ax J_1' = -a \sum_{m=0}^{\infty} \frac{(-1)^m (2m+1) x^{2m+1}}{2^{2m}(m+1)!\, m!}$$

$$\therefore\ -c_1 + c_0 x + \sum_{n=2}^{\infty} \left[(n-1)^2 c_{n+1} + c_{n-1}\right] x^n = -a \sum_{m=0}^{\infty} \frac{(-1)^m (2m+1)\, x^{2m+1}}{2^{2m}(m+1)!\, m!}$$

$$= -a\left(x + \sum_{m=1}^{\infty} \frac{(-1)^m (2m+1)\, x^{2m+1}}{2^{2m}(m+1)!\, m!} \right)$$

$\therefore\ c_1 = 0,\ c_0 = 1 = -a,\ \therefore\ a = -1$

Since only odd powers on right, even-powered coefficients on left are 0: $(n-1)^2 c_{n+1} + c_{n-1} = 0,\ n = 2, 4, \ldots$

$\therefore$ Since $c_1 = 0$, $c_1 = c_3 = c_5 = \ldots = 0$.

$$\therefore\ \sum_{n=3}^{\infty} \left[(n^2-1)\, c_{n+1} + c_{n-1}\right] x^n = \sum_{m=1}^{\infty} \frac{(-1)^m (2m+1)}{2^{2m}(m+1)!\, m!}\, x^{2m+1}$$

$\therefore$ let $n = 2m+1,\ m = 1, 2, 3, \ldots$

$$\sum_{m=1}^{\infty} \left[\left((2m+1)^2 - 1\right) c_{2m+2} + c_{2m}\right] x^{2m+1} = \sum_{m=1}^{\infty} \frac{(-1)^m (2m+1)}{2^{2m}(m+1)!\, m!}\, x^{2m+1}$$

$$\therefore\ \left[(2m+1)^2 - 1\right] c_{m+2} + c_{2m} = \frac{(-1)^m (2m+1)}{2^{2m}(m+1)!\, m!}, \quad m = 1, 2, 3, \ldots$$

This is equation (31) on p. 237 of the text.

Section 6.1 Definition of the Laplace Transform

Note that for $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$,

$$\mathcal{L}\{f(at)\} = \int_0^\infty e^{-st} f(at)\,dt \text{ , and}$$

$$\mathcal{L}\{f(t^2)\} = \int_0^\infty e^{-st} f(t^2)\,dt$$

So, what is between the brackets $\{\}$ only affects the function, $f()$, in the integrand. It does not affect the "t" part of e^{-st} or the t of dt. The "dt" just specifies what is the variable of integration.

So if $f(t) = \sin(t)$, $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}\sin(t)\,dt$, and $\mathcal{L}\{f(at)\} = \int_0^\infty e^{-st}\sin(at)\,dt$

Also, the "t" part of $\mathcal{L}\{f(t)\}$ is the same as the variable of integration, so $\mathcal{L}\{f(x)\} = \int_0^\infty e^{-sx} f(x)\,dx$

Problem # 23 states:

23. The Gamma Function. The gamma function is denoted by $\Gamma(p)$ and is defined by the integral

$$\Gamma(p+1) = \int_0^\infty e^{-x} x^p \, dx. \qquad (7)$$

The integral converges as $x \to \infty$ for all p. For $p < 0$ it is also improper at $x = 0$, because the integrand becomes unbounded as $x \to 0$. However, the integral can be shown to converge at $x = 0$ for $p > -1$.

(1) $p > 0$. There exists an integer n s.t. $p < n$

Consider $\int_0^\infty e^{-x} x^n \, dx = -e^{-x} x^n \Big|_0^\infty + n \int_0^\infty e^{-x} x^{n-1} \, dx$

$$= \underbrace{-\lim_{x \to \infty} \frac{x^n}{e^x}}_{=0 \text{ by L'Hopital's Rule}} - 0 + n \int_0^\infty e^{-x} x^{n-1} \, dx = n \int_0^\infty e^{-x} x^{n-1} \, dx$$

By induction, $\int_0^\infty e^{-x} x^n \, dx = n! \int_0^\infty e^{-x} \, dx$

$$= n! \left[-e^{-x} \right]_0^\infty = n!$$

$\therefore$ Since $e^{-x} x^p < e^{-x} x^n$, $\int_0^\infty e^{-x} x^p \, dx$ is bounded by a convergent integral, and by Th. 6.1.1, converges

(2) $p = 0$. $\int_0^\infty e^{-x} \, dx = = -e^{-x} \Big|_0^\infty = 1$

(3) $p<0$. $\int_0^\infty e^{-x}x^p\,dx = \int_0^1 e^{-x}x^p\,dx + \int_1^\infty e^{-x}x^p\,dx$

Consider $\int_1^\infty e^{-x}x^p\,dx = \int_1^\infty \frac{dx}{e^x x^{-p}}$

For $1 \le x$, $1 \le x^{-p}$, so $e^x \le e^x x^{-p} \Rightarrow \frac{1}{e^x x^{-p}} \le \frac{1}{e^x}$

$\therefore \int_1^\infty \frac{dx}{e^x x^{-p}} \le \int_1^\infty \frac{dx}{e^x} = -e^{-x}\Big|_0^\infty = 1$

$\therefore \int_1^\infty e^{-x}x^p\,dx$ converges for all $p<0$

$\therefore$ Convergence depends on $\int_0^1 e^{-x}x^p\,dx$

Note that $e^{-x} < e$ on $[0,1]$ since $1 < e^{x+1}$

$\therefore e^{-x}x^p < ex^p$ for $x \in (0,1]$.

$e\int_0^1 x^p\,dx = \frac{e}{p+1}\left[x^{p+1}\right]_0^1 = \frac{e}{p+1}\left[1 - \lim_{x\to 0} x^{p+1}\right]$

If $-1<p<0$, $0<p+1<1$, so $\lim_{x\to 0} x^{p+1} = 0$

$\therefore \int_0^1 ex^p\,dx$ converges for $-1<p<0$, and so

$\int_0^1 e^{-x}x^p\,dx$ converges for $-1<p<0$.

(1)-(3) $\Rightarrow \int_0^\infty e^{-x}x^p\,dx$ converges for all $p>-1$.

Pages 252-253: linearity of $\mathcal{L}^{-1}$

If $f(t)$ is continuous and $F(s) = \mathcal{L}\{f(t)\}$,

then by uniqueness, $\mathcal{L}^{-1}\{F(s)\} = f(t)$

$\therefore \mathcal{L}\{\mathcal{L}^{-1}\{F(s)\}\} = \mathcal{L}\{f(t)\} = F(s)$

and $\mathcal{L}^{-1}\{\mathcal{L}\{f(t)\}\} = \mathcal{L}^{-1}\{F(s)\} = f(t)$

Now suppose $F(s) = F_1(s) + \dots + F_n(s)$

Let $f(t) = \mathcal{L}^{-1}\{F(s)\}$, $f_1(t) = \mathcal{L}^{-1}\{F_1(s)\}$, ..., $f_n(t) = \mathcal{L}^{-1}\{F_n(s)\}$

and suppose each $f(t), f_1(t), \dots, f_n(t)$ are continuous.

$\therefore f(t) = \mathcal{L}^{-1}\{F_1(s) + \dots + F_n(s)\}$

By linearity of $\mathcal{L}$, $\mathcal{L}\{f_1(t) + \dots + f_n(t)\}$

$$= \mathcal{L}\{\mathcal{L}^{-1}\{F_1(s)\} + \dots + \mathcal{L}^{-1}\{F_n(s)\}\}$$

$$= \mathcal{L}\{\mathcal{L}^{-1}\{F_1(s)\}\} + \dots + \mathcal{L}\{\mathcal{L}^{-1}\{F_n(s)\}\}$$

$$= F_1(s) + \dots + F_n(s)$$

$$= F(s) = \mathcal{L}\{f(t)\}$$

$\therefore$ $\mathcal{L}\{f(t)\}$ and $\mathcal{L}\{f_1(t) + \dots + f_n(t)\}$ have the same transform, and so $f(t) = f_1(t) + \dots + f_n(t)$ by uniqueness of transforms, since $f, f_1, \dots, f_n$ are continuous.

$\therefore$ $\mathcal{L}^{-1}\{F_1(s)\} + \dots + \mathcal{L}^{-1}\{F_n(s)\} =$

$$f_1(t) + \dots + f_n(t) = f(t) = \mathcal{L}^{-1}\{F(s)\}$$

$$= \mathcal{L}^{-1}\{F_1(s) + \dots + F_n(s)\}$$

$\therefore$ $\mathcal{L}^{-1}$ is a linear operator.

Section 6.3 Step Functions

Example 3, p. 260

If the function $f(t)$ had been specified as

$$f(t) = \begin{cases} \sin(t) & 0 \le t < \frac{\pi}{4} \\ \cos\left(\frac{\pi}{4} - t\right) & \frac{\pi}{4} \le t \end{cases}$$

then $f(t)$ would be specified as

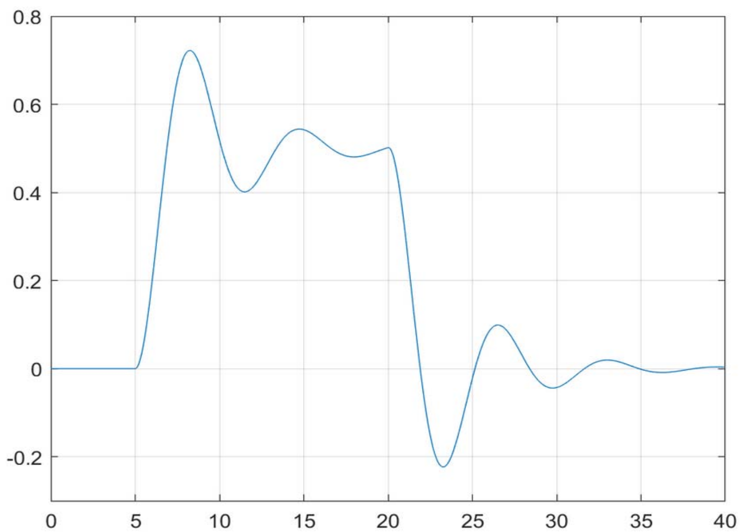
$$[u_0(t) - u_{\pi/4}(t)]\sin(t) + u_{\pi/4}\cos\left(t - \frac{\pi}{4}\right)$$

$\hookrightarrow$ clips $\sin(t)$ off at $t = \frac{\pi}{4}$

Section 6.4 Differential Equations with Discontinuous Forcing Functions

MATLAB code to display Example 1 is:

```
clear, clc
syms t
h(t) = 1/2 - (exp(-t/4)/2)*(cos(t*sqrt(15)/4) + (1/sqrt(15))*sin(t*sqrt(15)/4));
y(t) = heaviside(t-5)*h(t-5) - heaviside(t-20)*h(t-20);
fplot(y(t), [0,40])
ylim([-0.3, 0.8])
grid on
y_5 = y(5)
h1(t) = diff(h(t),t,1);
y1(t) = heaviside(t-5)*h1(t-5);
y1_5 = y1(5)
h_15 = vpa(h(15))
h1_15 = vpa(h1(15))
y2(t) = diff(y(t),t,2);
l = vpa(limit(y2(t),t,20,'left'))
r = vpa(limit(y2(t),t,20,'right'))
r-l
```



y_5 = 0

y1_5 = 0

h_15 = $0.50162057729687970867083762933591$

h1_15 = $0.011248565972730313813257298127588$

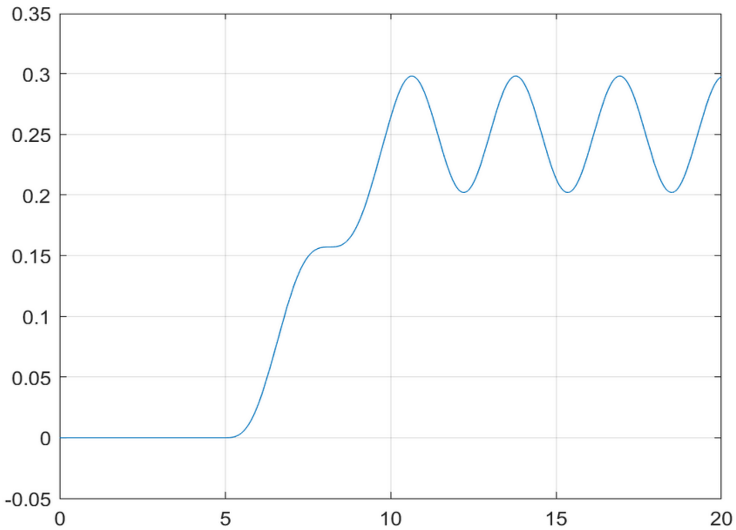
l = $-0.0072448602832448655774662783997024$

r = $-0.5072448602832448655774662783997$

ans = -0.5

MATLAB code for Example 2:

```
clear, clc
syms s t
d = s^2*(s^2+4)
partfrac(1/d)
h(t) = t/4 - (1/8)*sin(2*t);
y(t) = (1/5)*(heaviside(t-5)*h(t-5) - heaviside(t-10)*h(t-10));
fplot(y(t), [0,20])
grid on
ylim([-0.05, 0.35])
y1(t) = diff(y(t),t,1);
mx = vpasolve(y1(t)==0,t,[10, 11])
mn = vpasolve(y1(t)==0,t,[12, 13])
ymax = y(mx)
amp = (y(mx)-y(mn))/2
```



d = $s^2\ (s^2+4)$

ans =

$$\frac{1}{4\,s^2} - \frac{1}{4\ (s^2+4)}$$

mx = 10.64159265358979323846264338328

mn = 12.212388980384689857693965074919

ymax = 0.2979462137331569234446577203078

amp = 0.0479462137331569234446577203078

Chapter 7 Notes

Note Title 11/25/2019

7.4 Basic Theory of Systems of First-order Linear Equations

Analagous to Abel's Theorem (7.4.3), and more explicitly stated:

If $x^{(1)}, \ldots, x^{(n)}$ are solutions to the $n \times n$ system $x' = Px$ on an open interval I, and $W = W[x^{(1)}, \ldots, x^{(n)}](t)$, then

(1) If $x^{(1)}, \ldots, x^{(n)}$ are linearly dependent on I, then $W = 0$ at every point of I.

(2) If $x^{(1)}, \ldots, x^{(n)}$ are linearly independent solutions to $x' = Px$ on I, then $W \neq 0$ at every point of I.

Can't say: if $W = 0$ at every point of I, then $x^{(1)}, \ldots, x^{(n)}$ are dependent.

Example: $\begin{bmatrix} e^t & 1 \\ te^t & t \end{bmatrix}$, $W = 0$ at every point

yet $\begin{bmatrix} e^t \\ te^t \end{bmatrix}, \begin{bmatrix} 1 \\ t \end{bmatrix}$ are independent

This is #13 of section 7.3

Can say: If $W \neq 0$ at every point of I, then

$x^{(1)}, \ldots, x^{(n)}$ are independent.

For if they were dependent, then $W = 0$

at every point of I.

Note: Proof of (2): if $W = 0$ at some point $t_0 \in I$,

so that $c_1 x^{(1)}(t_0) + \ldots + c_n x^{(n)}(t_0) = 0$, then

$c_1 x^{(1)}(t) + \ldots + c_n x^{(n)}(t)$ is equivalent to

$y(t) = 0$ (and $\therefore y(t_0) = 0$), using the

uniqueness theorem. $\therefore$ This uses

continuity of $P(t)$, so all $p_{ij}(t)$.

The proof of (1) is a direct result of the

definition of linear dependence.

Note: The above statements list n solutions

for an $n \times n$ system. This allows use of

$W[x^{(1)}, \dots, x^{(n)}](t)$, which requires a square matrix. The above statements are also true for $K < n$ solutions, replacing linearly dependent for $W = 0$, and linearly independent for $W \neq 0$.

Thus: K solutions, $K < n$, are linearly independent on I if they are linearly independent for any one value of I.

Page 316 states n eigenvectors with n different eigenvalues are independent.

Proof: Consider $c_1 x^{(1)} + c_2 x^{(2)} = 0$ [1]

$\therefore A(c_1 x^{(1)} + c_2 x^{(2)}) = c_1 \lambda_1 x^{(1)} + c_2 \lambda_2 x^{(2)} = 0$ [2]

Multiplying [1] by λ_2,

$$c_1 \lambda_2 x^{(1)} + c_2 \lambda_2 x^{(2)} = 0 \quad [3]$$

Subtracting [3] from [2],

$c_1 (\lambda_1 - \lambda_2) x^{(1)} = 0$, and since $\lambda_1 \neq \lambda_2$,

$c_1 = 0$. Similarly, $c_2 = 0$.

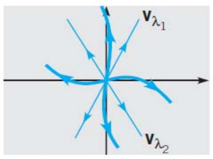
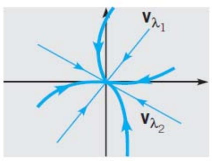
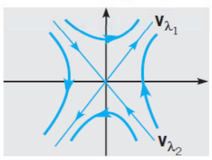
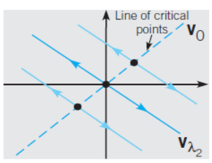
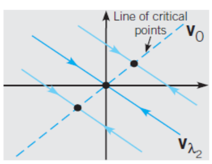
$\therefore x^{(1)}$ and $x^{(2)}$ are independent.

Similarly for $x^{(1)}, \dots, x^{(n)}$

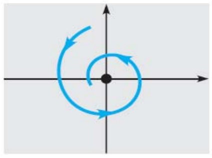
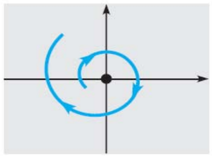
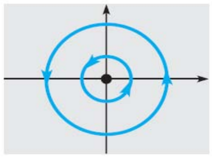
7.6 Complex-Valued Eigenvalues

From Brannon, Boyce, McKibben, 3E, pp. 165, 176

Phase portraits for $\mathbf{x}' = \mathbf{A}\mathbf{x}$ when $\mathbf{A}$ has distinct real eigenvalues.

Eigenvalues	Sample Phase Portrait	Type of Critical Point	Stability
$\lambda_1 \neq \lambda_2$ Both positive		(0, 0) is a **nodal source**.	Unstable
$\lambda_1 \neq \lambda_2$ Both negative		(0, 0) is a **nodal sink**.	Asymptotically stable
$\lambda_1 \neq \lambda_2$ Opposite signs		(0, 0) is a **saddle**.	Unstable
$\lambda_1 = 0$ and $\lambda_2 > 0$			
$\lambda_1 = 0$ and $\lambda_2 < 0$			

Phase portraits for $\mathbf{x}' = \mathbf{A}\mathbf{x}$ when $\mathbf{A}$ has complex eigenvalues.

Eigenvalues	Sample Phase Portrait	Type of Critical Point	Stability
$\lambda = \mu \pm i\nu$ $\mu < 0$		(0, 0) is a **spiral sink**.	Asymptotically stable
$\lambda = \mu \pm i\nu$ $\mu > 0$		(0, 0) is a **spiral source**.	Unstable
$\lambda = \mu \pm i\nu$ $\mu = 0$		(0, 0) is a **center**.	Stable

7.7 Fundamental Matrices

Page 330 at the top states that a fundamental matrix is nonsingular. Note that by def., a fundamental set of solutions means the set of vector solutions is independent at every point in the interval. Independent vector functions can be dependent at every point: for example, $\begin{bmatrix} e^t \\ te^t \end{bmatrix}, \begin{bmatrix} 1 \\ t \end{bmatrix}$. But a fundamental set of solutions to a linear system of differential equations cannot behave this way, by def.

Pages 330-331

Note that $\phi^{-1}(t_0)$ is really $[\phi(t_0)]^{-1}$

There is no such thing as $\phi^{-1}(t) = [\phi(t)]^{-1}$, since in general, $\phi(t)\,\phi^{-1}(t) \neq I$ for every t.

Page 331, Example 2

Original fundamental matrix from Example 1 is $\begin{bmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{bmatrix}$, and at $t=0$, $\begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}$

$\therefore$ Seeking $\begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Page 332 The Matrix exp(At) and Convergence

It is stated that it is possible to show each element of this matrix sum converges for all t as $n \to \infty$. To prove this, need to make definition of the norm of a matrix.

Definition of Norm of Matrix

If A is an $m \times n$ matrix of real or complex entries, the norm of A, denoted by $\|A\|$, is the nonnegative number $\|A\| = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} |a_{ij}| \right)$

Definition of Convergent Series of Matrices

Let $\{A_k\}$ be an infinite sequence of $m \times n$ matrices, whose entries are real or complex numbers. Let $a_{ij}^{(k)}$ be the ij-entry for A_k. Then $\sum_{k=1}^{\infty} A_k$ is convergent if and only if $\sum_{k=1}^{\infty} a_{ij}^{(k)}$ is convergent for every $1 \le i \le m$, $1 \le j \le n$. If $\sum_{k=1}^{\infty} A_k$ is convergent, its sum S is the matrix whose entries $S_{ij} = \sum_{k=1}^{\infty} a_{ij}^{(k)}$.

Lemma

Let A be an $m \times n$ matrix, B an $n \times p$ matrix, both of real or complex numbers.

Then $\|AB\| \le \|A\| \|B\|$.

Proof: By the above definition of norm,

$$\|AB\| = \sum_{i=1}^{m} \left(\sum_{j=1}^{p} |(AB)_{ij}| \right)$$

$$= \sum_{i=1}^{m} \left(\sum_{j=1}^{p} \left| \sum_{k=1}^{n} a_{ik} b_{kj} \right| \right) \quad \text{def. matrix multiplication}$$

$$\leq \sum_{i=1}^{m} \left(\sum_{j=1}^{p} \left[\sum_{k=1}^{n} |a_{ik} b_{kj}| \right] \right) \quad |a+b| \leq |a| + |b|$$

$$= \sum_{i=1}^{m} \left(\sum_{j=1}^{p} \left[\sum_{k=1}^{n} |a_{ik}| |b_{kj}| \right] \right) \quad |ab| = |a||b|$$

row i of A × column j of B

$$= \sum_{i=1}^{m} \Big(|a_{i1}||b_{11}| + |a_{i2}||b_{21}| + \dots + |a_{in}||b_{n1}|$$

$$+ |a_{i1}||b_{12}| + |a_{i2}||b_{22}| + \dots + |a_{in}||b_{n2}|$$

$$+ \vdots$$

$$+ |a_{i1}||b_{1p}| + |a_{i2}||b_{2p}| + \dots + |a_{in}||b_{np}| \Big)$$

$$\updownarrow \qquad \updownarrow \qquad \updownarrow$$

$$= \sum_{i=1}^{m} \left(|a_{i1}| \sum_{j=1}^{p} |b_{1j}| + |a_{i2}| \sum_{j=1}^{p} |b_{2j}| + \dots + |a_{in}| \sum_{j=1}^{p} |b_{nj}| \right)$$

$$= \sum_{i=1}^{m} \left(\sum_{k=1}^{n} \left[|a_{ik}| \sum_{j=1}^{p} |b_{kj}| \right] \right)$$

$$\leq \sum_{i=1}^{m} \left(\sum_{k=1}^{n} \left[|a_{ik}| \, \|B\| \right] \right) \quad \sum_{j=1}^{p} |b_{kj}| \leq \sum_{k=1}^{n} \sum_{j=1}^{p} |b_{kj}| = \|B\|$$

$$= \sum_{i=1}^{m} \left(\|B\| \sum_{k=1}^{n} |a_{ik}| \right) = \|B\| \sum_{i=1}^{m} \left(\sum_{k=1}^{n} |a_{ik}| \right)$$

$$= \|B\| \, \|A\| = \|A\| \, \|B\|$$

$\therefore \|AB\| \le \|A\|\,\|B\|$

Note: $\|A^2\| \le \|A\|^2$, and by induction,

$$\|A^k\| \le \|A\|^k \quad \text{for } k = 1, 2, 3, \ldots$$

Theorem *Test for Convergence of a Matrix Series*

Let $\{A_k\}$ be a sequence of $m \times n$ matrices.

