11.1 The Occurrence Of Two-Point Boundry Value Problems

Note Title 9/17/2021

1.

All conditions are zero $\Rightarrow$ <u>homogeneous</u>

2.

$y(1) = 1 \neq 0 \Rightarrow$ <u>nonhomogeneous</u>

3.

$y'' + 4y = \sin(x) \neq 0 \Rightarrow$ <u>nonhomogeneous</u>

4.

$-y'' + x^2 y - \lambda y = 0$, all conditions are zero $\Rightarrow$ <u>homogeneous</u>

5.

$-((1+x^2)y')' - \lambda y - 1 = 0$, all conditions zero $\Rightarrow$ <u>homogeneous</u>

6.

$-y'' - \lambda(1+x^2)y = 0$, all conditions zero $\Rightarrow$ <u>homogeneous</u>

7.

(a) As in Example 1 on p. 531, $\lambda > 0$, so

$y = c_1 \sin(\sqrt{\lambda}\, x) + c_2 \cos(\sqrt{\lambda}\, x)$

$y(0) = 0 \Rightarrow c_2 = 0. \quad \therefore \quad y(x) = c_1 \sin(\sqrt{\lambda}\, x)$

$y(\pi) + y'(\pi) = c_1 [\sin(\sqrt{\lambda}\, \pi) + \sqrt{\lambda} \cos(\sqrt{\lambda}\, \pi)] = 0$

Since $c_1 \neq 0$ for a nontrivial solution, and

if $\cos(\sqrt{\lambda}\, \pi) = 0$, then $\sin(\sqrt{\lambda}\, \pi) \neq 0$.

$\therefore$ Can divide by $\cos(\sqrt{\lambda}\, \pi)$ to get

$\tan(\sqrt{\lambda}\, \pi) = -\sqrt{\lambda}$. $\therefore$ There is a solution in

every interval $\left[\frac{(2n-1)\pi}{2}, \frac{(2n+1)\pi}{2}\right]$, $n = 1, 2, 3, \ldots$

$\therefore$ $\underline{y_n(x) = \sin(\sqrt{\lambda_n}\, x)}$, where $\underline{\tan(\sqrt{\lambda_n}\, \pi) = -\sqrt{\lambda_n}}$,

$n = 1, 2, 3, \ldots$

(b)

If $\lambda = 0$, then $y'' + \lambda y = y'' = 0 \Rightarrow y(x) = c_1 x + c_2$

$y(0) = 0 \Rightarrow c_2 = 0 \Rightarrow y = c_1 x.$

$y(\pi) + y'(\pi) = c_1 \pi + c_1 = c_1(\pi + 1) = 0 \Rightarrow c_1 = 0.$

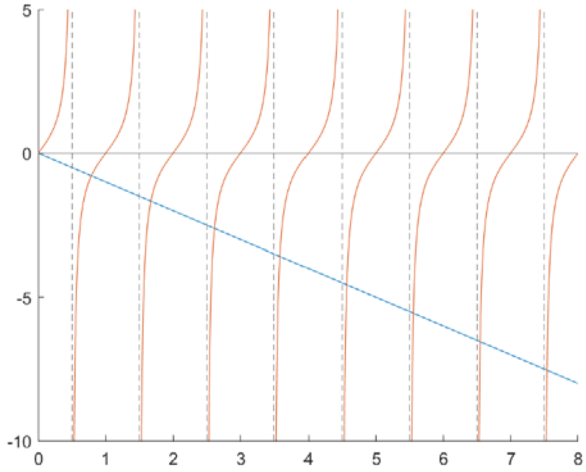
$\therefore \lambda = 0 \Rightarrow$ only trivial solution

$\therefore \lambda = 0$ is not an eigenvalue.

(c), (d)

Use MATLAB to solve $\tan(\sqrt{\lambda}\,\pi) = -\sqrt{\lambda}$

```
clear
syms x
figure
axis([0,8,-10,5])
hold on
yline(0)
fplot(-x)
fplot(tan(pi*x),[0,8])
for n = 1:8
    % start estimation just to right of (2n-1)/2
    a = vpasolve(tan(pi*x)==-x,x,(2*n-1)/2 + 0.1);
    a2 = a^2;
    sprintf('n=%d   a = %.5f   a^2 = %.5f',n,a,a2)
end
```




```
ans = 'n=1   a = 0.78764   a^2 = 0.62037'
ans = 'n=2   a = 1.67161   a^2 = 2.79427'
ans = 'n=3   a = 2.61621   a^2 = 6.84457'
ans = 'n=4   a = 3.58655   a^2 = 12.86336'
ans = 'n=5   a = 4.56859   a^2 = 20.87203'
ans = 'n=6   a = 5.55668   a^2 = 30.87667'
ans = 'n=7   a = 6.54824   a^2 = 42.87941'
ans = 'n=8   a = 7.54196   a^2 = 56.88117'
```

$\therefore \lambda_1 \approx 0.6204$, $\lambda_2 \approx 2.7943$

As n increases, the line $y=-x$ intersects $y=\tan(\pi x)$ at odd multiples of $\frac{1}{2}$, the asymptotes of $\tan(\pi x)$.

$\therefore \sqrt{\lambda_n} \approx \frac{(2n-1)}{2}$, so $\lambda_n \approx \frac{(2n-1)^2}{4}$

8.

(a)

As in Example 1 on p.531, $\lambda > 0$,

$y = c_1 \sin(\sqrt{\lambda}\, x) + c_2 \cos(\sqrt{\lambda}\, x)$

$y'(x) = c_1 \sqrt{\lambda} \cos(\sqrt{\lambda}\, x) - c_2 \sqrt{\lambda} \sin(\sqrt{\lambda}\, x)$

$y'(0) = 0 \Rightarrow c_1 = 0$. $\therefore y = c_2 \cos(\sqrt{\lambda}\, x)$

$y(1) + y'(1) = c_2 \cos(\sqrt{\lambda}) - c_2 \sqrt{\lambda} \sin(\sqrt{\lambda}) = 0$

If $\cos(\sqrt{\lambda}) = 0$, then $\sin(\lambda) \neq 0$, so $c_2 = 0$, trivial solution. $\therefore \cos(\sqrt{\lambda}) \neq 0$, so divide by $\cos(\sqrt{\lambda})$, $\therefore$ $y(1)+y'(1)=0 \Rightarrow \tan(\sqrt{\lambda}) = \frac{1}{\sqrt{\lambda}}$

There is a solution to $\tan(z) = \frac{1}{z}$ in every interval $\left[\frac{(2n-1)\pi}{2}, \frac{2n+1\pi}{2}\right]$, $n = 0,1,2,3,\ldots$

$\therefore$ $\underline{y_n(x) = \cos(\sqrt{\lambda_n}\, x)}$, $\underline{\tan(\sqrt{\lambda_n})} = \frac{1}{\sqrt{\lambda_n}}$

(b)

$y'' + \lambda y = 0 \Rightarrow y'' = 0$ for $\lambda = 0$. $\therefore$ $y = c_1 x + c_2$.

$y'(0) = 0 \Rightarrow c_1 = 0$. $\therefore$ $y(x) = c_2$

$y(1) + y'(1) = c_2 + 0 = 0 \Rightarrow c_2 = 0$.

$\therefore$ Only the trivial solution is possible if $\lambda = 0$

$\therefore$ $\underline{\lambda = 0 \text{ not}}$ an eigenvalue

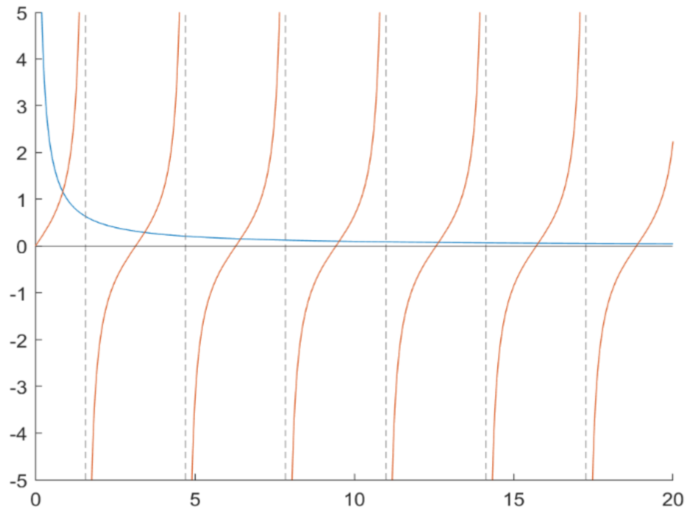
(c), (d)

Use MATLAB to solve $\tan(\sqrt{\lambda}) = \frac{1}{\sqrt{\lambda}}$

Code, values, and plot on next page.

```
clear
syms x
figure
axis([0,20,-5,5])
hold on
yline(0)
fplot(1/x)
fplot(tan(x),[0,20])
for n = 1:6
    % start estimation just to right of (n-1)*pi
    a = vpasolve(tan(x)==1/x,x,(n-1)*pi + 0.1);
    a2 = a^2;
    sprintf('n=%d    a = %.5f    a^2 = %.5f',n,a,a2)
end
```

```
ans = 'n=1    a = 0.86033    a^2 = 0.74017'
ans = 'n=2    a = 3.42562    a^2 = 11.73486'
ans = 'n=3    a = 6.43730    a^2 = 41.43881'
ans = 'n=4    a = 9.52933    a^2 = 90.80821'
ans = 'n=5    a = 12.64529    a^2 = 159.90329'
ans = 'n=6    a = 15.77128    a^2 = 248.73343'
```



$\therefore \underline{\underline{\lambda_1 \approx 0.7402}}, \quad \underline{\underline{\lambda_2 \approx 11.7349}}$

Note as n increases, $\frac{1}{x}$ intersects $\tan(x)$ at

$x = \sqrt{\lambda_n} = (n-1)\pi \quad \therefore \underline{\lambda_n \approx (n-1)^2 \pi^2}$

9.

(a)

As in Example 1 on p. 531, $\lambda > 0$,

$y(x) = c_1 \sin(\sqrt{\lambda}\,x) + c_2 \cos(\sqrt{\lambda}\,x)$

$y'(x) = c_1 \sqrt{\lambda} \cos(\sqrt{\lambda}\,x) - c_2 \sqrt{\lambda} \sin(\sqrt{\lambda}\,x)$

$y(0) - y'(0) = c_2 - c_1\sqrt{\lambda} = 0$, $\therefore c_2 = c_1 \sqrt{\lambda}$ [1]

$\therefore y(x) = c_1 [\sin(\sqrt{\lambda}\,x) + \sqrt{\lambda}\cos(\sqrt{\lambda}\,x)]$

$y'(x) = c_1 [\sqrt{\lambda}\cos(\sqrt{\lambda}\,x) - \lambda \sin(\sqrt{\lambda}\,x)]$

$y(1) = c_1 [\sin(\sqrt{\lambda}) + \sqrt{\lambda}\cos(\sqrt{\lambda})]$

$y'(1) = c_1 [\sqrt{\lambda}\cos(\sqrt{\lambda}) - \lambda \sin(\sqrt{\lambda})]$

$y(1) + y'(1) = c_1 [(1-\lambda)\sin(\sqrt{\lambda}) + 2\sqrt{\lambda}\cos(\sqrt{\lambda})] = 0$ [2]

For non-trivial solution, [1] $\Rightarrow c_1 \neq 0$.

If $\cos(\sqrt{\lambda}) = 0$, then $\sin(\sqrt{\lambda}) \neq 0$ and $\lambda \neq 1$.

$\therefore (1-\lambda)\sin(\sqrt{\lambda}) \neq 0$, so $\underline{\cos(\sqrt{\lambda}) \neq 0}$.

Also, if $\lambda = 1$, [2] is not satified. $\therefore \underline{\lambda \neq 1}$

$\therefore$ [2] becomes $\tan(\sqrt{\lambda}) = \dfrac{2\sqrt{\lambda}}{\lambda - 1}$

$$\therefore \boxed{\begin{array}{l} y_n(x) = K\left[\sin(\sqrt{\lambda_n}\, x) + \sqrt{\lambda}\cos(\sqrt{\lambda_n}\, x)\right., \text{ where} \\ \qquad \tan(\sqrt{\lambda_n}) = \dfrac{2\sqrt{\lambda_n}}{\lambda_n - 1}, \; K \text{ a constant} \end{array}}$$

(b)

If $\lambda = 0$, then $y'' = 0 \Rightarrow y(x) = c_1 x + c_2$

$y(0) - y'(0) = (c_2) - (c_1) = 0 \Rightarrow c_1 = c_2$

$y(1) + y'(1) = (c_1 + c_1) + (c_1) = 0 \Rightarrow c_1 = 0$, so $c_2 = 0$.

$\therefore \lambda = 0 \Rightarrow$ trivial solution.

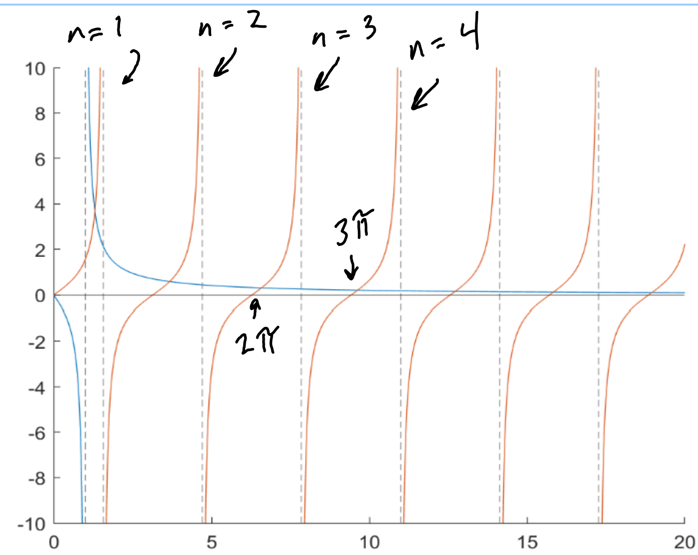
$\therefore \lambda = 0$ is <u>not</u> an eigenvalue.

(c), (d)

Use MATLAB to plot $y = \tan(x)$ and $y = \frac{2x}{x^2 - 1}$ for $x > 0$ and $x \neq 1$. The intersection points represent $\sqrt{\lambda_n}$. Code, plot on next page.

```
clear
syms x
f2 = 2*x/(x^2 - 1);
figure
axis([0,20,-10,10])
hold on
yline(0)
fplot(f2)
fplot(tan(x),[0,20])
for n = 1:8
    % start estimation just to left of (2n-1)*pi/2
    a = vpasolve(tan(x)==f2,x,(2*n-1)*pi/2 - 0.1);
    a2 = a^2;
    sprintf('n=%d   a = %.5f   a^2 = %.5f',n,a,a2)
end
```

```
ans = 'n=1   a = 1.30654    a^2 = 1.70705'
ans = 'n=2   a = 3.67319    a^2 = 13.49236'
ans = 'n=3   a = 6.58462    a^2 = 43.35722'
ans = 'n=4   a = 9.63168    a^2 = 92.76935'
ans = 'n=5   a = 12.72324    a^2 = 161.88086'
ans = 'n=6   a = 15.83411    a^2 = 250.71889'
ans = 'n=7   a = 18.95497    a^2 = 359.29094'
ans = 'n=8   a = 22.08166    a^2 = 487.59969'
```



As $n \to \infty$, $x \approx (n-1)\pi$, the intersection of the plots. $\therefore \sqrt{\lambda_n} \approx (n-1)\pi$

$$\therefore \underline{\lambda_1 \approx 1.7071}, \quad \underline{\lambda_2 = 13.4924}, \quad \underline{\lambda_n \approx (n-1)^2\pi^2}$$

10.

(a)

Assume $-\lambda > 0$, analogous to Example 1, p. 531.

$$\therefore y(x) = c_1 \sin(\sqrt{-\lambda}\, x) + c_2 \cos(\sqrt{-\lambda}\, x)$$

$$y'(x) = c_1 \sqrt{-\lambda} \cos(\sqrt{-\lambda}\, x) - c_2 \sqrt{-\lambda} \sin(\sqrt{-\lambda}\, x)$$

$y(0) + y'(0) = 0 \Rightarrow (c_2) + (c_1\sqrt{-\lambda}) = 0$, or $c_2 = -c_1\sqrt{-\lambda}$ [1]

$\therefore\ y(x) = c_1\left[\sin(\sqrt{-\lambda}\,x) - \sqrt{-\lambda}\cos(\sqrt{-\lambda}\,x)\right]$

$y(1) = 0 \Rightarrow c_1\left[\sin(\sqrt{-\lambda}) - \sqrt{-\lambda}\cos(\sqrt{-\lambda})\right] = 0$ [2]

For a nontrivial solution, using [1], $c_1 \neq 0$.

Note when $\cos(\sqrt{-\lambda}) = 0$, $\sin(\sqrt{-\lambda}) \neq 0$. This contradicts [2]. $\therefore \cos(\sqrt{-\lambda}) \neq 0$.

$\therefore$ Divide [2] by $c_1\cos(\sqrt{-\lambda})$ to get:

$\tan(\sqrt{-\lambda}) = \sqrt{-\lambda}$, where $-\lambda > 0$.

Since there is a solution to $\tan(x) = x$ in every interval $\left[-\frac{(2n-1)\pi}{2}, \frac{(2n-1)\pi}{2}\right]$, $n = 1, 2, 3, \ldots$

$$\boxed{\begin{array}{l} y_n(x) = \sin(\sqrt{-\lambda_n}) - \sqrt{-\lambda_n}\cos(\sqrt{-\lambda_n}\,x), \text{ where} \\ \qquad \tan(\sqrt{-\lambda_n}) = \sqrt{-\lambda_n}, \quad n = 1, 2, 3, \ldots \end{array}}$$

(b)

$\lambda = 0 \Rightarrow y'' = 0 \Rightarrow y(x) = c_1 x + c_2$

$\therefore\ y(0) + y'(0) = 0 \Rightarrow (c_2) + (c_1) = 0$, $c_2 = -c_1$

$\therefore\ y(x) = c_1(x-1) \Rightarrow y(1) = 0 \Rightarrow c_1 - c_1 = 0.$

$\therefore\ \lambda = 0$ is an eigenvalue with eigenfunction

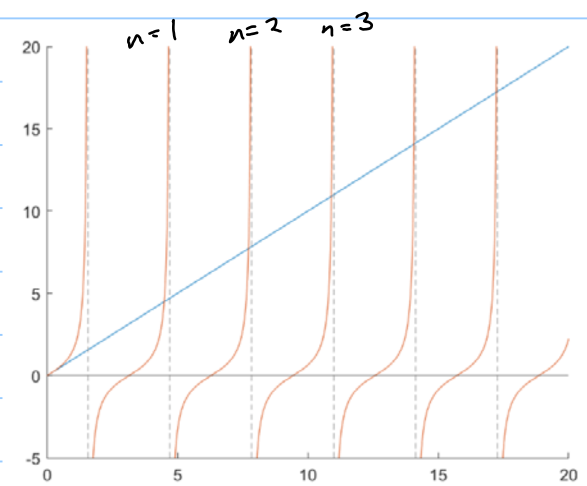
$$\underline{y(x) = x-1}.$$

(c), (d)

Use MATLAB to plot $y=\tan(x)$ and $y=x$. The intersection points represent $\sqrt{-\lambda_n}$.

Note $\tan'(x) = \sec^2(x)$, so $\tan'(0) = \sec^2(0) = 1$. So $\tan(x)$ only intersects $y=x$ at $x=0$ in $[-\frac{\pi}{2}, \frac{\pi}{2}]$. $\therefore$ Look for first intersection in $[\frac{\pi}{2}, \frac{3}{2}\pi]$ since excluding $-\lambda=0$.

```
clear
syms x
figure
axis([0,20,-5,20])
hold on
yline(0)
fplot(x)
fplot(tan(x),[0,20])
for n = 1:8
    % start estimation just to left of (2n+1)*pi/2
    a = vpasolve(tan(x)==x,x,(2*n+1)*pi/2 - 0.1);
    a2 = a^2;
    sprintf('n=%d    a = %.5f    a^2 = %.5f',n,a,a2)
end
```




```
ans = 'n=1    a = 4.49341    a^2 = 20.19073'
ans = 'n=2    a = 7.72525    a^2 = 59.67952'
ans = 'n=3    a = 10.90412    a^2 = 118.89987'
ans = 'n=4    a = 14.06619    a^2 = 197.85781'
ans = 'n=5    a = 17.22076    a^2 = 296.55441'
ans = 'n=6    a = 20.37130    a^2 = 414.98998'
ans = 'n=7    a = 23.51945    a^2 = 553.16465'
ans = 'n=8    a = 26.66605    a^2 = 711.07845'
```

As $n\to\infty$, $x_n \approx \frac{(2n+1)\pi}{2}$

$\therefore\ \sqrt{-\lambda_n} \approx \frac{(2n+1)\pi}{2}$

$\therefore \sqrt{-\lambda_1} \approx 4.4931, \sqrt{-\lambda_2} = 7.72525$

$\lambda_1 \approx \underline{-20.1907}, \lambda_2 \approx \underline{-59.6795}$

As $n \to \infty$, $\lambda_n \approx \underline{-\frac{(2n+1)^2 \pi^2}{4}}$

11.

(a)

$$(\mu(x)P(x)y')' = \mu' P y' + \mu P' y' + \mu P y''$$
$$= (\mu' P + \mu P')y' + \mu P y'' \quad [1]$$

Multiplying (30) by $\mu(x)$,

$$\mu P y'' + \mu Q y' + \mu R y = 0. \quad [2]$$

Equating coefficients of y' from [1], [2], we need

$\mu' P + \mu P' = \mu Q$, or $\underline{P\mu' = (Q - P')\mu}$

(6)

Assume $P(x) \neq 0$ for some interval I.

$\therefore \; P\mu' = (Q - P')\mu \Rightarrow P\mu' + P'\mu = Q\mu$

Or, $(P\mu)' = Q\mu = \dfrac{QP\mu}{P}$ for $x \in I$.

$\therefore$ For $x \in I$, $\dfrac{(P(x)\mu(x))'}{P(x)\mu(x)} = \dfrac{Q(x)}{P(x)}$

For $x_0, x \in I$, integrate to get

$$\int_{x_0}^{x} \frac{(P(s)\mu(s))'}{P(s)\mu(s)}\,ds = \int_{x_0}^{x} \frac{Q(s)}{P(s)}\,ds$$

$$\therefore \; \ln[P(x)\mu(x)] - \ln[P(x_0)\mu(x_0)] = \int_{x_0}^{x} \frac{Q(s)}{P(s)}\,ds$$

Or, $\ln\left[\dfrac{P(x)\mu(x)}{P(x_0)\mu(x_0)}\right] = \displaystyle\int_{x_0}^{x} \frac{Q(s)}{P(s)}\,ds$

$$\therefore \; \frac{P(x)\mu(x)}{P(x_0)\mu(x_0)} = \exp\left(\int_{x_0}^{x} \frac{Q(s)}{P(s)}\,ds\right)$$

Let $K = P(x_0)\mu(x_0)$, a constant

$\therefore \mu(x) = K \dfrac{1}{P(x)} \exp\left(\int_{x_0}^{x} \dfrac{Q(s)}{P(s)}\, ds\right)$, for $x, x_0 \in I$.

$\therefore \mu(x)$ is determined up to a constant, and we can choose the integrating factor to be just:

$$\boxed{\mu(x) = \frac{1}{P(x)} \exp\left(\int_{x_0}^{x} \frac{Q(s)}{P(s)}\, ds\right)}$$

where x_0, x are in an interval in which $P(x) \neq 0$.

12.

$P(x) = 1$, $Q(x) = -2x$ $\therefore$ integrating factor $\mu(x)$ is

$$\mu(x) = \exp\left(\int_{x_0}^{x} -2s\, ds\right) = \exp\left(-s^2\Big|_{x_0}^{x}\right) = \exp(-x^2 + x_0^2)$$

$$= K e^{-x^2}, \ K \text{ a constant}$$

$\therefore$ Choose $\underline{\mu(x) = e^{-x^2}}$.

$$\therefore e^{-x^2} y'' - 2x e^{-x^2} y' + \lambda e^{-x^2} y = 0$$

$$\therefore \underline{\left(e^{-x^2} y'\right)' + \lambda e^{-x^2} y = 0}$$

13.

$P(x) = x^2$, $Q(x) = x$ $\therefore \mu(x) = \frac{1}{x^2} \exp\left(\int_{x_0}^{x} \frac{s}{s^2}\, ds\right)$

$\therefore \mu(x) = \frac{1}{x^2} \exp\left(\ln\left(\frac{x}{x_0}\right)\right) = \frac{1}{x_0} \cdot \frac{1}{x}$, x_0 a constant.

$\therefore$ choose $\underline{\mu(x) = \frac{1}{x}}$

$$\therefore \frac{1}{x} x^2 y'' + \frac{1}{x} x y' + \frac{(x^2 - \nu^2)}{x} y = 0, \text{ or}$$

$$\underline{(xy')' + \frac{(x^2 - \nu^2)}{x} y = 0}$$

14.

$P(x) = x$, $Q(x) = (1-x)$

$$\therefore \mu(x) = \frac{1}{x} \exp\left(\int_{x_0}^{x} \frac{1-s}{s}\, ds\right) = \frac{1}{x} \exp\left(\int_{x_0}^{x} \frac{1}{s} - 1\, ds\right)$$

$$= \frac{1}{x} \exp\left(\ln(x) - x \underbrace{- \ln(x_0) + x_0}_{\text{a constant}}\right)$$

$$= \frac{1}{x}\left(c x e^{-x}\right), \text{ c a constant}$$

$$= c e^{-x}$$ $\therefore$ Choose $\underline{\mu(x) = e^{-x}}$

$$\therefore x e^{-x} y'' + (1-x) e^{-x} y' + \lambda e^{-x} y = 0, \text{ or}$$

$$\therefore \underline{(xe^{-x}y')' + \lambda e^{-x}y = 0}$$

15.

$P(x) = 1-x^2$, $Q(x) = -x$

$$\therefore \mu(x) = \frac{1}{1-x^2}\exp\left(\int_{x_0}^{x}\frac{-s}{1-s^2}\,ds\right) = \frac{1}{1-x^2}\exp\left(\frac{1}{2}\int_{x_0}^{x}\frac{-2s}{1-s^2}\,ds\right)$$

$$= \frac{1}{1-x^2}\exp\left(\frac{1}{2}\ln(1-x^2) - \underbrace{\frac{1}{2}\ln(1-x_0^2)}_{\text{a constant}}\right)$$

$$= c\,\frac{(1-x^2)^{\frac{1}{2}}}{1-x^2} = c(1-x^2)^{-\frac{1}{2}}, \text{ c a constant.}$$

$$\therefore \underline{\mu(x) = (1-x^2)^{-\frac{1}{2}}}$$

$$\therefore (1-x^2)^{-\frac{1}{2}}(1-x^2)y'' - x(1-x^2)^{-\frac{1}{2}}y' + \alpha(1-x^2)^{-\frac{1}{2}}y = 0$$

Or, $\underline{((1-x^2)^{\frac{1}{2}}y')' + \alpha(1-x^2)^{-\frac{1}{2}}y = 0}$

16.

Let $u(x,t) = X(x)T(t)$. $\therefore u_{xx} = X''T$, $u_{tt} = XT''$, $u_t = XT'$

$\therefore XT'' + cXT' + KXT = a^2X''T$. Dividing by a^2XT

$$\frac{1}{a^2}\frac{T''}{T} + \frac{c}{a^2}\frac{T'}{T} + \frac{k}{a^2} = \frac{X''}{X}$$

Dividing by a^2 keeps at least one side as simple as possible.

Since the left is an equation of t only, the right of x only, they must equal the same constant, call it $-\lambda$.

$$\therefore \frac{1}{a^2}\frac{T''}{T} + \frac{c}{a^2}\frac{T'}{T} + \frac{k}{a^2} = \frac{X''}{X} = -\lambda$$

$$\therefore \boxed{\begin{array}{l} X''(x) + \lambda X(x) = 0 \\ T''(t) + cT'(t) + (k + a^2\lambda)T(t) = 0 \end{array}}$$

17.

(a)

With $y(x) = s(x)u(x)$, $u(x) = \dfrac{y(x)}{s(x)}$, so $s(x) \neq 0$ at least in some interval encompassing $x=0$, $x=1$.

Since $y(0)=0$, $y(1)=0$, $\underline{u(0)=0}, u(1)=0$

$y' = su' + s'u$

$y'' = su'' + s'u' + s''u + s'u' = su'' + s''u + 2s'u'$

$\therefore (su'' + s''u + 2s'u') - 2(su' + s'u) + (1+\lambda)su = 0$

Dividing by su,

$$\frac{u''}{u} + \frac{s''}{s} + \frac{2s'u'}{su} - \frac{2u'}{u} - \frac{2s'}{s} + (1+\lambda) = 0 \qquad [1]$$

To get rid of $\frac{2s'u'}{su} - \frac{2u'}{u}$, we need $\frac{s'}{s} = 1$

Integrating, $\int \frac{s'}{s} = \int 1 \Rightarrow \ln(s) = x + C$, $s = Ke^x$

$\therefore$ choose $\underline{s(x) = e^x}$

(b)

In (a), with $s(x) = e^x$, [1] becomes

$$-\frac{u''}{u} = \frac{s''}{s} - \frac{2s'}{s} + (1+\lambda) = 1 - 2 + (1+\lambda) = \lambda$$

$\therefore u''(x) + \lambda u(x) = 0$, $u(0)=0$, $u(1)=0$

(1) Suppose $\lambda = 0$. $\therefore u(x) = c_1 x + c_2$

$u(0) = 0 \Rightarrow c_2 = 0$, $u(1) = 0 \Rightarrow c_1 = 0$.

$\therefore$ only trivial solution. $\therefore \underline{\lambda \neq 0}$

(2) Suppose $\lambda < 0$. Let $\lambda = -\omega^2$, $\omega > 0$.

$\therefore u'' - \omega^2 u = 0 \Rightarrow u = c_1 e^{\omega x} + c_2 e^{-\omega x}$

$u(0) = 0 \Rightarrow c_1 + c_2 = 0$, $u(1) = 0 \Rightarrow c_1 e^{\omega} + c_2 e^{-\omega} = 0$

$\therefore c_1 e^{2\omega} + \underbrace{c_2}_{=-c_1} = 0$, $c_1 e^{2\omega} - c_1 = c_1(e^{2\omega} - 1) = 0 \Rightarrow c_1 = 0$

$\therefore c_2 = 0$. $\therefore$ Only trivial solution.

$\therefore \lambda > 0$. $\therefore u(x) = c_1 \cos(\sqrt{\lambda}\, x) + c_2 \sin(\sqrt{\lambda}\, x)$

$u(0) = 0 \Rightarrow c_1 = 0$. $u(1) = 0 \Rightarrow c_2 \sin(\sqrt{\lambda}) = 0$.

For $c_2 \neq 0$ for nontrivial solution, $\sqrt{\lambda} = n\pi$, $n = 1, 2, 3, \ldots$

$\therefore u_n(x) = \sin(n\pi x)$

$\therefore y_n(x) = s(x) u_n(x) = e^{x} \sin(n\pi x)$, $n = 1, 2, 3, \ldots$

$\therefore$ $\boxed{\begin{array}{l} y_n(x) = e^{x} \sin(n\pi x), \quad n = 1, 2, 3, \ldots \\ \text{eigenvalues}: \lambda_n = n^2\pi^2, \quad n = 1, 2, 3, \ldots \end{array}}$

(c)

From the characteristic equation: $r^2 - 2r + (1+\lambda) = 0$,

$r = \dfrac{2 \pm \sqrt{4 - 4(1+\lambda)}}{2} = 1 \pm \sqrt{-\lambda}$

(1) $\lambda = 0$: $r = 1$, repeated root. $\therefore y = c_1 e^x + c_2 x e^x$

$y(0) = 0 \Rightarrow c_1 = 0$. $\therefore y(x) = c_2 x e^x$. $y(1) = 0 \Rightarrow c_2 = 0$.

$\therefore$ Only the trivial solution. $\therefore \lambda \neq 0$.

(2) $\lambda > 0$. $\therefore r$ is complex, $r = 1 \pm i\sqrt{\lambda} \Rightarrow$

$y = c_1 e^x \cos(\sqrt{\lambda}\, x) + c_2 e^x \sin(\sqrt{\lambda}\, x)$

$y(0) = 0 \Rightarrow c_1 = 0$. $\therefore y(x) = c_2 e^x \sin(\sqrt{\lambda}\, x)$

$y(1) = 0 \Rightarrow c_2 e \sin(\sqrt{\lambda}) = 0$. For $c_2 \neq 0$

(i.e., nontrivial solution), $\sin(\sqrt{\lambda}) = 0 \Rightarrow$

$\sqrt{\lambda} = n\pi$, $n = 1, 2, 3, \ldots$ $n \neq 0$ from (1) above.

$\therefore y_n(x) = \underline{e^x \sin(n\pi x)}$, $n = 1, 2, 3, \ldots$ and $\lambda_n = \underline{n^2 \pi^2}$

(3) $\lambda < 0$. Let $\lambda = -\omega^2$, $\omega > 0$.

$\therefore y(x) = c_1 e^{(1+\omega)x} + c_2 e^{(1-\omega)x}$

$y(0) = 0 \Rightarrow c_1 + c_2 = 0$

$y(1) = 0 \Rightarrow c_1 e^{1+w} + c_2 e^{1-w} = 0$, or $c_1 e^{2w} + c_2 = 0$

Subtracting, $c_1 - c_1 e^{2w} = 0$, or $c_1(1 - e^{2w}) = 0$.

Since $1 - e^{2w} \neq 0$ since $w > 0$, $c_1 = 0$.

$\therefore c_2 = 0$. $\therefore$ Only trivial solution.

$\therefore$ (1), (2), (3) $\Rightarrow$ $y_n(x) = e^x \sin(n\pi x)$, $\lambda_n = n^2\pi^2$, $n = 1, 2, 3, \ldots$

are the only non-trivial solutions.

18.

(a)

Let $y(x) = u(x)s(x)$, $u(x) = \frac{y(x)}{s(x)}$, $\therefore s(x) \neq 0$ in some interval encompassing $[0, L]$. $\therefore u(0) = 0$

$y' = u'(x)s(x) + u(x)s'(x)$

$\therefore y'(L) = 0 \Rightarrow u'(L)s(L) + u(L)s'(L) = 0$

$y'' = su'' + s'u' + s''u + s'u' = su'' + s''u + 2s'u'$

$\therefore (su'' + s''u + 2s'u') + 4(su' + us') + (4+9\lambda)su = 0$

Dividing by su,

$$\frac{u''}{u} + \frac{s''}{s} + \frac{2s'u'}{su} + \frac{4u'}{u} + \frac{4s'}{s} + (4+9\lambda) = 0 \quad [1]$$

To get rid of $\frac{2s'u'}{su} + \frac{4u'}{u}$, require it to be zero,

so $\frac{2s'}{s} + 4 = 0$, or $\frac{s'}{s} = -2$, $\therefore \underline{s(x) = e^{-2x}}$

$\therefore$ [1] becomes: $\frac{u''}{u} + \frac{s''}{s} = 4 - 9\lambda$

Since $\frac{s''}{s} = \frac{4e^{-2x}}{e^{-2x}} = 4$, $\frac{u''}{u} = -9\lambda$, $u'' + 9\lambda u = 0$ [2]

(1) $\lambda = 0$: $u'' = 0 \Rightarrow u = c_1 x + c_2$. $u(0) = 0 \Rightarrow c_2 = 0$

$$y'(L) = u'(L)e^{-2L} + u(L)(-2e^{-2L}) = 0$$

$$\Rightarrow (c_1)e^{-2L} + c_1 L(-2e^{-2L}) = 0 \Rightarrow c_1 - 2c_1 L = 0$$

$\Rightarrow c_1(1-2L) = 0$. Assuming $L \neq \frac{1}{2}$, then $c_1 = 0$.

$\therefore$ If $\underline{L = \frac{1}{2}}$, $\underline{\lambda = 0}$ is an eigenvalue, $\underline{y = xe^{-2x}}$

(2) $\lambda < 0$: Let $\lambda = -\omega^2$, $\omega > 0$. $\therefore u'' - 9\omega^2 u = 0$,

$$u(x) = c_1 e^{3\omega x} + c_2 e^{-3\omega x}, \quad u'(x) = c_1 3\omega e^{3\omega x} - c_2 3\omega e^{-3\omega x}$$

$$u(0) = 0 \Rightarrow c_2 = -c_1 \quad \therefore u(x) = c_1\left(e^{3\omega x} - e^{-3\omega x}\right),$$

or, $u(x) = K \sinh(3\omega x)$, $K = 2c_1$.

$$\therefore y(x) = s(x)u(x) = \underline{e^{-2x}} \sinh(3\omega x)$$

$$y'(x) = -2e^{-2x}\sinh(3\omega x) + 3\omega e^{-2x}\cosh(3\omega x)$$

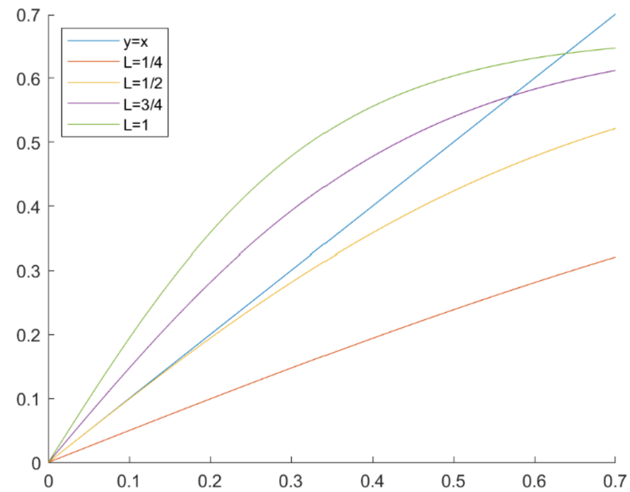
$$y'(L) = -2e^{-2L}\sinh(3\omega L) + 3\omega e^{-2L}\cosh(3\omega L) = 0$$

$$\therefore 2\sinh(3\omega L) = 3\omega\cosh(3\omega L)$$

Since $\cosh(3\omega L) \neq 0$, $\omega = \frac{2}{3}\tanh(3\omega L)$

Need to find out if the plot of $y = x$ (with a slope of 1) ever intersects the plot of $y = \frac{2}{3}\tanh(3Lx)$. Use MATLAB to plot.

```
clear
syms x L
x = 0:0.01:5;
figure
hold on
plot(x,x)
for L = [1/4,1/2,3/4,1]
    f = (2/3)*tanh(3*L*x);
    plot(x,f)
end
axis([0,0.7,0,0.7])
legend('y=x','L=1/4','L=1/2','L=3/4','L=1',...
       'Location','northwest')
```



This suggests for L big enough, $y = \frac{2}{3}\tanh(3Lw)$ will intersect $y = w$, giving a nonzero solution to $w = \frac{2}{3}\tanh(3Lw)$.

$\therefore$ Since $\frac{d}{dw}(w) = 1$, we need $\frac{d}{dw}\left(\frac{2}{3}\tanh(3Lw)\right) > 1$.

At $w = 0$, $\frac{2}{3}\tanh(3Lw) = 0$, but $-w^2 = \lambda < 0$, so $w = 0$ is not allowed.

$$\frac{d}{dw}\left(\frac{2}{3}\tanh(3Lw)\right) = \frac{2}{3}(3L)\operatorname{sech}^2(3Lw)$$
$$= 2L\operatorname{sech}^2(3Lw)$$

Since $\cosh(x) > 1$ for all $x \neq 0$, $\cosh^2(x) > 1$.

$\therefore$ $2L\operatorname{sech}^2(3Lw) > 1 \Rightarrow 2L > \frac{1}{\operatorname{sech}^2(3Lw)} = \cosh^2(3Lw) \geq 1$

$\therefore$ We want $2L > 1$, or $L > \frac{1}{2}$

If $2L \leq 1$, since $\operatorname{sech}^2(3Lw) \leq 1$, $2L\operatorname{sech}^2(3Lw) \leq 1$ so there will be no intersection with $y = w$ except at $w = 0$.

With $L > \frac{1}{2}$, there is a nonzero solution to

$$\omega = \frac{2}{3}\tanh(3L\omega).$$

$\therefore$ If $L \le \frac{1}{2}$, there are no negative eigenvalues.

If $L > \frac{1}{2}$, there is one nonzero eigenvalue,

$\lambda = -\omega^2$, where $\omega = \frac{2}{3}\tanh(3L\omega)$, and

$$y(x) = e^{-2x}\sinh(3\omega x)$$

(3) $\lambda > 0$. $\therefore u(x) = c_1\cos(3\sqrt{\lambda}x) + c_2\sin(3\sqrt{\lambda}x) = 0$

$u(0) = 0 \Rightarrow c_1 = 0$.

$\therefore y(x) = c_2 e^{-2x}\sin(3\sqrt{\lambda}x)$

$y'(L) = 0 \Rightarrow -2c_2 e^{-2L}\sin(3\sqrt{\lambda}L) + c_2(3\sqrt{\lambda})e^{-2L}\cos(3\sqrt{\lambda}L) = 0$

For a non-trivial solution, $c_2 \ne 0$,

$\therefore 2\sin(3\sqrt{\lambda}L) = 3\sqrt{\lambda}\cos(3\sqrt{\lambda}L)$ [3]

For $\lambda > 0$, $\sqrt{\lambda} > 0$. When $\sin(3\sqrt{\lambda}L) = 0$, $\cos(3\sqrt{\lambda}L) \ne 0$.

When $\cos(3\sqrt{\lambda}L) = 0$, $\sin(3\sqrt{\lambda}L) \ne 0$.

$\sin(3\sqrt{\lambda}L)$ and $\cos(3\sqrt{\lambda}L)$ must be the same sign.

$\therefore n\pi < 3\sqrt{\lambda}L < \frac{\pi}{2} + n\pi$, $n = 0, 1, 2, 3, \ldots$

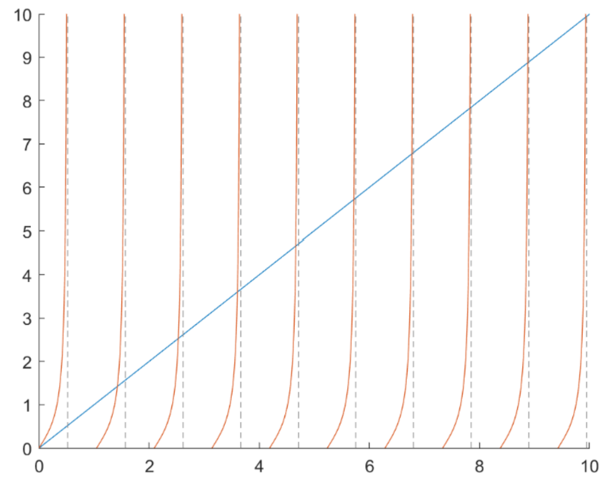
From [3], $\frac{2}{3}\tan(3\sqrt{\lambda_n}\,L) = \sqrt{\lambda_n}$, $\lambda_n > 0$

$y_n(x) = e^{-2x}\sin(3\sqrt{\lambda_n}\,x)$

Note, from MATLAB, the plot of $y = x$

intersects $y = \frac{2}{3}\tan(nx)$ at multiple points.

```
clear
syms x n
n = 3;  % arbitrary value
figure
hold on
fplot(x)
f = (2/3)*tan(n*x);
fplot(f)
axis([0,10,0,10])
```



Summary:

(1) $\lambda = 0$ is an eigenvalue only when $L = \frac{1}{2}$,

and $y(x) = x e^{-2x}$

(2) $\lambda < 0$: If $L \le \frac{1}{2}$, there is no eigenvalue.

If $L > \frac{1}{2}$, there is one nonzero eigenvalue,

$\lambda = -\omega^2$, where $\omega > 0$ and $\omega = \frac{2}{3}\tanh(3L\omega)$,

and $y(x) = e^{-2x}\sinh(3\omega x)$

(3) $\lambda > 0$: there are multiple eigenvalues, the solution to $\sqrt{\lambda_n} = \frac{2}{3}\tan(3\sqrt{\lambda_n}L)$, and

$y_n(x) = e^{-2x}\sin(3\sqrt{\lambda_n}\,x)$

(6)

The characteristic equation is $r^2 + 4r + (4+9\lambda) = 0$

$\therefore r = \dfrac{-4 \pm \sqrt{16 - 4(4+9\lambda)}}{2} = -2 \pm 3\sqrt{-\lambda}$

(1) $\lambda = 0$. Then $r = -2$, a repeated root, so

$y(x) = c_1 e^{-2x} + c_2 x e^{-2x}$ $\quad y(0) = 0 \Rightarrow c_1 = 0$

$\therefore y'(x) = c_2 e^{-2x} - 2c_2 x e^{-2x}$

$y'(L) = 0 \Rightarrow c_2 e^{-2L}(1 - 2L) = 0.$

For nontrivial solution, $c_2 \neq 0$, so $L = \frac{1}{2}$

$\therefore$ $\underline{\lambda = 0}$ is an eigenvalue when $\underline{L = \frac{1}{2}}$,

$\therefore y(x) \underline{=} x e^{-2x}$

(2) $\lambda < 0$. $\therefore$ Let $\lambda = -\omega^2$, $\omega > 0$.

$\therefore r = -2 \pm 3\omega$, $y(x) = e^{-2x}(c_1 e^{3\omega x} + c_2 e^{-3\omega x})$

$y(0) = 0 \Rightarrow c_1 + c_2 = 0$, $c_2 = -c_1$.

$K = 2c_1$

$\therefore y(x) = c_1 e^{-2x}(e^{3\omega x} - e^{-3\omega x}) = Ke^{-2x}\sinh(3\omega x)$

$y'(x) = -2Ke^{-2x}\sinh(3\omega x) + 3\omega Ke^{-2x}\cosh(3\omega x)$

$y'(L) = 0 \Rightarrow Ke^{-2L}[-2\sinh(3\omega L) + 3\omega\cosh(3\omega L)] = 0$

For nontrivial solution, $K \neq 0$,

$\therefore \omega = \frac{2}{3}\tanh(3\omega L)$

Does $y = \omega$ intersect $y = \frac{2}{3}\tanh(3\omega L)$?

Slope of $y = \omega$ is 1. $\frac{d}{d\omega}\left[\frac{2}{3}\tanh(3\omega L)\right] = 2L\,\text{sech}^2(3\omega L)$

$\text{sech}^2(3L\omega) \leq 1$ for $\omega \geq 0$ and $\text{sech}^2(3L\omega) = 1$

only when $\omega = 0$.

$\therefore$ If $2L \leq 1$, or $L \leq \frac{1}{2}$, $2L\,\text{sech}^2(3L\omega) \leq 1$, so

$\frac{2}{3}\tanh(3\omega L)$ will only intersect $y = \omega$ at $\omega = 0$.

$\therefore$ If $\underline{L \leq \frac{1}{2}}$, no $\underline{\text{eigenvalues}}$ for $\lambda < 0$.

If $L > \frac{1}{2}$, $2L > 1$. For very small ω,

$\operatorname{sech}^2(3L\omega) \approx 1$. $\therefore$ $2L\operatorname{sech}^2(3L\omega) > 1$

As $\omega \to \infty$, $2L\operatorname{sech}^2(3L\omega) \to 0$.

$\therefore$ For $L > \frac{1}{2}$, $y = \frac{2}{3}\tanh(3L\omega)$ intersects $y = \omega$

at one point.

$\therefore$ For $\underline{L > \frac{1}{2}}$, one $\underline{\text{eigenvalue}}$ for $\lambda < 0$:

$\lambda = -\omega^2$, where $\omega = \underline{\frac{2}{3}}\tanh(3\omega L)$

and $y(x) = \underline{e^{-2x}}\sinh(3\omega x)$

(3) $\lambda > 0$. $\therefore$ $r = -2 \pm 3i\sqrt{\lambda}$

$\therefore$ $y(x) = c_1 e^{-2x}\cos(3\sqrt{\lambda}\,x) + c_2 e^{-2x}\sin(3\sqrt{\lambda}\,x)$

$y(0) = 0 \Rightarrow c_1 = 0$. $\therefore$ $y(x) = c_2 e^{-2x}\sin(3\sqrt{\lambda}\,x)$

$y'(x) = -2c_2 e^{-2x}\sin(3\sqrt{\lambda}\,x) + 3\sqrt{\lambda}\,c_2 e^{-2x}\cos(3\sqrt{\lambda}\,x)$

$y'(L) = 0 \Rightarrow c_2 e^{-2L}\left[-2\sin(3\sqrt{\lambda}L) + 3\sqrt{\lambda}\cos(3\sqrt{\lambda}L)\right]$

For nontrivial solution, $c_2 \neq 0$, so

$2\sin(3\sqrt{\lambda}\,L) = 3\sqrt{\lambda}\cos(3\sqrt{\lambda}\,L)$

$\therefore$ $\sin(3\sqrt{\lambda}\,L)$ and $\cos(3\sqrt{\lambda}\,L)$ must be the same sign since $\sqrt{\lambda} > 0$, and neither can be zero.

$\therefore$ $\underline{\frac{2}{3}\tan(3\sqrt{\lambda}\,L)} = \sqrt{\lambda}$, and as shown in (a), this has multiple eigenvalue solutions for $\sqrt{\lambda}$

$$y_n(x) = e^{-2x}\sin(3\sqrt{\lambda_n}\,x)$$

19.

Rewrite as: $y'' + (1+\lambda)y' + \lambda y = 0$

Characteristic equation: $r^2 + (1+\lambda)r + \lambda = 0$

$$\therefore r = \frac{-1-\lambda \pm \sqrt{\lambda^2+2\lambda+1-4\lambda}}{2} = \frac{-1-\lambda \pm \sqrt{(\lambda-1)^2}}{2} = \frac{-1-\lambda \pm |\lambda-1|}{2}$$

$|\lambda-1| = \lambda-1$: $\quad r = \frac{-1-\lambda+(\lambda-1)}{2}, \frac{-1-\lambda-(\lambda-1)}{2} = -1, -\lambda$

$|\lambda-1| = 1-\lambda$: $\quad r = \frac{-1-\lambda+(1-\lambda)}{2}, \frac{-1-\lambda-(1-\lambda)}{2} = -\lambda, -1$

$\therefore\ y(x) = c_1 e^{-x} + c_2 e^{-\lambda x}$ [1]

$y'(x) = -c_1 e^{-x} - c_2 \lambda e^{-\lambda x}$

$y'(0) = 0 \Rightarrow -c_1 - c_2 \lambda = 0, \quad c_1 = -c_2 \lambda$ [2]

$y(1) = 0 \Rightarrow -c_2 \lambda e^{-1} + c_2 e^{-\lambda} = 0, \quad c_2 (e^{1-\lambda} - \lambda) = 0$ [3]

[2] $\Rightarrow c_2 = 0 \Rightarrow c_1 = 0$, $\therefore$ trivial solution.

$\therefore$ [3] $\Rightarrow e^{1-\lambda} - \lambda = 0$, or $\lambda = e^{1-\lambda}$, or $\lambda e^{\lambda} = e$

$\therefore\ \lambda > 0$. The plots of $y = e^{\lambda}$ and $y = \frac{e}{\lambda}$ intersect at only one point: $\underline{\lambda = 1}$

$\therefore\ r = -1, -1$, repeated roots.

$\therefore$ Consider $y(x) = c_1 e^{-x} + c_2 x e^{-x}$

$y'(x) = -c_1 e^{-x} + c_2 e^{-x} - c_2 x e^{-x} = -c_1 e^{-x} + c_2 e^{-x}(1-x)$

$y'(0) = -c_1 + c_2 = 0, \quad c_1 = c_2 \quad \therefore\ y(x) = c_1 e^{-x}(1+x)$

$y(1) = 0 \Rightarrow c_1 e^{-1}(2) = 0 \Rightarrow c_1 = 0.$

$\therefore$ only trivial solution.

$\therefore$ $\underline{\text{No real}}$ eigenvalues.

20.

Rewrite as: $x^2y'' - \lambda x y' + \lambda y = 0$

Assuming $y = x^r$ for the Euler equation as in

Section 5.4, p. 211, $y' = r x^{r-1}$, $y'' = r(r-1)x^{r-2}$

$\therefore r(r-1)x^r - \lambda r x^r + \lambda x^r = 0$, $r^2 + (-\lambda - 1)r + \lambda = 0$

$\therefore \quad r = \dfrac{\lambda + 1 \pm \sqrt{(\lambda+1)^2 - 4\lambda}}{2} = \dfrac{\lambda + 1 \pm \sqrt{(\lambda - 1)^2}}{2} = \dfrac{\lambda + 1 \pm |\lambda - 1|}{2}$

$|\lambda - 1| = \lambda - 1$: $\quad r = \dfrac{\lambda + 1 \pm (\lambda - 1)}{2} = \lambda, 1$

$|\lambda - 1| = 1 - \lambda$: $\quad r = \dfrac{\lambda + 1 \pm (1 - \lambda)}{2} = 1, \lambda$

$\therefore \quad y = x, x^\lambda$

$\therefore \quad y(x) = c_1 x + c_2 x^\lambda$, $\quad y'(x) = c_1 + c_2 \lambda x^{\lambda - 1}$

$y(1) = 0 \Rightarrow c_1 + c_2 = 0$, $\quad c_2 = -c_1$, $\quad \therefore y(x) = c_1(x - x^\lambda)$

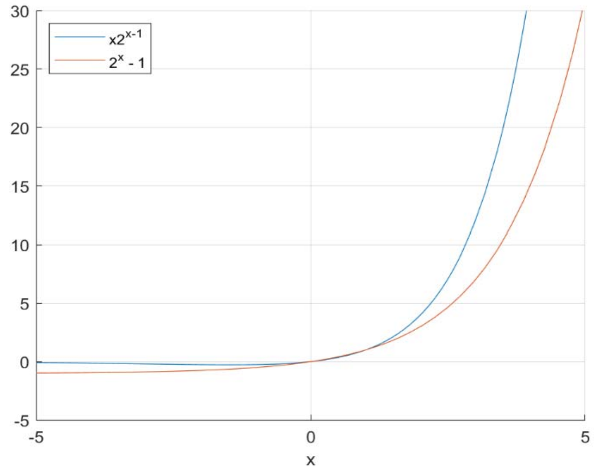
$\therefore c_1 \neq 0$, since otherwise the trivial solution.

