1.1 Some Basic Mathematical Models; Direction Fields

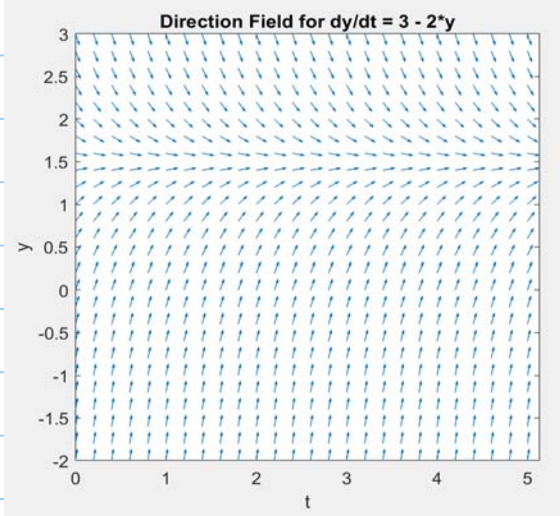
Note Title

11/20/2017

For #1-4, use this MATLAB code, changing S (dy/dt) and range of axes (meshgrid) as needed.

```
clear,clc;
Xmin = 0; Xmax = 5;
Ymin = -2; Ymax = 3;
[t, y] = meshgrid(Xmin:0.2:Xmax, Ymin:0.2:Ymax);
% S = dy/dt = slope at point (t,y)
S = 3 - 2*y;
L = sqrt(1 + S.^2);    %length of the slope vector
% Create a unit vector (hypotenuse) from L,S
% scale by 0.5, color blue
q = quiver(t,y, 1./L, S./L, 0.5, 'b');
%q.ShowArrowHead = 'off';
axis([Xmin Xmax Ymin Ymax]);
xlabel 't', ylabel 'y'
title 'Direction Field for dy/dt = 3 - 2*y'
```

1\.

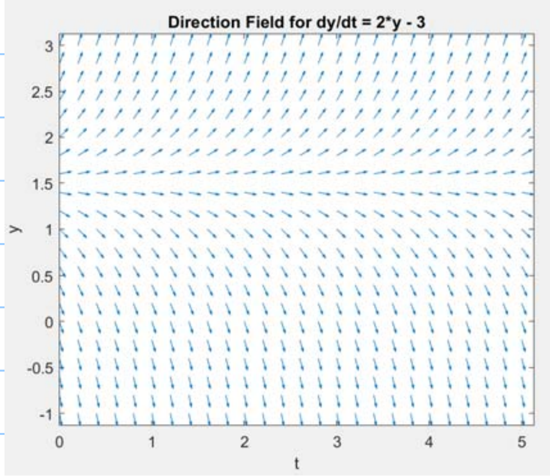


Note $y' = 0 \Rightarrow$

$3 - 2y = 0 \Rightarrow y = \frac{3}{2}$

$\therefore$ As $t \to \infty$,
$y \to 3/2$

2\.



For $y' = 0$, $2y - 3 = 0 \Rightarrow$

$y = 3/2$. As $t \to \infty$,

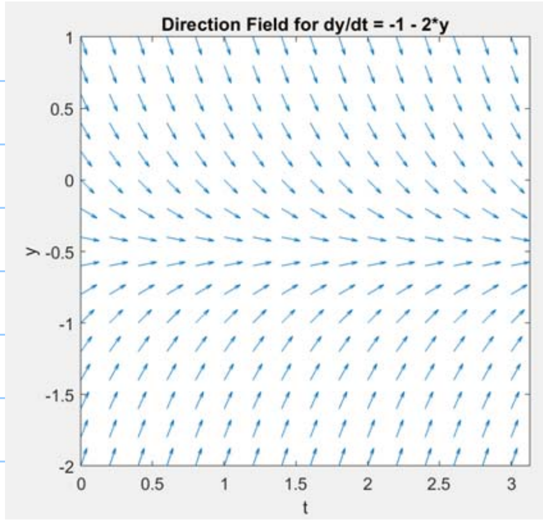
y diverges from $3/2$

3.

$y' = 0 \Rightarrow 2y = -1,$

$y = -\frac{1}{2}$

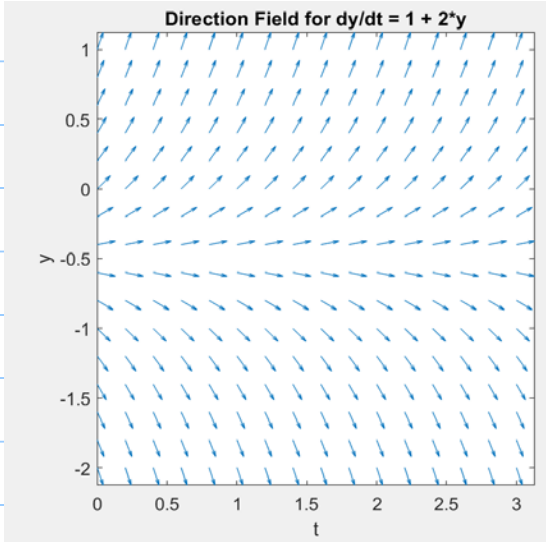
As $t \to \infty$, $y \to -\frac{1}{2}$



4.

$y' = 0 \Rightarrow y = -\frac{1}{2}$

$\therefore$ As $t \to \infty$, y diverges from $-\frac{1}{2}$



5.

As $t \to \infty$, $y' \to 0$, so $ay + b = 0$, $y = -\frac{b}{a} = \frac{2}{3}$.

$\therefore$ Let $a = -3$, $b = 2$. $\therefore \underline{\frac{dy}{dt} = 2 - 3y}$

Don't want $3y - 2$, because $a > 0$ creates a divergence, $a < 0$ creates a convergence.

6.

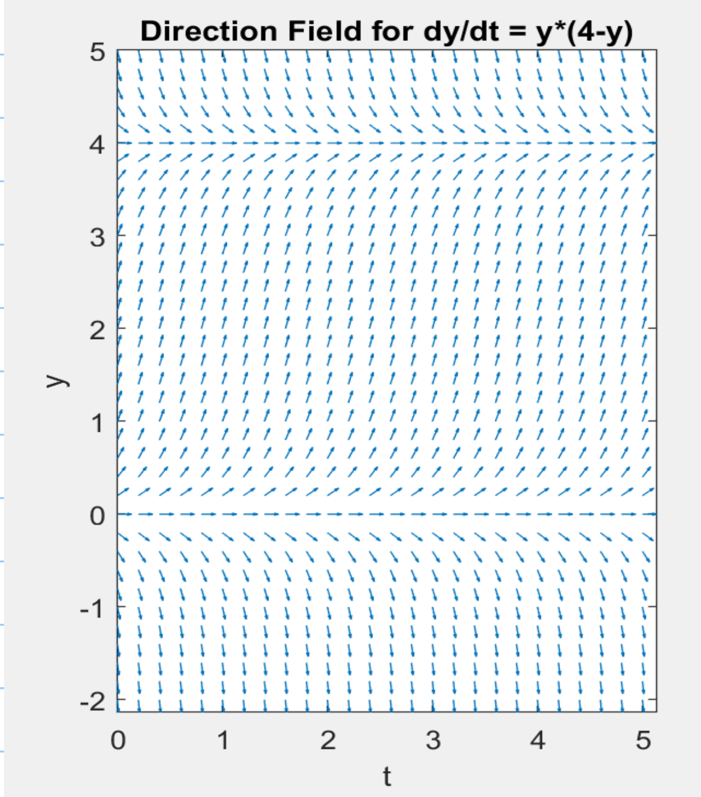
As $t \to \infty$, $y' = 0$ and want $y' = 0$ at $y = 2$.

$\therefore$ $y' = 0 \Rightarrow ay + b = 0$, $y = \frac{-b}{a} = 2$. $\therefore$ $a = 1, b = -2$

$a > 0$ ensures divergence ($a < 0 \Rightarrow$ convergence).

$\therefore$ $\underline{\frac{dy}{dt} = y - 2}$

7.



$y' = 0 \Rightarrow y(4-y) = 0$,

$\therefore$ $y = 0, 4$

$\therefore$ For $y(0) > 0$, $y \to 4$ as $t \to \infty$

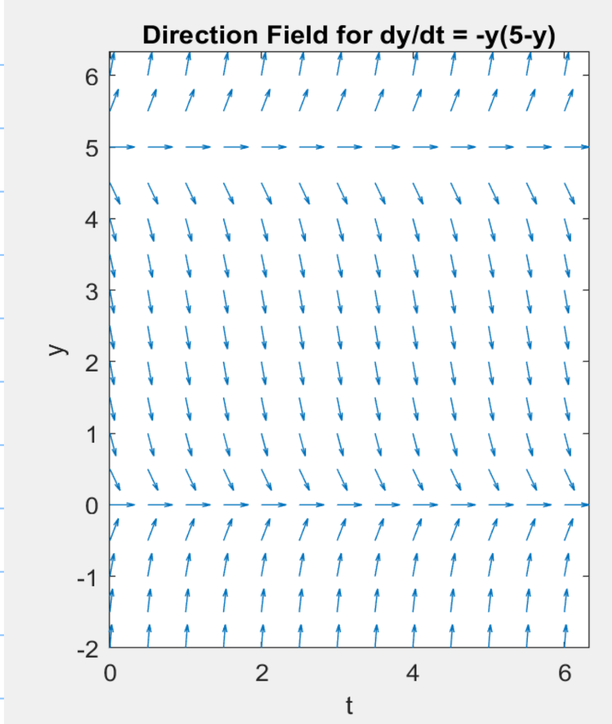
For $y(0) < 0$, $y \to -\infty$ as $t \to \infty$

8.

For $y' = 0$, $y = 0, 5$

For $y(0) < 5$, $y(t) \to 0$ as $t \to \infty$

For $y(0) > 5$, $y(t) \to \infty$ as $t \to \infty$



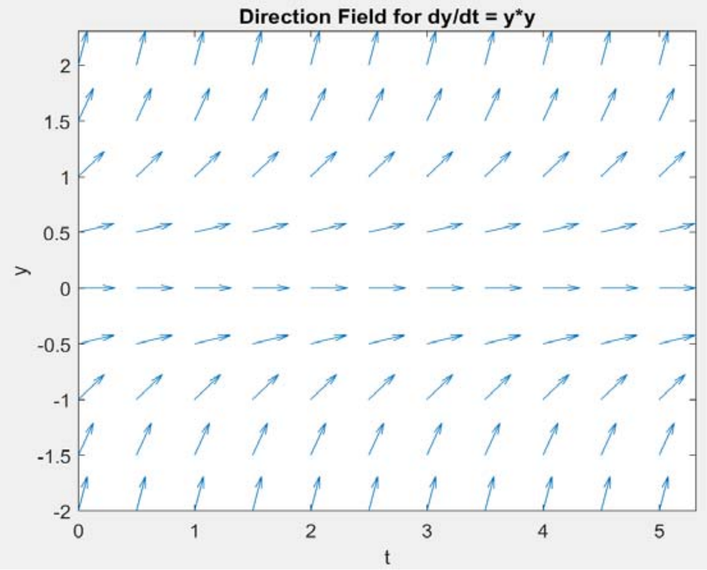
9.