If $\sum_{k=1}^{\infty} \|A_k\|$ converges, then $\sum_{k=1}^{\infty} A_k$ converges.

Proof: Since $|a_{ij}^{(k)}| \le \|A_k\|$ for $1 \le i \le m,\ 1 \le j \le n$, then by the comparison test for series, convergence of $\sum_{k=1}^{\infty} \|A_k\| \Rightarrow$ convergence of $\sum_{k=1}^{\infty} |a_{ij}^{(k)}|$ for each i, j, which is absolute convergence, and this implies convergence of $\sum_{k=1}^{\infty} a_{ij}^{(k)}$ for each i, j. By definition, this means $\sum_{k=1}^{\infty} A_k$ converges.

Theorem *Convergence of exp(A)*

If A is any square matrix of real or complex numbers, then $\sum_{k=0}^{\infty} \frac{A^k}{k!}$ converges.

Proof: Assume $A^0 = I$.

$$\left\| \frac{A^k}{k!} \right\| = \frac{\|A^k\|}{k!} \leq \frac{\|A\|^k}{k!}$$ as noted above.

Letting $x = \|A\|$, the series $\sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x$, and so converges. $\therefore$ By the comparison test, $\sum_{k=0}^{\infty} \left\| \frac{A^k}{k!} \right\|$ converges, and so by the theorem above, $\sum_{k=0}^{\infty} \frac{A^k}{k!}$ converges

By definition,
$$\exp(A) = \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

so $\exp(A)$ converges.

Page 341 - developing $\vec{x}^{(2)}$ solution

Given ρ is a double eigenvalue with just one eigenvector $\vec{\epsilon}$, so that $(A-\rho I)\vec{\epsilon} = \vec{0}$, [1]

assume $\vec{x}^{(2)} = \vec{\epsilon}\, t e^{\rho t} + \vec{n} e^{\rho t}$, $\vec{\epsilon}$ the same evector.

$$\therefore \frac{d}{dt}\vec{x}^{(2)} = \vec{\epsilon} e^{\rho t} + \vec{\epsilon} \rho t e^{\rho t} + \vec{n} \rho e^{\rho t}$$

$$A\vec{x}^{(2)} = A\vec{\epsilon}\, t e^{\rho t} + A\vec{n} e^{\rho t}$$

Equating like terms for all t,

$A\vec{\epsilon}\, t e^{\rho t} = \vec{\epsilon} \rho t e^{\rho t}$, or $A\vec{\epsilon} = \rho\vec{\epsilon}$,

or $(A-\rho I)\vec{\epsilon} = \vec{0}$, true from [1]

i.e., even if $\vec{\epsilon}$ was assumed an unknown vector, it is the vector associated with ρ.

$A\vec{n} e^{\rho t} = \vec{\epsilon} e^{\rho t} + \vec{n}\rho e^{\rho t}$, or $A\vec{n} = \vec{\epsilon} + \rho\vec{n}$,

or $(A-\rho I)\vec{n} = \vec{\epsilon}$ [2]

This always has a non-zero solution

for $\vec{n}$. To see this, note that $(A-\rho I)\vec{n}=\vec{\epsilon}$ is solvable only if the row reduced form of the augmented matrix $[A-\rho I, \vec{\epsilon}]$ has corresponding zeros. The fact that $(A-\rho I)^2\vec{n} = (A-\rho I)\vec{\epsilon} = \vec{0}$ means that there are rows in $A-\rho I$ that can serve as rows in the elementary matrix E; that produces the row reduced form of $[A-\rho I, \vec{\epsilon}]$, and these rows produce corresponding zeros in $[A-\rho I, \vec{\epsilon}]$.

Note: To find $\vec{x}^{(2)}$, just immediately jump to $(A-\rho I)\vec{n} = \vec{\epsilon}$ to find $\vec{n}$. Also note $\vec{n}$ is <u>not</u> an eigenvector, and so should not be scaled or normalized, like $\vec{\epsilon}$ can be.

8.1 the Euler or Tangent Line Method

On the bottom of p. 359, it is stated,

Then, making use of a Taylor polynomial with a remainder to expand ϕ about t_n, we obtain

$$\phi(t_n + h) = \phi(t_n) + \phi'(t_n)h + \frac{1}{2}\phi''(\overline{t_n})h^2, \tag{19}$$

This derives from use of the Lagrange form of the remainder in Taylor's Theorem. From Purcell,

21.8.2 Theorem. (Lagrange's Form of the Remainder). *If $f^{(n+1)}$ is continuous in $(b - r, b + r)$, where $0 \le r \le \infty$, and x is in this interval, then the Lagrange form of the remainder in Taylor's theorem (21.7.1) is*

$$R_n(x) = \frac{(x - b)^{n+1}}{(n + 1)!} \cdot f^{(n+1)}(\mu_n),$$

where μ_n is some number between b and x.

If you substitute $h = x - b$, then

$$R_n(b+h) = \frac{h^{n+1}}{(n+1)!} f^{(n+1)}(\mu_n)$$

If just taking the Taylor series to $n=1$, then

$R_1(b+h) = \frac{h^2}{2!} f''(\mu_1)$, hence the $\frac{h^2}{2}\phi''(\bar{t}_n)$ factor in equation (19).

On p. 361, equation (29)

$$1.0617 \cong \frac{19e^{3.8}(0.0025)}{2} \leq e_{20} \leq \frac{19e^{4}(0.0025)}{2} \cong 1.2967 \qquad (29)$$

comes from the fact that $n=20 \Rightarrow 20(0.05)=1$

$\therefore e^{4t_{20}} = e^{4(1)} = e^4$. $n=19 \Rightarrow 4(19)(0.05) = 3.8$

$\therefore e^{3.8}$. $t_{19} < \bar{t}_{19} < t_{19}+h = t_{20}$

Similarly for (30):

$$57.96 \cong \frac{19e^{7.8}(0.0025)}{2} \leq e_{40} \leq \frac{19e^{8}(0.0025)}{2} \cong 70.80 \qquad (30)$$

$t = 1.95 \Rightarrow n = \frac{1.95}{0.05} = 39$, $t = 2.00 \Rightarrow n = 40$

$\therefore 4\bar{t}_{40} \Rightarrow t_{39} < \bar{t}_{39} < t_{40} \Rightarrow 1.95 < \bar{t}_{39} < 2.00$

$\Rightarrow e^{4(1.95)} < e^{4(\bar{t}_{39})} < e^{4(2.00)}$

or, $e^{7.8} < e^{4(\bar{t}_{39})} < e^{8}$

as $e_{40} = \frac{19}{2} e^{4\bar{t}_{39}} h^2$, $t_{39} < \bar{t}_{39} < t_{40}$

8.4 Multistep Methods

Adams Methods

Second-order Adams-Bashforth formula

From the integration, and using $A = \frac{f_n - f_{n-1}}{h}$

and $B = \frac{f_{n-1} t_n - f_n t_{n-1}}{h}$,

$$y_{n+1} - y_n = \frac{A}{2}\left(t_{n+1}^2 - t_n^2\right) + B(\underbrace{t_{n+1} - t_n}_{=h})$$

$$= \frac{f_n - f_{n-1}}{2h}(t_{n+1} + t_n)(\underbrace{t_{n+1} - t_n}_{=h}) + \frac{(f_{n-1} t_n - f_n t_{n-1})h}{h}$$

$$= \frac{(f_n - f_{n-1})(t_{n+1} + t_n)}{2} + \frac{2(f_{n-1} t_n - f_n t_{n-1})}{2}$$

$$= \frac{f_n t_{n+1} + f_n t_n - f_{n-1} t_{n+1} - f_{n-1} t_n + 2 f_{n-1} t_n - 2 f_n t_{n-1}}{2}$$

$$= \frac{f_n (t_{n+1} + t_n - 2t_{n-1}) - f_{n-1}(t_{n+1} + t_n - 2t_n)}{2}$$

$$= \frac{f_n (\underbrace{t_{n+1} - t_{n-1}}_{=2h} + \underbrace{t_n - t_{n-1}}_{=h}) - f_{n-1}(\underbrace{t_{n+1} - t_n}_{=h})}{2}$$

$$= \frac{3h f_n - h f_{n-1}}{2} = \frac{3}{2} h f_n - \frac{1}{2} h f_{n-1}$$

MATLAB can be used to obtain the formulas for Adams-Bashforth methods. For example, for third order Adams-Bashforth, let

$P_2(t) = At^2 + Bt + C$. $\therefore$ need three points to determine A, B, C, using $P_2(t_n) = \phi'(t_n) = f(t_n, y_n)$

$$\therefore \begin{bmatrix} t_n^2 & t_n & 1 \\ t_{n-1}^2 & t_{n-1} & 1 \\ t_{n-2}^2 & t_{n-2} & 1 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} f(t_n, y_n) \\ f(t_{n-1}, y_{n-1}) \\ f(t_{n-2}, y_{n-2}) \end{bmatrix}$$

MATLAB can solve for A, B, C.

$$\therefore \quad y_{n+1} - y_n = \int_{t_n}^{t_{n+1}} At^2 + Bt + C \, dt$$

and MATLAB can solve for that as well. The Key is to reduce the number of variables for MATLAB in order to obtain a simplified formula.

$\therefore$ Use $t_{n-1} = t_n - h$, $t_{n-2} = t_n - 2h$.

The code is on the next page.

```
clear
syms t tn h fn fn1 fn2
A = [ tn^2,      tn,     1;
     (tn-h)^2,   tn-h,   1;
     (tn-2*h)^2, tn-2*h, 1];
B = [fn; fn1; fn2];
X = A\B  % solve AX = B
% X holds coefficients of the polynomial
% integrate the polynomial from tn to tn+h
int(X(1)*t^2 + X(2)*t + X(3), t, tn, tn+h)
```

X =

$$\begin{pmatrix} \frac{\mathrm{fn} - 2\,\mathrm{fn}_1 + \mathrm{fn}_2}{2\,h^2} \\ \frac{3\,\mathrm{fn}\,h - 4\,\mathrm{fn}_1\,h + \mathrm{fn}_2\,h - 2\,\mathrm{fn}\,\mathrm{tn} + 4\,\mathrm{fn}_1\,\mathrm{tn} - 2\,\mathrm{fn}_2\,\mathrm{tn}}{2\,h^2} \\ \frac{2\,\mathrm{fn}\,h^2 + \mathrm{fn}\,\mathrm{tn}^2 - 2\,\mathrm{fn}_1\,\mathrm{tn}^2 + \mathrm{fn}_2\,\mathrm{tn}^2 - 3\,\mathrm{fn}\,h\,\mathrm{tn} + 4\,\mathrm{fn}_1\,h\,\mathrm{tn} - \mathrm{fn}_2\,h\,\mathrm{tn}}{2\,h^2} \end{pmatrix}$$

ans =

$$\frac{h\ (23\,\mathrm{fn} - 16\,\mathrm{fn}_1 + 5\,\mathrm{fn}_2)}{12}$$

Amazingly, MATLAB does all the algebraic simplification.

For 4th order Adams-Bashforth,

```
clear
syms t tn h fn fn1 fn2 fn3
A = [ tn^3,       tn^2,       tn,     1;
     (tn-h)^3,   (tn-h)^2,   tn-h,   1;
     (tn-2*h)^3, (tn-2*h)^2, tn-2*h, 1;
     (tn-3*h)^3, (tn-3*h)^2, tn-3*h, 1];
B = [fn; fn1; fn2; fn3];
X = A\B  % solve AX = B
% X holds coefficients of the polynomial
% integrate the polynomial from tn to tn+h
int(X(1)*t^3 + X(2)*t^2 + X(3)*t + X(4), t, tn, tn+h)
```

X =

$$\begin{pmatrix} \frac{\mathrm{fn} - 3\,\mathrm{fn}_1 + 3\,\mathrm{fn}_2 - \mathrm{fn}_3}{6\,h^3} \\ \frac{2\,\mathrm{fn}\,h - 5\,\mathrm{fn}_1\,h + 4\,\mathrm{fn}_2\,h - \mathrm{fn}_3\,h - \mathrm{fn}\,\mathrm{tn} + 3\,\mathrm{fn}_1\,\mathrm{tn} - 3\,\mathrm{fn}_2\,\mathrm{tn} + \mathrm{fn}_3\,\mathrm{tn}}{2\,h^3} \\ \frac{11\,\mathrm{fn}\,h^2 - 18\,\mathrm{fn}_1\,h^2 + 9\,\mathrm{fn}_2\,h^2 - 2\,\mathrm{fn}_3\,h^2 + 3\,\mathrm{fn}\,\mathrm{tn}^2 - 9\,\mathrm{fn}_1\,\mathrm{tn}^2 + 9\,\mathrm{fn}_2\,\mathrm{tn}^2 - 3\,\mathrm{fn}_3\,\mathrm{tn}^2 - 12\,\mathrm{fn}\,h\,\mathrm{tn} + 30\,\mathrm{fn}_1\,h\,\mathrm{tn} - 24\,\mathrm{fn}_2\,h\,\mathrm{tn} + 6\,\mathrm{fn}_3\,h\,\mathrm{tn}}{6\,h^3} \\ \frac{6\,\mathrm{fn}\,h^3 - \mathrm{fn}\,\mathrm{tn}^3 + 3\,\mathrm{fn}_1\,\mathrm{tn}^3 - 3\,\mathrm{fn}_2\,\mathrm{tn}^3 + \mathrm{fn}_3\,\mathrm{tn}^3 + 6\,\mathrm{fn}\,h\,\mathrm{tn}^2 - 11\,\mathrm{fn}\,h^2\,\mathrm{tn} - 15\,\mathrm{fn}_1\,h\,\mathrm{tn}^2 + 18\,\mathrm{fn}_1\,h^2\,\mathrm{tn} + 12\,\mathrm{fn}_2\,h\,\mathrm{tn}^2 - 9\,\mathrm{fn}_2\,h^2\,\mathrm{tn} - 3\,\mathrm{fn}_3\,h\,\mathrm{tn}^2 + 2\,\mathrm{fn}_3\,h^2\,\mathrm{tn}}{6\,h^3} \end{pmatrix}$$

ans =

$$\frac{h\ (55\,\mathrm{fn} - 59\,\mathrm{fn}_1 + 37\,\mathrm{fn}_2 - 9\,\mathrm{fn}_3)}{24}$$

Backward Differentiation Formulas

Let $P_1(t) = At + B$, where $P_1(t) \approx \phi(t)$, and $\phi'(t) = f(t, \phi)$

$$\therefore P_1'(t_n) \approx \phi'(t_n) = f(t_n, \phi(t_n)) \approx f(t_n, y_n)$$

$$\therefore P_1'(t_{n+1}) = f(t_{n+1}, y_{n+1})$$

From $\left.\begin{aligned} At_n + B &= y_n \\ At_{n+1} + B &= y_{n+1} \end{aligned}\right\}$ $A(t_{n+1} - t_n) = Ah = y_{n+1} - y_n$

$$\therefore A = \frac{y_{n+1} - y_n}{h}$$

$$\therefore B = y_n - At_n = y_n - \frac{y_{n+1} - y_n}{h} t_n$$

$$= \frac{y_n h - y_{n+1} t_n + y_n t_n}{h} = \frac{y_n(h + t_n) - y_{n+1} t_n}{h}$$

$$= \frac{y_n t_{n+1} - y_{n+1} t_n}{h}$$

Since $P_1(t) = At + B$, $P_1'(t) = A$. $\therefore P_1'(t_{n+1}) = A$

$\therefore P_1'(t_{n+1}) = f(t_{n+1}, y_{n+1}) = \dfrac{y_{n+1} - y_n}{h}$, so

$$y_{n+1} = y_n + hf(t_{n+1}, y_{n+1})$$

which is just the backward Euler method.

A 3rd order backward differentiation formula is:

```
clear
syms t tn h yp1 yn yn1 yn2 fp1
E = [ (tn+h)^3,    (tn+h)^2,   tn+h,  1;
          tn^3,        tn^2,     tn,  1;
      (tn-h)^3,    (tn-h)^2,   tn-h,  1;
    (tn-2*h)^3, (tn-2*h)^2, tn-2*h,  1];
F = [yp1; yn; yn1; yn2];
x = E\F;  % solve Ex = F
% x holds coefficients of the polynomial
eqn = 3*x(1)*(tn + h)^2 + 2*x(2)*(tn + h) + x(3) == fp1;
yp1 = simplify(solve(eqn, yp1))
```

yp1 =

$$\frac{18\,yn}{11} - \frac{9\,yn_1}{11} + \frac{2\,yn_2}{11} + \frac{6\,fp_1\,h}{11}$$

The solve part is $3At_{n+1}^2 + 2Bt_{n+1} + C = f_{n+1}(t_{n+1}, y_{n+1})$

since $P_3'(t) = \phi'(t) = f(t,y)$, $P_3 = At^3 + Bt^2 + Ct + D$

And 4th order backward differentiation formula:

```
clear
syms t tn h yp1 yn yn1 yn2 yn3 fp1
E = [  (tn+h)^4,   (tn+h)^3,   (tn+h)^2,   tn+h,  1;
           tn^4,       tn^3,       tn^2,     tn,  1;
       (tn-h)^4,   (tn-h)^3,   (tn-h)^2,   tn-h,  1;
     (tn-2*h)^4, (tn-2*h)^3, (tn-2*h)^2, tn-2*h,  1;
     (tn-3*h)^4, (tn-3*h)^3, (tn-3*h)^2, tn-3*h,  1];
F = [yp1; yn; yn1; yn2; yn3];
x = E\F;  % solve Ex = F
% x holds coefficients of the polynomial
eqn = 4*x(1)*(tn + h)^3 + 3*x(2)*(tn + h)^2 + 2*x(3)*(tn + h) + x(4) == fp1;
yp1 = simplify(solve(eqn, yp1))
```

$x(1) = A$
$x(2) = B$
$x(3) = C$
$x(4) = D$
$x(5) = E$

yp1 =

$$\frac{48\,yn}{25} - \frac{36\,yn_1}{25} + \frac{16\,yn_2}{25} - \frac{3\,yn_3}{25} + \frac{12\,fp_1\,h}{25}$$

Note: $x(i)$ are in terms of $y_{n+1}, y_n, y_{n-1}, y_{n-2}, y_{n-3}$

Here, $P_4'(t_{n+1}) = 4At_{n+1}^3 + 3Bt_{n+1}^2 + 2Ct_{n+1} + D = f_{n+1}(t_{n+1}, y_{n+1})$

8.6 More on Errors; Stability

Vertical Asymptotes

Page 381: $y' = 1 + y^2$, $y(0) = 1$

$$\int_1^y \frac{dy}{1+y^2} = \int_0^t 1\,dt \Rightarrow \arctan(y)\Big|_1^y = t\Big|_0^t$$

$$\therefore \arctan(y) - \frac{\pi}{4} = t - 0, \quad y = \underline{\tan\left(t + \frac{\pi}{4}\right)}$$

For $y' = y^2$, $y(0) = 1$, $\displaystyle\int_1^y \frac{dy}{y^2} = \int_0^t 1\,dt$

$$\therefore -\frac{1}{y}\Big|_1^y = t\Big|_0^t$$

$$\therefore -\frac{1}{y} + 1 = t, \quad 1 - t = \frac{1}{y}, \quad \underline{y = \frac{1}{1-t}}$$

For $y(0.9) = 14.305$

$$\arctan(y)\Big|_{14.305}^{y} = t\Big|_{0.9}^{t}$$

$= 1.501$

$$\therefore \arctan(y) - \arctan(14.305) = t - 0.9$$

$$\arctan(y) = t + 0.601, \quad y = \underline{\tan(t + 0.601)}$$

$$-\frac{1}{y}\Big|_{14.305}^{y} = t\Big|_{0.9}^{t}, \quad -\frac{1}{y} + 0.06991 = t - 0.9$$

$$\therefore\ 0.96991 - t = \frac{1}{y},\quad y = \underline{\frac{1}{0.96991 - t}}$$

Example 3

On page 385, the statement is made,

$\mathbf{x}^{(2)}(t_n)$. The solution of the initial value problem with these data at t_n involves not only $e^{-\sqrt{10}\pi t}$ but also $e^{\sqrt{10}\pi t}$. Because the error in the data at t_n is small, the latter function appears with a very

In general the solution is:

$$\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ \sqrt{10}\,\pi \end{bmatrix} e^{\sqrt{10}\pi t} + c_2 \begin{bmatrix} 1 \\ -\sqrt{10}\,\pi \end{bmatrix} e^{-\sqrt{10}\pi t}$$

The solution to the initial value problem with

$\vec{x}(t_n) = \begin{bmatrix} y(t_n) \\ y'(t_n) \end{bmatrix}$, used as a means to

analyze error and stability going forward from t_n, will $\therefore$ involve $e^{\sqrt{10}\pi t}$ and $e^{-\sqrt{10}\pi t}$,

as opposed to the more simple solution with

$$\vec{x}(0) = \begin{bmatrix} 1 \\ -\sqrt{10}\,\pi \end{bmatrix}$$

Chapter 9 Notes

Note Title 6/23/2020

9.1 The Phase Plane: Linear Systems

On p.393-395, for complex eigenvalues, $\lambda \pm i\mu$, it is stated that through a linear transformation, every 2x2 system with complex eigenvalues can be converted to $\vec{x}' = \begin{bmatrix} \lambda & \mu \\ -\mu & \lambda \end{bmatrix} \vec{x}$.

To prove such a linear transformation exists, let $\lambda + i\mu$ be the one eigenvalue, and let $\vec{y} = \vec{r} + i\vec{m}$ be the associated eigenvector, $\vec{r}$ the real part, $i\vec{m}$ the imaginary part, so $\vec{r}$ and $\vec{m}$ are real-valued 2x1 vectors.

Let A be the 2x2 matrix of real coefficients.

$$\therefore \quad A\vec{y} = (\lambda + i\mu)\vec{y} = (\lambda + i\mu)(\vec{r} + i\vec{m})$$

$$= \lambda\vec{r} - \mu\vec{m} + i(\lambda\vec{m} + \mu\vec{r}) \qquad [1]$$

But $A\vec{y} = A(\vec{r} + i\vec{m}) = A\vec{r} + iA\vec{m}$ $\qquad [2]$

Equating the real and imaginary parts of [1], [2],

$$A\vec{r} = \lambda\vec{r} - \mu\vec{m} \qquad [3]$$

$$A\vec{m} = \mu\vec{r} + \lambda\vec{m} \qquad [4]$$

Consider $[\vec{r}\ \ \vec{m}]$, a 2×2 matrix.