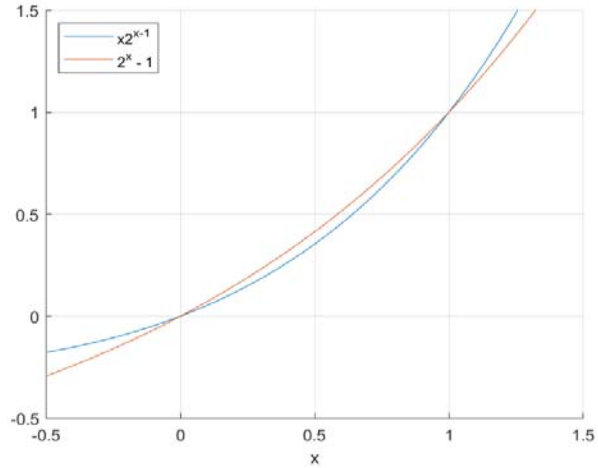
$y(2) - y'(2) = 0 \Rightarrow c_1(2 - 2^\lambda) - c_1[1 - \lambda 2^{\lambda - 1}] = 0$

$\therefore \quad 1 - 2^\lambda + \lambda 2^{\lambda - 1} = 0$, or $\lambda 2^{\lambda - 1} = 2^\lambda - 1$

Using MATLAB to plot $y = \lambda 2^{\lambda-1}$ and $y = 2^{\lambda} - 1$.

```
clear
syms x
f1 = x*2^(x-1);
f2 = 2^x - 1;
figure
hold on
grid on
fplot(f1)
fplot(f2)
xlabel('x')
axis([-5,5,-5,30])
legend('x2^x^-^1','2^x - 1',...
       'Location','northwest')
% zoom in on intersection
figure
hold on
grid on
fplot(f1)
fplot(f2)
xlabel('x')
axis([-0.5,1.5,-0.5,1.5])
legend('x2^x^-^1','2^x - 1',...
       'Location','northwest')
```





$\therefore \lambda = 0, 1$

(1) $\underline{\lambda = 0}$. Two separate roots. $\therefore y(x) = \underline{x - 1}$

(2) $\lambda = 1$. Repeated roots. $y = x$, $y = x\ln(x)$

$\therefore y(x) = c_1 x + c_2 x \ln(x)$ $\quad y(1) = 0 \Rightarrow c_1 = 0$

$\therefore y(x) = c_2 x \ln(x)$ $\quad y'(x) = c_2 \ln(x) + c_2$, $x > 0$

$y(1) = 0 \Rightarrow c_2 = 0$, $\therefore$ trivial solution.

$\therefore \underline{\lambda = 1}$ is <u>not an</u> eigenvalue.

Summary: $\underline{\lambda = 0}$ is the only eigenvalue,

eigenfunction: $y(x) = \underline{x-1}$.

21.

(a)

Characteristic equation: $r^2 + \lambda = 0$, $r = \pm i\sqrt{\lambda}$

Given that $\lambda > 0$, $\therefore$ $y(x) = c_1 \cos(\sqrt{\lambda}\, x) + c_2 \sin(\sqrt{\lambda}\, x)$

$2y(0) + y'(0) = 0 \Rightarrow 2(c_1) + (c_2\sqrt{\lambda}) = 0 \Rightarrow c_1 = -\frac{c_2\sqrt{\lambda}}{2}$

$\therefore$ $y(x) = c_2\left[-\frac{\sqrt{\lambda}}{2}\cos(\sqrt{\lambda}\, x) + \sin(\sqrt{\lambda}\, x)\right]$

For nontrivial solution, $c_2 \neq 0$, for otherwise $c_1 = c_2 = 0$.

$\therefore$ $y(1) = 0 \Rightarrow -\frac{\sqrt{\lambda}}{2}\cos(\sqrt{\lambda}) + \sin(\sqrt{\lambda}) = 0$, or

$\underline{2\sin(\sqrt{\lambda}) - \sqrt{\lambda}\cos(\sqrt{\lambda}) = 0}$

(b)

From (a), $2\sin(\sqrt{\lambda}) = \sqrt{\lambda}\cos(\sqrt{\lambda})$

Suppose $\sin(\sqrt{\lambda}) = 0$. Then $\cos(\sqrt{\lambda}) \neq 0 \Rightarrow \sqrt{\lambda} = 0$.

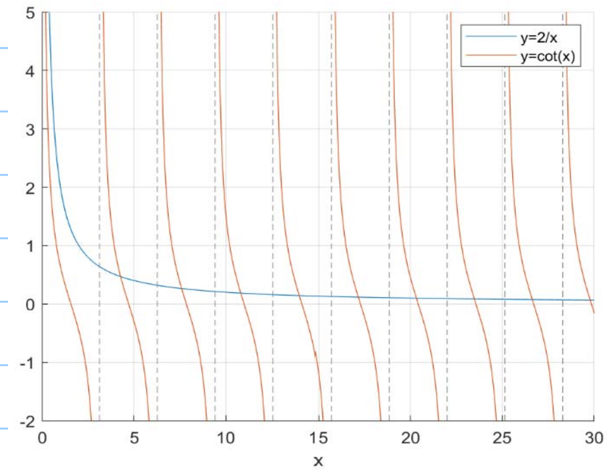
$\therefore \lambda = 0$ is the only solution, and not positive.

Now assume $\sin(\sqrt{\lambda}) \neq 0$. $\therefore \lambda \neq 0$.

$$\therefore \quad \frac{2}{\sqrt{\lambda}} = \cot(\sqrt{\lambda})$$

The graph of $y(\lambda) = \frac{2}{\sqrt{\lambda}}$ intersects $y(\lambda) = \cot(\sqrt{\lambda})$ at an infinite number of points, so that there is an infinite number of values for λ for which $\frac{2}{\sqrt{\lambda}} = \cot(\sqrt{\lambda})$. Using MATLAB,

```
clear
syms x
y1 = 2/x;
y2 = cot(x);
figure
hold on
grid on
fplot(y1)
fplot(y2)
axis([0,30,-2,5])
xlabel('x')
legend('y=2/x','y=cot(x)')
```



The plots of $y = \frac{2}{x}$ and $y = \cot(x)$ is a little more explicit than the plots of $y = \frac{2}{\sqrt{x}}$, $y = \cot(\sqrt{x})$.

(c)

Using MATLAB, adding the below code to the code above, and using the plots as a guide to guess an initial value for the solution.

```
x1 = vpasolve(2/x==cot(x),x,4);
x2 = vpasolve(2/x==cot(x),x,8);
sprintf('x1^2 = %.5f',x1^2)
sprintf('x2^2 = %.5f',x2^2)
```

```
ans = 'x1^2 = 18.27376'

ans = 'x2^2 = 57.70751'
```

In the above, $x = \sqrt{\lambda}$. $\therefore \lambda = x^2$.

$\therefore \lambda_1 = \underline{18.27376}$, $\lambda_2 = \underline{57.70751}$

As $\lambda_n \to \infty$, $\frac{2}{\sqrt{\lambda_n}} \to 0$. $\therefore \cot(\sqrt{\lambda_n}) = 0$ which means $\sqrt{\lambda_n} = (2n+1)\frac{\pi}{2}$. $2n+1$ is used since there is no intersection of $y = \frac{2}{x}$ with $y = \cot(x)$ for $(0, \pi)$, and $\therefore$ for $n = 0$. $\cot(x) = 0$ for odd multiples of $\frac{\pi}{2}$.

$\therefore \lambda_n \approx \frac{(2n+1)^2\pi^2}{4}$ as $n \to \infty$.

(d)

Let $\lambda = -\omega$, $\omega > 0$. Characteristic equation becomes $r^2 - \omega = 0$, $r = \pm\sqrt{\omega}$

$\therefore\ y(x) = c_1 \cosh(\sqrt{\omega}\, x) + c_2 \sinh(\sqrt{\omega}\, x)$

$2y(0) + y'(0) = 0 \Rightarrow 2c_1 + \sqrt{\omega}\, c_2 = 0, \quad c_1 = -\frac{\sqrt{\omega}}{2} c_2$

$\therefore\ y(x) = c_2 \left[-\frac{\sqrt{\omega}}{2} \cosh(\sqrt{\omega}\, x) + \sinh(\sqrt{\omega}\, x) \right], \quad c_2 \neq 0$

since that would imply $c_1 = 0$.

$y(1) = 0 \Rightarrow -\frac{\sqrt{\omega}}{2} \cosh(\sqrt{\omega}) + \sinh(\sqrt{\omega}) = 0$, or

$\underline{2\sinh(\sqrt{\omega}) - \sqrt{\omega} \cosh(\sqrt{\omega}) = 0}, \quad \omega = -\lambda > 0$

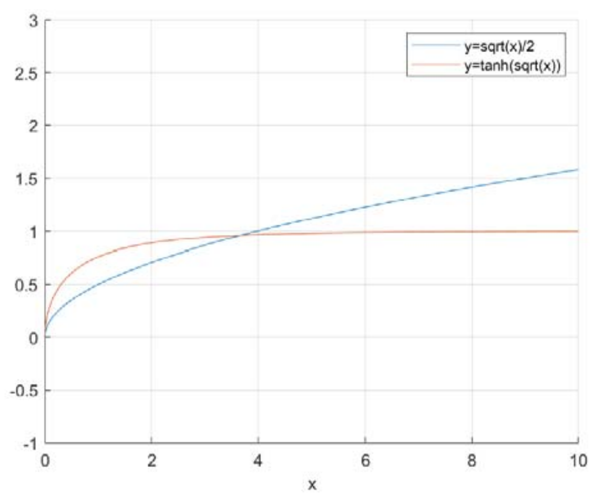
(e)

From (d), and since $\cosh(x) \geq 1$ for all x,

dividing by $\cosh(\sqrt{\omega})$, $\tanh(\sqrt{\omega}) = \frac{\sqrt{\omega}}{2}$

Using MATLAB to solve, and noting $\omega \neq 0$,

```
clear
syms x
y1 = sqrt(x)/2;
y2 = tanh(sqrt(x));
figure
hold on
grid on
fplot(y1)
fplot(y2)
axis([0,10,-1,3])
xlabel('x')
legend('y=sqrt(x)/2','y=tanh(sqrt(x))')
w = vpasolve(y1==y2,x,4);
sprintf('lambda = %.5f',-w)
```



$\therefore\ \underline{\lambda \simeq -3.66726}$

```
ans = 'lambda = -3.66726'
```

Note that as $x \to \infty$, $\sqrt{x} \to \infty$, whereas $\tanh(\sqrt{x}) \to 1$. $\therefore$ $\frac{\sqrt{x}}{2}$ and $\tanh(\sqrt{x})$ intersect at only one point for $x > 0$.

22.

(a)

Characteristic equation: $r^2 + \lambda = 0$, $r = \pm i\sqrt{\lambda}$

Given that $\lambda > 0$, $\therefore$ $y(x) = c_1 \cos(\sqrt{\lambda}\, x) + c_2 \sin(\sqrt{\lambda}\, x)$

$\alpha y(0) + y'(0) = 0 \Rightarrow \alpha(c_1) + (c_2\sqrt{\lambda}) = 0$ [1]

$\therefore$ $c_2 = -\dfrac{\alpha c_1}{\sqrt{\lambda}}$ $\quad\therefore$ $y(x) = c_1\left[\cos(\sqrt{\lambda}\, x) - \frac{\alpha}{\sqrt{\lambda}} \sin(\sqrt{\lambda}\, x)\right]$

$c_1 \neq 0$ for otherwise $c_2 = 0$ by [1].

$\therefore$ $y(1) = 0 \Rightarrow \sqrt{\lambda} \cos(\sqrt{\lambda}) = \alpha \sin(\sqrt{\lambda})$ [2]

If $\sin(\sqrt{\lambda}) = 0$, then $\cos(\sqrt{\lambda}) \neq 0$, so that

$\sqrt{\lambda} \cos(\sqrt{\lambda}) = 0 \Rightarrow \sqrt{\lambda} = 0$, and so λ is not positive.

$\therefore$ λ is such that $\sin(\sqrt{\lambda}) \neq 0$. Dividing [2] by

$\sin(\sqrt{\lambda})$ yields $\cot(\sqrt{\lambda}) = \frac{\alpha}{\sqrt{\lambda}}$. If $\alpha = 0$, there is an infinite number of values for λ in which $\cot(\sqrt{\lambda}) = 0$, namely $\lambda_n = \frac{(2n+1)^2 \pi^2}{4}$. For $\alpha \neq 0$, #21 (b) above shows there is an infinite number of intersection points between $y = \cot(x)$ and $y = \frac{1}{x}$.

There is an infinite sequence of values for $\lambda_n > 0$ for which $\frac{\alpha}{\sqrt{\lambda_n}} = \cot(\sqrt{\lambda_n})$, with corresponding eigenfunction $y(x) = \cos(\sqrt{\lambda_n}\, x) - \frac{\alpha}{\sqrt{\lambda_n}} \sin(\sqrt{\lambda_n}\, x)$.

(b)

Consider the cases for $\lambda = 0$, $\lambda < 0$.

(1) $\lambda = 0$: $\therefore y'' = 0 \Rightarrow y(x) = c_1 x + c_2$.

$y(1) = 0 \Rightarrow c_1 + c_2 = 0$. $\therefore y(x) = c_1 x - c_1$

$\alpha y(0) + y'(0) = 0 \Rightarrow \alpha(-c_1) + c_1 = 0 \Rightarrow c_1(1-\alpha) = 0$.

A nontrivial solution means $c_1 \neq 0$, $\therefore \alpha = 1$.

$\therefore$ If $\alpha < 1$, $\lambda = 0$ is not an eigenvalue.

(2) $\lambda < 0$: Let $\lambda = -\omega$, $\omega > 0$. $\therefore$ $y'' - \omega\lambda = 0$ $\Rightarrow$

$$y(x) = c_1 \cosh(\sqrt{\omega}\, x) + c_2 \sinh(\sqrt{\omega}\, x).$$

$$y'(x) = c_1 \sqrt{\omega} \sinh(\sqrt{\omega}\, x) + c_2 \sqrt{\omega} \cosh(\sqrt{\omega}\, x)$$

$\therefore$ $\alpha y(0) + y'(0) = 0 \Rightarrow \alpha c_1 + c_2 \sqrt{\omega} = 0$

$\therefore$ $c_2 = \dfrac{-\alpha c_1}{\sqrt{\omega}}$, and $\therefore$ $c_1 \neq 0$ for a nontrivial solution.

$\therefore$ $y(x) = c_1 \left[\cosh(\sqrt{\omega}\, x) - \dfrac{\alpha}{\sqrt{\omega}} \sinh(\sqrt{\omega}\, x) \right]$

$$y(1) = 0 \Rightarrow \cosh(\sqrt{\omega}) = \frac{\alpha}{\sqrt{\omega}} \sinh(\sqrt{\omega})$$

Dividing by $\cosh(\sqrt{\omega})$ as $\cosh(x) \neq 0$, for all x, $\sqrt{\omega} = \alpha \tanh(\sqrt{\omega})$. To solve for $\omega > 0$, look at the derivative (the slope) of each side.

$$\frac{d}{d\omega}(\sqrt{\omega}) = \frac{1}{2\sqrt{\omega}}, \quad \frac{d}{d\omega}\left(\alpha \tanh(\sqrt{\omega})\right) = \frac{\alpha}{2\sqrt{\omega}} \sec^2(\sqrt{\omega})$$

$\therefore$ $\dfrac{1}{2\sqrt{\omega}} = \dfrac{\alpha}{2\sqrt{\omega}} \sec^2(\sqrt{\omega})$, or $\alpha = \dfrac{1}{\sec^2(\sqrt{\omega})} = \cosh^2(\sqrt{\omega})$

Since $\cosh(x) > 1$ for all $x \neq 0$, $\cosh^2(\sqrt{\omega}) > 1$.

$\therefore \alpha > 1$, which conficts with $\alpha < 1$.

$\therefore$ $\underline{\lambda < 0}$ is $\underline{\text{not an eigen}}$ value $\underline{\text{for } \alpha < 1}$.

$\therefore$ If $\alpha < 1$, $\lambda = 0$ and $\lambda < 0$ are not eigenvalues.

$\therefore$ If $\underline{\alpha < 1}$, the $\underline{\text{only real eigenvalues}}$ are $\underline{\lambda > 0}$.

(3) From (9), with $\lambda > 0$, $\alpha = \sqrt{\lambda_n} \cot(\sqrt{\lambda_n})$,

as $\alpha \to 1$ from the left (so λ is positive),

$\sqrt{\lambda_n} \cot(\sqrt{\lambda_n}) \to 1$, so $\tan(\sqrt{\lambda_n}) \to \sqrt{\lambda_n}$. Since

$\tan(x) = x$ only for $x = 0$, then $\lambda_n \to 0$.

$\therefore$ As $\underline{\alpha \to 1^-, \lambda_n} \to 0^+$.

(c)

As in (6) above, $\lambda = 0 \Rightarrow y(x) = c_1 x + c_2$, $y(1) = 0 \Rightarrow$

$c_2 = -c_1$ so $y(x) = c_1 x - c_1$. $\alpha y(0) + y'(0) = 0 \Rightarrow$

$\alpha(-c_1) + c_1 = 0$, or $c_1(1-\alpha) = 0$. $c_1 \neq 0$ for a

nontrivial solution, so $\underline{\alpha = 1}$

(d)

Let $\lambda = -\omega$, $\omega > 0$, so λ is negative.

From the characteristic equation, $r^2 - \omega = 0$, $r = \pm\sqrt{\omega}$,

$y(x) = c_1 \cosh(\sqrt{\omega}\, x) + c_2 \sinh(\sqrt{\omega}\, x)$.

$\therefore y'(x) = c_1\sqrt{\omega} \sinh(\sqrt{\omega}\, x) + c_2 \sqrt{\omega} \cosh(\sqrt{\omega}\, x)$

$\alpha y(0) + y'(0) = 0 \Rightarrow \alpha c_1 + c_2\sqrt{\omega} = 0$, $\therefore c_2 = -\frac{\alpha c_1}{\sqrt{\omega}}$,

and $c_1 \neq 0$ for a nontrivial solution.

$\therefore y(x) = c_1\left[\cosh(\sqrt{\omega}\, x) - \frac{\alpha}{\sqrt{\omega}} \sinh(\sqrt{\omega}\, x)\right]$

$\therefore y(1) = 0 \Rightarrow \sqrt{\omega}\cosh(\sqrt{\omega}) = \alpha \sinh(\sqrt{\omega})$

Dividing by $\alpha\cosh(\sqrt{\omega})$ since $\alpha > 1$ and $\cosh(x) > 0$

for all x, $\frac{\sqrt{\omega}}{\alpha} = \tanh(\sqrt{\omega})$.

Taking derivatives of both sides with respect to ω,

$\frac{1}{\alpha 2\sqrt{\omega}} = \frac{1}{2\sqrt{\omega}} \operatorname{sech}^2(\sqrt{\omega})$, or $\cosh^2(\sqrt{\omega}) = \alpha$.

$\therefore \cosh(\sqrt{\omega}) = \sqrt{\alpha}$. Since $\cosh(x) > 1$ for all $x > 0$,

and since $\cosh(x)$ is a montonically increasing function, and since $\sqrt{\alpha} > 1$, there is exactly one value s.t. $\cosh(\sqrt{\omega}) = \sqrt{\alpha}$: $\omega = [\cosh^{-1}(\sqrt{\alpha})]^2$

$\therefore \lambda = -\omega = -[\cosh^{-1}(\sqrt{\alpha})]^2$

$\therefore$ There is exactly one negative eigenvalue for $\alpha > 1$.

Note as α increases, $\sqrt{\alpha}$ increases. $\therefore \cosh(\sqrt{\omega})$ increases $\Rightarrow \cosh^{-1}(\sqrt{\omega})$ increases $\Rightarrow \omega$ increases $\Rightarrow \lambda = -\omega$ decreases (becomes more negative).

23.

Given $\phi_m'' + \lambda_m \phi_m = 0$ and $\phi_n'' + \lambda_n \phi_n = 0$, then

$$\phi_m'' \phi_n + \lambda_m \phi_m \phi_n = 0$$

$$\phi_n'' \phi_m + \lambda_n \phi_n \phi_m = 0$$

Integrating from 0 to L,

$$\int_0^L \phi_m''(x)\,\phi_n(x)\,dx + \lambda_m \int_0^L \phi_m(x)\phi_n(x)\,dx = 0 \qquad [1]$$

$$\int_0^L \phi_n''(x)\phi_m(x)\,dx + \lambda_n \int_0^L \phi_m(x)\phi_n(x)\,dx = 0 \qquad [2]$$

Note that $\dfrac{d}{dx}\left[\phi_m'(x)\phi_n(x)\right] = \phi_m''\phi_n + \phi_m'\phi_n'$

$$\therefore \ \phi_m''\phi_n = \left[\phi_m'\phi_n\right]' - \phi_m'\phi_n' \qquad [3]$$

Similarly, $\dfrac{d}{dx}\left[\phi_m\phi_n'\right] = \phi_n''\phi_m + \phi_m'\phi_n'$

$$\therefore \ \phi_n''\phi_m = \left[\phi_m\phi_n'\right]' - \phi_m'\phi_n' \qquad [4]$$

Integrating [3] and [4],

$$\int_0^L \phi_m''(x)\,\phi_n(x)\,dx = \phi_m'(x)\phi_n(x)\Big|_0^L - \int_0^L \phi_m'(x)\,\phi_n'(x)\,dx$$

$$= \phi_m'(L)\overset{=0}{\phi_n(L)} - \phi_m'(0)\overset{=0}{\phi_n(0)} - \int_0^L \phi_m'(x)\phi_n'(x)\,dx$$

$$\therefore \int_0^L \phi_m''(x)\phi_n(x)\,dx = -\int_0^L \phi_m'(x)\,\phi_n'(x)\,dx \qquad [1']$$

$$\int_0^L \phi_n''(x)\phi_m(x)\,dx = \phi_m(x)\phi_n'(x)\Big|_0^L - \int_0^L \phi_m'(x)\phi_n'(x)\,dx$$

$$= \overset{=0}{\phi_m(L)}\phi_n'(L) - \overset{=0}{\phi_m(0)}\phi_n'(0) - \int_0^L \phi_m'(x)\phi_n'(x)\,dx$$

$$\therefore \int_0^L \phi_n''(x)\,\phi_m(x)\,dx = -\int_0^L \phi_m'(x)\,\phi_n'(x)\,dx \qquad [2']$$

Substituting [1'] into [1] and [2'] into [2],

$$-\int_0^L \phi_m'(x)\phi_n'(x)\,dx + \lambda_m \int_0^L \phi_m(x)\phi_n(x)\,dx = 0 \qquad [1'']$$

$$-\int_0^L \phi_m'(x)\phi_n'(x)\,dx + \lambda_n \int_0^L \phi_m(x)\phi_n(x)\,dx = 0 \qquad [2'']$$

Subtracting [2''] from [1''],

$$(\lambda_m - \lambda_n)\int_0^L \phi_m(x)\phi_n(x)\,dx = 0 \qquad [5]$$

Since $\lambda_m - \lambda_n \neq 0$, dividing [5] by $\lambda_m - \lambda_n$,

$$\underline{\int_0^L \phi_m(x)\,\phi_n(x)\,dx = 0}$$

24.

(a)

Assuming $\lambda > 0$, let $\lambda = \omega^4$, $\omega > 0$.

Characteristic equation: $r^4 - \omega^4 = 0$, $(r^2+\omega^2)(r^2-\omega^2)=0$,

$r = \pm\omega, \pm i\omega$. $\therefore$ $y(x) = c_1 e^{\omega x} + c_2 e^{-\omega x} + c_3 \cos(\omega x) + c_4 \sin(\omega x)$

$y''(x) = c_1 \omega^2 e^{\omega x} + c_2 \omega^2 e^{-\omega x} - c_3 \omega^2 \cos(\omega x) - c_4 \omega^2 \sin(\omega x)$

$y(0) = 0 \Rightarrow c_1 + c_2 + c_3 = 0$ [1]

$y''(0) = 0 \Rightarrow c_1 + c_2 - c_3 = 0$ [2]

$y(L) = 0 \Rightarrow c_1 e^{\omega L} + c_2 e^{-\omega L} + c_3 \cos(\omega L) + c_4 \sin(\omega L) = 0$ [3]

$y''(L) = 0 \Rightarrow c_1 e^{\omega L} + c_2 e^{-\omega L} - c_3 \cos(\omega L) - c_4 \sin(\omega L) = 0$ [4]

Adding [1], [2] $\Rightarrow$ $c_2 = -c_1$.

Adding [3], [4] $\Rightarrow$ $2c_1\left(e^{\omega L} - e^{-\omega L}\right) = 0$ [5]

$c_1 \neq 0$: Then by [5], $e^{\omega L} - e^{-\omega L} = 0$, or $e^{2\omega L} = 1$

This implies $2\omega L = 0$, which is impossible since $\omega \neq 0$ and presumably $L \neq 0$.

$\therefore$ $c_1 = 0$. Then by [1], [2], $c_2 = 0$, $c_3 = 0$

For nontrivial solution, $c_4 \neq 0$. $\therefore$ $y(x) = c_4 \sin(\omega x)$

$y(L) = 0 \Rightarrow \sin(\omega L) = 0 \Rightarrow \omega L = n\pi$, $n = 1, 2, 3, \ldots$

This also is consistent with $y''(L) = 0$

$$\therefore \boxed{y_n(x) = \sin\left(\frac{n\pi x}{L}\right),\ n = 1, 2, 3, \ldots \text{ and } \lambda_n = \frac{n^4\pi^4}{L^4}}$$

$$\underline{\lambda_1} = \frac{1^4\pi^4}{L^4} = \underline{\frac{97.409}{L^4}} \qquad \underline{\lambda_2} = \frac{2^4\pi^4}{L^4} = \underline{\frac{1{,}558.545}{L^4}}$$

(b)

As in (a), $y(x) = c_1 e^{\omega x} + c_2 e^{-\omega x} + c_3 \cos(\omega x) + c_4 \sin(\omega x)$

where $\lambda = \omega^4$, $\omega > 0$. This time use cosh(), sinh().

$$\therefore\ y(x) = c_1\cosh(\omega x) + c_2\sinh(\omega x) + c_3\cos(\omega x) + c_4\sin(\omega x)$$

$$y'(x) = c_1\omega\sinh(\omega x) + c_2\omega\cosh(\omega x) - c_3\omega\sin(\omega x) + c_4\omega\cos(\omega x)$$

$$y''(x) = c_1\omega^2\cosh(\omega x) + c_2\omega^2\sinh(\omega x) - c_3\omega^2\cos(\omega x) - c_4\omega^2\sin(\omega x)$$

$$\therefore\ y(0) = 0 \Rightarrow c_1 + c_3 = 0$$

$$y''(0) = 0 \Rightarrow c_1 - c_3 = 0$$

$$\therefore\ c_1 = c_3 = 0$$

$$\therefore\ y(L) = 0 \Rightarrow c_2\sinh(\omega L) + c_4\sin(\omega L) = 0 \quad [1]$$

$$y'(L) = 0 \Rightarrow c_2\cosh(\omega L) + c_4\cos(\omega L) = 0 \quad [2]$$

If $c_2 = 0$, then add [1], [2] to get:

$c_4[\sin(\omega L) + \cos(\omega L)] = 0$. Since sin() and cos() can't both be zero, then $c_4 = 0 \Rightarrow$ trivial solution.

$\therefore\ c_2 \neq 0$.

Similarly, if $c_4 = 0$, [1], [2] yield:

$c_2[\sinh(\omega L) + \cosh(\omega L)] = 0 \Rightarrow c_2 = 0$, as $\omega, L > 0$.

$\therefore\ c_4 \neq 0$

Multiply [1] by $\cos(\omega L)$, [2] by $\sin(\omega L)$, then subtract.

$$\therefore\ c_2\left[\cos(\omega L)\sinh(\omega L)-\sin(\omega L)\cosh(\omega L)\right]=0$$

$$\therefore\ \cos(\omega L)\sinh(\omega L)=\sin(\omega L)\cosh(\omega L) \qquad [3]$$

For $\omega>0$, $L>0$, $\sinh(\omega L)>0$. $\therefore$ Divide [1] by $\sinh(\omega L)$

to get $c_2=-c_4\dfrac{\sin(\omega L)}{\sinh(\omega L)}$

$$\therefore\ y(x)=c_4\left[-\frac{\sin(\omega L)}{\sinh(\omega L)}\sinh(\omega x)+\sin(\omega x)\right]$$

$$\therefore\ \boxed{y_n(x)=\sin(\omega_n x)-\frac{\sin(\omega_n L)}{\sinh(\omega_n L)}\sinh(\omega_n x),\ \text{where}\quad \lambda_n=\omega_n^4 \text{ and } \cos(\omega_n L)\sinh(\omega_n L)=\sin(\omega_n L)\cosh(\omega_n L)}$$

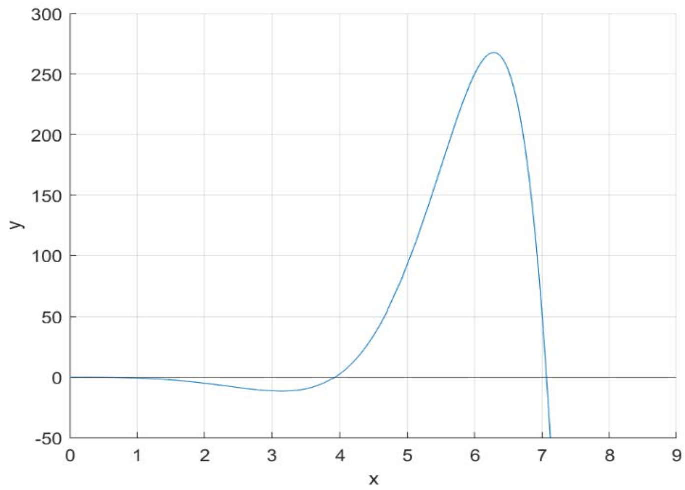
To find λ_1 and λ_2, use MATLAB to find the roots of $y=\cos(x)\sinh(x)-\sin(x)\cosh(x)$.

These points will be $\omega_n L=x$, so $\omega_n=\frac{x}{L}$, and $\therefore\ \lambda=x^4/L^4$. MATLAB code on next page.

The plot indicates estimation starting points for the vpasolve function.

```
clear
syms x
y = cos(x)*sinh(x)-sin(x)*cosh(x);
fplot(y)
grid on
yline(0)  % x-axis
axis([0,9,-50,300])
xlabel('x');ylabel('y')

% solve for first 2 roots
x1 = vpasolve(y==0, x, 4);
x2 = vpasolve(y==0, x, 7);
sprintf('x1^4 = %.4f',x1^4)
sprintf('x2^4 = %.4f',x2^4)
```




```
ans = 'x1^4 = 237.7211'

ans = 'x2^4 = 2496.4874'
```

$$\therefore \lambda_1 \approx \frac{237.72}{L^4}, \quad \lambda_2 \approx \frac{2496.49}{L^4}$$

(c)

As in (b), using $\lambda = \omega^4$, $\omega > 0$.

$$y(x) = c_1 \cosh(\omega x) + c_2 \sinh(\omega x) + c_3 \cos(\omega x) + c_4 \sin(\omega x) \quad [1]$$

$$y'(x) = c_1 \omega \sinh(\omega x) + c_2 \omega \cosh(\omega x) - c_3 \omega \sin(\omega x) + c_4 \omega \cos(\omega x) \quad [2]$$

$$y''(x) = c_1 \omega^2 \cosh(\omega x) + c_2 \omega^2 \sinh(\omega x) - c_3 \omega^2 \cos(\omega x) - c_4 \omega^2 \sin(\omega x) \quad [3]$$

$$y'''(x) = c_1 \omega^3 \sinh(\omega x) + c_2 \omega^3 \cosh(\omega x) + c_3 \omega^3 \sin(\omega x) - c_4 \omega^3 \cos(\omega x) \quad [4]$$

$$y(0) = 0 \Rightarrow c_1 + c_3 = 0 \quad \therefore c_3 = -c_1 \quad [1']$$

$y'(0) = 0 \Rightarrow c_2 + c_4 = 0 \quad \therefore \; c_4 = -c_2$ [1'']

$\therefore \; y''(L) = 0 \Rightarrow c_1 \cosh(\omega L) + c_2 \sinh(\omega L) + c_1 \cos(\omega L) + c_2 \sin(\omega L) = 0$ [5]

$y'''(L) = 0 \Rightarrow c_1 \sinh(\omega L) + c_2 \cosh(\omega L) - c_1 \sin(\omega L) + c_2 \cos(\omega L) = 0$ [6]

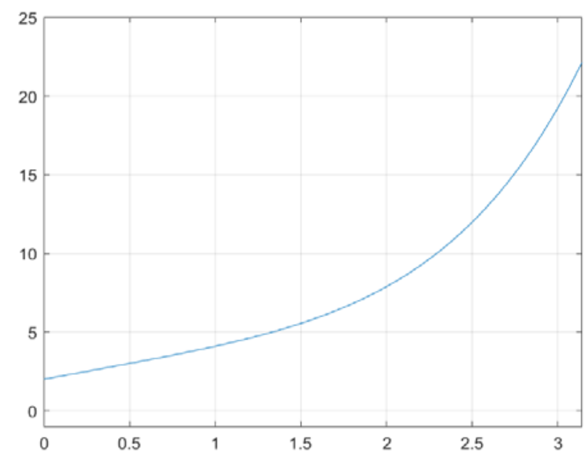
Suppose $c_1 = 0$.

$$\therefore \quad c_2 \left[\sinh(\omega L) + \cosh(\omega L) + \sin(\omega L) + \cos(\omega L) \right] = 0$$

This simplifies to $c_2 \left[e^{\omega L} + \sin(\omega L) + \cos(\omega L) \right] = 0$.

For $x \geq 0$, $e^x + \sin(x) + \cos(x) > 0$ as a MATLAB plot shows:

```
clear
syms x
y = exp(x) + sin(x) + cos(x);
fplot(y)
grid on
axis([0,pi,-1,25])
```



$\therefore \; c_1 = 0 \Rightarrow c_2 = 0$, and $\therefore \; c_3 = c_4 = 0$.

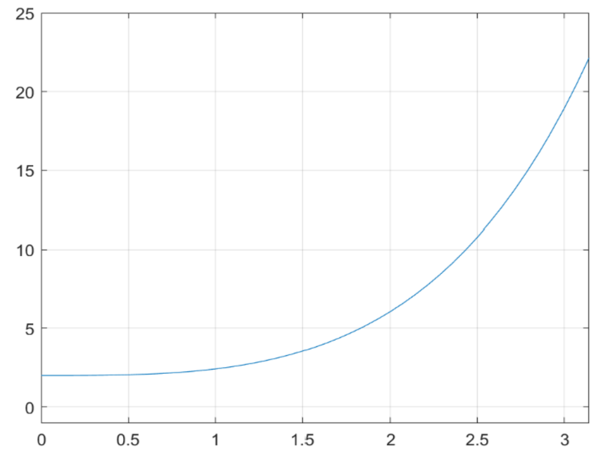
Suppose $c_2 = 0$:

$$\therefore \quad c_1 \left[\cosh(\omega L) + \sinh(\omega L) + \cos(\omega L) - \sin(\omega L) \right] = 0$$

or, $c_1 \left[e^{\omega L} + \cos(\omega L) - \sin(\omega L) \right] = 0$

A similar MATLAB plot shows $e^x + \cos(x) - \sin(x) > 0$ for $x \geq 0$.

```
clear
syms x
y = exp(x) - sin(x) + cos(x);
fplot(y)
grid on
axis([0,pi,-1,25])
```



$\therefore \; c_2 = 0 \Rightarrow c_1 = 0 \Rightarrow c_3 = c_4 = 0.$

$\therefore$ Neither c_1 or c_2 is zero.

Rewrite [5], [6] as:

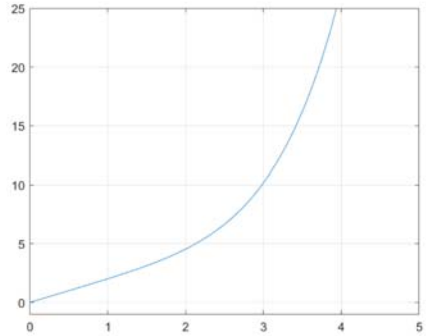
$$y''(L) = 0 \Rightarrow c_1\left[\cosh(\omega L) + \cos(\omega L)\right] + c_2\left[\sinh(\omega L) + \sin(\omega L)\right] = 0 \quad [5']$$

$$y'''(L) = 0 \Rightarrow c_1\left[\sinh(\omega L) - \sin(\omega L)\right] + c_2\left[\cosh(\omega L) + \cos(\omega L)\right] = 0 \quad [6']$$

From [5'], $\displaystyle c_2 = -c_1 \frac{\left[\cosh(\omega L) + \cos(\omega L)\right]}{\left[\sinh(\omega L) + \sin(\omega L)\right]}$ [7]

Note that for $x > 0$, $\sinh(x) + \sin(x) > 0$

```
clear
syms x
y = sinh(x) + sin(x);
fplot(y)
grid on
axis([0,5,-1,25])
```



Now, to delete one c_2 term, multiply [5] by $\cos(\omega L)$ and [6] by $\sin(\omega L)$, and subtract to get:

$$c_1\cosh(\omega L)\cos(\omega L) + c_2\sinh(\omega L)\cos(\omega L) + c_1\cos^2(\omega L) + c_2\cos(\omega L)\sin(\omega L) = 0 \quad [5'']$$

$$c_1\sinh(\omega L)\sin(\omega L) + c_2\cosh(\omega L)\sin(\omega L) - c_1\sin^2(\omega L) + c_2\cos(\omega L)\sin(\omega L) = 0 \quad [6'']$$

Subtracting [5''] - [6''],

$$c_1\left[\cosh(\omega L)\cos(\omega L) - \sinh(\omega L)\sin(\omega L)\right] + c_2\left[\sinh(\omega L)\cos(\omega L) - \cosh(\omega L)\sin(\omega L)\right] + c_1 = 0 \quad [8]$$

Substituting [7] for c_2, [8] becomes,

$$c_1\left[\cosh(\omega L)\cos(\omega L) - \sinh(\omega L)\sin(\omega L)\right] + c_1 - c_1\frac{\left[\cosh(\omega L) + \cos(\omega L)\right]}{\left[\sinh(\omega L) + \sin(\omega L)\right]}\left[\sinh(\omega L)\cos(\omega L) - \cosh(\omega L)\sin(\omega L)\right] = 0 \quad [8']$$

Divide by c_1 since $c_1 \neq 0$, and multiply out by $\left[\sinh(\omega L) + \sin(\omega L)\right]$ to get:

$$\begin{aligned}
&\sinh(\omega L)\cosh(\omega L)\cos(\omega L) - \sinh^2(\omega L)\sin(\omega L) + \sinh(\omega L) \\
&+ \cosh(\omega L)\cos(\omega L)\sin(\omega L) - \sinh(\omega L)\sin^2(\omega L) + \sin(\omega L) \\
&- \cosh(\omega L)\sinh(\omega L)\cos(\omega L) + \cosh^2(\omega L)\sin(\omega L) \\
&- \sinh(\omega L)\cos^2(\omega L) + \cosh(\omega L)\sin(\omega L)\cos(\omega L) = 0 \quad [8'']
\end{aligned}$$

Simplifying using $\cosh^2(x) - \sinh^2(x) = 1$, $\sin^2(x) + \cos^2(x) = 1$,

$2\sin(\omega L) + 2\cosh(\omega L)\sin(\omega L)\cos(\omega L) = 0$, or

$$\sin(\omega L)\left[1 + \cosh(\omega L)\cos(\omega L)\right] = 0 \qquad [8'']$$

Suppose $\sin(\omega L) = 0 \Rightarrow \omega L = n\pi$, $n = 1, 2, 3, \ldots$

then $\cos(\omega L) = \pm 1$

(a) $\sin(\omega L) = 0$, $\cos(\omega L) = 1$. Then [6'], [7] become

$$\sinh(\omega L) - \left[\frac{\cosh(\omega L) + 1}{\sinh(\omega L)}\right]\left[\cosh(\omega L) + 1\right] = 0, \text{ or}$$

$$\sinh^2(\omega L) - \cosh^2(\omega L) - 2\cosh(\omega L) - 1 = 0, \text{ or}$$

$$-\cosh(\omega L) = 1, \text{ which is impossible.}$$

(b) $\sin(\omega L) = 0$, $\cosh(\omega L) = -1$. Then [6'], [7] become

$$\sinh(\omega L) - \left[\frac{\cosh(\omega L) - 1}{\sinh(\omega L)}\right]\left[\cosh(\omega L) - 1\right] = 0, \text{ or}$$

$$\sinh^2(\omega L) - \cosh^2(\omega L) + 2\cosh(\omega L) - 1 = 0, \text{ or}$$

$$\cosh(\omega L) = 1, \text{ which is only true for } \omega L = 0.$$

$\therefore$ (a), (b) $\Rightarrow \sin(\omega L) \neq 0$ for $\omega L \neq 0$.

$\therefore$ [8'''] becomes $1 + \cosh(\omega L)\cos(\omega L) = 0$

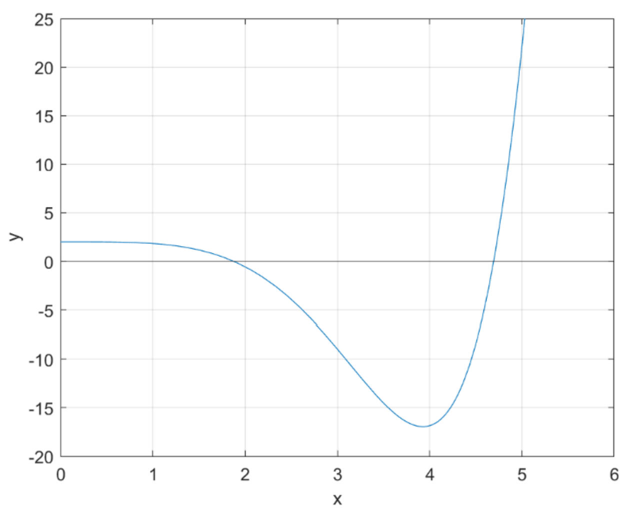
$\therefore$ Using [1], [1'], [1''], [7], and inverting [7] to resemble the answer in the back of the book,

$$c_1 = -c_2 \frac{[\sinh(\omega L) + \sin(\omega L)]}{[\cosh(\omega L) + \cos(\omega L)]},$$

$$\boxed{\begin{gathered} y_n(x) = \sin(\omega_n x) - \sinh(\omega_n x) - \frac{[\sinh(\omega L) + \sin(\omega L)]}{[\cosh(\omega L) + \cos(\omega L)]}[\cos(\omega_n x) - \cosh(\omega_n x)] \\ \text{where } \lambda_n = \omega_n^4 \text{ and } \omega_n \text{ satisfies } 1 + \cos(\omega_n L)\cosh(\omega_n L) = 0 \end{gathered}}$$

Use MATLAB to find λ_1, λ_2, using $1 + \cos(x)\cosh(x) = 0$, and noting $\omega_n L = x$, so $\lambda_n = \omega_n^4 = \frac{x^4}{L^4}$.

```
clear
syms x
y = 1 + cosh(x)*cos(x);
fplot(y)
grid on
yline(0)  % x-axis
xlabel('x'); ylabel('y')
axis([0,6,-20,25])
x1 = vpasolve(y==0,x,2);
x2 = vpasolve(y==0,x,4.7);
sprintf('x1^4 = %.4f',x1^4)
sprintf('x2^4 = %.4f',x2^4)
```




```
ans = 'x1^4 = 12.3624'

ans = 'x2^4 = 485.5188'
```

$\therefore \lambda_1 \approx \dfrac{12.3624}{L^4}$, $\lambda_2 \approx \dfrac{485.5188}{L^4}$

25.

(a)

With $u(x,t) = X(x)T(t)$ and $\frac{E}{\rho} u_{xx} = u_{tt}$,

$u_{xx} = X''(x)T(t)$, $u_{tt} = X(x)T''(t)$

$\therefore \frac{E}{\rho} X''(x)T(t) = X(x)T''(t)$. Dividing by $\frac{E}{\rho} X(x)T(t)$,

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{\frac{E}{\rho}T(t)}.$$ Since the left side

depends only on x and the right side only on t, they must equal the same constant, call it $-\lambda$.

$\therefore \dfrac{X''}{X} = -\lambda$ and $\dfrac{T''}{\frac{E}{\rho} T} = -\lambda$.

$\therefore X'' + \lambda X = 0$ and $T'' + \lambda \frac{E}{\rho} T = 0$

(6)

Using $u(x,t) = X(x)T(t)$ and $EA u_x(L,t) + m u_{tt}(L,t) = 0$,

$u_x = X'T$, $u_{tt} = XT''$, then

$$EA X'(L)T(t) + mX(L)T''(t) = 0 \qquad [1]$$

$u(0,t) = 0 \Rightarrow X(0)T(t) = 0$. Assuming a nontrivial solution, so $T(t) \neq 0$ for some t, then $X(0)T(t) = 0 \Rightarrow$ $X(0) = 0$.

Dividing [1] by $EA X(L)T(t)$,

$$\frac{X'(L)}{X(L)} + \frac{m}{EA}\frac{T''(t)}{T(t)} = 0. \qquad [2]$$

From (a), $T'' = -\lambda \frac{E}{\rho} T$. Substituting this into [2],

$$\frac{X'(L)}{X(L)} + \frac{m}{EA}\left(\frac{-\lambda E T}{\rho}\right)\cdot\frac{1}{T} = 0, \text{ or}$$

$$\frac{X'(L)}{X(L)} - \frac{\lambda m}{A\rho} = 0, \text{ or } X'(L) - \frac{\lambda m}{A\rho} X(L) = 0$$

$\therefore X'(L) - \frac{\lambda m L}{\rho A L} X(L) = 0$, and letting $\gamma = \frac{m}{\rho A L}$,

$$\underline{X'(L) - \gamma \lambda L X(L) = 0}$$

(c)

Equations (38) and (40) are:

$X'' + \lambda X = 0$ [38]

$X(0) = 0, \quad X'(L) - \gamma\lambda L X(L) = 0$ [40]

(1) $\lambda = 0$. $\therefore X(x) = c_1 x + c_2$ from [38].

$X(0) = 0 \Rightarrow c_2 = 0$. From $X'(L) - \gamma\lambda L X(L) = 0$ with

$\lambda = 0$, $X'(L) = 0 \Rightarrow c_1 = 0$. $\therefore$ only trivial

solution. $\therefore \lambda \neq 0$.

(2) $\lambda < 0$. $\therefore$ Let $\lambda = -\omega^2$, $\omega > 0$.

$\therefore$ [38] $\Rightarrow$ $X(x) = c_1 \sinh(\omega x) + c_2 \cosh(\omega x)$

$X(0) = 0 \Rightarrow c_2 = 0$. $\therefore$ $X(x) = c_1 \sinh(\omega x)$

[40] $\Rightarrow$ $c_1 \left[\omega \cosh(\omega L) + \gamma \omega^2 L \sinh(\omega L)\right] = 0$

$c_1 \neq 0$ for a nontrivial solution.

$\therefore$ $\omega \cosh(\omega L) = -\gamma \omega^2 L \sinh(\omega L)$

This is impossible, since $\omega L > 0$ so $\omega \cosh(\omega L) > 0$,

but $-\gamma \omega^2 L \sinh(\omega L) < 0$ since $\gamma > 0, \omega > 0, L > 0$,

and $\sinh(\omega L) > 0$ for $\omega L > 0$.

$\therefore$ $\lambda \not< 0$.

(3) $\therefore$ $\lambda > 0$. Letting $\lambda = \omega^2$, $\omega > 0$, then

$X(x) = c_1 \cos(\omega x) + c_2 \sin(\omega x)$

$X(0) = 0 \Rightarrow c_1 = 0$. $\therefore$ $X(x) = c_2 \sin(\omega x)$

[40] $\Rightarrow$ $c_2 \left[\omega \cos(\omega L) - \gamma \lambda L \sin(\omega L)\right] = 0$

$c_2 \neq 0$ for a nontrivial solution.

$\therefore\ \omega\cos(\omega L)-\gamma\lambda L\sin(\omega L)=0$, or, $\lambda=\omega^2$, $\sqrt{\lambda}=\omega$

$$\cos(\sqrt{\lambda}L)-\gamma L\sqrt{\lambda}\sin(\sqrt{\lambda}L)=0$$

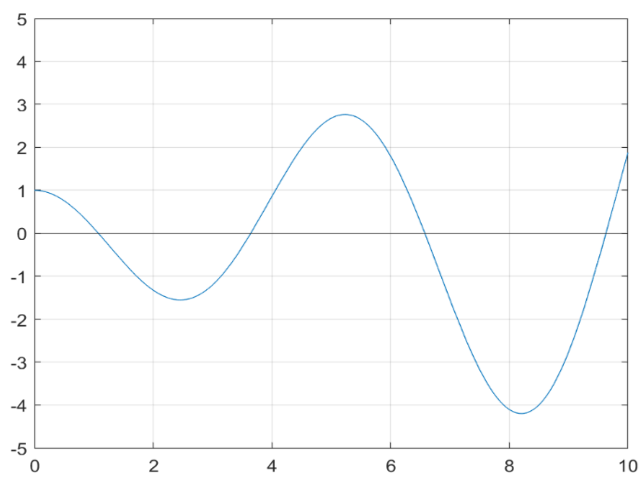
$$\therefore\ \boxed{\begin{array}{c} y_n(x)=\sin(\sqrt{\lambda_n}\,x),\ \text{where } \lambda_n \text{ satisfies} \\ \cos(\sqrt{\lambda_n}L)-\gamma L\sqrt{\lambda_n}\sin(\sqrt{\lambda_n}L)=0 \end{array}}$$

(d)

Use MATLAB to solve $\cos(x)-\gamma x\sin(x)=0$,

where $x=\sqrt{\lambda_n}L$, so $\frac{x}{L}=\sqrt{\lambda_n}$, and $\lambda_n=\frac{x^2}{L^2}$

```
clear
syms x
g = 0.5;
y = cos(x) - g*x*sin(x);
fplot(y)
grid on
yline(0)  % x-axis
axis([0,10,-5,5])
x1 = vpasolve(y==0,x,1);
x2 = vpasolve(y==0,x,3.5);
sprintf('x1^2 = %.4f',x1^2)
sprintf('x2^2 = %.4f',x2^2)
```




```
ans = 'x1^2 = 1.1597'

ans = 'x2^2 = 13.2758'
```

$$\therefore\ \lambda_1=\underline{\underline{\frac{1.1597}{L^2}}},\quad \lambda_2=\underline{\frac{13.2758}{L^2}}$$

11.2 Sturm-Liouville Boundary Value Problems

Note Title 10/12/2021

1.

$\lambda = 0$: $y = c_1 x + c_2$ $y(0) = 0 \Rightarrow c_2 = 0$. $y'(1) = 0 \Rightarrow c_1 = 0$.

$\therefore \lambda = 0$ not an eigenvalue

$\lambda > 0$: $y(x) = c_1 \cos(\sqrt{\lambda}\, x) + c_2 \sin(\sqrt{\lambda}\, x)$

$y(0) = 0 \Rightarrow c_1 = 0$ $y'(1) = c_2 \sqrt{\lambda} \cos(\sqrt{\lambda}) = 0$.

For $c_2 \neq 0$, $\sqrt{\lambda_n} = (2n-1)\frac{\pi}{2}$, $n = 1, 2, 3, \ldots$

$\therefore y_n(x) = K_n \sin(\sqrt{\lambda_n}\, x)$, $\lambda_n = \frac{(2n-1)^2 \pi^2}{4}$, $n = 1, 2, 3, \ldots$

$\lambda < 0$: $y(x) = c_1 \cosh(\sqrt{-\lambda}\, x) + c_2 \sinh(\sqrt{-\lambda}\, x)$

$y(0) = 0 \Rightarrow c_1 = 0$. $y'(1) = 0 \Rightarrow c_2 \sqrt{-\lambda} \cosh(\sqrt{-\lambda}) = 0$

and since $\cosh(z) \neq 0$, $c_2 = 0$.

$\therefore \lambda < 0$ is not an eigenvalue.