$y' = 0 \Rightarrow y = 0$ is equilibrium

As $t \to \infty$,

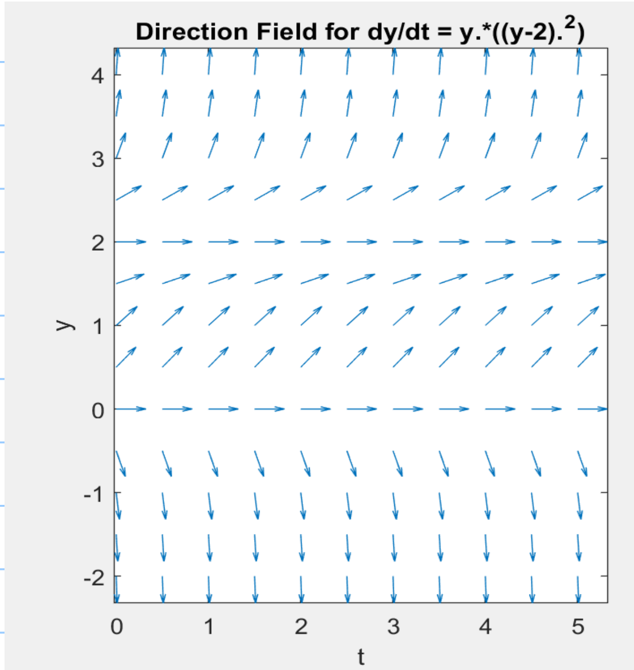
if $y(0) < 0$, $y(t) \to 0$

if $y(0) > 0$, $y(t) \to \infty$



10.

$y' = 0 \Rightarrow y = 0, 2$ are equilibrium solutions



As $t \to \infty$:

if $y(0) > 2$, $y(t) \to \infty$

if $0 < y(t) < 2$, $y(t) \to 2$

if $y(t) < 0$, $y(t) \to -\infty$

11.

Equilibrium is $y(t) = 2$

$\therefore$ (c), (j) possible.

For (c), if $y(0) = 2.1$, $y' = 0.1 > 0$ inconsistent with plot.

For (j), $y(0) = 2.1 \Rightarrow y' = -0.1$

and if $y(0) = 1.9$, $y' = +0.1$, both consistent with plot of direction field.

$\therefore$ (j) : $y' = 2 - y$

12. Equilibrium $y(t) = 2$

$\therefore$ (c), (j) possible

If (j), for $y(0) = 2.1$,

$y' = -0.1$, inconsistent with plot

For (c), $y' = y - 2$, if $y(0) = 2.1$, $y' = +0.1$

if $y(0) = 1.9$, $y' = -0.1$

$\therefore$ consistent with plot.

$\therefore$ (c) : $y' = y - 2$

13. Equilibrium is $y(t) = -2$

$\therefore$ Possibilities:

(b) : $y' = 2 + y$

(g) : $y' = -2 - y$

if (b), for $y(0) = -1.9$, $y' = 2 + (-1.9) = +0.1$
inconsistent with plot

if (g), for $y(0) = -1.9$, $y' = -2 - (-1.9) = -0.1$

$y(0) = -2.1$, $y' = -2 - (-2.1) = +0.1$

both consistent with plot.

$\therefore$ (g): $y' = -2 - y$

14.

Equilibrium is $y(t) = -2$

$\therefore$ Possibilities:

(b): $y' = 2 + y$

(g): $y' = -2 - y$

if (g), for $y(0) = -1.9$, $y' = -2 - (-1.9) = -0.1$
inconsistent with plot

if (b), for $y(0) = -1.9$, $y' = 2 + (-1.9) = +0.1$

for $y(0) = -2.1$, $y' = 2 + (-2.1) = -0.1$

both consistent with plot.

$\therefore$ (b): $y' = 2 + y$

15.

Two equilibrium solutions:

$y = 3, \; y = 0$

$\therefore$ Possibilities:

(e) $y' = y(y-3)$

(h) $y' = y(3-y)$

if (e), for $y(0) = 3.1$, $y' = 3.1(3.1-3) > 0$

inconsistent with plot

if (h), for $y(0) = 3.1$, $y' = 3.1(3-3.1) < 0$

for $y(0) = 1$, $y' = 1(3-1) > 0$

for $y(0) = -1$, $y' = (-1)(3-(-1)) < 0$

all consistent with plot.

$\therefore$ (h): $y' = y(3-y)$

16.

Two equilibrium solutions:

$y = 0, \; y = 3$

$\therefore$ Possibilities:

(e) $y' = y(y-3)$

(h) $y' = y(3-y)$

if (h), for $y(0)=4$, $y' = 4(y-4) < 0$

inconsistent with plot

if (e), for $y(0)=4$, $y' = 4(4-3) > 0$

for $y(0)=2$, $y' = 2(2-3) < 0$

for $y(0)=-1$, $y' = (-1)(-1-3) > 0$

all consistent with plot

$\therefore$ <u>(e)</u>: $y' = y(y-3)$

17.

(a)

Let $A(t)$ = amount of chemical, in grams, at time t.

Amount flowing in $= (0.01\,g/gal)(300\,gal/h)$

Amount flowing out $= \left(\frac{A(t)\,g}{1{,}000{,}000\,gal}\right)(300\,gal/h)$

Let Δt = time lapse in hours from t to $t=0$

$$\therefore\ A(t) = A(0) + (0.01)(300)\Delta t - \left(\frac{300}{1{,}000{,}000}\right)A(t)\,\Delta t$$

$$\text{or, } A(t) = A(0) + 3\Delta t - \left(\frac{3}{10{,}000}\right)A(t)\,\Delta t \quad [1]$$

$$\therefore\ \frac{A(t) - A(0)}{\Delta t} = 3 - \left(\frac{3}{10{,}000}\right)A(t)$$

$$\text{or } \frac{\Delta A(t)}{\Delta t} = 3 - \left(\frac{3}{10{,}000}\right)A(t)$$

$\therefore$ as limit $\Delta t \to 0$,

$$\frac{dA(t)}{dt} = 3 - \left(\frac{3}{10{,}000}\right)A(t) \quad [2]$$

(5)

From [1],

$$A(t) + \left(\frac{3}{10{,}000}\right)A(t)\,\Delta t = A(0) + 3\Delta t$$

$$\therefore\ A(t) = \frac{A(0) + 3\Delta t}{1 + \left(\frac{3}{10{,}000}\right)\Delta t} = \frac{\frac{A(0)}{\Delta t} + 3}{\frac{1}{\Delta t} + \frac{3}{10{,}000}}$$

$$\therefore\ \text{As } \Delta t \to \infty,\ A(t) \to \frac{3}{\frac{3}{10{,}000}} = 10{,}000$$

$\therefore$ $A(t)$ will approach $\underline{10{,}000 \text{ grams}}$

This does $\underline{\text{not}}$ depend on $A(0)$, since

$$\lim_{\Delta t \to \infty} \frac{A(0)}{\Delta t} = 0$$

(c)

$$\text{Concentration} = \frac{\text{amount}}{\text{volume}} = \frac{A(t) \text{ grams}}{1{,}000{,}000 \text{ gallons}}$$

From [2],

$$\frac{d}{dt}\left(\frac{A(t)}{10^6}\right) = \frac{3}{10^6} - \left(\frac{3}{10{,}000}\right)\frac{A(t)}{10^6}$$

or, letting $c(t) = \dfrac{A(t)}{10^6}$,

$$\underline{\frac{d}{dt}c(t) = \frac{3}{10^6} - \frac{3}{10^4}c(t)}$$

where $c(0) = \dfrac{A(0) \text{ grams}}{10^6 \text{ gallons}}$

18.

Surface area $= S = 4\pi R^2$, $R =$ sphere radius. $\therefore R = \dfrac{\sqrt{S}}{2\sqrt{\pi}}$

$$\text{Volume} = V = \frac{4}{3}\pi R^3 = (4\pi R^2)\frac{R}{3} = \frac{SR}{3} = \frac{S\sqrt{S}}{6\sqrt{\pi}} = \frac{S^{3/2}}{6\sqrt{\pi}}$$

$$\therefore \quad V = \frac{S^{3/2}}{6\sqrt{\pi}}, \quad \text{or} \quad (36\pi V^2)^{\frac{1}{3}} = S$$

Evaporation rate $= \dfrac{\Delta V}{\Delta t} =$ change in volume$/\Delta t$

$= -c\,S(t)$, $c > 0$, a constant

using a negative sign since the volume decreases over time for a positive surface area.

$$\therefore \quad \frac{d}{dt}V(t) = -c\,S(t) = -c\sqrt[3]{36\pi}\;V(t)^{2/3}$$

Let $K = c\sqrt[3]{36\pi}$, another constant

$$\therefore \quad \frac{d}{dt}V(t) = -K\,V(t)^{2/3}$$

19.

Let $T(t)$ = temperature of object at time t,

so $T(0)$ = starting temp.

$$\therefore \frac{T(t)-T(0)}{\Delta t} \approx \frac{dT(t)}{dt} = -0.05\left(T(t)-70\right)$$

using "-0.05" to represent cooling.

$$\therefore \underline{\frac{dT}{dt} = -0.05T + 35}, \quad T \text{ in } {}^{\circ}F, \; t \text{ in minutes}$$

20.

(a)

Let $A(t)$ = amount of drug, in mg, at time t (hrs).

$$\therefore A(t_2) = A(t_1) + \left(5\,\frac{mg}{cm^3}\right)\left(100\,\frac{cm^3}{hr}\right)\Delta t - 0.4\,A(t_2)\,\Delta t$$

or, $\dfrac{A(t_2) - A(t_1)}{\Delta t} \approx \dfrac{dA}{dt} = \underline{\underline{500 - 0.4A}}$

(b)

From (a), $A(t_2) = A(t_1) + 500\,\Delta t - 0.4A(t_2)\,\Delta t$

or, $A(t_2) + 0.4A(t_2)\,\Delta t = A(t_1) + 500\,\Delta t$

$A(t_2)\left(1 + 0.4\,\Delta t\right) = A(t_1) + 500\,\Delta t$

$$A(t_2) = \frac{A(t_1) + 500\,\Delta t}{1 + 0.4\,\Delta t} = \frac{\frac{A(t_1)}{\Delta t} + 500}{\frac{1}{\Delta t} + 0.4}$$

We want $t_2 \to \infty$, so $\Delta t = t_2 - t_1 \to \infty$

$\therefore$ as $t_2 \to \infty$, $A(t_2) \to \dfrac{500}{0.4} = 1250$

since $\dfrac{A(t_1)}{\Delta t} \to 0$, $\dfrac{1}{\Delta t} \to 0$.

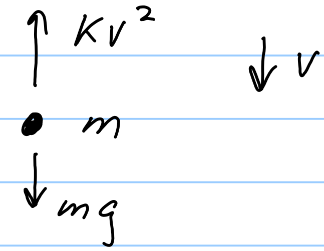
$\therefore$ After long time, amount $\to$ $\underline{\underline{1250\text{ mg}}}$

More simply, at equilibrium, $\dfrac{dA}{dA} = 0 = 500 - 0.4A$

$\therefore A = \dfrac{500}{0.4} = 1250$

21.

(a)



$$m\frac{dv}{dt} = mg - Kv^2$$

K a constant, $K > 0$.