Let $P = [\vec{r}\ \ \vec{m}]$. $\therefore A[\vec{r}\ \ \vec{m}] = AP$

Note $\lambda\vec{r} - \mu\vec{m}$ is a 2×1 matrix, as is $\mu\vec{r} + \lambda\vec{m}$, and can be viewed as 2×1 column vectors. Note that we can write

$$\lambda\vec{r} - \mu\vec{m} = \overset{2\times 2}{[\vec{r}\ \ \vec{m}]}\overset{2\times 1}{\begin{bmatrix}\lambda \\ -\mu\end{bmatrix}} \quad \text{and}$$

$$\mu\vec{r} + \lambda\vec{m} = \overset{2\times 2}{[\vec{r}\ \ \vec{m}]}\overset{2\times 1}{\begin{bmatrix}\mu \\ \lambda\end{bmatrix}}$$

$\therefore$ By combining [3], [4], we get

$$A[\vec{r}\ \ \vec{m}] = [A\vec{r}\ \ A\vec{m}] = [\lambda\vec{r} - \mu m\ \ \ \mu\vec{r} + \lambda\vec{m}]$$

$$= [\vec{r}\ \ \vec{m}]\begin{bmatrix}\lambda & \mu \\ -\mu & \lambda\end{bmatrix}$$

$$\therefore AP = P\begin{bmatrix}\lambda & \mu \\ -\mu & \lambda\end{bmatrix}$$

If $\det(P) \neq 0$, then P^{-1} exists, and

$\therefore \quad P^{-1}AP = \begin{bmatrix} \lambda & \mu \\ -\mu & \lambda \end{bmatrix}$

$\therefore$ if $\vec{z}\,' = A\vec{z}$ is the original problem,

let $\vec{x} = P^{-1}\vec{z}$, where $P = [\vec{r}\ \vec{m}]$. $\therefore \vec{z} = P\vec{x}$

$\therefore \vec{z}\,' = P\vec{x}\,' = A\vec{z} = AP\vec{x}$. $\therefore \vec{x}\,' = P^{-1}AP\vec{x}$

$\therefore \vec{x}\,' = \begin{bmatrix} \lambda & \mu \\ -\mu & \lambda \end{bmatrix}\vec{x}$

After solving for $\vec{x}$, $\vec{z} = P\vec{x}$.

$\begin{bmatrix} \lambda & \mu \\ -\mu & \lambda \end{bmatrix} = P^{-1}AP$ is the linear transformation.

Note: P^{-1} always exists (i.e., $\det(P) \neq 0$)

<u>Proof</u>: We assume $(A - rI)\vec{x} = \vec{0}$ has a non-zero solution and so $\det(A - rI) = 0$, for otherwise $\vec{x} = \vec{0}$ is the only solution.

$\therefore$ For a 2×2 system, $A - rI$ has rank 1, and in solving for the eigenvector,

$(A - rI)\vec{x} = 0$, r the complex eigenvalue, we can row reduce $(A-rI)$ to a matrix which has row 2 as a zero row since $\text{rank}(A-rI) = 1$, and the pivot in row 1, column 1 is nonzero, since $\overset{\text{real}}{a_{11}} - \overset{\text{complex}}{r}$ is nonzero complex. $\therefore$ if $\vec{\epsilon}$ is the eigenvector,

$\text{RREF}(A-rI)\begin{bmatrix} z \\ 1 \end{bmatrix} = 0$, i.e., we set the "free variable" to 1, and z must be complex.

$\therefore \vec{\epsilon} = \begin{bmatrix} z \\ 1 \end{bmatrix}$ must be complex.

If $\vec{\epsilon}$ were real, $A\vec{\epsilon}$ would be real, but $A\vec{\epsilon} = r\vec{\epsilon}$ must be complex since r is complex and $\vec{\epsilon}$ is of the form $\begin{bmatrix} z \\ 1 \end{bmatrix}$.

$\therefore r\vec{\epsilon} = \begin{bmatrix} rz \\ r \end{bmatrix}$. Even if rz is real, r is complex. $\therefore$ $\vec{\epsilon}$ is complex of form $\begin{bmatrix} z \\ 1 \end{bmatrix}$

$\therefore \vec{\epsilon} = \begin{bmatrix} \text{real}(z) \\ 1 \end{bmatrix} + i \begin{bmatrix} \text{imaginary}(z) \\ 0 \end{bmatrix}$

and $\text{imaginary}(z) \neq 0$ since $\vec{\epsilon}$ is complex.

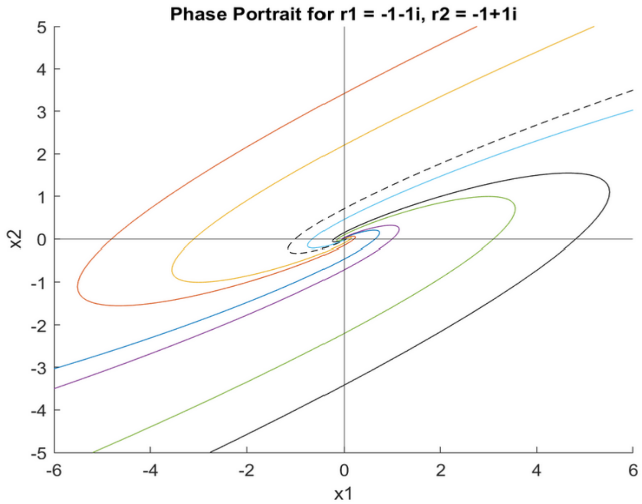
$\therefore P = \begin{bmatrix} real(z) & imaginary(z) \\ 1 & 0 \end{bmatrix}$

so $\det(P) = imaginary(z) \neq 0$.

$\therefore$ P^{-1} always exists.

To see the effect of $P^{-1}AP$, consider problem #5 of section 9.1. $A = \begin{bmatrix} 1 & -5 \\ 1 & -3 \end{bmatrix}$ with $r = -1+i, -1-i$

and eigenvectors $\begin{bmatrix} 2+i \\ 1 \end{bmatrix}, \begin{bmatrix} 2-i \\ 1 \end{bmatrix}$. As shown in #5, the phase diagram is:



Here, $P = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$ using the $\begin{bmatrix} 2+i \\ 1 \end{bmatrix}$ eigenvector

The linear transformation is $P^{-1}AP$

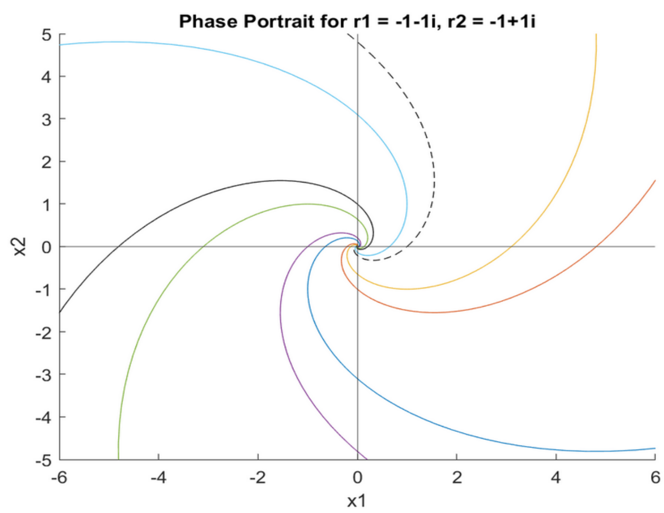
and this is $\begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} \lambda & \mu \\ -\mu & \lambda \end{bmatrix}$ using

$r = -1 + i = \lambda + i\mu$, the corrrsponding eigenvalue.

$\therefore$ Using $A = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$ in the same MATLAB

code used to generate the above phase

portrait for #5,



$\therefore$ $P^{-1}AP$ stretches and flips the above phase

portrait to produce the portrait for A.

On p.409, it states that if $\vec{x}_0 \neq \vec{0}$ is a critical point, the substitution $\vec{u} = \vec{x} - \vec{x}_0$ can be made in $\vec{x}\,' = f(\vec{x})$. Since $\frac{d\vec{u}}{dt} = \frac{d\vec{x}}{dt}$,

and note that $f(\vec{x}_0) = \vec{0}$, $\therefore f(\vec{x}) = f(\vec{u} + \vec{x}_0)$

$\therefore$ let $g(\vec{u}) = f(\vec{u} + \vec{x}_0)$. $\therefore g(\vec{0}) = f(\vec{x}_0) = \vec{0}$

$\therefore \vec{u}\,' = g(\vec{u})$ and $g(\vec{0}) = \vec{0}$

=

On pp. 410-411, Theorem 9.3.2

"Continuous partial derivatives up to order two" allows a Taylor expansion of $F(x,y)$, $G(x,y)$ s.t. $\frac{\eta_1(x,y)}{\|\vec{x} - \vec{x}_0\|} \to 0$, $\frac{\eta_2(x,y)}{\|\vec{x} - \vec{x}_0\|} \to 0$ as $(x,y) \to (x_0, y_0)$,

which fulfills the criteria for the definition of a locally linear system on p. 409. A simple proof of the Taylor theorem requires this.

Note also the assumption of $\det J \neq 0$ means the critical point (x_0, y_0) is a locally isolated critical point since $J\vec{x} = \vec{0}$ has only $\vec{x} = \vec{0}$ as the zero solution. This is also a criterion of the definition on p. 409.

=

p. 413

1. If $\gamma^2 - 4\omega^2 > 0$, then the eigenvalues are real, unequal, and negative.
   The critical point $(0, 0)$ is an asymptotically stable node of the linear system (17) and of the locally linear system (8).
2. If $\gamma^2 - 4\omega^2 = 0$, then the eigenvalues are real, equal, and negative.
   The critical point $(0, 0)$ is an asymptotically stable (proper or improper) node of the linear system (17). It may be either an asymptotically stable node or spiral point of the locally linear system (8).
3. If $\gamma^2 - 4\omega^2 < 0$, then the eigenvalues are complex with negative real part.
   The critical point $(0, 0)$ is an asymptotically stable spiral point of the linear system (17) and of the locally linear system (8).

If γ is small, then $\gamma^2 - 4\omega^2 < 0$ since $4\omega^2$ is likely larger than γ. $\therefore$ Condition (3) holds

If γ is large enough, $\gamma^2 > 4\omega^2$ so $\gamma^2 - 4\omega^2 > 0$ and $\therefore$ condition (1) holds.

=

p. 413

$$\begin{pmatrix} u \\ v \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ r_1 \end{pmatrix} e^{r_1 t} + C_2 \begin{pmatrix} 1 \\ r_2 \end{pmatrix} e^{r_2 t}, \qquad (21)$$

The eigenvectors $\begin{bmatrix} 1 \\ r_1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ r_2 \end{bmatrix}$ are determined from

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \omega^2 & -\gamma \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}, \qquad (18)$$

$\therefore$ From $A = \begin{bmatrix} 0 & 1 \\ \omega^2 & -\gamma \end{bmatrix}$, $\det(A - \lambda I) = \lambda^2 + \gamma\lambda + \omega^2 = 0$

Since $\det(A - \lambda I) = 0$, $A - \lambda I = \begin{bmatrix} -\lambda & 1 \\ \omega^2 & -\gamma - \lambda \end{bmatrix}$ has rank 1, and so can just look at row 1 to determine eigenvectors.

$\therefore \begin{bmatrix} -r_1 & 1 \end{bmatrix}\begin{bmatrix} \ \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ has solution $\begin{bmatrix} 1 \\ r_1 \end{bmatrix}$

and $\begin{bmatrix} -r_2 & 1 \end{bmatrix}\begin{bmatrix} \ \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ has solution $\begin{bmatrix} 1 \\ r_2 \end{bmatrix}$

$\therefore$ The general solution is $\begin{bmatrix} u \\ v \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ r_1 \end{bmatrix} e^{r_1 t} + C_2 \begin{bmatrix} 1 \\ r_2 \end{bmatrix} e^{r_2 t}$

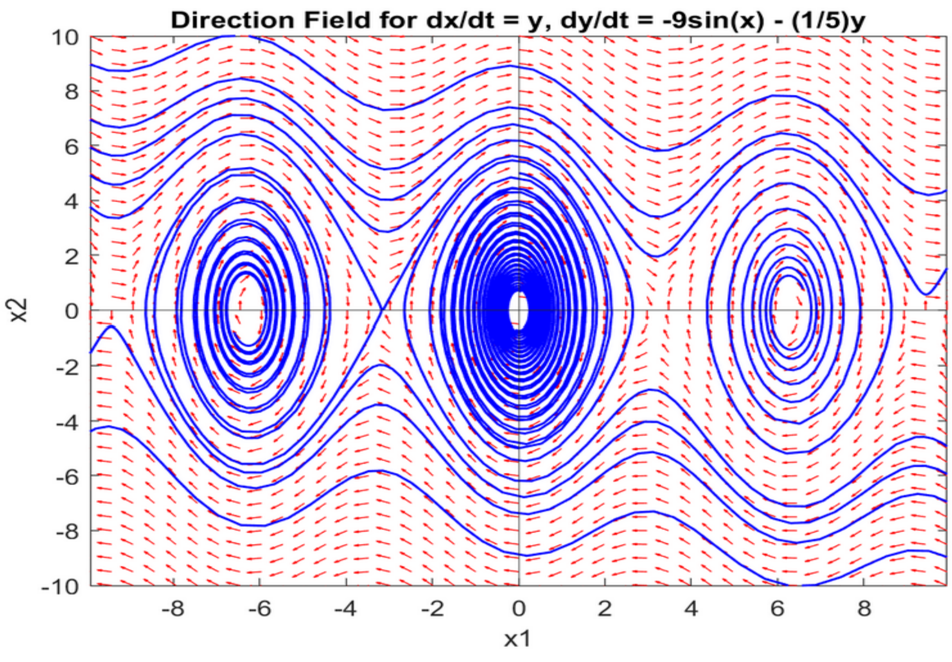
Now assume $r_2 < 0 < r_1$. $\therefore$ with $C_1 = 0$, as $t \to \infty$ the path lies along $\begin{bmatrix} 1 \\ r_2 \end{bmatrix}$, which has slope

$\frac{r_2}{1} = r_2$, which is negative. Similarly for $C_2 = 0$, as $t \to \infty$, the path lies along $\begin{bmatrix} 1 \\ r_1 \end{bmatrix}$, which has slope $\frac{r_1}{1} = r_1$, which is positive.

=

p. 414, Example 4,

Using MATLAB and ode45,

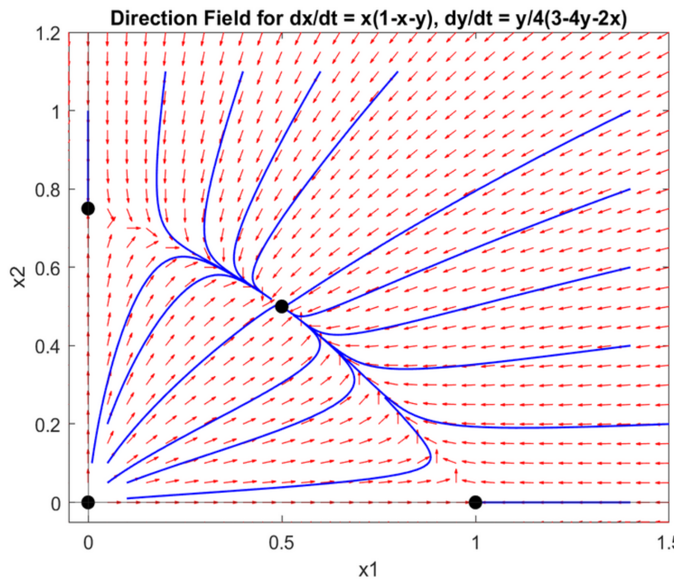


Code on next page.

```
clear
% set plot boundaries around critical points
Xmin = -3*pi - 0.5; Xmax = 3*pi + 0.5;
Ymin = -10; Ymax = 10;
% set how fine to make the grid of values
Step = 0.5;

% total # points on x and y axes
Nx = round((Xmax-Xmin)/Step, 0) + 1;
Ny = round((Ymax-Ymin)/Step, 0) + 1;
% set the actual x and y coords
x = linspace(Xmin, Xmax, Nx);  % a row vector
y = linspace(Ymin, Ymax, Ny);  % a row vector
% preallocate memory for coord, slope arrays
%  to be used in quiver, must be same dimension
xcoord = zeros(Nx, Ny);
ycoord = zeros(Nx, Ny);
dx = zeros(Nx, Ny);
dy = zeros(Nx, Ny);
for i = 1:Nx
    for j = 1:Ny
        xcoord(i,j) = x(i);
        ycoord(i,j) = y(j);
        [hz, vt, ls] = DirFieldVec(x(i), y(j));
        % make slope vector a unit vector
        dx(i,j) = hz/ls;
        dy(i,j) = vt/ls;
    end
end
% plot the slope vectors at (xcoord, ycoord)
quiver(xcoord, ycoord, dx, dy, 0.5, 'r')
xline(0);  % show x and y axes
yline(0);
axis([Xmin, Xmax, Ymin, Ymax])
xlabel 'x1', ylabel 'x2'
st1 = 'Direction Field for ';
st2 = ' dx/dt = y, dy/dt = -9sin(x) - (1/5)y';
title(strcat(st1,st2))

% use ode45 to plot trajectories
hold on
% ode45 uses a function handle
% x = z(1), y = z(2)
f = @(t,z) [z(2);
            -9*sin(z(1)) - 0.2*z(2)];
tspan = [0,20];  % time interval for plot
% xy0 is several initial values as [x;y] points
% split points into 2 arrays for display purposes
A = [0, -10, -10, -10, -10, -10, -3.141;
     5,   7,   9,   6,   5, 3.87,  0];
B = [ 0, 10, 10, 10, 10;
     -5, -7, -9, -6, -5];
xy0 = [A,B];
for Col = 1:size(xy0, 2)  % # of columns of xy0
    [t,z] = ode45(f, tspan, xy0(:,Col));
    plot(z(:,1), z(:,2), 'b', 'LineWidth', 1)
end

function [hz, vt, ls] = DirFieldVec(x, y)
% Supply hz = dx/dt, vt = dy/dt from problem
% Output horiz, vert components of slope vector
% ls = length of the slope vector
hz = y;
vt = -9*sin(x) - 0.2*y;
ls = sqrt(hz^2 + vt^2);
end
```

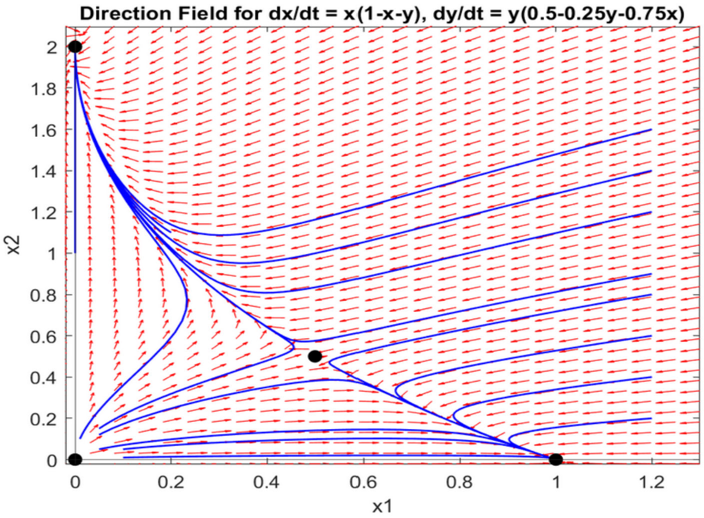
9.4 Competing Species

Example 1, p. 421




```
clear
% set plot boundaries around critical points
Xmin = -0.05; Xmax = 1.5;
Ymin = -0.05; Ymax = 1.2;
% set how fine to make the grid of values
Step = 0.05;

% total # points on x and y axes
Nx = round((Xmax-Xmin)/Step, 0) + 1;
Ny = round((Ymax-Ymin)/Step, 0) + 1;
% set the actual x and y coords
x = linspace(Xmin, Xmax, Nx);  % a row vector
y = linspace(Ymin, Ymax, Ny);  % a row vector
% preallocate memory for coord, slope arrays
%  to be used in quiver, must be same dimension
xcoord = zeros(Nx, Ny);
ycoord = zeros(Nx, Ny);
dx = zeros(Nx, Ny);
dy = zeros(Nx, Ny);
for i = 1:Nx
    for j = 1:Ny
        xcoord(i,j) = x(i);
        ycoord(i,j) = y(j);
        [hz, vt, ls] = DirFieldVec(x(i), y(j));
        % make slope vector a unit vector
        dx(i,j) = hz/ls;
        dy(i,j) = vt/ls;
    end
end
% plot the slope vectors at (xcoord, ycoord)
quiver(xcoord, ycoord, dx, dy, 0.5, 'r')
xline(0);  % show x and y axes
yline(0);
axis([Xmin, Xmax, Ymin, Ymax])
xlabel 'x1', ylabel 'x2'
st1 = 'Direction Field for ';
st2 = ' dx/dt = x(1-x-y), dy/dt = y/4(3-4y-2x)';
title(strcat(st1,st2))

% use ode45 to plot trajectories
hold on
% ode45 uses a function handle
% x = z(1), y = z(2)
f = @(t,z) [z(1)*(1 - z(1) - z(2));...
            z(2)/4*(3 - 4*z(2) - 2*z(1))];
tspan = [0, 12];  % time interval for plot
% xy0 is several initial values as [x;y] points
% split points into 2 arrays for display purposes
A = [1.5, 1.4, 1.4, 1.4, 1.4, 0.8, 0.6, 0.4;
     0.2, 0.4, 0.6, 0.8, 1.0, 1.1, 1.1, 1.1];
B = [1.4,  0,  0.2, 0.1,  0.05, 0.05, 0.05, 0.1,  0.01;
      0,  1.0, 1.1, 0.05, 0.05, 0.1,  0.2,  0.01, 0.1];
xy0 = [A,B];
for Col = 1:size(xy0, 2)  % # of columns of xy0
    [t,z] = ode45(f, tspan, xy0(:,Col));
    plot(z(:,1), z(:,2), 'b', 'LineWidth', 1)
end
plot(1,0,'k','Marker','.', 'MarkerSize', 25)
plot(0,0.75,'k','Marker','.', 'MarkerSize', 25)
plot(0,0,'k','Marker','.', 'MarkerSize', 25)
plot(0.5,0.5,'k','Marker','.', 'MarkerSize', 25)

function [hz, vt, ls] = DirFieldVec(x, y)
% Supply hz = dx/dt, vt = dy/dt from problem
% Output horiz, vert components of slope vector
% ls = length of the slope vector
hz = x*(1 - x - y);
vt = y/4*(3 - 4*y - 2*x);
ls = sqrt(hz^2 + vt^2);
end
```

Example 2, p. 421-424




```
clear
% set plot boundaries around critical points
Xmin = -0.02; Xmax = 1.3;
Ymin = -0.02; Ymax = 2.1;
% set how fine to make the grid of values
Step = 0.05;

% total # points on x and y axes
Nx = round((Xmax-Xmin)/Step, 0) + 1;
Ny = round((Ymax-Ymin)/Step, 0) + 1;
% set the actual x and y coords
x = linspace(Xmin, Xmax, Nx);  % a row vector
y = linspace(Ymin, Ymax, Ny);  % a row vector
% preallocate memory for coord, slope arrays
%  to be used in quiver, must be same dimension
xcoord = zeros(Nx, Ny);
ycoord = zeros(Nx, Ny);
dx = zeros(Nx, Ny);
dy = zeros(Nx, Ny);
for i = 1:Nx
    for j = 1:Ny
        xcoord(i,j) = x(i);
        ycoord(i,j) = y(j);
        [hz, vt, ls] = DirFieldVec(x(i), y(j));
        % make slope vector a unit vector
        dx(i,j) = hz/ls;
        dy(i,j) = vt/ls;
    end
end
% plot the slope vectors at (xcoord, ycoord)
quiver(xcoord, ycoord, dx, dy, 0.5, 'r')
xline(0);  % show x and y axes
yline(0);
axis([Xmin, Xmax, Ymin, Ymax])
xlabel 'x1', ylabel 'x2'
st1 = 'Direction Field for ';
st2 = ' dx/dt = x(1-x-y), dy/dt = y(0.5-0.25y-0.75x)';
title(strcat(st1,st2))

% use ode45 to plot trajectories
hold on
% ode45 uses a function handle
% x = z(1), y = z(2)
f = @(t,z) [z(1)*(1 - z(1) - z(2));...
            z(2)*(0.5 - 0.25*z(2) - 0.75*z(1))];
tspan = [0, 20];  % time interval for plot
% xy0 is several initial values as [x;y] points
% split points into 2 arrays for display purposes
A = [1.2, 1.2, 1.2, 1.2, 1.2, 1.2, 1.2, 1.2;
     0.2, 0.4, 0.6, 0.8, 0.9, 1.2, 1.4, 1.6];
B = [ 0,  0.2, 0.1,  0.05, 0.05, 0.05, 0.1,  0.01;
     1.0, 1.1, 0.05, 0.05, 0.12, 0.15, 0.01, 0.1];
xy0 = [A,B];
for Col = 1:size(xy0, 2)  % # of columns of xy0
    [t,z] = ode45(f, tspan, xy0(:,Col));
    plot(z(:,1), z(:,2), 'b', 'LineWidth', 1)
end
plot(1,0,'k','Marker','.', 'MarkerSize', 25)
plot(0,2,'k','Marker','.', 'MarkerSize', 25)
plot(0,0,'k','Marker','.', 'MarkerSize', 25)
plot(0.5,0.5,'k','Marker','.', 'MarkerSize', 25)

function [hz, vt, ls] = DirFieldVec(x, y)
% Supply hz = dx/dt, vt = dy/dt from problem
% Output horiz, vert components of slope vector
% ls = length of the slope vector
hz = x*(1 - x - y);
vt = y*(0.5 - 0.25*y - 0.75*x);
ls = sqrt(hz^2 + vt^2);
end
```

pp. 424-426

Only 4 possibilities: non-intersecting and intersecting lines in Quadrant I, and two possibilities for each.