$\therefore y_n(x) = K_n \sin(\sqrt{\lambda_n}\, x)$, $\lambda_n = \frac{(2n-1)^2 \pi^2}{4}$, $n = 1, 2, 3, \ldots$

Since $r(x) = 1$, to normalize, $K_n^2 \int_0^1 \sin^2(\sqrt{\lambda_n}\, x)\, dx = 1$.

$$K_n^2 \int_0^1 \sin^2(\sqrt{\lambda_n} x)\,dx = K_n^2 \int_0^1 \left[\frac{1}{2} - \frac{1}{2}\cos(2\sqrt{\lambda_n}\,x)\right] dx$$

$$= K_n^2 \left[\frac{x}{2} - \frac{\sin(2\sqrt{\lambda_n}\,x)}{4\sqrt{\lambda_n}}\right]_0^1 = K_n^2\left[\frac{1}{2} - \frac{\sin(2\sqrt{\lambda_n})}{4\sqrt{\lambda_n}}\right]$$

But $\sin(2\sqrt{\lambda_n}) = \sin\left(2\,(2n-1)\frac{\pi}{2}\right) = \sin[(2n-1)\pi] = 0$

$$\therefore K_n^2 \int_0^1 \sin^2(\sqrt{\lambda_n}\,x)\,dx = \frac{K_n^2}{2} = 1 \Rightarrow K_n = \sqrt{2}$$

$$\therefore \boxed{y_n(x) = \sqrt{2}\,\sin\left[\frac{(2n-1)}{2}\pi x\right],\quad n = 1, 2, 3, \ldots}$$

2.

$\lambda = 0$: $y(x) = c_1 x + c_2$ $\quad y'(0) = 0 \Rightarrow c_1 = 0$ $\quad y(1) = 0 \Rightarrow c_2 = 0$

$\therefore \lambda = 0$ is not an eigenvalue.

$\lambda < 0$: $y(x) = c_1\cosh(\sqrt{-\lambda}\,x) + c_2\sinh(\sqrt{-\lambda}\,x)$

$$y'(0) = c_2\cosh(\sqrt{-\lambda}) = 0 \Rightarrow c_2 = 0$$

$$y(1) = c_1\cosh(\sqrt{-\lambda}) = 0 \Rightarrow c_1 = 0$$

$\therefore \lambda < 0$ is not an eigenvalue

$\lambda > 0$: $y(x) = c_1\cos(\sqrt{\lambda}\,x) + c_2\sin(\sqrt{\lambda}\,x)$

$$y'(0) = c_2\sqrt{\lambda}\cos(0) = 0 \Rightarrow c_2 = 0$$

$$y(1) = c_1\cos(\sqrt{\lambda}) = 0 \Rightarrow \sqrt{\lambda_n} = \frac{(2n-1)}{2}\pi, \; n = 1,2,3,\dots$$

$$\therefore \; y_n(x) = K_n \cos\left[\frac{(2n-1)\pi}{2} x\right], \; n = 1,2,3,\dots$$

$$\cos^2(z) = \frac{1+\cos(2z)}{2}$$

$$K_n^2 \int_0^1 \cos^2\left[\frac{(2n-1)\pi}{2} x\right] dx = K_n^2 \int_0^1 \left[\frac{1}{2} + \frac{\cos[(2n-1)\pi x]}{2}\right] dx$$

$$= K_n^2 \left[\frac{x}{2} + \frac{\sin[(2n-1)\pi x]}{2(2n-1)\pi}\right]_0^1 = K_n^2\left[\frac{1}{2}\right] = 1$$

$$\therefore \; K_n = \sqrt{2}$$

$$\therefore \; \boxed{y_n(x) = \sqrt{2}\cos\left[\frac{(2n-1)}{2}\pi x\right], \; n = 1,2,3,\dots}$$

3.

$$\lambda = 0: \quad y(x) = c_1 x + c_2, \quad y'(0) = 0 \Rightarrow c_1 = 0$$

$$\therefore \; y_0(x) = K_0. \quad \int_0^1 K_0^2\, dx = 1. \quad \therefore \text{ Choose } K_0 = 1$$

$$\therefore \; \underline{y_0(x) = 1}$$

$$\lambda < 0: \quad y(x) = c_1\cosh(\sqrt{-\lambda}\,x) + c_2\sinh(\sqrt{-\lambda}\,x)$$

$$y'(0) = 0 \Rightarrow c_2 = 0 \quad \therefore \; y(x) = c_1\cosh(\sqrt{-\lambda}\,x)$$

$y'(1)=0 \Rightarrow c_1\sqrt{-\lambda}\sinh(\sqrt{-\lambda})=0 \Rightarrow c_1=0$ since

$\sinh(z)\neq 0$ for $z\neq 0$. $\therefore \lambda<0$ not an eigenvalue

$\lambda>0$: $y(x)=c_1\cos(\sqrt{\lambda}\,x)+c_2\sin(\sqrt{\lambda}\,x)$

$y'(0)=0 \Rightarrow c_2=0 \quad y'(1)=0 \Rightarrow -c_1\sqrt{\lambda}\sin(\sqrt{\lambda})=0$

$\therefore \sqrt{\lambda}=n\pi,\ n=1,2,3,\ldots$

$\therefore y_n(x)=K_n\cos(n\pi x)$

$\cos(2\alpha)=\cos^2\alpha-\sin^2\alpha=2\cos^2(\alpha)-1$

$$K_n^2\int_0^1\cos^2(n\pi x)\,dx = K_n^2\int_0^1 \frac{1}{2}+\frac{\cos(2n\pi x)}{2}\,dx =$$

$$K_n^2\left[\frac{x}{2}+\frac{\sin(2n\pi x)}{4n\pi}\right]_0^1 = K_n^2\left(\frac{1}{2}\right)=1 \Rightarrow K_n=\sqrt{2}$$

$\therefore y_n(x)=\sqrt{2}\cos(n\pi x),\ n=1,2,3,\ldots$

$$\therefore \boxed{y_0(x)=1,\ \ y_n=\sqrt{2}\cos(n\pi x),\ n=1,2,3,\ldots}$$

4.

From # 8 of Section 11.1, $y_n(x)=K_n\cos(\sqrt{\lambda_n}\,x)$, where

$\tan(\sqrt{\lambda_n})=\frac{1}{\sqrt{\lambda_n}}$.

$$\therefore K_n^2\int_0^1\cos^2(\sqrt{\lambda_n}\,x)\,dx = K_n^2\int_0^1\frac{1}{2}+\frac{\cos(2\sqrt{\lambda_n}\,x)}{2}\,dx =$$

$$K_n^2\left[\frac{x}{2}+\frac{\sin(2\sqrt{\lambda_n}x)}{4\sqrt{\lambda_n}}\right]_0^1 = K_n^2\left[\frac{1}{2}+\frac{\sin(2\sqrt{\lambda_n})}{4\sqrt{\lambda_n}}\right]=1$$

Note $\sin(2\sqrt{\lambda_n}) = 2\sin(\sqrt{\lambda_n})\cos(\sqrt{\lambda_n})$ and

from $\tan(\sqrt{\lambda_n}) = \frac{1}{\sqrt{\lambda_n}}$, $\frac{\sin(\sqrt{\lambda_n})}{\cos(\sqrt{\lambda_n})} = \frac{1}{\sqrt{\lambda_n}}$

$$\therefore \frac{\sin(2\sqrt{\lambda_n})}{4\sqrt{\lambda_n}} = \frac{2\sin(\sqrt{\lambda_n})\cos(\sqrt{\lambda_n})}{4}\cdot\frac{\sin(\sqrt{\lambda_n})}{\cos(\sqrt{\lambda_n})} = \frac{\sin^2(\sqrt{\lambda_n})}{2}$$

$$\therefore K_n^2\left[\frac{1}{2}+\frac{\sin(2\sqrt{\lambda_n})}{4\sqrt{\lambda_n}}\right] = K_n^2\left[\frac{1}{2}+\frac{\sin^2\sqrt{\lambda_n}}{2}\right]=1$$

$$\therefore K_n^2 = \frac{2}{1+\sin^2(\sqrt{\lambda_n})},\quad K_n = \frac{\sqrt{2}}{[1+\sin^2(\sqrt{\lambda_n})]^{1/2}}$$

$$\therefore \boxed{y_n(x) = \frac{\sqrt{2}}{[1+\sin^2(\sqrt{\lambda_n})]^{1/2}}\cos(\sqrt{\lambda_n}x),\ \text{where } \tan(\sqrt{\lambda_n}) = \frac{1}{\sqrt{\lambda_n}}}$$

5.

From #17 of Section 11.1, $y_n(x) = K_n e^x \sin(n\pi x)$, $n=1,2,3,\ldots$

where $\lambda_n = n^2\pi^2$.

Note that $y''-2y'+(1+\lambda)y$ is not in the standard

Sturm-Liouville form: $(p(x)y')' - q(x)y + \lambda r(x)y$

From # 11 of Section 11.1, we need the integrating factor of $\mu(x) = \frac{1}{P(x)} \exp\left(\int \frac{Q}{P}\right)$, where, in this case, $P(x) = 1$, $Q(x) = -2$. $\therefore \mu(x) = \exp\left(\int -2\right) = e^{-2x}$

$\therefore e^{-2x} y'' - 2e^{-2x} + e^{-2x} y + \lambda e^{-2x} y = e^{-2x}\left(y'' - 2y' + (1+\lambda) y\right)$

So $\left(e^{-2x} y'\right)' + e^{-2x} y + \lambda e^{-2x} y = 0$, so $p(x) = e^{-2x}$, $-q(x) = e^{-2x}$, $\underline{r(x) = e^{-2x}}$.

$\therefore$ In evaluating the normalized eigenfunction,

$$\int_0^1 r(x)\, y_n^2(x)\, dx = 1, \text{ or } \int_0^1 e^{-2x} K_n^2 e^{2x} \sin^2(n\pi x)\, dx = 1,$$

$$\therefore K_n^2 \int_0^1 \sin^2(n\pi x)\, dx = 1$$

$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha = 1 - 2\sin^2\alpha$

$$\therefore K_n^2 \int_0^1 \left[\frac{1}{2} - \frac{1}{2}\cos(2n\pi x)\right] dx = K_n^2 \left[\frac{x}{2} - \frac{\sin(2n\pi x)}{4n\pi}\right]_0^1$$

$$= \frac{1}{2} K_n^2 = 1 \Rightarrow \underline{K_n = \sqrt{2}}$$

$$\therefore \boxed{y_n = \sqrt{2}\, e^{x} \sin(n\pi x), \quad n = 1, 2, 3, \ldots}$$

The normalized eigenfunctions of #1 above are:

$$\phi_n(x) = \sqrt{2}\sin\left[\frac{(2n-1)}{2}\pi x\right], \quad n=1,2,3,\ldots$$

The coefficients are $a_n = \int_0^1 r(x)\, f(x)\, \phi_n(x)\, dx$, $n=1,2,3$

From #1 above, $r(x)=1$. $\therefore$ $a_n = \int_0^1 f(x)\sqrt{2}\sin\left[\frac{(2n-1)\pi}{2}x\right]dx$

6.

$$a_n = \sqrt{2}\int_0^1 \sin\left[\frac{(2n-1)\pi}{2}x\right]dx = \sqrt{2}\left[-\frac{2}{(2n-1)\pi}\cos\left[\frac{(2n-1)\pi}{2}x\right]\right]_0^1$$

$$= \sqrt{2}\left[0 + \frac{2}{(2n-1)\pi}\right] = \underline{\frac{2\sqrt{2}}{(2n-1)\pi}}, \quad n=1,2,3,\ldots$$

7.

$$a_n = \sqrt{2}\int_0^1 x\sin\left[\frac{(2n-1)\pi}{2}x\right]dx \qquad \int x\sin(\alpha x) = -\frac{x\cos(\alpha x)}{\alpha} + \frac{\sin(\alpha x)}{\alpha^2}$$

$$= \sqrt{2}\left[-\frac{2}{(2n-1)\pi}x\cos\left[\frac{(2n-1)\pi x}{2}\right] + \frac{4}{(2n-1)^2\pi^2}\sin\left[\frac{(2n-1)\pi}{2}x\right]\right]_0^1$$

$$= \sqrt{2}\left[\frac{4}{(2n-1)^2\pi^2}\underbrace{\sin\left[\frac{(2n-1)\pi}{2}\right]}_{=(-1)^{n-1}} - 0\right], \quad n=1,2,3,\ldots$$

$$= \underline{\frac{4\sqrt{2}}{(2n-1)^2\pi^2}(-1)^{n-1}}, \quad n=1,2,3,\ldots$$

8.

$$a_n = \sqrt{2}\int_0^{1/2} \sin\left[\frac{(2n-1)\pi}{2}x\right]dx = \sqrt{2}\left[-\frac{2}{(2n-1)\pi}\cos\left[\frac{(2n-1)\pi x}{2}\right]\right]_0^{1/2}$$

$$= \sqrt{2}\left[-\frac{2}{(2n-1)\pi}\cos\left[\frac{(2n-1)\pi}{4}\right] + \frac{2}{(2n-1)\pi}\right], \quad n=1,2,3,\dots$$

$$= \underline{\frac{2\sqrt{2}}{(2n-1)\pi}\left(1-\cos\left[\frac{(2n-1)\pi}{4}\right]\right)}, \quad n=1,2,3,\dots$$

9.

$$a_n = \sqrt{2}\int_0^{\frac{1}{2}} 2x\sin\left[\frac{(2n-1)\pi}{2}x\right]dx + \sqrt{2}\int_{1/2}^{1} \sin\left[\frac{(2n-1)\pi}{2}x\right]dx$$

$$\int x\sin(\alpha x) = -\frac{x\cos(\alpha x)}{\alpha} + \frac{\sin(\alpha x)}{\alpha^2}$$

$$= 2\sqrt{2}\left[-\frac{2}{(2n-1)\pi}x\cos\left[\frac{(2n-1)\pi x}{2}\right] + \frac{4}{(2n-1)^2\pi^2}\sin\left[\frac{(2n-1)\pi}{2}x\right]\right]_0^{1/2}$$

$$-\frac{2\sqrt{2}}{(2n-1)\pi}\cos\left[\frac{(2n-1)\pi}{2}x\right]\Bigg|_{1/2}^{1}$$

$$= 2\sqrt{2}\left[-\frac{\cos\left[\frac{(2n-1)\pi}{4}\right]}{(2n-1)\pi} + \frac{4}{(2n-1)^2\pi^2}\sin\left[\frac{(2n-1)\pi}{4}\right] - 0\right]$$

cancel

$$-\frac{2\sqrt{2}}{(2n-1)\pi}\left[0 - \cos\left[\frac{(2n-1)\pi}{4}\right]\right]$$

$$= \frac{8\sqrt{2}}{(2n-1)^2\pi^2} \sin\left[\frac{(2n-1)\pi}{4}\right], \quad n = 1, 2, 3, \ldots$$

The normalized eigenfunctions of #4 above are:

$$\phi_n(x) = \frac{\sqrt{2}}{[1+\sin^2(\sqrt{\lambda_n})]^{1/2}} \cos(\sqrt{\lambda_n}\, x), \text{ where } \tan(\sqrt{\lambda_n}) = \frac{1}{\sqrt{\lambda_n}},$$

$n = 1, 2, 3, \ldots$ and $\lambda_n > 0$.

The coefficients are $a_n = \int_0^1 r(x) f(x) \phi_n(x)\, dx$, $n = 1, 2, 3$.

From #4 above, $r(x) = 1$

$$\therefore a_n = \frac{\sqrt{2}}{[1+\sin^2(\sqrt{\lambda_n})]^{1/2}} \int_0^1 f(x) \cos(\sqrt{\lambda_n}\, x)\, dx$$

10.

$$\int_0^1 \cos(\sqrt{\lambda_n}\, x)\, dx = \left.\frac{\sin(\sqrt{\lambda_n}\, x)}{\sqrt{\lambda_n}}\right|_0^1 = \frac{\sin(\sqrt{\lambda_n})}{\sqrt{\lambda_n}}$$

$$\therefore \boxed{a_n = \frac{\sqrt{2}\sin(\sqrt{\lambda_n})}{\sqrt{\lambda_n}\,[1+\sin^2(\sqrt{\lambda_n})]^{1/2}}, \quad n = 1, 2, 3, \ldots}$$

where $\tan(\sqrt{\lambda_n}) = \frac{1}{\sqrt{\lambda_n}}$

11.

$$\int_0^1 x\cos(\sqrt{\lambda_n}x)\,dx = \frac{x\sin(\sqrt{\lambda_n}x)}{\sqrt{\lambda_n}}\Bigg|_0^1 - \int_0^1 \frac{\sin(\sqrt{\lambda_n}x)}{\sqrt{\lambda_n}}\,dx$$

$$= \frac{\sin(\sqrt{\lambda_n})}{\sqrt{\lambda_n}} + \frac{\cos(\sqrt{\lambda_n}x)}{\lambda_n}\Bigg|_0^1$$

$$= \frac{\sin(\sqrt{\lambda_n})}{\sqrt{\lambda_n}} + \frac{\cos(\sqrt{\lambda_n})}{\lambda_n} - \frac{1}{\lambda_n}$$

Using $\sqrt{\lambda_n}\tan(\sqrt{\lambda_n}) = 1$ from #4 above, $\sqrt{\lambda_n} = \dfrac{\cos(\sqrt{\lambda_n})}{\sin(\sqrt{\lambda_n})}$

$$\int_0^1 x\cos(\sqrt{\lambda_n})\,dx = \frac{\sin^2(\sqrt{\lambda_n})}{\cos(\sqrt{\lambda_n})} + \frac{\sin^2(\sqrt{\lambda_n})}{\cos(\sqrt{\lambda_n})} - \frac{\sin^2(\sqrt{\lambda_n})}{\cos^2(\sqrt{\lambda_n})}$$

$$= \frac{\sin^2(\sqrt{\lambda_n})}{\cos^2(\sqrt{\lambda_n})}\left[2\cos(\sqrt{\lambda_n}) - 1\right] = \frac{2\cos(\sqrt{\lambda_n}) - 1}{\lambda_n}$$

$$\therefore \quad \boxed{a_n = \frac{\sqrt{2}}{\left[1 + \sin^2(\sqrt{\lambda_n})\right]^{1/2}}\left(\frac{2\cos(\sqrt{\lambda_n}) - 1}{\lambda_n}\right)}$$

where $\tan(\sqrt{\lambda_n}) = \dfrac{1}{\sqrt{\lambda_n}}$, $n = 1, 2, 3, \ldots$

12.

$$\int_0^1 (1-x)\cos(\sqrt{\lambda_n}x)\,dx = \int_0^1 \cos(\sqrt{\lambda_n})\,dx - \int_0^1 x\cos(\sqrt{\lambda_n}x)\,dx$$

$$= \underbrace{\frac{\sin(\sqrt{\lambda_n})}{\sqrt{\lambda_n}}}_{\text{from \#10}} - \underbrace{\frac{2\cos(\sqrt{\lambda_n}) - 1}{\lambda_n}}_{\text{from \#11}}$$

$$= \frac{\sqrt{\lambda_n}\sin(\sqrt{\lambda_n}) - 2\cos(\sqrt{\lambda_n}) + 1}{\lambda_n} = \frac{\cos(\sqrt{\lambda_n}) - 2\cos(\sqrt{\lambda_n}) + 1}{\lambda_n}$$

using $\sqrt{\lambda_n} = \dfrac{\cos(\sqrt{\lambda_n})}{\sin(\sqrt{\lambda_n})}$ from #4 above

$$= \frac{1 - \cos(\sqrt{\lambda_n})}{\lambda_n}$$

$$\therefore \boxed{a_n = \frac{\sqrt{2}}{[1 + \sin^2(\sqrt{\lambda_n})]^{1/2}} \left(\frac{1 - \cos(\sqrt{\lambda_n})}{\lambda_n} \right)}$$ where $\tan(\sqrt{\lambda_n}) = \frac{1}{\sqrt{\lambda_n}}$, $n = 1, 2, 3, \ldots$

13.

$$\int_0^1 f(x)\cos(\sqrt{\lambda_n}x)\,dx = \int_0^{1/2} \cos(\sqrt{\lambda_n}x)\,dx = \frac{\sin(\sqrt{\lambda_n}x)}{\sqrt{\lambda_n}}\Bigg|_0^{1/2} = \frac{\sin\left(\frac{\sqrt{\lambda_n}}{2}\right)}{\sqrt{\lambda_n}}$$

$$\therefore \boxed{a_n = \frac{\sqrt{2}}{[1 + \sin^2(\sqrt{\lambda_n})]^{1/2}} \left(\frac{\sin(\sqrt{\lambda_n}/2)}{\sqrt{\lambda_n}} \right)}$$ where $\tan(\sqrt{\lambda_n}) = \frac{1}{\sqrt{\lambda_n}}$, $n = 1, 2, 3, \ldots$

14.

$L[y]$ must be in the form $-(p(x)y')' + q(x)y$

Using problem #11 of section 11.1, p. 533, an integrating factor of $-e^x$ is used, so

$L[y] = -(-e^x y')' + 2e^x y$, so $p(x) = -e^x$, $q(x) = 2e^x$

$\therefore L[y] = e^x y'' + e^x y' + 2e^x y$

$\therefore y'' + y' + 2y$ is not of that form and cannot be made into the proper form without some integrating factor. $\therefore$ Not self-adjoint.

Another way to show this is to define $L[y] = y'' + y' + 2y$ and let $u(x), v(x)$ be such that $u(0) = u(1) = 0$, $v(0) = v(1) = 0$.

Now do a direct computation to show $\int_0^1 L[u]v \neq \int_0^1 uL[v]$

Integration by parts is used to reduce the derivatives of u down to just $u(x)$, and then using the boundary conditions to cancel some terms.

$$\therefore \int_0^1 L[u]v\,dx = \int_0^1 (u'' + u' + 2u)v\,dx = \int_0^1 u''v + u'v + 2uv\,dx$$

$$= \int_0^1 u''v\,dx + \int_0^1 u'v\,dx + \int_0^1 2uv\,dx$$

$$= \int_0^1 v\, d(u') + \int_0^1 v\, d(u) + \int_0^1 2uv\, dx$$

integration by parts   integration by parts

$$= vu'\Big|_0^1 - \int_0^1 u'\, d(v) + vu\Big|_0^1 - \int_0^1 u\, d(v) + \int_0^1 2uv\, dx$$

$$= \overset{=0}{v(1)}\,u'(1) - \overset{=0}{v(0)}\,u'(0) + \overset{=0}{v(1)}\,\overset{=0}{u(1)} - \overset{=0}{v(0)}\,\overset{=0}{u(0)}$$

$$- \int_0^1 u'v'\, dx - \int_0^1 uv'\, dx + \int_0^1 2uv\, dx$$

$$= -\int_0^1 v'\, d(u) - \int_0^1 v'u\, dx + \int_0^1 2uv\, dx$$

integration by parts

$$= -v'u\Big|_0^1 + \int_0^1 uv''\, dx - \int_0^1 v'u\, dx + \int_0^1 2uv\, dx$$

$$= -v'(1)\,\overset{=0}{u(1)} + v'(0)\,\overset{=0}{u(0)} + \int_0^1 (v'' - v' + 2v)u\, dx$$

$$= \int_0^1 (v'' - v' + 2v)u\, dx \neq \int_0^1 (v'' + v' + 2v)u = \int_0^1 u\, L[v]\, dx$$

$\therefore \int_0^1 L[u]\, v\, dx \neq \int_0^1 u\, L[v]\, dx.$ $\therefore$ Not self-adjoint

15.

Note $(1+x^2)y'' + 2xy' + y = ((1+x^2)y')' + y$, $p(x) = -(1+x^2)$, $q(x) = 1$

$\therefore L[y] = (1+x^2)y'' + 2xy' + y = -(p(x)y')' + q(x)y$

so this is the required form and the boundary conditions are separated. $\therefore$ This satifies the conditions for Lagrange's identity, and so

$(L[u], v) - (u, L[v]) = 0$, and is <u>self-adjoint</u>

Alternatively, a direct computation can be performed.

Let $p(x) = 1 + x^2$, $q(x) = 2x$, $L[y] = py'' + qy' + y$

$$u'(0) = 0,\ u(1) + 2u'(1) = 0 \qquad [1]$$

$$v'(0) = 0,\ v(1) + 2v'(1) = 0 \qquad [2]$$

Note: $p'(x) = 2x = q(x)$

$$\therefore \int_0^1 L[u]v\,dx = \int_0^1 (pu'' + qu' + u)v\,dx$$

$$= \int_0^1 pu''v\,dx + \int_0^1 qu'v\,dx + \int_0^1 uv\,dx$$

$$= \int_0^1 pv\,d(u') + \int_0^1 qv\,d(u) + \int_0^1 uv\,dx$$

integration by parts integration by parts

$$= pvu'\Big|_0^1 - \int_0^1 u'\,d(pv) + qvu\Big|_0^1 - \int_0^1 u\,d(qv) + \int_0^1 uv\,dx$$

$$= \overset{=2}{p(1)}\,v(1)\,u'(1) - p(0)\,v(0)\,\overset{=0}{u'(0)} + \overset{=2}{q(1)}\,v(1)\,u(1) - \overset{=0}{q(0)}\,v(0)\,u(0)$$

$$- \int_0^1 u'(p'v + pv')\,dx - \int_0^1 u(\overset{=2}{q'}v + qv')\,dx + \int_0^1 uv\,dx$$

$$= v(1)\left[2u'(1) + 2u(1)\right] = v(1)\left[u(1) + \overset{=0 \text{ by } [1]}{2u'(1) + u(1)}\right]$$

$$- \int_0^1 p'vu'\,dx - \int_0^1 pv'u'\,dx - 2\int_0^1 uv\,dx - \int_0^1 quv'\,dx + \int_0^1 uv\,dx$$

$$= v(1)u(1)$$

$$- \int_0^1 p'v\,d(u) - \int_0^1 pv'\,d(u) - \int_0^1 qv'u\,dx - \int_0^1 uv\,dx$$

integration by parts

integration by parts

$$= v(1)u(1)$$

$$- p'vu\Big|_0^1 + \int_0^1 u(p''v + p'v')\,dx - pv'u\Big|_0^1 + \int_0^1 u(p'v' + pv'')\,dx$$

$$- \int_0^1 qv'u\,dx - \int_0^1 uv\,dx$$

$$= v(1)u(1) - \overset{=2}{p'(1)}v(1)u(1) + \overset{=0}{p'(0)}v(0)u(0) - \overset{=2}{p(1)}v'(1)u(1) + \overset{=1}{p(0)}v'(0)u(0)$$

$$+ \int_0^1 \overset{=2}{p''}uv\,dx + \int_0^1 up'v'\,dx + \int_0^1 up'v'\,dx + \int_0^1 upv''\,dx$$

add

$$- \int_0^1 uqv'\,dx - \int_0^1 uv\,dx$$

$$\overbrace{= -u(1)\left[v(1)+2v'(1)\right] = 0 \text{ by } [2]}$$

$$= -v(1)u(1) - 2v'(1)u(1) + \underbrace{v'(0)u(0)}_{=0 \text{ by } [2]}$$

$$+ \int_0^1 uv\,dx - \int_0^1 uqv'\,dx + 2\int_0^1 u\overset{p'=q}{p'}v'\,dx + \int_0^1 upv''\,dx$$

$$= \int_0^1 uv\,dx + \int_0^1 uqv'\,dx + \int_0^1 upv''\,dx$$

$$= \int_0^1 u(pv'' + qv' + v)\,dx = \int_0^1 u\,L[v]\,dx$$

$$\therefore \int_0^1 L[u]\,v\,dx = \int_0^1 u\,L[v]\,dx$$

$\therefore$ yes, <u>self-adjoint</u>

16.

Let $L[y] = y'' + y = -(-y')' + y$, so $p(x) = -1$, $q(x) = 1$, using the format on p. 535 of the text.

From $y(0) - y'(1) = 0$ and $y'(0) - y(1) = 0$,

$\therefore u(0) = u'(1),\ v(0) = v'(1),\ u'(0) = u(1),\ v'(0) = v(1)$

$\therefore$ Using Equation (5) on p. 536,

$$\int_0^1 \left(L[u]v - uL[v] \right) dx = -p(x)\left[u'(x)v(x) - u(x)v'(x) \right]\Big|_0^1$$

$$= \left[\overset{=u(0)}{u'(1)}v(1) - u(1)\overset{=v(0)}{v'(1)} \right] - \left[\overset{=u(1)}{u'(0)}v(0) - u(0)\overset{=v(1)}{v'(0)} \right]$$

$$= u(0)v(1) - u(1)v(0) - u(1)v(0) + u(0)v(1)$$

$$= 2u(0)v(1) - 2u(1)v(0)$$

There is no guarantee this latter quantity is zero using the boundary conditions.

$\therefore$ Not self-adjoint

17.

Let $L[y] = (1+x^2)y'' + 2xy' + y = -\left[-(1+x^2)y'\right]' + y$,

so let $p(x) = -(1+x^2)$, $q(x) = 1$ using the format on p. 535 of the text.

From $y(0) - y'(1) = 0$ and $y'(0) + 2y(1) = 0$, using $L[u]$ and $L[v]$, $u(0) = u'(1)$, $u'(0) + 2u(1) = 0$ [1]

$$v(0) = v'(1), \quad v'(0) + 2v(1) = 0 \quad [2]$$

Using Equation (5) on p. 536 of the text,

$$\int_0^1 \left(L[u]v - uL[v]\right)dx = -p(x)\left[u'(x)v(x) - u(x)v'(x)\right]\Big|_0^1$$

$$= (1+x^2)\left[u'(x)v(x) - u(x)v'(x)\right]\Big|_0^1$$

$$= 2\left[u'(1)v(1) - u(1)v'(1)\right] - \left[u'(0)\overset{=v'(1)}{v(0)} - \overset{=u'(1)}{u(0)}\,v'(0)\right]$$

$$u'(1)\left[\underbrace{2v(1) + v'(0)}_{=0 \text{ by } [2]}\right] - v'(1)\left[\underbrace{2u(1) + u'(0)}_{=0 \text{ by } [1]}\right]$$

$$= \underline{0}$$

$\therefore$ yes, <u>self-adjoint</u>

18.

Let $L[y] = -(-y')' = y''$. Let $p(x) = -1$, $q(x) = 0$ using the format on p. 535 of the text. Using $L[u]$ and $L[v]$, and from the boundary conditions,

$$u(0) = 0,\ u(\pi) + u'(\pi) = 0,\ v(0) = 0,\ v(\pi) + v'(\pi) = 0.$$

Using Equation (5) on p. 536 of the text,

$$\int_0^\pi \left(L[u]v - uL[v]\right)dx = -p(x)\left[u'(x)v(x) - u(x)v'(x)\right]\Big|_0^\pi$$

$$= [\underbrace{u'(\pi)}_{}^{=-u(\pi)} v(\pi) - u(\pi) \overset{=-v(\pi)}{v'(\pi)}] - [u'(0)\overset{=0}{v(0)} - \overset{=0}{u(0)} v'(0)]$$

$$= -u(\pi)v(\pi) + u(\pi)v(\pi) = \underline{0}.$$

$\therefore$ yes, self-adjoint

19.

Assuming $u(x)$ and $v(x)$ satisfy [2] below:

$$\alpha_1 y(0) + \alpha_2 y'(0) = 0 \quad , \quad \beta_1 y(1) + \beta_2 y'(1) = 0 \quad [2]$$

since α_1 and α_2 can't both be zero, and β_1 and β_2 can't both be zero, if $\alpha_2 = 0$, then $\alpha_1 \neq 0$, and if $\beta_2 = 0$, then $\beta_1 \neq 0$, and if $\alpha_2 = 0$ and $\beta_2 = 0$, then $\alpha_1 \neq 0$ and $\beta_1 \neq 0$.

$\therefore$ If $\alpha_1 \neq 0$, then $-u'(0)v(0) + u(0)v'(0) =$

$$\frac{\alpha_2}{\alpha_1} u'(0)v'(0) - \frac{\alpha_2}{\alpha_1} u'(0)v'(0) = 0$$

If $\alpha_1 = 0$, then $\alpha_2 \neq 0$, then $-u'(0)v(0) + u(0)v'(0) =$

$$\frac{\alpha_1}{\alpha_2} u(0)v(0) - \frac{\alpha_1}{\alpha_2} u(0)v(0) = 0$$

$\therefore$ in any case $p(0)[u'(0)v(0) - u(0)v'(0)] = p(0)[0] = 0$

If $\beta_1 \neq 0$, then $-u'(1)v(1) + u(1)v'(1) =$

$$\frac{\beta_2}{\beta_1}u'(1)v'(1) - \frac{\beta_2}{\beta_1}u'(1)v'(1) = 0$$

If $\beta_1 = 0$, then $\beta_2 \neq 0$, so $-u'(1)v(1) + u(1)v'(1) =$

$$\frac{\beta_1}{\beta_2}u(1)v(1) - \frac{\beta_1}{\beta_2}u(1)v(1) = 0$$

$\therefore$ in any case $p(1)[u'(1)v(1) - u(1)v'(1)] = p(1)[0] = 0$

$\therefore$ $p(x)[u'(x)v(x) - u(x)v'(x)]\Big|_0^1$ becomes $p(1)[0] - p(0)[0] = \underline{0}$

20.

(b)

$$W[\phi_1, \phi_2](0) = \begin{vmatrix} \phi_1(0) & \phi_2(0) \\ \phi_1'(0) & \phi_2'(0) \end{vmatrix} = \phi_1(0)\phi_2'(0) - \phi_1'(0)\phi_2(0)$$

The boundary condition for $x=0$ is $\alpha_1 y(0) + \alpha_2 y'(0) = 0$

As shown in #19 above, if $\alpha_1 = 0$, then $\alpha_2 \neq 0$, so

$$\phi_1(0)\phi_2'(0) - \phi_1'(0)\phi_2(0) = -\frac{\alpha_1}{\alpha_2}\phi_1(0)\phi_2(0) + \frac{\alpha_1}{\alpha_2}\phi_1(0)\phi_2(0) = 0$$

If $\alpha_1 \neq 0$, then

$$\phi_1(0)\phi_2'(0) - \phi_1'(0)\phi_2(0) = -\frac{\alpha_2}{\alpha_1}\phi_1'(0)\phi_2'(0) + \frac{\alpha_2}{\alpha_1}\phi_1'(0)\phi_2'(0) = 0$$

$\therefore$ $\underline{W[\phi_1, \phi_2](0) = 0}$

(c)

Theorem 3.2.7 states that $W[\phi_1, \phi_2]$ is either zero everywhere, or zero nowhere. Since $W[\phi_1, \phi_2](0) = 0$ from (b), then $W[\phi_1, \phi_2]$ is zero everywhere on $[0,1]$. $\therefore$ By Theorem 4.1.3, ϕ_1 and ϕ_2 are linearly dependent, a contradiction.

21.

(a)

Since $-(p\phi')' + q\phi = \lambda r\phi$, then $-(p\phi')'\phi + q\phi^2 = \lambda r\phi^2$

$$\therefore \lambda\int_0^1 r\phi^2\,dx = \int_0^1 \left[-(p\phi')'\phi + q\phi^2\right]dx$$

$$= \int_0^1 -(p\phi')'\phi\,dx + \int_0^1 q\phi^2\,dx \quad [1]$$

Integrating the first integral by parts,

$$\int_0^1 -(p\phi')'\phi\,dx = \int_0^1 -\phi\,d(p\phi')dx = -\phi p\phi'\Big|_0^1 + \int_0^1 p(\phi')^2\,dx$$

$$= \int_0^1 p(\phi')^2\,dx - \phi(1)p(1)\phi'(1) + \phi(0)p(0)\phi'(0) \quad [2]$$

From $\beta_1\phi(1) + \beta_2\phi'(1) = 0$, $\phi'(1) = -\dfrac{\beta_1}{\beta_2}\phi(1)$, $\beta_2 \neq 0$

From $\alpha_1\phi(0) + \alpha_2\phi'(0) = 0$, $\phi'(0) = -\dfrac{\alpha_1}{\alpha_2}\phi(0)$, $\alpha_2 \neq 0$

Substituting these into [2],

$$\int_0^1 -(p\phi')'\phi\,dx = \int_0^1 p(\phi')^2\,dx + \frac{\beta_1}{\beta_2}p(1)\phi^2(1) - \frac{\alpha_1}{\alpha_2}p(0)\phi^2(0)$$

Substituting this into [1], and adding the two integrals,

$$\boxed{\lambda\int_0^1 r\phi^2\,dx = \int_0^1 [p(\phi')^2 + q\phi^2]\,dx + \frac{\beta_1}{\beta_2}p(1)\phi^2(1) - \frac{\alpha_1}{\alpha_2}p(0)\phi^2(0)}$$

If $\alpha_2 = 0$, then $\alpha_1 \neq 0$, so $\alpha_1\phi(0) = 0 \Rightarrow \phi(0) = 0$.

$\therefore$ [2] becomes $\int_0^1 p(\phi')^2\,dx - \phi(1)p(1)\phi'(1)$

$$\therefore \lambda\int_0^1 r\phi^2\,dx = \int_0^1 [p(\phi')^2 + q\phi^2]\,dx + \frac{\beta_1}{\beta_2}p(1)\phi^2(1)$$

If $\beta_2 = 0$, then $\beta_1 \neq 0$, so $\beta_1\phi(1) = 0 \Rightarrow \phi(1) = 0$.

$\therefore$ [2] becomes $\int_0^1 p(\phi')^2\,dx + \phi(0)p(0)\phi'(0)$

$$\therefore \lambda\int_0^1 r\phi^2\,dx = \int_0^1 [p(\phi')^2 + q\phi^2]\,dx - \frac{\alpha_1}{\alpha_2}p(0)\phi^2(0)$$

(b)

$\frac{\beta_1}{\beta_2} \geq 0$, $\frac{-\alpha_1}{\alpha_2} \geq 0$, and $p(x) > 0$ as stated on p. 535 of the text, so $\frac{\beta_1}{\beta_2}p(1)\phi^2(1) - \frac{\alpha_1}{\alpha_2}p(0)\phi^2(0) \geq 0$

Since $q(x) \geq 0$, $q\phi^2 \geq 0$ on $[0,1]$. $p(x) > 0$ on $[0,1]$,

so $p(\phi')^2 \geq 0$ on $[0,1]$. $\therefore$ $p(\phi')^2 + q\phi^2 \geq 0$ on $[0,1]$.

$\therefore$ $\int_0^1 [p(\phi')^2 + q\phi^2]\,dx \geq 0$.

$\therefore$ $\int_0^1 [p(\phi')^2 + q\phi^2]\,dx + \frac{\beta_1}{\beta_2} p(1)\phi^2(1) - \frac{\alpha_1}{\alpha_2} p(0)\phi^2(0) \geq 0$.

From the text on p. 535, $r(x) > 0$ on $[0,1]$, and is continuous. $\phi(x) \neq 0$ for some point in $[0,1]$ since it is a non-trivial solution. $\therefore$ $r\phi^2 \geq 0$ on $[0,1]$, and $r\phi^2 > 0$ for at least some point in $[0,1]$ and is continuous. $\therefore$ $\int_0^1 r\phi^2\,dx > 0$.

$\therefore$

$$\lambda = \frac{\int_0^1 [p(\phi')^2 + q\phi^2]\,dx + \frac{\beta_1}{\beta_2} p(1)\phi^2(1) - \frac{\alpha_1}{\alpha_2} p(0)\phi^2(0)}{\int_0^1 r\phi^2\,dx} \geq 0$$

since the numerator is nonnegative and the denominator is positive.

(c)

If $\alpha_1 = \beta_1 = 0$ and $q(x) = 0$ on $[0,1]$, then the numerator in the formula in (b) becomes $\int_0^1 p(x)(\phi'(x))^2\,dx$. This is nonnegative on $[0,1]$ since $p(x) > 0$ on $[0,1]$. So if $\phi'(x) = 0$ on $[0,1]$, then the numerator is zero. If $\phi'(x_0) \neq 0$ for some $x_0 \in [0,1]$, since $\phi'(x)$ is continuous, then $p(\phi')^2 > 0$ for some interval in $[0,1]$, and then the numerator is strictly positive.

So if $\alpha_1 = \beta_1 = 0$ and $q(x) = 0$ on $[0,1]$, λ may be strictly positive, but it may not be if ϕ is a non-zero constant function.

On the other hand, if $\alpha_1 \neq 0$ or $\beta_1 \neq 0$ or $q(x_0) \neq 0$ for some $x_0 \in [0,1]$, then λ is strictly positive.

In any of these cases, the value of λ depends on $\phi'(x)$ on $(0,1)$.

(1) If $\phi' = 0$ on $[0,1]$, then $\phi(x)$ is a constant, and since it is a nontrivial continuous solution, $\phi^2(x) > 0$

$\therefore$ $-\frac{\alpha_1}{\alpha_2} p(0)\phi^2(0) > 0$ if $\alpha_1 \neq 0$, and $\frac{\beta_1}{\beta_2} p(1)\phi^2(1) > 0$ if $\beta_1 \neq 0$. $\therefore$ $\lambda > 0$. If $\alpha_1 = \beta_1 = 0$ but $q(x_0) \neq 0$ for some $x_0 \in [0,1]$, since $q(x)$ is continuous and $q(x) \geq 0$ on $[0,1]$, then $q(x_0)\phi^2(x_0) > 0$ for some interval in $[0,1]$ and $q(x)\phi^2(x) \geq 0$ otherwise.

$\therefore$ $\int_0^1 [p(\phi')^2 + q\phi^2]\,dx = \int_0^1 q\phi^2\,dx > 0$ and so λ is strictly positive.

(2) If $\phi'(x_0) \neq 0$ for some $x_0 \in [0,1]$, since $p(x) > 0$ for all $x \in [0,1]$, $p(x_0)(\phi'(x_0))^2 > 0$ for some interval in $[0,1]$ since both factors are continuous. Also, $q(x_0)\phi^2(x_0) \geq 0$.

$\therefore$ $p(x_0)(\phi'(x_0))^2 + q(x_0)\phi^2(x_0) > 0$ for some interval in $[0,1]$, and $p(\phi')^2 + q\phi^2 \geq 0$ otherwise.

$\therefore \int_0^1 [p(\phi')^2 + q\phi^2]\,dx > 0$, and so λ is strictly positive.

$\therefore$ (1) & (2) mean if $\alpha_1 \neq 0$ or $\beta_1 \neq 0$ or $q(x_0) \neq 0$ for some $x_0 \in [0,1]$, then $\lambda > 0$ under the conditions of (6).

If $\alpha_1 = \beta_1 = 0$ and $q(x) = 0$ on $[0,1]$ and $\phi(x)$ is a constant function on $[0,1]$, then $\lambda = 0$.

22.

Note that here $L[y] = -(py')' + qy$.

Let $y = u + iv$, where u, v are real valued.

$\therefore L[u+iv] = -(p(u'+iv'))' + q(u+iv)$

$= -(pu' + ipv')' + qu + iqv = -(pu')' - i(pv')' + qu + iqv$

$= -(pu')' + qu + i[-(pv')' + qv] = L[u] + iL[v]$

Now consider $y(x) = u(x) + iv(x)$, $z(x) = r(x) + is(x)$,

where u, v, r, s are real valued.

$$\therefore \left(L[y], z\right) - \left(y, L[z]\right) = \int_0^1 L[y]\,\bar{z}\,dx - \int_0^1 y\,\overline{L[z]}\,dx$$

$$= \int_0^1 \left(L[u] + iL[v]\right)\left(r - is\right)dx - \int_0^1 (u + iv)\overline{\left(L[r] + iL[s]\right)}\,dx$$

Note: $\overline{L[r] + iL[s]} = L[r] - iL[s]$

$$= \int_0^1 L[u]r + iL[v]r - iL[u]s + L[v]s \; dx$$

$$- \int_0^1 uL[r] - iuL[s] + ivL[r] + vL[s] \; dx$$

$$= \int_0^1 L[u]r\,dx - \int_0^1 uL[r]dx \; + i\int_0^1 L[v]r\,dx - i\int_0^1 vL[r]\,dx$$

$$- i\int_0^1 L[u]s\,dx + i\int_0^1 uL[s]dx + \int_0^1 L[v]s\,dx - \int_0^1 vL[s]\,dx$$

$$= \underbrace{\left(L[u], r\right) - \left(u, L[r]\right)}_{=0} + i\Big[\underbrace{\left(L[v], r\right) - \left(v, L[r]\right)}_{=0}\Big]$$

$$- i\Big[\underbrace{\left(L[u], s\right) - \left(u, L[s]\right)}_{=0}\Big] + \underbrace{\left(L[v], s\right) - \left(v, L[s]\right)}_{=0}$$

$$= \underline{0}$$

$\therefore \left(L[y], z\right) - \left(y, L[z]\right) = 0$, y, z complex valued.

23.

(a)

$$L[\phi(x)] = \lambda r(x)\phi(x) = \lambda r\,[U + iV] = \lambda r U + i\lambda r V$$

$$L[\phi(x)] = L[U(x) + iV(x)] = L[u] + iL[v]$$

Equating real and imaginary parts,

$$\underline{L[u] = \lambda r U}\;,\quad \underline{L[v] = \lambda r V}$$

(b)

Theorem 11.2.3 says there is only one linearly independent eigenfunction to the eigenvalue λ. Since $U(x)$ and $V(x)$ are eigenfunctions of λ, they must be dependent.

(c)

Since $U(x)$ and $V(x)$ are dependent, $V(x) = cU(x)$, some constant c. $\therefore\ \phi(x) = U(x) + icU(x)$

$\therefore\ \phi(x) = (1+ic)\,u(x)$

24.

Similar to an Euler equation of section 5.4, let $y = x^r$, r real.

$\therefore\ x^2 y'' - \lambda x y' + \lambda y = 0$ becomes $x^2 r(r-1)x^{r-2} - \lambda x r x^{r-1} + \lambda x^r =$

$x^r[r(r-1) - \lambda r + \lambda] = 0$, or $r^2 - (1+\lambda)r + \lambda = 0$.

$$\therefore\ r = \frac{1+\lambda \pm \sqrt{(1+\lambda)^2 - 4\lambda}}{2} = \frac{1+\lambda \pm \sqrt{(\lambda-1)^2}}{2} = \frac{\lambda+1 \pm |\lambda-1|}{2}$$

$\therefore\ r = 1, \lambda$. $\therefore\ y = x, x^\lambda$ solve the equation, but the boundary conditions are not satisfied, as $y = c_1 x + c_2 x^\lambda$, for $\lambda \neq 1$, implies $y(1) = c_1 + c_2 = 0$,

$\therefore\ c_1 = -c_2$, $\therefore\ y = c_2(x^\lambda - x)$, so $y(2) = c_2(2^\lambda - 2) = 0$

$\Rightarrow c_2 = 0$ as $\lambda \neq 1$. For $\lambda = 1$, then have repeated roots $\Rightarrow y = c_1 x + c_2 x \ln(x)$. $y(1) = 0 \Rightarrow c_1 = 0$

and $y(2)=0 \Rightarrow c_2(2)\ln(2)=0 \Rightarrow c_2=0$.

$\therefore$ For λ real, no nontrivial solution.

If λ is complex, then $r=1,\ a+ib$, so

$$y(x)=c_1 x + c_2 x^a \cos(b\ln(x)) + c_3 x^a \sin(b\ln(x))$$

$$y(1)=0 \Rightarrow c_1=0$$

$$y(2)=0 \Rightarrow c_2 2^a \cos(b\ln(2)) + c_3 2^a \sin(b\ln(2))=0$$

or $\quad c_2 \cos(b\ln(2)) + c_3 \sin(b\ln(2))=0$

There are numerous possibilities, such as

$$\lambda = \underline{a + \frac{n\pi}{\ln(2)} i}, \quad a \text{ any real number}, \ n=1,2,3,\dots$$

with $c_2=0$.

$$\therefore\ y(x) = x^a \sin\left(\frac{n\pi}{\ln(2)}\ln(x)\right), \quad n=1,2,3,\dots, \ a \text{ real}.$$

$\therefore$ Eigenvalues exist but are not real.

25.

$\lambda = 0 \Rightarrow y^{(4)} = 0, \quad y(x) = c_1 x^3 + c_2 x^2 + c_3 x + c_4$

$\lambda > 0$: let $y = e^{rx}$. $\therefore r^4 + \lambda r^2 = r^2(r^2 + \lambda) = 0 \Rightarrow r = 0, 0, \pm i\sqrt{\lambda}$

repeated roots

$\therefore y(x) = c_1 + c_2 x + c_3 \cos(\sqrt{\lambda}x) + c_4 \sin(\sqrt{\lambda}x)$

$\lambda < 0$: let $y = e^{rx}$, $-\mu = \lambda$, $\mu > 0$. $\therefore r^4 - \mu r^2 = r^2(r^2 - \mu) = 0$

$\Rightarrow r = 0, 0, \pm\sqrt{\mu}$. $\therefore y(x) = c_1 + c_2 x + c_3 e^{\sqrt{-\lambda}x} + c_4 e^{-\sqrt{-\lambda}x}$

repeated roots

(a)

$\lambda = 0$: $y(x) = c_1 x^3 + c_2 x^2 + c_3 x + c_4$

$y(0) = 0 \Rightarrow \underline{c_4 = 0}$. $y''(0) = 0 \Rightarrow \underline{c_2 = 0}$

$\therefore y(x) = c_1 x^3 + c_3 x$, $y(L) = 0 \Rightarrow c_1 L^2 + c_3 = 0$

$y''(L) = 0 \Rightarrow 6c_1 L = 0 \Rightarrow \underline{c_1 = 0}$ $\therefore \underline{c_3 = 0}$

$\therefore \lambda = 0$ $\underline{\text{not}}$ an eigenvalue

$\lambda > 0$: $y(x) = c_1 + c_2 x + c_3 \cos(\sqrt{\lambda}x) + c_4 \sin(\sqrt{\lambda}x)$

$y''(x) = -c_3 \lambda \cos(\sqrt{\lambda}x) - c_4 \lambda \sin(\sqrt{\lambda}x)$

$y(0)=0 \Rightarrow c_1 + c_3 = 0$

$y''(0)=0 \Rightarrow c_3 = 0. \therefore c_1 = 0$

$y(L)=0 \Rightarrow c_2 L + c_4 \sin(\sqrt{\lambda} L) = 0$

$y''(L)=0 \Rightarrow -c_4 \lambda \sin(\sqrt{\lambda} L) = 0$

$\therefore c_4 \neq 0$ for a nontrivial solution.

$\therefore \sqrt{\lambda} L = n\pi, \quad \lambda = \frac{n^2\pi^2}{L^2}, \quad n = 1,2,3,\dots$

$\therefore c_4 \sin(\sqrt{\lambda} L) = 0 \Rightarrow c_2 L = 0 \Rightarrow c_2 = 0$

$\therefore y(x) = \sin\left(\frac{n\pi x}{L}\right), \quad \lambda = \frac{n^2\pi^2}{L^2}, \quad n = 1,2,3,\dots$

smallest eigenvalue: $\frac{\pi^2}{L^2}$

with eigenfunction $y(x) = \sin\left(\frac{\pi x}{L}\right)$

$\lambda < 0$: $y(x) = c_1 + c_2 x + c_3 e^{\sqrt{-\lambda} x} + c_4 e^{-\sqrt{-\lambda} x}$

$y''(x) = -c_3 \lambda e^{\sqrt{-\lambda} x} - c_4 \lambda e^{-\sqrt{-\lambda} x}$

$y(0) = 0 \Rightarrow c_1 + c_3 + c_4 = 0$

$y''(0) = 0 \Rightarrow -c_3 - c_4 = 0 \quad \therefore c_1 = 0$

$y(L) = 0 \Rightarrow c_2 L + c_3 e^{\sqrt{-\lambda} L} + c_4 e^{-\sqrt{-\lambda} L} = 0$

$y''(L) = 0 \Rightarrow -c_3 e^{\sqrt{-\lambda}L} - c_4 e^{-\sqrt{-\lambda}L} = 0 \quad \therefore \; c_2 = 0$

$\therefore \; c_3 + c_4 = 0$

$c_3 e^{\sqrt{-\lambda}L} + c_4 e^{-\sqrt{-\lambda}L} = 0 \Rightarrow c_3 e^{2\sqrt{-\lambda}L} + \underbrace{c_4}_{=-c_3} = 0$

$\therefore \; c_3\left(e^{2\sqrt{-\lambda}L} - 1\right) = 0$

For non-trivial solution, $c_3 \neq 0$. $\therefore \; e^{2\sqrt{-\lambda}L} = 1$

$\therefore \; L = 0$ or $\sqrt{-\lambda} = 0$, both impossible.

$\therefore \; \lambda < 0$ not an eigenvalue.

$\therefore$ smallest eigenvalue: π^2/L^2 with $y(x) = \sin\left(\frac{\pi x}{L}\right)$

(b)

$\lambda = 0: \quad y(x) = c_1 x^3 + c_2 x^2 + c_3 x + c_4$

$y(0) = 0 \Rightarrow c_4 = 0 \quad y''(0) = 0 \Rightarrow c_2 = 0$

$\therefore \; y(x) = c_1 x^3 + c_3 x \quad y(L) = 0 \Rightarrow c_1 L^2 + c_3 = 0$

$y'(L) = 0 \Rightarrow 3c_1 L^2 + c_3 = 0 \quad \therefore \; y'(L) - y(L) = 2c_1 L^2 = 0$

$\therefore \; c_1 = 0$, and $\therefore \; c_3 = 0$

$\therefore \; \lambda = 0$ not an eigenvalue

$\lambda > 0: \ y(x) = c_1 + c_2 x + c_3 \cos(\sqrt{\lambda} x) + c_4 \sin(\sqrt{\lambda} x)$

$y'(x) = c_2 - c_3 \sqrt{\lambda} \sin(\sqrt{\lambda} x) + c_4 \sqrt{\lambda} \cos(\sqrt{\lambda} x)$

$y''(x) = -c_3 \sqrt{\lambda} \cos(\sqrt{\lambda} x) - c_4 \lambda \sin(\sqrt{\lambda} x)$

$y(0) = 0 \Rightarrow c_1 + c_3 = 0 \quad y''(0) = 0 \Rightarrow \underline{c_3 = 0}. \ \therefore \underline{c_1 = 0}$

$y(L) = 0 \Rightarrow c_2 L + c_4 \sin(\sqrt{\lambda} L) = 0$ [61]

$y'(L) = 0 \Rightarrow c_2 + c_4 \sqrt{\lambda} \cos(\sqrt{\lambda} L) = 0$ [62]

$\therefore -c_4 \sqrt{\lambda} L \cos(\sqrt{\lambda} L) + c_4 \sin(\sqrt{\lambda} L) = 0$

$c_4 \neq 0$ for otherwise $c_2 = 0$ from [61]

$\therefore \sin(\sqrt{\lambda} L) = \sqrt{\lambda} L \cos(\sqrt{\lambda} L)$

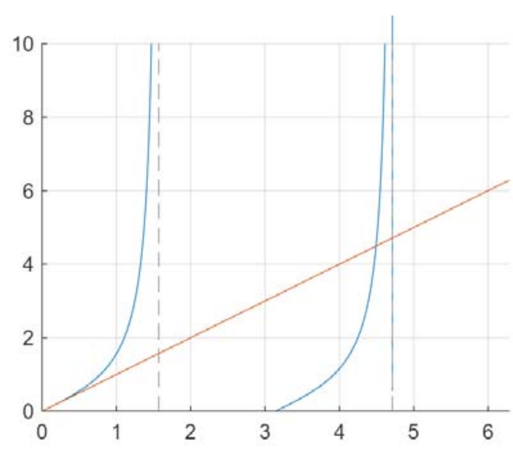
$\sin(x)$ and $\cos(x)$ can't both be zero.