(b)

At equilibrium, $\frac{dv}{dt} = 0$, so $mg - Kv^2 = 0$,

or, $v^2 = \frac{mg}{K}$, $\underline{v = \sqrt{\frac{mg}{K}}}$

(c)

From (b), $49 = \sqrt{\frac{10(9.8)}{K}}$, $K = \frac{98}{49^2} = \frac{2}{49}$

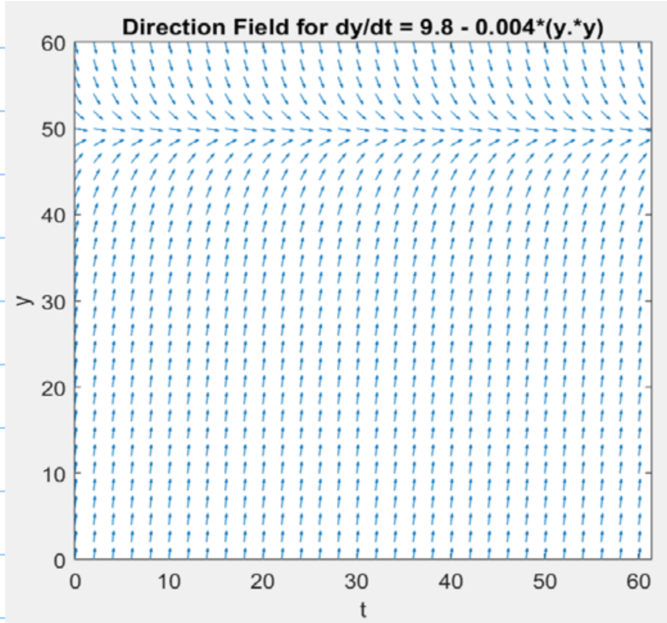
$\underline{K = 0.04}$ $\frac{\text{Newtons} \cdot \text{sec}^2}{m^2}$

(d)

Use $\frac{dv}{dt} = g - \frac{K}{m}v^2 = 9.8 - 0.004v^2$

Using MATLAB,

```
clear,clc;
Xmin = 0; Xmax = 60;
Ymin = -10; Ymax = 60;
[t, y] = meshgrid(Xmin:2:Xmax, Ymin:2:Ymax);
% S = dy/dt = slope at point (t,y)
S = 9.8 - 0.004*(y.*y);
L = sqrt(1 + S.^2);    %length of the slope vector
% Create a unit vector (hypotenuse) from L,S
% scale by 0.5, color blue
q = quiver(t,y, 1./L, S./L, 0.5, 'b');
%q.ShowArrowHead = 'off';
axis equal tight
%axis([Xmin Xmax Ymin Ymax]);
xlabel 't', ylabel 'y'
title 'Direction Field for dy/dt = 9.8 - 0.004*(y.*y)'
```

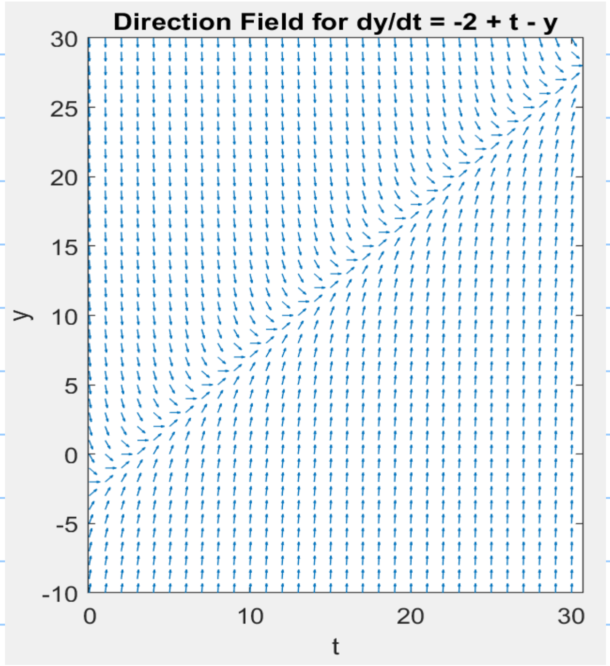


Direction field pretty much the same as Fig. 1.1.3, with same terminal velocity. The slopes of this field are "steeper", so that the terminal velocity is reached faster.

Using MATLAB code, using #22 as an example, changing S (dy/dt) and range of axes as needed.

```
clear,clc;
Xmin = 0; Xmax = 30;
Ymin = -10; Ymax = 30;
[t, y] = meshgrid(Xmin:1:Xmax, Ymin:1:Ymax);
% S = dy/dt = slope at point (t,y)
S = -2 + t - y;
L = sqrt(1 + S.^2);    %length of the slope vector
% Create a unit vector (hypotenuse) from L,S
% scale by 0.5, color blue
q = quiver(t,y, 1./L, S./L, 0.5, 'b');
%q.ShowArrowHead = 'off';
axis equal tight
%axis([Xmin Xmax Ymin Ymax]);
xlabel 't', ylabel 'y'
title 'Direction Field for dy/dt = -2 + t - y'
```

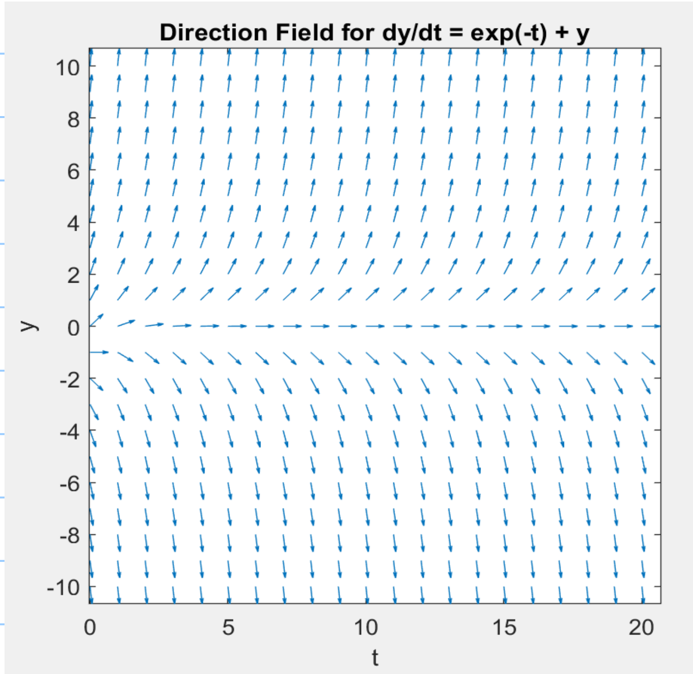
22.



(a) as $t \to \infty$, y approaches asymptotically the line $y = t - 3$.

(b) Doesn't depend on value of y at $t=0$!

23.



As $t \to \infty$,

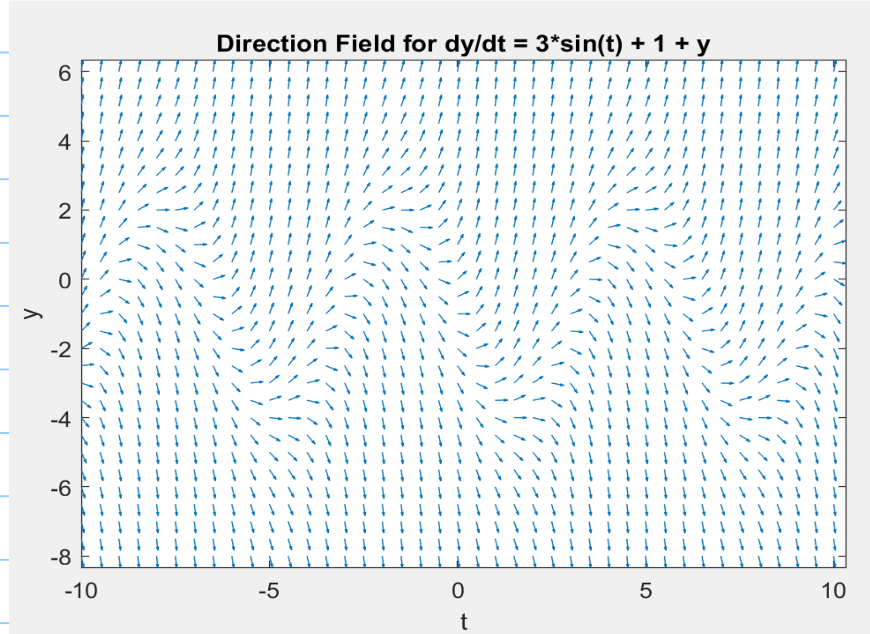
(a) if $y(0) > 0$, $y \to +\infty$

(b) if $y(0) = 0$, $y \to 0$

(c) if $y(0) < 0$, $y \to -\infty$

24.

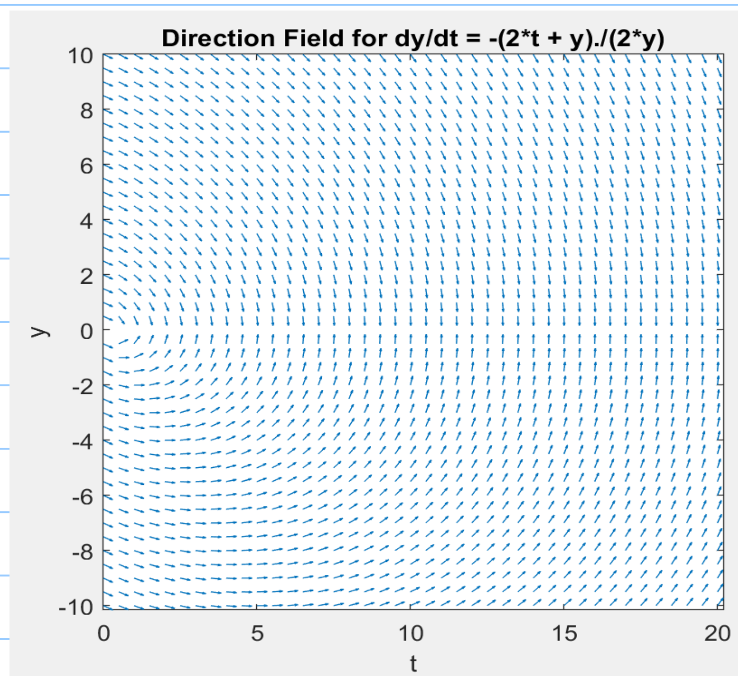
As $t \to \infty$,
$y \to +\infty, -\infty$,
or oscillates,
depending on
initial y.



25.

As $t \to \infty$, $y \to 0$

$y(0)=0$ is not allowed.



1.