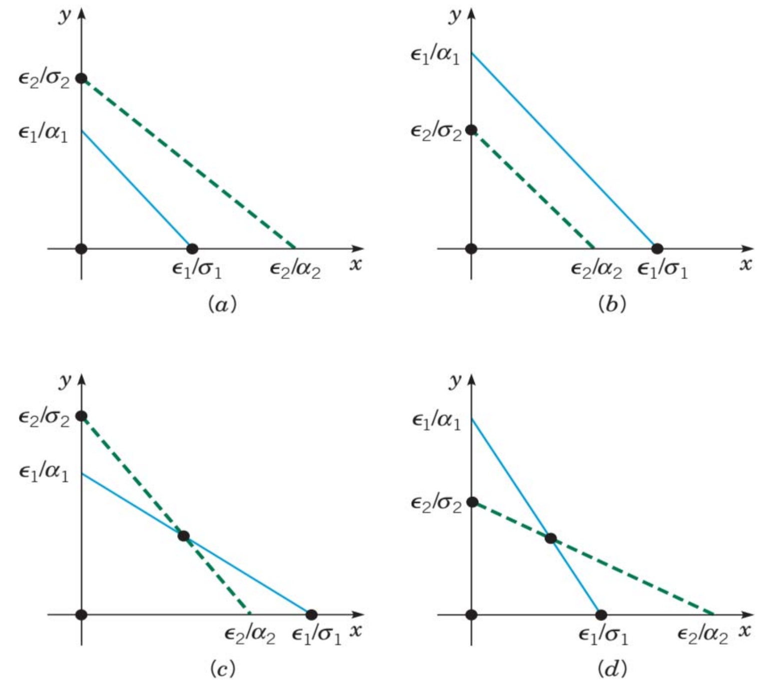


FIGURE 9.4.5 The four cases for the competing-species system (2). The x-nullcline is the solid blue line, and the y-nullcline is the dashed green line.

$$\epsilon_1 - \sigma_1 x - \alpha_1 y = 0 \Rightarrow y = -\frac{\sigma_1}{\alpha_1}x + \frac{\epsilon_1}{\alpha_1}$$

$$\epsilon_2 - \sigma_2 y - \alpha_2 x = 0 \Rightarrow y = -\frac{\alpha_2}{\sigma_2}x + \frac{\epsilon_2}{\sigma_2}$$

Also, $\begin{bmatrix} \sigma_1 & \alpha_1 \\ \alpha_2 & \sigma_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix}$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{\sigma_1\sigma_2 - \alpha_1\alpha_2}\begin{bmatrix} \sigma_2 & -\alpha_1 \\ -\alpha_2 & \sigma_1 \end{bmatrix}\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix}$$

For eigenvalues of: $\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -\sigma_1 X & -\alpha_1 X \\ -\alpha_2 Y & -\sigma_2 Y \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}.$

$$\det \begin{bmatrix} -\sigma_1 X - r & -\alpha_1 X \\ -\alpha_2 Y & -\sigma_2 Y - r \end{bmatrix} =$$

$$(r+\sigma_1 x)(r+\sigma_2 y) - \alpha_1\alpha_2 xy =$$

$$r^2 + (\sigma_1 x + \sigma_2 y) r + \sigma_1\sigma_2 xy - \alpha_1\alpha_2 xy = 0$$

Or, $r^2 + (\sigma_1 x + \sigma_2 y) r + (\sigma_1\sigma_2 - \alpha_1\alpha_2) xy = 0$

$$\therefore\ r = \frac{-(\sigma_1 x + \sigma_2 y) \pm \sqrt{(\sigma_1 x + \sigma_2 y)^2 - 4(\sigma_1\sigma_2 - \alpha_1\alpha_2)xy}}{2}$$

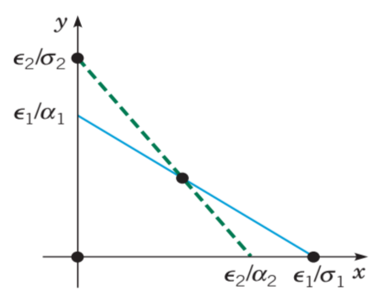
Note: $b^2 - 4ac = \sigma_1^2 x^2 + 2\sigma_1\sigma_2 xy + \sigma_2^2 y^2$

$$- 4\sigma_1\sigma_2 xy + 4\alpha_1\alpha_2 xy$$

$$= (\sigma_1 x - \sigma_2 y)^2 + 4\alpha_1\alpha_2 xy \geq 0$$

$\therefore\ b^2 - 4ac \geq 0$ so r is never complex.

For (c):



(c)

$\dfrac{\epsilon_1}{\sigma_1} > \dfrac{\epsilon_2}{\alpha_2}$ (x-axis) $\therefore\ \epsilon_2\sigma_1 - \epsilon_1\alpha_2 < 0$

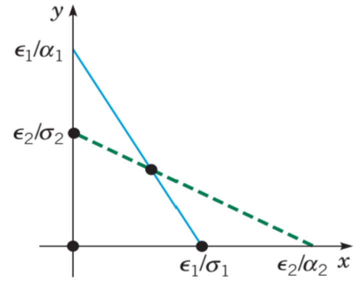
$\dfrac{\epsilon_2}{\sigma_2} > \dfrac{\epsilon_1}{\alpha_1}$ (y-axis) $\therefore\ \epsilon_1\sigma_2 - \epsilon_2\alpha_1 < 0$

Using $X = \dfrac{\epsilon_1\sigma_2 - \epsilon_2\alpha_1}{\sigma_1\sigma_2 - \alpha_1\alpha_2}, \quad Y = \dfrac{\epsilon_2\sigma_1 - \epsilon_1\alpha_2}{\sigma_1\sigma_2 - \alpha_1\alpha_2}.$ and $x > 0,\ y > 0$,

this means $\sigma_1\sigma_2 - \alpha_1\alpha_2 < 0$

$\therefore$ Intersection point is saddle point as $r_1 < 0 < r_2$.

For (d):



(d)

$\frac{\epsilon_2}{\alpha_2} > \frac{\epsilon_1}{\sigma_1}$ (x-axis)

$\therefore \epsilon_2\sigma_1 - \epsilon_1\alpha_2 > 0$

$\frac{\epsilon_1}{\alpha_1} > \frac{\epsilon_2}{\sigma_2}$ (y-axis)

$\therefore \epsilon_1\sigma_2 - \epsilon_2\alpha_1 > 0$

Using $X = \dfrac{\epsilon_1\sigma_2 - \epsilon_2\alpha_1}{\sigma_1\sigma_2 - \alpha_1\alpha_2}, \quad Y = \dfrac{\epsilon_2\sigma_1 - \epsilon_1\alpha_2}{\sigma_1\sigma_2 - \alpha_1\alpha_2}.$ and $X > 0,\ Y > 0$,

this means $\sigma_1\sigma_2 - \alpha_1\alpha_2 > 0$

$\therefore$ Intersection point is a stable node

as $r_1 < r_2 < 0$.

The other critical points: $(0,0)$, $(\frac{\epsilon_1}{\sigma_1}, 0)$, $(0, \frac{\epsilon_2}{\sigma_2})$

come from solving $x(\epsilon_1 - \sigma_1 x - \alpha_1 y) = 0$ and

$y(\epsilon_2 - \sigma_2 y - \alpha_2 x) = 0$

From: $$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \epsilon_1 - 2\sigma_1 X - \alpha_1 Y & -\alpha_1 X \\ -\alpha_2 Y & \epsilon_2 - 2\sigma_2 Y - \alpha_2 X \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}. \qquad (35)$$

$(0,0)$: $J(0,0) = \begin{bmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{bmatrix}$.

$\therefore$ Eigenvalues: $r_1 = \epsilon_1$, $r_2 = \epsilon_2$, both positive

$\therefore$ unstable node

$\left(\frac{\epsilon_1}{\sigma_1}, 0\right)$: $J\left(\frac{\epsilon_1}{\sigma_1}, 0\right) = \begin{bmatrix} -\epsilon_1 & -\alpha_1\epsilon_1/\sigma_1 \\ 0 & \epsilon_2 - \alpha_2\epsilon_1/\sigma_1 \end{bmatrix}$

Eigenvalues: $(r+\epsilon_1)(r-\epsilon_2+\alpha_2\epsilon_1/\sigma_1) = 0$

$\therefore\ r_1 = -\epsilon_1,\ r_2 = \dfrac{\epsilon_2\sigma_1 - \epsilon_1\alpha_2}{\sigma_1}$

From above, $\epsilon_2\sigma_1 - \epsilon_1\alpha_2 > 0$, so

$r_1 < 0 < r_2$ $\therefore$ saddle point, unstable

$\left(0, \frac{\epsilon_2}{\sigma_2}\right)$: $J\left(0, \frac{\epsilon_2}{\sigma_2}\right) = \begin{bmatrix} \epsilon_1 - \epsilon_2\alpha_1/\sigma_2 & 0 \\ -\alpha_2\epsilon_2/\sigma_2 & -\epsilon_2 \end{bmatrix}$

Eigenvalues: $(r+\epsilon_2)(r-\epsilon_1+\epsilon_2\alpha_1/\sigma_2) = 0$

$\therefore\ r_1 = -\epsilon_2,\ r_2 = \dfrac{\epsilon_1\sigma_2 - \epsilon_2\alpha_1}{\sigma_2}$

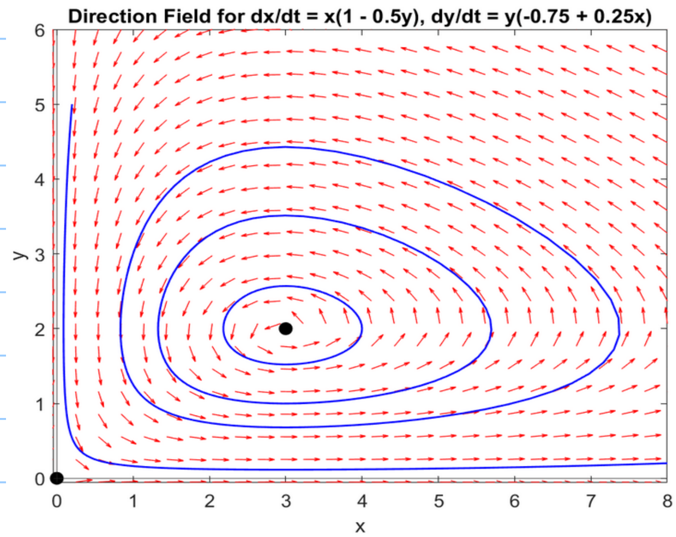
From above, $\epsilon_1\sigma_2 - \epsilon_2\alpha_1 > 0$.

$\therefore\ r_1 < 0 < r_2$ $\therefore$ saddle point, unstable.

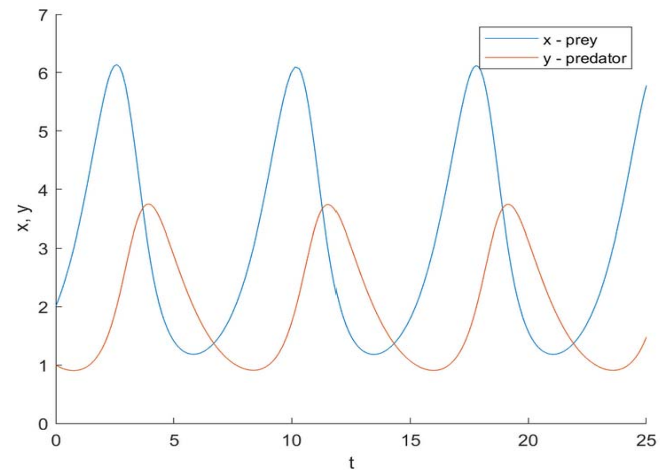
Note: Did not consider the unlikely scenario of the two nullclines being the same line, so infinite intersection points.

9.5 Predator-Prey Equations

Example 1, p.429



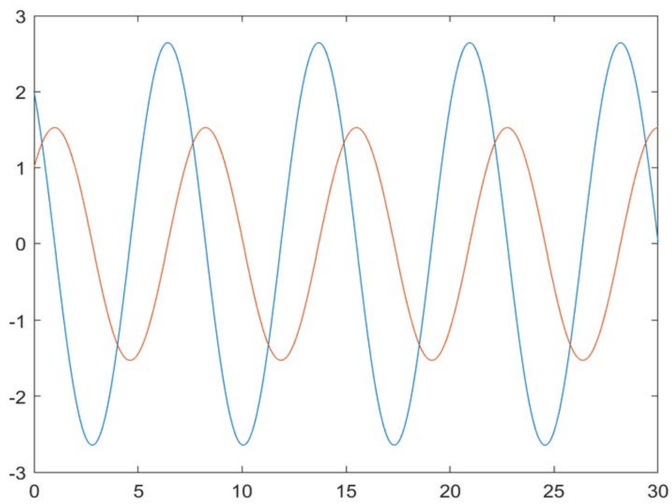

```
clear
% set plot boundaries around critical points
Xmin = -0.05; Xmax = 8.0;
Ymin = -0.05; Ymax = 6.0;
% set how fine to make the grid of values
Step = 0.3;

% total # points on x and y axes
Nx = round((Xmax-Xmin)/Step, 0) + 1;
Ny = round((Ymax-Ymin)/Step, 0) + 1;
% set the actual x and y coords
x = linspace(Xmin, Xmax, Nx);  % a row vector
y = linspace(Ymin, Ymax, Ny);  % a row vector
% preallocate memory for coord, slope arrays
%  to be used in quiver, must be same dimension
xcoord = zeros(Nx, Ny);
ycoord = zeros(Nx, Ny);
dx = zeros(Nx, Ny);
dy = zeros(Nx, Ny);
for i = 1:Nx
    for j = 1:Ny
        xcoord(i,j) = x(i);
        ycoord(i,j) = y(j);
        [hz, vt, ls] = DirFieldVec(x(i), y(j));
        % make slope vector a unit vector
        dx(i,j) = hz/ls;
        dy(i,j) = vt/ls;
    end
end
% plot the slope vectors at (xcoord, ycoord)
quiver(xcoord, ycoord, dx, dy, 0.5, 'r')
xline(0);  % show x and y axes
yline(0);
axis([Xmin, Xmax, Ymin, Ymax])
xlabel 'x', ylabel 'y'
st1 = 'Direction Field for ';
st2 = ' dx/dt = x(1 - 0.5y), dy/dt = y(-0.75 + 0.25x)';
title(strcat(st1,st2))

% use ode45 to plot trajectories
hold on
% ode45 uses a function handle
% x = z(1), y = z(2)
f = @(t,z) [z(1)*(1 - 0.5*z(2));...
            z(2)*(-0.75 + 0.25*z(1))];
tspan = [0, 8];  % time interval for plot
% xy0 is several initial values as [x;y] points
% split points into 2 arrays for display purposes
A = [3.0, 4.0;
     1.0, 2.0];
B = [1.0, 0.2;
     3.0, 5.0];
xy0 = [A,B];
% more ode45 points => smoother plot
opts = odeset('Refine', 8);
for Col = 1:size(xy0, 2)  % # of columns of xy0
    [t,z] = ode45(f, tspan, xy0(:,Col), opts);
    plot(z(:,1), z(:,2), 'b', 'LineWidth', 1)
end
% plot critical points
plot(0,0,'k','Marker','.', 'MarkerSize', 25)
plot(3,2,'k','Marker','.', 'MarkerSize', 25)
```




```
% plot individual species over time
tspan = [0, 25];
% initial condition x(0) = 2, y(0) = 1
[t,z] = ode45(f, tspan, [2,1], opts);
figure
hold on
plot(t,z(:,1))
plot(t,z(:,2))
xlabel 't', ylabel 'x, y'
legend('x - prey', 'y - predator')

function [hz, vt, ls] = DirFieldVec(x, y)
% Supply hz = dx/dt, vt = dy/dt from problem
% Output horiz, vert components of slope vector
% ls = length of the slope vector
hz = x*(1 - 0.5*y);
vt = y*(-0.75 + 0.25*x);
ls = sqrt(hz^2 + vt^2);
end
```

The nonlinear solution to $x(0)=2$, $y(0)=1$ is plotted using ode45. The linear plot is shown below. Note the prey (blue) and predator (red) go negative, which is not reality.

```
clear
A = sym([0, -3/2;
         1/2, 0]);
[evec,eval] = eig(A);
t = 0:0.1:30;
evec1 = evec(:,1);
x = evec1*exp(eval(1,1)*t);
u = real(x);
v = imag(x);
B = [u(:,1), v(:,1)];
C = linsolve(B,[2;1]);
p = C(1,1)*u + C(2,1)*v;
x1 = p(1,:);
x2 = p(2,:);
plot(t,x1)
hold on
plot(t,x2)
```



p. 432

For $r = \pm i\sqrt{ac}$ and $A = \begin{bmatrix} 0 & -\alpha c/\gamma \\ \gamma a/\alpha & 0 \end{bmatrix}$

$\therefore$ For $r = i\sqrt{ac}$, $A - rI = \begin{bmatrix} -i\sqrt{ac} & -\alpha c/\gamma \\ \gamma a/\alpha & -i\sqrt{ac} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$

$\therefore -i\sqrt{ac}\,x - \frac{\alpha c}{\gamma} y = 0$, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} i\alpha\sqrt{c}/\gamma \\ \sqrt{a} \end{bmatrix}$

$\therefore \vec{X}(t) = e^{i\sqrt{ac}\,t} \begin{bmatrix} i\alpha\sqrt{c}/\gamma \\ \sqrt{a} \end{bmatrix}$

$$= \left[\cos(\sqrt{ac}\,t) + i\sin(\sqrt{ac}\,t)\right]\begin{bmatrix} i\alpha\sqrt{c}/\gamma \\ \sqrt{a}\end{bmatrix}$$

$$= \begin{bmatrix} -\alpha\sqrt{c}/\gamma \sin(\sqrt{ac}\,t) \\ \sqrt{a}\cos(\sqrt{ac}\,t)\end{bmatrix} + i\begin{bmatrix} \alpha\sqrt{c}/\gamma \cos(\sqrt{ac}\,t) \\ \sqrt{a}\sin(\sqrt{ac}\,t)\end{bmatrix}$$

Taking real and imaginary parts, using Th. 7.4.5,

$$u(t) = K_1(-\alpha\sqrt{c}/\gamma)\sin(\sqrt{ac}\,t) + K_2(\alpha\sqrt{c}/\gamma)\cos(\sqrt{ac}\,t)$$

$$v(t) = K_1\sqrt{a}\cos(\sqrt{ac}\,t) + K_2\sqrt{a}\sin(\sqrt{ac}\,t)$$

For $u(t)$, let

$$R_1\cos(\delta_1) = K_2(\alpha\sqrt{c}/\gamma), \quad R_1\sin(\delta_1) = K_1(\alpha\sqrt{c}/\gamma)$$

$$\therefore R_1^2 = (K_1^2 + K_2^2)(\alpha^2 c/\gamma^2), \quad \tan(\delta_1) = \frac{K_1}{K_2}$$

$$R_1 = \sqrt{K_1^2 + K_2^2}\,\frac{\alpha}{\sqrt{c}}\,\frac{c}{\gamma}$$

$$\text{Let } K = \sqrt{K_1^2 + K_2^2}\,\frac{\alpha}{\sqrt{c}}$$

$$\therefore u(t) = \frac{c}{\gamma}K\cos(\sqrt{ac}\,t + \delta_1)$$

For $v(t)$, let

$$R_2\cos(\delta_2) = K_2\sqrt{a}, \quad R_2\sin(\delta_2) = K_1\sqrt{a}$$

$$\therefore R_2^2 = (K_1^2 + K_2^2)a, \quad \tan(\delta_2) = \frac{K_1}{K_2} = \tan(\delta_1)$$

$$R_2 = \sqrt{K_1^2 + K_2^2}\,\sqrt{a} = \sqrt{K_1^2 + K_2^2}\,\frac{\alpha}{\sqrt{c}}\,\frac{\sqrt{c}}{\alpha}\,\sqrt{a}$$

$$= K\,\frac{\sqrt{c}}{\alpha}\,\sqrt{a} = K\sqrt{c}\,\frac{\sqrt{a}}{a}\,\frac{a}{\alpha} = K\sqrt{\frac{c}{a}}\,\frac{a}{\alpha}$$

$$\therefore\ v(t) = \frac{a}{\alpha}\sqrt{\frac{c}{a}}\,K \sin\left(\sqrt{ac} + \delta_2\right)$$

Noting $\delta_1 = \delta_2 = \mathrm{Arctan}\left(\frac{K_1}{K_2}\right)$, and noting $K = \sqrt{K_1^2 + K_2^2}\,\frac{\alpha}{\sqrt{c}}$, where K_1, K_2 are constants determined by initial conditions,

$$\therefore\ \boxed{\begin{aligned} u(t) &= \frac{c}{\gamma} K \cos\left(\sqrt{ac}\,t + \delta\right) \\ v(t) &= \frac{a}{\alpha}\sqrt{\frac{c}{a}}\,K \sin\left(\sqrt{ac}\,t + \delta\right) \end{aligned}}$$

Example 2, pp. 439-440

The domain Ω could actually be $-\pi < x < \pi$, and V would still be positive definite. $\dot{V}$ remains zero for all x, y. Perhaps chose $(-\frac{\pi}{2}, \frac{\pi}{2})$ which corresponds to $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, a more physically realistic interval.