$\therefore \tan(\sqrt{\lambda} L) = \sqrt{\lambda} L$

Using MATLAB to solve $\tan(x) = x$,

```
clear
syms x
s = vpasolve(tan(x)==x,x,4)
figure
hold on
fplot(tan(x))
fplot(x)
grid on
axis([0,2*pi,0,10])
```

s = 4.4934094579090641753078809272803



$\therefore \sqrt{\lambda} L \approx 4.4934$, $\lambda \approx \frac{20.19}{L^2}$ is the smallest λ.

$y(x) = c_2 x + c_4 \sin(\sqrt{\lambda} x)$

From [62], $c_2 = -c_4 \sqrt{\lambda} \cos(\sqrt{\lambda} L)$

$\therefore y(x) = c_4 \left[\sin(\sqrt{\lambda} x) - x \sqrt{\lambda} \cos(\sqrt{\lambda} L \right]$

$\therefore$ The fundamental solution, up to a multiplicative constant is:

$$\boxed{y(x) = \sin(\sqrt{\lambda} x) - x \sqrt{\lambda} \cos(\sqrt{\lambda} L), \quad \lambda \approx \frac{20.19}{L^2}}$$

$\lambda < 0$: $y(x) = c_1 + c_2 x + c_3 e^{\sqrt{-\lambda} x} + c_4 e^{-\sqrt{-\lambda} x}$

$y'(x) = c_2 + c_3 \sqrt{-\lambda} e^{\sqrt{-\lambda} x} - c_4 \sqrt{-\lambda} e^{-\sqrt{-\lambda} x}$

$y''(x) = -c_3 \lambda e^{\sqrt{-\lambda} x} - c_4 \lambda e^{-\sqrt{-\lambda} x}$

$y(0) = 0 \Rightarrow c_1 + c_3 + c_4 = 0$

$y''(0) = 0 \Rightarrow -c_3 - c_4 = 0$. $\therefore \underline{c_1 = 0}$, $c_4 = -c_3$

$y(L) = 0 \Rightarrow c_2 L + c_3 e^{\sqrt{-\lambda} L} - c_3 e^{-\sqrt{-\lambda} L} = 0$ [63]

$y'(L) = 0 \Rightarrow c_2 + c_3 \sqrt{-\lambda} e^{\sqrt{-\lambda} L} + c_3 \sqrt{-\lambda} e^{-\sqrt{-\lambda} L} = 0$ [64]

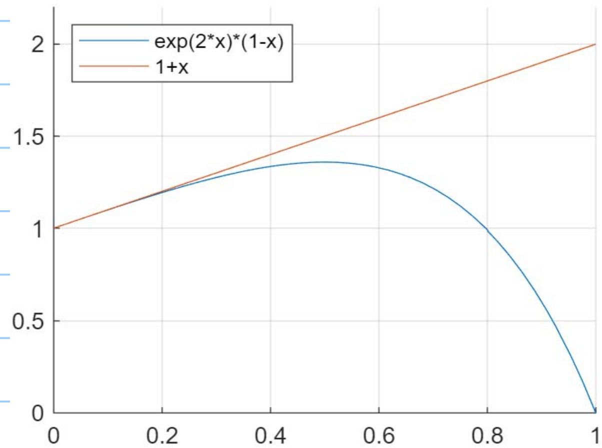
Adding $-L[64] + [63]$ yields

$$c_3 e^{\sqrt{-\lambda}L}(1 - L\sqrt{-\lambda}) - c_3 e^{-\sqrt{-\lambda}L}(1 + L\sqrt{-\lambda}) = 0$$

if $c_3 \neq 0$, $\quad e^{2\sqrt{-\lambda}L}(1 - L\sqrt{-\lambda}) = 1 + L\sqrt{-\lambda} \quad [65]$

However, as shown with MATLAB, equality of [65] only holds for $L\sqrt{-\lambda} = 0$, which is impossible.

```
clear
syms x
figure
hold on
fplot(exp(2*x)*(1-x))
fplot(1+x)
grid on
axis([0,1,0,2.2])
legend('exp(2*x)*(1-x)','1+x',...
       'Location','northwest')
```



$\therefore\ c_3 = 0$, and $\therefore\ c_4 = 0$, and $\therefore\ c_2 = 0$.

$\therefore\ \lambda < 0$ is <u>not</u> an eigenvalue.

Note: $\dfrac{d}{dx}\left[e^{2x}(1-x)\right]\Big|_{x=0} = e^{2x}(1-2x)\Big|_{x=0} = 1$

$\dfrac{d^2}{dx^2}\left[e^{2x}(1-x)\right] = -4xe^{2x}$, so for $x > 0$,

the second derivative is negative, so the slope of $e^{2x}(1-x)$ decreases from 1 at $x=0$.

(c)

$\lambda=0$: $y(x) = c_1x^3 + c_2x^2 + c_3x + c_4$

$y(0)=0 \Rightarrow \underline{c_4=0}$ $\quad y'(0)=0 \Rightarrow \underline{c_3=0}$

$\therefore y(x) = c_1x^3 + c_2x^2$ $\quad y(L)=0 \Rightarrow c_1L + c_2 = 0$

$y'(L)=0 \Rightarrow 3c_1L + 2c_2 = 0$ $\quad 3c_1L + 3c_2 = 0$

Subtracting, $\underline{c_2=0}$, $\therefore \underline{c_1=0}$

$\therefore$ $\underline{\lambda=0}$ $\underline{\text{not}}$ an eigenvalue.

$\lambda>0$: $y(x) = c_1 + c_2x + c_3\cos(\sqrt{\lambda}x) + c_4\sin(\sqrt{\lambda}x)$

$y'(x) = c_2 - c_3\sqrt{\lambda}\sin(\sqrt{\lambda}x) + c_4\sqrt{\lambda}\cos(\sqrt{\lambda}x)$

$y(0)=0 \Rightarrow c_1 + c_3 = 0$ $\quad y'(0)=0 \Rightarrow c_2 + \sqrt{\lambda}c_4 = 0$ [c1]

$y(L)=0 \Rightarrow c_1 - \sqrt{\lambda}Lc_4 - c_1\cos(\sqrt{\lambda}L) + c_4\sin(\sqrt{\lambda}L) = 0$

$\therefore c_1(1-\cos(\sqrt{\lambda}L)) = c_4(\sqrt{\lambda}L - \sin(\sqrt{\lambda}L))$ [c2]

$y'(L)=0 \Rightarrow -\sqrt{\lambda}c_4 + c_1\sqrt{\lambda}\sin(\sqrt{\lambda}L) + c_4\sqrt{\lambda}\cos(\sqrt{\lambda}L) = 0$

or $-c_4 + c_1\sin(\sqrt{\lambda}L) + c_4\cos(\sqrt{\lambda}L) = 0$

$$\therefore\ c_1 \sin(\sqrt{\lambda}\, L) = c_4\left[1 - \cos(\sqrt{\lambda}\, L)\right] \quad [c3]$$

From [c1], $c_1 = 0 \Rightarrow c_3 = 0$, $c_4 = 0 \Rightarrow c_2 = 0$

From [c3], if $c_1 = 0$, then for nontrivial solution,

$1 = \cos(\sqrt{\lambda}\, L)$, so $\sqrt{\lambda} = n(2\pi)$, $n = 1, 2, 3, \ldots$

But then from [c2], $0 = c_4(n 2\pi)$, so $c_4 = 0$

yielding a trivial solution.

$\therefore\ \underline{c_1 \neq 0}$

From [c3], if $\sqrt{\lambda}\, L = (2n-1)\pi$, $n = 1, 2, 3, \ldots$, then

$0 = 2c_4 \Rightarrow c_4 = 0$. If $\sqrt{\lambda}\, L = n 2\pi$, $n = 1, 2, 3, \ldots$, then

[c2] $\Rightarrow 0 = c_4(n 2\pi) \Rightarrow c_4 = 0$.

$\therefore$ Suppose $c_4 \neq 0$. $\therefore \sin(\sqrt{\lambda}\, L) \neq 0$ because

otherwise $\sqrt{\lambda}\, L = n\pi \Rightarrow c_4 = 0$.

$\therefore$ From [c3], $c_1 = \dfrac{c_4\left[1 - \cos(\sqrt{\lambda}\, L)\right]}{\sin(\sqrt{\lambda}\, L)}$ [c4]

Substituting into [c2],

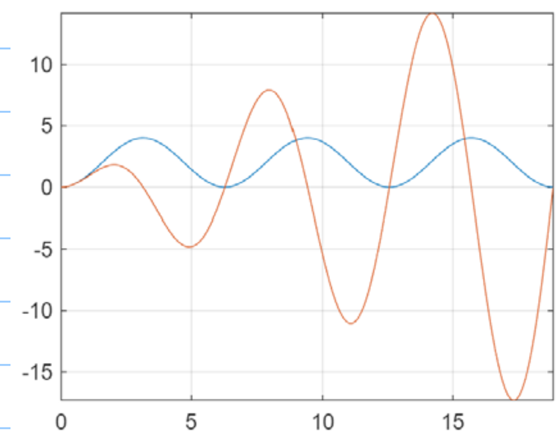
$$c_4\left[1 - \cos(\sqrt{\lambda}\, L)\right]^2 = c_4\left[\sqrt{\lambda}\, L \sin(\sqrt{\lambda}\, L) - \sin^2(\sqrt{\lambda}\, L)\right]$$

Since $c_4 \neq 0$, and using $\sin^2(\sqrt{\lambda}L) + \cos^2(\sqrt{\lambda}L) = 1$,

$$2 - 2\cos(\sqrt{\lambda}L) = \sqrt{\lambda}L\sin(\sqrt{\lambda}L)$$

One set of solutions is $n2\pi$, $n = 1, 2, 3, \ldots$, a contradiction to $c_4 \neq 0$. Other solutions are found via MATLAB

```
clear
syms x
fplot(2-2*cos(x))
hold on
fplot(x*sin(x))
grid on
xlim([0,6*pi])
vpasolve(2-2*cos(x) == x*sin(x),x,3*pi - 0.1)
```




```
ans = 8.9868189158181283506157618545606
```

$\therefore$ The smallest value is $\sqrt{\lambda}L \approx 8.9868$ (which is greater than 2π), the smallest positive zero.

$$\therefore \lambda \approx \frac{(8.9868)^2}{L^2} = \frac{80.7626}{L^2}$$

Assigning $c_1 = 1$, then $c_3 = -1$, $c_4 = 0.2226$ from [c4], and $c_2 = -\sqrt{\lambda}c_4 = -\frac{(8.9868)}{L}(0.2226) = -2.0001/L$

$$\therefore y(x) = 1 - \frac{2.0001x}{L} - \cos\left(\frac{8.9868x}{L}\right) + 0.2226\sin\left(\frac{8.9868x}{L}\right)$$

If $c_4=0$, then from [c1] $c_2=0$ and from [c2], $1-\cos(\sqrt{\lambda}L)=0$. $\therefore \sqrt{\lambda}L = n2\pi$, $n=1,2,3,\ldots$

[c3] is also satisfied. Smallest eigenvalue is

$$\boxed{\lambda = \frac{4\pi^2}{L^2}, \text{ and } y(x) = 1-\cos\left(\frac{2\pi x}{L}\right)}$$

Note that this is the smallest eigenvalue compared to the one assuming $c_4 \neq 0$.

$\lambda<0$: $y(x) = c_1 + c_2 x + c_3 e^{\sqrt{-\lambda}x} + c_4 e^{-\sqrt{-\lambda}x}$

$$y'(x) = c_2 + c_3\sqrt{-\lambda}\, e^{\sqrt{-\lambda}x} - c_4\sqrt{-\lambda}\, e^{-\sqrt{-\lambda}x}$$

$$y(0)=0 \Rightarrow c_1 + c_3 + c_4 = 0$$

$$y'(0)=0 \Rightarrow c_2 + c_3\sqrt{-\lambda} - c_4\sqrt{-\lambda} = 0$$

$$y(L)=0 \Rightarrow c_1 + c_2 L + c_3 e^{\sqrt{-\lambda}L} + c_4 e^{-\sqrt{-\lambda}L} = 0$$

$$y'(L)=0 \Rightarrow c_2 + c_3\sqrt{-\lambda}\, e^{\sqrt{-\lambda}L} - c_4\sqrt{-\lambda}\, e^{-\sqrt{-\lambda}L} = 0$$

Or,

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & a & -a \\ 1 & L & e^{aL} & e^{-aL} \\ 0 & 1 & ae^{aL} & -ae^{-aL} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Using MATLAB,

```
clear
syms a L
A = [1, 0,        1,            1;
     0, 1,        a,           -a;
     1, L,    exp(a*L),     exp(-a*L);
     0, 1, a*exp(a*L), -a*exp(-a*L)]
rrefA = rref(A)
B = [0,0,0,0]';
C = linsolve(A,B)
```

A =

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & a & -a \\ 1 & L & e^{La} & e^{-La} \\ 0 & 1 & a\,e^{La} & -a\,e^{-La} \end{pmatrix}$$

rrefA =

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

C =

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$\therefore$ $c_1 = c_2 = c_3 = c_4 = 0$, and the only solution is the trivial solution. $\therefore$ $\underline{\lambda < 0}$ is $\underline{\text{not}}$ an eigenvalue.

26.

As with #25,

$\lambda = 0 \Rightarrow y^{(4)} = 0$, $y(x) = c_1 x^3 + c_2 x^2 + c_3 x + c_4$

$\lambda > 0$: let $y = e^{rx}$. $\therefore$ $r^4 + \lambda r^2 = r^2(r^2 + \lambda) = 0 \Rightarrow r = 0, 0, \pm i\sqrt{\lambda}$

repeated roots

$\therefore$ $y(x) = c_1 + c_2 x + c_3 \cos(\sqrt{\lambda}\, x) + c_4 \sin(\sqrt{\lambda}\, x)$

$\lambda < 0$: let $y = e^{rx}$, $-\mu = \lambda$, $\mu > 0$. $\therefore$ $r^4 - \mu r^2 = r^2(r^2 - \mu) = 0$

$\Rightarrow r = 0, 0, \pm\sqrt{\mu}$. $\therefore$ $y(x) = c_1 + c_2 x + c_3 e^{\sqrt{-\lambda}\, x} + c_4 e^{-\sqrt{-\lambda}\, x}$

repeated roots

$\lambda = 0$: $y(0) = 0 \Rightarrow \underline{c_4 = 0}$. $y'(0) = 0 \Rightarrow \underline{c_3 = 0}$

$y'''(L) + \lambda y'(L) = 6c_1 = 0 \Rightarrow \underline{c_1 = 0}$. $\therefore y(x) = c_2 x^2$

$y''(L) = 0 \Rightarrow 2c_2 = 0 \Rightarrow \underline{c_2 = 0}$

$\therefore$ Trivial solution. $\therefore \lambda = 0$ is <u>not</u> an eigenvalue.

$\lambda < 0$: $y(0) = 0 \Rightarrow c_1 + c_3 + c_4 = 0$ [1]

$y'(0) = 0 \Rightarrow c_2 + \sqrt{-\lambda}\, c_3 - \sqrt{-\lambda}\, c_4 = 0$ [2]

$y''(L) = 0 \Rightarrow -\lambda c_3 e^{\sqrt{-\lambda} L} - \lambda c_4 e^{-\sqrt{-\lambda} L} = 0$ [3]

$$y'''(L) + \lambda y'(L) = 0 \Rightarrow -\lambda\sqrt{-\lambda}\, c_3 e^{\sqrt{-\lambda} L} + \lambda\sqrt{-\lambda}\, c_4 e^{-\sqrt{-\lambda} L}$$
$$+ \lambda c_2 + \lambda\sqrt{-\lambda}\, c_3 e^{\sqrt{-\lambda} L} - \lambda\sqrt{-\lambda}\, c_4 e^{-\sqrt{-\lambda} L} = 0$$

or, $\lambda c_2 = 0 \Rightarrow \underline{c_2 = 0}$

$\therefore$ [2] becomes $c_3 - c_4 = 0$, or $c_3 = c_4$

[3] reduces to $c_3 e^{2\sqrt{-\lambda} L} + c_4 = 0$, $c_3(e^{2\sqrt{-\lambda} L} + 1) = 0$,

so $\therefore \underline{c_3 = 0} \Rightarrow \underline{c_4 = 0}$. $\therefore$ From [1], $\underline{c_1 = 0}$

$\therefore$ Trivial solution. $\therefore \lambda < 0$ is <u>not</u> an eigenvalue.

$\lambda > 0$: $y(x) = c_1 + c_2 x + c_3 \cos(\sqrt{\lambda}\, x) + c_4 \sin(\sqrt{\lambda}\, x)$

$y(0)=0 \Rightarrow c_1 + c_3 = 0 \quad \therefore \quad c_3 = -c_1$

$y'(0)=0 \Rightarrow c_2 + \sqrt{\lambda}\, c_4 = 0 \quad \therefore \quad c_2 = -\sqrt{\lambda}\, c_4$

$\therefore \; y(x) = c_1 - \sqrt{\lambda}\, c_4 x - c_1 \cos(\sqrt{\lambda}\, x) + c_4 \sin(\sqrt{\lambda}\, x) \quad [4]$

$y''(L) = c_1 \lambda \cos(\sqrt{\lambda} L) - c_4 \lambda \sin(\sqrt{\lambda} L) = 0 \quad [5]$

$y'''(L) = -c_1 \lambda \sqrt{\lambda} \sin(\sqrt{\lambda} L) - c_4 \lambda \sqrt{\lambda} \cos(\sqrt{\lambda} L)$

$y'(L) = -\sqrt{\lambda}\, c_4 + c_1 \sqrt{\lambda} \sin(\sqrt{\lambda} L) + c_4 \sqrt{\lambda} \cos(\sqrt{\lambda} L)$

$\therefore \; y'''(L) + \lambda y'(L) = 0 \Rightarrow -\lambda \sqrt{\lambda}\, c_4 = 0 \Rightarrow \underline{c_4 = 0}$

$\therefore$ [4] becomes $y(x) = c_1 - c_1 \cos(\sqrt{\lambda}\, x)$

[5] becomes $c_1 \lambda \cos(\sqrt{\lambda} L) = 0$. For nontrivial

solution, $c_1 \neq 0$. $\therefore \cos(\sqrt{\lambda} L) = 0 \Rightarrow \sqrt{\lambda} L = (2n-1)\frac{\pi}{2}$

for $n = 0, 1, 2, \ldots$ Smallest value is $n=0$.

$\therefore \sqrt{\lambda} = \frac{\pi}{2L}$. $\therefore \lambda = \frac{\pi^2}{4L^2}$, $\underline{y(x) = 1 - \cos\left(\frac{\pi x}{2L}\right)}$

27.

(a)

Let $c(x,t) = X(x)T(t)$. $\therefore$ $c_t + v c_x = D c_{xx}$ becomes

$XT' + vX'T = DX''T$. Dividing by XT,

$$\frac{T'}{T} + v\frac{X'}{X} = D\frac{X''}{X}, \text{ or } D\frac{X''}{X} - v\frac{X'}{X} = \frac{T'}{T}$$

The left side is a function of x only and the right side is a function of t only, and so each side is equal to the same real constant. To get an equation in which the coefficient of X'' is one, divide by Δ. $\therefore \frac{X''}{X} - \left(\frac{v}{\Delta}\right)\frac{X'}{X} = \frac{T'}{\Delta T} = -\lambda$, λ some real constant.

$$\therefore \quad \underline{X'' - \left(\frac{v}{\Delta}\right)X' + \lambda X = 0} \quad \text{and} \quad \underline{T' + \lambda\Delta T = 0}$$

For the boundary conditions, $X(0)T(t)=0$, $X'(L)T(t)=0$. Disallowing the trivial solution of $c(x,t)=0$ for all t, $T(t) \neq 0$ for all t, and so $\underline{X(0)=0}$, $\underline{X'(L)=0}$.

Let $u = e^{-(v/\Delta)x}$. Multiply the equation in x by this factor to get $e^{-(v/\Delta)x}X'' - (v/\Delta)e^{-(v/\Delta)x}X' + \lambda e^{-(v/\Delta)x}X = 0$

This can be written as $\left(e^{-(v/\Delta)x}X'(x)\right)' + \lambda e^{-(v/\Delta)x}X = 0$

Letting $p(x) = e^{-(v/\Delta)x}$ and $r(x) = e^{-(v/\Delta)x}$, this can be written as $\underline{(p(x)X')' + \lambda r(x)X = 0}$, the Sturm-Liouville form.

(6)

Try to convert $X'' - \left(\frac{v}{D}\right)X' + \lambda X = 0$ to the more usual form: $y'' + \lambda y = 0$. From $X'' - \left(\frac{v}{D}\right)X' + \left(\mu^2 + \frac{v^2}{4D^2}\right)X = 0$ [1] use approach of #17 section 11.1. Let $X = s(x)u(x)$, and try to get a form of $u'' + (\text{constant})u = 0$.

Since $u(x) = \frac{X(x)}{s(x)}$, $u(0) = \frac{X(0)}{s(0)} = 0$. $\therefore$ Need to find an $s(x)$ s.t. $s(0) \neq 0$.

$X' = s'u + su'$, $X'' = s''u + 2s'u' + su''$. $\therefore$ [1] becomes

$$s''u + 2s'u' + su'' - \left(\frac{v}{D}\right)s'u - \left(\frac{v}{D}\right)su' + \left(\mu^2 + \frac{v^2}{4D^2}\right)su = 0$$

Dividing by su,

$$\frac{s''}{s} + \frac{2s'u'}{su} + \frac{u''}{u} - \left(\frac{v}{D}\right)\frac{s'}{s} - \left(\frac{v}{D}\right)\frac{u'}{u} + \left(\mu^2 + \frac{v^2}{4D^2}\right) = 0 \quad [2]$$

To get to the $u'' + (\ )u = 0$ form, need to get rid of the terms with u'. $\therefore$ set $2\frac{s'u'}{su} - \left(\frac{v}{D}\right)\frac{u'}{u} = 0$

This will be true if $2\frac{s'}{s} = \frac{v}{D}$, or $\frac{s'}{s} = \frac{v}{2D}$

$\therefore$ Let $\underline{s(x) = e^{\frac{vx}{2D}}}$. $\therefore$ [2] becomes

$$\frac{u''}{u} + \frac{s''}{s} - \left(\frac{v}{D}\right)\frac{s'}{s} + \left(\mu^2 + \frac{v^2}{4D^2}\right) = 0, \text{ or}$$

$$\frac{u''}{u} + \frac{v^2}{4D^2} - \frac{v^2}{2D^2} + \lambda = 0, \text{ or } \frac{u''}{u} + \lambda - \frac{v^2}{4D^2} = 0, \text{ or}$$

$$u'' + \left(\lambda - \frac{v^2}{4D^2}\right)u = 0 \quad , \quad u(0) = 0, \qquad [3]$$

$$u'(L) = \frac{s(L)\underbrace{X'(L)}_{=0} - X(L)s'(L)}{s(L)^2} = -\frac{X(L)s'(L)}{s(L)^2} = -\frac{s(L)u(L)s'(L)}{s(L)^2}$$

$$= -\frac{u(L)s'(L)}{s(L)} = -\left(\frac{v}{2D}\right)u(L)$$

$$\therefore u'(L) = -\frac{v}{2D}u(L) \qquad [4]$$

Now evaluate $u'' + \mu^2 u = 0$ for $\mu^2 = 0, > 0, < 0,$

where $\mu^2 = \lambda - \frac{v^2}{4D^2}$, and $\mu > 0$.

$\mu^2 = 0$: $\therefore u(x) = c_1 x + c_2 \quad u(0) = 0 \Rightarrow c_2 = 0$

$$u(L) = c_1 L, \quad u'(L) = c_1 = -\frac{v}{2D}(c_1 L) = -\frac{vL}{2D}c_1$$

$\therefore c_1\left(1 + \frac{vL}{2D}\right) = 0 \Rightarrow c_1 = 0$ since $v, L, D > 0$.

$\therefore \mu^2 = 0$, or $\lambda = \frac{v^2}{4D^2}$ is <u>not</u> an eigenvalue.

$\mu^2 < 0$: Let $-\omega^2 = \mu^2$, where $\omega > 0$. $\therefore u'' - \omega^2 u = 0$

$\therefore u(x) = c_1 \cosh(\omega x) + c_2 \sinh(\omega x)$.

$u(0) = 0 \Rightarrow \underline{c_1 = 0}$. $\therefore u(L) = c_2 \sinh(\omega L)$

$u'(L) = -\frac{V}{2D}[c_2 \sinh(\omega L)]$ and $u'(x) = c_2 \omega \cosh(\omega x)$

$\therefore u'(L) = c_2 \omega \cosh(\omega L)$

$\therefore -\frac{V}{2D}[c_2 \sinh(\omega L)] = c_2 \omega \cosh(\omega L)$

If $c_2 \neq 0$, then $-\frac{V}{2D} \sinh(\omega L) = \omega \cosh(\omega L)$

But $\omega > 0$ and $\cosh(\omega L) > 0$, so $\omega \cosh(\omega L) > 0$

But $\sinh(\omega L) > 0$ since $\omega > 0$ and $L > 0$.

$\therefore -\frac{V}{2D} \sinh(\omega L) < 0$, so $-\frac{V}{2D} \sinh(\omega L) \neq \omega \cosh(\omega L)$

$\therefore \underline{\underline{c_2 = 0}}$.

$\therefore \mu^2 < 0$ is <u>not</u> an eigenvalue

$\mu^2 > 0$: $\therefore u(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$

$u(0) = 0 \Rightarrow c_1 = 0$ $\quad u(L) = c_2 \sin(\mu L)$

$u'(x) = c_2 \mu \cos(\mu x)$, so $u'(L) = c_2 \mu \cos(\mu L)$

But $u'(L) = -\frac{V}{2D} u(L) = -c_2 \frac{V}{2D} \sin(\mu L)$

$\therefore -c_2 \frac{V}{2D} \sin(\mu L) = c_2 \mu \cos(\mu L)$

For nontrivial solution, $c_2 \neq 0$.

$\therefore \mu \cos(\mu L) = -\frac{V}{2D} \sin(\mu L)$ [5]

If $\cos(\mu L) = 0$, then [5] $\Rightarrow \sin(\mu L) = 0$

But $\cos(\mu L)$ and $\sin(\mu L)$ can't both be zero. $\therefore \cos(\mu L) \neq 0$.

$\therefore -\frac{2D\mu}{V} = \tan(\mu L)$

With that μ, $u(x) = \sin(\mu x)$ solves $u'' + \mu^2 u = 0$ up to a multiplicative constant.

$\therefore X(x) = s(x)u(x) = \underline{e^{Vx/2D} \sin(\mu x)}$, where

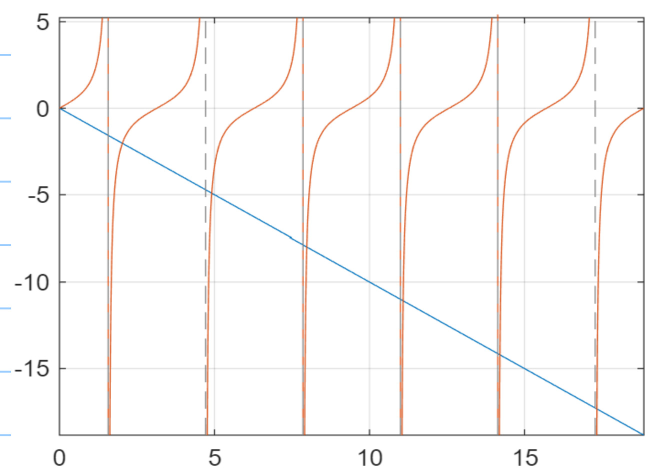
$-\frac{2D\mu}{V} = \underline{\tan(\mu L)}$

(c)

Let $x = \mu L$. $\therefore \mu = \frac{x}{L}$. $\therefore$ Graph $\tan(x) = -\frac{2D}{VL}x$,

or $\tan(x) = -mx$, $m > 0$. Using MATLAB,

```
clear
syms x
m = 1;   % arbitrary
y1 = -m*x;
y2 = tan(x);
fplot(y1)
hold on
fplot(y2)
xlim([0,6*pi])
grid on
```



No matter what value for $m > 0$, the line $y = -mx$ intersects $y = \tan(x)$ an infinite number of times for $x > 0$, and as $x > 0$, the intersection point is $x_n \approx (2n-1)\frac{\pi}{2}$, one of the asymptotes for $\tan(x)$.

$\therefore \mu_n = \frac{x_n}{L} \approx (2n-1)\frac{\pi}{2L}$ for large n.

(d)

From (a), $r(x) = e^{-Vx/D}$. From (b), $X_n(x) = e^{Vx/2D}\sin(\mu_n x)$

$$\therefore X_n^2(x) = e^{vx/D}\sin^2(\mu_n x). \quad \therefore r(x)X_n^2(x) = \sin^2(\mu_n x)$$

$$\therefore \int_0^L r(x)\, X_n^2(x)\,dx = \int_0^L \sin^2(\mu_n x)\,dx = \tfrac{1}{2}x - \tfrac{1}{4\mu_n}\sin(2\mu_n x)\Big|_0^L$$

$$= \frac{1}{2}L - \frac{1}{4\mu_n}\underbrace{\sin(2\mu_n L)}_{= 2\sin(\mu_n L)\cos(\mu_n L)}$$

$$= \frac{L}{2} - \frac{1}{4\mu_n}\sin(\mu_n L)\left[2\cos(\mu_n L)\right]$$

$-2\frac{D\mu}{v} = \tan(\mu L) = \frac{\sin(\mu L)}{\cos(\mu L)}$

$\Rightarrow 2\cos(\mu L) = -\frac{v\sin(\mu L)}{D\mu}$

$$= \frac{L}{2} - \frac{1}{4\mu_n}\sin(\mu_n L)\left[-\frac{v\sin(\mu_n L)}{D\mu_n}\right]$$

$$\therefore \boxed{\int_0^L r(x)\, X_n^2(x)\,dx = \frac{L}{2} + \frac{v}{4D\mu_n^2}\sin^2(\mu_n L)}$$

(e)

From (a), $T' + \lambda D T = 0$. $\therefore T(t) = e^{-\lambda D t}$ is a fundamental solution. $\therefore C_n(x,t) = e^{-\lambda_n D t} e^{vx/2D}\sin(\mu_n x)$

where $\lambda_n = \mu_n^2 + \frac{v^2}{4D^2}$ and $-\frac{2D\mu_n}{v} = \tan(\mu_n L)$

$$\therefore c(x,t) = \sum_{n=1}^{\infty} a_n C_n(x,t) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n D t} e^{vx/2D}\sin(\mu_n x) \quad [6]$$

$$\therefore c(x,0) = f(x) = \sum_{n=1}^{\infty} a_n e^{vx/2D}\sin(\mu_n x) = \sum_{n=1}^{\infty} a_n X_n(x)$$

Multiplying by $r(x)\, X_m(x)$ and integrating,

$$\int_0^L f(x)\, r(x)\, X_m(x)\, dx = \int_0^L \sum_{n=1}^{\infty} a_n r(x)\, X_m(x)\, X_n(x)\, dx$$

$$= \sum_{n=1}^{\infty} a_n \int_0^L r(x)\, X_m(x)\, X_n(x)\, dx$$

orthogonality $\Rightarrow$ for $m \neq n$, $\int_0^L r(x) X_m(x) X_n(x) dx = 0$

$$= a_m \int_0^L r(x)\, X_m^2(x)\, dx$$

Note that Theorems 11.2.1, 11.2.2, 11.2.3 do not depend on the interval of $[0,1]$. $0 \le x \le L$ yields the same results. Using (d) above,

$$\therefore \int_0^L f(x) \underbrace{r(x)}_{= e^{-vx/D}} \underbrace{X_m(x)}_{= e^{vx/2D}\sin(\mu_m x)}\, dx = a_m \left[\frac{L}{2} + \frac{v}{4D\mu_m^2} \sin^2(\mu_m L) \right]$$

$$\therefore \int_0^L f(x)\, e^{-vx/2D} \sin(\mu_m x)\, dx = a_m \left[\frac{2DL\mu_m^2 + v\sin^2(\mu_m L)}{4D\mu_m^2} \right]$$

$\therefore$

$$\boxed{\begin{gathered} c(x,t) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n D t} e^{vx/2D} \sin(\mu_n x), \\ \text{where } \lambda_n = \mu_n^2 + \frac{v^2}{4D^2},\quad -\frac{2D}{v}\mu_n = \tan(\mu_n L), \\ \text{and } a_n = \frac{4D\mu_n^2 \int_0^L f(x)\, e^{-vx/2D} \sin(\mu_n x)\, dx}{2DL\mu_n^2 + v\sin^2(\mu_n L)} \end{gathered}}$$

(f)

For $n = 1, 2, 3, \ldots$, first determine μ_n from $-\frac{2D}{V}\mu_n = \tan(\mu_n L)$

From (c), $\mu_n \approx (2n-1)\frac{\pi}{2L}$. $\therefore$ Use MATLAB's "vpasolve"

to estimate μ_n using $(2n-1)\frac{\pi}{2L} + \frac{0.1}{n^2}$ as a seed.

Then compute $\lambda_n = \mu_n^2 + \frac{V^2}{4D^2}$.

To compute a_n, first note that, from section 6.5,

since $0 < 3 < L$, $\int_{-\infty}^{\infty} \delta(x-3)\, g(x)\, dx = \int_0^L \delta(x-3)\, g(x)\, dx =$

$g(3)$. $\therefore$ for $f(x) = \delta(x-3)$, $\int_0^L \delta(x-3)\, e^{-Vx/2D} \sin(\mu_n x)\, dx =$

$e^{-3V/2D} \sin(3\mu_n)$. $\therefore$ $a_n = \dfrac{4D\mu_n^2 e^{-3V/2D} \sin(3\mu_n)}{2DL\mu_n^2 + V\sin^2(\mu_n L)}$

$\therefore$ With μ_n, λ_n, a_n determined, $c(x,t)$ can be

computed using $\sum_{n=1}^{N} a_n e^{-\lambda_n D t} e^{Vx/2D} \sin(\mu_n x)$,

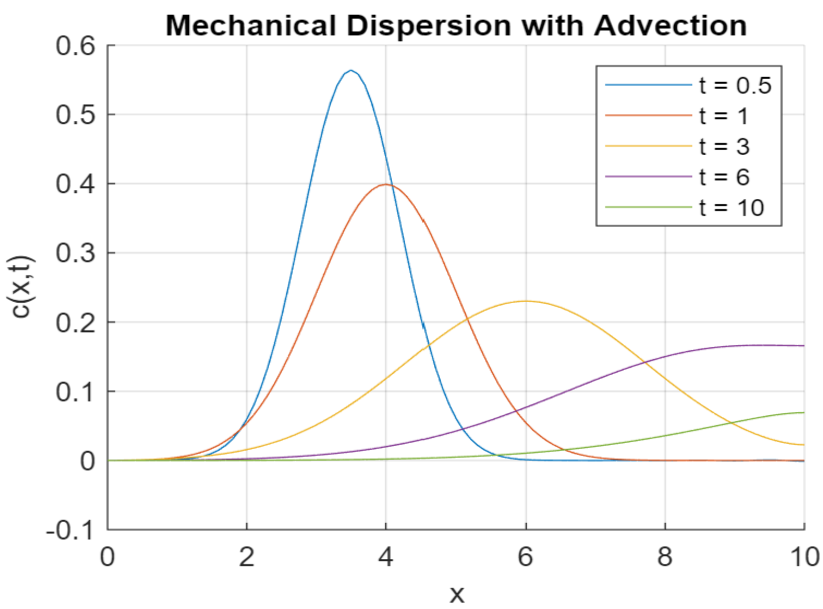
where N determines the number of terms.

First plot $c(x,t)$ vs. x for several t values.

Then plot $c(x,t)$ vs. t for several x values.

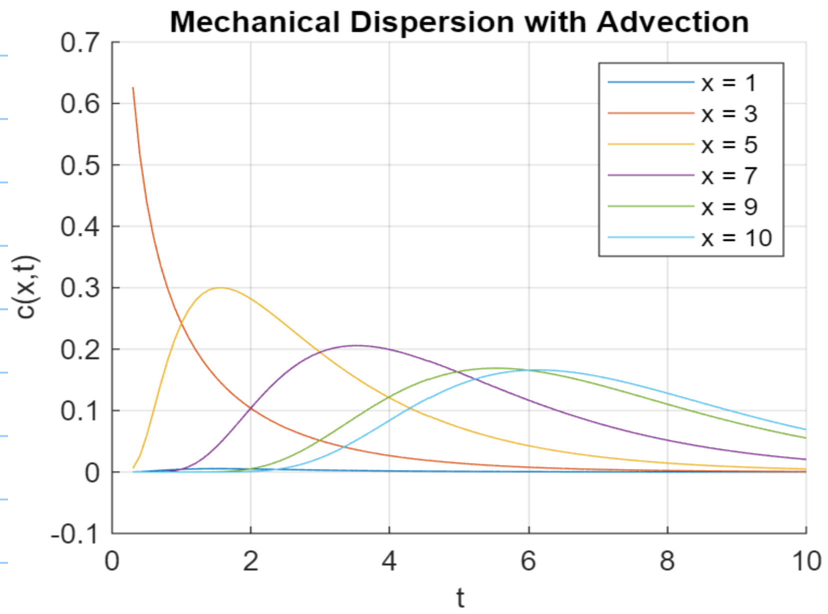
Using MATLAB,

```
clear
v = 1; D = 0.5; L = 10;
x = 0:0.1:L;
T = [0.5,1,3,6,10];
N = 20;  % number of terms
syms u
y1 = -2*D*u/v;
y2 = tan(u*L);
figure
hold on
for t = T
    c_sum = zeros(size(x,1),1);  % initialize
    for n = 1:N
        seed = (2*n-1)*pi/(2*L) + 0.1/n^2;
        u_n = vpasolve(y1==y2,u,seed);
        l_n = u_n^2 + v^2/(4*D^2);  % lambda
          num = 4*D*u_n^2*exp(-3*v/(2*D))*sin(3*u_n);
          den = 2*D*L*u_n^2 + v*(sin(u_n*L))^2;
        a_n = num/den;
        c_n = a_n*exp(-l_n*D*t)*exp(v*x/(2*D)).*sin(u_n*x);
        c_sum = c_sum + c_n;
    end
    plot(x,c_sum)
end
grid on
xlabel('x'); ylabel('c(x,t)')
title('Mechanical Dispersion with Advection')
legend('t = 0.5','t = 1','t = 3','t = 6','t = 10')
```



Now plot c(x,t) vs. t for several x values.

```
clear
v = 1; D = 0.5; L = 10;
t = 0.3:0.1:10;
X = [1,3,5,7,9,10];
N = 30;  % number of terms
syms u
y1 = -2*D*u/v;
y2 = tan(u*L);
figure
hold on
for x = X
    c_sum = zeros(size(t,1),1);  % initialize
    for n = 1:N
        seed = (2*n-1)*pi/(2*L) + 0.1/n^2;
        u_n = vpasolve(y1==y2,u,seed);
        l_n = u_n^2 + v^2/(4*D^2);  % lambda
          num = 4*D*u_n^2*exp(-3*v/(2*D))*sin(3*u_n);
          den = 2*D*L*u_n^2 + v*(sin(u_n*L))^2;
        a_n = num/den;
        c_n = a_n*exp(-l_n*D*t)*exp(v*x/(2*D))*sin(u_n*x);
        c_sum = c_sum + c_n;
    end
    plot(t,c_sum)
end
grid on
xlabel('t'); ylabel('c(x,t)')
title('Mechanical Dispersion with Advection')
legend('x = 1','x = 3','x = 5','x = 7','x = 9','x = 10')
```



(g)

As time evolves, the solution gradually moves to the right, due to $v=1$, and also disperses, as seen by the maximum "hump" broadening out, and decreasing in height. With time, the concentration decreases and the solute moves to the right.

28.

(a)

Using $c(x,t) = C_0 + u(x,t)$, $c_t = u_t$, $c_x = u_x$, $c_{xx} = u_{xx}$

as C_0 is a constant. $\therefore$ $c_t + Vc_x = D c_{xx} \Rightarrow u_t + Vu_x = Du_{xx}$

$c(0,t) = c_0 + u(0,t) = c_0 \Rightarrow u(0,t) = 0$

$c_x(L,t) = u_x(L,t) = 0$

$$c(x,0) = c_0 + u(x,0) = 0 \implies u(x,0) = -c_0$$

$$\therefore \boxed{\begin{array}{l} u_t + vu_x = Du_{xx}, \quad 0<x<L, \ t>0 \\ u(0,t) = 0, \ t>0 \qquad u_x(L,t) = 0, \ t>0 \\ u(x,0) = -c_0, \quad 0<x<L \end{array}}$$

(b)

This is identical to #27 above, where $f(x)$ in #27 is the constant $-c_0$ in this case. $\therefore$ The arguments in #27(a)-(e) are identical, so that

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n D t} e^{vx/2D} \sin(\mu_n x),$$

where $\lambda_n = \mu_n^2 + \frac{v^2}{4D^2}$, $-\frac{2D}{v}\mu_n = \tan(\mu_n L)$,

and $a_n = \dfrac{4D\mu_n^2 \int_0^L (-c_0) e^{-vx/2D} \sin(\mu_n x)\,dx}{2DL\mu_n^2 + v\sin^2(\mu_n L)}$

$$= \frac{-4D\mu_n^2 c_0 \int_0^L e^{-vx/2D} \sin(\mu_n x)\,dx}{2DL\mu_n^2 + v\sin^2(\mu_n L)}$$

The integral can be evaluated. Using a table of integrals: $\int e^{au}\sin bu\,du = \frac{e^{au}}{a^2+b^2}(a\sin bu - b\cos bu) + C$

$$\therefore \int_0^L e^{-vx/2D}\sin(\mu_n x)\,dx = \frac{e^{-vx/2D}}{\frac{v^2}{4D^2}+\mu_n^2}\left[-\frac{v}{2D}\sin(\mu_n x) - \mu_n\cos(\mu_n x)\right]_0^L$$

$$= \frac{e^{-vL/2D}}{\frac{v^2}{4D^2}+\mu_n^2}\left[-\frac{v}{2D}\sin(\mu_n L) - \mu_n\cos(\mu_n L)\right] + \frac{\mu_n}{\frac{v^2}{4D^2}+\mu_n^2}$$

$$= \frac{\frac{2D\mu_n}{2D} - e^{-vL/2D}\left[\frac{v\sin(\mu_n L) + 2D\mu_n\cos(\mu_n L)}{2D}\right]}{\frac{v^2+4D^2\mu_n^2}{4D^2}}$$

$$= \frac{8D^3\mu_n - 4D^2 e^{-vL/2D}\left[v\sin(\mu_n L) + 2D\mu_n\cos(\mu_n L)\right]}{2D(v^2+4D^2\mu_n^2)}$$

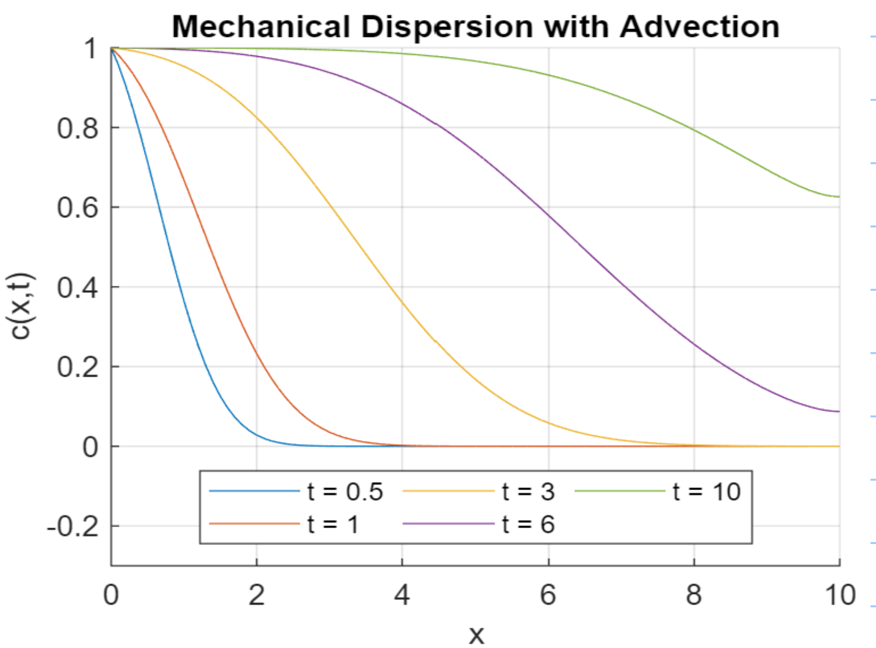
$$= \frac{2D\left(2D\mu_n - e^{-vL/2D}\left[v\sin(\mu_n L) + 2D\mu_n\cos(\mu_n L)\right]\right)}{v^2+4D^2\mu_n^2}$$

$$\therefore \quad a_n = \frac{8D^2\mu_n^2 c_0\left(e^{-vL/2D}\left[v\sin(\mu_n L) + 2D\mu_n\cos(\mu_n L)\right] - 2D\mu_n\right)}{(v^2+4D^2\mu_n^2)\left[2DL\mu_n^2 + v\sin^2(\mu_n L)\right]}$$

(c)

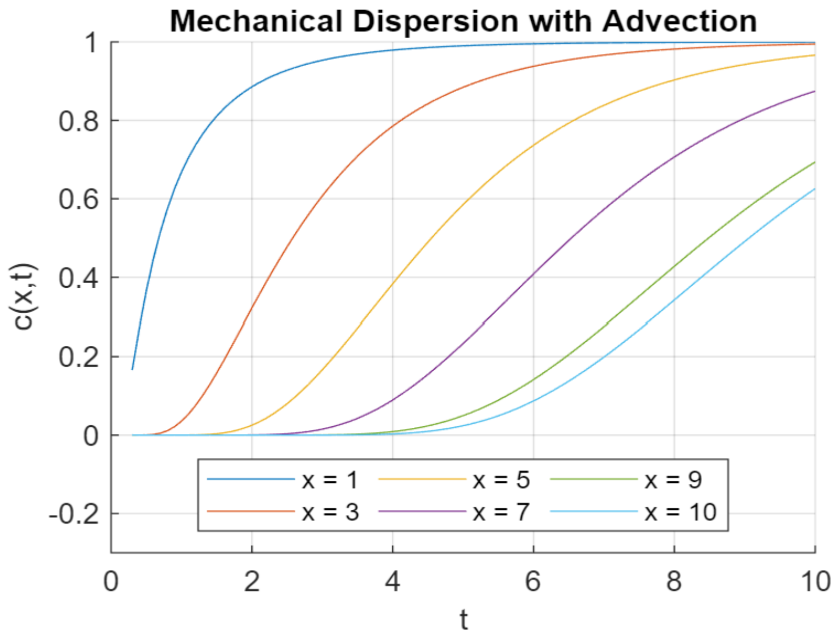
Using MATLAB, and similar code from #27(f), and using $c(x,t) = c_0 + u(x,t)$,

```
clear
v = 1; D = 0.5; c0 = 1; L = 10;
x = 0:0.1:L;
T = [0.5,1,3,6,10];
N = 25;  % number of terms
syms u
y1 = -2*D*u/v;
y2 = tan(u*L);
figure
hold on
for t = T
    c_sum = c0 + zeros(size(x,1),1);  % initialize
    for n = 1:N
        seed = (2*n-1)*pi/(2*L) + 0.1/n^2;
        u_n = vpasolve(y1==y2,u,seed);
        l_n = u_n^2 + v^2/(4*D^2);  % lambda
          num1 = 8*D^2*u_n^2*c0;
          num2 = v*sin(u_n*L) + 2*D*u_n*cos(u_n*L);
          num3 = exp(-v*L/(2*D))*num2 - 2*D*u_n;
          num = num1*num3;
          den1 = v^2 + 4*D^2*u_n^2;
          den2 = 2*D*L*u_n^2 + v*(sin(u_n*L))^2;
        a_n = num/(den1*den2);
        c_n = a_n*exp(-l_n*D*t)*exp(v*x/(2*D)).*sin(u_n*x);
        c_sum = c_sum + c_n;
    end
    plot(x,c_sum)
end
grid on
xlabel('x'); ylabel('c(x,t)')
title('Mechanical Dispersion with Advection')
ylim([-0.3,1])
legend('t = 0.5','t = 1','t = 3','t = 6','t = 10', ...
       'Location','south','NumColumns',3)
```



Now plot c(x,t) vs. t

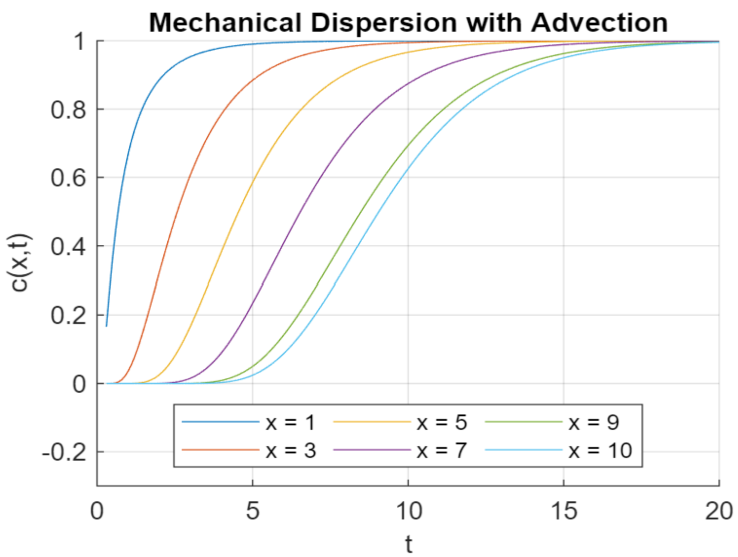
```
clear
v = 1; D = 0.5; c0 = 1; L = 10;
t = 0.3:0.1:10;
X = [1,3,5,7,9,10];
N = 30;  % number of terms
syms u
y1 = -2*D*u/v;
y2 = tan(u*L);
figure
hold on
for x = X
    c_sum = c0 + zeros(size(t,1),1);  % initialize
    for n = 1:N
        seed = (2*n-1)*pi/(2*L) + 0.1/n^2;
        u_n = vpasolve(y1==y2,u,seed);
        l_n = u_n^2 + v^2/(4*D^2);  % lambda
          num1 = 8*D^2*u_n^2*c0;
          num2 = v*sin(u_n*L) + 2*D*u_n*cos(u_n*L);
          num3 = exp(-v*L/(2*D))*num2 - 2*D*u_n;
          num = num1*num3;
          den1 = v^2 + 4*D^2*u_n^2;
          den2 = 2*D*L*u_n^2 + v*(sin(u_n*L))^2;
        a_n = num/(den1*den2);
        c_n = a_n*exp(-l_n*D*t)*exp(v*x/(2*D))*sin(u_n*x);
        c_sum = c_sum + c_n;
    end
    plot(t,c_sum)
end
grid on
xlabel('t'); ylabel('c(x,t)')
title('Mechanical Dispersion with Advection')
ylim([-0.3,1])
legend('x = 1','x = 3','x = 5','x = 7','x = 9','x = 10', ...
       'Location','south','NumColumns',3)
```



(d)

At $t = 15$ secs, virtually all parts of the column have reached a concentration of $c_0 = 1$.

Over time, as material is added at $x=0$, the flow to the right causes the concentration of material to increase. The added material is added faster than the flow can take it away so that all parts of the column attain a concentration of $c_0 = 1$. The plot below shows $c(x,t)$ vs. t for various segments.



1.

Convert to $-y'' = 2y + x$ and look at $y'' + \lambda y = 0$.

Here, $\mu = 2$, $r(x) = 1$, $f(x) = x$

$\lambda = 0$: $y = c_1 x + c_2$. $y(0) = 0 \Rightarrow c_2 = 0$. $y(1) = 0 \Rightarrow c_1 = 0$.

$\therefore$ $\lambda = 0$ doesn't yield a nontrivial solution.

$\lambda < 0$: $y = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}$. $y(0) = 0 \Rightarrow c_1 + c_2 = 0$

$y(1) = 0 \Rightarrow c_1 e^{\sqrt{-\lambda}} + c_2 e^{-\sqrt{-\lambda}} = 0$, or $c_1 e^{2\sqrt{-\lambda}} + c_2 = 0$.