(a)

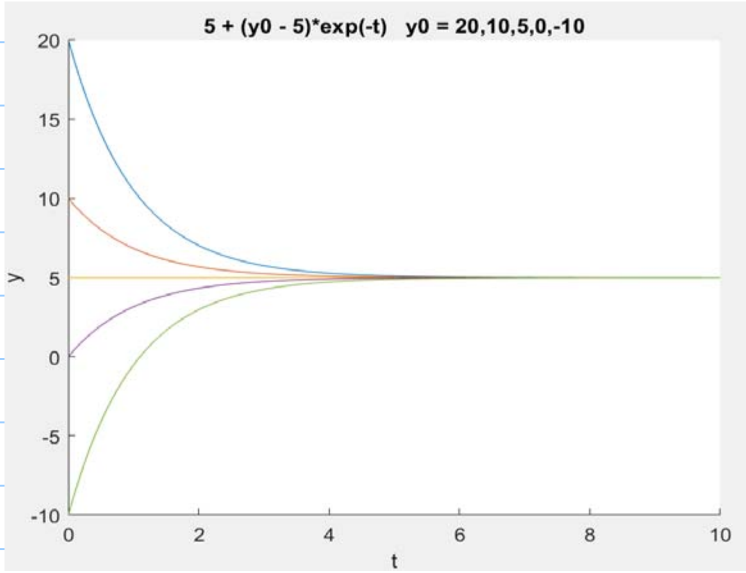
$$\frac{dy/dt}{y-5} = -1 \text{, integrating: } \ln|y-5| = -t + C$$

$$\therefore y-5 = \pm e^{-t}e^{C} \Rightarrow y = 5 + Ke^{-t}$$

$$\therefore y(0) = 5 + Ke^{0} \Rightarrow K = y_0 - 5$$

$$\therefore y(t) = 5 + (y_0 - 5)e^{-t}$$

```
clear,clc;
t = 0:0.1:10;
y0 =[20, 10, 5, 0, -10];
hold on
for n = 1:5
    eqn = 5 + (y0(n) - 5)*exp(-t);
    plot(t, eqn)
end
hold off
xlabel 't', ylabel 'y'
title '5 + (y0 - 5)*exp(-t)   y0 = 20,10,5,0,-10'
```



MATLAB code

Except for equilibrium, $y_0 = 5$, all solutions are exponential, tending toward equilibrium as $t \to \infty$.

(b)

As in (a), or using text general solution

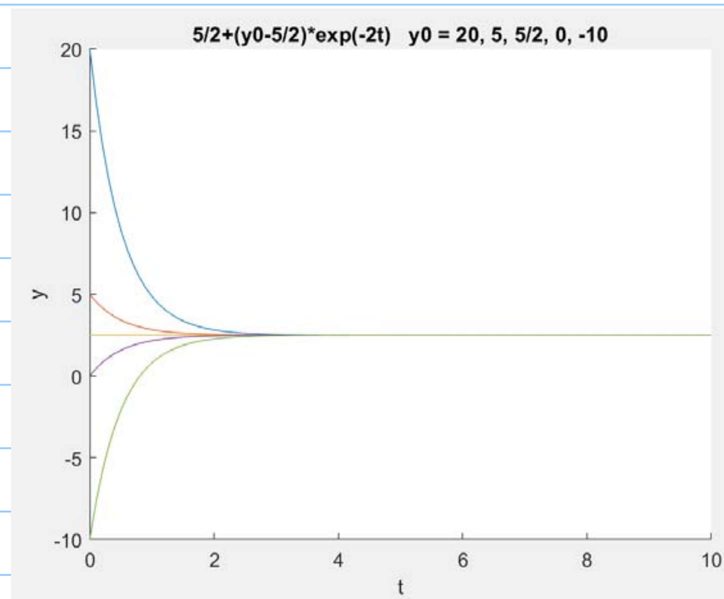
$$y(t) = \frac{b}{a} + \left(y_0 - \frac{b}{a}\right) e^{at} \quad \text{for} \quad \frac{dy}{dt} = ay - b$$

$$a = -2, \; b = -5$$

$$\therefore \; y(t) = \frac{5}{2} + \left(y_0 - \frac{5}{2}\right) e^{-2t}$$

```
clear,clc;
% dy/dt = ay - b
t = 0:0.1:10;
a = -2;
b = -5;
y0 =[20, 5, 5/2, 0, -10];
hold on
for n = 1:5
    eqn = b/a + (y0(n) - b/a)*exp(a*t);
    plot(t, eqn)
end
hold off
xlabel 't', ylabel 'y'
title '5/2 + (y0-5/2)*exp(-2t)  y0 = 20,5,5/2,0,-10'
```

MATLAB code



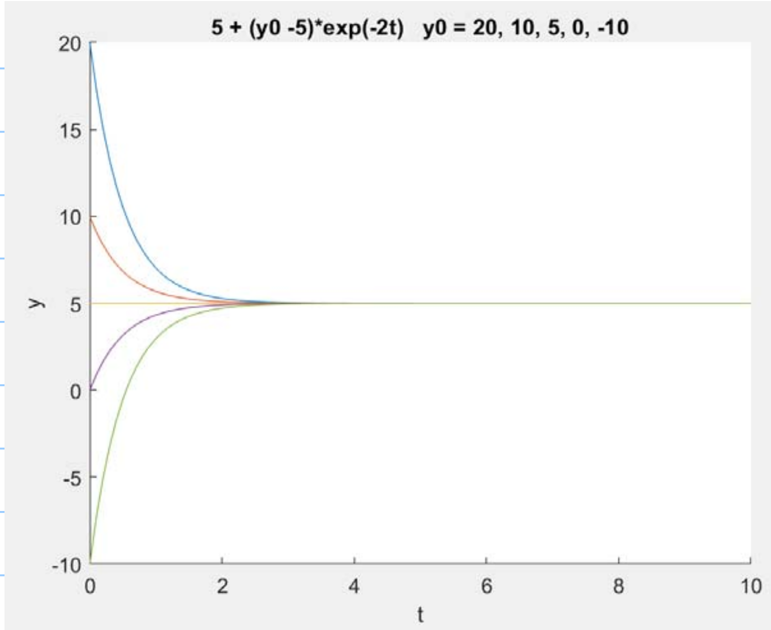
All solutions are exponential, tending toward equilibrium of $y = 5/2$ as $t \to \infty$.

(c)

$$a = -2, \; b = -10 \quad \text{using} \quad \frac{dy}{dt} = ay - b$$

$$\therefore \; y(t) = 5 + (y_0 - 5) e^{-2t}$$

```
clear,clc;
% dy/dt = ay - b
t = 0:0.1:10;
a = -2;
b = -10;
y0 =[20, 10, 5, 0, -10];
hold on
for n = 1:5
    eqn = b/a + (y0(n) - b/a)*exp(a*t);
    plot(t, eqn)
end
hold off
xlabel 't', ylabel 'y'
title '5 + (y0-5)*exp(-2t)  y0 = 20, 10, 5, 0, -10'
```



MATLAB code

All solutions exponential, tending toward equilibrium of $y = 5$ as $t \to \infty$.

(b) & (c) approach equilibrium faster than (a).

2.

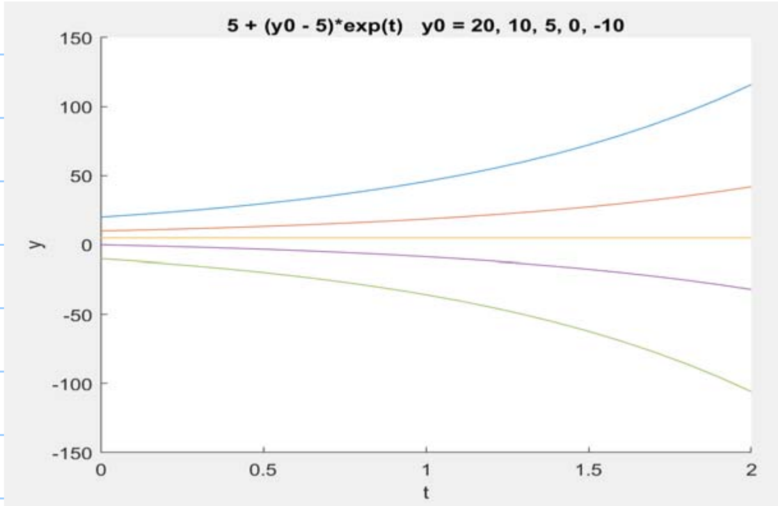
(a)

For $\frac{dy}{dt} = ay - b$, $a = 1$, $b = 5$

$$\therefore y(t) = 5 + (y_0 - 5)e^{t}$$

All solution diverge exponentially away from equilibrium $y_0 = 5$

```
clear,clc;
% dy/dt = ay - b
t = 0:0.1:2;
a = 1;
b = 5;
y0 =[20, 10, 5, 0, -10];
hold on
for n = 1:5
    eqn = b/a + (y0(n) - b/a)*exp(a*t);
    plot(t, eqn)
end
hold off
xlabel 't', ylabel 'y'
title '5 + (y0 - 5)*exp(t)  y0 = 20, 10, 5, 0, -10'
```

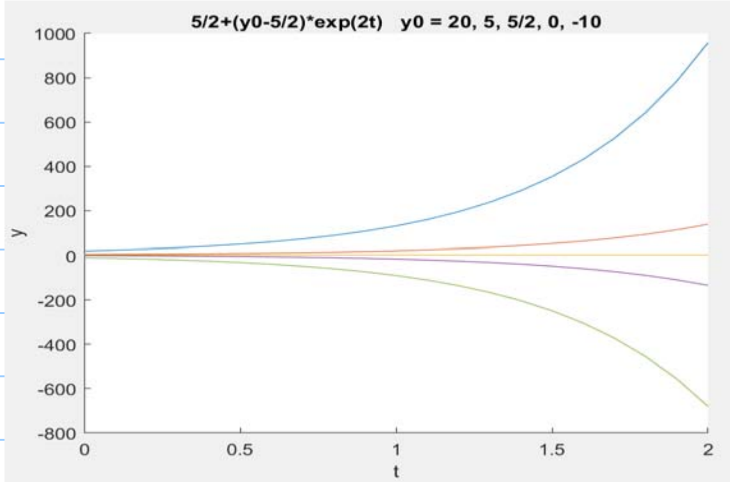


(b)

For $dy/dt = ay - b$, $a = 2$, $b = 5$, $\frac{b}{a} = \frac{5}{2}$

$$\underline{y(t) = \frac{5}{2} + \left(y_0 - \frac{5}{2}\right)e^{2t}}$$

```
clear,clc;
% dy/dt = ay - b
t = 0:0.1:2;
a = 2;
b = 5;
y0 =[20, 5, 5/2, 0, -10];
hold on
for n = 1:5
    eqn = b/a + (y0(n) - b/a)*exp(a*t);
    plot(t, eqn)
end
hold off
xlabel 't', ylabel 'y'
title '5/2+(y0-5/2)*exp(2t)  y0 = 20, 5, 5/2, 0, -10'
```



MATLAB code

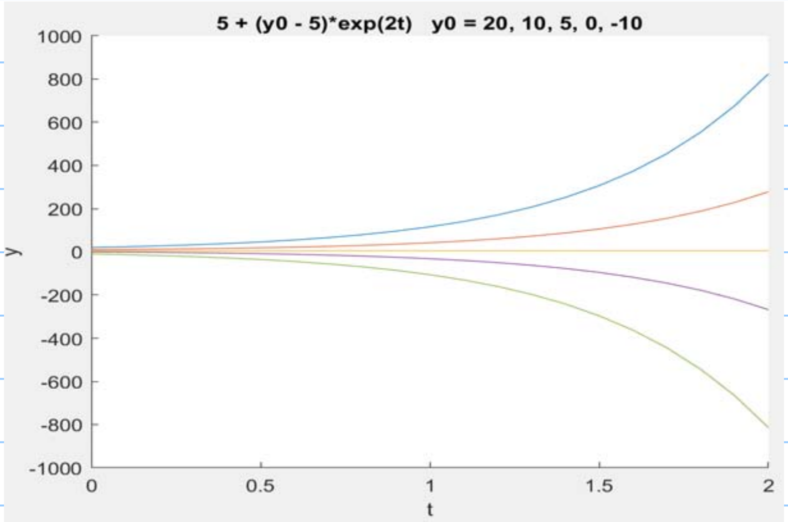
All solution diverge away from equilibrium, $y = \frac{5}{2}$, and faster than in (a)

(c)

For $\frac{dy}{dt} = ay - b$, $a = 2$, $b = 10$, $\frac{b}{a} = 5$

$$\therefore \; y(t) = 5 + (y_0 - 5)e^{2t}$$

```
clear,clc;
% dy/dt = ay - b
t = 0:0.1:2;
a = 2;
b = 10;
y0 =[20, 10, 5, 0, -10];
hold on
for n = 1:5
    eqn = b/a + (y0(n) - b/a)*exp(a*t);
    plot(t, eqn)
end
hold off
xlabel 't', ylabel 'y'
title '5 +(y0 - 5)*exp(2t)  y0 = 20, 10, 5, 0, -10'
```



MATLAB code

All solutions diverge from equilibrium, $y = 5$, and faster than in (a)

3.