Theorem 9.64

The function

$$V(x, y) = ax^2 + bxy + cy^2 \tag{14}$$

is positive definite if, and only if,

$$a > 0 \text{ and } 4ac - b^2 > 0 \tag{15}$$

and is negative definite if, and only if,

$$a < 0 \text{ and } 4ac - b^2 > 0. \tag{16}$$

Proof: Note $V(0,0) = 0$. Assume $a, c > 0$ or $a, c < 0$.

Otherwise, $4ac - b^2 > 0$ is false.

$$\therefore V = a\left[x^2 + \frac{by}{a}x + \frac{c}{a}y^2\right]$$

$$= a\left[\left(x + \frac{by}{2a}\right)^2 - \frac{b^2y^2}{4a^2} + \frac{c}{a}y^2\right]$$

$$= a\left[\left(x+\frac{by}{2a}\right)^2 + y^2\left(\frac{4ac-b^2}{4a^2}\right)\right] \quad [1]$$

$$= c\left[\left(y+\frac{bx}{2c}\right)^2 + x^2\left(\frac{4ac-b^2}{4c^2}\right)\right] \quad [2]$$

(1) $4ac-b^2>0$

$$\left(x+\frac{by}{2a}\right)^2 \geq 0 \text{ for all } x, y$$

If $(x,y)\neq\vec{0}$, Then $x\neq 0$ or $y\neq 0$.

If $y\neq 0$, $y^2\left(\frac{4ac-b^2}{4a^2}\right)>0 \Rightarrow$

$$\left[\left(x+\frac{by}{2a}\right)^2 + y^2\left(\frac{4ac-b^2}{4a^2}\right)\right] > 0$$

$\therefore$ (i) $a>0 \Rightarrow V$ is positive definite

(ii) $a<0 \Rightarrow V$ is negative definite

If $y=0$, then $x\neq 0 \Rightarrow V=ax^2$.

$\therefore$ (i) $a>0 \Rightarrow ax^2>0 \Rightarrow V$ is positive definite

(ii) $a<0 \Rightarrow ax^2<0 \Rightarrow V$ is negative definite

Summary:

$a>0$ and $4ac-b^2>0 \Rightarrow V$ is positive definite

$a<0$ and $4ac-b^2>0 \Rightarrow V$ is negative definite

(2) $V(x,y)$ is positive definite

Consider the points $(x,0)$ or $(0,y)$, so

$V(x,0) > 0$ or $V(0,y) > 0$.

$\therefore$ $V = ax^2$ or $V = cy^2$. $\therefore$ $a > 0$ and $c > 0$

From [1], $\left(x + \frac{by}{2a}\right)^2 + y^2\left(\frac{4ac - b^2}{4a^2}\right) > 0$ [3]

for $(x,y) \neq (0,0)$.

If $b = 0$, then clearly $4ac - b^2 > 0$

If $b \neq 0$, for (x,y) s.t. $x + \frac{by}{2a} = 0$, then

[3] becomes $y^2\left(\frac{4ac - b^2}{4a^2}\right) > 0$, so

$4ac - b^2 > 0$

$\therefore$ V positive definite $\Rightarrow$ $a > 0$, $c > 0$, and

$4ac - b^2 > 0$.

(3) $V(x,y)$ is negative definite

Consider the points $(x,0)$ or $(0,y)$, so

$V(x,0) < 0$ or $V(0,y) < 0$.

$\therefore V = ax^2$ or $V = cy^2$. $\therefore a < 0$ and $c < 0$.

From [1], $\left(x + \frac{by}{2a}\right)^2 + y^2\left(\frac{4ac - b^2}{4a^2}\right) > 0$ [3]

for $(x, y) \neq (0, 0)$.

If $b = 0$, then clearly $4ac - b^2 > 0$

If $b \neq 0$, for (x, y) s.t. $x + \frac{by}{2a} = 0$, then

[3] becomes $y^2\left(\frac{4ac - b^2}{4a^2}\right) > 0$, so

$4ac - b^2 > 0$

$\therefore V$ negative definite $\Rightarrow a < 0$, $c < 0$, and

$4ac - b^2 > 0$.

Example 1 $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} x + y - x(x^2+y^2) \\ -x + y - y(x^2+y^2) \end{pmatrix}$. $= \begin{bmatrix} F(x,y) \\ G(x,y) \end{bmatrix}$

(a) $(0,0)$ is the only critical point.

$x+y = x(x^2+y^2)$ [1]

$-x+y = y(x^2+y^2)$ [2]

Suppose (x,y) is a critical point and $(x,y) \neq (0,0)$.

Then x or y is not zero.

Suppose $x \neq 0$. $\therefore \frac{x+y}{x} = x^2+y^2$, from [1]

$\therefore -x+y = y\left(\frac{x+y}{x}\right) \Rightarrow -x^2+xy = xy+y^2$

$\Rightarrow -x^2 = y^2 \Rightarrow x = y = 0$

Suppose $y \neq 0$. $\therefore \frac{-x+y}{y} = x^2+y^2$, from [2]

$\therefore x+y = x\left(\frac{-x+y}{y}\right) \Rightarrow xy+y^2 = -x^2+xy$

$\Rightarrow y^2 = -x^2 \Rightarrow x = y = 0.$

$\therefore (0,0)$ is the only critical point.

(b) System is locally linear since [1], [2] have continuous partial derivatives up to order 2, and using Theorem 9.3.2 on page 410.

$$J = \begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix} = \begin{bmatrix} 1-3x^2-y^2 & 1-2xy \\ -1-2xy & 1-x^2-3y^2 \end{bmatrix}$$

(c) $\therefore J[0,0] = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

Eigenvalues : $(\lambda-1)^2+1=0 \Rightarrow \lambda^2-2\lambda+2=0$

$$\Rightarrow \lambda = \frac{2 \pm \sqrt{4-8}}{2} = \underline{1 \pm i}$$

(d) Since real part is $1>0$, unstable spiral point.

(e) To obtain $d\theta/dt$,

$$x = r\cos\theta \quad , \quad y = r\sin\theta$$

$$\therefore dx/dt = \frac{dr}{dt}\cos(\theta) - r\sin(\theta)\frac{d\theta}{dt}$$

$$dy/dt = \frac{dr}{dt}\sin(\theta) + r\cos(\theta)\frac{d\theta}{dt}$$

$$\therefore y\frac{dx}{dt} = r\frac{dr}{dt}\sin(\theta)\cos(\theta) - r^2\sin^2\theta\frac{d\theta}{dt}$$

$$x\frac{dy}{dt} = r\frac{dr}{dt}\sin(\theta)\cos(\theta) + r^2\cos^2\theta\frac{d\theta}{dt}$$

$$\therefore\ y\frac{dx}{dt} - x\frac{dy}{dt} = -r^2(\sin^2\theta + \cos^2\theta)\frac{d\theta}{dt}$$
$$= -r^2\frac{d\theta}{dt}$$

Also, from $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} x + y - x(x^2+y^2) \\ -x + y - y(x^2+y^2) \end{pmatrix}.$ (4)

$$y\frac{dx}{dt} = xy + y^2 - xy(x^2+y^2)$$
$$x\frac{dy}{dt} = -x^2 + xy - xy(x^2+y^2)$$

$$\therefore\ y\frac{dx}{dt} - x\frac{dy}{dt} = y^2 + x^2 = r^2$$

$$\therefore\ -r^2\frac{d\theta}{dt} = r^2 \Rightarrow \frac{d\theta}{dt} = -1 \Rightarrow \theta(t) = -t + t_0$$

(f) $\dfrac{dr}{r(1-r^2)} = dt$

Using MATLAB,

```
clear
syms r
partfrac(1/(r*(1-r^2)))
```

```
ans =
```

$$\frac{1}{r} - \frac{1}{2\,(r+1)} - \frac{1}{2\,(r-1)}$$

$$\therefore \int \frac{1}{r} - \frac{1}{2(r+1)} - \frac{1}{2(r-1)}\,dr = \int dt$$

$$\therefore\ \ln(r) - \frac{1}{2}\ln(r+1) - \frac{1}{2}\ln(r-1) = t + c$$

$$\therefore \ln \frac{r}{\sqrt{r^2-1}} = t + c, \quad \frac{r}{\sqrt{r^2-1}} = Ke^{t}$$

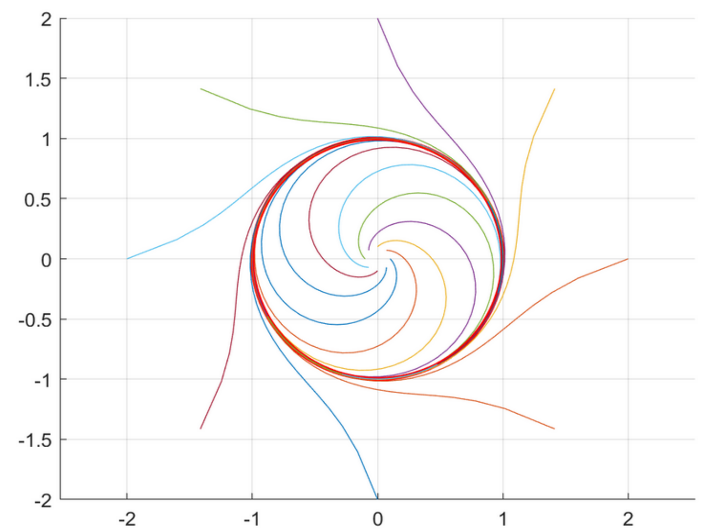
$$\frac{r^2}{r^2-1} = Ke^{2t}, \quad r^2 = Ke^{2t} r^2 - Ke^{2t}$$

$$Ke^{2t} = r^2\left(Ke^{2t}-1\right), \quad r^2 = \frac{Ke^{2t}}{Ke^{2t}-1}$$

Or, $r^2 = \dfrac{1}{1 - \frac{1}{K}e^{-2t}}$, or $\underline{r = \dfrac{1}{\sqrt{1 + C_0 e^{-2t}}}}$

(g) MATLAB code

```
clear
figure
hold on
grid on
t = 0:0.1:5;
for p = [0.1, 1, 2]
    for a = [0, pi/4, pi/2, 3*pi/4, pi,...
             5*pi/4, 3*pi/2, 7*pi/4]
        r = 1./(sqrt(1+((1/p^2)-1)*exp(-2*t)));
        th = a - t;
        x = r.*cos(th);
        y = r.*sin(th);
        if p == 1
            plot(x,y, 'r', 'LineWidth', 2)
        else
            plot(x,y)
        end
    end
end
axis equal
```



Example 2

Equation 21: $$r' = \mu(1 - r^2\cos^2\theta)\, r \sin^2\theta. \tag{21}$$

Using $x = r\cos\theta$, $y = r\sin\theta$, $r^2 = x^2 + y^2$,

$$\therefore\ 2r\frac{dr}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \qquad [3]$$

Using $x = u$, $y = u' = x'$ and

$$u'' - \mu(1-u^2)u' + u = 0, \qquad (17)$$

$$x' = y$$

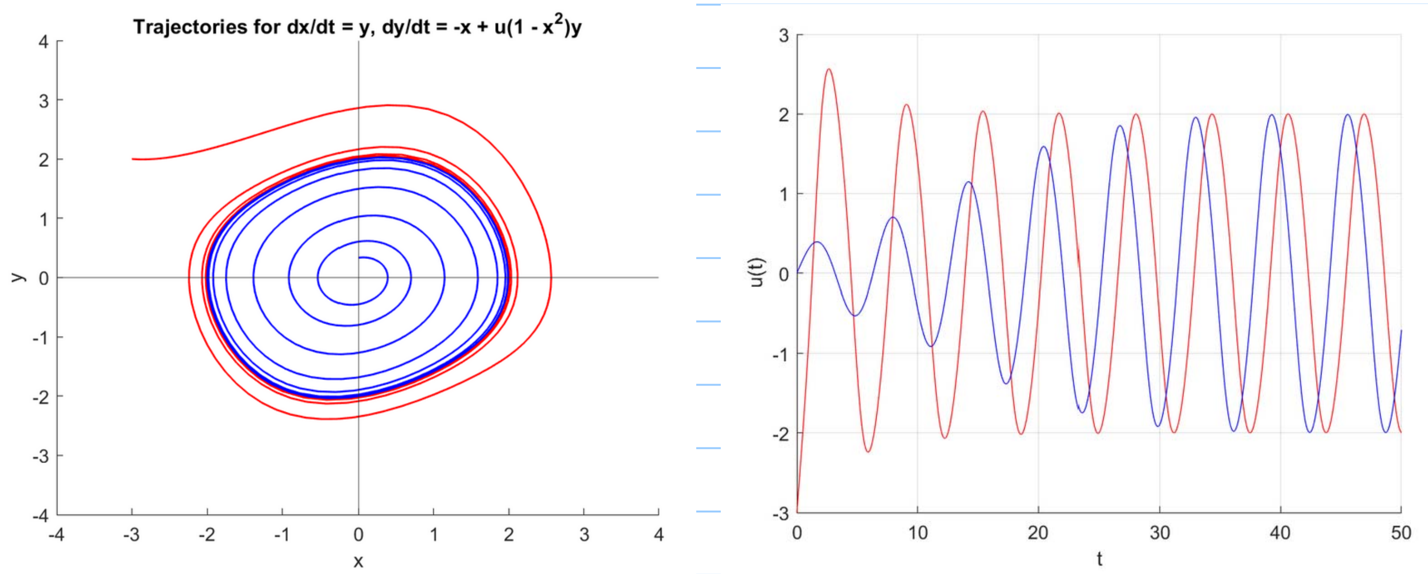
$$y' = -x + \mu(1-x^2)y$$

$\therefore$ From [3],

$$r\frac{dr}{dt} = x(y) + y\left[-x + \mu(1-x^2)y\right] = \mu(1-x^2)y^2$$

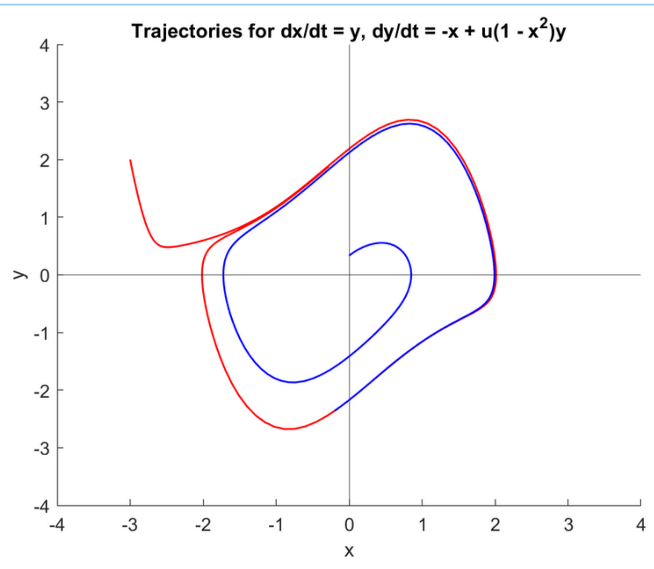
$$= \mu(1 - r^2\cos^2\theta)r^2\sin^2\theta$$

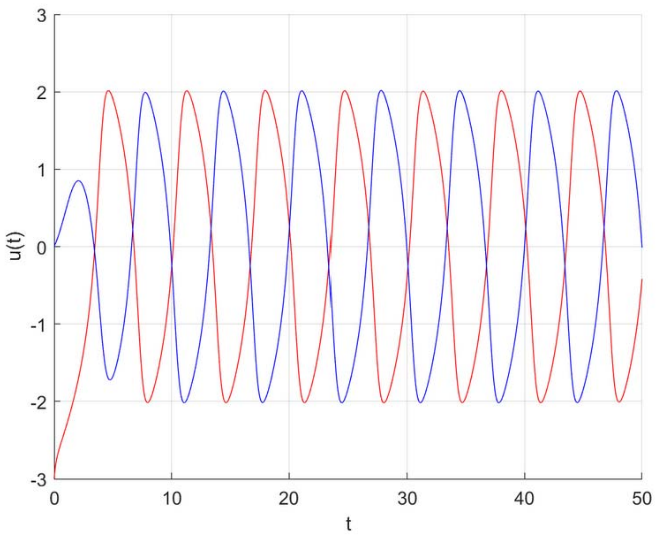
$$\therefore\ \frac{dr}{dt} = \underline{\mu(1-r^2\cos^2\theta)r\sin^2\theta}$$



```
clear
% set plot boundaries around critical points
Xmin = -4.0; Xmax = 4.0;
Ymin = -4.0; Ymax = 4.0;
u = 0.2;

figure
axis([Xmin, Xmax, Ymin, Ymax])
xline(0);  % show x and y axes
yline(0);
xlabel 'x', ylabel 'y'
st1 = 'Trajectories for ';
st2 = ' dx/dt = y, dy/dt = -x + u(1 - x^2)y';
title(strcat(st1,st2))

% use ode45 to plot trajectories
hold on
% ode45 uses a function handle
% x = z(1), y = z(2)
f = @(t,z) [z(2);...
            -z(1) + u*(1 - z(1)^2)*z(2)];
tspan = [0, 50];  % time interval for plot
% xy0 is several initial values as [x;y] points
xy0 = [-3.0, 0;
        2.0, 1/3];
% more ode45 points => smoother plot
opts = odeset('Refine', 8);
[t,z] = ode45(f, tspan, xy0(:,1), opts);
plot(z(:,1), z(:,2), 'r', 'LineWidth', 1)
[t,z] = ode45(f, tspan, xy0(:,2), opts);
plot(z(:,1), z(:,2), 'b', 'LineWidth', 1)

% plot u = x(t) over time
tspan = [0, 50];
figure
grid on
hold on
[t,z] = ode45(f, tspan, [-3,2], opts);
plot(t,z(:,1),'r')
[t,z] = ode45(f, tspan, [0,1/3], opts);
plot(t,z(:,1),'b')
xlabel 't', ylabel 'u(t)'
```

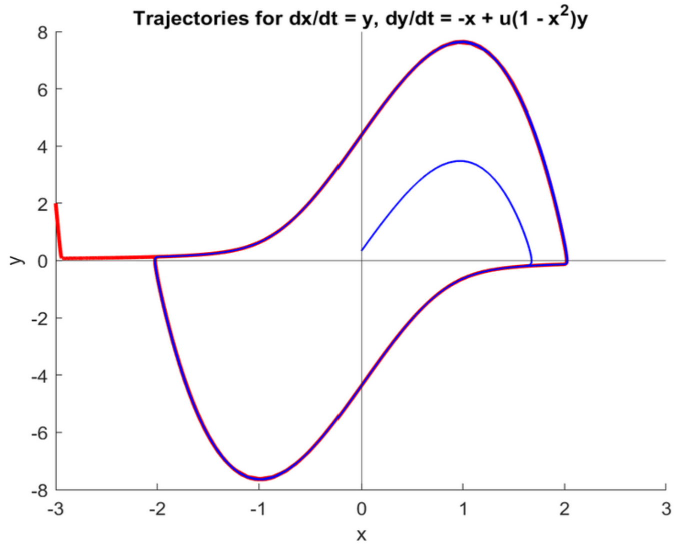
For $u = 1.0$

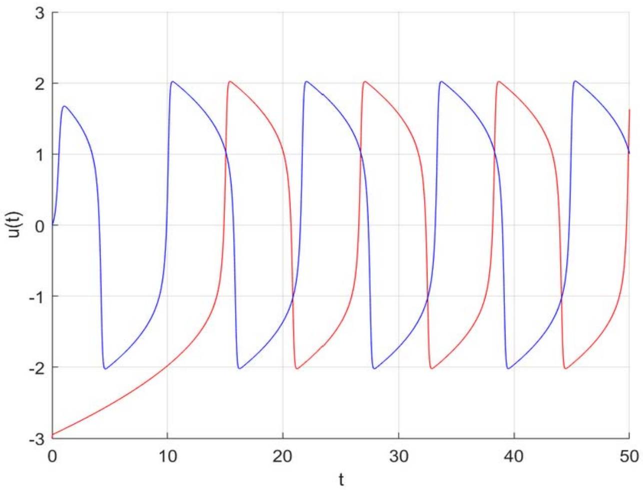




```
clear
% set plot boundaries around critical points
Xmin = -4.0; Xmax = 4.0;
Ymin = -4.0; Ymax = 4.0;
u = 1.0;

figure
axis([Xmin, Xmax, Ymin, Ymax])
xline(0);  % show x and y axes
yline(0);
xlabel 'x', ylabel 'y'
st1 = 'Trajectories for ';
st2 = ' dx/dt = y, dy/dt = -x + u(1 - x^2)y';
title(strcat(st1,st2))

% use ode45 to plot trajectories
hold on
% ode45 uses a function handle
% x = z(1), y = z(2)
f = @(t,z) [z(2);...
            -z(1) + u*(1 - z(1)^2)*z(2)];
tspan = [0, 10];  % time interval for plot
% xy0 is several initial values as [x;y] points
xy0 = [-3.0, 0;
        2.0, 1/3];
% more ode45 points => smoother plot
opts = odeset('Refine', 8);
[t,z] = ode45(f, tspan, xy0(:,1), opts);
plot(z(:,1), z(:,2), 'r', 'LineWidth', 1)
[t,z] = ode45(f, tspan, xy0(:,2), opts);
plot(z(:,1), z(:,2), 'b', 'LineWidth', 1)

% plot u = x(t) over time
tspan = [0, 50];
figure
grid on
hold on
[t,z] = ode45(f, tspan, [-3,2], opts);
plot(t,z(:,1),'r')
[t,z] = ode45(f, tspan, [0,1/3], opts);
plot(t,z(:,1),'b')
xlabel 't', ylabel 'u(t)'
```

For $u = 5.0$





```
clear
% set plot boundaries around critical points
Xmin = -3.0; Xmax = 3.0;
Ymin = -8.0; Ymax = 8.0;
u = 5.0;

figure
axis([Xmin, Xmax, Ymin, Ymax])
xline(0);  % show x and y axes
yline(0);
xlabel 'x', ylabel 'y'
st1 = 'Trajectories for ';
st2 = ' dx/dt = y, dy/dt = -x + u(1 - x^2)y';
title(strcat(st1,st2))

% use ode45 to plot trajectories
hold on
% ode45 uses a function handle
% x = z(1), y = z(2)
f = @(t,z) [z(2);...
            -z(1) + u*(1 - z(1)^2)*z(2)];
tspan = [0, 50];  % time interval for plot
% xy0 is several initial values as [x;y] points
xy0 = [-3.0, 0;
        2.0, 1/3];
% more ode45 points => smoother plot
opts = odeset('Refine', 8);
[t,z] = ode45(f, tspan, xy0(:,1), opts);
plot(z(:,1), z(:,2), 'r', 'LineWidth', 2)
[t,z] = ode45(f, tspan, xy0(:,2), opts);
plot(z(:,1), z(:,2), 'b', 'LineWidth', 1)

% plot u = x(t) over time
tspan = [0, 50];
figure
grid on
hold on
[t,z] = ode45(f, tspan, [-3,2], opts);
plot(t,z(:,1),'r')
[t,z] = ode45(f, tspan, [0,1/3], opts);
plot(t,z(:,1),'b')
xlabel 't', ylabel 'u(t)'
```