$\therefore$ $c_1 e^{2\sqrt{-\lambda}} - c_1 = c_1 \left(e^{2\sqrt{-\lambda}} - 1\right) = 0$. As $\lambda \neq 0$, $e^{2\sqrt{-\lambda}} \neq 1$.

$\therefore$ $c_1 = 0$, and $\therefore$ $c_2 = 0$. $\therefore$ $\lambda < 0$ not an eigenvalue.

$\lambda > 0$: $y = c_1 \cos(\sqrt{\lambda}\, x) + c_2 \sin(\sqrt{\lambda}\, x)$. $y(0) = 0 \Rightarrow c_1 = 0$

$y(1) = 0 \Rightarrow c_2 \sin(\sqrt{\lambda}) = 0$. $\therefore$ $\sqrt{\lambda} = n\pi$, $n = 1, 2, 3, \ldots$

$\therefore$ $y(x) = K_n \sin(n\pi x)$, $n = 1, 2, 3, \ldots$

To normalize, $K_n^2 \int_0^1 \sin^2(n\pi x)\,dx = 1$, use $\int \sin^2 u \, du = \frac{1}{2}u - \frac{1}{4}\sin 2u + C$

$$\therefore K_n^2 \int_0^1 \frac{1}{n\pi} \sin^2(n\pi x)\, d(n\pi x) = \frac{K_n^2}{n\pi}\left[\frac{1}{2} n\pi x - \frac{1}{4}\sin(2n\pi x)\right]_0^1$$

$$= \frac{K_n^2}{n\pi}\left[\frac{1}{2} n\pi\right] = \frac{K_n^2}{2} = 1.$$

$$\therefore K_n = \sqrt{2}. \quad \therefore \phi_n(x) = \sqrt{2}\sin(n\pi x), \ \underline{\lambda_n = n^2\pi^2}$$

$$\therefore \phi(x) = \sum_{n=1}^{\infty} b_n \sqrt{2}\sin(n\pi x) \quad [1]$$

Note $\lambda_n = n^2\pi^2 \neq 2 = \mu$. $\therefore b_n = \frac{c_n}{\lambda_n - \mu}$, where

$$c_n = \int_0^1 f(x)\phi_n(x) = \int_0^1 x\sqrt{2}\sin(n\pi x)\, dx$$

use $\int u \sin u \, du = \sin u - u\cos u + C$

$$= \frac{\sqrt{2}}{n^2\pi^2}\int_0^1 (n\pi x)\sin(n\pi x)\, d(n\pi x)$$

$$= \frac{\sqrt{2}}{n^2\pi^2}\left[\sin(n\pi x) - (n\pi x)\cos(n\pi x)\right]_0^1$$

$$= \frac{\sqrt{2}}{n^2\pi^2}\left[-n\pi\cos(n\pi)\right] = -\frac{\sqrt{2}}{n\pi}(-1)^n = \frac{\sqrt{2}}{n\pi}(-1)^{n+1}$$

$$\therefore b_n = \frac{\frac{\sqrt{2}}{n\pi}(-1)^{n+1}}{n^2\pi^2 - 2} = \frac{\sqrt{2}(-1)^{n+1}}{n\pi(n^2\pi^2 - 2)}$$

$\therefore$ From [1],

$$\boxed{\phi(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi(n^2\pi^2 - 2)}\sin(n\pi x)}$$

2.

Convert to $-y'' = 2y + x$, so $\mu = 2$, $f(x) = x$, $r(x) = 1$.

Look at $y'' + \lambda y = 0$, $y(0) = 0$, $y'(1) = 0$.

From Section 11.2, problem #1, $\lambda_n = \dfrac{(2n-1)^2 \pi^2}{4}$, $n = 1, 2, 3, \ldots$,

and normalized $\underline{y_n(x) = \sqrt{2}}\,\sin\left[\dfrac{(2n-1)\pi x}{2}\right]$, $n = 1, 2, 3, \ldots$

Note $\lambda_n \neq 2 = \mu$.

$$\therefore\ y(x) = \sum_{n=1}^{\infty} b_n y_n(x), \quad b_n = \frac{c_n}{\lambda_n - \mu}, \text{ and}$$

$$c_n = \int_0^1 f(x)\, y_n(x)\, dx = \sqrt{2}\int_0^1 x \sin\left[\frac{(2n-1)\pi x}{2}\right] dx$$

$$= \frac{4\sqrt{2}}{(2n-1)^2 \pi^2}(-1)^{n-1}, \quad n = 1, 2, 3, \ldots \qquad \text{Note: } (-1)^{n-1} = (-1)^{n+1}$$

using Problem #7 of Section 11.2.

$$\therefore\ b_n = \frac{4\sqrt{2}(-1)^{n+1}}{(2n-1)^2 \pi^2} \cdot \left[\frac{1}{\frac{(2n-1)^2 \pi^2}{4} - 2}\right]$$

$$= \frac{\sqrt{2}(-1)^{n+1}}{(n-\frac{1}{2})^2 \pi^2} \cdot \frac{1}{(n-\frac{1}{2})^2 \pi^2 - 2}$$

$$\therefore\ \boxed{y(x) = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin\left[(n-\frac{1}{2})\pi x\right]}{(n-\frac{1}{2})^2 \pi^2 \left[(n-\frac{1}{2})^2 \pi^2 - 2\right]}}$$

3.

Convert to $-y'' = 2y + x$, $\mu = 2$, $r(x) = 1$, $f(x) = x$.

Look at $y'' + \lambda y = 0$, $y'(0) = 0$, $y'(1) = 0$.

From #3, Section 11.2, $\lambda_n = n^2\pi^2$, $n = 1, 2, 3, \ldots$,

normalized $y_0 = 1$, $y_n(x) = \sqrt{2}\cos(n\pi x)$, $n = 1, 2, 3, \ldots$

$\therefore$ Here, $\lambda_1 = 0$, $y_1(x) = 1$, $\lambda_n = (n-1)^2\pi^2$, $y_n(x) = \sqrt{2}\cos[(n-1)\pi x]$ [1]

for $n = 2, 3, 4, \ldots$ Note $\lambda_1 - \mu = 0 - 2 \neq 0$, and

$\lambda_n - \mu = (n-1)^2\pi^2 - 2 \neq 0$ for $n = 2, 3, 4, \ldots$

$\therefore y(x) = \sum_{n=1}^{\infty} b_n y_n(x)$, $b_n = \dfrac{c_n}{\lambda_n - \mu}$ where $c_n = \int_0^1 f(x) y_n(x)\,dx$

$c_1 = \int_0^1 x\,dx = \frac{1}{2}$. $b_1 = \dfrac{1/2}{0-2} = -\frac{1}{4}$. $\therefore \underline{b_1 y_1 = -\frac{1}{4}}$ [2]

For $n > 1$, $c_n = \sqrt{2}\int_0^1 x\cos[(n-1)\pi x]\,dx$

$$= \underbrace{\sqrt{2}\,\frac{x\sin[(n-1)\pi x]}{(n-1)\pi}\Big|_0^1}_{=0} - \sqrt{2}\int_0^1 \frac{\sin[(n-1)\pi x]}{(n-1)\pi}\,dx$$

$$= \sqrt{2}\,\frac{\cos[(n-1)\pi x]}{(n-1)^2\pi^2}\Big|_0^1 = \frac{\sqrt{2}}{(n-1)^2\pi^2}\left[(-1)^{n+1} - 1\right], \ n = 2, 3, \ldots$$

Note for $n = 3, 5, 7, \ldots$, $c_n = 0$ and $\therefore b_n = 0$.

For $n = 2, 4, 6, \ldots$, $C_n = \dfrac{-2\sqrt{2}}{(n-1)^2\pi^2}$.

$\therefore$ For $n = 1, 2, 3, \ldots$, $C_n = \dfrac{-2\sqrt{2}}{(2n-1)^2\pi^2}$

$$\therefore y(x) = b_1 y_1(x) + \sum_{n=2,4,6,\ldots}^{\infty} b_n y_n$$

$$b_{2,4,6,\ldots} = \frac{C_{2,4,6,\ldots}}{\lambda_{2,4,6,\ldots} - \mu} = \frac{\frac{-2\sqrt{2}}{(2n-1)^2\pi^2}}{(2n-1)^2\pi^2 - 2}, \quad n = 1, 2, 3, \ldots$$

$$= \frac{-2\sqrt{2}}{(2n-1)^2\pi^2\left[(2n-1)^2\pi^2 - 2\right]}, \quad n = 1, 2, 3, \ldots \quad [3]$$

and $y_{2,4,6,\ldots}(x) = \sqrt{2}\cos\left[(2n-1)\pi x\right]$, $n = 1, 2, 3, \ldots$

$\therefore$ From [1], [2], [3],

$$\therefore \boxed{y(x) = -\frac{1}{4} - 4\sum_{n=1}^{\infty} \frac{\cos\left[(2n-1)\pi x\right]}{(2n-1)^2\pi^2\left[(2n-1)^2\pi^2 - 2\right]}}$$

4.

Convert to $-y'' = 2y + x$, $\mu = 2$, $r(x) = 1$, $f(x) = x$

Look at $y'' + \lambda y = 0$, $y'(0) = 0$, $y'(1) + y(1) = 0$ to get the normalized eigenfunctions $y_n(x)$. From Section 11.2, #4, $y_n(x) = \dfrac{\sqrt{2}}{[1+\sin^2(\sqrt{\lambda_n})]^{1/2}}\cos(\sqrt{\lambda_n}\,x)$, where $\tan(\sqrt{\lambda_n}) = \dfrac{1}{\sqrt{\lambda_n}}$

From Section 11.1, #8, $\lambda_n \neq 2$ for $n = 1, 2, 3, \ldots$

$\therefore\ y(x) = \sum_{n=1}^{\infty} b_n y_n(x)$, where $b_n = \dfrac{c_n}{\lambda_n - \mu}$, and

$$c_n = \int_0^1 f(x)\, y_n(x)\, dx = \frac{\sqrt{2}}{[1+\sin^2(\sqrt{\lambda_n})]^{1/2}} \int_0^1 x\cos(\sqrt{\lambda_n}\,x)\,dx$$

From Section 11.2, #11, $c_n = \dfrac{\sqrt{2}}{[1+\sin^2(\sqrt{\lambda_n})]^{1/2}}\left(\dfrac{2\cos(\sqrt{\lambda_n})-1}{\lambda_n}\right)$

$$\therefore\ b_n y_n(x) = \frac{c_n}{\lambda_n - 2}\, y_n(x) =$$

$$\frac{1}{(\lambda_n-2)} \cdot \frac{\sqrt{2}}{[1+\sin^2(\sqrt{\lambda_n})]^{1/2}}\left[\frac{2\cos(\sqrt{\lambda_n})-1}{\lambda_n}\right]\frac{\sqrt{2}}{[1+\sin^2(\sqrt{\lambda_n})]^{1/2}}\cos(\sqrt{\lambda_n}\,x)$$

$$= \frac{2}{\lambda_n(\lambda_n-2)} \cdot \frac{[2\cos(\sqrt{\lambda_n})-1]}{[1+\sin^2(\sqrt{\lambda_n})]}\cos(\sqrt{\lambda_n}\,x)$$

$\therefore$

$$\boxed{y(x) = 2\sum_{n=1}^{\infty} \frac{[2\cos(\sqrt{\lambda_n})-1]}{\lambda_n(\lambda_n-2)[1+\sin^2(\sqrt{\lambda_n})]}\cos(\sqrt{\lambda_n}\,x), \quad \text{where } \tan(\sqrt{\lambda_n}) = \frac{1}{\sqrt{\lambda_n}}}$$

5.

Convert to $-y'' = 2y + 1 - |1-2x|$, $\mu = 2$, $r(x) = 1$, $f(x) = 1 - |1-2x|$

Look at $y'' + \lambda y = 0$, $y(0) = 0$, $y(1) = 0$ to get the normalized eigenfunctions $y_n(x)$. From Example #1 of Section 11.2, $\lambda_n = n^2\pi^2$, $y_n(x) = \sqrt{2}\sin(n\pi x)$, $n = 1, 2, 3, \ldots$

Note $\lambda_n \neq 2 = \mu$ for $n = 1, 2, 3, \ldots$

$\therefore$ $y(x) = \sum_{n=1}^{\infty} b_n y_n(x)$, where $b_n = \dfrac{c_n}{\lambda_n - \mu}$, and

$$c_n = \int_0^1 f(x)\, y_n(x)\, dx = \sqrt{2}\int_0^1 \left(1 - |1-2x|\right)\sin(n\pi x)\, dx$$

≥ 0 on $[0, 1/2]$
≤ 0 on $[1/2, 1]$

$$= 2\sqrt{2}\int_0^{\frac{1}{2}} x\sin(n\pi x)\,dx + 2\sqrt{2}\int_{1/2}^1 (1-x)\sin(n\pi x)\,dx$$

$$= 2\sqrt{2}\int_0^{1/2} x\sin(n\pi x)\,dx + 2\sqrt{2}\int_{1/2}^1 \sin(n\pi x)\,dx - 2\sqrt{2}\int_{1/2}^1 x\sin(n\pi x)\,dx$$

Use MATLAB to compute c_n:

```
clear
syms x n
i1 = int(x*sin(n*pi*x),x,0,1/2);
i2 = int(sin(n*pi*x),x,1/2,1);
i3 = int(x*sin(n*pi*x),x,1/2,1);
cn = simplify(2*sqrt(2)*(i1 + i2 - i3))
```

cn =

$$-\frac{2\sqrt{2}\left(\sin(\pi n) - 2\sin\left(\frac{\pi n}{2}\right)\right)}{n^2\pi^2}$$

Since $\sin(n\pi)=0$, $c_n = \dfrac{4\sqrt{2}\sin\left(\frac{n\pi}{2}\right)}{n^2\pi^2}$

$\therefore \quad b_n y_n(x) = \dfrac{c_n}{\lambda_n - \mu} = \dfrac{4\sqrt{2}\sin\left(\frac{n\pi}{2}\right)\sqrt{2}\sin(n\pi x)}{n^2\pi^2(n^2\pi^2-2)}$

$\therefore$

$$y(x) = \sum_{n=1}^{\infty} b_n y_n(x) = 8\sum_{n=1}^{\infty} \frac{\sin(n\pi/2)}{n^2\pi^2(n^2\pi^2-2)}\sin(n\pi x)$$

In all problems 6-9, convert to $-y'' = \mu y + f(x)$.

Note $r(x)=1$.

6.

Look at $y''+\lambda y=0$, $y(0)=0$, $y'(1)=0$. From Section 11.2,

#1, $y_n(x) = \sqrt{2}\sin\left[\frac{(2n-1)}{2}\pi x\right]$, $\lambda_n = \frac{(2n-1)^2\pi^2}{4}$, $n=1,2,3,\ldots$

(1) A unique solution exists for $\mu \neq \frac{(2n-1)^2\pi^2}{4}$.

Let $c_n = \int_0^1 f(x)y_n(x)\,dx$, $n=1,2,3,\ldots$, and let

$b_n = \dfrac{c_n}{\lambda_n-\mu}$, $n=1,2,3,\ldots$

$$\therefore \; y(x) = \sqrt{2} \sum_{n=1}^{\infty} b_n \sin\left[\frac{(2n-1)\pi x}{2}\right] \quad [1]$$

(2) If $\mu = \lambda_m$ for some m, then if $\int_0^1 f(x) y_m(x)\, dx = 0$, then the above series [1] yields a non unique solution in which b_m is arbitrary.

7.

Look at $y'' + \lambda y = 0$, $y'(0) = 0$, $y(1) = 0$. From Section 11.2, # 2,

$$y_n(x) = \sqrt{2} \cos\left[\frac{(2n-1)\pi x}{2}\right], \quad \lambda_n = \frac{(2n-1)^2 \pi^2}{4}, \quad n = 1, 2, 3, \ldots$$

(1) A unique solution exists for $\mu \neq \frac{(2n-1)^2 \pi^2}{4}$.

Let $c_n = \int_0^1 f(x) y_n(x)\, dx$, $n = 1, 2, 3, \ldots$, and let

$$b_n = \frac{c_n}{\lambda_n - \mu}, \quad n = 1, 2, 3, \ldots$$

$$\therefore \; y(x) = \sqrt{2} \sum_{n=1}^{\infty} b_n \cos\left[\frac{(2n-1)\pi x}{2}\right] \quad [1]$$

(2) If $\mu = \lambda_m$ for some m, then if $\int_0^1 f(x) y_m(x)\, dx = 0$, then the above series [1] yields a non unique solution in which b_m is arbitrary.

8.

Look at $y'' + \lambda y = 0$, $y'(0) = 0$, $y'(1) = 0$. From Section 11.2, #3,

$y_0(x) = 1$, $\lambda_0 = 0$, $y_n(x) = \sqrt{2}\cos(n\pi x)$, $\lambda_n = n^2\pi^2$, $n = 1,2,3,\dots$

(1) A unique solution exists for $\mu \neq \lambda_n$, $n = 0,1,2,3,\dots$

Let $c_n = \int_0^1 f(x)\, y_n(x)\, dx$, $n = 0,1,2,3,\dots$ and let

$$b_n = \frac{c_n}{\lambda_n - \mu}, \quad n = 0,1,2,3,\dots$$

$$\therefore\ y(x) = b_0 + \sqrt{2}\sum_{n=1}^{\infty} b_n \cos(n\pi x) \quad [1]$$

(2) If $\mu = \lambda_m$ for some m, then if $\int_0^1 f(x)\, y_m(x)\, dx = 0$,

then the above series [1] yields a non unique solution in which b_m is arbitrary.

9.

Look at $y'' + \lambda y = 0$, $y'(0) = 0$, $y'(1) + y(1) = 0$. From Section 11.2, #4,

$$y_n(x) = \frac{\sqrt{2}}{[1 + \sin^2(\sqrt{\lambda_n})]^{1/2}}\cos(\sqrt{\lambda_n}\, x), \text{ where } \tan(\sqrt{\lambda_n}) = \frac{1}{\sqrt{\lambda_n}},\ n = 1,2,3,\dots$$

and $\lambda_n > 0$.

(1) A unique solution exists for $\mu \neq \lambda_n$, $n = 1,2,3,\dots$

Let $c_n = \int_0^1 f(x)\, y_n(x)\, dx$, $n = 1, 2, 3, \ldots$, and let

$$b_n = \frac{c_n}{\lambda_n - \mu}, \quad n = 1, 2, 3, \ldots$$

$$\therefore\ y(x) = \sqrt{2} \sum_{n=1}^{\infty} \frac{b_n \cos(\sqrt{\lambda_n}\, x)}{[1 + \sin^2(\sqrt{\lambda_n})]^{1/2}} \qquad [1]$$

(2) If $\mu = \lambda_m$ for some m, then if $\int_0^1 f(x)\, y_m(x)\, dx = 0$, then the above series [1] yields a non unique solution in which b_m is arbitrary.

10.

Convert to $-y'' = \pi^2 y - a - x$. $\therefore\ \mu = \pi^2$, $r(x) = 1$, $f(x) = -a - x$

Look at $y'' + \lambda y = 0$. As with #1 above, $\lambda_n = n^2 \pi^2$,

$y_n(x) = \sqrt{2} \sin(n\pi x)$ is the normalized eigenfunction, $n = 1, 2, 3, \ldots$

A unique solution exists for $\lambda_n \neq \mu$, but $\lambda_1 = \pi^2 = \mu = \pi^2$.

$\therefore$ A nonunique solution exists only if

$$\int_0^1 f(x)\,y_1(x)\,dx = 0, \quad \text{or} \quad \int_0^1 (-a-x)\sqrt{2}\sin(\pi x)\,dx = 0$$

Using MATLAB to compute,

```
clear
syms x a
g = sqrt(2)*(-a-x)*sin(pi*x);
i = int(g,x,0,1)
```

i =

$$-\frac{\sqrt{2}\ (1+2\,a)}{\pi}$$

$$\therefore\ -\frac{\sqrt{2}(1+2a)}{\pi} = 0 \iff \underline{a = -\tfrac{1}{2}}.$$

$$\therefore\ y(x) = \sum_{n=1}^{\infty} b_n y_n(x) = b_1 y_1(x) + \sum_{n=2}^{\infty} \frac{c_n}{\lambda_n - \mu} y_n(x)$$

$y_1(x) = \sqrt{2}\sin(\pi x)$ and b_1 is arbitrary.

For $n>1$, $c_n = \int_0^1 f(x)\,y_n(x)\,dx = \int_0^1 \left(\tfrac{1}{2} - x\right)\sqrt{2}\sin(n\pi x)\,dx$

Using MATLAB,

```
clear
syms x n
g = sqrt(2)*(1/2-x)*sin(n*pi*x);
cn = int(g,x,0,1)
```

cn =

$$-\frac{\sqrt{2}\,\sin(\pi\,n) - n\left(\frac{\pi\ \sqrt{2}}{2} + \frac{\pi\ \sqrt{2}\,\cos(\pi\,n)}{2}\right)}{n^2\,\pi^2}$$

Since $\sin(n\pi) = 0$, $c_n = \dfrac{\sqrt{2}}{2n\pi} + \dfrac{\sqrt{2}(-1)^n}{2n\pi}$

$$\therefore\ b_n = \frac{c_n}{\lambda_n - \mu} = \frac{\sqrt{2} + \sqrt{2}(-1)^n}{2n\pi(n^2\pi^2 - \pi^2)} = \frac{\sqrt{2}\,(1+(-1)^n)}{2n\pi^3(n^2-1)},\ n>1$$

For $n>1$, $1+(-1)^n = 2, 0, 2, 0, 2, \ldots$

$$\therefore\ b_2, b_4, b_6, \ldots = \frac{\sqrt{2}\,(2)}{2n\pi^3(n^2-1)} = \frac{\sqrt{2}}{2n\pi^3(4n^2-1)},\ n = 1,2,3,\ldots$$

$$\therefore\ b_{2n}\, y_{2n}(x) = \frac{\sqrt{2}}{2n\pi^3(4n^2-1)} \cdot \sqrt{2}\sin(2n\pi x) = \frac{\sin(2n\pi x)}{n\pi^3(4n^2-1)}$$

$$\therefore\ y(x) = c\sin(\pi x) + \sum_{n=2,4,6,\ldots}^{\infty} b_n y_n(x) = c\sin(\pi x) + \sum_{n=1}^{\infty} b_{2n} y_{2n}$$

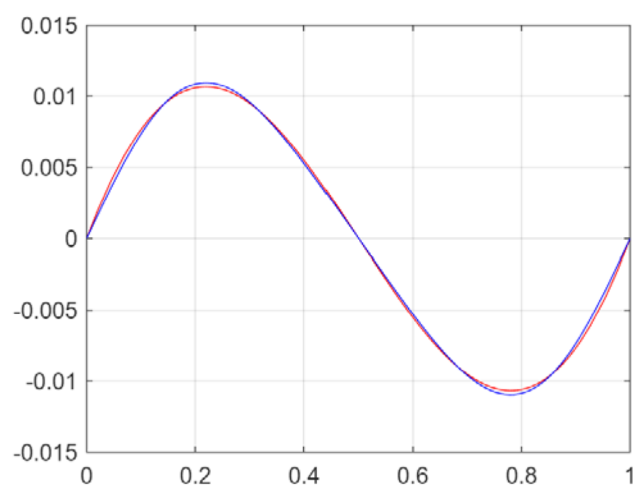
$$= c\sin(\pi x) + \sum_{n=1}^{\infty} \frac{\sin(2n\pi x)}{n\pi^3(4n^2-1)}, \quad c \text{ a constant}$$

$$\therefore\ \boxed{a = -\frac{1}{2},\ y(x) = c\sin(\pi x) + \sum_{n=1}^{\infty} \frac{\sin(2n\pi x)}{n\pi^3(4n^2-1)}}$$

Note this is the same function as given in the back of the book: $y = c\sin(\pi x) + \frac{1}{2\pi^2}\cos(\pi x) + \frac{1}{\pi^2}\left(x - \frac{1}{2}\right)$.

The following MATLAB code shows this, ignoring the $c\sin(\pi x)$ term, using just two terms.

```
clear
syms x
figure
y1 = (1/(2*pi^2))*cos(pi*x) + (1/pi^2)*(x-1/2);
fplot(y1,[0,1],'r')  % book solution
ylim([-0.015,0.015])
y2 = 0;
for n = 1:2
    y2 = y2 + sin(2*n*pi*x)/(n*pi^3*(4*n^2-1));
end
hold on
fplot(y2,[0,1],'b')  % plot 2 terms
grid on
```



To obtain $y = c\sin(\pi x) + \frac{1}{2\pi^2}\cos(\pi x) + \frac{1}{\pi^2}(x - \frac{1}{2})$, use the method of undetermined coefficients for the nonhomogeneous term.

Homogeneous term: $y'' + \pi^2 y = 0 \Rightarrow y_c = c_1\cos(\pi x) + c_2\sin(\pi x)$.

Nonhomogeneous term: $y'' + \pi^2 y = a + x$.

Let $y_p(x) = Ax + B$, $\therefore y_p'' + \pi^2 y_p = \pi^2 Ax + B\pi^2 = a + x$

$\therefore \pi^2 A = 1$, $A = \frac{1}{\pi^2}$, $B\pi^2 = a$, $B = \frac{a}{\pi^2}$

$\therefore y = c_1\cos(\pi x) + c_2\sin(\pi x) + \frac{1}{\pi^2}x + \frac{a}{\pi^2}$

$y(0) = 0 \Rightarrow c_1 + \frac{a}{\pi^2} = 0$, $c_1 = -\frac{a}{\pi^2}$

$y(1) = 0 \Rightarrow \frac{a}{\pi^2} + \frac{1}{\pi^2} + \frac{a}{\pi^2} = 0 \Rightarrow 2a + 1 = 0 \Rightarrow a = -\frac{1}{2}$.

$\therefore c_1 = \frac{1}{2\pi^2}$, $B = -\frac{1}{2\pi^2}$

$\therefore y(x) = \frac{1}{2\pi^2}\cos(\pi x) + c\sin(\pi x) + \frac{1}{\pi^2}(x - \frac{1}{2})$,

c any constant.

11.

Convert to $-y'' = 4\pi^2 y - a - x$, $\mu = 4\pi^2$, $r(x) = 1$, $f(x) = -a - x$

Look at $y'' + \lambda y = 0$. As with #1 above, $\lambda_n = n^2\pi^2$,

$y_n(x) = \sqrt{2}\sin(n\pi x)$ is the normalized eigenfunction, $n = 1,2,3,\ldots$

A unique solution exists for $\lambda_n \neq \mu = 4\pi^2$. But for

$n = 2$, $\lambda_n = 4\pi^2$. $\therefore$ A solution exists only if

$\int_0^1 f(x)\, y_2(x)\, dx = 0$, or $\int_0^1 (-a-x)\sqrt{2}\sin(2\pi x)\, dx = 0$

Using MATLAB to compute,

```
clear
syms x a
g = sqrt(2)*(-a-x)*sin(2*pi*x);
i = int(g,x,0,1)
```

```
i =
```

$$\frac{\sqrt{2}}{2\pi}$$

Since $\frac{\sqrt{2}}{2}\pi \neq 0$, there is no value of a which will

satisfy $\int_0^1 f(x)\, y_2(x)\, dx = 0$.

$\therefore$ There is no solution.

12.

Convert to $-y'' = \pi^2 y - a$, $\mu = \pi^2$, $r(x) = 1$, $f(x) = -a$.

Look at $y'' + \lambda y = 0$. As with #3 above, $\lambda_0 = 0$,

$\lambda_n = n^2\pi^2$, $n = 1,2,3,\ldots$, and the normalized eigenfunctions are $y_0 = 1$, $y_n(x) = \sqrt{2}\cos(n\pi x)$, $n = 1, 2, 3, \ldots$

A unique solution exists for $\lambda_n \neq \mu$, and this is true except for $\lambda_1 = \pi^2 = \mu$. $\therefore$ A nonunique solution exists if $\int_0^1 f(x) y_1(x)\,dx = 0$, or $\int_0^1 (-a)\sqrt{2}\cos(\pi x)\,dx = 0$

But $-\sqrt{2}\,a\int_0^1 \cos(\pi x)\,dx = -\frac{\sqrt{2}\,a}{\pi}\sin(\pi x)\Big|_0^1 = 0$, for all values of a.

Solving the homogeneous $y'' + \pi^2 y = 0 \Rightarrow$

$$y_c(x) = C_1\cos(\pi x) + C_2\sin(\pi x).$$

For the nonhomogeneous term, let $y_p(x) = Ka$, K a constant. $\therefore$ $y_p'' + \pi^2 y_p = K\pi^2 a = a \Rightarrow K = \frac{1}{\pi^2}$

$$\therefore\ y(x) = y_c + y_p = C_1\cos(\pi x) + C_2\sin(\pi x) + \frac{a}{\pi^2}$$

$$y'(x) = -C_1\pi\sin(\pi x) + C_2\pi\cos(\pi x)$$

$y'(0) = 0 \Rightarrow C_2 = 0$ $\quad y'(1) = 0 \Rightarrow -C_1\pi\sin(\pi) = 0$, so C_1 is undetermined.

$$\therefore\ \boxed{y(x) = C\cos(\pi x) + \frac{a}{\pi^2},\ \ C \text{ and } a \text{ arbitrary constants}}$$

13.

Convert to $-y'' = \pi^2 y - a + \cos(\pi x)$, $\mu = \pi^2$, $r(x) = 1$, $f(x) = -a + \cos(\pi x)$.

Look at $y'' + \lambda y = 0$. As with #1 above, $\lambda_n = n^2\pi^2$,

$y_n(x) = \sqrt{2}\sin(n\pi x)$ is the normalized eigenfunction, $n = 1, 2, 3, \ldots$

A unique function exists if $\lambda_n \neq \mu$, and this is true

except for $\lambda_1 = \pi^2 = \mu$. $\therefore$ A nonunique solution

exists if $\int_0^1 f(x) y_1(x)\,dx = 0$, or $\int_0^1 (-a + \cos(\pi x))\sqrt{2}\sin(\pi x)\,dx = 0$.

Using MATLAB to compute,

```
clear
syms x a
g = sqrt(2)*(-a+cos(pi*x))*sin(pi*x);
i = int(g,x,0,1)
```

i =

$$-\frac{2\sqrt{2}\,a}{\pi}$$

$\therefore \int_0^1 f(x) y_1(x)\,dx = 0$ if $\underline{a = 0}$

Homogeneous solution: $y'' + \pi^2 y = 0 \Rightarrow y_c(x) = c_1\cos(\pi x) + c_2\sin(\pi x)$

Nonhomogeneous: Let $y_p(x) = K_1 x\cos(\pi x) + K_2 x\sin(\pi x)$

Using MATLAB,

```
clear
syms x k1 k2 p
yp = k1*x*cos(pi*x) + k2*x*sin(pi*x);
d = diff(yp,x,2) + p^2*yp;
collect(d,[cos(pi*x),sin(pi*x)])
```

using p for π for clarity.

ans = $\left(2\pi k_2 - \pi^2 k_1 x + k_1 p^2 x\right)\cos(\pi x) + \left(-2\pi k_1 - \pi^2 k_2 x + k_2 p^2 x\right)\sin(\pi x)$

$\therefore$ Equating coefficients with $-\cos(\pi x)$,

$-2\pi K_1 - \pi^2 K_2 x + K_2 \pi^2 x = 0 \Rightarrow K_1 = 0$

$\therefore 2\pi K_2 = -1, \quad K_2 = -\frac{1}{2\pi}$

$\therefore y_p(x) = -\frac{1}{2\pi} x \sin(\pi x)$

$y = y_c + y_p = c_1 \cos(\pi x) + c_2 \sin(\pi x) - \frac{x}{2\pi}\sin(\pi x)$

$y(0) = 0 \Rightarrow c_1 = 0$

$y(1) = 0 \Rightarrow c_2 \sin(\pi) - \frac{\sin(\pi)}{2\pi} = 0$, so c_2 is not determined.

$\therefore$ $\boxed{a = 0, \quad y(x) = c\sin(\pi x) - \frac{x}{2\pi}\sin(\pi x), \quad c \text{ a constant}}$

14.

Given $f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x)$, then for each $\phi_m(x)$, $m = 1, 2, 3, \ldots$,

$r(x)\phi_m(x) f(x) = \sum_{n=1}^{\infty} c_n r(x) \phi_n(x) \phi_m(x)$. Assuming term-by-term integration,

$$\int_0^1 r(x)\,\phi_m(x)\,f(x)\,dx = \int_0^1 \sum_{n=1}^{\infty} c_n\, r(x)\,\phi_n(x)\,\phi_m(x)\,dx$$

$$= \sum_{n=1}^{\infty} c_n \underbrace{\int_0^1 r(x)\phi_n(x)\,\phi_m(x)\,dx}_{=0,\ m\neq n,\ =1,\ m=n}$$

$$= c_m$$

But since $f(x)=0$ on $[0,1]$, then

$$\int_0^1 r(x)\,\phi_m(x)\,f(x)\,dx = 0.$$

$\therefore$ $\underline{c_m = 0}$ for $m=1,2,3,\ldots$

15.

$$L[y] = L[u+v] = L[u] + L[v] = 0 + f(x) = f(x)$$

$\therefore$ $u+v$ satisfies the differential equation.

$$\alpha_1 y(0) + \alpha_2 y'(0) = \alpha_1 (u+v)(0) + \alpha_2 (u+v)'(0)$$

$$= \alpha_1 u(0) + \alpha_1 v(0) + \alpha_2 u'(0) + \alpha_2 v'(0)$$

$$= [\alpha_1 u(0) + \alpha_2 u'(0)] + [\alpha_1 v(0) + \alpha_2 v'(0)]$$

$$= a + 0 = a$$

$$\beta_1 y(1) + \beta_2 y'(1) = \beta_1 (u+v)(1) + \beta_2 (u+v)'(1)$$

$$= \beta_1 u(1) + \beta_1 v(1) + \beta_2 u'(1) + \beta_2 v'(1)$$

$$= [\beta_1 u(1) + \beta_2 u'(1)] + [\beta_1 v(1) + \beta_2 v'(1)]$$

$$= b + 0 = b$$

$\therefore$ $u+v$ also satisfies the boundary conditions.

16.

Given $y(x) = c_1 \sin(\pi x) + \cos(\pi x) + x$

$$y''(x) = -c_1 \pi^2 \sin(\pi x) - \pi^2 \cos(\pi x)$$

$$\therefore\ y'' + \pi^2 y = -c_1 \pi^2 \sin(\pi x) - \pi^2 \cos(x)$$

$$+ \pi^2 (c_1 \sin(\pi x) + \cos(\pi x) + x)$$

$$= \pi^2 x$$

Also, $y(0) = c_1 \sin(\pi \cdot 0) + \cos(\pi \cdot 0) + 0$

$= 0 + 1 + 0 = 1$

$y(1) = c_1 \sin(\pi \cdot 1) + \cos(\pi \cdot 1) + 1$

$= 0 + (-1) + 1 = 0$

$\therefore$ $y(x) = c_1 \sin(\pi x) + \cos(x) + x$ satisfies the D.E. and the boundary conditions.

$\therefore$ $y(x)$ satisfies both the D.E. and boundary problems.

The only way to split the solution up as in #15 is to let $u(x) = c_1 \sin(\pi x)$ or $u(x) = c_1 \sin(\pi x) + \cos(x)$, where $L[u] = 0$.

However, for $u(x) = c_1 \sin(\pi x)$, $u(0) \neq 1$

For $u(x) = c_1 \sin(\pi x) + \cos(\pi x)$, $u(1) = -1 \neq 0$.

$\therefore$ The homogeneous solution to the D.E. doesn't satisfy the boundary conditions.

17.

Given: $y(x)$ is a solution to $y'' + py' + qy = 0$, $y(0) = a$, $y(1) = b$,

and $y = u + v$.

$\therefore (u+v)'' + p(u+v)' + q(u+v) = 0 \Rightarrow$

$u'' + v'' + pu' + pv' + qu + qv = 0 \Rightarrow$

$u'' + pu' + qu = -(v'' + pv' + qv) = g(x)$

$u(0) = y(0) - v(0) = a - a = 0$

$u(1) = y(1) - v(1) = b - b = 0$

$\therefore u(x)$ satisfies $u'' + pu' + qu = g(x)$, and $u(0) = 0$, $u(1) = 0$.

$v(x)$ can be a linear equation s.t. $v(0) = a$, $v(1) = b$.

$\therefore$ Let $v(x) = c_1 x + c_2$. $v(0) = a \Rightarrow c_2 = a$.

$\therefore v(1) = c_1 + a = b$, $\Rightarrow c_1 = b - a$. $\therefore v(x) = \underline{(b-a)x + a}$

18.

Let $v(x)$ be the function s.t. $v(0)=1$, $v(1)+v'(1)=-2$,

but doesn't solve an D.E., yet to be determined.

For ease, let $v(x) = Ax+B$, as in #17.

$\therefore$ $v(0) = B = 1$, so $v(x) = Ax+1$. $v'(x) = A$.

$\therefore$ $v(1)+v'(1) = (A+1)+A = -2$, $A = -\frac{3}{2}$.

$\therefore$ $\underline{v(x) =} -\frac{3}{2}x+1$.

To find $u(x)$, we want $L[u+v] = 2-4x$ using

$L[y] = y''+2y$. $\therefore$ $L[u]+L[v] = 2-4x$,

$L[v] = L\left[-\frac{3}{2}x+1\right] = \left(-\frac{3}{2}x+1\right)''+2\left(-\frac{3}{2}x+1\right) = -3x+2$

$\therefore$ $L[u]+(-3x+2) = 2-4x$. $\therefore$ $L[u] = -x$

$\therefore$ $u''+2u = -x$. Also, $u(0)=0$, $u(1)+u'(1)=0$

$\therefore$ $(u+v)(0) = u(0)+v(0) = 0+1$

$(u+v)(1)+(u+v)'(1) = u(1)+v(1)+u'(1)+v'(1)$

$$= u(1)+u'(1)+\left[v(1)+v'(1)\right] = 0+\left[-\tfrac{1}{2}-\tfrac{3}{2}\right] = -2$$

$\therefore$ The new problem is:

$$\underline{u''+2u=-x}, \quad \underline{u(0)=0}, \quad u(1)+u'(1)=0$$

This was solved in Example #1 of the text. Using standard methods, let $u_c(x) = c_1\cos(\sqrt{2}x)+c_2\sin(\sqrt{2}x)$,

$u_p(x) = Ax+B$. $\therefore u_p''+2u_p = 2Ax+2B = -x$, $B=0, A=-\frac{1}{2}$.

$$\therefore u(x) = c_1\cos(\sqrt{2}x)+c_2\sin(\sqrt{2}x)-\frac{1}{2}x$$

$$u(0)=0 \Rightarrow c_1=0. \quad \therefore u(x) = c_2\sin(\sqrt{2}x)-\frac{1}{2}x$$

$$u(1)+u'(1) = c_2\sin(\sqrt{2})-\frac{1}{2}+c_2\sqrt{2}\cos(\sqrt{2})-\frac{1}{2} = 0$$

$$\therefore c_2 = \frac{1}{\sin(\sqrt{2})+\sqrt{2}\cos(\sqrt{2})}$$

$$\therefore u(x) = \frac{\sin(\sqrt{2}x)}{\sin(\sqrt{2})+\sqrt{2}\cos(\sqrt{2})}-\frac{1}{2}x$$

$$\therefore y = u+v = \underline{\frac{\sin(\sqrt{2}x)}{\sin(\sqrt{2})+\sqrt{2}\cos(\sqrt{2})}}-2x+1$$

19.

Using $r(x)u_t = (p(x)u_x)_x - q(x)u + F(x,t)$,

$r(x)=1$, $p(x)=1$, $q(x)=0$, $F(x,t)=-x$, $f(x)=\sin\left(\frac{\pi x}{2}\right)$

Using $u(x,t)=y(x)T(t)$, the homogeneous equation is

$-y'' = \lambda X$, or $y''(x)+\lambda y(x)=0$, $y(0)=0$, $y'(1)=0$

From #2 above, $\lambda_n = \frac{(2n-1)^2\pi^2}{4}$ and normalized

$y_n(x) = \sqrt{2}\sin\left[\frac{(2n-1)\pi x}{2}\right]$, $n=1,2,3,\ldots$

$\therefore$ Assume $u(x,t) = \sum_{n=1}^{\infty} b_n(t)\, y_n(x)$

From $\frac{F(x,t)}{r(x)} = \sum_{n=1}^{\infty} \gamma_n(t)\, y_n(x)$, $\gamma_n(t) = \int_0^1 F(x,t)\, y_n(x)\,dx$

$\therefore \gamma_n(t) = \int_0^1 (-x)\sqrt{2}\sin\left[\frac{(2n-1)\pi x}{2}\right]dx$ Using MATLAB,

```
clear
syms x n
f= -sqrt(2)*x*sin((2*n-1)*pi*x/2);
i = int(f,x,0,1)
```

i =

$$\frac{\sqrt{2}\cos(\pi n)}{\frac{\pi^2}{4} - \pi^2 n + \pi^2 n^2} - \frac{\sqrt{2}\sin(\pi n)}{\frac{\pi}{2} - \pi n}$$

Since $\sin(n\pi)=0$, $\gamma_n(t) = \frac{4\sqrt{2}\cos(n\pi)}{\pi^2(2n-1)^2} = \frac{4\sqrt{2}(-1)^n}{\pi^2(2n-1)^2}$

$\therefore \gamma_n(t)$ is not a function of t.

From $b_n'(t) + \lambda_n b_n(t) = \gamma_n$, $\gamma_n = \dfrac{4\sqrt{2}\,(-1)^n}{\pi^2 (2n-1)^2}$

$$b_n'(t) + \lambda_n b_n(t) = \gamma_n \,, \quad b_n(0) = B_n,$$

where $B_n = \displaystyle\int_0^1 f(x)\, y_n(x)\, dx = \int_0^1 \sin\left(\frac{\pi x}{2}\right) \sqrt{2} \sin\left[\frac{(2n-1)\pi x}{2}\right] dx$

Using MATLAB,

```
clear
syms x n
f= sin(pi*x/2);
yn = sqrt(2)*sin((2*n-1)*pi*x/2);
i = int(f*yn,x,0,1)
```

i =

$$\begin{cases} -\dfrac{\sqrt{2}}{2} & \text{if } n = 0 \\ \dfrac{\sqrt{2}}{2} & \text{if } n = 1 \\ -\dfrac{\sqrt{2}\,\sin(\pi n)\,(-1+2n)}{2n\pi\,(-1+n)} & \text{if } n \neq 0 \wedge n \neq 1 \end{cases}$$

$\therefore$ $B_1 = \dfrac{\sqrt{2}}{2}$, $B_n = 0$, $n \geq 2$ since $\sin(n\pi) = 0$

$\therefore$ $b_n'(t) + \lambda_n b_n(t) = \gamma_n$, $b_1(0) = \dfrac{\sqrt{2}}{2}$, $b_n(0) = 0$, $n \geq 2$

$$\left[e^{\lambda_n t} b_n(t)\right]' = \gamma_n e^{\lambda_n t}, \quad \int_0^t \left[e^{\lambda_n s} b_n(s)\right]' ds = \gamma_n \int_0^t e^{\lambda_n s}\, ds$$

$$e^{\lambda_n t} b_n(t) - \overset{= B_n}{b_n(0)} = \frac{\gamma_n}{\lambda_n}\left(e^{\lambda_n t} - 1\right)$$

$\therefore$ $b_n(t) = B_n e^{-\lambda_n t} + \dfrac{\gamma_n}{\lambda_n}\left(1 - e^{-\lambda_n t}\right)$, $\lambda_n = \dfrac{(2n-1)^2 \pi^2}{4}$, $\gamma_n = \dfrac{4\sqrt{2}\,(-1)^n}{\pi^2 (2n-1)^2}$

$$\therefore\ b_1(t) = \frac{\sqrt{2}}{2} e^{-\frac{\pi^2}{4} t} + \frac{4\gamma_1}{\pi^2}\left(1 - e^{-\frac{\pi^2}{4} t}\right)$$

$$= \frac{4\gamma_1}{\pi^2} + \left[\frac{\sqrt{2}}{2} - \frac{4\gamma_1}{\pi^2}\right] e^{-\frac{\pi^2}{4} t}, \quad \gamma_1 = -\frac{4\sqrt{2}}{\pi^2}$$

$$b_n(t) = \frac{4\gamma_n}{(2n-1)^2\pi^2}\left(1 - e^{-\frac{(2n-1)^2\pi^2}{4}t}\right), \quad n = 2,3,4,\ldots, \quad \gamma_n = \frac{4\sqrt{2}(-1)^n}{\pi^2(2n-1)^2}$$

$$\therefore \boxed{\begin{aligned} u(x,t) &= \sqrt{2}\left[\frac{4\gamma_1}{\pi^2} + \left(\frac{\sqrt{2}}{2} - \frac{4\gamma_1}{\pi^2}\right)\right]e^{-\frac{\pi^2}{4}t}\sin\left(\frac{\pi x}{2}\right) \\ &+ \sqrt{2}\sum_{n=2}^{\infty}\frac{4\gamma_n}{(2n-1)^2\pi^2}\left(1 - e^{-\frac{(2n-1)^2\pi^2}{4}t}\right)\sin\left[\frac{(2n-1)\pi x}{2}\right] \\ \gamma_n &= \frac{4\sqrt{2}(-1)^n}{\pi^2(2n-1)^2}, \quad n = 1,2,3,\ldots \end{aligned}}$$

20.

Using $r(x)u_t = (p(x)u_x)_x - q(x)u + F(x,t)$, $r(x) = 1$, $p(x) = 1$,

$q(x) = 0$, $F(x,t) = e^{-t}$, and $f(x) = 1 - x$.

Using $u(x,t) = y(x)T(t)$, the homogeneous equation is

$$y''(x) + \lambda y(x) = 0, \quad y'(0) = 0, \quad y(1) + y'(1) = 0$$

From #4 above, normalized $y_n(x) = \dfrac{\sqrt{2}}{[1 + \sin^2(\sqrt{\lambda_n})]^{1/2}}\cos(\sqrt{\lambda_n}\,x)$

where $\tan(\sqrt{\lambda_n}) = \dfrac{1}{\sqrt{\lambda_n}}$

$\therefore$ Assume $u(x,t) = \sum_{n=1}^{\infty} b_n(t)y_n(x)$

From $\dfrac{F(x,t)}{r(x)} = \sum_{n=1}^{\infty}\gamma_n(t)y_n(x)$, $\gamma_n(t) = \int_0^1 F(x,t)y_n(x)\,dx$,

$$\gamma_n(t) = \frac{\sqrt{2}\, e^{-t}}{[1+\sin^2(\sqrt{\lambda_n})]^{1/2}} \int_0^1 \cos(\sqrt{\lambda_n}\, x)\, dx = \frac{\sqrt{2}\, e^{-t} \sin(\sqrt{\lambda_n})}{\sqrt{\lambda_n}\, [1+\sin^2(\sqrt{\lambda_n})]^{1/2}}$$

From $b_n'(t) + \lambda_n b_n(t) = \gamma_n(t)$, $b_n(0) = B_n$, where

$$B_n = \int_0^1 f(x)\, y_n(x)\, dx = \frac{\sqrt{2}}{[1+\sin^2(\sqrt{\lambda_n})]^{1/2}} \int_0^1 (1-x)\cos(\sqrt{\lambda_n}\, x)\, dx$$

Using MATLAB,

```
clear
syms x zn
f = (1-x)*cos(zn*x);
i = int(f,x,0,1)
```

i =

$$-\frac{\cos(zn)-1}{zn^2}$$

Here, $zn = \sqrt{\lambda_n}$

$$\therefore B_n = \frac{\sqrt{2}}{[1+\sin^2(\sqrt{\lambda_n})]^{1/2}} \frac{[1-\cos(\sqrt{\lambda_n})]}{\lambda_n}$$

$$\therefore [e^{\lambda_n t} b_n(t)]' = \gamma_n(t) e^{\lambda_n t}, \quad \int_0^t [e^{\lambda_n s} b_n(s)]' ds = \int_0^t \gamma_n(s) e^{\lambda_n s} ds$$

$$e^{\lambda_n t} b_n(t) - \overset{= B_n}{b_n(0)} = \frac{\sqrt{2}\sin(\sqrt{\lambda_n})}{\sqrt{\lambda_n}\,[1+\sin^2(\sqrt{\lambda_n})]^{1/2}} \int_0^t e^{-s} e^{\lambda_n s}\, ds$$

$$\int_0^t e^{(\lambda_n - 1)s}\, ds = \frac{e^{(\lambda_n-1)t} - 1}{\lambda_n - 1}$$

$$\therefore e^{\lambda_n t} b_n(t) = \frac{\sqrt{2}\,[1-\cos(\sqrt{\lambda_n})]}{\lambda_n\,[1+\sin^2(\sqrt{\lambda_n})]^{1/2}} + \frac{\sqrt{2}\sin(\sqrt{\lambda_n})\,[e^{(\lambda_n-1)t}-1]}{\sqrt{\lambda_n}\,(\lambda_n-1)\,[1+\sin^2(\sqrt{\lambda_n})]^{1/2}}$$

For simplicity, let $a_n = \dfrac{\sqrt{2}\,[1-\cos(\sqrt{\lambda_n})]}{\lambda_n[1+\sin^2(\sqrt{\lambda_n})]^{1/2}}$, and let $c_n = \dfrac{\sqrt{2}\sin(\sqrt{\lambda_n})}{\sqrt{\lambda_n}\,[1+\sin^2(\sqrt{\lambda_n})]^{1/2}}$

$$\therefore\ e^{\lambda_n t} b_n(t) = a_n + c_n \frac{[e^{\lambda_n t}e^{-t} - 1]}{\lambda_n - 1}$$

$$\therefore\ b_n(t) = a_n e^{-\lambda_n t} + c_n\left(\frac{e^{-t} - e^{-\lambda_n t}}{\lambda_n - 1}\right)$$

$\therefore\ u(x,t) = \sum_{n=1}^{\infty} b_n(t)\, y_n(x)$, where $y_n(x) = \dfrac{\sqrt{2}\cos(\sqrt{\lambda_n}x)}{[1+\sin^2(\sqrt{\lambda_n})]^{\frac{1}{2}}}$

$$\therefore\ \boxed{u(x,t) = \sqrt{2}\sum_{n=1}^{\infty}\left[a_n e^{-\lambda_n t} + \frac{c_n}{\lambda_n - 1}\left(e^{-t} - e^{-\lambda_n t}\right)\right]\frac{\cos(\sqrt{\lambda_n}x)}{[1+\sin^2(\lambda_n)]^{1/2}}}$$

where $\tan(\sqrt{\lambda_n}) = \dfrac{1}{\sqrt{\lambda_n}}$,

$a_n = \dfrac{\sqrt{2}\,[1-\cos(\sqrt{\lambda_n})]}{\lambda_n[1+\sin^2(\sqrt{\lambda_n})]^{1/2}}$, $c_n = \dfrac{\sqrt{2}\sin(\sqrt{\lambda_n})}{\sqrt{\lambda_n}\,[1+\sin^2(\sqrt{\lambda_n})]^{1/2}}$

21.

Using $r(x)u_t = (p(x)u_x)_x - q(x)u + F(x,t)$, $r(x)=1$, $p(x)=1$,

$q(x)=0$, $F(x,t) = 1-|1-2x|$, $f(x)=0$.

Using $u(x,t) = y(x)T(t)$, the homogeneous equation is

$y''(x) + \lambda y(x) = 0$, $y(0)=0$, $y(1)=0$. From #5 above, normalized $y_n(x) = \sqrt{2}\sin(n\pi x)$, $\lambda_n = n^2\pi^2$, $n = 1,2,3,\ldots$

$\therefore$ Assume $u(x,t) = \sum_{n=1}^{\infty} b_n(t) y_n(x)$

From $\frac{F(x,t)}{r(x)} = \sum_{n=1}^{\infty} \gamma_n(t) y_n(x)$, $\gamma_n(t) = \int_0^1 F(x,t) y_n(x)\,dx$,

$$\therefore \gamma_n(t) = \int_0^1 \big(1 - |1-2x|\big)\sqrt{2}\sin(n\pi x)\,dx$$

$\hookrightarrow \geq 0$ on $[0, 1/2]$
≤ 0 on $[1/2, 1]$

$$= \sqrt{2}\int_0^{1/2} 2x\sin(n\pi x)\,dx + \sqrt{2}\int_0^1 (2-2x)\sin(n\pi x)\,dx$$

From #5 above, $\gamma_n(t) = \frac{4\sqrt{2}\sin\left(\frac{n\pi}{2}\right)}{n^2\pi^2}$, $n = 1,2,3,\ldots$

Note $\gamma_n(t)$ is not a function of t.