(a)

$$\frac{dy}{dt} = -a\left(y - \frac{b}{a}\right) \Rightarrow \frac{dy/dt}{y - \frac{b}{a}} = -a \, , \; y \neq \frac{b}{a}$$

Integrating: For $y - \frac{b}{a} > 0$, $\ln\left(y - \frac{b}{a}\right) = -at + C_1$

For $y - \frac{b}{a} < 0$, $\frac{-dy/dt}{-(y-\frac{b}{a})} = -a \Rightarrow$

$$\ln\left[-\left(y-\frac{b}{a}\right)\right] = -at + C_2 \quad , \quad C_1, C_2 \text{ constants}$$

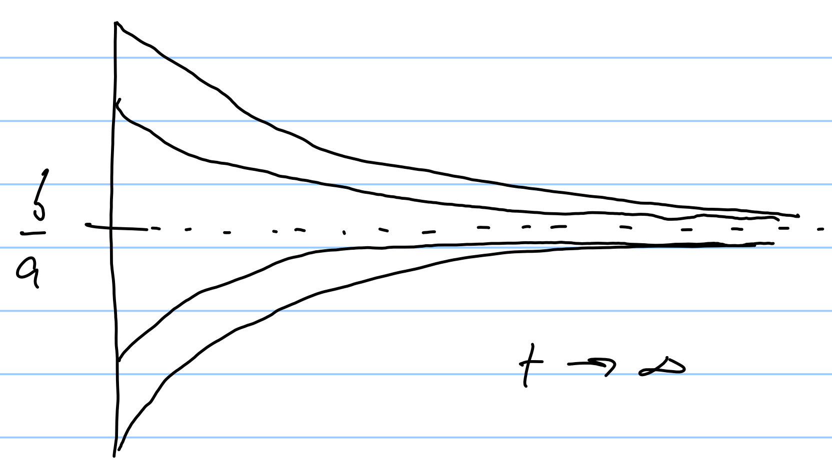
$\therefore$ $y - \frac{b}{a} > 0$: $y - \frac{b}{a} = e^{-at}e^{C_1}$

$y - \frac{b}{a} < 0$: $y - \frac{b}{a} = -e^{-at}e^{C_2}$

As $e^{C_1} > 0$ and $-e^{C_2} < 0$ are constants, then a general solution is $\underline{y(t) = \frac{b}{a} + ce^{-at}}$

c a constant. And this solution also works for $y = \frac{b}{a}$, using $c = 0$.

(6)



All solutions exponential, moving toward $y = \frac{b}{a}$, the equilibrium, as $t \to \infty$.

(c)

(i) As a increases, the y-intercept, $\frac{b}{a}+c$, is closer to c, and the equilibrium value, $\frac{b}{a}$, is smaller, and as $t \to \infty$, $y(t)$ moves to $y=\frac{b}{a}$ faster due to e^{-at}

(ii) As b increases, y-intercept, $\frac{b}{a}+c$, is farther from c, equilibrium value, $\frac{b}{a}$, is greater, no change in speed toward $\frac{b}{a}$ as $t \to \infty$ as there is no change in e^{-at}.

(iii) Equilibrium, $y=\frac{b}{a}$, is unchanged, but as a increases, speed to get to equilibrium, from e^{-at}, is faster as t increases.

4.

(a)

From the general solution, $y(t) = \frac{b}{a} + \left(y_0 - \frac{b}{a}\right)e^{at}$,

$\underline{y_e = \frac{b}{a}}$, since $y_0 = \frac{b}{a}$ eliminates e^{at}.

(b)

From (a), $Y(t) = y(t) - y_e = \left(y_0 - \frac{b}{a}\right)e^{at}$

$\therefore \frac{d}{dt}Y(t) = a\left(y_0 - \frac{b}{a}\right)e^{at} = aY(t)$

$\therefore \underline{\frac{dY}{dt} = aY}$

5.

(a) $\frac{dy/dt}{y} = a$. $\therefore \ln(y) = at + C_1$, $y = e^{at}e^{C_1}$

C_1 a constant. Now let $c = e^{C_1}$

or $\underline{y_1(t) = ce^{at}}$

(b)

$$\frac{d}{dt}(y_1 + K) = a(y_1 + K) - b$$

But $\frac{d}{dt}(y_1 + K) = \frac{d}{dt}(y_1) + \frac{d}{dt}(K) = ay_1 + 0 = ay_1$

$\therefore ay_1 = a(y_1 + K) - b = ay_1 + ak - b$

$\therefore b = aK, \ \underline{K = \frac{b}{a}}$

(c)

From (6), $y = ce^{at} + \frac{b}{a}$

Equation (17) is $y = \frac{b}{a} + ce^{at}$

6.

$$\frac{dy_1}{dt} = -ay_1, \quad \frac{dy_1/dt}{y_1} = -a$$

Integrating, $\ln(y_1) = -at + C_1$, C_1 a constant

$\therefore\ y_1(t) = ce^{-at}$, c a constant

Now let $y = y_1 + k$

$$\therefore\ \frac{d}{dt}(y_1+k) = -a(y_1+k)+b$$

$$\frac{dy_1}{dt} + \frac{d(k)}{dt} = \frac{dy_1}{dt} + 0 = \frac{dy_1}{dt} = -ay_1$$

$$\therefore\ -ay_1 = -a(y_1+k)+b,$$

or $0 = -ak + b$, $k = \frac{b}{a}$

$$\therefore\ y(t) = y_1 + k = ce^{-at} + \frac{b}{a}$$

$$\underline{y(t) = ce^{-at} + \frac{b}{a}}$$

7.

should be

$$\frac{dp}{dt} = \frac{p}{2} - 450$$

(a)

$p(t) = 900 + ce^{t/2}$, t is in months

$$p(0) = 850 = 900 + ce^{0/2} = 900 + c \Rightarrow c = -50$$

$$\therefore\ p(t) = 900 - 50e^{t/2}$$

Population extinct when $p(t) = 0$.

$\therefore\ 0 = 900 - 50e^{t/2}$, $18 = e^{t/2}$,

$\ln(18) = t/2$, $\underline{t = 2\ln(18) \approx 5.78 \text{ months}}$

(b)

From $p(t) = 900 + ce^{t/2}$, t in months,

$p(0) = p_0 = 900 + c$, $c = p_0 - 900$

$\therefore\ p(t) = 900 + (p_0 - 900)e^{t/2}$

Extinct $\Rightarrow 0 = 900 + (p_0 - 900)e^{t/2}$,

or $\dfrac{900}{900 - p_0} = e^{t/2}$, $0 < p_0 < 900$

$\therefore\ \ln\left(\dfrac{900}{900 - p_0}\right) = t/2$, $t = \underline{2\ln\left(\dfrac{900}{900 - p_0}\right)}$ months

(c)

From (b), $p(t) = 900 + (p_0 - 900)e^{t/2}$

1 year = 12 months, and $p(12) = 0$

$\therefore\ p(12) = 0 = 900 + (p_0 - 900)\, e^{12/2}$

or, $\quad 0 = 900 + p_0 e^6 - 900 e^6$,

$$\therefore\ \frac{900 e^6 - 900}{e^6} = p_0$$

$$\therefore\ p_0 = 900\left(1 - \frac{1}{e^6}\right) \simeq 897.8$$

So $\underline{p_0 \le 897 \text{ rabbits}}$

8.

(a)

As in Example 2, the solution to the intial value problem is $V(t) = 49\left(1 - e^{-t/5}\right)$

Limiting velocity: $\lim_{t \to \infty} 49\left(1 - e^{-t/5}\right) = 49$ m/sec

Let K = time to reach 98% of 49 m/sec

$\therefore\ V(K) = (0.98)\, 49 = 49\left(1 - e^{-K/5}\right)$

$\therefore \quad e^{-K/5} = 1 - 0.98 = 0.02,$

$-K/5 = \ln(0.02), \quad K = -5\ln(0.02)$

or, $K = 5\ln\left(\frac{1}{.02}\right) = 5\ln(50) \approx \underline{\underline{19.56 \text{ secs}}}$

(b)

$\frac{d\,x(t)}{dt} = v(t) = 49 - 49e^{-t/5}$, and $x(0) = 0$.

Integrating, $x(t) = 49t + 5(49)e^{-t/5} + c$,

c a constant

$x(0) = 0 \Rightarrow 49(0) + 5(49)(1) + c = 0, \quad c = -245$

$\therefore \; x(t) = 49t + 245e^{-t/5} - 245$

From (a), time of falling is: $5\ln(50)$

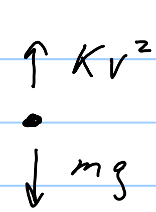
$\therefore \; x(5\ln(50)) = 245\ln(50) + 245\,e^{-\ln(50)} - 245$

$= 245\ln(50) + \frac{245}{50} - 245$

$\approx \underline{\underline{718.3 \text{ meters}}}$

9.