The value of r, for the nonzero critical point on p. 456 is derived as follows, using MATLAB

```
clear
syms r n
J = [    -10,                  10,                  0;
           1,                  -1,          -sqrt(8*(r-1)/3);
      sqrt(8*(r-1)/3), sqrt(8*(r-1)/3),         -8/3];
p = det(J - n*eye(3));
p = collect(-3*p,n)  % positve integer leading coefficient
dp = diff(p,n,1);  % find rel max and min of p
c = coeffs(dp,n,'all')
rdp = roots(c)  % values of n where derivative is 0
% the two values are a function of r
pr2 = subs(p,n,rdp(2));  % take p value of rel min
% pr2 is a function of r
s = vpa(solve(pr2==0),8)  % value of r where rel min is 0
p = subs(p,r,1.4);  % example plot
fplot(p)
xlim([-15,5])
ylim([-100,500])
grid on
```

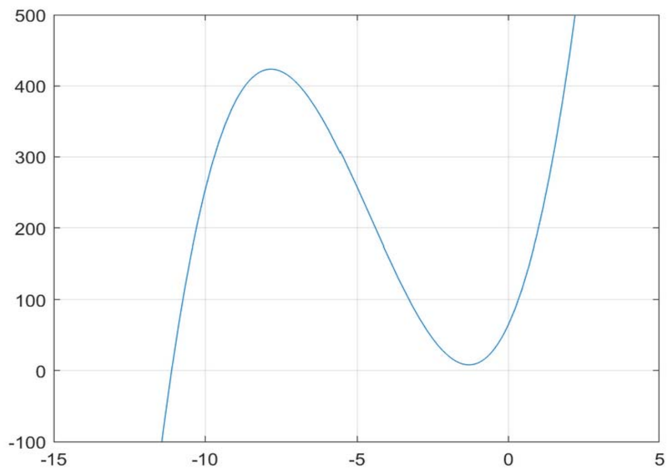
p = $3\,n^3 + 41\,n^2 + (8\,r + 80)\,n + 160\,r - 160$

c = $\begin{pmatrix} 9 & 82 & 8\,r + 80 \end{pmatrix}$

rdp =

$$\begin{pmatrix} -\dfrac{\sqrt{\dfrac{3844}{81} - \dfrac{32\,r}{9}}}{2} - \dfrac{41}{9} \\ \dfrac{\sqrt{\dfrac{3844}{81} - \dfrac{32\,r}{9}}}{2} - \dfrac{41}{9} \end{pmatrix}$$

s = 1.3456172

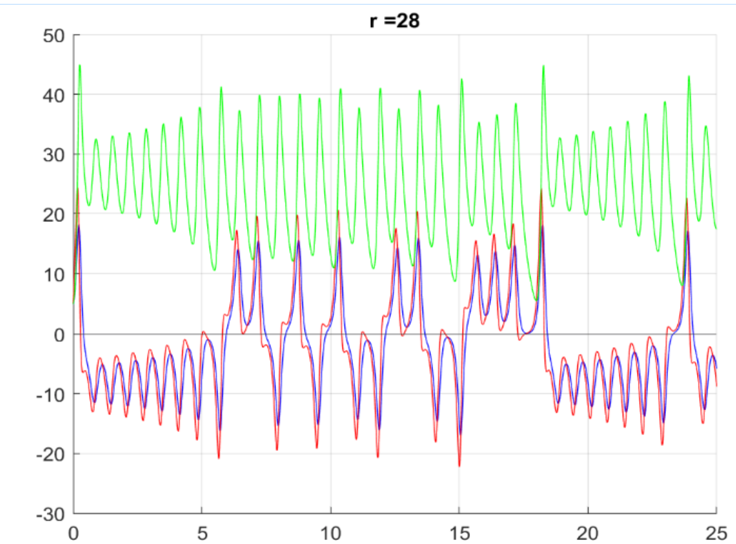


Using $r = 1.4$, the relative min is just above the $y = 0$ axis.

Note pr2(r) is a monotonically increasing function on $\left[0, \frac{3844}{9(32)}\right]$, seen by looking at its expression in MATLAB. Thus, there is only one zero of pr2 on the interval the interval for which the roots of $p(r)$ are real.

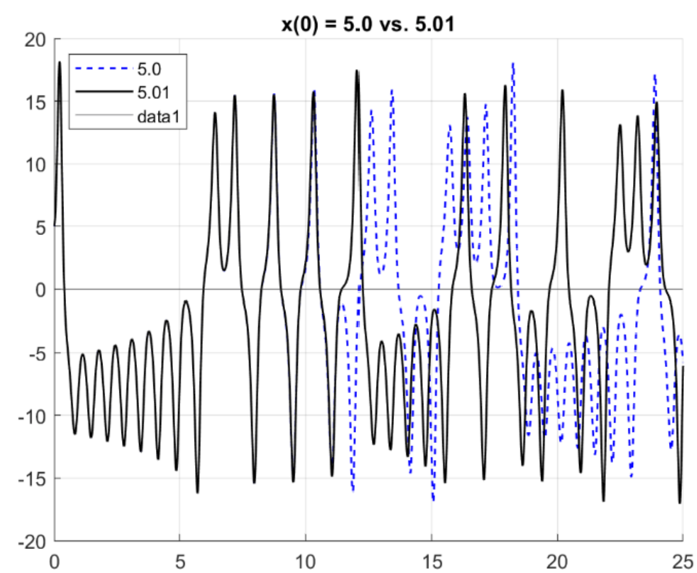
Using MATLAB, a plot of the x, y, z vs. t is shown below. Same plot as Figure 9.8.2 on p. 457

```
clear
s = 10; b = 8/3;
r = 28;
cp = [5,5,5];

% use ode45 to plot trajectories
% ode45 uses a function handle
% x = z(1), y = z(2), z = z(3)
f = @(t,z) [s*(-z(1) + z(2));...
            r*z(1) - z(2) - z(1)*z(3);...
            -b*z(3) + z(1)*z(2)];
tspan = [0, 25];  % time interval for plot
% more ode45 points => smoother plot
opts = odeset('Refine', 10);
[t,z] = ode45(f, tspan, cp, opts);
figure
hold on
grid on
plot(t,z(:,1), 'b')
plot(t,z(:,2), 'r')
plot(t,z(:,3), 'g')
title(strcat('r =  ',num2str(r)))
yline(0)
```



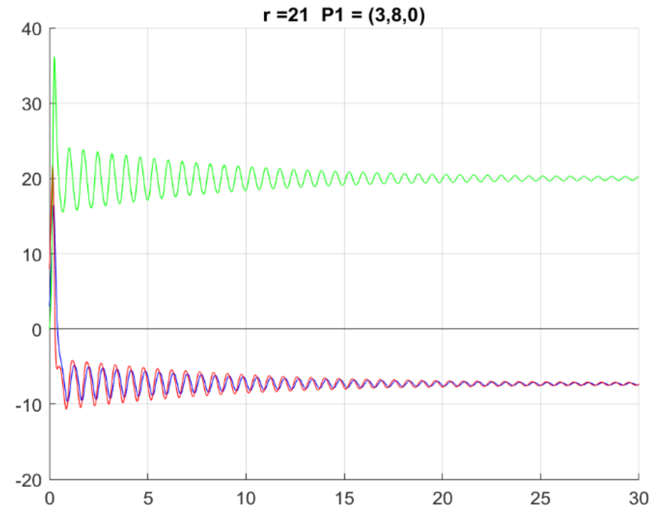
And Figure 9.8.3 on p. 458

```
clear
s = 10; b = 8/3;
r = 28;
cp = [5,5,5];

% use ode45 to plot trajectories
% ode45 uses a function handle
% x = z(1), y = z(2), z = z(3)
f = @(t,z) [s*(-z(1) + z(2));...
            r*z(1) - z(2) - z(1)*z(3);...
            -b*z(3) + z(1)*z(2)];
tspan = [0, 25];  % time interval for plot
% more ode45 points => smoother plot
opts = odeset('Refine', 10);
[t,z] = ode45(f, tspan, cp, opts);
figure
hold on
grid on
plot(t,z(:,1), '--b', 'LineWidth', 1)
cp = [5.01,5,5];
[t,z] = ode45(f, tspan, cp, opts);
plot(t,z(:,1), 'k', 'LineWidth', 1)
title('x(0) = 5.0 vs. 5.01')
legend('5.0', '5.01', 'Location', 'northwest')
yline(0)
```



For Figure 9.8.4 (a), initial point = (3,8,0)

```
clear
s = 10; b = 8/3;
r = 21;
cp = [3,8,0];

% use ode45 to plot trajectories
% ode45 uses a function handle
% x = z(1), y = z(2), z = z(3)
f = @(t,z) [s*(-z(1) + z(2));...
            r*z(1) - z(2) - z(1)*z(3);...
            -b*z(3) + z(1)*z(2)];
tspan = [0, 30];  % time interval for plot
% more ode45 points => smoother plot
opts = odeset('Refine', 10);
[t,z] = ode45(f, tspan, cp, opts);
figure
hold on
grid on
plot(t,z(:,1), 'b')
plot(t,z(:,2), 'r')
plot(t,z(:,3), 'g')
title(strcat('r =  ',num2str(r),'  P1 = (3,8,0)'))
yline(0)
```



9.8.4 (b), initial point (5,5,10)

```
clear
s = 10; b = 8/3;
r = 21;
ip = [5,5,10];  % initial point

% use ode45 to plot trajectories
% ode45 uses a function handle
% x = z(1), y = z(2), z = z(3)
f = @(t,z) [s*(-z(1) + z(2));...
            r*z(1) - z(2) - z(1)*z(3);...
            -b*z(3) + z(1)*z(2)];
tspan = [0, 50];  % time interval for plot
% more ode45 points => smoother plot
opts = odeset('Refine', 10);
[t,z] = ode45(f, tspan, ip, opts);
figure
hold on
grid on
plot(t,z(:,1), 'b')
plot(t,z(:,2), 'r')
plot(t,z(:,3), 'g')
title(strcat('r =  ',num2str(r),'  ip = (5,5,10)'))
yline(0)
```

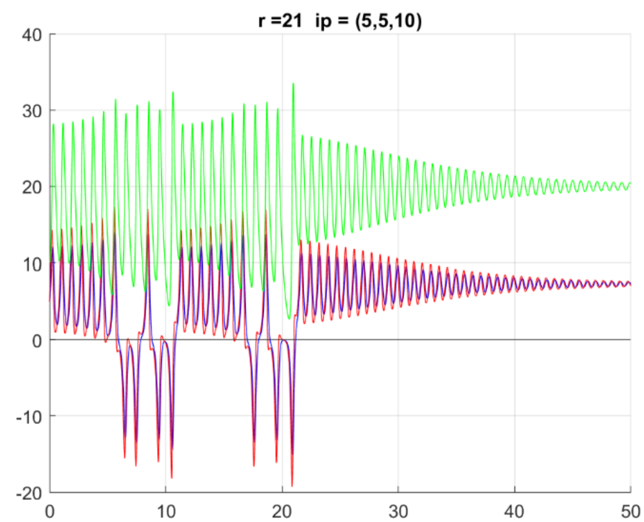


Figure 9.8.4 (c), initial point = (5,5,5)

```
clear
s = 10; b = 8/3;
r = 21;
ip = [5,5,5];  % initial point

% use ode45 to plot trajectories
% ode45 uses a function handle
% x = z(1), y = z(2), z = z(3)
f = @(t,z) [s*(-z(1) + z(2));...
            r*z(1) - z(2) - z(1)*z(3);...
            -b*z(3) + z(1)*z(2)];
tspan = [0, 150];  % time interval for plot
% more ode45 points => smoother plot
opts = odeset('Refine', 10);
[t,z] = ode45(f, tspan, ip, opts);
figure
hold on
grid on
plot(t,z(:,1), 'b')
plot(t,z(:,2), 'r')
plot(t,z(:,3), 'g')
title(strcat('r =  ',num2str(r),'  ip = (5,5,5)'))
yline(0)
```

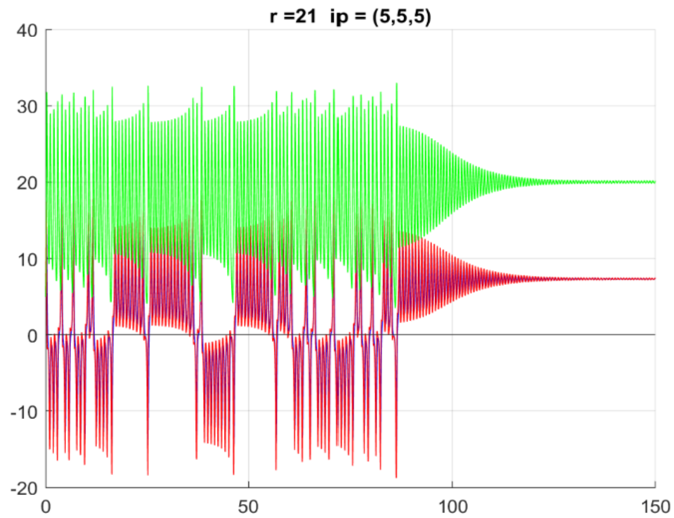


Figure 9.8.5 on p. 459

```
clear
s = 10; b = 8/3;
r = 28;
ip = [5,5,5];  % initial point

% use ode45 to plot trajectories
% ode45 uses a function handle
% x = z(1), y = z(2), z = z(3)
f = @(t,z) [s*(-z(1) + z(2));...
            r*z(1) - z(2) - z(1)*z(3);...
            -b*z(3) + z(1)*z(2)];
tspan = [0, 25];  % time interval for plot
% more ode45 points => smoother plot
opts = odeset('Refine', 10);
[t,z] = ode45(f, tspan, ip, opts);
figure
hold on
grid on
plot(z(:,1),z(:,2), 'b')
xlabel('x'); ylabel('y')
title(strcat('r =  ',num2str(r)))
```

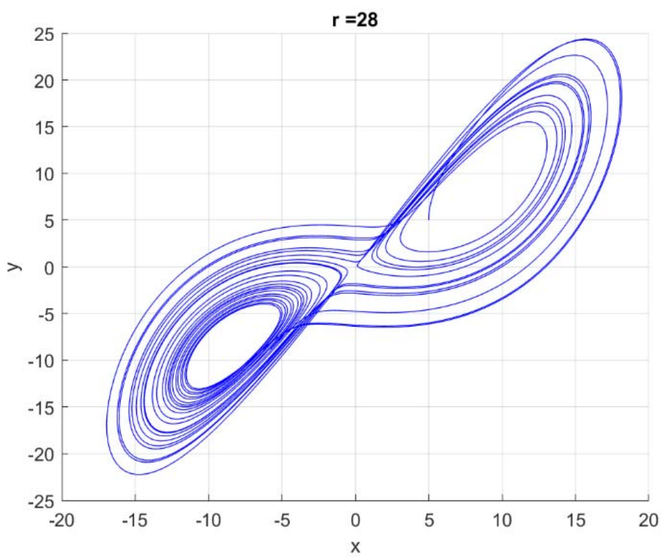
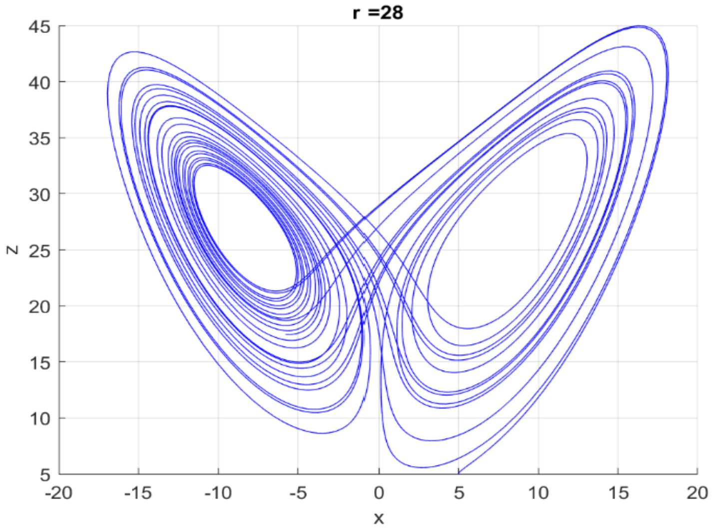


Figure 9.8.6 on p. 459

```
clear
s = 10; b = 8/3;
r = 28;
ip = [5,5,5];  % initial point

% use ode45 to plot trajectories
% ode45 uses a function handle
% x = z(1), y = z(2), z = z(3)
f = @(t,z) [s*(-z(1) + z(2));...
            r*z(1) - z(2) - z(1)*z(3);...
            -b*z(3) + z(1)*z(2)];
tspan = [0, 25];  % time interval for plot
% more ode45 points => smoother plot
opts = odeset('Refine', 10);
[t,z] = ode45(f, tspan, ip, opts);
figure
hold on
grid on
plot(z(:,1),z(:,3), 'b')
xlabel('x'); ylabel('z')
title(strcat('r =  ',num2str(r)))
```



10.1 Two - Point Boundary Value Problems

On page 467, $y = c_1 \cosh(\mu x) + c_2 \sinh(\mu x) \Leftrightarrow$

$$y = a_1 e^{\mu x} + a_2 e^{-\mu x}$$

(1) if $y = c_1 \cosh(\mu x) + c_2 \sinh(\mu x)$, then

$$y = c_1 \frac{e^{\mu x} + e^{-\mu x}}{2} + c_2 \frac{e^{\mu x} - e^{-\mu x}}{2}$$

$$= \left(\frac{c_1 + c_2}{2}\right) e^{\mu x} + \left(\frac{c_1 - c_2}{2}\right) e^{-\mu x}$$

$\therefore$ Choose $a_1 = \frac{c_1 + c_2}{2}$, $a_2 = \frac{c_1 - c_2}{2}$ [1]

Or,

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

Let $T = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$. T^{-1} exists

$\therefore$ Given any $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$, since $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = T^{-1} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$

is a solution, then $T \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ is a

solution.

More explicitly, given any a_1, a_2, since

$y = (a_1 + a_2)\cosh(\mu x) + (a_1 - a_2)\sinh(\mu x)$ is a solution,

then $y = a_1[\cosh(\mu x) + \sinh(\mu x)] + a_2[\cosh(\mu x) - \sinh(\mu x)]$

$$= a_1\left[\frac{e^{\mu x} + e^{-\mu x}}{2} + \frac{e^{\mu x} - e^{-\mu x}}{2}\right] + a_2\left[\frac{e^{\mu x} + e^{-\mu x}}{2} - \frac{e^{\mu x} - e^{-\mu x}}{2}\right]$$

$= a_1 e^{\mu x} + a_2 e^{-\mu x}$ is a solution.

(2) if $y = a_1 e^{\mu x} + a_2 e^{-\mu x}$ is a solution for all a_1 and a_2, then

given any c_1 and c_2, since

$y = \left(\frac{c_1 + c_2}{2}\right) e^{\mu x} + \left(\frac{c_1 - c_2}{2}\right) e^{-\mu x}$ is a solution,

where $a_1 = \frac{c_1 + c_2}{2}$, $a_2 = \frac{c_1 - c_2}{2}$,

then $y = c_1\left(\frac{e^{\mu x} + e^{-\mu x}}{2}\right) + c_2\left(\frac{e^{\mu x} - e^{-\mu x}}{2}\right)$

$$= c_1 \cosh(\mu x) + c_2 \sinh(\mu x)$$

is a solution.

$\blacksquare$

Eigenvalues of boundary value problem are real.

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(L) = 0, \quad L \text{ is real.}$$

Proof: Let $\lambda = \mu^2$. $\therefore$ $y = K_1 e^{i\mu x} + K_2 e^{-i\mu x}$ is a general solution to $y'' + \lambda y = 0$.

(a) Nontrivial solutions exist $\Leftrightarrow$ $e^{i\mu x} - e^{-i\mu x} = 0, \forall x$

(1) Suppose $y(x) = K_1 e^{i\mu x} + K_2 e^{-i\mu x}$ is nontrivial

$\therefore$ $y(0) = K_1 e^{i\mu(0)} + K_2 e^{-i\mu(0)} = K_1 + K_2 = 0$

$\therefore$ $K_1 \neq 0$, $K_2 \neq 0$ and $K_2 = -K_1$

$\therefore$ $y(L) = K_1 e^{i\mu L} - K_1 e^{-i\mu L} = 0 \Rightarrow e^{2i\mu L} = 1$

$\therefore$ $2i\mu L = 0 \Rightarrow \mu = 0$.

$\therefore$ $e^{i\mu x} - e^{-i\mu x} = 0$

(2) Suppose $e^{i\mu x} - e^{-i\mu x} = 0$ for all x. $\therefore$ Let $x = L$.

$\therefore$ $e^{i\mu L} - e^{-i\mu L} = 0$. Let $K_1 \neq 0$ and $K_2 = -K_1$

$\therefore$ $y(x) = K_1 e^{i\mu x} + K_2 e^{-i\mu x}$ is nontrivial,

$y(0) = 0$ and $y(L) = 0$ and $y(x)$ is

a nontrivial solution to $y'' + \lambda y = 0$

(b) Let $\mu = v + i\sigma$, where v and σ are real.

$\therefore e^{i\mu x} - e^{-i\mu x} = e^{i(v+i\sigma)x} - e^{-i(v+i\sigma)x}$

$= e^{ivx}e^{-\sigma x} - e^{-ivx}e^{\sigma x} = 0$ from (a)

$\Rightarrow e^{2ivx} = e^{2\sigma x}$

$\therefore e^{ivx} = e^{\sigma x}$ for all x.

$\therefore e^{ivL} = e^{\sigma L}$, $\cos(vL) + i\sin(vL) = e^{\sigma L}$

Equating real and imaginary parts,

$\sin(vL) = 0 \Rightarrow v = 0, \pm\frac{\pi}{L}, \pm\frac{2\pi}{L}, \ldots$, or

$v = \frac{n\pi}{L}$, $n = 0, \pm 1, \pm 2, \ldots$

$\therefore \cos(vL) = \cos(n\pi) = \pm 1 = e^{\sigma L}$

Since $e^{\sigma L} > 0$, $e^{\sigma L} = 1 \Rightarrow \sigma L = 0 \Rightarrow \sigma = 0$

$\therefore \mu = v + i\sigma = v$, so μ is real. $\therefore \lambda = \mu^2$ is real.

$\therefore$ If $y(x) = K_1 e^{i\mu x} + K_2 e^{-i\mu x}$ is a nontrivial solution satisfying the boundary conditions, then $\lambda = \mu^2$ is real.