From $b_n'(t) + \lambda_n b_n(t) = \gamma_n(t)$, $b_n(0) = B_n$, where

$$B_n = \int_0^1 f(x) y_n(x)\,dx = 0 \text{ since } f(x) = 0$$

$$\therefore \left[e^{\lambda_n t} b_n(t)\right]' = \gamma_n(t) e^{\lambda_n t}, \quad \int_0^t \left[e^{\lambda_n s} b_n(s)\right]' ds = \int_0^t \gamma_n(s) e^{\lambda_n s}\,ds$$

$$\therefore e^{\lambda_n t} b_n(t) - \overbrace{b_n(0)}^{= B_n = 0} = \frac{4\sqrt{2}\sin\left(\frac{n\pi}{2}\right)}{n^2\pi^2}\left[\frac{e^{\lambda_n t} - 1}{\underbrace{\lambda_n}_{= n^2\pi^2}}\right]$$

$$\therefore b_n(t) = \frac{4\sqrt{2}\sin\left(\frac{n\pi}{2}\right)}{n^2\pi^2}\left[\frac{1 - e^{-\lambda_n t}}{n^2\pi^2}\right]$$

$$= \frac{4\sqrt{2}\sin\left(\frac{n\pi}{2}\right)}{n^4\pi^4}\left(1 - e^{-n^2\pi^2 t}\right)$$

$$\therefore\ u(x,t) = \sum_{n=1}^{\infty} b_n(t)\, y_n(x) = \sum_{n=1}^{\infty} \frac{4\sqrt{2}\sin\left(\frac{n\pi}{2}\right)}{n^2\pi^4}\left(1-e^{-n^2\pi^2 t}\right)\sqrt{2}\sin(n\pi x)$$

$$\boxed{= 8\sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi}{2}\right)}{n^4\pi^4}\left(1-e^{-n^2\pi^2 t}\right)\sin(n\pi x)}$$

22.

Using $r(x)u_t = (p(x)u_x)_x - q(x)u + F(x,t)$, $r(x)=1$, $p(x)=1$,

$q(x)=0$, $F(x,t)=e^{-t}(1-x)$, $f(x)=0$.

Using $u(x,t) = y(x)T(t)$, the homogeneous equation is

$y''(x) + \lambda_n y(x) = 0$, $y(0)=0$, $y'(1)=0$. From #2 above,

the normalized $y_n(x) = \sqrt{2}\sin\left[\frac{(2n-1)}{2}\pi x\right]$, $\lambda_n = \frac{(2n-1)^2\pi^2}{4}$, $n=1,2,3,\ldots$

$\therefore$ Assume $u(x,t) = \sum_{n=1}^{\infty} b_n(t)\, y_n(x)$

From $\frac{F(x,t)}{r(x)} = \sum_{n=1}^{\infty} \gamma_n(t)\, y_n(x)$, $\gamma_n(t) = \int_0^1 F(x,t)\, y_n(x)\, dx$,

$$\therefore\ \gamma_n(t) = \int_0^1 e^{-t}(1-x)\sqrt{2}\sin\left[\frac{(2n-1)\pi x}{2}\right] dx$$

This integration was done in #6, 7 in Section 11.2,

and can also be done using MATLAB.

```
clear
syms x n
f = (1-x)*sin((2*n-1)*pi*x/2);
i = int(f,x,0,1)
```

i =

$$\frac{4\cos(\pi\, n) + \pi\,(-2+4\,n)}{\pi^2 - 4\,\pi^2\,n + 4\,\pi^2\,n^2}$$

$$\therefore \int_0^1 (1-x)\sin\left[\frac{(2n-1)\pi x}{2}\right]dx = \frac{4(-1)^n + \pi(4n-2)}{\pi^2(2n-1)^2}$$

$$\therefore \gamma(t) = \frac{\sqrt{2}\left[4(-1)^n + \pi(4n-2)\right]}{\pi^2(2n-1)^2}e^{-t}$$

From $b_n'(t) + \lambda_n b_n(t) = \gamma_n(t)$, $b_n(0) = B_n$, where

$$B_n = \int_0^1 f(x)y_n(x)\,dx = 0 \text{ since } f(x) = 0$$

$$\therefore \left[e^{\lambda_n t}b_n(t)\right]' = \gamma_n(t)e^{\lambda_n t}, \quad \int_0^t \left[e^{\lambda_n s}b_n(s)\right]'ds = \int_0^t \gamma_n(s)e^{\lambda_n s}ds$$

$$\therefore e^{\lambda_n t}b_n(t) - \overset{=B_n=0}{b_n(0)} = \frac{\sqrt{2}\left[4(-1)^n + \pi(4n-2)\right]}{\pi^2(2n-1)^2}\int_0^t e^{(\lambda_n-1)s}ds$$

$$\therefore b_n(t) = \frac{\sqrt{2}\left[4(-1)^n + \pi(4n-2)\right]}{\pi^2(2n-1)^2}\cdot\frac{e^{(\lambda_n-1)t}-1}{(\lambda_n-1)}\cdot e^{-\lambda_n t}$$

$$= \frac{\sqrt{2}\left[4(-1)^n + \pi(4n-2)\right]}{\pi^2(2n-1)^2}\,\frac{\left(e^{-t} - e^{-\frac{(2n-1)^2\pi^2}{4}t}\right)}{\left[\frac{(2n-1)^2\pi^2}{4} - 1\right]}$$

$$\therefore u(x,t) = \sum_{n=1}^{\infty} b_n(t)y_n(x) =$$

$$\boxed{2\sum_{n=1}^{\infty}\frac{\left[4(-1)^n + \pi(4n-2)\right]}{\pi^2(2n-1)^2}\,\frac{\left(e^{-t} - e^{-\frac{(2n-1)^2\pi^2}{4}t}\right)}{\left[\frac{(2n-1)^2\pi^2}{4} - 1\right]}\sin\left[\frac{(2n-1)\pi x}{2}\right]}$$

23.

(a)

$$(p(x)v')' - q(x)v = pv'' + p'v' - qv$$

$$= p(u_{xx} - w_{xx}) + p'(u_x - w_x) - q(x)v$$

$$= pu_{xx} + p'u_x - (pw_{xx} + p'w_x + q(x)v)$$

$$= (pu_x)_x - [(pw_x)_x + q(x)v]$$

$$= r(x)u_t + q(x)u - F(x) - [(pw_x)_x + q(x)v]$$

$$= r(x)\frac{\partial}{\partial t}(u-v) + q(x)(u-v) - F(x) - (pw_x)_x$$

$$= r(x)w_t + q(x)w - (pw_x)_x - F(x)$$

$$\therefore \underbrace{(p(x)v')' - q(x)v}_{= -F(x)} + F(x) = -(pw_x)_x + q(x)w + r(x)w_t$$

$$\therefore 0 = (-pw_x)_x + q(x)w + r(x)w_t$$

Or, $(p(x)w_x)_x - q(x)w = r(x)w_t$

$$w(0,t) = u(0,t) - v(0) = T_1 - T_1 = 0$$

$$w(1,t) = u(1,t) - v(1) = T_2 - T_2 = 0$$

$$w(x,0) = u(x,0) - v(x) = f(x) - v(x)$$

$$\therefore \boxed{\begin{array}{l} (p(x)w_x)_x - q(x)w = r(x)w_t \\ w(0,t) = 0, \quad w(1,t) = 0, \quad w(x,0) = f(x) - v(x) \end{array}}$$

(3)

[1] Let $v(x)$ be the solution to:

$$(p(x)v')' - q(x)v = -F(x),$$

$$v'(0) - h_1 v(0) = T_1, \quad v_x(1) + h_2 v(1) = T_2$$

[2] Let $w(x,t) = u(x,t) - v(x)$. If $w(x,t)$ satisfies

$$(p(x)w_x)_x - q(x)w = r(x)w_t$$

$$w_x(0,t) - h_1 w(0,t) = 0, \quad w_x(1,t) + h_2 w(1,t) = 0$$

$$w(x,0) = f(x) - v(x)$$

then $u_x(0,t) - h_1 u(0,t) = w_x(0,t) + v'(0) - h_1 w(0,t) - h_1 v(0)$

$$= 0 + T_1 = T_1 \qquad [3]$$

$$u_x(1,t) + h_2 u(1,t) = w_x(1,t) + v'(1) + h_2 w(1,t) + v(1)$$

$$= 0 + T_2 = T_2 \qquad [4]$$

$$u(x,0) = w(x,0) + v(x) = f(x) - v(x) + v(x) = f(x) \quad [5]$$

and $r(x)\, u_t = r(x) \dfrac{\partial}{\partial t}\left(w(x,t) + v(x)\right) = r(x)\, w_t$

$$(p(x) u_x)_x - q(x) u + F(x) =$$

$$(p(x)(w_x + v'))_x - q(x)(w + v) + F(x) =$$

$$(p(x) w_x + p(x) v')_x - q(x) w - q(x) v + F(x) =$$

$$(p(x) w_x)_x + (p(x) v')' - q(x) w - q(x) v + F(x) =$$

$$\underbrace{(p(x) w_x)_x - q(x) w}_{r(x)\, w_t} + \underbrace{(p(x) v')' - q(x) v}_{(-F(x))} + F(x) = r(x)\, w_t$$

$$\therefore\ r(x)\, u_t = (p(x) u_x)_x - q(x) u + F(x)$$

$\therefore$ Find $v(x)$ from [1] and $w(x,t)$ from [2] to get $u(x,t) = w(x,t) + v(x)$ which satisfies [3], [4], [5].

24.

Here, $p(x)=1$, $q(x)=0$, $r(x)=1$, $F(x)=-2$, $f(x)=x^2-2x+2$,

(1) First solve $v''(x)=2$, $v(0)=1$, $v(1)=0$

$\therefore v(x)=x^2+c_1x+c_2$, $v(0)=1 \Rightarrow c_2=1$, $v(1)=0 \Rightarrow c_1=-2$

$\therefore v(x)=x^2-2x+1$

(2) Solve $w_{xx}=w_t$, $w(0,t)=0$, $w(1,t)=0$, $w(x,0)=f(x)-v(x)$,

$\therefore w(x,0)=(x^2-2x+2)-(x^2-2x+1)=1$

Consider $y''(x)=\lambda y(x)$, $y(0)=0$, $y(1)=0$

$\therefore \lambda_n=n^2\pi^2$, normalized $y_n(x)=\sqrt{2}\sin(n\pi x)$

using Example 1 from section 11.2.

Assume $w(x,t)=\sum_{n=1}^{\infty} b_n(t)\,y_n(x)$

From $w_t=w_{xx}+G(x,t)$, and $G(x,t)=0$,

$$\gamma_n(t)=\int_0^1 G(x,t)\,y_n(x)\,dx=\int_0^1 0\cdot y_n(x)\,dx=0$$

$\therefore$ From $b_n'(t)+\lambda_n b_n(t)=\gamma_n(t)$,

$b_n'(t) + \lambda_n b_n(t) = 0$, and $b_n(0) = B_n$, where

$$B_n = \int_0^1 w(x,0)\, y_n(x)\, dx = \sqrt{2}\int_0^1 \sin(n\pi x)\, dx$$

$$= -\frac{\sqrt{2}}{n\pi}\cos(n\pi x)\Big|_0^1 = -\frac{\sqrt{2}}{n\pi}\left[(-1)^n - 1\right] = \frac{\sqrt{2}}{n\pi}\left[1-(-1)^n\right]$$

$$\therefore \frac{b_n'(t)}{b_n(t)} = -n^2\pi^2, \quad \int_0^t \frac{b_n'(s)}{b_n(s)}\, ds = -n^2\pi^2 \int_0^t ds$$

$$\log\left(b_n(t)\right) - \log\big(\underbrace{b_n(0)}_{=B_n}\big) = -n^2\pi^2 t$$

$$b_n(t) = B_n e^{-n^2\pi^2 t} = \frac{\sqrt{2}}{n\pi}\left[1-(-1)^n\right] e^{-n^2\pi^2 t}$$

$$\therefore w(x,t) = \frac{2}{n\pi}\left[1-(-1)^n\right] e^{-n^2\pi^2 t}\sin(n\pi x)$$

(3) $u(x,t) = v(x) + w(x,t) = x^2 - 2x + 1 + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\left[1-(-1)^n\right]}{n} e^{-n^2\pi^2 t}\sin(n\pi x)$

Note that for $n = 1, 2, 3, \ldots$, $1-(-1)^n = 2, 0, 2, 0, \ldots$

$\therefore$ Odd terms are 2 and even terms are 0.

$\therefore$ Just count the odd terms by replacing n with $(2n-1)$ and count by $n = 1, 2, 3, \ldots$

$$\therefore \boxed{u(x,t) = x^2 - 2x + 1 + \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{e^{-(2n-1)^2\pi^2 t}}{(2n-1)}\sin\left[(2n-1)\pi x\right]}$$

25.

$r(x)=1,\ p(x)=1,\ q(x)=0,\ F(x)=-\pi^2\cos(\pi x),\ f(x)=\cos\left(\frac{3\pi x}{2}\right)-\cos(\pi x)$

(1) Let $v(x)$ be the solution to: $(p(x)v')'-q(x)v=-F(x)$,

$\therefore\ v''=\pi^2\cos(\pi x),\quad v'(0)=0,\ v(1)=1$

Homogeneous: $v_c''=0 \Rightarrow v_c(x)=Ax+B$

Nonhomogeneous: Let $v_p(x)=C\cos(\pi x)$. $\therefore v_p''=-C\pi^2\cos(\pi x)$

$\therefore\ C=-1.$

$\therefore\ v(x)=Ax+B-\cos(\pi x),\quad v'=A-\pi\sin(\pi x)$

$\therefore\ v'(0)=A=0.\quad \therefore\ v(x)=B-\cos(\pi x)$

$v(1)=B+1=1 \Rightarrow B=0$

$\therefore\ \underline{v(x)=-\cos(\pi x)}$

(2) Let $w(x,t)=u(x,t)-v(x)$

$\therefore$ Solve $(p(x)w_x)_x-q(x)w=r(x)w_t$, or

$w_{xx}=w_t,\quad w_x(0,t)=0,\ w(1,t)=0,\ w(x,0)=f(x)-v(x)$,

or $w(x,0)=\cos\left(\frac{3\pi x}{2}\right)-\cos(\pi x)+\cos(\pi x)$

$\therefore\ w(x,0) = \cos\left(\frac{3\pi x}{2}\right)$

$\therefore$ Assume $w(x,t) = \sum_{n=1}^{\infty} b_n(t)\, y_n(x)$, and consider

$-(p(x)y'(x))' + q(x)y = \lambda r(x) y(x)$, or $-y'' = \lambda y$, or

$y'' + \lambda y = 0, \quad y'(0) = 0, \quad y(1) = 0$

From Problem #2, Section 11.2, the normalized

$y_n(x) = \sqrt{2}\cos\left[\frac{(2n-1)}{2}\pi x\right], \quad \lambda_n = \frac{(2n-1)^2\pi^2}{4}, \quad n = 1, 2, 3, \ldots$

From $w_t = w_{xx} + G(x,t)$ with $G(x,t) = 0$,

$\gamma_n(t) = \int_0^1 G(x,t)\, y_n(x)\, dx = 0$

$\therefore$ From $b_n'(t) + \lambda_n b_n(t) = \gamma_n(t) = 0, \quad b_n'(t) = -\lambda_n b_n(t)$

$\frac{b_n'(t)}{b_n(t)} = -\frac{(2n-1)^2\pi^2}{4}, \quad \int_0^t \frac{b_n'(s)}{b_n(s)}\, ds = -\lambda_n \int_0^t ds = -\lambda_n t$

$\therefore\ b_n(t) = b_n(0)\, e^{-\lambda_n t}$, where

$b_n(0) = \beta_n = \int_0^1 r(x)\, w(x,0)\, y_n(x)\, dx$

$= \int_0^1 \cos\left(\frac{3\pi x}{2}\right) \sqrt{2}\cos\left[\frac{(2n-1)\pi x}{2}\right] dx$

Using MATLAB to do the integration,

```
clear
syms x n
f1 = cos(3*pi*x/2);
f2 = sqrt(2)*cos((2*n-1)*pi*x/2);
i = int(f1*f2,x,0,1)
```

i =

$$\begin{cases} \frac{\sqrt{2}}{2} & \text{if } n = -1 \lor n = 2 \\ -\frac{3\sqrt{2}\sin(\pi n)}{2\pi(2+n-n^2)} & \text{if } n \neq -1 \land n \neq 2 \end{cases}$$

$\therefore$ since $\sin(n\pi)=0$, $B_2 = \frac{\sqrt{2}}{2}$, $B_n = 0$ for $n = 1, 3, 4, 5, 6, \ldots$

$$\therefore b_2(t) = \frac{\sqrt{2}}{2} e^{-\lambda_2 t} = \frac{\sqrt{2}}{2} e^{-\frac{9\pi^2}{4}t}$$

$$\therefore w(x,t) = \sum_{n=1}^{\infty} b_n(t) y_n(x) = b_2(t) y_2(x)$$

$$= \frac{\sqrt{2}}{2} e^{-\frac{9\pi^2}{4}t} \left[\sqrt{2}\cos\left(\frac{3\pi x}{2}\right)\right] = e^{-\frac{9\pi^2}{4}t}\cos\left(\frac{3\pi x}{2}\right)$$

(3) $u(x,t) = w(x,t) + v(x)$

$$\therefore \boxed{u(x,t) = e^{-\frac{9\pi^2}{4}t}\cos\left(\frac{3\pi x}{2}\right) - \cos(\pi x)}$$

26.

(a)

$$r(x)XT'' = (p(x)X'T)' - q(x)XT = T(p(x)X')' - q(x)XT$$

Dividing by $r(x)XT$,

$$\frac{T''(t)}{T(t)} = \frac{(p(x)X')'}{r(x)X} - \frac{q(x)}{r(x)}$$

Since the left side is a function of t only, and the right side of x only, each side must equal the same constant.

$$\therefore \frac{(p(x)X')'}{r(x)X} - \frac{q(x)}{r(x)} = -\lambda, \text{ or } (p(x)X')' - q(x)X = -\lambda r(x)X$$

$$\therefore -(p(x)X')' + q(x)X = \lambda r(x)X$$

$u_x(x,t) = X'(x)T(t)$

$\therefore u_x(0,t) - h_1 u(0,t) = 0$ becomes $X'(0)T(t) - h_1 X(0)T(t) = 0$.

Assuming $T(t) \neq 0$ for any t, then $X'(0) - h_1 X(0) = 0$

Also, $u_x(1,t) + h_2 u(1,t) = 0$ becomes $X'(1)T(t) + h_2 X(1)T(t) = 0$

Again dividing by $T(t)$, then $X'(1) + h_2 X(1) = 0$

Note: the assumptions of $X(x) \neq 0$ and $T(t) \neq 0$ were made in order to find nonzero solutions $u(x,t) = X(x)T(t)$.

(b)

From $u(x,t) = \sum_{n=1}^{\infty} b_n(t)\phi_n(x)$ and

$r(x)u_{tt} = (p(x)u_x)_x - q(x)u + F(x,t),$ [0]

$r(x)u_{tt} = r(x)\sum_{n=1}^{\infty} b_n''(t)\phi_n(x)$ [1]

$$(p(x)u_x)_x = \frac{\partial}{\partial x}\left(p(x)\sum_{n=1}^{\infty} b_n(t)\phi_n'(x)\right)$$

$$= \frac{\partial}{\partial x}\sum_{n=1}^{\infty} b_n(t)p(x)\phi_n'(x) = \sum_{n=1}^{\infty} b_n(t)\frac{\partial}{\partial x}(p(x)\phi_n'(x))$$

$$= \sum_{n=1}^{\infty} b_n(t)(p(x)\phi_n'(x))'$$

$$\therefore (p(x)u_x)_x - q(x)u = \sum_{n=1}^{\infty}\left[b_n(t)(p(x)\phi_n'(x))' - q(x)b_n(t)\phi_n(x)\right]$$

$$= \sum_{n=1}^{\infty} b_n(t)\left[(p(x)\phi_n'(x))' - q(x)\phi_n(x)\right]$$ [2]

Since each $\phi_n(x)$ is a solution to, from (a),

$$-(p(x)X')' + q(x)X = \lambda r(x)X, \quad X'(0) - h_1X(0) = 0,$$

$$\text{and} \quad X'(1) + h_2X(1) = 0,$$

then [2] becomes

$$(p(x)u_x)_x - q(x)u = -r(x)\sum_{n=1}^{\infty} b_n(t)\lambda_n\phi_n(x) \qquad [3]$$

Let $\dfrac{F(x,t)}{r(x)} = \sum_{n=1}^{\infty} \gamma_n(t)\phi_n(x)$, where

$$\gamma_n(t) = \int_0^1 r(x)\,\frac{F(x,t)}{r(x)}\phi_n(x)\,dx = \int_0^1 F(x,t)\phi_n(x)\,dx \qquad [4]$$

$$\therefore F(x,t) = r(x)\sum_{n=1}^{\infty}\gamma_n(t)\phi_n(x)$$

$\therefore$ [0] becomes:

$$r(x)\sum_{n=1}^{\infty} b_n''(t)\phi_n(x) = -r(x)\sum_{n=1}^{\infty} b_n(t)\lambda_n\phi_n(x) + r(x)\sum_{n=1}^{\infty}\gamma_n(t)\phi_n(x)$$

Dividing by $r(x)$ and rearranging,

$$\sum_{n=1}^{\infty}\left[b_n''(t) + \lambda_n b_n(t) - \gamma_n(t)\right]\phi_n(x) = 0$$

If this is to be true for all $0<x<1$ for each n,

then $b_n''(t) + \lambda_n b_n(t) = \gamma_n(t)$ for each n. [5]

To determine $b_n(t)$ completely, initial conditions are needed: $b_n(0) = \alpha_n$, $b_n'(0) = \beta_n$

From $u(x,t) = \sum_{n=1}^{\infty} b_n(t)\phi_n(x)$, $u(x,0) = f(x) = \sum_{n=1}^{\infty} b_n(0)\phi_n(x)$

$\therefore\ f(x) = \sum_{n=1}^{\infty} \alpha_n \phi_n(x)$, so $\alpha_n = \int_0^1 r(x) f(x) \phi_n(x)\,dx$ [6]

Also, $u_t(x,0) = g(x) = \sum_{n=1}^{\infty} b_n'(0)\phi_n(x) = \sum_{n=1}^{\infty} \beta_n \phi_n(x) dx$

$$\therefore\ \beta_n = \int_0^1 r(x) g(x) \phi_n(x)\,dx \quad [7]$$

In summary from [4], [5], [6], [7],

$$b_n''(t) + \lambda_n b_n(t) = \gamma_n(t) \quad , \quad n = 1,2,3,\ldots$$

$$\alpha_n = b_n(0), \quad \beta_n = b_n'(0),$$

where $\gamma_n(t) = \int_0^1 F(x,t)\phi_n(x)\,dx$

$$\alpha_n = \int_0^1 r(x) f(x) \phi_n(x)\,dx$$

$$\beta_n = \int_0^1 r(x) g(x) \phi_n(x)\,dx$$

27.

(a)

Since the $\vec{e}^{(i)}$ are orthonormal, they form a basis in C^n, C = complex numbers. $\therefore$ There exist

$b_j \in \mathbb{C}$ s.t. $\vec{b} = \sum_{j=1}^{n} b_j \vec{e}^{(j)}$

$$\therefore (\vec{b}, \vec{e}^{(i)}) = \left(\sum_{j=1}^{n} b_j \vec{e}^{(j)}, \vec{e}^{(i)}\right)$$

$$= \sum_{j=1}^{n} \left(b_j \vec{e}^{(j)}, \vec{e}^{(i)}\right)$$

$$= b_i, \text{ since } (\vec{e}^{(j)}, \vec{e}^{(i)}) = 0 \text{ for } j \neq i,$$

$$\text{and } (\vec{e}^{(i)}, \vec{e}^{(i)}) = 1$$

(b)

From $\vec{x} = \sum_{i=1}^{n} a_i \vec{e}^{(i)}$, $A\vec{x} - \mu\vec{x} = \vec{b}$ becomes

$$A\left(\sum_{i=1}^{n} a_i \vec{e}^{(i)}\right) - \mu\left(\sum_{i=1}^{n} a_i \vec{e}^{(i)}\right) = \sum_{i=1}^{n} b_i \vec{e}^{(i)}$$

Note that $A\vec{e}^{(i)} = \lambda_i \vec{e}^{(i)}$ by definition.

$$\therefore A a_i \vec{e}^{(i)} = a_i \lambda_i \vec{e}^{(i)}$$

$\therefore A\vec{x} - \mu\vec{x} = \vec{b}$ becomes

$$\sum_{i=1}^{n} a_i \lambda_i \vec{e}^{(i)} - \sum_{i=1}^{n} \mu a_i \vec{e}^{(i)} = \sum_{i=1}^{n} b_i \vec{e}^{(i)}$$

$$\therefore \sum_{i=1}^{n} (a_i \lambda_i - \mu a_i - b_i) \vec{\epsilon}^{(i)} = \vec{0}$$

Since $\vec{\epsilon}^{(i)}$ are independent,

$$a_i \lambda_i - \mu a_i - b_i = 0,$$

$$\therefore a_i = \frac{b_i}{\lambda_i - \mu}, \text{ assuming } \mu \neq \lambda_i \text{ for any } i.$$

$$\therefore \vec{x} = \sum_{i=1}^{n} \left(\frac{b_i}{\lambda_i - \mu} \right) \vec{\epsilon}^{(i)} = \sum_{i=1}^{n} \left(\frac{(\vec{b}, \vec{\epsilon}^{(i)})}{\lambda_i - \mu} \right) \vec{\epsilon}^{(i)}$$

(c)

Equation (13) is $\phi(x) = \sum_{i=1}^{\infty} \frac{c_i}{\lambda_i - \mu} \phi_i(x)$, and has the same form as in (b).

The eigenfunctions $\phi_i(x)$ are analogous to the eigenvectors $\vec{\epsilon}^{(i)}$, effectively spanning the space of real valued functions. λ_i are the corresponding eigenvalues. c_i are like $b_i = (\vec{b}, \vec{\epsilon}^{(i)})$, which are coefficients for the nonhomogeneous term.

28.

(a) Look at $y'' = -f(x)$. From the homogeneous problem, $y_c'' = 0 \Rightarrow y_c(x) = c_1 + c_2 x = c_1(1) + c_2 x$

Using variation of parameters, assume

$$y_p(x) = c_1(x)(1) + c_2(x)x = c_1(x) + c_2(x)\,x$$

$$\therefore\ y_p' = c_1'(x) + c_2(x) + c_2'(x)\,x$$

Require terms of $c_1'(x)$ and $c_2'(x)$ be zero.

$$\therefore\ c_1'(x) + c_2'(x)\,x = 0$$

$$\therefore\ y_p' = c_2(x)$$

$$\therefore\ y_p'' = c_2'(x) = -f(x)$$

In matrix form, $\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_1'(x) \\ c_2'(x) \end{bmatrix} = \begin{bmatrix} 0 \\ -f(x) \end{bmatrix}$

$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -x \\ 0 & 1 \end{bmatrix} \quad \therefore\ \begin{bmatrix} c_1'(x) \\ c_2'(x) \end{bmatrix} = \begin{bmatrix} 1 & -x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -f(x) \end{bmatrix}$$

$$\therefore\ c_1'(x) = x f(x), \quad c_2'(x) = -f(x)$$

$$\therefore\ c_1(x) = \int_0^x s f(s)\,ds, \quad c_2(x) = \int_0^x -f(s)\,ds$$

$$\therefore\ y_p(x) = c_1(x) + x\, c_2(x) = \int_0^x s f(s)\,ds - x \int_0^x f(s)\,ds$$

$$= \int_0^x (s-x) f(s)\,ds = -\int_0^x (x-s) f(s)\,ds$$

$$\therefore\ \phi(x) = y_c(x) + y_p(x) = C_1 + C_2 x - \int_0^x (x-s) f(s)\,ds$$

(b)

From (a), $\phi(0) = C_1 + C_2 \cdot 0 - \underbrace{\int_0^0 (0-s) f(s)\,ds}_{=0} = 0 \Rightarrow C_1 = 0$

$\phi(1) = C_1 + C_2 \cdot 1 - \int_0^1 (1-s) f(s)\,ds = 0 \Rightarrow C_2 = \int_0^1 (1-s) f(s)\,ds$

(c)

From (b) with $C_1 = 0$, $C_2 = \int_0^1 (1-s) f(s)\,ds$, and (a),

$$\phi(x) = x \int_0^1 (1-s) f(s)\,ds - \int_0^x (x-s) f(s)\,ds$$

$$= x\int_0^x (1-s) f(s)\,ds + x\int_x^1 (1-s) f(s)\,ds - \int_0^x (x-s) f(s)\,ds$$

$$= \underbrace{\int_0^x x f(s)\,ds} - \int_0^x xs f(s)\,ds + \int_x^1 x(1-s) f(s)\,ds - \underbrace{\int_0^x x f(s)\,ds} + \int_0^x s f(s)\,ds$$

cancel

$$= -\int_0^x xs f(s)\,ds + \int_0^x s f(s)\,ds + \int_x^1 x(1-s) f(s)\,ds$$

$$= \int_0^x (s - xs) f(s)\,ds + \int_x^1 x(1-s) f(s)\,ds$$

$$= \int_0^x s(1-x) f(s)\,ds + \int_x^1 x(1-s) f(s)\,ds$$

(d)

Using (c) and the definition of $G(x,s)$,

$$\phi(x) = \int_0^x s(1-x) f(s)\,ds + \int_x^1 x(1-s) f(s)\,ds$$

$$= \int_0^x G(x,s) f(s)\,ds + \int_x^1 G(x,s) f(s)\,ds$$

$$= \int_0^1 G(x,s) f(s)\,ds$$

29.

Homogeneous problem: $y''+y=0$, $y_c = c_1\cos(x) + c_2\sin(x)$.

Nonhomogeneous problem: $y''+y = -f(x)$.

Let $y_p(x) = c_1(x)\cos(x) + c_2(x)\sin(x)$ [1]

$\therefore y_p' = c_1'(x)\cos(x) - c_1(x)\sin(x) + c_2'(x)\sin(x) + c_2(x)\cos(x)$

Require terms with $c_1'(x)$ and $c_2'(x)$ vanish, so

$c_1'(x)\cos(x) + c_2'(x)\sin(x) = 0.$ [2]

$\therefore y_p' = -c_1(x)\sin(x) + c_2(x)\cos(x)$

$y_p'' = -c_1'(x)\sin(x) - c_1(x)\cos(x) + c_2'(x)\cos(x) - c_2(x)\sin(x)$

Using [1], $y_p''+y_p = -c_1'(x)\sin(x) + c_2'(x)\cos(x) = -f(x)$ [3]

[2] and [3] in matrix form become:

$$\begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix} \begin{bmatrix} c_1'(x) \\ c_2'(x) \end{bmatrix} = \begin{bmatrix} 0 \\ -f(x) \end{bmatrix}$$

$$\begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}^{-1} = \begin{bmatrix} \cos(x) & -\sin(x) \\ \sin(x) & \cos(x) \end{bmatrix}$$

$$\therefore \begin{bmatrix} c_1'(x) \\ c_2'(x) \end{bmatrix} = \begin{bmatrix} \cos(x) & -\sin(x) \\ \sin(x) & \cos(x) \end{bmatrix} \begin{bmatrix} 0 \\ -f(x) \end{bmatrix} = \begin{bmatrix} \sin(x) f(x) \\ -\cos(x) f(x) \end{bmatrix}$$

$$\therefore \quad c_1'(x) = \sin(x) f(x) \qquad c_2'(x) = -\cos(x) f(x)$$

$$\therefore \quad c_1(x) = \int_0^x \sin(s) f(s)\, ds \qquad c_2(x) = -\int_0^x \cos(s) f(s)\, ds$$

$$\therefore \phi(x) = y_c(x) + y_p(x)$$

$$= c_1 \cos(x) + c_2 \sin(x)$$

$$+ \cos(x) \int_0^x \sin(s) f(s)\, ds - \sin(x) \int_0^x \cos(s) f(s)\, ds$$

$$= c_1 \cos(x) + c_2 \sin(x) + \int_0^x \left[\cos(x)\sin(s) - \sin(x)\cos(s)\right] f(s)\, ds$$

$$= c_1 \cos(x) + c_2 \sin(x) + \int_0^x \sin(s-x) f(s)\, ds$$

$$\therefore \phi(0) = 0 \Rightarrow c_1 = 0 \qquad \phi(1) = 0 \Rightarrow c_2 \sin(1) + \int_0^1 \sin(s-1) f(s)\, ds = 0$$

$$\therefore \quad c_2 = -\frac{\int_0^1 \sin(s-1) f(s)\, ds}{\sin(1)} = \frac{\int_0^1 \sin(1-s) f(s)\, ds}{\sin(1)}$$

$$\therefore\ \phi(x) = \frac{\sin(x)}{\sin(1)}\int_0^1 \sin(1-s)\, f(s)\, ds + \int_0^x \sin(s-x)\, f(s)\, ds$$

For $0 \le x \le 1$, $\int_0^1 \sin(1-s) f(s)\, ds = \int_0^x \sin(1-s) f(s)\, ds + \int_x^1 \sin(1-s) f(s)\, ds$

$$\therefore\ \phi(x) = \int_0^x \frac{\sin(x)\sin(1-s)\, f(s)}{\sin(1)}\, ds + \int_0^x \frac{\sin(1)}{\sin(1)} \sin(s-x)\, f(s)\, ds$$

$$+ \int_x^1 \frac{\sin(x)}{\sin(1)} \sin(1-s)\, f(s)\, ds$$

$$= \int_0^x \frac{[\sin(x)\sin(1-s) + \sin(1)\sin(s-x)]}{\sin(1)}\, f(s)\, ds$$

$$+ \int_x^1 \frac{\sin(x)\sin(1-s)}{\sin(1)}\, f(s)\, ds$$

Note: $\sin(x)\sin(1-s) + \sin(1)\sin(s-x) =$

$$\sin(x)\sin(1)\cos(s) - \sin(x)\cos(1)\sin(s)$$

cancel

$$+ \sin(1)\sin(s)\cos(x) - \sin(1)\cos(s)\sin(x)$$

$$= \sin(s)\left[\sin(1)\cos(x) - \cos(1)\sin(x)\right] = \sin(s)\sin(1-x)$$

$$\therefore\ \phi(x) = \int_0^x \frac{\sin(s)\sin(1-x)}{\sin(1)}\, f(s)\, ds + \int_x^1 \frac{\sin(x)\sin(1-s)}{\sin(1)}\, f(s)\, ds$$

Letting $G(x,s) = \begin{cases} \dfrac{\sin(s)\sin(1-x)}{\sin(1)}, & 0 \le s \le x \\ \dfrac{\sin(x)\sin(1-s)}{\sin(1)}, & x \le s \le 1 \end{cases}$

(and noting $\lim_{s \to x^-} G(x,s) = \lim_{s \to x^+} G(x,s)$)

then
$$\phi(x) = \int_0^x G(x,s) f(s) ds + \int_x^1 G(x,s) f(s) ds$$

$$\therefore \quad \underline{\phi(x) = \int_0^1 G(x,s) f(s) ds}$$

30.

(a)

For # 28, $p(x) = 1$, $q(x) = 0$, $\alpha_1 = 1$, $\alpha_2 = 0$, $\beta_1 = 1$, $\beta_2 = 0$

$\therefore L[y] = -y''$ $\therefore L[y] = 0 \Rightarrow y_c(x) = c_1 + c_2 x$

$y_c(0) = 0 \Rightarrow c_1 = 0 \Rightarrow \underline{y_1(x) = x}$, setting $c_2 = 1$ for simplicity. $y_c(1) = 0 \Rightarrow c_1 + c_2 = 0$, $\therefore c_1 = -c_2$.

$\therefore\ y_2(x) = -1 + x$, setting $c_2 = 1$ for simplicity.

$$\therefore\ W[y_1, y_2](x) = \begin{vmatrix} x & -1+x \\ 1 & 1 \end{vmatrix} = x - (-1+x) = 1 \neq 0$$

$$\therefore\ p(x)W[y_1, y_2](x) = 1$$

$$\therefore\ G(x,s) = \begin{cases} -\dfrac{y_1(s)\,y_2(x)}{p(x)\,W[y_1, y_2](x)}, & 0 \le s \le x \\[2ex] -\dfrac{y_1(x)\,y_2(s)}{p(x)\,W[y_1, y_2](x)}, & x \le s \le 1 \end{cases}$$

$$\Rightarrow \begin{cases} -\dfrac{(s)(-1+x)}{1}, & 0 \le s \le x \\[2ex] -\dfrac{(x)(-1+s)}{1}, & x \le s \le 1 \end{cases}$$

$$\Rightarrow \begin{cases} s(1-x), & 0 \le s \le x \\ x(1-s), & x \le s \le 1 \end{cases}$$

This is the same Green's function as in #28.

(6)

For #29, $p(x) = 1$, $q(x) = -1$, $\alpha_1 = 1$, $\alpha_2 = 0$, $\beta_1 = 1$, $\beta_2 = 0$

$\therefore L[y] = -y'' - y$. As in #29, the solution to

$L[y] = 0$ is $y_c(x) = c_1 \cos(x) + c_2 \sin(x)$

For $y_c(0) = 0$, $c_1 = 0$, so $\underline{y_1(x) = \sin(x)}$,

assigning $\underline{c_2 = 1}$ for simplicity.

For $y_c(1) = 0$, $c_1 \cos(1) + c_2 \sin(1) = 0$

If $c_1 = 0$, then $c_2 = 0$ since $\sin(1) \neq 0$.

If $c_2 = 0$, then $c_1 = 0$ since $\cos(1) \neq 0$.

$\therefore \tan(1) = -\frac{c_1}{c_2}$ and for simplicity, set $\underline{c_2 = 1}$,

$\underline{y_2(x) =} \sin(x) - \tan(1)\cos(x)$

$$\therefore W[y_1, y_2](x) = \begin{vmatrix} \sin(x) & \sin(x) - \tan(1)\cos(x) \\ \cos(x) & \cos(x) + \tan(1)\sin(x) \end{vmatrix}$$

$$= \sin(x)\cos(x) + \tan(1)\sin^2(x) - [\sin(x)\cos(x) - \tan(1)\cos^2(x)]$$

$$= \tan(1)[\sin^2(x) + \cos^2(x)] = \tan(1) \neq 0.$$

$\therefore p(x) W[y_1, y_2](x) = \tan(1)$

$$\therefore \quad G(x,s) = \begin{cases} \dfrac{-y_1(s)\,y_2(x)}{p(x)\,W[y_1,y_2](x)} & 0 \le s \le x \\ \dfrac{-y_1(x)\,y_2(s)}{p(x)\,W[y_1,y_2](x)} & x \le s \le 1 \end{cases}$$

$$\Rightarrow \begin{cases} \dfrac{-\sin(s)\left[\sin(x) - \tan(1)\cos(x)\right]}{\tan(1)} & 0 \le s \le x \\ \dfrac{-\sin(x)\left[\sin(s) - \tan(1)\cos(s)\right]}{\tan(1)} & x \le s \le 1 \end{cases}$$

$$\Rightarrow \begin{cases} -\dfrac{1}{\sin(1)}\left[\sin(s)\left[\sin(x)\cos(1) - \sin(1)\cos(x)\right]\right] & 0 \le s \le x \\ -\dfrac{1}{\sin(1)}\left[\sin(x)\left[\sin(s)\cos(1) - \sin(1)\cos(s)\right]\right] & x \le s \le 1 \end{cases}$$

use $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$$\Rightarrow \begin{cases} -\dfrac{1}{\sin(1)}\left[\sin(s)\sin(x-1)\right] & 0 \le s \le x \\ -\dfrac{1}{\sin(1)}\left[\sin(x)\sin(s-1)\right] & x \le s \le 1 \end{cases}$$

$$\Rightarrow \begin{cases} \dfrac{\sin(s)\sin(1-x)}{\sin(1)} & 0 \le s \le x \\ \dfrac{\sin(x)\sin(1-s)}{\sin(1)} & x \le s \le 1 \end{cases}$$

This is the same Green's function in #29.

(c)

$$p(x)\,W[y_1,y_2](x) = p(x)\left[y_1(x)\,y_2'(x) - y_1'(x)\,y_2(x)\right]$$

$$= p y_1 y_2' - p y_1' y_2$$

$$\therefore \frac{d}{dx}\left(p y_1 y_2' - p y_1' y_2\right) = p' y_1 y_2' + p y_1' y_2' + p y_1 y_2''$$

cancel

$$- p' y_1' y_2 - p y_1'' y_2 - p y_1' y_2'$$

$$= p' y_1 y_2' - p' y_1' y_2 + p y_1 y_2'' - p y_1'' y_2$$

$$= p' y_2' y_1 + p y_2'' y_1 - p' y_1' y_2 - p y_1'' y_2$$

$$= (p y_2')' y_1 - (p y_1')' y_2$$

$$= (p y_2')' y_1 - q y_2 y_1 - (p y_1')' y_2 + q y_1 y_2$$

$$= -y_1 L[y_2] + y_2 L[y_1] \qquad L[y] = -(p y')' + q y$$

$$= 0 + 0 = 0$$

$$\therefore \frac{d}{dx}\left[p(x)\,W[y_1,y_2](x)\right] = 0 \Rightarrow$$

$$p(x)\,W[y_1,y_2](x) = \underline{\text{constant}}$$

From the definition of $G(x,s)$, it's assumed constant $\neq 0$.

(d)

Let $p(x)W[y_1, y_2](x) = K$, a constant

$\therefore$ $KG(x,s) = \begin{cases} -y_1(s)y_2(x), & 0 \le s \le x & [1] \\ -y_1(x)y_2(s), & x \le s \le 1 & [2] \end{cases}$

For $KG(s,x)$, just switch the variables:

$KG(s,x) = \begin{cases} -y_1(x)y_2(s), & 0 \le x \le s & [3] \\ -y_1(s)y_2(x), & s \le x \le 1 & [4] \end{cases}$

Consider the three possibilities for x: $x=0$, $x=1$, and $0<x<1$. Note for all 3 cases, $0 \le s \le 1$.

$x=0$: [1] becomes $s=0$, $-y_1(0)y_2(0)$

[2] becomes $0 \le s \le 1$, $-y_1(0)y_2(s)$

$\therefore$ $KG(x,s) = KG(0,s) = \begin{cases} -y_1(0)y_2(0), & s=0 \\ -y_1(0)y_2(s), & 0 \le s \le 1 \end{cases}$

[3] becomes $0 \le s \le 1$, $-y_1(0)y_2(s)$

[4] becomes $s \le 0$, so $s=0$, $-y_1(0)y_2(0)$

$\therefore$ $KG(s,x) = KG(s,0) = \begin{cases} -y_1(0)y_2(s), & 0 \le s \le 1 \\ -y_1(0)y_2(0), & s=0 \end{cases}$

$\therefore$ For $x=0$, $KG(x,s) = KG(s,x)$

$x=1$: [2] becomes $s=1$, $-y_1(1)y_2(s) = -y_1(1)y_2(1)$

[1] becomes $0 \le s \le 1$, $-y_1(s)\,y_2(1)$

$$\therefore KG(x,s) = KG(1,s) = \begin{cases} -y_1(s)\,y_2(1), & 0 \le s \le 1 \\ -y_1(1)\,y_2(1), & s=1 \end{cases}$$

[3] becomes $s=1$, $-y_1(1)y_2(1)$

[4] becomes $0 \le s \le 1$, $-y_1(s)\,y_2(1)$

$$\therefore KG(s,x) = KG(s,1) = \begin{cases} -y_1(1)\,y_2(1), & s=1 \\ -y_1(s)\,y_2(1), & 0 \le s \le 1 \end{cases}$$

$\therefore KG(s,x) = KG(x,s)$ for $x=1$

$0<x<1$: [3] becomes $0<x \le s \le 1$, or $x \le s \le 1$, $-y_1(x)y_2(s)$

[4] becomes $s \le x < 1$, or $0 \le s \le x$, $-y_1(s)y_2(x)$

$$\therefore KG(s,x) = \begin{cases} -y_1(s)\,y_2(x), & 0 \le s \le x \\ -y_1(x)\,y_2(s), & x \le s \le 1 \end{cases}$$

This is the same as $KG(x,s)$ using [1], [2].

$\therefore KG(x,s) = KG(s,x)$ for $0<x<1$

$\therefore$ For all 3 cases, $KG(x,s) = KG(s,x) \Rightarrow \underline{G(x,s) = G(s,x)}$

(e)

From (c), let $K = p(x)W[y_1, y_2](x)$, a constant.

$$\therefore G(x,s) = \begin{cases} -y_1(s)y_2(x)/K, & 0 \le s \le x \\ -y_1(x)y_2(s)/K, & x \le s \le 1 \end{cases} \quad \begin{matrix}[1]\\[2]\end{matrix}$$

From $\phi(x) = \int_0^1 G(x,s)f(s)ds = \int_0^x G(x,s)f(s)ds + \int_x^1 G(x,s)f(s)ds$

$$= -\frac{y_2(x)}{K}\int_0^x y_1(s)f(s)ds - \frac{y_1(x)}{K}\int_x^1 y_2(s)f(s)ds$$

$$= -\frac{y_2(x)}{K}\int_0^x y_1(s)f(s)ds + \frac{y_1(x)}{K}\int_1^x y_2(s)f(s)ds \quad [3]$$

Fundamental Theorem of Calculus

$$\therefore \phi'(x) = -\frac{y_2'(x)}{K}\int_0^x y_1(s)f(s)ds - \frac{y_2(x)}{K}y_1(x)f(x)$$

cancel

$$+\frac{y_1'(x)}{K}\int_1^x y_2(s)f(s)ds + \frac{y_1(x)}{K}y_2(x)f(x)$$

$$= -\frac{y_2'(x)}{K}\int_0^x y_1(s)f(s)ds + \frac{y_1'(x)}{K}\int_1^x y_2(s)f(s)ds \quad [4]$$

$$\phi''(x) = -\frac{y_2''(x)}{K}\int_0^x y_1(s)f(s)ds - \frac{y_2'(x)}{K}y_1(x)f(x)$$

[5]

$$+\frac{y_1''(x)}{K}\int_1^x y_2(s)f(s)ds + \frac{y_1'(x)}{K}y_2(x)f(x)$$

Boundary Conditions:

(1) $\phi(0) = -\frac{y_2(0)}{K}\underbrace{\int_0^0 y_1(s)f(s)\,ds}_{=0} - \frac{y_1(0)}{K}\int_0^1 y_2(s)f(s)\,ds$ from [3]

$$\therefore \alpha_1\phi(0) = \alpha_1 y_1(0)\left[-\frac{1}{K}\int_0^1 y_2(s)f(s)\,ds\right]$$

$\phi'(0) = -\frac{y_2'(0)}{K}\underbrace{\int_0^0 y_1(s)f(s)\,ds}_{=0} - \frac{y_1'(0)}{K}\int_0^1 y_2(s)f(s)\,ds$ from [4]

$$\therefore \alpha_2\phi'(0) = \alpha_2 y_1'(0)\left[-\frac{1}{K}\int_0^1 y_2(s)f(s)\,ds\right]$$

$$\therefore \alpha_1\phi(0)+\alpha_2\phi'(0) = \underbrace{\left[\alpha_1 y_1(0)+\alpha_2 y_1'(0)\right]}_{=0}\left[-\frac{1}{K}\int_0^1 y_2(s)f(s)\,ds\right]$$

$= 0$, since $y_1(x)$ satisfies boundary conditions at $x=0$.

(2) $\phi(1) = -\frac{y_2(1)}{K}\int_0^1 y_1(s)f(s)\,ds + \frac{y_1(1)}{K}\underbrace{\int_1^1 y_2(s)f(s)\,ds}_{=0}$ from [3]

$$\therefore \beta_1\phi(1) = \beta_1 y_2(1)\left[-\frac{1}{K}\int_0^1 y_1(s)f(s)\,ds\right]$$

$\phi'(1) = -\frac{y_2'(1)}{K}\int_0^1 y_1(s)f(s)\,ds + \frac{y_1'(1)}{K}\underbrace{\int_1^1 y_2(s)f(s)\,ds}_{=0}$ from [4]

$$\therefore \beta_2\phi'(1) = \beta_2 y_2'(1)\left[-\frac{1}{K}\int_0^1 y_1(s)f(s)\,ds\right]$$

$$\therefore \beta_1\phi(1)+\beta_2\phi'(1) = \underbrace{\left[\beta_1 y_2(1)+\beta_2 y_2'(1)\right]}_{=0}\left[-\frac{1}{K}\int_0^1 y_1(s)f(s)\,ds\right]$$

$= 0$, since $y_2(x)$ satisfies boundary conditions at $x=1$.

$\therefore$ (1) & (2) $\Rightarrow$ $\phi(x)$ satisfies the boundary conditions.

Differential Equation:

$$L[\phi(x)] = -\left(p(x)\phi'(x)\right)' + q(x)\phi(x)$$

$$= -p(x)\phi''(x) - p'(x)\phi'(x) + q(x)\phi(x)$$

$$= -p(x)y_2''(x)\left[-\frac{1}{K}\int_0^x y_1(s)f(s)\,ds\right] + p(x)y_2'(x)y_1(x)f(x)/K$$

from [5]

$$-p(x)y_1''(x)\left[\frac{1}{K}\int_1^x y_2(s)f(s)\,ds\right] - p(x)y_1'(x)y_2(x)f(x)/K$$

$$-p'(x)y_2'(x)\left[-\frac{1}{K}\int_0^x y_1(s)f(s)\,ds\right] - p'(x)y'(x)\left[\frac{1}{K}\int_1^x y_2(s)f(s)\,ds\right]$$

from [4]

$$+q(x)y_2(x)\left[-\frac{1}{K}\int_0^x y_1(s)f(s)\,ds\right] + q(x)y_1(x)\left[\frac{1}{K}\int_1^x y_2(s)f(s)\,ds\right]$$

from [3]

$$= \left[-p(x)y_2''(x) - p'(x)y_2'(x) + q(x)y_2(x)\right]\left[-\frac{1}{K}\int_0^x y_1(s)f(s)\,ds\right]$$

$$+\, p(x)\underbrace{\left[y_1(x)y_2'(x) - y_2(x)y_1'(x)\right]}_{=W[y_1,y_2](x)}\frac{f(x)}{K}$$

$$+\left[-p(x)y_1''(x) - p'(x)y'(x) + q(x)y_1(x)\right]\left[\frac{1}{K}\int_1^x y_2(s)f(s)\,ds\right]$$

$$= \underbrace{\left[-\left(p(x)y_2'(x)\right)' + q(x)y_2(x)\right]}_{=L[y_2(x)]}\left[-\frac{1}{K}\int_0^x y_1(s)f(s)\,ds\right]$$

$$+\ p(x)\underbrace{W[y_1,y_2](x)}_{=K}\ \frac{f(x)}{K}$$

$$+\ \underbrace{\left[-(p(x)y_1'(x))' + q(x)y_1(x)\right]}_{=L[y_1(x)]}\left[\frac{1}{K}\int_1^x y_2(s)f(s)\,ds\right]$$

$$= \underbrace{L[y_1(x)]}_{=0} + \frac{K}{K}f(x) + \underbrace{L[y_1(x)]}_{=0}$$

$$= 0 + f(x) + 0 = f(x)$$

$$\therefore\ \underline{\underline{L[\phi(x)] = f(x)}}$$

31.

To find $y_1(x)$ and $y_2(x)$, look at $L[y(x)] = -y''(x) = 0$.

$\therefore\ y(x) = c_1 + c_2 x$, $y'(x) = c_2$. $\therefore\ y'(0) \Rightarrow c_2 = 0$

$\therefore\ y_1(x) = c_1$ or $\underline{y_1(x) = 1}$. $\therefore\ L[y_1] = 0$ and $y_1'(0) = 0$

$y(1) = 0 \Rightarrow c_1 + c_2 = 0$, or $c_1 = -c_2$, $\therefore\ y_2(x) = c_2x - c_2$,

or $\underline{y_2(x) = x - 1}$. $\therefore\ L[y_2] = 0$ and $y_2(1) = 0$.

$$W[y_1,y_2](x) = \begin{vmatrix} 1 & x-1 \\ 0 & 1 \end{vmatrix} = 1 \neq 0.$$

Here, $p(x)=1$, $q(x)=0$ $\therefore$ $p(x)W[y_1,y_2](x)=1$.

$$\therefore\ G(x,s)=\begin{cases}-y_1(s)y_2(x) = -(x-1) = 1-x, & 0\le s\le x\\ -y_1(x)y_2(s) = -(s-1) = \underline{1-s}, & \underline{x\le s\le 1}\end{cases}$$

$$\therefore\ \phi(x)=\int_0^1 G(x,s)f(s)ds = \int_0^x G(x,s)f(s)ds + \int_x^1 G(x,s)f(s)ds$$

$$= \int_0^x \underline{(1-x)f(s)ds} + \int_x^1 (1-s)f(s)\,ds$$

32.

$L[y(x)] = -y''(x) = 0 \Rightarrow y(x) = c_1 + c_2 x$

$y(0)=0 \Rightarrow c_1 = 0$. $\therefore$ let $\underline{y_1(x)=x}$, so $y_1(0)=0$

$y(1)+y'(1)=0 \Rightarrow (c_1+c_2)+(c_2)=0 \Rightarrow 2c_2 = -c_1$

$\therefore$ let $c_2=1$, $c_1=-2$, $\underline{y_2(x)=x-2}$, so $y_2(1)+y_2'(1)=0$.

$$W[y_1,y_2](x)=\begin{vmatrix} x & x-2\\ 1 & 1\end{vmatrix} = 2 \neq 0.$$

Here, $p(x)=1$, $q(x)=0$. $\therefore$ $p(x)W[y_1,y_2](x)=2$

$$\therefore\ G(x,s)=\begin{cases}\dfrac{-y_1(s)y_2(x)}{2} = \dfrac{-s(x-2)}{2} = \dfrac{s(2-x)}{2}, & 0\le s\le x\\[2ex] \dfrac{-y_1(x)y_2(s)}{2} = \dfrac{-x(s-2)}{2} = \underline{\dfrac{x(2-s)}{2}}, & \underline{x\le s\le 1}\end{cases}$$

33.