(a)



Look at forces: $m\dfrac{dv}{dt} = mg - Kv^2$

$$\therefore \frac{dv}{dt} = g - \frac{K}{m}v^2 = 9.8 - \frac{K}{10}v^2$$

Limiting velocity is when $\dfrac{dv}{dt} = 0$.

$$\therefore 0 = 9.8 - \frac{K}{10}49^2, \text{ for } 10\text{ Kg mass}$$

$$\therefore 98 = K\,49^2, \quad K = \frac{98}{49^2} = \frac{49(2)}{49^2} = \frac{2}{49}$$

$$\therefore \frac{dv}{dt} = 9.8 - \frac{2}{490}v^2 = \frac{2}{490}\left[9.8\left(\frac{490}{2}\right) - v^2\right]$$

$$= \frac{1}{245}\left[4.9(490) - v^2\right]$$

$$\therefore \underline{\frac{dv}{dt} = \frac{1}{245}\left[49^2 - v^2\right]}$$

(b)

$$\frac{dv/dt}{49^2 - v^2} = \frac{1}{245}$$

Using a table of integrals,

$$\int \frac{du}{a^2 - u^2} = \frac{1}{2\,a} \ln \frac{a+u}{a-u} + C = \frac{1}{a} \tanh^{-1} \frac{u}{a} + C, \quad \text{if } u^2 < a^2.$$

Since $v(0) = 0$ and limiting velocity is 49,

$0 \le v < 49$, $\therefore$ $v^2 < 49^2$

$$\therefore \int \frac{dv/dt}{49^2 - v^2} = \frac{1}{49} \tanh^{-1}\left(\frac{v}{49}\right) = \int \frac{dt}{245} = \frac{t}{245} + C$$

Using $v(0) = 0$, $\frac{1}{49} \tanh^{-1}(0) = 0 = \frac{0}{245} + C \Rightarrow C = 0$

$$\therefore \tanh^{-1}\left(\frac{v}{49}\right) = \frac{49}{245} t = \frac{t}{5}$$

$$\therefore \frac{v}{49} = \tanh\left(\frac{t}{5}\right), \text{ or } v(t) = 49 \tanh\left(\frac{t}{5}\right)$$

$$\therefore \underline{v(t) = 49 \tanh\left(\frac{t}{5}\right) \text{ m/sec}}$$

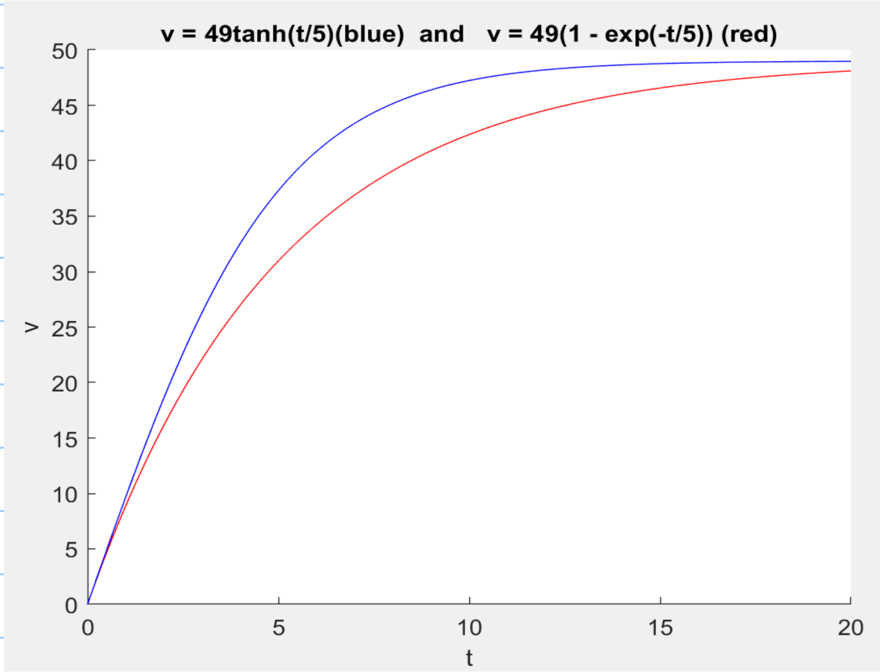
(c)

Solution from Example 2 on p.13 of text

is $v(t) = 49(1 - e^{-t/5})$ for $\frac{dv}{dt} = 9.8 - \frac{v}{5}$

Using MATLAB code:

```
clear,clc;
t = 0:0.1:20;
hold on
eqn = 49*(1 - exp(-t/5));
plot(t, eqn, 'r')
eqn = 49*tanh(t/5);
plot(t, eqn, 'b')
hold off
xlabel 't', ylabel 'v'
title 'v = 49tanh(t/5)(blue)  and  v = 49(1 - exp(-t/5)) (red)'
```



With drag proportional to v^2 (plot in blue), the terminal velocity is reached faster.

(d)

Basically, The same shaped curve, but reaching limiting velocity faster.

(e)

From (6), $v(t) = 49\tanh\left(\frac{t}{5}\right) = \frac{dx(t)}{dt}$

$\therefore$ integrating, and using $x(0) = 0$

$$x(t) = \int 49\tanh\left(\frac{t}{5}\right) dt$$

Using:

$$\int \tanh u \, du = \ln \cosh u + C.$$

$$= 5(49)\ln\left[\cosh\left(\frac{t}{5}\right)\right] + C$$

$\cosh(0) = 1$, so $x(0) = 245\ln[1] + C = 0$, $C = 0$

$\therefore$ $x(t) = 245\ln\left[\cosh\left(\frac{t}{5}\right)\right]$ meters

(f)

From (e), $300 = 245\ln\left[\cosh\left(\frac{t}{5}\right)\right]$

$\therefore$ $e^{300/245} = \cosh\left(\frac{t}{5}\right)$,

$t = 5\cosh^{-1}\left[e^{300/245}\right] = 5\cosh^{-1}(3.402) = 9.48\text{ s}$

$\therefore$ $T \approx 9.48$ sec

10.

(a)

$$\frac{dQ/dt}{Q} = -r, \quad \ln Q = -rt + C, \quad Q = e^{-rt}e^{C},$$

$$\therefore Q(t) = Ke^{-rt}, \quad Q(0) = K = 100\,mg$$

$$\therefore Q(t) = 100\,e^{-rt}$$

Let t be in days

$$\therefore Q(7) = 82.04 = 100\,e^{-r(7)}, \quad e^{-7r} = 0.8204$$

$$\therefore -7r = \ln(0.8204) = -0.19796,$$

$$\underline{r \doteqdot 0.02828}\,/day$$

(b)

From (a), $\underline{Q(t) = 100\,e^{-0.02828t}}$, t in days

(c)

$$50 = 100\,e^{-rt}, \quad \tfrac{1}{2} = e^{-rt}, \quad \ln(\tfrac{1}{2}) = -rt,$$

$$t = -\frac{\ln(\frac{1}{2})}{r} = \frac{\ln(2)}{r} = \frac{0.6931}{0.02828} \approx 24.51$$

$\therefore$ half-life $\approx$ <u>24.51</u> days

11.

$\frac{Q'}{Q} = -r$. $\therefore$ Integrating, $\ln Q(t) = -rt + C$,

$\therefore Q(t) = e^{-rt}e^{C}$, or $Q(t) = Ke^{-rt}$, $K = e^{C}$, constant

$Q(0) = K$, $\therefore \frac{1}{2}Q(0) = \frac{1}{2}K$.

$\therefore \frac{1}{2}K = Ke^{-rt}$, t = half-life.

$\therefore \frac{1}{2} = e^{-rt}$, $\ln(\frac{1}{2}) = -rt$,

and since $\ln(\frac{1}{2}) = -\ln(2)$,

$\therefore \underline{\ln(2) = rt}$, t = half-life

12.

(a)

$$\frac{du/dt}{u-T} = -K, \text{ integrating} \Rightarrow \ln(u-T) = -Kt + C_1$$

C_1 a constant

$\therefore u(t) - T = e^{-Kt} e^{C_1} = c e^{-Kt}$, c a constant

$\therefore u(t) = T + c e^{-Kt}$

$\therefore u(0) = T + c$, $c = u_0 - T$

$\therefore u(t) = \underline{T + (u_0 - T) e^{-Kt}}$

(b)

$$u(\tau) - T = \frac{(u_0 - T)}{2} = (u_0 - T) e^{-K\tau}$$

$\therefore \frac{1}{2} = e^{-K\tau}$, $\ln(\frac{1}{2}) = -K\tau$, or $\ln(2) = K\tau$

$$\therefore \underline{\tau = \frac{\ln(2)}{K}}$$

13.

(a)

$$\frac{dQ}{dt} = -\frac{Q}{RC} + \frac{V}{R} = \frac{VC - Q}{RC} = -\frac{Q - VC}{RC}$$

$$\therefore \frac{dQ/dt}{Q - VC} = -\frac{1}{RC} \Rightarrow \ln(Q - VC) = -\frac{t}{RC} + K_1$$

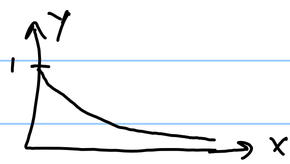
$$\therefore Q - VC = e^{-t/RC} e^{K_1} = K e^{-t/RC}, \quad K = e^{K_1} \text{ a constant}$$

$$\therefore Q(t) = VC + K e^{-t/RC}$$

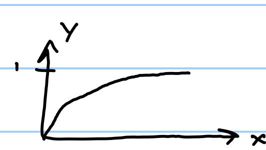
$$Q(0) = 0 = VC + K, \quad K = -VC$$

$$\therefore Q(t) = VC\left(1 - e^{-t/RC}\right)$$

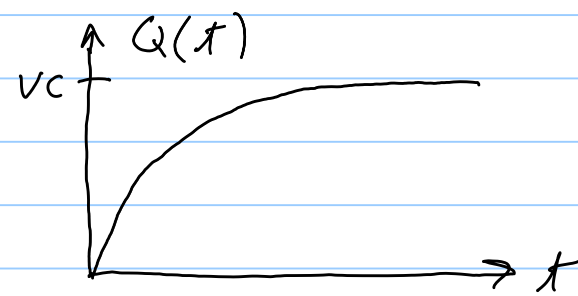
e^{-x} :



$1 - e^{-x}$:



$\therefore$



$Q(t)$ approaches VC asymptotically as $t \to \infty$

(b)

As in (a), as $t \to \infty$, $\underline{Q_L = VC}$

(c)

From (a), $Q(t) = VC + Ke^{-t/RC}$ is the general solution. $\therefore$ when $V = 0$,

$Q(t) = Ke^{-t/RC}$ [1]

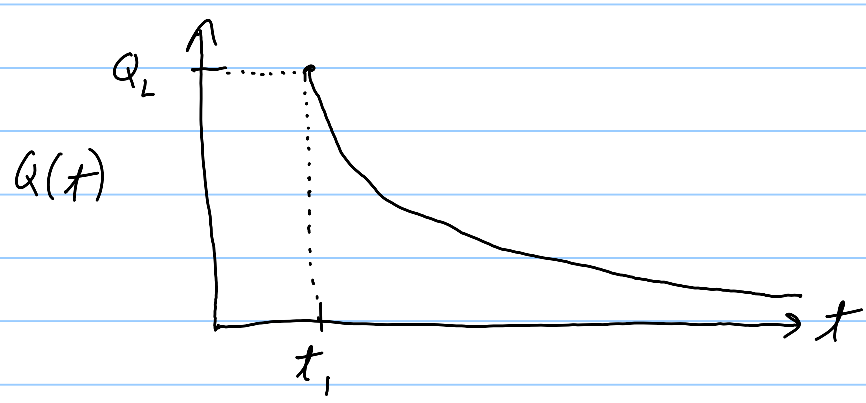
$Q(t_1) = Q_L = Ke^{-t_1/RC}$, $\therefore \ln(Q_L) = \ln(K) - t_1/RC$

$\therefore \ln(K) = \ln(Q_L) + \frac{t_1}{RC}$

$\therefore K = Q_L e^{t_1/RC}$

From [1], $Q(t) = Q_L e^{(t_1 - t)/RC} = Q_L e^{-(t-t_1)/RC}$ as $t \geq t_1$

$\therefore \underline{\underline{Q(t) = Q_L e^{-(t-t_1)/RC}}}$



as $t \to \infty$, $Q(t) \to 0$

14.

(a)

$Q(t)$ = amount in grams at time t.