Equation (21) states:

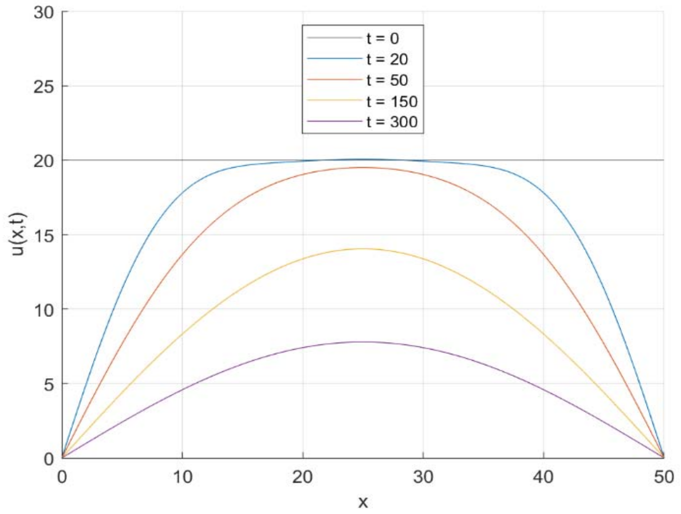
$$c_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx. \tag{21}$$

Note that $f(x)$ is defined over $0 \le x \le L$, and (21) has made an odd extension for $f(x)$ defined on $-L \le x \le L$. $\therefore$ $f(x)\sin\left(\frac{n\pi x}{L}\right)$ is an even function, justifying $\frac{2}{L}\int_0^L \cdots$.

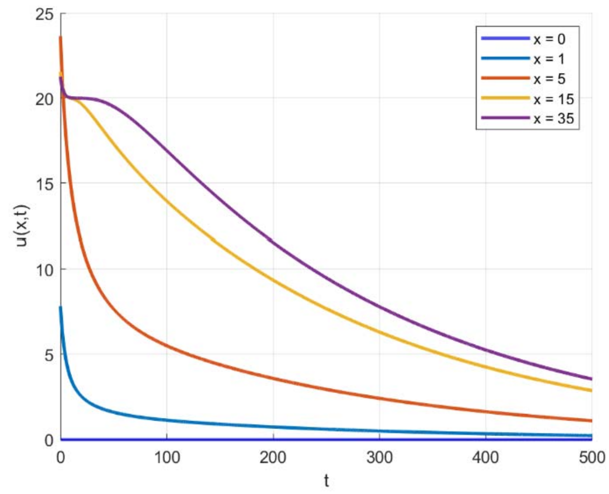
Note that the odd extension redefines $f(0) = 0$. In Example (1), $f(x)$ is defined on $0 < x < 50$, so the odd extension and Fourier series mean $f(0) = 0$, yielding the Gibbs behavior at $x = 0$ as that is a point of discontinuity.

Using MATLAB, Example 1 plots on next page.

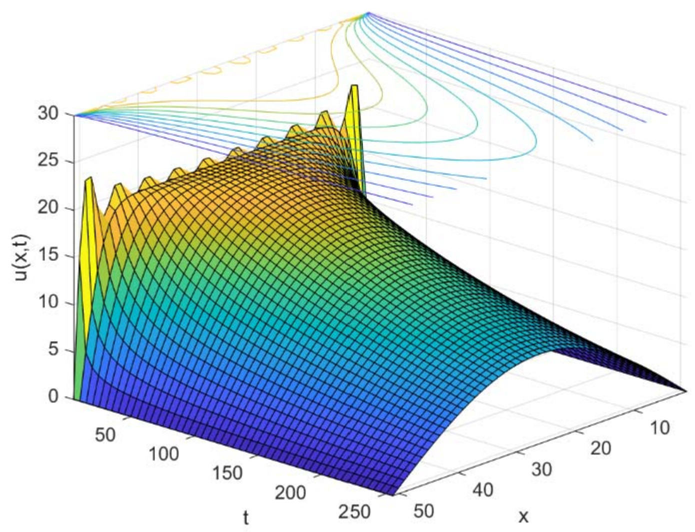
```
clear
a = pi^2/2500;
x = 0:0.01:50;
figure
ylim([0,30])  % make room for legend
yline(20)     % f(x) at t = 0
hold on
grid on
for t = [20,50,150,300]
    sum = 0;
    for n = [1,3,5]
        e = (80/pi)*exp(-n^2*a*t);
        s = sin(n*pi*x/50);
        sum = sum + e*s/n;
    end
    plot(x,sum)
end
xlabel('x')
ylabel('u(x,t)')
legend('t = 0', 't = 20', 't = 50',...
       't = 150', 't = 300','Location',"north")
```




```
clear
a = pi^2/2500;
t = 0:0.1:500;
figure
yline(0,'b','LineWidth',2)    % u(x,t) at x = 0
hold on
grid on
for x = [1,5,15,25]
    sum = 0;
    for n = [1,3,5,7,9]       % first 5 terms
        e = (80/pi)*exp(-n^2*a*t);
        s = sin(n*pi*x/50);
        sum = sum + e*s/n;
    end
    plot(t,sum,'LineWidth',2)
end
xlabel('t')
ylabel('u(x,t)')
legend('x = 0', 'x = 1', 'x = 5',...
       'x = 15', 'x = 35','Location',"northeast")
```




```
clear
a = pi^2/2500;
% skip 5 on t array to spread out colors
[x,t] = meshgrid(0:50, 0:5:250);
u = zeros(size(x,1), size(x,2));
for n = 1:2:19      % first 10 terms
    e = (80/pi)*exp(-n^2*a*t);
    s = sin(n*pi*x/50);
    u = u + e.*s/n;  % u(x,t) to 10 terms
end

sc = surfc(u);   % plot surface and contours
ax = gca;        % these lines move contour
ax.ZLim(2) = 30; %  plot to z = 30
sc(2).ZLocation = 'zmax';

xlabel('x')
ylabel('t')
% relabel y-ticks since skipped x5 in meshgrid
yticklabels({50,100,150,200,250})
zlabel('u(x,t)')

view([132.8 26.5]) % from experimentation
```



Note that in section 10.1, you generally solved a problem involving a specific value for λ. Since λ has many possible values, you obtain many possible solutions. $\therefore$ Using superposition, you use these possible solutions to come up with a series expression for the initial value function.

10.6 Other Heat Conduction Problems

In Example 1, $u(x,0) - v(x) = 60 - 2x - (20+x) = 40 - 3x$.

$$\therefore c_n = \frac{2}{L}\int_0^L (40-3x)\sin\left(\frac{n\pi x}{L}\right)dx \ , \quad L = 30$$

$$= \frac{1}{15}\int_0^{30} 40\sin\left(\frac{n\pi x}{30}\right)dx - \frac{1}{15}\int_0^{30} 3x\sin\left(\frac{n\pi x}{30}\right)dx$$

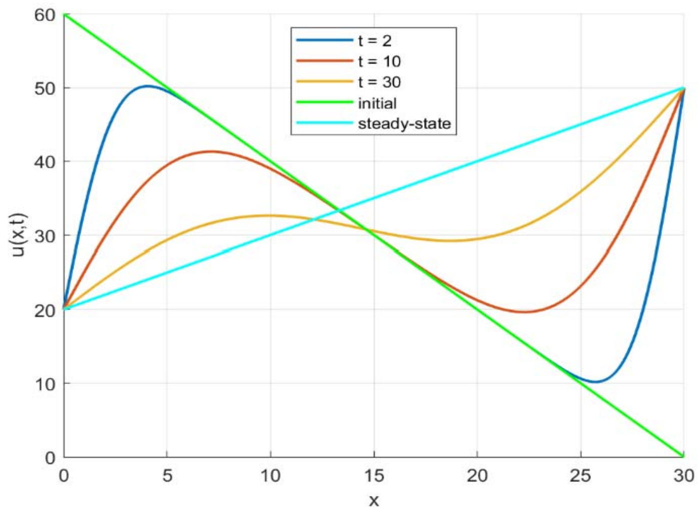
$$= \frac{8}{3}\left[-\frac{30}{n\pi}\cos\left(\frac{n\pi x}{30}\right)\right]_0^{30} - \frac{1}{5}\left[-\frac{30}{n\pi}x\cos\left(\frac{n\pi x}{30}\right) + \underbrace{\frac{30^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{30}\right)}_{=0}\right]_0^{30}$$

$$= \frac{80}{n\pi}\left[1 - \cos(n\pi)\right] + \frac{6}{n\pi}\left[30\cos(n\pi)\right]$$

$$\therefore\ c_n = \frac{80}{n\pi} + \frac{100}{n\pi}\cos(n\pi),\quad n = 1, 2, 3, \ldots$$

Using MATLAB,

```
clear
a2 = 1;
L = 30;          % length of rod
x = 0:0.01:L;    % extent of plot
u0 = 60 - 2*x;   % initial condition
v = 20 + x;      % steady-state temp
Nterms = 20;     % # of terms in series
figure
hold on
grid on
for t = [2,10,30]
    w= 0;
    for n = 1:Nterms
        coef = (80 + 100*cos(n*pi))/(n*pi);
        e = exp(-n^2*pi^2*a2*t/L^2);
        s = sin(n*pi*x/L);
        w = w + coef*e*s;   % Fourier sum
    end
    u = w + v;
    plot(x,u,'LineWidth',1.5)
end
plot(x,u0,'g','LineWidth',1.5) % init condition
plot(x,v,'c','LineWidth',1.5)  % steady-state temp
xlabel('x')
ylabel('u(x,t)')
legend('t = 2','t = 10','t = 30',...
       'initial', 'steady-state',...
       'Location','north')
```



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Case I: $\lambda < 0$, $\therefore -\mu^2 < 0$, when $\mu > 0$.

Let $X(x) = K_1 e^{\mu x} + K_2 e^{-\mu x}$

$\therefore X' = K_1 \mu e^{\mu x} - K_2 \mu e^{-\mu x}$

$\therefore X'(0) = K_1\mu - K_2\mu = 0 \Rightarrow K_1 = K_2$

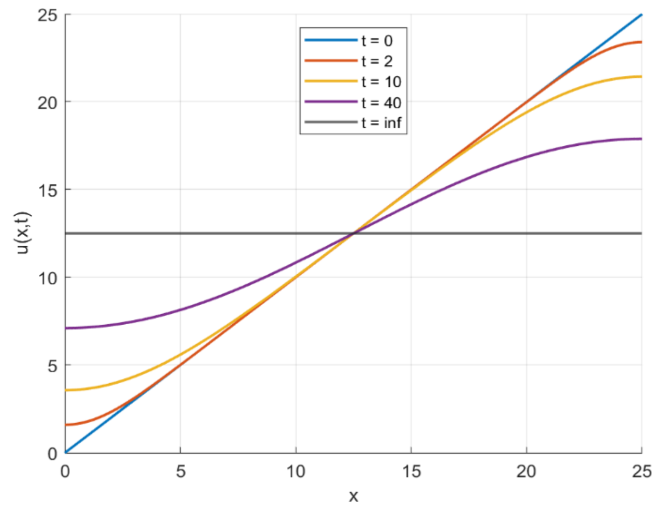
$\therefore X'(L) = K_1 \mu e^{\mu L} - K_1 \mu e^{-\mu L} = 0 \Rightarrow K_1 e^{2\mu L} = K_1$

$\Rightarrow e^{2\mu L} = 1$ or $K_1 = 0$ Since $\mu L \neq 0$, $\therefore K_1 = 0$

$\therefore K_1 = K_2 = 0$, trivial solution.

Example 2, pp. 500 - 501

```
clear
a2 = 1;
L = 25;          % length of rod
x = 0:0.01:L;    % extent of plot
N = 5;           % # of terms
figure
hold on
grid on
plot(x,x,'LineWidth',1.5)    % t = 0
c0 = 25;
for t = [2,10,40]
    u = c0/2;
    for n = 1:2:2*N    % just odd terms
        coef = -100/(n^2*pi^2);
        e = exp(-n^2*pi^2*a2*t/L^2);
        c = cos(n*pi*x/L);
        u = u + coef*e*c;  % Fourier sum
    end
    plot(x,u,'LineWidth',1.5)
end
yline(c0/2,'LineWidth',1.5)  % steady-state temp
xlabel('x')
ylabel('u(x,t)')
legend('t = 0', 't = 2','t = 10','t = 40',...
       't = inf','Location','north')
```



10.7 The Wave Equation: Vibrations of an Elastic String

Example 1, pp. 507 - 509

Use MATLAB to compute c_n

```
clear
syms n x
L = 30;
s = sin(n*pi*x/L);
f1 = (x/10);
f2 = (30-x)/20;
cn = (2/L)*(int(f1*s,x,0,10) + int(f2*s,x,10,30))
```

cn =

$$\frac{-180\,\sigma_2\,\sigma_1{}^2 + 30\,n\,\pi\,\sigma_1 + 90\,\sigma_2}{15\,n^2\,\pi^2} + \frac{30\,\sigma_2 - 10\,\pi\,n\,\sigma_1}{5\,n^2\,\pi^2}$$

where

$$\sigma_1 = \cos\left(\frac{\pi\,n}{3}\right)$$

$$\sigma_2 = \sin\left(\frac{\pi\,n}{3}\right)$$

$$\therefore\ c_n = \frac{-180\,\sigma_2\sigma_1^2 + 30n\pi\sigma_1 + 90\sigma_2 + 90\sigma_2 - 30n\pi\sigma_1}{15n^2\pi^2}$$

$$= \frac{180\,\sigma_2\left(1-\sigma_1^2\right)}{15n^2\pi^2} = \frac{180\sin\left(\frac{n\pi}{3}\right)\left[1-\cos^2\left(\frac{n\pi}{3}\right)\right]}{15n^2\pi^2}$$

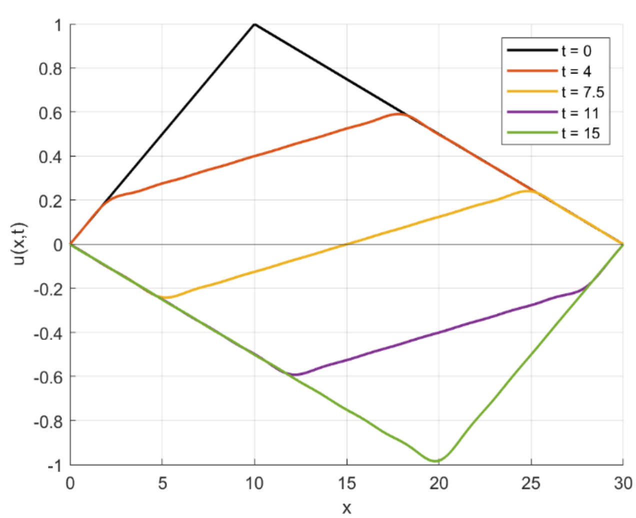
$$= 12\sin^3\left(\frac{n\pi}{3}\right)$$

Note for $n \neq$ multiple of 3, $\sin\left(\frac{n\pi}{3}\right) = \pm\frac{\sqrt{3}}{2}$ so $\sin^2\left(\frac{n\pi}{3}\right) = \frac{3}{4}$. $\therefore\ \sin^3\left(\frac{n\pi}{3}\right) = \frac{3}{4}\sin\left(\frac{n\pi}{3}\right)$ for all n.

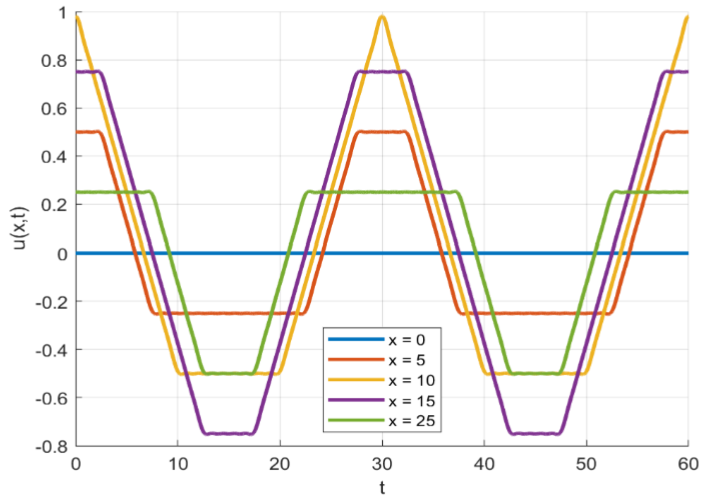
$$\therefore\ c_n = \frac{12\sin^3\left(\frac{n\pi}{3}\right)}{n^2\pi^2} = \frac{9\sin\left(\frac{n\pi}{3}\right)}{n^2\pi^2},\quad n = 1,2,3,\ldots$$

Using MATLAB to plot:

```
clear
a = 2;
L = 30;          % length of rod
x = 0:0.01:L;    % extent of plot
                 % initial condition
ic = x/10.*heaviside(10-x) +...
     ((30-x)/20).*heaviside(x-10);
N = 25;          % # of terms
figure
hold on
grid on
plot(x,ic,'k','LineWidth',1.5)   % t = 0
for t = [4,7.5,11,15]
    u = zeros(1,length(x));  % initialize
    for n = 1:N
        coef = 9*sin(n*pi/3)/(n^2*pi^2);
        sc = sin(n*pi*x/L)*cos(n*pi*a*t/L);
        u = u + coef*sc;  % Fourier sum
    end
    plot(x,u,'LineWidth',1.5)
end
yline(0)  % x-axis
xlabel('x')
ylabel('u(x,t)')
legend('t = 0', 't = 4','t = 7.5','t = 11',...
       't = 15','Location','northeast')
```



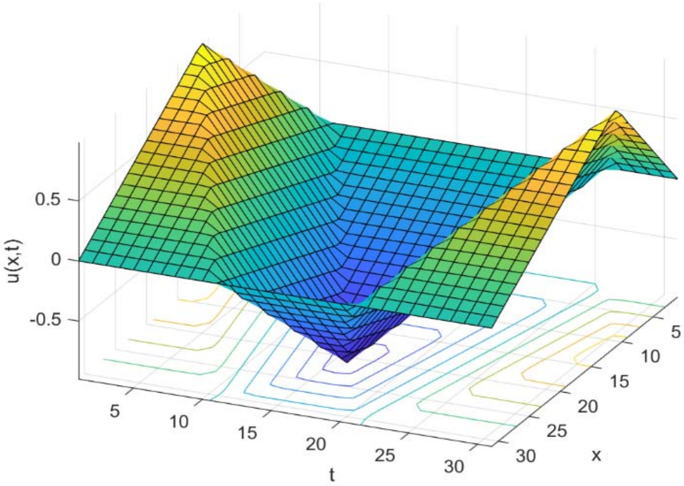
```
clear
a = 2;
L = 30;          % length of rod
t = 0:0.01:60; % 2 periods
N = 25;          % # of terms
figure
hold on
grid on
for x = [0,5,10,15,25]
    u = zeros(1,length(t));
    for n = 1:N
        coef = 9*sin(n*pi/3)/(n^2*pi^2);
        sc = sin(n*pi*x/L)*cos(n*pi*a*t/L);
        u = u + coef*sc;  % Fourier sum
    end
    plot(t,u,'LineWidth',2)
end
xlabel('t')
ylabel('u(x,t)')
legend('x = 0', 'x = 5', 'x = 10',...
       'x = 15', 'x = 25','Location','south')
```




```
clear
a = 2;
L = 30;          % length of rod
N = 25;          % numbrr of termd
[x,t] = meshgrid(0:30, 0:30);
u = zeros(size(x,1), size(x,2));
for n = 1:N
    coef = 9*sin(n*pi/3)/(n^2*pi^2);
    sc = sin(n*pi*x/L).*cos(n*pi*a*t/L);
    u = u + coef*sc;  % Fourier sum
end

sc = surfc(u);   % plot surface and contours
xlabel('x')
ylabel('t')
zlabel('u(x,t)')

view([-245.76 34.69])  % after interacting trial-and-error
```



10.8 Laplace's Equation

Example 1, pp. 517-518

$$c_n \sinh\left(\frac{n\pi a}{b}\right) = \frac{2}{b}\int_0^b f(y)\sin\left(\frac{n\pi y}{b}\right)dy$$

Using MATLAB,

```
clear
syms n y
a = 3;
b = 2;
f1 = y;
f2 = 2-y;
s = sin(n*pi*y/b);
kn = (2/b)*(int(f1*s,y,0,1) + int(f2*s,y,1,b));
cn = simplify(kn/sinh(n*pi*a/b))
```

cn =

$$-\frac{4\sin(\pi n) - 8\sin\left(\frac{\pi n}{2}\right)}{n^2\pi^2\sinh\left(\frac{3\pi n}{2}\right)}$$

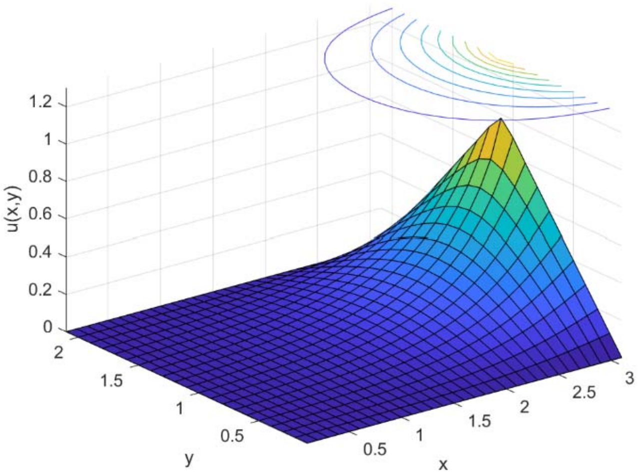
As $\sin(n\pi) = 0$ for $n = 1, 2, 3, \ldots$, $c_n = \dfrac{8\sin\left(\frac{n\pi}{2}\right)}{n^2\pi^2\sinh\left(\frac{3n\pi}{2}\right)}$

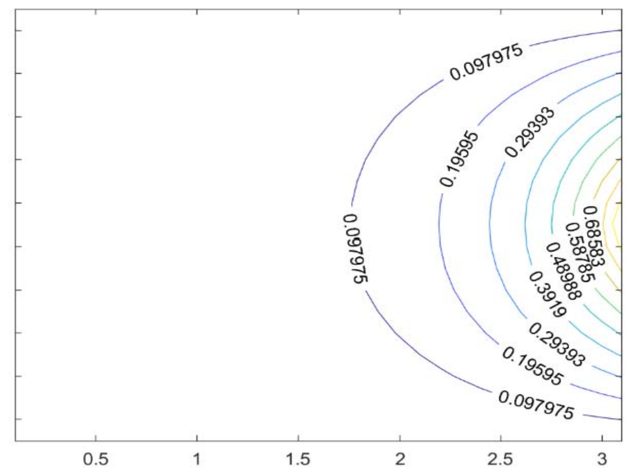
```
clear
a = 3;
b = 2;
[x,y] = meshgrid(0:0.1:a, 0:0.1:b);
u = zeros(size(x,1), size(x,2));
for n = 1:20      % first 20 terms
    cn = 8*sin(n*pi/2)/(n^2*pi^2*sinh(3*n*pi/2));
    sh = sinh(n*pi*x/b);
    s = sin(n*pi*y/b);
    u = u + cn*sh.*s;  % u(x,y) to 20 terms
end

sc = surfc(u);    % plot surface and contours
ax = gca;         % these lines move contour
ax.ZLim(2) = 1.1; %  plot to z = 1.3
sc(2).ZLocation = 'zmax';

xlabel('x')
ylabel('y')
% relabel x-ticks, y-ticks
xticklabels({0.5,1.0,1.5,2.0,2.5,3.0})
yticklabels({0.5,1.0,1.5,2.0})
zlabel('u(x,y)')

contour(u,9,'ShowText','on')
xticklabels({0.5,1.0,1.5,2.0,2.5,3.0})
yticklabels('')
```





Dirichlet Problem for a Circle

On p. 520, for equation (36),

$$u(a,\theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty} a^n (c_n \cos(n\theta) + k_n \sin(n\theta)) = f(\theta) \qquad (36)$$

note that $f(\theta)$ has period 2π on $[0, 2\pi]$. The extension to $[-2\pi, 2\pi]$ allows a Fourier series, but it still has a period of 2π. $\therefore$ Could integrate over $[-\pi, \pi]$ or $[0, 2\pi]$. Note $L = \pi$, so $\frac{n\pi\theta}{L} = \frac{n\pi\theta}{\pi} = n\theta$. $\therefore$ trigonometric terms for $f(\theta)$ are in $n\theta$, not $\frac{n\pi\theta}{2\pi} = \frac{n\theta}{2}$.