$L[y] = -(y')' - y = 0. \quad \therefore p(x) = 1, \ q(x) = -1.$

$y'' + y = 0 \Rightarrow y(x) = c_1 \cos(x) + c_2 \sin(x)$

$y'(0) = 0 \Rightarrow -c_1 \sin(0) + c_2 \cos(0) = 0 \Rightarrow c_2 = 0$

$\therefore$ Choose $\underline{y_1(x) = \cos(x)}$

$y(1) = 0 \Rightarrow c_1 \cos(1) + c_2 \sin(1) = 0$. Choose $c_2 = 1, c_1 = -\tan(1)$

$\therefore \underline{y_2(x) = -\tan(1)\cos(x) + \sin(x)}$

$$W[y_1, y_2](x) = \begin{vmatrix} \cos(x) & -\tan(1)\cos(x) + \sin(x) \\ -\sin(x) & \tan(1)\sin(x) + \cos(x) \end{vmatrix} = 1 \neq 0$$

$\therefore p(x) W[y_1, y_2](x) = 1$

$$\therefore G(x,s) = \begin{cases} -y_1(s) y_2(x) = -\cos(s)(-\tan(1)\cos(x) + \sin(x)) \\ -y_1(x) y_2(s) = -\cos(x)(-\tan(1)\cos(s) + \sin(s)) \end{cases}$$

Note: $\cos(s)[\tan(1)\cos(x) - \sin(x)] = \cos(s)\left[\dfrac{\sin(1)\cos(x) - \cos(1)\sin(x)}{\cos(1)}\right]$

$$= \frac{\cos(s)\sin(1-x)}{\cos(1)}$$

and $\cos(x)[\tan(1)\cos(s) - \sin(s)] = \cos(x)\left[\dfrac{\sin(1)\cos(s) - \cos(1)\sin(s)}{\cos(1)}\right]$

$$= \frac{\cos(x)\sin(1-s)}{\cos(1)}$$

$$\therefore \quad G(x,s) = \begin{cases} \cos(s)\sin(1-x)/\cos(1), & 0 \le s \le x \\ \cos(x)\sin(1-s)/\cos(1), & x \le s \le 1 \end{cases}$$

34.

$L[y] = -y'' = 0 \Rightarrow y(x) = c_1 + c_2 x \quad p(x) = 1,\ q(x) = 0$

$y(0) = 0 \Rightarrow c_1 = 0. \therefore$ Choose $y_1(x) = x$

$y'(1) = 0 \Rightarrow c_2 = 0. \therefore$ Choose $y_2(x) = 1$

$W[y_1, y_2](x) = \begin{vmatrix} x & 1 \\ 1 & 0 \end{vmatrix} = -1. \quad p(x)W[y_1,y_2](x) = -1$

$$\therefore G(x,s) = \begin{cases} y_1(s)y_2(x) \\ y_1(x)y_2(s) \end{cases} = \begin{matrix} (s)(1) \\ (x)(1) \end{matrix} = \begin{cases} s, & 0 \le s \le x \\ x, & x \le s \le 1 \end{cases}$$

35.

(a)

Given $\phi(x) = \sum_{n=1}^{\infty} \frac{c_n}{\lambda_n - \mu} \phi_n(x)$, where $\mu \neq \lambda_n$ and

$L[\phi_n(x)] = \lambda_n r(x) \phi_n(x)$, $L[y] = -(py')' + qy = \mu r(x) y + f(x)$

$\alpha_1 \phi_n(0) + \alpha_2 \phi'(0) = 0$, $\beta_1 \phi_n(1) + \beta_2 \phi_n'(1) = 0$, and

$c_n = \int_0^1 f(x) \phi_n(x) dx$, $n = 1, 2, 3, \ldots$

Write c_n as $c_n = \int_0^1 f(s) \phi_n(s) ds$

$$\therefore \quad \phi(x) = \sum_{n=1}^{\infty} \frac{c_n}{\lambda_n - \mu} \phi_n(x) = \sum_{n=1}^{\infty} \frac{\int_0^1 f(s) \phi_n(s) ds}{\lambda_n - \mu} \phi_n(x)$$

$$= \sum_{n=1}^{\infty} \frac{\phi_n(x)}{\lambda_n - \mu} \int_0^1 \phi_n(s) f(s) ds$$

$$= \sum_{n=1}^{\infty} \int_0^1 \frac{\phi_n(x)\,\phi_n(s)}{\lambda_n - \mu} f(s)\,ds$$ since the integral doesn't depend on x

$$= \int_0^1 \sum_{n=1}^{\infty} \frac{\phi_n(x)\,\phi_n(s)}{\lambda_n - \mu} f(s)\,ds$$ linearity of integrals

$$= \int_0^1 G(x,s,\mu)\, f(s)\,ds, \quad \text{if } G(x,s,\mu) = \sum_{n=1}^{\infty} \frac{\phi_n(x)\,\phi_n(s)}{\lambda_n - \mu}$$

$\therefore$ The two forms for $\phi(x)$ are the same if $G(x,s,\mu)$ is defined by $G(x,s,\mu) = \sum_{n=1}^{\infty} \frac{\phi_n(x)\,\phi_n(s)}{\lambda_n - \mu}$

(b)

Assume $G(x,s,\mu) = \sum_{i=1}^{\infty} a_i(x,\mu)\,\phi_i(s)$. Need to find $a_i(x,\mu)$

Use orthonormality of ϕ_i with respect to $r(x)$.

$$\therefore\ G(x,s,\mu)\, r(s)\,\phi_j(s) = \sum_{i=1}^{\infty} a_i(x,\mu)\, r(s)\,\phi_j(s)\,\phi_i(s)$$

$$\therefore\ \int_0^1 G(x,s,\mu)\, r(s)\,\phi_j(s)\,ds = \sum_{i=1}^{\infty} a_i(x,\mu) \underbrace{\int_0^1 r(s)\,\phi_j(s)\,\phi_i(s)\,ds}_{=\,\delta_{ij}}$$

$\therefore \int_0^1 G(x,s,\mu) r(s) \phi_j(s)\, ds = a_j(x,\mu)$

From (a), the form for $G(x,s,\mu)$ is $\sum_{n=1}^{\infty} \frac{\phi_n(x)}{\lambda_n - \mu} \phi_n(s)$

This suggests $a_i(x,\mu) = \frac{\phi_i(x)}{\lambda_i - \mu}$, or $(\lambda_i - \mu) a_i(x,\mu) = \phi_i(x)$, or $\lambda_i a_i(x,\mu) = \mu a_i(x,\mu) + \phi_i(x)$. Trying to get this into Sturm-Liouville form, as the solution to the nonhomogeneous equation was of form $\sum \frac{c_i}{\lambda_i - \mu} \phi_i$, multiply both sides by $r(x)$ to get:

$$\lambda_i r(x) a_i(x,\mu) = \mu r(x) a_i(x,\mu) + r(x)\phi_i(x)$$

$\therefore$ Let $a_i(x,\mu) = \sum_{j=1}^{\infty} b_j' \phi_j(x)$. The $\phi_j(x)$ satisfy the boundary conditions, so need to find the coefficients b_j' so that $\phi(x) = a_i(x,\mu)$ satisfies $L[\phi(x)] = \mu r(x)\phi(x) + r(x)\phi_i(x)$, [1] where $r(x)\phi_i(x)$ serves the purpose of $f(x)$ in the Sturm-Liouville form $L[y] = \mu r(x) y + f(x)$.

Note that the solution to [1] is $a_i(x,\mu) = \sum_{j=1}^{\infty} \frac{c_j}{\lambda_j - \mu} \phi_j(x)$, where $c_j = \int_0^1 r(x)\phi_i(x)\phi_j(x)\,dx = \delta_{ij}$.

So $c_j = 0$ except for $j=i$ in which case $c_j = 1$.

$\therefore\ a_i(x,\mu) = \frac{1}{\lambda_i - \mu}\phi_i(x)$, as desired.

To show $b_j^i = \frac{\delta_{ij}}{\lambda_i - \mu}$, note that

$$L[a_i(x,\mu)] = \sum_{j=1}^{\infty} b_j^i L[\phi_j(x)] = \sum_{j=1}^{\infty} b_j^i \lambda_j r(x)\phi_j(x)$$

$\therefore$ [1] becomes

$$\sum_{j=1}^{\infty} b_j^i \lambda_j r(x)\phi_j(x) = \mu r(x)\sum_{j=1}^{\infty} b_j^i \phi_j(x) + r(x)\phi_i(x)$$

Dividing by $r(x)$ and rearranging,

$$\sum_{j=1}^{\infty} b_j^i (\lambda_j - \mu)\phi_j(x) = \phi_i(x)$$

Since the $\phi_j(x)$ are independent, $b_j^i(\lambda_j - \mu) = 0$ for j except $j=i$, in which case $b_j^i(\lambda_i - \mu) = 1$.

Since $\lambda_i - \mu \neq 0$ for all i, $b_j^i = \frac{\delta_{ij}}{\lambda_i - \mu}$

Summary:

The function $a_i(x,\mu)$ for a given μ, defined as the solution $\phi(x)$ to $L[\phi] = \mu r(x)\phi(x) + r(x)\phi_i(x)$

and satisfying the given boundary conditions, where $\phi_i(x)$, λ_i are eigenfunctions and corresponding eigenvectors to $L[y] = \lambda r(x) y$, is $a_i(x,\mu) = \dfrac{\phi_i(x)}{\lambda_i - \mu}$.

The function

$$G(x,s,\mu) = \sum_{i=1}^{\infty} a_i(x,\mu)\phi_i(s) = \sum_{i=1}^{\infty} \frac{\phi_i(x)\phi_i(s)}{\lambda_i - \mu}$$

which yields a Green's function integral

$$\phi(x) = \int_0^1 G(x,s,\mu) f(s)\,ds$$

solves $L[y] = \mu r(x) y + f(x)$ and the given boundary conditions, as shown in (a).

36.

Problem #28 is $-\dfrac{d^2y}{dx^2} = f(x)$, $y(0)=0$, $y(1)=0$

The solution from #28(c) is:

$$\phi(x) = \int_0^x s(1-x) f(s)\,ds + \int_x^1 x(1-s) f(s)\,ds$$

The current problem uses s as the independent variable. Switching to s as the independent variable and t as the "dummy" variable in the above definition,

$$\phi(s) = \int_0^s t(1-s)f(t)\,dt + \int_s^1 s(1-t)f(t)\,dt$$

This can be written as:

$$\phi(s) = \begin{cases} \int_0^s t(1-s)f(t)\,dt, & 0 \le t \le s \\ \int_s^1 s(1-t)f(t)\,dt, & s \le t \le 1 \end{cases} \qquad [1]$$

Now consider a point $x \in (0,1)$ and the impulse function $\delta(t-x)$. $\delta(t-x)$ is defined on an interval $[x-\tau, x+\tau]$, where τ is small enough so that $[x-\tau, x+\tau] \subset (0,1)$.

Note $\delta(t-x) = \frac{1}{2\tau}$ on $[x-\tau, x+\tau]$.

Consider the cases of $0 < x < s$, $s < x < 1$

ϕ is now a function of both s and x

(1) $0 < x < s$

Note on the interval $(x+\tau, 1]$, $\delta(t-x) = 0$.

$$\therefore \int_s^1 s(1-t) f(t)\,dt = \int_s^1 s(1-t)\,\delta(t-x)\,dt = 0$$

since $\delta(t-x) = 0$ on $[s, 1]$.

$\therefore$ From [1],

$$\phi(s,x) = \int_0^s t(1-s) f(t)\,dt = \int_0^s t(1-s)\,\delta(t-x)\,dt$$

$$= \int_{x-\tau}^{x+\tau} t(1-s)\frac{1}{2\tau}\,dt = \frac{(1-s)}{2\tau}\left.\frac{t^2}{2}\right|_{t=x-\tau}^{t=x+\tau}$$

$$= \frac{(1-s)}{2\tau}\left[\frac{(x+\tau)^2 - (x-\tau)^2}{2}\right] = \frac{(1-s)}{2\tau}\,4\tau x = x(1-s)$$

$$\therefore \underline{\phi(s,x) = x(1-s)}, \quad 0 < x < s$$

(2) $s < x < 1$

Note on the interval $[0, x-\tau)$, $\delta(t-x) = 0$

$$\therefore \int_0^s t(1-s) f(t)\,dt = \int_0^s t(1-s)\,\delta(t-x)\,dt = 0$$

since $\delta(t-x) = 0$ on $[0, s]$

$\therefore$ From [1],

$$\phi(s,x) = \int_s^1 s(1-t) f(t)\,dt = \int_s^1 s(1-t)\,\delta(t-x)\,dt$$

$$= \int_{x-\tau}^{x+\tau} s(1-t)\frac{1}{2\tau}\,dt = \frac{s}{2\tau}\left[-\frac{(1-t)^2}{2}\right]\Bigg|_{t=x-\tau}^{t=x+\tau}$$

$$= \frac{s}{2\tau}\left[\frac{-(1-(x+\tau))^2 + (1-(x-\tau))^2}{2}\right]$$

$$= \frac{s}{2\tau}\left[\frac{1-2(x-\tau)+(x-\tau)^2 - [1-2(x+\tau)+(x+\tau)^2]}{2}\right]$$

$$= \frac{s}{2\tau}\left[\frac{1-2x+2\tau+x^2-2x\tau+\tau^2-1+2x+2\tau-x^2-2\tau x-\tau^2}{2}\right]$$

$$= \frac{s}{2\tau}\left[\frac{4\tau-4x\tau}{2}\right] = \frac{s(4\tau)(1-x)}{4\tau} = s(1-x)$$

$$\therefore \underline{\phi(s,x)} = s(1-x), \quad s<x<1$$

$\therefore$ (1) and (2) indicate

$$\phi(s,x) = \begin{cases} x(1-s), & 0<x<s \\ s(1-x), & s<x<1 \end{cases}$$

This is the same definition as $G(x,s)$ in #28 (with x and s interchanged). More clearly,

$$\phi(a,t) = \begin{cases} t(1-a), & 0<t<a \\ a(1-t), & a<t<1 \end{cases}$$

Note that at $t=a$, both formulas are the same, so you could, in the definition, state $0<t\le a$, $a\le t<1$.

11.4 Singular Sturm-Liouville Problems

Note Title 1/10/2022

1.

Use the eigenfunctions from $-(xy')' = \lambda x y$, where $r(x) = x$. As discussed in Example 2 on page 560, eigenfunctions for the homogeneous problem $-(xy')' = \lambda x y$, with the given boundary conditions, are $\phi_n(x) = J_0(\sqrt{\lambda_n}\, x)$, where λ_n are determined as the roots of $J_0(\sqrt{\lambda}) = 0$.

Since it's given $\mu \neq \lambda_n$ for any $n = 1, 2, 3, \dots$, then using the development in Section 11.3, p. 546,

assume $$\frac{f(x)}{r(x)} = \sum_{n=1}^{\infty} c_n J_0(\sqrt{\lambda_n}\, x)$$

Since the $J_0(\sqrt{\lambda_n}\, x)$ are not normalized, multiply both sides by $r(x) J_0(\sqrt{\lambda_m}\, x) = x J_0(\sqrt{\lambda_m}\, x)$, and integrate.

$$\therefore \int_0^1 f(x) J_0(\sqrt{\lambda_m}\, x)\, dx = \sum_{n=1}^{\infty} c_n \int_0^1 x J_0(\sqrt{\lambda_n}\, x) J_0(\sqrt{\lambda_m}\, x)\, dx$$

$$= c_m \int_0^1 x \left[J_0(\sqrt{\lambda_m}\, x)\right]^2 dx$$

because of orthogonality of $J_0(\sqrt{\lambda_n}\, x)$.

$$\therefore \quad c_n = \frac{\int_0^1 f(x) J_0(\sqrt{\lambda_n}\, x)\, dx}{\int_0^1 x \left[J_0(\sqrt{\lambda_n}\, x)\right]^2 dx}$$

$\therefore$ As with Section 11.3, p. 546,

$$\underline{y(x) = \sum_{n=1}^{\infty} \frac{c_n}{\lambda_n - \mu} J_0(\sqrt{\lambda_n}\, x)}, \text{ where } c_n, \lambda_n \text{ are as described above.}$$

2.

First note $p(x) = x$, $r(x) = x$, $q(x) = 0$. As shown on p. 536 of the text, $\int_0^1 (L[u]v - uL[v])\, dx = (u'(1)v(1) - u(1)v'(1)) - \lim_{\epsilon \to 0^+} \epsilon[u'(\epsilon)v(\epsilon) - u(\epsilon)v'(\epsilon)] = 0$, since u, v, u', v' are bounded as $x \to 0^+$.

$=0$ $=0$

$\therefore$ Eigenfunctions $\phi_n(x)$ exist which are solutions to $-(xy')' = \lambda_n x y$, and ϕ_n, ϕ_m are orthogonal relative to x for $n \neq m$, and they satisfy the boundary conditions.

(a)

For $\lambda = 0$, $-(xy')' = \lambda xy$ becomes $-(xy')' = 0$.

For $y = \phi_0(x) = 1$, $y' = 0$. $\therefore$ $xy' = 0$, so $-(xy')' = 0$.

$\therefore$ $\phi_0(x) = 1$ is an eigenfunction for $\lambda_0 = 0$.

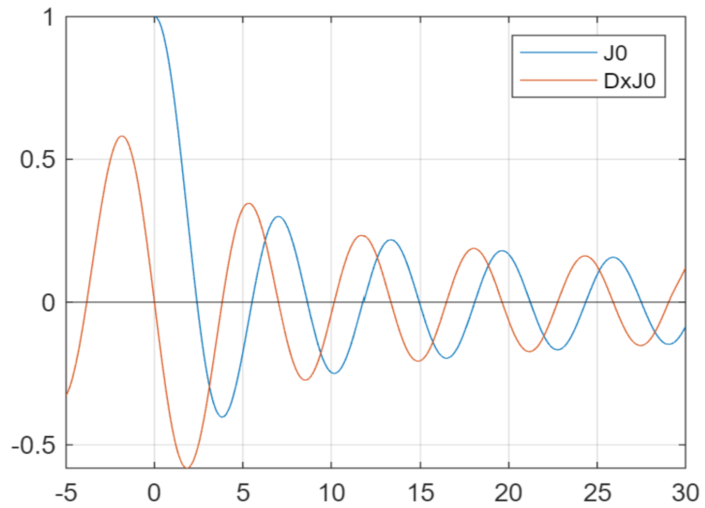
For $\lambda > 0$, p. 557 of the text showed the general solution to $-(xy')' = \lambda xy$ is $y = c_1 J_0(\sqrt{\lambda} x) + c_2 Y_0(\sqrt{\lambda} x)$, which is also discussed on pp. 230-233 of the text.

To satisfy the boundary condition at $x = 0$, set $c_2 = 0$ as discussed on p. 557. $\therefore$ $\phi_n(x) = J_0(\sqrt{\lambda_n} x)$ is the eigenfunction for $\lambda_n > 0$, and $J_0(\sqrt{\lambda_n} x)$ satisfies the boundary condition at $x = 1$, or $J_0'(\lambda_n) = 0$

(b)

Using MATLAB,

```
clear
syms x
N = 100;   % number of terms for J0
% construct J0 for N terms
J0 = 1;
for m = 1:N
    f = factorial(m);
    J0 = J0 + (-1)^m*x^(2*m)/(4^m*f^2);
end
fplot(J0,[0,30])
hold on
fplot(diff(J0,x,1))  % derivative of J0
grid on
yline(0)
legend('J0','DxJ0')
```

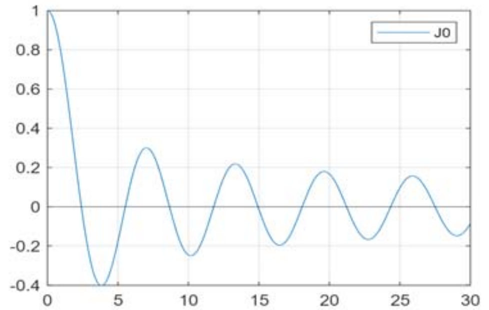


With $x=1$, $J_0'(\sqrt{\lambda}\,x) = J_0'(\sqrt{\lambda})$, so plotting $J_0'(x)$ in MATLAB shows $J_0'(x)$ crossing $y=0$ multiple times

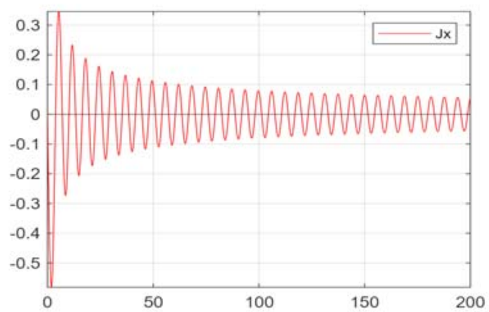
MATLAB has a function "besselj" that can also be used.

```
clear
syms x
J0 = besselj(0,x);
Jx = diff(J0,x,1); % d/dx J0
fplot(J0,[0,30])
grid on
yline(0)
legend('J0')

fplot(Jx,[0,200],'r')
grid on
yline(0)
legend('Jx')
```



$J_0'(x)$ oscillates as $x \to +\infty$



$J_0'(x)$ plotted on $[0, 200]$

(c)

From the comment made at the beginning of the problem, $\int_0^1 L[u]v - uL[v]\,dx = 0$.

$\therefore \int_0^1 L[u]v\,dx = \int_0^1 uL[v]\,dx$

$\therefore \int_0^1 L[\phi_m]\phi_n\,dx = \int_0^1 \phi_m L[\phi_n]\,dx$

$\therefore \int_0^1 (\lambda_m x \phi_m)\phi_n\,dx = \int_0^1 \phi_m(\lambda_n x \phi_n)\,dx$

$\therefore (\lambda_m - \lambda_n)\int_0^1 x\phi_m\phi_n\,dx = 0$ Note: true for $m=0$

Since $\lambda_m \neq \lambda_n$, $\int_0^1 x\phi_m\phi_n\,dx = 0$

(d)

As noted in (a), $\phi_n(x) = J_0(\sqrt{\lambda_n}\,x)$ is the eigenfunction solution for $\underline{\lambda_n > 0}$ that satisfies the boundary conditions at $x=0, x=1$, where λ_n are determined from $J_0'(\lambda_n)=0$.

However, in this problem, $\phi_0(x)=1$ is an eigenfunction for $\lambda=0$.

$\therefore$ Assume $y=\phi(x)=\sum_{n=0}^{\infty} b_n \phi_n(x)$, [1]

where $\phi_0(x)=1$ and $\phi_n(x)=J_0(\sqrt{\lambda_n}\,x)$ for $n>0$.

Consider $\frac{f(x)}{r(x)} = \sum_{n=0}^{\infty} c_n \phi_n(x)$

Multiply both sides by $r(x)\phi_m(x)$ and integrate.

$$\therefore \int_0^1 f(x)\phi_m(x)\,dx = \sum_{n=0}^{\infty} c_n \int_0^1 x\,\phi_n(x)\phi_m(x)\,dx$$

$$= c_m \int_0^1 x\,[\phi_m(x)]^2\,dx \quad \text{using (c)}$$

Note for $m=0$, $\phi_m(x)=1$, so

$$\int_0^1 f(x)\,dx = c_0\int_0^1 x\,dx = \tfrac{1}{2}c_0. \qquad [2]$$

For $m>0$,

$$c_m = \frac{\int_0^1 f(x)\,J_0(\sqrt{\lambda_m}\,x)\,dx}{\int_0^1 x\left[J_0(\sqrt{\lambda_m})\right]^2 dx}$$

From [1], $-(xy')' = L[y] = \mu xy + x\,\frac{f(x)}{x}$ [3]

and $L[y] = L[\phi(x)] = \sum_{n=0}^{\infty} b_n L[\phi_n(x)] = \sum_{n=0}^{\infty} b_n \lambda_n x \phi_n(x)$

From [3],
$$\sum_{n=0}^{\infty} b_n \lambda_n x \phi_n(x) = \mu x \sum_{n=0} b_n \phi_n(x) + x \sum_{n=0}^{\infty} c_n \phi_n(x)$$

Rearranging,
$$\sum_{n=0}^{\infty} \left[b_n (\lambda_n - \mu) - c_n \right] x \phi_n(x) = 0$$

$\therefore$ For all values of x,
$$\sum_{n=0}^{\infty} \left[b_n (\lambda_n - \mu) - c_n \right] = 0$$

For $n=0$, $b_0(0-\mu) - c_0 = 0$, so $b_0 = -\frac{c_0}{\mu} = -\frac{2\int_0^1 f(x)dx}{\mu}$

For $n>0$, $b_n = \frac{c_n}{\lambda_n - \mu}$ from [2]

$$\boxed{\begin{aligned} \therefore\ y = \phi(x) &= -\frac{c_0}{\mu} + \sum_{n=1}^{\infty} \frac{c_n}{\lambda_n - \mu} J_0(\sqrt{\lambda_n}\, x) \\ \text{where } c_0 &= 2\int_0^1 f(x)dx \\ c_n &= \frac{\int_0^1 f(x) J_0(\sqrt{\lambda_n}\, x) dx}{\int_0^1 x \left[J_0(\sqrt{\lambda_n}\, x) \right]^2 dx} \\ \text{and } \lambda_n &\text{ is from } J_0'(\sqrt{\lambda_n}) = 0 \end{aligned}}$$

3.

(a)

$-(xy')' + \frac{K^2}{x} y = \lambda x y$ becomes $-xy'' - y' + \frac{K^2}{x} y = \lambda x y$,

or $-x^2 y'' - x y' + K^2 y - \lambda x^2 y = 0$, or

$x^2 y'' + x y' + (\lambda x^2 - K^2) y = 0$ [1]

Now consider y a function of t: $y(x(t))$

Using $t = \sqrt{\lambda} x$, $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = \frac{dy}{dx} \cdot \frac{1}{\sqrt{\lambda}}$

$$\therefore \frac{d^2 y}{dt^2} = \frac{1}{\sqrt{\lambda}} \frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{1}{\sqrt{\lambda}} \frac{d^2 y}{dx^2} \cdot \frac{dx}{dt} = \frac{1}{\lambda} \frac{d^2 y}{dx^2}$$

Substituting these into [1],

$$\left(\frac{t}{\sqrt{\lambda}}\right)^2 \left(\lambda \frac{d^2 y}{dt^2}\right) + \left(\frac{t}{\sqrt{\lambda}}\right)\left(\sqrt{\lambda} \frac{dy}{dt}\right) + \left[\lambda \left(\frac{t}{\sqrt{\lambda}}\right)^2 - K^2\right] y(t) = 0$$

Or, $t^2 y'' + t y' + [t^2 - K^2] y = 0$

This is Bessel's equation of order K.

(b)

As stated in (a), $J_K(t)$ is a solution to $t^2y'' + ty' + (t^2-K^2)y=0$. Let t_n be a solution to $J_K(t_n)=0$. Using $t=\sqrt{\lambda}\,x$, $J_K(\sqrt{\lambda}x)$ is a solution to $-(xy')' + \frac{K^2}{x}y = \lambda xy$, and so, at the boundary of $x=1$, $t_n = \sqrt{\lambda_n}(1) = \sqrt{\lambda_n}$, and so $J_K(\sqrt{\lambda_n}) = 0$. $\therefore$ The $\sqrt{\lambda_n}$ are zeros to $J_K(t)=0$, so the eigenvalues λ_n are the squares of $\sqrt{\lambda_n}$, the zeros of $J_K(\sqrt{\lambda})$.

Since we consider $\lambda_n > 0$, they are the positive zeros of $J_K(\sqrt{\lambda})$. $\therefore$ $J_K(\sqrt{\lambda_n}\,x)$ are the corresponding eigenfunctions solving $-(xy')' + \frac{K^2}{y}y = \lambda xy$ and the boundary conditions.

(c)

This is a self-adjoint Sturm-Liouville problem, and so orthogonality follows as a result.

Specifically, let $L[y] = -(xy')' + \frac{k^2}{x}y = \lambda x y$, so that $p(x) = x$, $q(x) = \frac{k^2}{x}$, and $r(x) = x$.

$\therefore$ If $y = u(x)$ and $y = v(x)$ are solutions obeying the boundary conditions, $\int_0^1 (L[u]v - uL[v])\,dx =$

$-p(x)\left[u'(x)v(x) - u(x)v'(x)\right]\Big|_0^1 =$

$-p(1)\left[u'(1)v(1) - u(1)v'(1)\right] + \lim_{\epsilon \to 0^+} p(\epsilon)\left[u'(\epsilon)v(\epsilon) - u(\epsilon)v'(\epsilon)\right]$

In this problem, $u(1) = 0$, $v(1) = 0$, and $p(\epsilon) = \epsilon$, and u, v, u', v' are bounded as $x \to 0^+$.

$\therefore\ [u'(1)\underbrace{v(1)}_{=0} - \underbrace{u(1)}_{=0}v'(1)] = 0$, and

$\lim_{\epsilon \to 0^+} \epsilon\left[u'(\epsilon)v(\epsilon) - u(\epsilon)v'(\epsilon)\right] = 0$

The latter follows using the definition of limit:

Given an $\alpha > 0$, choose $\delta > 0$ s.t. $\delta < \frac{\alpha}{2mn}$, where $M = \max\{|\text{bound of } u|, |\text{bound of } v|\}$ on $[0, \epsilon]$,

and $M' = \max\{|\text{bound of } u'|, |\text{bound of } v'|\}$ on $[0, \epsilon]$.

$\therefore$ if $\epsilon < \delta < \frac{\alpha}{2MM'}$, then

$$|\epsilon[u'(\epsilon)v(\epsilon) - u(\epsilon)v'(\epsilon)] - 0| \leq$$ using triangle inequality

$$|\epsilon u'(\epsilon)v(\epsilon)| + |\epsilon u(\epsilon)v'(\epsilon)| < \delta M'M + \delta MM'$$

$$= 2\delta MM' < 2\left(\frac{\alpha}{2MM'}\right)MM' = \alpha$$

$\therefore$ Using $\phi_m(x)$ for $u(x)$, $\phi_n(x)$ for $v(x)$,

for $m \neq n$, $\int_0^1 (L[\phi_m]\phi_n - \phi_m L[\phi_n])dx = 0$,

$$\therefore \int_0^1 ([\lambda_m x \phi_m]\phi_n - \phi_m[\lambda_n x \phi_n])dx = 0$$

Or, $(\lambda_m - \lambda_n)\int_0^1 x\phi_m\phi_n\,dx = 0$.

Since $\lambda_m - \lambda_n \neq 0$ for $m \neq n$, $\underline{\int_0^1 x\phi_m\phi_n dx = 0}$

(d)

Here, the $\phi_n(x)$ are eigenfunctions satisfying the orthogonality condition in (c). They are not normalized so $\int_0^1 x[\phi_n(x)]^2 dx$ isn't necessarily equal to 1. Multiply $f(x) = \sum_{n=1}^{\infty} a_n \phi_n(x)$ by $x\phi_m(x)$, and integrate. $\therefore$

$$\int_0^1 x f(x) \phi_m(x)\, dx = \sum_{n=1}^{\infty} a_n \int_0^1 x \phi_n \phi_m \, dx$$

Using orthogonality from (c), all terms on the right side are 0, except for when $n = m$.

$$\therefore \int_0^1 x f(x) \phi_m(x) dx = a_m \int_0^1 x [\phi_m(x)]^2 dx$$

$$\therefore a_m = \frac{\int_0^1 x f(x) \phi_m(x) dx}{\int_0^1 x [\phi_m(x)]^2 dx} = \underline{\underline{\frac{\int_0^1 x f(x) J_\kappa(\sqrt{\lambda_m} x) dx}{\int_0^1 x [J_\kappa(\sqrt{\lambda_m} x)]^2 dx}}}$$

(e)

Assume $y = \phi(x) = \sum_{n=1}^{\infty} b_n \phi_n(x)$, [1]

where $\phi_n(x) = J_\kappa(\sqrt{\lambda_n} x)$ and λ_n are from $J_\kappa(\sqrt{\lambda}) = 0$ as shown in (b), where $L[y] = -(xy')' + \frac{\kappa^2}{x} y = \lambda x y$

Consider $\frac{f(x)}{r(x)} = \sum_{n=1}^{\infty} c_n \phi_n(x)$, where $r(x) = x$

Multiplying both sides by $r(x)\phi_m(x)$ and integrating,

$$\int_0^1 f(x)\phi_m(x)\,dx = \sum_{n=1}^{\infty} c_n \int_0^1 x\,\phi_n(x)\phi_m(x)\,dx = c_m \int_0^1 x\,\phi_m^2(x)\,dx$$

using orthogonality of ϕ_n, ϕ_m from (c).

$$\therefore\quad c_n = \frac{\int_0^1 f(x)\phi_n(x)\,dx}{\int_0^1 x\,\phi_n^2(x)\,dx}$$

$\therefore$ From [1], $L[y] = L[\phi(x)] = \sum_{n=1}^{\infty} b_n L[\phi_n(x)] = \sum_{n=1}^{\infty} b_n \lambda_n x \phi_n(x)$

From $L[y] = \mu x y + f(x)$,

$$\sum_{n=1}^{\infty} b_n \lambda_n x \phi_n(x) = \mu x \sum_{n=1}^{\infty} b_n \phi_n(x) + \sum_{n=1}^{\infty} c_n \phi_n(x)$$

$$\therefore\quad \sum_{n=1}^{\infty} \left[b_n \lambda_n - \mu b_n - c_n \right] x\,\phi_n(x) = 0, \quad \text{or}$$

$$\sum_{n=1}^{\infty} \left[b_n(\lambda_n - \mu) - c_n \right] x\,\phi_n(x) = 0$$

To be true for all $x \in [0,1]$, $b_n(\lambda_n - \mu) - c_n = 0$,

or, $b_n = \frac{c_n}{\lambda_n - \mu}$, $n = 1, 2, 3, \ldots$

Using $\phi_n(x) = J_K(\sqrt{\lambda_n}\, x)$,

$$\boxed{\begin{array}{l} y = \phi(x) = \displaystyle\sum_{n=1}^{\infty} \frac{c_n}{\lambda_n - \mu} J_K(\sqrt{\lambda_n}\, x), \text{ where} \\ \lambda_n \text{ are positive zeros of } J(\sqrt{\lambda}) = 0, \\ c_n = \dfrac{\int_0^1 f(x) J_K(\sqrt{\lambda_n}\, x)\, dx}{\int_0^1 x J_K^2(\sqrt{\lambda_n}\, x)\, dx} \end{array}}$$

4.

(a)

Let $L[y] = -((1-x^2)y')' = \lambda y$. $\therefore$ $p(x) = 1-x^2$, $q(x) = 0$, $r(x) = 1$

using the standard Sturm-Liouville form.

From Lagrange's identity (section 11.2, p. 536), if

$y = u(x)$ and $y = v(x)$ are two solutions, then

$$\int_0^1 (L[u]v - uL[v])\,dx = -p(x)[u'(x)v(x) - u(x)v'(x)]\Big|_0^1$$

$\therefore$ In this problem with $p(x) = (1-x^2)$ and

$u(0) = v(0) = 0$,

$$\int_0^1 (L[u]v - uL[v])dx = \lim_{\epsilon \to 1^-} -(1-\epsilon^2)[u'(\epsilon)v(\epsilon) - u(\epsilon)v'(\epsilon)]$$
$$+ (1-0^2)[u'(0)\underbrace{v(0)}_{=0} - \underbrace{u(0)}_{=0}v'(0)]$$

$$= \lim_{\epsilon \to 1^-} -(1-\epsilon^2)[u'(\epsilon)v(\epsilon) - u(\epsilon)v'(\epsilon)] = 0$$

since $u(\epsilon), v(\epsilon), u'(\epsilon), v'(\epsilon)$ are bounded as $\epsilon \to 1^-$

and $\lim_{\epsilon \to 1^-} (1-\epsilon^2) = 0$.

For a ϵ-δ type proof of this, see #3(c) above.

$\therefore \int_0^1 L[u]v - uL[v]\,dx = 0$ for this problem.

Now using $u(x) = \phi_m(x)$ and $v(x) = \phi_n(x)$, $m \neq n$,

$$\int_0^1 L[\phi_m]\phi_n - \phi_m L[\phi_n]\,dx = \int_0^1 (\lambda_m \phi_m)\phi_n - \phi_m(\lambda_n \phi_n)\,dx$$

$$= (\lambda_m - \lambda_n)\int_0^1 \phi_m \phi_n\,dx = 0.$$ Since $\lambda_m \neq \lambda_n$

for $m \neq n$, divide by $\lambda_m - \lambda_n \neq 0$ to get:

$$\underline{\int_0^1 \phi_m(x)\,\phi_n(x)\,dx = 0}$$

(b)

Let $y = \phi(x)$ be a solution and assume $\phi(x) = \sum_{n=1}^{\infty} b_n \phi_n(x)$,

where $\phi_n(x) = P_{2n-1}(x)$, the orthogonal Legendre eigenfunctions

from the homogeneous $-((1-x^2)y')' = \lambda y$

$\therefore\ L[\phi(x)] = \sum_{n=1}^{\infty} b_n L[\phi_n(x)] = \sum_{n=1}^{\infty} b_n \lambda_n \phi_n(x)$ [1]

Also consider $\frac{f(x)}{r(x)} = f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x)$, since $r(x) = 1$.

Multiplying both sides by $\phi_m(x)$ and integrating,

$$\int_0^1 f(x)\phi_m(x)\,dx = \sum_{n=1}^{\infty} c_n \int_0^1 \phi_n(x)\phi_m(x)\,dx = c_m \int_0^1 \phi_m^2(x)\,dx$$

using orthogonality of $\phi_n(x)$ shown in (a).

$$\therefore\ c_m = \frac{\int_0^1 f(x)\phi_m(x)\,dx}{\int_0^1 \phi_m^2(x)\,dx} = \frac{\int_0^1 f(x) P_{2m-1}(x)\,dx}{\int_0^1 P_{2m-1}^2(x)\,dx}$$

$\therefore$ From $L[y] = \mu y + f(x)$ and [1] for $y = \phi(x)$,

$$\sum_{n=1}^{\infty} b_n \lambda_n \phi_n(x) = \mu \sum_{n=1}^{\infty} b_n \phi_n(x) + \sum_{n=1}^{\infty} c_n \phi_n(x)$$

Rearranging, $\sum_{n=1}^{\infty} [b_n(\lambda_n - \mu) - c_n]\phi_n(x) = 0$ [2]

For [2] to be true for all x, then $b_n(\lambda_n - \mu) - c_n = 0$

Or, $b_n = \dfrac{c_n}{\lambda_n - \mu}$ for $n = 1, 2, 3, \ldots$

$\therefore$

$$\boxed{\begin{array}{l} y(x) = \displaystyle\sum_{n=1}^{\infty} \frac{c_n}{\lambda_n - \mu} P_{2n-1}(x), \text{ where} \\ \\ \lambda_n = 2n(2n-1),\ P_{2n-1}(x) \text{ are the corresponding} \\ \quad \text{Legendre polynomial eigenfunctions, and} \\ c_n = \dfrac{\int_0^1 f(x) P_{2n-1}(x)\,dx}{\int_0^1 P_{2n-1}^2(x)\,dx} \end{array}}$$

5.

(a)

$$(1-x^2)y'' - xy' + \lambda y = 0 \iff -(1-x^2)y'' + xy' = \lambda y$$

$$\iff \frac{-(1-x^2)y''}{(1-x^2)^{1/2}} + \frac{x\ y'}{(1-x^2)^{1/2}} = \frac{\lambda y}{(1-x^2)^{1/2}}$$

$$\iff -(1-x^2)^{1/2}y'' - [(1-x^2)^{1/2}]'y' = \lambda(1-x^2)^{-1/2}y$$

$$\iff \underline{-((1-x^2)^{1/2}y')'} = \lambda(1-x^2)^{-1/2}y$$

(b)

Let $L[y] = -((1-x^2)^{1/2}y')' = \lambda(1-x^2)^{-1/2}y$, so $p(x) = (1-x^2)^{1/2}$, $q(x) = 0$, and $r(x) = (1-x^2)^{-1/2}$ in standard Sturm-Liouville form.

From the Lagrange identity, if $y = u(x)$ and $y = v(x)$

are solutions to $L[y] = \lambda(1-x^2)^{-1/2}y$, then

$$\int_{-1}^{1} (L[u]v - uL[v])\,dx = \lim_{\substack{\alpha \to 1^- \\ \beta \to -1^+}} \int_{\beta}^{\alpha} (L[u]v - uL[v])\,dx$$

$$= \lim_{\substack{\alpha \to 1^- \\ \beta \to -1^+}} p(x)\left[u'(x)v(x) - u(x)v'(x)\right]\Big|_{\beta}^{\alpha}$$

see text pp. 535-536, using integration by parts on Sturm-Liouville form

$$= \lim_{\alpha \to 1^-} p(\alpha)\left[u'(\alpha)v(\alpha) - u(\alpha)v'(\alpha)\right]$$

$$- \lim_{\beta \to -1^+} p(\beta)\left[u'(\beta)v(\beta) - u(\beta)v'(\beta)\right] \qquad [1]$$

Let $M = \max\{|\text{bound of } u|, |\text{bound of } v|\}$ on $[0,\alpha]$

$M' = \max\{|\text{bound of } u'|, |\text{bound of } v'|\}$ on $[0,\alpha]$

$N = \max\{|\text{bound of } u|, |\text{bound of } v|\}$ on $[\beta, 0]$

$N' = \max\{|\text{bound of } u'|, |\text{bound of } v'|\}$ on $[\beta, 0]$

$\therefore$ Given any $\epsilon_\alpha > 0$ and $\epsilon_\beta > 0$, choose $\delta_\alpha = \frac{\epsilon_\alpha}{4MM'}$ and $\delta_\beta = \frac{\epsilon_\beta}{4NN'}$. Then if $0 < 1-\alpha < \delta_\alpha$ and $0 < \beta - (-1) < \delta_\beta$, then

(a) $\alpha < 1 \Rightarrow \alpha + 1 < 2 \Rightarrow 1-\alpha^2 = (1-\alpha)(1+\alpha) < 2\delta_\alpha$

$$\therefore \left| \overset{=1-\alpha^2}{\rho(\alpha)} [u'(\alpha)v(\alpha) - u(\alpha)v'(\alpha)] - 0 \right| \le$$

$$|\rho(\alpha)| \left(|u'(\alpha)v(\alpha)| + |u(\alpha)v'(\alpha)| \right) \quad \text{triangle inequality}$$

$$\le (1-\alpha^2)(m'm + mm') \le 2\delta_\alpha(2mm')$$

$$= 4mm'\delta_\alpha = 4mm'\left(\frac{\epsilon_\alpha}{4mm'}\right) = \epsilon_\alpha$$

$$\therefore \lim_{\alpha \to 1^-} \rho(\alpha)[u'(\alpha)v(\alpha) - u(\alpha)v'(\alpha)] = 0$$

(b) $0 < \beta - (-1) < \delta_\beta \iff 0 < \beta + 1 < \delta_\beta$

Also, $-\beta < 1 \Rightarrow 1 - \beta < 2$. $\therefore 1 - \beta^2 < 2\delta_\beta$

$$\therefore \left| \overset{=1-\beta^2}{\rho(\beta)} [u'(\beta)v(\beta) - u(\beta)v'(\beta) - 0 \right| \le$$

$$(1-\beta^2)\left(|u'(\beta)v(\beta)| + |u(\beta)v'(\beta)| \right) \quad \text{triangle inequality}$$

$$\le 2\delta_\beta(NN' + NN') = 4NN'\delta_\beta = 4NN'\left(\frac{\epsilon_\beta}{4NN'}\right)$$

$$= \epsilon_\beta.$$

$$\therefore \lim_{\beta \to -1^+} \rho(\beta)[u'(\beta)v(\beta) - u(\beta)v'(\beta)] = 0$$

$\therefore$ From [1], (a), and (b),

$$\int_{-1}^{1} (L[u]v - uL[v])\,dx = 0 \quad [2]$$

$\therefore$ For this problem, solutions to $L[y] = \lambda(1-x^2)^{-1/2}y$ that satisfy the boundary conditions at $x=-1$, $x=1$ obey [2] $\Rightarrow$ self adjoint problem

(c)

Eigenfunctions in a self adjoint Sturm-Liouville problem are orthogonal relative to $r(x)$.

More specifically, given $L[y] = \lambda r(x) y$, where $r(x) = (1-x^2)^{-1/2}$ in this problem, from (b), if $T_m(x)$ and $T_n(x)$ are two solutions, then

$$\int_{-1}^{1} \left(L[T_m(x)]\, T_n(x) - T_m(x)\, L[T_n(x)] \right) dx = 0$$

$$\Rightarrow \int_{-1}^{1} \left[\lambda_m r(x) T_m(x)\, T_n(x) - T_m(x)\, \lambda_n r(x) T_n(x) \right] dx = 0$$

$$\Rightarrow (\lambda_m - \lambda_n) \int_{-1}^{1} r(x)\, T_m(x) T_n(x)\, dx = 0$$

Since $\lambda_m \neq \lambda_n$ for $m \neq n$, then dividing by $\lambda_m - \lambda_n$,

$$\int_{-1}^{1} r(x)\, T_m(x)\, T_n(x)\, dx = 0$$

Since $r(x) = (1-x^2)^{-1/2}$,

$$\int_{-1}^{1} \frac{T_m(x)\, T_n(x)}{(1-x^2)^{1/2}}\, dx = 0, \text{ for } m \neq n$$

1.

(a)

Suppose $u(x,y) = X(x)Y(y)$

$\therefore u_{xx} + u_{yy} = 0 \Rightarrow$

$X''Y + XY'' = 0$, or

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda$$

$\therefore X'' + \lambda X = 0, \quad Y'' - \lambda Y = 0$ [0]

$\therefore$ Along the left side, $y = 2x$ for $0 \le x < 1$

Along the right side, $y = 2x - 4$ for $2 \le x < 3$

(1) Suppose $\lambda = 0$. Then $X(x) = c_1 + c_2 x$, $Y(y) = K_1 + K_2 y$

$\therefore$ Bottom: $u(x,0) = 0 \Rightarrow (c_1 + c_2 x)(K_1) = 0, \ 0 \le x \le 2$ [1]

Left: $u(x,2x) = 0 \Rightarrow (c_1 + c_2 x)(K_1 + 2K_2 x) = 0, \ 0 \le x < 1$ [2]

Right: $u(x, 2x-4) \Rightarrow (c_1 + c_2 x)(K_1 + 2K_2 x - 4K_2) = 0, \ 2 \le x < 3$ [3]

From [1], at $x=0$, $c_1K_1=0$

and at $x=2$, $c_1K_1+2c_2k_1=0$. $\therefore c_2k_1=0$

$\therefore$ If $K_1\neq 0$, $c_1=0$ and $c_2=0 \Rightarrow X(x)=0$, so

$u(x,y)=0$, the trivial solution. $\therefore \underline{K_1=0}$

$\therefore Y(y)=K_2y$, and $K_2\neq 0$ for nontrivial solution.

From [2], using $K_1=0$, at $x=1$, $(c_1+c_2)(2K_2)=0 \Rightarrow c_1+c_2=0$

since $K_2\neq 0$. At $x=\frac{1}{2}$, $(c_1+\frac{c_2}{2})K_2=0 \Rightarrow c_1+\frac{c_2}{2}=0$

Subtracting, $\frac{c_2}{2}=0 \Rightarrow c_2=0$ and $\therefore c_1=0$.

$\therefore X(x)=0$ so $u(x,y)=0$, a trivial solution.

$\therefore \lambda=0 \Rightarrow$ no nontrivial solutions.

(2) Suppose $\lambda>0$. Let $\lambda=\mu^2$, $\mu>0$.

From [0], $X''+\mu^2X=0 \Rightarrow X(x)=K_1\cos(\mu x)+K_2\sin(\mu x)$

Along the bottom border, $0\leq x\leq 2$,

$K_1\cos(\mu x)+K_2\sin(\mu x)=0 \Rightarrow -K_1\cos(\mu x)=K_2\sin(\mu x)$

At $x=0$, $-K_1=0$. $\therefore X(x)=K_2\sin(\mu x)$

The only way for $X(x)=0$ for $0 \le x \le 2$ is for

$K_2 = 0$. $\therefore$ $K_1 = K_2 = 0 \Rightarrow X(x) = 0$ for $0 \le x \le 2$.

$\therefore$ $u(x,y) = 0$.

$\therefore$ $\lambda > 0 \Rightarrow$ no nontrivial solutions.

(3) Suppose $\lambda < 0$. Let $\lambda = -\mu^2$, $\mu > 0$.

From [0], $Y'' + \mu^2 Y = 0 \Rightarrow Y(y) = c_1 \cos(\mu y) + c_2 \sin(\mu y)$

At the bottom border, $y = 0$, So $Y(0) = C_1$. $\therefore c_1 = 0$.

$Y(y) = c_2 \sin(\mu y)$ and $c_2 \neq 0$.

From [1], $y = 2x$ for $0 \le x < 1$ along the left border.

$\therefore$ $Y(2x) = C_2 \sin(2\mu x) = 0$ for $0 \le x < 1$.

The only way for $c_2 \sin(2\mu x) = 0$ for every $x \in (0,1)$

is for $c_2 = 0$. $\therefore$ $c_1 = c_2 = 0$ and $Y(y) = 0$

$\therefore$ $u(x,y) = 0$, so $\lambda < 0 \Rightarrow$ no nontrivial solutions.

$\therefore$ (1), (2), (3) $\Rightarrow$ no nontrivial solutions of form

$u(x,y) = X(x)Y(y)$.

Note: for the problem $Z'' + \lambda Z = 0$, $Z(0) = Z(L) = 0$ on $[0, L]$,

λ is real, as demonstrated in Problem #23 of Section 10.1, p. 468. That problem used $[0, \pi]$, but the proof works for $[0, L]$, $L > 0$.

$\therefore$ The above analysis was restricted to the cases of $\lambda = 0$, $\lambda > 0$, $\lambda < 0$.

(b)

(1) Bottom on $0 \le x \le 2$, $y = 0$. $\therefore \xi = x - \frac{1}{2}y = x$, $\eta = y$.

$\therefore$ $\underline{0 \le \xi \le 2,\ \eta = 0}$, same bottom border

Left border on $0 \le x \le 1$, $y = 2x$, $0 \le y \le 2$

$\therefore$ $\xi = x - \frac{1}{2}y = x - \frac{1}{2}(2x) = 0$, $\eta = y$, $0 \le \eta \le 2$.

$\therefore$ $\underline{\xi = 0,\ 0 \le \eta \le 2}$, left border of square

Top border on $1 \le x \le 3$, $y = 2$.

$\therefore\ \epsilon = x - \frac{1}{2}y = x - \frac{1}{2}(2) = x - 1.\quad n = y \Rightarrow n = 2.$

$\therefore\ 1 \le x \le 3 \Rightarrow 1 \le \epsilon + 1 \le 3 \Rightarrow 0 \le \epsilon \le 2.$

$\therefore\ \underline{\underline{0 \le \epsilon \le 2,\ n = 2}}$, top border of square

Right border on $2 \le x \le 3$, $y = 2x - 4$, $0 \le y \le 2$

$\therefore\ \epsilon = x - \frac{1}{2}y = x - \frac{1}{2}(2x-4) = x - (x-2) = 2,\ n = y$

$\therefore\ \underline{\underline{\epsilon = 2,\ 0 \le n \le 2}}$, right border of square.

(2) Consider $u(\epsilon, n) = u(\epsilon(x,y), n(x,y)) = u(x - \frac{1}{2}y, y)$

Given $u_{xx} + u_{yy} = 0$ and $\epsilon = x - \frac{1}{2}y,\ n = y$

$$\therefore\ \frac{\partial \epsilon}{\partial x} = 1,\quad \frac{\partial \epsilon}{\partial y} = -\frac{1}{2},\quad \frac{\partial n}{\partial x} = 0,\quad \frac{\partial n}{\partial y} = 1$$

$$\therefore\ \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \epsilon}\cdot\overbrace{\frac{\partial \epsilon}{\partial x}}^{=1} + \frac{\partial u}{\partial n}\cdot\overbrace{\frac{\partial n}{\partial x}}^{=0} = u_\epsilon$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(u_\epsilon) = \frac{\partial u_\epsilon}{\partial \epsilon}\cdot\overbrace{\frac{\partial \epsilon}{\partial x}}^{=1} + \frac{\partial u_\epsilon}{\partial n}\cdot\overbrace{\frac{\partial n}{\partial x}}^{=0} = u_{\epsilon\epsilon}$$

$$\therefore\ u_{xx} = u_{\epsilon\epsilon}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \epsilon}\cdot\overbrace{\frac{\partial \epsilon}{\partial y}}^{=-\frac{1}{2}} + \frac{\partial u}{\partial n}\cdot\overbrace{\frac{\partial n}{\partial y}}^{=1} = -\frac{1}{2}u_\epsilon + u_n$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(-\tfrac{1}{2}u_\epsilon + u_n\right) = \frac{\partial}{\partial \epsilon}\left(-\tfrac{1}{2}u_\epsilon\right)\overbrace{\frac{\partial \epsilon}{\partial y}}^{=-\frac{1}{2}} + \frac{\partial}{\partial n}\left(-\tfrac{1}{2}u_\epsilon\right)\cdot\overbrace{\frac{\partial n}{\partial y}}^{=1}$$

$$+ \frac{\partial}{\partial \epsilon}(u_n)\overbrace{\frac{\partial \epsilon}{\partial y}}^{=-\frac{1}{2}} + \frac{\partial}{\partial n}(u_n)\overbrace{\frac{\partial n}{\partial y}}^{=1}$$

$$= \frac{1}{4}u_{\epsilon\epsilon} - \frac{1}{2}u_{\epsilon n} - \frac{1}{2}u_{n\epsilon} + u_{nn}$$

$$\therefore\ u_{yy} = \frac{1}{4}u_{\epsilon\epsilon} - u_{\epsilon n} + u_{nn}$$

assuming continuity of $u_{n\epsilon}$ and $u_{\epsilon n}$.

$$\therefore\ u_{xx} + u_{yy} = u_{\epsilon\epsilon} + \frac{1}{4}u_{\epsilon\epsilon} - u_{\epsilon n} + u_{nn}$$

$$= \frac{5}{4}u_{\epsilon\epsilon} - u_{\epsilon n} + u_{nn}$$

$$\therefore\ u_{xx} + u_{yy} = 0 \Rightarrow \underline{\frac{5}{4}u_{\epsilon\epsilon} - u_{\epsilon n} + u_{nn} = 0}$$

(3) From (1), the borders in the xy-plane translate to the square borders in the ϵn-plane.