$\frac{dQ}{dt}$ = change in amount per hour

Rate flowing in $= (0.01\ g/gal)(300\ gal/hr) = 3g/hr$

Rate flowing out $= \left(\frac{Q(t)}{1{,}000{,}000}\ \frac{g}{gal}\right)\left(300\ gal/hr\right)$

$= \frac{3}{10{,}000}\,Q(t)\ g/hr$

$$\therefore\ \frac{dQ}{dt} = 3 - \frac{3}{10{,}000}\,Q(t)\ ,\quad Q(0) = 0$$

$$\text{or,}\quad \frac{dQ}{dt} = \underline{3\left(1 - 10^{-4}Q(t)\right),\quad Q(0) = 0}$$

(b)

$$\frac{dQ}{dt} = -3\times10^{-4}\left(Q(t) - 10^{4}\right)$$

$$\frac{dQ/dt}{Q - 10^{4}} = -3\times10^{-4}$$

$$\ln(Q - 10^4) = (-3\times 10^{-4})t + C$$

$$Q - 10^4 = K e^{(-3\times 10^{-4})t}, \quad K = e^C, \text{ a constant}$$

$$\therefore Q(t) = 10^4 + K e^{(-3\times 10^{-4})t}$$

$$Q(0) = 0 \Rightarrow K = -10^4$$

$$\therefore \underline{Q(t) = 10^4\left(1 - e^{(-3\times 10^{-4})t}\right)}, \quad t \text{ in hours}$$

$$1 \text{ year} = (365 \text{ days})(24 \text{ hrs/day}) = 8760 \text{ hrs.}$$

$$\therefore Q(1\,\text{year}) = 10^4\left(1 - e^{(-3\times 10^{-4})(8760)}\right)$$

$$\approx \underline{9277.77 \text{ grams}}$$

(c)

Rate (grams/hr) flowing in: $0\ g/hr$

Rate flowing out: $\frac{3}{10{,}000} Q(t)\ g/hr$ as in (a)

$$\therefore \underline{\frac{dQ}{dt} = -\frac{3}{10{,}000} Q(t), \quad Q(0) = 9277.77 \text{ grams}}$$

(d)

$$\frac{dQ/dt}{Q} = -\frac{3}{10{,}000}, \quad \ln Q = -\frac{3}{10{,}000}t + C_1$$

$$\therefore \; Q(t) = Ke^{-\frac{3}{10{,}000}t}, \quad K = e^{C_1}, \text{ a constant}$$

$$Q(0) = 9277.77 = K$$

$$\therefore \; \underline{Q(t) = 9277.77\, e^{-\frac{3}{10{,}000}t}}, \quad t \text{ in hours}$$

1 year = 8760 hrs.

$$\therefore \; Q(1\,yr) = 9277.77\, e^{-\frac{3}{10{,}000}(8760)}$$

$$\approx \underline{670.07 \text{ grams}}$$

(e)

$$10 = 9277.77\, e^{-\frac{3}{10{,}000}t}$$

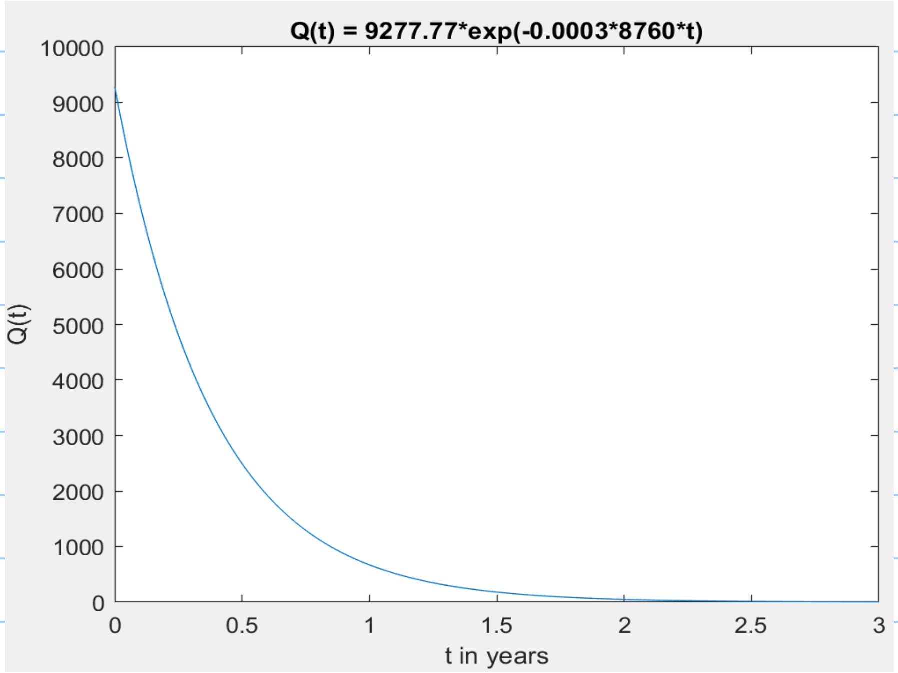
$$\ln\left(\frac{10}{9277.77}\right) \approx -6.83279 = -\frac{3}{10{,}000}t$$

$$t = \frac{68327.9}{3} = 22775.97 \text{ hrs}$$

$$= \frac{22775.97 \text{ hrs}}{8760 \text{ hr/yr}} \approx \underline{2.6 \text{ yrs}}$$

MATLAB code, plotting years on horizontal axis, using 8760 hrs = 1yr

```
clear,clc;
t = 0:0.01:3;
eqn = 9277.77*exp(-.0003*8760*t);
plot(t, eqn)
xlabel 't in years', ylabel 'Q(t)'
title 'Q(t) = 9277.77*exp(-.0003*8760*t)'
```



1.3 Classification of Differential Equations

Note Title 12/6/2017

1.

Second order, from term $\frac{d^2y}{dt^2}$

Linear, since terms in y (not "t") are linear.

2.

Second order, from $\frac{d^2y}{dt^2}$ term

Nonlinear, due to $y^2\frac{d^2y}{dt^2}$ term

3.

Fourth order, from $\frac{d^4y}{dt^4}$ term

Linear, as terms in y are linear

4.

Second order, from $\frac{d^2y}{dt^2}$ term

Nonlinear, due to $\sin(t+y)$ term.

5.

$y_1'(t) = e^t$, $y_1''(t) = e^t$ $\therefore y_1'' - y_1 = e^t - e^t = \underline{0}$

$y_2'(t) = \sinh(t)$, $y_2''(t) = D_t(\sinh(t)) = \cosh(t)$

$\therefore y_2'' - y_2 = \cosh(t) - \cosh(t) = \underline{0}$

6.

(a) $y_1'(t) = -3e^{-3t}$ $y_1'' = 9e^{-3t}$

$\therefore y_1'' + 2y_1' - 3y_1 = 9e^{-3t} + (2)(-3e^{-3t}) - 3(e^{-3t})$

$= 9e^{-3t} - 9e^{-3t} = \underline{\underline{0}}$

(b) $y_2'(t) = e^t$ $y_2''(t) = e^t$

$\therefore y_2'' + 2y_2' - 3y_2 = e^t + 2e^t - 3e^t = \underline{0}$

7.

$y' = 3 + 2t$, $ty' = 3t + 2t^2$

$\therefore ty' - y = (3t + 2t^2) - (3t + t^2) = \underline{t^2}$

8.

(a) $y_1' = \frac{1}{3}$, $y_1'' = y_1''' = y_1'''' = 0$

$$\therefore y_1'''' + 4y_1''' + 3y_1 = (0) + 4(0) + 3\left(\frac{t}{3}\right) = \underline{t}$$

(b) $y_2' = -e^{-t} + \frac{1}{3}$, $y_2'' = e^{-t}$, $y_2''' = -e^{-t}$, $y_2'''' = e^{-t}$

$$\therefore y_2'''' + 4y_2''' + 3y_2 = (e^{-t}) + 4(-e^{-t}) + 3(e^{-t} + t/3)$$

$$= 4e^{-t} - 4e^{-t} + t = \underline{t}$$

9.

(a) $y_1' = -2t^{-3}$, $y_1'' = 6t^{-4}$

$$\therefore t^2 y_1'' + 5t\, y_1' + 4y_1 = t^2\left(\frac{6}{t^4}\right) + 5t\left(\frac{-2}{t^3}\right) + 4\left(\frac{1}{t^2}\right)$$

$$= \frac{6}{t^2} - \frac{10}{t^2} + \frac{4}{t^2} = \underline{0}$$

(b) $y_2' = \frac{1}{t^3} - \frac{2\ln t}{t^3}$, $y_2'' = \frac{-3}{t^4} - \frac{2}{t^4} + \frac{6\ln t}{t^4}$

$$\therefore t^2 y_2'' + 5t\, y_2' + 4y_2 =$$

$$t^2\left(-\frac{5}{t^4} + \frac{6\ln t}{t^4}\right) + 5t\left(\frac{1}{t^3} - \frac{2\ln t}{t^3}\right) + 4\left(\frac{\ln t}{t^2}\right)$$

$$= -\frac{5}{t^2} + \frac{6\ln t}{t^2} + \frac{5}{t^2} - \frac{10\ln t}{t^2} + \frac{4\ln t}{t^2} = \underline{0}$$

10.

$$y' = e^{t^2}\left(e^{-t^2}\right) + 2te^{t^2}\int_0^t e^{-s^2}ds + 2te^{t^2}$$

$$\left[\text{as } \frac{d}{dt}\int_0^t e^{-s^2}ds = e^{-t^2} \text{ by The Fundamental Theorem of Calculus}\right]$$

$$= 1 + 2te^{t^2}\left(\int_0^t e^{-s^2}ds + 1\right)$$

$$\therefore\ y' - 2ty = 1 + 2te^{t^2}\left(\int_0^t e^{-s^2}ds + 1\right) - 2t\left(e^{t^2}\int_0^t e^{-s^2}ds + e^{t^2}\right)$$

$$= 1 + \underbrace{2te^{t^2}\int_0^t e^{-s^2}ds}_{\text{cancel}} + \underbrace{2te^{t^2}}_{\text{cancel}}$$

$$- 2te^{t^2}\int_0^t e^{-s^2}ds - 2te^{t^2}$$

$$= \underline{1}$$

11.

$$y' + 2y = \left(re^{rt}\right) + 2\left(e^{rt}\right) = e^{rt}(r+2) = 0 \Rightarrow \underline{r = -2}$$

12.

$$y'' + y' - 6y = (r^2 e^{rt}) + (re^{rt}) - 6(e^{rt})$$

$$= e^{rt}(r^2 + r - 6) = 0$$

$$= e^{rt}(r+3)(r-2) = 0 \Rightarrow \underline{r = 2, -3}$$

13.

$$y''' - 3y'' + 2y' = (r^3 e^{rt}) - 3(r^2 e^{rt}) + 2(re^{rt})$$

$$= e^{rt}(r^3 - 3r^2 + 2r)$$

$$= e^{rt}(r)(r-2)(r-1) = 0$$

$$\Rightarrow \underline{r = 0, 2, 1}$$

14.

$$y' = rt^{r-1}, \quad y'' = r(r-1)t^{r-2}$$

$$\therefore t^2 y'' + 4t y' + 2y = t^2(r)(r-1)t^{r-2} + 4t(r)t^{r-1} + 2(t^r)$$

$$= t^r[r^2 - r + 4r + 2]$$

$= t^r[(r+2)(r+1)] = 0$

$\Rightarrow \underline{r = -1, -2}$

15.

$y' = rt^{r-1}, \quad y'' = r(r-1)t^{r-2}$

$\therefore t^2 y'' - 4ty' + 4y = t^2(r)(r-1)t^{r-2} - 4t(r)t^{r-1} + 4t^r$

$= t^r[r^2 - r - 4r + 4]$

$= t^r[(r-4)(r-1)] = 0$

$\Rightarrow \underline{r = 1, 4}$

16.