Appendix A

Derivation of the Heat Conduction Equation

The term absorbtion is not clearly defined. It means the temperature of the material. So from heat (energy transfer), the temperature increases by $\Delta Q = c m \Delta T$, where c = specific heat, m = mass, ΔT = change in temperature from ΔQ.

Note $H(x,t)$, the rate of heat transfer, is defined in this discussion as positive if coming from left-to-right.

On p. 524, equation (14) states,

$$u_x(0,t) - h_1 u(0,t) = 0, \quad t > 0, \tag{14}$$

So, if $h_1 \to \infty$, the rate of flow out of the bar increases greatly as $u(0,t) \to 0$, meaning the boundary is at absolute zero.

On p. 525, dividing equation (9) by $\Delta x \Delta t$ yields,

$$\kappa A\big(u_x(x_0+\Delta x,t)-u_x(x_0,t)\big)\Delta t$$
$$= s\rho A\big(u(x_0+\theta\Delta x,t+\Delta t)-u(x_0+\theta\Delta x,t)\big)\Delta x. \qquad (9)$$

$$\frac{K(x_0+\Delta x)A(x_0+\Delta x)\left[u_x(x_0+\Delta x,t)\right]}{\Delta x} =$$

$$s(x_0+\Delta x)\rho(x_0+\Delta x)A(x_0+\Delta x)\frac{\left[u(x_0+\theta\Delta x, t+\Delta t)-u(x_0+\theta\Delta x, t)\right]}{\Delta t}$$

$\therefore$ As $\Delta x \to 0$, $\Delta t \to 0$,

$$(KAu_x)_x = s\rho A\, u_t$$

since s, ρ, A are assumed continuous functions of x.

Chapter 11 Notes

Note Title 9/17/2021

11.1 The Occurrence of Two-Point Boundary Value Problems

Page 530, it states,

$$u_x(0,t) - h_1 u(0,t) = 0, \quad u_x(L,t) + h_2 u(L,t) = 0. \tag{8}$$

the temperature there. Usually, h_1 and h_2 are nonnegative constants, but in some cases they may be negative or depend on t. The boundary conditions (2) are obtained in the limit as $h_1 \to \infty$ and $h_2 \to \infty$. The other important limiting case, $h_1 = h_2 = 0$, gives the boundary conditions for insulated ends.

As $h_1 \to \infty$ and $h_2 \to \infty$, rewrite (8) as

$$\frac{u_x(0,t)}{h_1} = u(0,t) \qquad \frac{u_x(L,t)}{h_2} = u(L,t)$$

$\therefore$ As $h_1 \to \infty$, $u(0,t) \to 0$, as $h_2 \to \infty$, $u(L,t) \to 0$.

The other limiting case, $h_1 = h_2 = 0$, yields from (8),

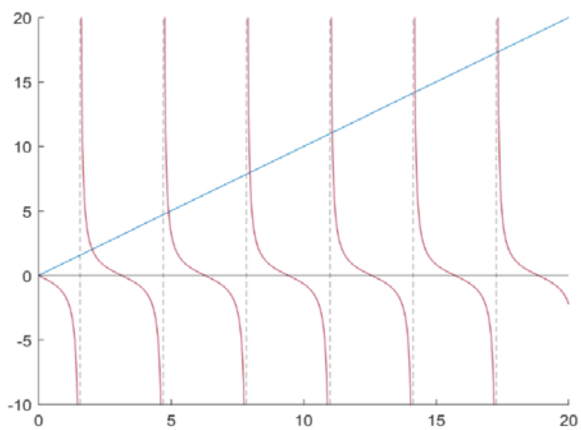
$$u_x(0,t) = 0, \qquad u_x(L,t) = 0.$$

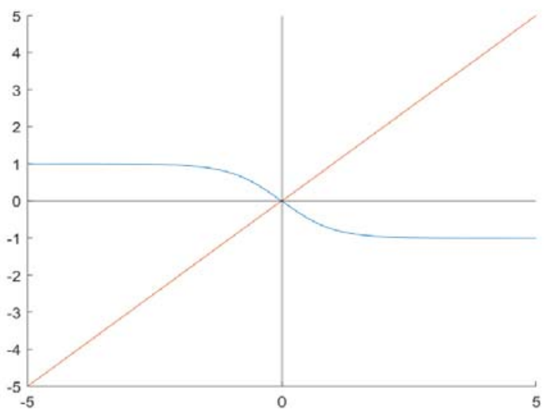
Example 1

```
clear
syms x
figure
axis([0,20,-10,20])
hold on
xline(0)
yline(0)
fplot(x)
for n = 1:6
    fplot(-tan(x-n*pi),[0,20])
    % start estimation just to right of pi/2
    a = vpasolve(tan(x)==-x,x,(2*n-1)*pi/2 + 0.1);
    n,a
end

figure
hold on
fplot(-tanh(x))
fplot(x)
xline(0)
yline(0)
```

```
n = 1
a = 2.0287578381104342235769711247347
n = 2
a = 4.9131804394348836888378206685945
n = 3
a = 7.9786657124132407552457812070142
n = 4
a = 11.085538406497022543376124600405
n = 5
a = 14.207436725191188359768502149593
n = 6
a = 17.336377923983360670860109208644
```





Lagrange's identity

$$\int_0^1 L[u]v\,dx = \int_0^1 [-(pu')'v + quv]\,dx$$

$$= \int_0^1 -v\,d(pu') + \int_0^1 quv\,dx$$

$\int r\,ds = rs - \int s\,dr$

$$= -pu'v\Big|_0^1 + \int_0^1 pu'\,d(v) + \int_0^1 quv\,dx$$

$$= -pu'v\Big|_0^1 + \int_0^1 pv'\,d(u) + \int_0^1 quv\,dx$$

$$= -pu'v\Big|_0^1 + pv'u\Big|_0^1 - \int_0^1 u\,d(pv') + \int_0^1 quv\,dx$$

$$= -pu'v\Big|_0^1 + puv'\Big|_0^1 + \int_0^1 -(pv')'u\,dx + \int_0^1 qvu\,dx$$

$$= -pu'v\Big|_0^1 + puv'\Big|_0^1 + \int_0^1 [-(pv')' + qv]u\,dx$$

$$= -pu'v\Big|_0^1 + puv'\Big|_0^1 + \int_0^1 u\,L[v]\,dx$$

$$\therefore \int_0^1 (L[u]v - u\,L[v])\,dx = p(-u'v + uv')\Big|_0^1 \quad [5]$$

The latter is Lagrange's identity.

[5] becomes:

$$p(1)\left[-u'(1)v(1)+u(1)v'(1)\right]-p(0)\left[-u'(0)v(0)+u(0)v'(0)\right]$$

Assuming $u(x)$ and $v(x)$ satisfy [2] below:

$$\alpha_1 y(0)+\alpha_2 y'(0)=0 \quad , \quad \beta_1 y(1)+\beta_2 y'(1)=0 \qquad [2]$$

If $\alpha_2 \neq 0$, then $-u'(0)v(0)+u(0)v'(0)=$

$$\frac{\alpha_1}{\alpha_2}u(0)v(0)-\frac{\alpha_1}{\alpha_2}u(0)v(0)=0$$

If $\alpha_1 \neq 0$, then $-u'(0)v(0)+u(0)v'(0)=$

$$\frac{\alpha_2}{\alpha_1}u'(0)v'(0)-\frac{\alpha_2}{\alpha_1}u'(0)v'(0)=0$$

If $\beta_2 \neq 0$, then $-u'(1)v(1)+u(1)v'(1)=$

$$\frac{\beta_1}{\beta_2}u(1)v(1)-\frac{\beta_1}{\beta_2}u(1)v(1)=0$$

If $\beta_1 \neq 0$, then $-u'(1)v(1)+u(1)v'(1)=$

$$\frac{\beta_2}{\beta_1}u'(1)v'(1)-\frac{\beta_2}{\beta_1}u'(1)v'(1)=0$$

$\therefore$ [5] becomes: $p(1)[0]-p(0)[0]=0$

Self-Adjoint

Note the definition refers only to the linear operator and boundary conditions, not the condition of the differential equation (i.e., $L[y]=0$).

Example 4, p. 542

Show that $y''+\lambda y=0$, $y(-1)-y(1)=0$, $y'(-1)-y'(1)=0$ is self-adjoint. Here, $p(x)=1$.

$$\therefore \int_{-1}^{1}\left(L[u]v-uL[v]\right)dx = u(x)v'(x)-u'(x)v(x)\Big|_{-1}^{1}$$

$$= u(1)v'(1)-u'(1)v(1)-u(-1)v'(-1)+u'(-1)v(-1)$$

$$+u(1)v'(-1)-u(1)v'(-1)$$

$$= u(1)\underbrace{[v'(1)-v'(-1)]}_{=0} + v'(1)\underbrace{[u(1)-u(-1)]}_{=0}$$

$$-u'(1)v(1)+u'(-1)v(-1)$$

$$= -u'(1)v(1)+u'(-1)v(-1)$$

$$+u'(-1)v(1)-u'(-1)v(1)$$

$$= \underbrace{[u'(-1)-u'(1)]}_{=0}v(1) + u'(-1)\underbrace{[v(-1)-v(1)]}_{=0}$$

$$= 0$$

$\therefore \int_{-1}^{1} (L[u]v - uL[v])dx = 0$, so $(L[u], v) = (u, L[v])$

Solutions:

$\lambda = 0 \Rightarrow \phi(x) = c_1 x + c_2$, $\phi(-1) - \phi(1) = 0 =$

$-c_1 + c_2 - (c_1 + c_2) = 0 \Rightarrow -2c_1 = 0 \Rightarrow c_1 = 0.$

$\therefore \phi_0(x) = 1$ is a solution.

$\lambda < 0 \Rightarrow \phi(x) = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}$

$\phi(-1) - \phi(1) = 0 \Rightarrow c_1 e^{-\sqrt{-\lambda}} + c_2 e^{\sqrt{-\lambda}} - c_1 e^{\sqrt{-\lambda}} - c_2 e^{-\sqrt{-\lambda}} = 0,$

or $c_1 + c_2 e^{2\sqrt{-\lambda}} - c_1 e^{2\sqrt{-\lambda}} - c_2 = 0$ [1]

$\phi'(-1) - \phi'(1) = 0 \Rightarrow c_1\sqrt{-\lambda}\, e^{-\sqrt{-\lambda}} - c_2\sqrt{-\lambda}\, e^{\sqrt{-\lambda}} - c_1\sqrt{-\lambda}\, e^{\sqrt{-\lambda}} + c_2\sqrt{-\lambda}\, e^{-\sqrt{-\lambda}} = 0,$

or $c_1 - c_2 e^{2\sqrt{-\lambda}} - c_1 e^{2\sqrt{-\lambda}} + c_2 = 0$ [2]

Adding [1], [2], $2c_1 - 2c_1 e^{2\sqrt{-\lambda}} = 0$, or

$c_1 (1 - e^{2\sqrt{-\lambda}}) = 0$. Since $\sqrt{-\lambda} \neq 0$, $c_1 = 0$.

$\therefore$ [1] $\Rightarrow c_2 (e^{2\sqrt{-\lambda}} - 1) = 0 \Rightarrow c_2 = 0$

$\therefore$ $\underline{\lambda < 0}$ is $\underline{\text{not}}$ an eigenvalue.

$\lambda > 0 \Rightarrow \phi(x) = c_1 \cos(\sqrt{\lambda}\, x) + c_2 \sin(\sqrt{\lambda}\, x)$

$\phi(-1) - \phi(1) = 0 \Rightarrow$

$c_1 \cos(\sqrt{\lambda}) - c_2 \sin(\sqrt{\lambda}) - c_1 \cos(\sqrt{\lambda}) - c_2 \sin(\sqrt{\lambda}) = 0,$

or $c_2 \sin(\sqrt{\lambda}) = 0.$

$\phi'(-1) - \phi'(1) = 0 \Rightarrow$

$c_1 \sqrt{\lambda} \sin(\sqrt{\lambda}) + c_2 \sqrt{\lambda} \cos(\sqrt{\lambda}) + c_1 \sqrt{\lambda} \sin(\sqrt{\lambda}) - c_2 \sqrt{\lambda} \cos(\sqrt{\lambda}) = 0,$

or $c_1 \sin(\sqrt{\lambda}) = 0$

$\therefore$ If $\sin(\sqrt{\lambda}) \neq 0$, $c_1 = c_2 = 0$, so trivial solution.

$\therefore \sin(\sqrt{\lambda}) = 0 \Rightarrow \sqrt{\lambda} = n\pi$, $n = 1, 2, 3, \ldots$

$\therefore \underline{\lambda = n^2\pi^2}$, $n = 1, 2, 3, \ldots$

$\therefore \phi(x) = c_1 \cos(n\pi x) + c_2 \sin(n\pi x)$

$\therefore$ For each $\lambda_n = n^2\pi^2$, $\phi_n(x) = \cos(n\pi x)$ or

$\phi_n(x) = \sin(n\pi x)$ are independent solutions.

$\therefore \lambda_n$ is not simple.

Note that Chapter 10 showed that <u>any</u> piecewise continuous function f on $[0, L]$, with piecewise continuous f' on $[0, L]$, can be expressed as a convergent Fourier series, either as a cosine series $\left(a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx \,,\ n = 0, 1, 2, \ldots \right)$ or as a sine series $\left(a_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx \,,\ n = 0, 1, 2, 3, \ldots \right)$, or as a combination of cosine and sine functions, depending upon how $f(x)$ is extended on $[-L, L]$.

In this section, $f(x)$ is expressed as a series of eigenfunctions, which are found by solving a <u>related</u> differential equation, $L[y] = \lambda r(x) y$, <u>not</u> the original equation, $L[y] = \mu r(x) y + f(x)$.

The eigenfunctions are just independent orthonormal functions used to express the solution in series form, as a "nice" solution may not be able to be found. $r(x)$ from the original problem is used in the related $L[y] = \lambda r(x) y$. This yields λ_n.

To get c_n, $c_n = \int_0^l f(x)\phi_n(x)\,dx = \int_0^l r(x)\frac{f(x)}{r(x)}\phi_n(x)\,dx$.

Once c_n and λ_n are obtained, b_n is computed using $(\lambda_n - \mu) b_n - c_n = 0$, $n = 1, 2, 3, \ldots$

$\therefore \phi(x) = \sum_{n=1}^{\infty} b_n \phi_n(x)$ is the solution to the original problem.

$\therefore$ (1) First put problem in form $L[y] = -(p(x)y')' + q(x)y$

(2) Solve $L[y] = \lambda r(x) y$ to get $\lambda_n, \phi_n(x)$, which use the original problem's boundary conditions

(3) Get c_n from $c_n = \int_0^l f(x)\phi_n(x)\,dx$

(4) obtain b_n from $(\lambda_n - \mu) b_n - c_n = 0$.

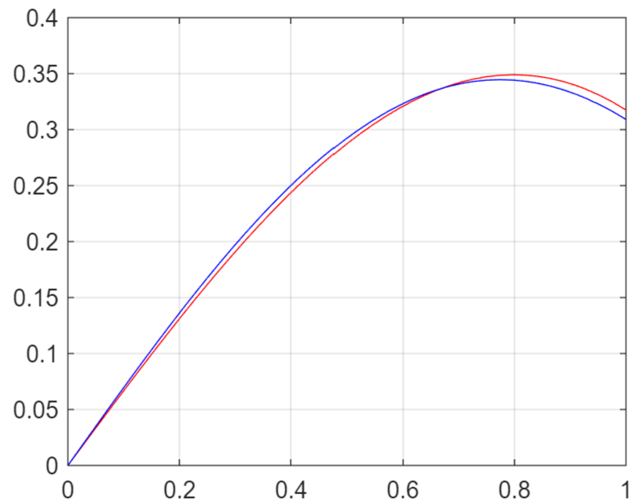
$\therefore \phi(x) = \sum_{n=1}^{\infty} b_n \phi_n(x)$ is the solution to

$$L[y] = \mu r(x) y + f(x)$$

Example 1, pp. 547-549

Use "vpasolve" to obtain λ. Using MATLAB,

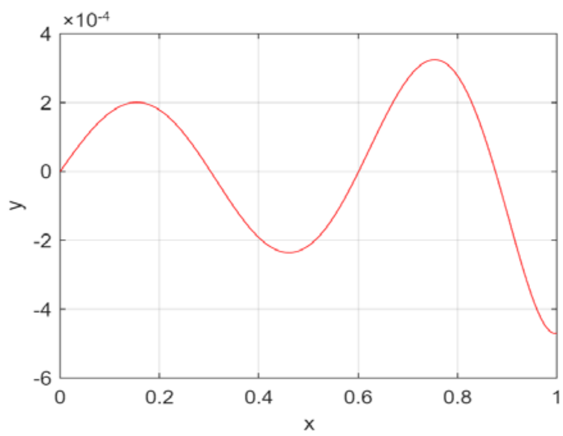
```
clear
syms x
figure
sq2 = sqrt(2);
y1 = -x/2 + sin(sq2*x)/(sin(sq2)+sq2*cos(sq2));
fplot(y1,[0,1],'r')  % exact solution
ylim([0,0.4])
% solve for lambda
u1 = vpasolve(-tan(x)==x, x, pi/2 + 0.1);
u2 = u1^2;
y = 4*sin(u1)*sin(u1*x)/(u2*(u2-2)*(1+cos(u1)^2));
hold on
fplot(y,[0,1],'b')  % plot just one term
grid on
```

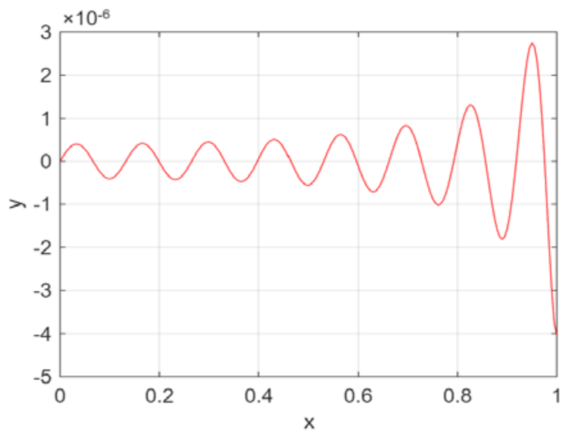



```
clear
x = 0:0.005:1;
% get exact solution
sq2 = sqrt(2);
y_ex = -x/2 + sin(sq2*x)/(sin(sq2)+sq2*cos(sq2));
y_est = SolEst(3);
plot(x,y_est - y_ex,'r')
grid on
ylabel('y'); xlabel('x');

y_est = SolEst(15);
plot(x,y_est - y_ex,'r')
grid on
ylabel('y'); xlabel('x');

function y = SolEst(NumTerms)
syms z
x= 0:0.005:1;
y = 0;
for n = 1:NumTerms
    % start estimation just to right of pi/2
    e = (2*n-1)*pi/2;
    u1 = vpasolve(-tan(z)==z, z, e + 0.1);
    u2 = u1^2;
    y = y + 4*sin(u1)*sin(u1*x)/...
            (u2*(u2-2)*(1+cos(u1)^2));
end
end
```





Note Case IIa on p. 547 states no solution to the nonhomogeneous equation exists. If such a piecewise continuously differentiable function did exist, then by Theorem 11.2.4, it could be represented as a linear combination of the normalized eigenfunctions $\phi_n(x)$ derived by the process on page 546. Since there is no such combination, the function doesn't exist.

11.5 Further Remarks on the Method of Separation of Variables

Example 1

$T(t) = K_1 \sin(\lambda a t) + K_2 \cos(\lambda a t)$ Note that one boundary condition is $u_t(r,0) = 0$ which implies $T'(0) = 0 \Rightarrow K_1 \lambda a \cos(0) = 0 \Rightarrow K_1 = 0$.

$\therefore$ $T(t) = K_2 \cos(\lambda a t)$ so $u_n(r,t) = J_0(\lambda_n r)\cos(\lambda_n a t)$

Also, to see the variable substitution better, let $r(\epsilon) = \frac{\epsilon}{\lambda}$, $S(\epsilon) = R(r) \circ r(\epsilon) = R(\frac{\epsilon}{\lambda})$

$\therefore \frac{dS}{d\epsilon} = \frac{dR}{dr} \cdot \frac{dr}{d\epsilon} = R'(\frac{1}{\lambda})$,

$$\frac{d^2S}{d\epsilon^2} = \frac{1}{\lambda}\frac{d}{d\epsilon}R' = \frac{1}{\lambda}\frac{dR'}{dr} \cdot \frac{dr}{d\epsilon} = \frac{1}{\lambda}\frac{d^2R}{dr^2}\left(\frac{1}{\lambda}\right) = \frac{1}{\lambda^2}R''$$

$\therefore r^2R'' + rR' + \lambda^2 r^2 R = 0$ becomes

$$r^2\lambda^2 S''(\epsilon) + r\lambda S'(\epsilon) + \lambda^2 r^2 S(\epsilon) =$$

$$\epsilon^2 S'' + \epsilon S + \epsilon^2 S = 0$$

$\therefore S(\epsilon) = c_1 J_0(\epsilon) + c_2 Y_0(\epsilon)$,

$\therefore R(\frac{\epsilon}{\lambda}) = c_1 J_0(\epsilon) + c_2 Y_0(\epsilon)$

Since $r = \epsilon/\lambda$, $\epsilon = r\lambda$

$\therefore R(r) = c_1 J_0(r\lambda) + c_2 Y_0(r\lambda)$

Note $r^2R'' + rR + \lambda^2 r^2 R = 0$ is $rR'' + R + \lambda^2 r R = 0$,

or $-(rR')' = \lambda^2 r R$, so "$r(r) = r$" in Sturm-Liouville

form.

11.6 Series of Orthogonal Functions: Mean Convergence

Example 1 - using MATLAB,

```
clear
syms x
f = 1;
Nmax = 20;      % number in series
N = [1 Nmax];   % array for x-axis
Rn = [1 Nmax]; % array for mean square error
Sn = 0;         % initialize for partial sums
for m = 1:Nmax  % compute partial sums, error
    N(m) = m;
    coef = 2*(1-cos(m*pi))/(m*pi);
    Sn = Sn + coef*sin(m*pi*x);   % next partial sum
    Rn(m) = vpaintegral((f-Sn)^2,x,0,1);  % mean square error
end

plot(N,Rn,'b*','LineWidth',1.5)
grid on
xlabel('N');ylabel('Rn')
```