Highest derivative: all are terms are second degree.

and individual terms are added.

$\therefore$ <u>2nd order, linear</u>

17.

All terms are 4th degree, and individual terms are added. $\therefore$ 4th order, linear

18.

Highest derivative is from u_{xx}, $\therefore$ 2nd order

The multiplication of $(u)(u_x)$ makes nonlinear

19.

(a) $\dfrac{\partial u_1}{\partial x} = -\sin x \cosh y$, $\dfrac{\partial^2 u_1}{\partial x^2} = -\cos x \cosh y$

$\dfrac{\partial u_1}{\partial y} = \cos x \sinh y$, $\dfrac{\partial^2 u_1}{\partial y^2} = \cos x \cosh y$

$$\therefore (u_1)_{xx} + (u_1)_{yy} = -\cos x \cosh y + \cos x \cosh y$$
$$= \underline{0}$$

(b) $\dfrac{\partial u_2}{\partial x} = \dfrac{2x}{x^2+y^2}$, $\dfrac{\partial^2 u_2}{\partial x^2} = \dfrac{(x^2+y^2)(2) - 2x(2x)}{(x^2+y^2)^2}$

$$\frac{\partial u_2}{\partial y} = \frac{2y}{x^2+y^2}, \quad \frac{\partial^2 u_2}{\partial y^2} = \frac{(x^2+y^2)(2) - 2y(2y)}{(x^2+y^2)^2}$$

$$\therefore (u_2)_{xx} + (u_2)_{yy} = \frac{(x^2+y^2)(2) - 4x^2 + (x^2+y^2)(2) - 4y^2}{(x^2+y^2)^2}$$

$$= \frac{2x^2 + 2y^2 - 4x^2 + 2x^2 + 2y^2 - 4y^2}{(x^2+y^2)^2}$$

$$= \underline{0}$$

20.

(a) $(u_1)_x = e^{-\alpha^2 t}\cos x, \quad (u_1)_{xx} = -e^{-\alpha^2 t}\sin x$

$(u_1)_t = -\alpha^2 e^{-\alpha^2 t}\sin x$

$\therefore \underline{\alpha^2 (u_1)_{xx}} = -\alpha^2 e^{-\alpha^2 t}\sin x = \underline{(u_1)_t}$

(b) $(u_2)_x = \lambda e^{-\alpha^2\lambda^2 t}\cos\lambda x, \quad (u_2)_{xx} = -\lambda^2 e^{-\alpha^2\lambda^2 t}\sin\lambda x$

$(u_2)_t = -\alpha^2\lambda^2 e^{-\alpha^2\lambda^2 t}\sin\lambda x$

$\therefore \alpha^2 (u_2)_{xx} = -\alpha^2\lambda^2 e^{-\alpha^2\lambda^2 t}\sin\lambda x = \underline{(u_2)_t}$

21.

(a) $(u_1)_x = \lambda \cos(\lambda x) \sin(a\lambda t)$

$(u_1)_{xx} = -\lambda^2 \sin(\lambda x) \sin(a\lambda t)$

$(u_1)_t = a\lambda \sin(\lambda x) \cos(a\lambda t)$

$(u_1)_{tt} = -a^2\lambda^2 \sin(\lambda x) \sin(a\lambda t)$

$\therefore \underline{a^2 (u_1)_{xx}} = -a^2\lambda^2 \sin(\lambda x)\sin(a\lambda t) = \underline{(u_1)_{tt}}$

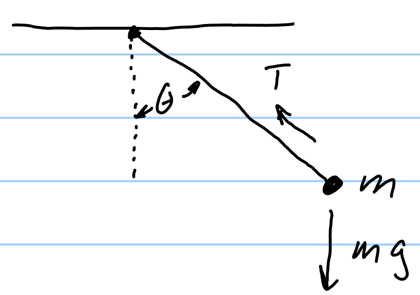
(b) $(u_2)_x = \cos(x-at)$, $(u_2)_{xx} = -\sin(x-at)$

$(u_2)_t = -a\cos(x-at)$, $(u_2)_{tt} = -a^2\sin(x-at)$

$\therefore \underline{a^2(u_2)_{xx}} = -a^2 \sin(x-at) = \underline{(u_2)_{tt}}$

22.

(a)

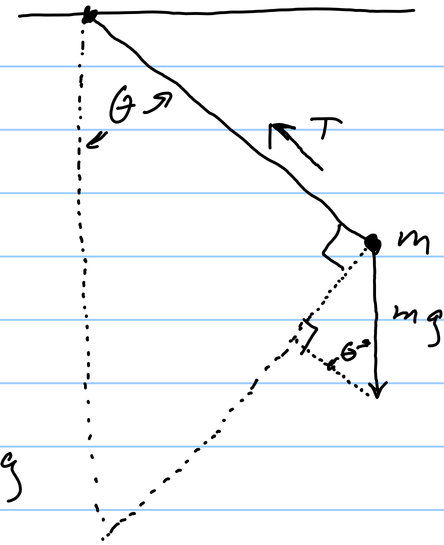


m = mass

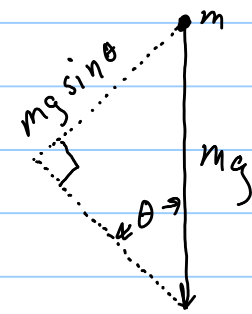
T = tension in rod

$g = 9.8\ m/sec^2$

(5)

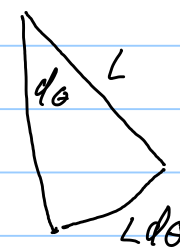


The component of gravity perpendicular to the rod is $mg\sin\theta$



Note for an arc of radius L, sweeping an angle of $d\theta$, the length of the arc is:

$$L\,d\theta$$



$\therefore$ The instantaneous linear velocity along the arc is $L\dfrac{d\theta}{dt}$ and the instantaneous acceleration is $L\dfrac{d^2\theta}{dt^2}$

$\therefore$ From Newton's $\vec{F} = m\vec{a}$, or in scalar terms

$$-mg\sin\theta = m\left(L\frac{d^2\theta}{dt^2}\right)$$

Note: Assign direction to right of vertical as positive, to left of vertical as negative $\therefore$ if $\theta > 0$, the force component, $mg\sin\theta$, points to the left. If $\theta < 0$, it points to right. $\therefore$ Need a minus sign in front of $mg\sin\theta$.

(c)

Cancelling m, $g \sin\theta = -L \frac{d^2\theta}{dt^2}$, or

$$\frac{d^2\theta}{dt} + \frac{g}{L}\sin\theta = 0$$

23.

(a)

Kinetic energy is $\frac{1}{2}mv^2$

As shown in 22(b), the instantaneous velocity along an arc is $L\frac{d\theta}{dt}$.

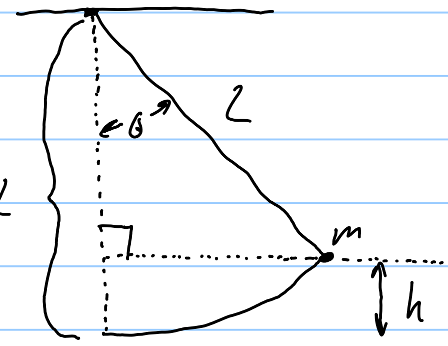
$$\therefore v^2 = \left(L\frac{d\theta}{dt}\right)^2 = L^2\left(\frac{d\theta}{dt}\right)^2$$

$$\therefore \frac{1}{2}mv^2 = \frac{1}{2}mL^2\left(\frac{d\theta}{dt}\right)^2$$

(b)

Potential energy is mgh, h = height above

reference point at which potential energy $=0$.
Let the bottom of the arc be the reference point. From the figure,



$h = L - L\cos\theta$

$\therefore V = mgh$

$= mg(L - L\cos\theta)$

$= \underline{mgL(1-\cos\theta)}$

(c)

$$E = \frac{1}{2} m L^2 \left(\frac{d\theta}{dt}\right)^2 + mgL(1-\cos\theta)$$

$$\frac{dE}{dt} = mL^2 \left(\frac{d\theta}{dt}\right)\left(\frac{d^2\theta}{dt^2}\right) + mgL \frac{d}{dt}(1-\cos\theta)$$

$$= mL^2 \frac{d\theta}{dt} \cdot \frac{d^2\theta}{dt^2} + (mgL\sin\theta)\frac{d\theta}{dt}$$

Setting $\frac{dE}{dt} = 0$ and cancelling m,

$$0 = L^2 \frac{d\theta}{dt} \cdot \frac{d^2\theta}{dt^2} + gL \sin\theta \frac{d\theta}{dt}$$

Cancelling L and assuming $\frac{d\theta}{dt} \neq 0$, which occurs only at extremes of swing,

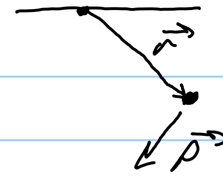
$$0 = \frac{d^2\theta}{dt^2} + \frac{g}{L} \sin\theta$$

24.

(a)

The angular momentum of a point mass m is defined as $\vec{r} \times \vec{p}$, where $\vec{r}$ is the instantaneous position vector, and $\vec{p}$ is the instantaneous linear momentum vector.

As shown in 22(b), the instantaneous velocity of m is $L \frac{d\theta}{dt}$, so its instantaneous momentum is $mL\frac{d\theta}{dt} = \|\vec{p}\|$



$\|\vec{r} \times \vec{p}\| = \|\vec{r}\|\|\vec{p}\| \sin\phi$, ϕ = angle between $\vec{r}$, $\vec{p}$.

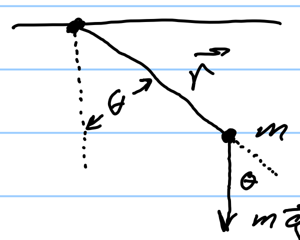
In the pendulum case, $\phi = 90°$, and in this case $\|\vec{r}\| = L$. $\therefore$ $\|\vec{r} \times \vec{p}\| = \|\vec{r}\|\|\vec{p}\| = L\|\vec{p}\|$

$\therefore$ Angular momentum $= (L)\left(mL\frac{d\theta}{dt}\right) = \underline{mL^2\frac{d\theta}{dt}}$

(b)

The moment of the gravitation force, or torque, is $\vec{r} \times \vec{F}$, and the magnitude is

$\|\vec{r} \times \vec{F}\| = \|\vec{r}\|\|\vec{F}\| \sin\phi$, ϕ = angle between $\vec{r}$ and $\vec{F}$.



Here, $\|\vec{r}\| = L$, $\|\vec{F}\| = mg$

and θ is the angle between $\vec{r}$ and $m\vec{g}$.

$\therefore$ $\|\vec{r} \times \vec{F}\| = Lmg\sin\theta$

Using the right hand rule, the scalar torque value is: $-Lmg\sin\theta$ in the above diagram,

assigning positive to right of vertical, so $\sin\theta > 0$

From (a), $M = mL^2 \dfrac{d\theta}{dt}$

$$\therefore\ -Lmg\sin\theta = \frac{d}{dt}M = \frac{d}{dt}\left(mL^2\frac{d\theta}{dt}\right) = mL^2\frac{d^2\theta}{dt^2}$$

Cancelling L and m, $-g\sin\theta = L\dfrac{d^2\theta}{dt^2}$,

or $$\underline{\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0}$$