$\therefore\ u(x,0) = 0$ on $0 \le x \le 2,\ y=0$ translates to

$$\underline{u(\epsilon,0) = 0 \text{ on } 0 \le \epsilon \le 2,\ n=0}$$

More directly, $u(\epsilon,n) = u(x - \frac{1}{2}y, y) =$

$$u(x - \tfrac{1}{2}(0), 0) = u(x,0) = 0 = u(\epsilon, 0)$$

since $x = \epsilon$ on $0 \le x \le 2 \iff 0 \le \epsilon \le 2$

$u(x,y) = 0$ on $0 \le x \le 1$, $y = 2x$ translates to

$\underline{u(0,n) = 0}$ on $\epsilon = 0$, $0 \le n \le 2$

More directly, $u(\epsilon, n) = u(x - \frac{1}{2}y, y) =$

$u(x - \frac{1}{2}(2x), y) = u(0, y) = u(0, n)$

$u(x,y) = 0$ on $2 \le x \le 3$, $y = 2x - 4$ translates to

$\underline{u(2,n) = 0}$ on $\epsilon = 2$, $0 \le n \le 2$

More directly, $u(\epsilon, n) = u(x - \frac{1}{2}y, y) =$

$u(x - \frac{1}{2}(2x-4), y) = u(2, y) = u(2, n) = 0$

$u(x,y) = f(x)$ on $1 \le x \le 3$, $y = 2$ translates to

$\underline{u(\epsilon, 2) = f(\epsilon + 1)}$ on $0 \le \epsilon \le 2$, $y = 2$

More directly, for $\epsilon = x - \frac{1}{2}y$, $y = 2$,

then $\epsilon = x - 1$, so $x = \epsilon + 1$.

$\therefore f(x) \Rightarrow f(\epsilon + 1)$ on $0 \le \epsilon \le 2$

$u(\epsilon, n) = u(x - \frac{1}{2}y, y) = u(x - 1, y) = u(\epsilon, 2)$

(c)

If $u(\epsilon, n) = U(\epsilon)V(n)$, then

$$\frac{5}{4}U''V - U'V' + UV'' = 0$$

Dividing by UV doesn't indicate a way to separate:

$$\frac{5}{4}\frac{U''}{U} - \frac{U'V'}{UV} + \frac{V''}{V} = 0$$

There is no way to separate the U and V functions in order to create separate ordinary differential equations, unless $U' = 0$ or $V' = 0$. $\therefore U(\epsilon) = c$ or $V(n) = c$, c a constant. If $U(\epsilon) = c$, then to adhere to the boundary conditions, $c = 0 \Rightarrow$ trivial solution. If $V(n) = c$, then $c = 0$ using the lower boundary, and $\therefore$ again, a trivial solution.

2.

Since there is radial symmetry, Laplace's equation in polar coordinates becomes $u_{rr} + \frac{1}{r} u_r = \frac{1}{a^2} u_{tt}$

As in Example 1 of the text, using $u(r,t) = R(r)T(t)$,

$$r^2 R'' + rR' + \lambda^2 r^2 R = 0 \quad [1]$$

$$T'' + \lambda^2 a^2 T = 0 \quad [2]$$

$$\therefore T(t) = K_1 \cos(\lambda a t) + K_2 \sin(\lambda a t)$$

$$u(r,0) = 0 \Rightarrow T(0) = 0 \Rightarrow K_1 = 0$$

$$\therefore T(t) = K_2 \sin(\lambda a t)$$

As in Example 1, using the boundedness of $u(r,t)$,

the solution to [1] is $R(r) = c_1 J_0(\lambda r)$

From $u(1,t) = 0$ for $t \geq 0$, $R(1) = 0 \Rightarrow J_0(\lambda) = 0$

and $\therefore$ the infinite eigenvalues λ_n derive from

$J_0(\lambda_n) = 0.$

$\therefore u_n(r,t) = J_0(\lambda_n r)\sin(\lambda_n a t)$

$\therefore$ Let $u(r,t) = \sum_{n=1}^{\infty} c_n J_0(\lambda_n r)\sin(\lambda_n a t)$

$\cos(0) = 1$

$\therefore u_t(r,0) = \sum_{n=1}^{\infty} c_n \lambda_n a J_0(\lambda_n r) = g(r), \quad 0 \le r \le 1$

Since [1] is $-(rR')' = \lambda^2 r R$ in Sturm-Liouville form, the $J_0(\lambda_n r)$ are orthogonal relative to r,

so $\sum_{n=1}^{\infty} c_n \lambda_n a \int_0^1 r J_0(\lambda_n r) J_0(\lambda_m r)\,dr = \int_0^1 r g(r) J_0(\lambda_m r)\,dr$

$\therefore c_m \lambda_m a \int_0^1 r J_0^2(\lambda_m r)\,dr = \int_0^1 r g(r) J_0(\lambda_m r)\,dr$

$$\therefore c_m = \frac{\int_0^1 r g(r) J_0(\lambda_m r)\,dr}{\lambda_m a \int_0^1 r J_0^2(\lambda_m r)\,dr}$$

$\therefore$ $u(r,t) = \sum_{n=1}^{\infty} c_n J_0(\lambda_n r)\sin(\lambda_n a t)$, where

λ_n is from $J_0(\lambda_n) = 0$ and

$$c_n = \frac{\int_0^1 r g(r) J_0(\lambda_n r)\,dr}{\lambda_n a \int_0^1 r J_0^2(\lambda_n r)\,dr}$$

3.

Since there is radial symmetry, Laplace's equation in polar coordinates for the circular membrane becomes

$$u_{rr} + \frac{1}{r}u_r = \frac{1}{a^2}u_{tt}, \quad 0<r<1, \ t>0 \quad [1]$$

Let $v(r,t)$ be the solution to [1] s.t. $v(1,t)=0$, $t \geq 0$,

$$v(r,0) = f(r), \quad v_t(r,0) = 0, \quad 0 \leq r \leq 1$$

Let $w(r,t)$ be the solution to [1] s.t. $w(1,t)=0$, $t \geq 0$,

$$w(r,0) = 0, \quad w_t(r,0) = g(r), \quad 0 \leq r \leq 1$$

Now let $u(r,t) = v(r,t) + w(r,t)$

$$\therefore \ u_{rr} + \frac{1}{r}u_r = v_{rr} + w_{rr} + \frac{1}{r}v_r + \frac{1}{r}w_r$$

$$= \frac{1}{a^2}v_{tt} + \frac{1}{a^2}w_{tt} = \frac{1}{a^2}u_{tt}$$

$$u(1,t) = v(1,t) + w(1,t) = 0 + 0 = 0$$

$$u(r,0) = v(r,0) + w(r,0) = f(r) + 0 = f(r)$$

$$u_t(r,0) = v_t(r,0) + w_t(r,0) = 0 + g(r) = g(r)$$

$\therefore$ $u(r,t)$ satisfies [1] as well as the boundary and initial conditions.

From Example #1 on p. 563-564,

$$v(r,t) = \sum_{n=1}^{\infty} c_n J_0(\lambda_n r)\cos(\lambda_n a t), \text{ where } \lambda_n \text{ are from } J_0(\lambda_n)=0 \text{ and } c_n = \frac{\int_0^1 r f(r) J_0(\lambda_n r)\,dr}{\int_0^1 r J_0^2(\lambda_n r)\,dr}$$

and from #2 above,

$$w(r,t) = \sum_{n=1}^{\infty} K_n J_0(\lambda_n r)\sin(\lambda_n a t), \text{ where } \lambda_n \text{ are from } J_0(\lambda_n)=0, \text{ and } K_n = \frac{\int_0^1 r g(r) J_0(\lambda_n r)\,dr}{\lambda_n a \int_0^1 r J_0^2(\lambda_n r)\,dr}$$

$\therefore$

$$\boxed{\begin{gathered} u(r,t) = \sum_{n=1}^{\infty} J_0(\lambda_n r)\left[c_n\cos(\lambda_n a t) + K_n \sin(\lambda_n a t)\right], \\ \text{where } \lambda_n \text{ are from } J_0(\lambda_n)=0, \text{ and} \\ c_n = \frac{\int_0^1 r f(r) J_0(\lambda_n r)\,dr}{\int_0^1 r J_0^2(\lambda_n r)\,dr}, \quad K_n = \frac{\int_0^1 r g(r) J_0(\lambda_n r)\,dr}{\lambda_n a \int_0^1 r J_0^2(\lambda_n r)\,dr} \end{gathered}}$$

4.

Let $u(r,\theta,t) = R(r)\Theta(\theta)T(t)$

$\therefore\ u_{rr} = R''\Theta T,\ u_r = R'\Theta T,\ u_{\theta\theta} = R\Theta''T,\ u_{tt} = R\Theta T''$

$\therefore\ R''\Theta T + \frac{1}{r}R'\Theta T + \frac{1}{r^2}R\Theta''T = \frac{1}{a^2}R\Theta T''$

Dividing by $R\Theta T$,

$$\frac{R''}{R} + \frac{R'}{rR} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = \frac{1}{a^2}\frac{T''}{T} \qquad [1]$$

Assume $\Theta(\theta)$ is periodic, such as $\Theta(\theta+2\pi) = \Theta(\theta)$.

Also, as shown in problem #9 of Section 10.8, p. 521, the solution to $\Theta'' + \lambda\Theta = 0$ is periodic only if λ is real. Case I on p. 519 shows $\lambda \geq 0$ for periodic eigenvalues.

The right side of [1] is a function of t only, the left side of r and θ but not t. $\therefore$ For a fixed r and θ, as t varies, T''/a^2T is a fixed number. Call it K_1. $\therefore$ The left side is also equal to this constant as r and θ vary.

$\therefore$ [1] becomes $\frac{R''}{R} + \frac{R'}{rR} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = K_1$, or

$$r^2\frac{R''}{R} + \frac{rR'}{R} - K_1 r^2 = -\frac{\Theta''}{\Theta} \qquad [2]$$

The left side is a function of r only, the right side of θ only, and so each equals the same constant. Call it K_2. $\therefore$ [2] becomes $\Theta'' + K_2\Theta = 0$. From the comments above, assuming Θ is periodic, K_2 is real and $K_2 \geq 0$. If $K_2 = 0$, then $\Theta(\theta) = c_1 + c_2\theta$. $\Theta(\theta) = \Theta(\theta + T) \Rightarrow c_2 = 0$. And if $\Theta(\theta) = c_1$, then $u(r,\theta,t)$ is not a function of θ, so $u_{\theta\theta} = 0$. $\therefore$ Reject $K_2 = 0$. Also, if $K_2 < 0$, $\Theta(\theta) = c_1 e^{\sqrt{-K_2}\,\theta} + c_2 e^{-\sqrt{-K_2}\,\theta}$, which is not periodic.

$\therefore K_2 > 0$. $\therefore$ Assign $K_2 = n^2$, $n > 0$.

$\therefore \underline{\Theta'' + n^2\Theta = 0}$, $n > 0$.

Now [1] becomes $\dfrac{R''}{R} + \dfrac{R'}{rR} - \dfrac{n^2}{r^2} = \dfrac{T''}{a^2 T}$

Since the left side is a function of r only and the right side of t only, they must equal the same constant. Call it K_3.

If you assume $T(t)$ is periodic, then $K_3 < 0$, similar to the reasons given above for $\Theta(\theta)$.

Then assign $K_3 = -\lambda^2$, $\lambda > 0$, and then

$T'' + \lambda^2 a^2 T = 0$, and $r^2 R'' + rR' + (\lambda^2 r^2 - n^2)R = 0$

If $K_3 = 0$, then [1] becomes $r^2 R'' + rR' - n^2 R = 0$, so $R(r) = c_1 r^n + c_2 r^{-n}$. Assuming $R(r)$ is bounded, as $r \to 0^+$, $c_2 = 0$, and if $R(1) = 0$ (i.e., unit circular membrane), then $c_1 = 0$.

$\therefore K_3 = 0$ yields a trivial solution.

If $K_3 > 0$, let $K_3 = \lambda^2$, $\lambda > 0$. Then [1] becomes
$T'' - \lambda^2 a^2 T = 0$, so $T(t) = c_1 e^{\lambda a t} + c_2 e^{-\lambda a t}$
and $r^2 R'' + rR' - (\lambda^2 r^2 + n^2) R = 0$. The latter
is a regular singular point at $r = 0$, and
further assumptions are necessary to analyze
if $K_3 > 0$ leads to any contradictions.

$\therefore$ Assuming the periodicity of $\Theta(\theta)$ and $T(t)$,
with $u(r,\theta,t) = R(r)\Theta(\theta)T(t)$, then

$$\boxed{\begin{aligned} &r^2 R'' + rR' + (\lambda^2 r^2 - n^2) R = 0 \\ &T'' + \lambda^2 a^2 T = 0 \\ &\Theta'' + n^2 \Theta = 0 \end{aligned}}$$

5.

(a)

With $u(r,\theta,z) = R(r)\,\Theta(\theta)\,Z(z)$, $u_r = R'\Theta Z$, $u_{rr} = R''\Theta Z$,

$u_{\theta\theta} = R\Theta'' Z$, $u_{zz} = R\Theta Z''$.

$$\therefore\ R''\Theta Z + \frac{1}{r}R'\Theta Z + \frac{1}{r^2}R\Theta'' Z + R\Theta Z'' = 0$$

Dividing by $R\Theta Z$,

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} + \frac{Z''}{Z} = 0. \qquad [0]$$

Rearranging, $\dfrac{R''}{R} + \dfrac{1}{r}\dfrac{R'}{R} + \dfrac{1}{r^2}\dfrac{\Theta''}{\Theta} = -\dfrac{Z''}{Z}$ [0']

The left side is a function of (r,θ), the right a function of z only, and so both sides must equal the same constant, call it K_1

$$\therefore\ Z'' + K_1 Z = 0, \qquad [1]$$

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = K_1 \text{, or after rearranging,}$$

$$r^2\frac{R''}{R} + r\frac{R'}{R} - r^2 K_1 = -\frac{\Theta''}{\Theta}$$

The left side is a function of r only, the right side of θ only, and so both equal the same constant, call it K_2.

$\therefore\ \Theta'' + K_2\Theta = 0$ and

$$r^2R'' + rR' - (r^2K_1 + K_2)R = 0 \qquad [2]$$

We can assume Θ is periodic: $\Theta(\theta) = \Theta(\theta + P)$

$\therefore$ If $K_2 = 0$, $\Theta(\theta) = c_1 + c_2\theta$. $\Theta(0) = \Theta(P) \Rightarrow c_2 = 0$.

$\Theta(\theta) = c_1 \Rightarrow \Theta'' = 0$, so $u_{\theta\theta} = 0$. $\therefore$ Exclude $K_2 = 0$.

If $K_2 < 0$, then $\Theta(\theta) = c_1 e^{\sqrt{-K_2}\,\theta} + c_2 e^{-\sqrt{-K_2}\,\theta}$, which is not periodic. $\therefore$ Exclude $K_2 < 0$.

$\therefore\ K_2 > 0$. Let $K_2 = n^2$, $n > 0$.

$\therefore\ \Theta'' + n^2\Theta = 0$. If the period is 2π, so that $\Theta(0) = \Theta(2\pi)$, then $\Theta(\theta) = c_1\cos(n\theta) + c_2\sin(n\theta)$

and n must be an integer, $n = 1, 2, 3, 4, \ldots$

[2] now becomes $r^2 R'' + rR' - (r^2 K_1 + n^2) R = 0$ [2']

If $K_1 = 0$, then $R(r) = c_1 r^n + c_2 r^{-n}$. Assuming $R(r)$ is bounded, $c_2 = 0$ since $r^{-n} \to \infty$ as $r \to 0^+$.

Requiring $R(a) =$ constant at the boundary of $r = a \Rightarrow c_1 = 0$. $\therefore$ Exclude $K_1 = 0$.

If $K_1 > 0$, the solution to [2'] involves "modified Bessel functions", which violate the assumption that $R(r)$ is bounded.

$\therefore$ Exclude $K_1 < 0$. $\therefore K_2 < 0$, call it $K_2 = -\lambda^2$, $\lambda > 0$.

$\therefore$ [2'] becomes $r^2 R'' + rR' + (\lambda^2 r^2 - n^2) R = 0$

and [1] becomes $Z'' - \lambda^2 Z = 0$.

$\therefore$ Assuming periodicity in θ and boundedness in r, then

$$\boxed{\begin{aligned} &r^2R'' + rR' + (\lambda^2r^2 - n^2)R = 0 \\ &\Theta'' + n^2\Theta = 0 \\ &Z'' - \lambda^2 Z = 0 \end{aligned}}$$

(b)

If $u(r,\theta,z)$ is independent of θ, then $u_{\theta\theta} = 0$, and $u(r,\theta,z) = R(r)Z(z)$. The equation [0'] in (a) becomes $\dfrac{R''}{R} + \dfrac{1}{r}\dfrac{R'}{R} = -\dfrac{Z''}{Z} = K_1$ [0'']

$\therefore$ Consider $rR'' + R' = K_1$, or $(rR')' = K_1$

If $K_1 = 0$, then $rR' = c_1$, a constant.

$\therefore$ $R'(r) = \dfrac{c_1}{r}$, $R(r) = c_2 + c_1 \ln(r)$

This is unbounded as $r \to 0^+$, so $c_1 = 0$.

$R(r) =$ a constant means $u(r,\theta,z)$ is also not dependent on r, so $u_r = u_{rr} = 0$, which seems unacceptable. $\therefore$ Exclude $K_1 = 0$.

If $K_1 > 0$, let $K_1 = \mu^2$, $\mu > 0$. Then

$r^2 R'' + rR' - \mu^2 r^2 R = 0$, which is a

modified Bessel function described in (a),

which violates bounded considerations.

$\therefore K_1 < 0$, let $K_1 = -\lambda^2$, $\lambda > 0$,

$\therefore$ [0''] becomes $\boxed{\begin{array}{l} Z'' - \lambda^2 Z = 0 \\ r^2 R'' + rR' + \lambda^2 r^2 R = 0 \end{array}}$

6.

Independence of $\theta \Rightarrow u_{rr} + \frac{1}{r} u_r + u_{zz} = 0$. Steady

state means $t \to \infty$, so the is no u_t term.

Let $u(r,z) = R(r)Z(z)$, so $u(1,z) = R(1)Z(z) = 0$ for all

z means $R(1) = 0$.

$\therefore R''Z + \frac{1}{r} R'Z + RZ'' = 0$. Dividing by RZ and rearranging, $\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} = \frac{-Z''}{Z} = K_1$, where K_1 is a separation constant as both sides must equal the same constant, as the left side is a function of r only, the right of z only.

(1) If $K_1 = 0$, then $rR'' + R' = 0 \Rightarrow (rR')' = 0 \Rightarrow$

$R(r) = c_1 + c_2 \ln(r)$. $R(1) = 0 \Rightarrow c_1 = 0$.

Since $\ln(r) \to -\infty$ as $r \to 0^+$, then $c_2 = 0$.

$\therefore$ Trivial solution so exclude $K_1 = 0$.

(2) $K_1 > 0$. $\therefore Z'' + K_1 Z = 0 \Rightarrow Z(z) = c_1 \cos(\sqrt{K_1}\, z) + c_2 \sin(\sqrt{K_1}\, z)$

For $u(r,z) = R(r)Z(z) \to 0$ as $z \to \infty$, $c_1 = c_2 = 0$.

This means a trivial solution, so exclude $K_1 > 0$.

(3) $\therefore K_1 < 0$, so let $K_1 = -\lambda^2$, $\lambda > 0$.

$\therefore Z'' - \lambda^2 Z = 0$ [1]

$r^2 R'' + rR' + \lambda^2 r^2 R = 0$ [2]

As shown in Example 1 on p. 563 of the text, the solution to [2] is $R(r) = J_0(\lambda r)$, as a fundamental solution, and the eigenvalues arise from $R(1) = 0 \Rightarrow J_0(\lambda) = 0$. This is an infinite set of eigenvalues as discussed on p. 557-558. Since [2] can be written in Sturm-Liouville form: $-(rR')' = \lambda^2 r R$, the $J_0(\lambda_n)$ are orthogonal relative to r.

$\therefore R_n(r) = J_0(\lambda_n r)$, λ_n from $J_0(\lambda_n) = 0$.

From [1], the solution is $Z(z) = c_1 e^{\lambda z} + c_2 e^{-\lambda z}$

$\therefore u_n(r,z) = R_n(r) Z_n(z) = J_0(\lambda_n r)\left[c_1 e^{\lambda_n z} + c_2 e^{-\lambda_n z}\right]$

The requirement that $u_n(r,z) \to 0$ as $z \to \infty$ means $c_1 = 0$.

$\therefore u_n(r,z) = J_0(\lambda_n r) e^{-\lambda_n z}$ are fundamental solutions.

$\therefore$ Let $u(r,z) = \sum_{n=1}^{\infty} c_n J_0(\lambda_n r) e^{-\lambda_n z}$

$$\therefore\ u(r,0) = f(r) = \sum_{n=1}^{\infty} c_n J_0(\lambda_n r)$$

$$\therefore\ r f(r) J_0(\lambda_m r) = \sum_{n=1}^{\infty} c_n r J_0(\lambda_n r) J_0(\lambda_m r)$$

$$\therefore\ \int_0^1 r f(r) J_0(\lambda_m r)\,dr = \sum_{n=1}^{\infty} c_n \int_0^1 r J_0(\lambda_n r) J_0(\lambda_m r)\,dr$$

$$= c_m \int_0^1 r J_0^2(\lambda_m r)\,dr$$

using orthogonality of J_0 relative to r.

$$\therefore\ \boxed{\begin{gathered} u(r,z) = \sum_{n=1}^{\infty} c_n J_0(\lambda_n r) e^{-\lambda_n z}, \text{ where} \\ J_0(\lambda_n) = 0 \text{ and} \\ c_n = \frac{\int_0^1 r f(r) J_0(\lambda_n r)\,dr}{\int_0^1 r J_0^2(\lambda_n r)\,dr} \end{gathered}}$$

7.

(a)

$V_{rr} + \frac{1}{r} V_r + \frac{1}{r^2} V_{\theta\theta} + k^2 V = 0$ becomes

$R''\Theta + \frac{1}{r} R'\Theta + \frac{1}{r^2} R\Theta'' + k^2 R\Theta = 0$, or

$r^2 R''\Theta + rR'\Theta + R\Theta'' + k^2 r^2 R\Theta = 0$

Dividing by $R\Theta$ and rearranging,

$r^2 \frac{R''}{R} + r\frac{R'}{R} + k^2 r^2 = -\frac{\Theta''}{\Theta} = c_1$

c_1 is a constant as the left side is a function of r only and the right side of θ only, so both sides must equal the same constant.

$\therefore \ \Theta'' + c_1\Theta = 0.$

Assume Θ is periodic, so $\theta(0) = \theta(2\pi)$. This ensures continuity, e.g. of a circular membrane or rod, etc.

(1) $c_1 = 0 \Rightarrow \Theta = a_1 + a_2\theta$. $\Theta(0) = \Theta(2\pi) \Rightarrow a_2 = 0$.

$\Theta(\theta) = c_1$ is periodic, and we don't know if $\Theta(0) = 0$.

$\therefore \Theta(\theta) = 1$ is a possible solution.

(2) $c_1 < 0$. $\therefore$ Let $c_1 = -\mu^2$, $\mu > 0$. $\therefore \Theta(\theta) = a_1 e^{\mu\theta} + a_2 e^{-\mu\theta}$.

This is not a periodic function. $\therefore$ Exclude $c_1 < 0$.

$\therefore C_1 > 0$. $\therefore$ Let $c_1 = \lambda^2$, $\lambda > 0$.

$$\therefore \quad \underline{\underline{\Theta'' + \lambda^2 \Theta = 0}} \quad , \quad \lambda \geq 0$$

Also, $r^2 \frac{R''}{R} + r\frac{R'}{R} + K^2 r^2 = \lambda^2$, or

$$\underline{\underline{r^2 R'' + rR' + (K^2 r^2 - \lambda^2) R = 0}}$$

(b)

Let $v(r,\theta)$ be the solution and assume $v(r,\theta) = R(r)\Theta(\theta)$.

From (a), $\Theta'' + \lambda^2 \Theta = 0 \Rightarrow \Theta = c_1 \cos(\lambda\theta) + c_2 \sin(\lambda\theta)$. For a period of 2π, $\lambda \underline{\underline{= 0, 1, 2, 3, \ldots}}$ Note for each λ, the constants can be different. $\therefore$ It is better to write

$\Theta(\theta) = a_n \cos(n\theta) + b_n \sin(n\theta)$, $n = 0, 1, 2, 3, \ldots$, using n for λ.

From (a), $r^2 R'' + rR' + (k^2 r^2 - n^2)R = 0$ [1]

Using the suggestion of a variable substitution in Section 11.4, problem #3, let $t = kr$, and consider $R(r(t))$. $\therefore \dfrac{dR}{dt} = \dfrac{dR}{dr} \cdot \dfrac{dr}{dt} = R'(r)\left(\dfrac{1}{k}\right)$

$$\frac{d^2R}{dt^2} = \frac{d}{dt}\left(R'(r) \cdot \frac{1}{k}\right) = \frac{1}{k}\left(\frac{dR'}{dr} \cdot \frac{dr}{dt}\right) = \frac{1}{k}\left(R'' \cdot \frac{1}{k}\right) = \frac{1}{k^2} R''$$

$$\therefore \frac{dR}{dr} = R'(r) = k\frac{dR}{dt}, \quad \frac{d^2R}{dr^2} = R''(r) = k^2 \frac{d^2R}{dt^2}$$

$\therefore$ [1] becomes $\left(\dfrac{t}{k}\right)^2 k^2 \dfrac{d^2R}{dt^2} + \left(\dfrac{t}{k}\right) k \dfrac{dR}{dt} + \left(k^2 \cdot \dfrac{t^2}{k^2} - n^2\right) R(r(t)) = 0$

Or, $t^2 \dfrac{d^2R}{dt^2} + t\dfrac{dR}{dt} + (t^2 - n^2)R(r(t)) = 0$

This is a Bessel equation of order n, and the solution, from Section 11.4, #3, is $R(r(t)) = R\left(\dfrac{t}{k}\right) = J_n(t)$.

Using $t = kr$, $R(r) = J_n(kr)$, $n = 0, 1, 2, 3, \ldots$

$\therefore\ V_n(r,\theta) = J_n(kr)\left[a_n\cos(n\theta) + b_n\sin(n\theta)\right]$

That is, one solution is $J_n(kr)\cos(n\theta)$ and another is $J_n(kr)\sin(n\theta)$. Note these functions are orthogonal as they act like $c_n\cos(n\theta)$ and $d_n\sin(n\theta)$.

$$\therefore \int_0^{2\pi} c_n\cos(n\theta)\, d_n\sin(n\theta)\,d\theta = c_n d_n \int_0^{2\pi}\cos(n\theta)\sin(n\theta)\,d\theta = 0$$

as $J_n(kr)$ does not depend on θ.

$$\therefore \text{Let } V(r,\theta) = \sum_{n=0}^{\infty} V_n(r,\theta) = \sum_{n=0}^{\infty} J_n(kr)\left[a_n\cos(n\theta) + b_n\sin(n\theta)\right]$$

One boundary condition is $V(c,\theta) = f(\theta)$

$$\therefore V(c,\theta) = \sum_{n=0}^{\infty} J_n(kc)\left[a_n\cos(n\theta) + b_n\sin(n\theta)\right] = f(\theta)$$

This is a Fourier series, so write this as

$$V(r,\theta) = \frac{a_0}{2}J_0(kr) + \sum_{n=1}^{\infty} J_n(kr)\left[a_n\cos(n\theta) + b_n\sin(n\theta)\right] \quad [2]$$

$$\therefore f(\theta) = V(c,\theta) = \frac{a_0}{2}J_0(kc) + \sum_{n=1}^{\infty} J_n(kc)\left[a_n\cos(n\theta) + b_n\sin(n\theta)\right] \quad [3]$$

Using an even extension of $f(\theta)$ on $[-2\pi, 2\pi]$, and using $\int_0^{2\pi}\sin(n\theta)\cos(m\theta)\,d\theta = 0$, all integers $n,m > 0$

$$\int_0^{2\pi} \sin(n\theta)\sin(m\theta)\,d\theta = 0, \quad m \neq n, \quad m,n>0$$
$$= \pi, \quad m=n, \quad m,n>0$$

$$\int_0^{2\pi} \cos(n\theta)\cos(m\theta)\,d\theta = 0, \quad m \neq n, \quad m,n>0$$
$$= \pi, \quad m=n, \quad m,n>0$$

$\therefore$ Multiply [3] by $\cos(m\theta)$ and integrate, $m \geq 1$,

$$\int_0^{2\pi} f(\theta)\cos(m\theta)\,d\theta = \frac{a_0}{2} J_0(Kc)\overbrace{\int_0^{2\pi} \cos(m\theta)\,d\theta}^{=0} +$$
$$J_m(Kc)\, a_m \pi$$

$$\therefore \quad a_m = \frac{1}{\pi J_m(Kc)} \int_0^{2\pi} f(\theta)\cos(m\theta)\,d\theta, \quad m \geq 1$$

Multiply [3] by $\sin(m\theta)$ and integrate, $m \geq 1$,

$$\int_0^{2\pi} f(\theta)\sin(m\theta)\,d\theta = \frac{a_0}{2} J_0(Kc)\overbrace{\int_0^{2\pi} \sin(m\theta)\,d\theta}^{=0} +$$
$$J_m(Kc)\, b_m \pi$$

$$\therefore \quad b_m = \frac{1}{\pi J_m(Kc)} \int_0^{2\pi} f(\theta)\sin(m\theta)\,d\theta, \quad m \geq 1$$

Integrating [3],

$$\int_0^{2\pi} f(\theta)\,d\theta = \frac{a_0}{2} J_0(Kc)\overbrace{\int_0^{2\pi} d\theta}^{=2\pi} +$$
$$\sum_{n=1}^{\infty} J_n(Kc)\left[a_n \overbrace{\int_0^{2\pi} \cos(n\theta)\,d\theta}^{=0} + b_n \overbrace{\int_0^{2\pi} \sin(n\theta)\,d\theta}^{=0}\right]$$

$$= a_0 \pi J_0(Kc)$$

$$\therefore a_0 = \frac{1}{\pi J_0(kc)} \int_0^{2\pi} f(\theta)\cos(m\theta)\,d\theta, \quad m=0$$

$$\therefore \boxed{\begin{aligned} &V(r,\theta) = \frac{a_0}{2} J_0(kr) + \sum_{n=1}^{\infty} J_n(kr)\left[a_n\cos(n\theta) + b_n\sin(n\theta)\right], \\ &\text{where} \\ &\qquad a_n = \frac{1}{\pi J_n(kc)} \int_0^{2\pi} f(\theta)\cos(n\theta)\,d\theta, \quad n=0,1,2,\ldots \\ &\qquad b_n = \frac{1}{\pi J_n(kc)} \int_0^{2\pi} f(\theta)\sin(n\theta)\,d\theta, \quad n=1,2,3,\ldots \end{aligned}}$$

8.

Assume $u(r,t) = R(r)T(t)$

From $u_{rr} + \frac{1}{r}u_r = \frac{1}{\alpha^2}u_t$, $\quad R''T + \frac{1}{r}R'T = \frac{1}{\alpha^2}RT'$

Dividing by RT, $\quad \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} = \frac{1}{\alpha^2}\frac{T'}{T}$

As the left side is a function of r only, the right side of t only, they both must equal the same constant. Call it K. $\therefore$ $T' - K\alpha^2 T = 0$

and $r^2 R'' + rR' - Kr^2 R = 0$

(1) $K=0$. Then $T(t) = $ constant, so $u(r,t)$ is a function of r only and independent of time. This makes no physical sense, so exclude $K=0$. Also, for $R(r)$, $rR'' + R' = (rR')' = 0 \Rightarrow rR' = c_1$, a constant $\Rightarrow R(r) = c_1 \ln(r) + c_2$. $R(1) = 0 \Rightarrow c_2 = 0$.

But as $r \to 0^+$, $R(r)$ becomes unbounded, so $u(r,t)$ is unbounded as $r \to 0^+$, which also makes no physical sense.

(2) $K>0$. Let $K = \mu^2$, $\mu > 0$. $\therefore$ $T' - \mu^2\alpha^2 T = 0 \Rightarrow$ $T(t) = c_1 e^{\mu^2\alpha^2 t}$. $\therefore$ $u(r,t) = R(r) c_1 e^{\mu^2\alpha^2 t}$, so that as $t \to \infty$, $u(r,t)$ becomes unbounded.

$\therefore$ Exclude $K > 0$.

$\therefore$ $K < 0$. Let $K = -\lambda^2$, $\lambda > 0$. $\therefore$ $T' + \lambda^2 \alpha^2 T = 0 \Rightarrow$

$T(t) = e^{-\lambda^2 \alpha^2 t}$ is a fundamental solution.

$\therefore$ $r^2 R'' + rR' + \lambda^2 r^2 R = 0$ As shown in Example 1,

p. 563, $R(r) = J_0(\lambda r)$ using boundedness of $u(r,t)$.

The boundary condition $u(1,t) = 0 \Rightarrow R(1) = J_0(\lambda) = 0$,

which yields eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$

$\therefore$ $\phi_n(r,t) = J_0(\lambda_n r) e^{-\lambda_n^2 \alpha^2 t}$

$\therefore$ Assume $u(r,t) = \sum_{n=1}^{\infty} c_n J_0(\lambda_n r) e^{-\lambda^2 \alpha^2 t}$

The boundary condition $u(r,0) = f(r)$, $0 \le r \le 1$

means $f(r) = \sum_{n=1}^{\infty} c_n J_0(\lambda_n r)$

Using $r^2 R'' + rR' + \lambda^2 r^2 R = 0$ as $-(rR')' = \lambda^2 r R$

in Sturm-Liouville form, the $J_0(\lambda_n r)$ are

orthogonal relative to r, so that

$$\int_0^1 r f(r) J_0(\lambda_m)\, dr = \sum_{n=1}^{\infty} c_n \underbrace{\int_0^1 r J_0(\lambda_n r) J_0(\lambda_m r)\, dr}^{= 0 \text{ except when } n = m}$$

$$\therefore\quad c_m = \frac{\int_0^1 r f(r) J_0(\lambda_m r)\,dr}{\int_0^1 r J_0^2(\lambda_m r)\,dr}$$

$$\therefore\quad \boxed{\begin{aligned} &u(r,t) = \sum_{n=1}^{\infty} c_n J_0(\lambda_n r) e^{-\lambda^2 \alpha^2 t}, \text{ where} \\ &\lambda_n \text{ are from } J_0(\lambda_n) = 0 \text{ and} \\ &c_n = \frac{\int_0^1 r f(r) J_0(\lambda_n r)\,dr}{\int_0^1 r J_0^2(\lambda_n r)\,dr} \end{aligned}}$$

9.

(a)

Let $u(\rho, \theta, \phi) = P(\rho)\Theta(\theta)\phi(\phi)$

$\therefore$ Laplace's equation becomes,

$\rho^2 P''\Theta\phi + 2\rho P'\Theta\phi + \csc^2(\phi) P\Theta''\phi + P\Theta\phi'' + \cot(\phi) P\Theta\phi' = 0$

Dividing by $P\Theta\phi$, and rearranging,

$$\rho^2 \frac{P''}{P} + 2\rho \frac{P'}{P} = -\csc^2(\phi)\frac{\Theta''}{\Theta} - \frac{\phi''}{\phi} - \cot(\phi)\frac{\phi'}{\phi} \quad [1]$$

The left is a function of ρ only, the right side of Θ and ϕ but not ρ. $\therefore$ Both sides must equal the same constant, call it K.

$\therefore \rho^2 \frac{P''}{P} + 2\rho\frac{P'}{P} = K$, or $\rho^2 P'' + 2\rho P' = KP$, or $(\rho^2 P')' = KP$.

If $K=0$, then $(\rho^2 P')' = 0 \Rightarrow \rho^2 P' = c_1$, $P(\rho) = \frac{c_1}{\rho} + c_2$.

$\therefore$ As $\rho \to 0^+$, P becomes unbounded. $\therefore$ Set $c_1 = 0$.

$\therefore$ $P(\rho)$ is a constant, so $u(\rho,\theta,\phi)$ doesn't depend on ρ. $\therefore$ Exclude $K=0$.

$\therefore$ Consider $\rho^2 P + 2\rho P' - KP = 0$, an Euler equation.

Let $P = \rho^\alpha$. $\therefore$ $\alpha(\alpha-1)\rho^\alpha + 2\alpha\rho^\alpha - K\rho^\alpha = 0$, or

$\alpha^2 + \alpha - K = 0$, or $\alpha = \frac{-1 \pm \sqrt{1+4K}}{2}$, so α is real if $1+4K \geq 0$. If $\sqrt{1+4K} < 1$, then both roots are negative. $\therefore P = c_1\rho^{\alpha_1} + c_2\rho^{\alpha_2}$ is unbounded, $\alpha_1, \alpha_2 < 0$, as $\rho \to 0^+$. $\therefore$ Set $\sqrt{1+4K} > 1 \Rightarrow K > 0$.

$\therefore$ Set $K = \mu^2$, $\mu > 0$ $\therefore$ $\rho^2 P'' + 2\rho P' - \mu^2 P = 0$ [2]

Now [1] becomes $-\csc^2(\phi)\dfrac{\Theta''}{\Theta} - \dfrac{\Phi''}{\Phi} - \cot(\phi)\dfrac{\Phi'}{\Phi} = \mu^2$

After rearranging,

$$-\frac{\Phi''}{\Phi} - \cot(\phi)\frac{\Phi'}{\Phi} - \mu^2 = \csc^2(\phi)\frac{\Theta''}{\Theta}, \text{ or}$$

$$-\sin^2(\phi)\frac{\Phi''}{\Phi} - \sin^2(\phi)\cot(\phi)\frac{\Phi'}{\Phi} - \mu^2\sin^2(\phi) = \frac{\Theta''}{\Theta} \quad [3]$$

The left side is a function of ϕ only, the right side of θ only, and so they equal the same constant, call it c. $\therefore$ $\Theta'' - c\Theta = 0$. [4]

It is assumed $u(\rho, \theta, \phi)$ is periodic in θ, so

$\Theta(0) = \Theta(2\pi)$. $\therefore$ $c = 0 \Rightarrow \Theta(\theta) = c_1\theta + c_2$.

$\Theta(0) = \Theta(2\pi) \Rightarrow c_1 = 0$, so $\Theta(\theta) = c_2$, which means

$U(r,\theta,\phi)$ is not a function of θ. $\therefore$ Exclude $c=0$.

If $c>0$, let $c=\lambda^2$, $\lambda>0$. $\therefore$ [4] $\Rightarrow \Theta(\theta)=c_1 e^{\lambda\theta}+c_2 e^{-\lambda\theta}$

But this is not periodic in 2π. $\therefore$ Exclude $c>0$.

$\therefore$ $c<0$, call it $c=-\lambda^2$, $\lambda>0$. $\therefore$ [4] becomes

$$\underline{\Theta''+\lambda^2\Theta=0} \quad [5]$$

Θ periodic in 2π means $\lambda=1,2,3,\dots$ as

$$\Theta(\theta)=c_1\cos(\lambda\theta)+c_2\sin(\lambda\theta).$$

Using [3] with $-\lambda^2$,

$$-\sin^2(\phi)\frac{\Phi''}{\Phi}-\underbrace{\sin^2(\phi)\cot(\phi)}_{=\sin(\phi)\cos(\phi)}\frac{\Phi'}{\Phi}-\mu^2\sin^2(\phi)=-\lambda^2,$$

or, $\sin^2(\phi)\Phi''+\sin(\phi)\cos(\phi)\Phi'+\mu^2\sin^2(\phi)\Phi=\lambda^2\Phi$

or, $\sin^2(\phi)\Phi''+\sin(\phi)\cos(\phi)\Phi'+(\mu^2\sin^2(\phi)-\lambda^2)\Phi=0$ [6]

In summary,

$$\boxed{\begin{array}{ll} \rho^2 P''+2\rho P'-\mu^2 P=0, & \rho>0 \\ \Theta''+\lambda^2\Theta=0, & 0<\theta<2\pi \\ \sin^2(\phi)\Phi''+\sin(\phi)\cos(\phi)\Phi'+(\mu^2\sin^2(\phi)-\lambda^2)\Phi=0, & 0<\phi<\pi \end{array}}$$

(b)

$u(\rho,\theta,\phi)$ becomes $u(\rho,\phi) = P(\rho)\Phi(\phi)$ and

$\rho^2 u_{\rho\rho} + 2\rho u_\rho \; \csc^2(\phi) u_{\theta\theta} + u_{\phi\phi} + \cot(\phi)\, u_\phi = 0$

becomes $\rho^2 u_{\rho\rho} + 2\rho u_\rho + u_{\phi\phi} + \cot(\phi)\, u_\phi = 0$ as

$u_\theta = 0$ and $\therefore$ $u_{\theta\theta} = 0$.

$\therefore \quad \rho^2 P''\Phi + 2\rho P'\Phi + P\Phi'' + \cot(\phi) P\Phi' = 0$

Dividing by $P\Phi$ and rearranging,

$$\rho^2 \frac{P''}{P} + 2\rho \frac{P'}{P} = -\frac{\Phi''}{\Phi} - \cot(\phi)\frac{\Phi'}{\Phi}$$

Reasoning as in (a),

$$\rho^2 \frac{P''}{P} + 2\rho \frac{P'}{P} = \mu^2 = -\frac{\Phi''}{\Phi} - \cot(\phi)\frac{\Phi'}{\Phi}, \quad \mu > 0$$

$\therefore \quad \rho^2 P'' + 2\rho P' - \mu^2 P = 0$

$\Phi'' + \cot(\phi)\Phi' + \mu^2\Phi = 0 \qquad$ [7]

Multiplying [7] by $\sin^2(\phi)$ and noting

$\sin^2(\phi)\cot(\phi) = \sin(\phi)\cos(\phi)$,

$$\sin^2(\phi)\,\Phi'' + \sin(\phi)\cos(\phi)\,\Phi' + \mu^2\sin^2(\phi)\,\Phi = 0$$

(c)

As $\cos(\phi)$ is continuous and decreasing on $0 < \phi < \pi$, use $\phi = \arccos(s)$ for the change of variables, so $\Phi(\phi(s))$ is the function to consider, $-1 < s < 1$

Note: $\dfrac{d\phi}{ds} = \dfrac{-1}{\sqrt{1-s^2}}$. $\dfrac{d^2\phi}{ds^2} = -s(1-s^2)^{-3/2}$

here, $\Phi' = \dfrac{d\Phi}{d\phi}$

$$\frac{d\Phi}{ds} = \frac{d\Phi}{d\phi}\cdot\frac{d\phi}{ds} = \frac{-\Phi'}{\sqrt{1-s^2}} \quad \therefore\ \Phi' = -\sqrt{1-s^2}\,\frac{d\Phi}{ds} \quad [8]$$

$$\frac{d^2\Phi}{ds^2} = \frac{d}{ds}\left(\frac{d\Phi}{d\phi}\cdot\frac{d\phi}{ds}\right) = \frac{d^2\Phi}{d\phi^2}\cdot\frac{d\phi}{ds}\cdot\frac{d\phi}{ds} + \frac{d\Phi}{d\phi}\cdot\frac{d^2\phi}{ds^2}$$

$$= \frac{\Phi''}{1-s^2} + \Phi'\left[-s(1-s^2)^{-3/2}\right]$$

use [8]

$$\therefore\ (1-s^2)\frac{d^2\Phi}{ds^2} = \Phi'' - \frac{s}{\sqrt{1-s^2}}\,\Phi' = \Phi'' + s\,\frac{d\Phi}{ds}$$

$$\therefore\ \Phi'' = (1-s^2)\frac{d^2\Phi}{ds^2} - s\,\frac{d\Phi}{ds} \quad [9]$$

From $s = \cos(\phi)$, $s^2 = \cos^2(\phi) = 1 - \sin^2(\phi)$.

$\therefore \sin^2(\phi) = 1 - s^2$ [10]

$\therefore \sin(\phi) = \sqrt{1-s^2}$ [11]

as $\sin(\phi) > 0$ for $0 < \phi < \pi$.

Substituting [8], [9], [10], [11] into

$$\sin^2(\phi)\, \Phi'' + \sin(\phi)\cos(\phi)\, \Phi' + \mu^2 \sin^2(\phi)\, \Phi = 0$$

$$(1-s^2)\left[(1-s^2)\frac{d^2\Phi}{ds^2} - s\frac{d\Phi}{ds}\right] + \sqrt{1-s^2}\, s\left[-\sqrt{1-s^2}\frac{d\Phi}{ds}\right] + \mu^2(1-s^2)\Phi = 0$$

Dividing by $1-s^2$ as $1-s^2 \neq 0$ for $-1 < s < 1$,

$$\underline{(1-s^2)\frac{d^2\Phi}{ds^2} - 2s\frac{d\Phi}{ds} + \mu^2\Phi = 0}$$

10.

Steady-state means $u_t = 0$, where t is for time.

$\therefore$ Independence of θ means:

$$\rho^2 u_{\rho\rho} + 2\rho u_\rho + u_{\phi\phi} + \cos(\phi)\, u_\phi = 0, \quad \rho > 0, \quad 0 < \phi < \pi \quad [0]$$

and $\therefore$ from 9(b) using $u(\rho, \phi) = P(\rho)\Phi(\phi)$, and

Using the variable change $\phi = \arccos(s)$, $-1 < s < 1$ for $0 < \phi < \pi$, $u(\rho, \phi(s)) = P(\rho)\Phi(\phi(s))$, or $u(\rho, s) = P(\rho)\Phi(s)$,

$$\rho^2 P'' + 2\rho P' - \mu^2 P = 0 \quad , \mu > 0 \qquad [1]$$

$$(1-s^2)\Phi'' - 2s\Phi' + \mu^2\Phi = 0 \qquad [2]$$

Here, $\Phi' = d\Phi/ds$

Assume $P(\rho) = \rho^\alpha$ $\therefore$ [1] $\Rightarrow \alpha(\alpha-1)\rho^\alpha + 2\alpha\rho^\alpha - \mu^2\rho^\alpha = 0$.

$\therefore \alpha^2 + \alpha - \mu^2 = 0 \Rightarrow \alpha = \dfrac{-1 \pm \sqrt{1+4\mu^2}}{2}$

Reject the negative root since that would make ρ^α unbounded as $\rho \to 0^+$. $\therefore$ $\alpha = \dfrac{-1 + \sqrt{1+4\mu^2}}{2}$,

$2\alpha + 1 = \sqrt{1+4\mu^2}$, $4\alpha^2 + 4\alpha + 1 = 1 + 4\mu^2$, $\mu^2 = \alpha(\alpha+1)$,

and note $\alpha > 0$.

$\therefore$ Change [1], [2] to

$$\rho^2 P'' + 2\rho P' - \alpha(\alpha+1)P = 0 \qquad [1']$$

$$(1-s^2)\Phi'' - 2s\Phi' + \alpha(\alpha+1)\Phi = 0 \qquad [2']$$

Note: ρ^α solves [1']

From Section 5.3, Problem #17, the solutions to [2']
are, with a radius of convergence of $|s| < 1$:

$$\phi_1(s) = 1 + \sum_{m=1}^{\infty} (-1)^m \frac{\alpha(\alpha-2)(\alpha-4)\cdots[\alpha-(2m-2)](\alpha+1)(\alpha+3)\cdots[\alpha+(2m-1)]}{(2m)!} s^{2m}$$

$$\phi_2(s) = s + \sum_{m=1}^{\infty} (-1)^m \frac{(\alpha-1)(\alpha-3)\cdots[\alpha-(2m-1)](\alpha+2)(\alpha+4)\cdots(\alpha+2m)}{(2m+1)!} s^{2m+1}$$

It is shown above that $\alpha > 0$ for ρ^α to be bounded.
$\therefore$ If $\alpha = 1, 2, 3, \ldots$, then ρ^α is bounded as $\rho \to 0^+$ and, as shown in Section 5.3, Problem #18, $\phi_1(s)$ and $\phi_2(s)$ are finite polynomials. The Legendre polynomial $P_n(s)$ is $\phi_1(s)$ for n even, and $\phi_2(s)$ for n odd. In addition, by definition, $P_n(1) = 1$.
Since $\phi_1(s)$ has only even powers of s in its series, $\phi_1(1) = \phi_1(-1)$. Since $\phi_2(s)$ has an odd power of s in every term, $\phi_2(-1) = -\phi_2(1)$. $\therefore$ $P_n(1) = P_n(-1) = 1$ for n even, and $P_n(1) = 1$, $P_n(-1) = -1$ for n odd.

$\therefore P_n(s)$ is defined and bounded at $s = \pm 1$.

$\therefore P_n(s)$ converge for $-1 \le s \le 1$.

$\therefore$ We can use $P_n(\cos(\phi))$ for $0 \le \phi \le \pi$, using $s = \cos(\phi)$

There are infinitely many $P_n(s)$, and as shown in Section 5.3, Problem #22, the P_n are orthogonal:

$\int_{-1}^{1} P_n(s) P_m(s)\,ds = 0,\ m \ne n$, and $\int_{-1}^{1} P_n^2(s)\,ds = \frac{2}{2n+1}$.

$\therefore u_n(\rho, s) = \rho^n P_n(s)$, $n = 1, 2, 3, \ldots$ are solutions to [0]

and can serve like basis vectors or eigenfunctions.

Switching back using $s = \cos(\phi)$, $u_n(\rho, \phi) = \rho^n P_n[\cos(\phi)]$

$\therefore$ Assume $u(\rho, \phi) = \sum_{n=1}^{\infty} c_n \rho^n P_n[\cos(\phi)]$, $\rho \ge 0$, $0 \le \phi \le \pi$

Using the boundary condition, $u(1, \phi) = f(\phi)$, $0 \le \phi \le \pi$

$$\therefore \quad f(\phi) = \sum_{n=1}^{\infty} c_n P_n[\cos(\phi)]$$

To use orthogonality, which is in s, not ϕ, write

$$f(\arccos(s)) = \sum_{n=1}^{\infty} c_n P_n(s)$$

$$\therefore \quad f(\arccos(s)) P_m(s) = \sum_{n=1}^{\infty} c_n P_n(s) P_m(s)$$

$$\therefore \int_{-1}^{1} f(\arccos(s))\, P_m(s)\, ds = \sum_{n=1}^{\infty} c_n \int_{-1}^{1} P_n(s)\, P_m(s)\, ds$$

$$= c_m \int_{-1}^{1} P_m^2(s)\, ds = c_m \left(\frac{2}{2m+1}\right)$$

$$\therefore c_m = \frac{2m+1}{2} \int_{-1}^{1} f(\arccos(s))\, P_m(s)\, ds$$

$\therefore$

$$\boxed{\begin{array}{c} u(\rho, \phi) = \displaystyle\sum_{n=1}^{\infty} c_n \rho^n P_n(\cos(\phi)), \quad \rho \geq 0,\ 0 \leq \phi \leq \pi \\ \text{where } P_n(\,) \text{ are Legendre polynomials,} \\ \text{and } \quad c_n = \dfrac{2n+1}{2} \displaystyle\int_{-1}^{1} f(\arccos(s))\, P_n(s)\, ds \end{array}}$$

11.6 Series of Orthogonal Functions: Mean Convergence

Note Title 1/31/2022

1.

Using MATLAB,

```
clear
syms x
f = 1;  % the function
n = 1;
Sn = (4/pi)*sin(pi*x);  % first partial sum
Rn = vpaintegral((f-Sn)^2,x,0,1);  % 1st error
while Rn >= 0.02
    n = n + 2;  % only look at odd terms
    coef = 2*(1-cos(n*pi))/(n*pi);
    Sn = Sn + coef*sin(n*pi*x);  % next partial sum
    Rn = vpaintegral((f-Sn)^2,x,0,1);  % next error
end
n,sprintf('%.4f',Rn)  % display n, error
```

```
n = 21
ans = '0.0184'
```

$\therefore$ $\underline{n = 21}$

2.

Note that $\phi_m(x) = \sqrt{2}\sin(m\pi x)$ are the normalized eigenfunctions to $-(y')' = \lambda y$, $y(0) = 0$, $y(1) = 0$ as discussed on p. 569 and on p. 538-539, Example 1.

$\therefore$ $r(x) = 1$ in the Sturm-Liouville equation.

(a)

$$\text{Let } f(x) = x = \sum_{m=1}^{\infty} b_m \phi_m(x) = \sqrt{2} \sum_{m=1}^{\infty} b_m \sin(m\pi x)$$

$$\therefore \int_0^1 x \sin(n\pi x)\,dx = \sqrt{2} \sum_{m=1}^{\infty} b_m \int_0^1 \sin(m\pi x)\sin(n\pi x)\,dx$$

Using MATLAB,

```
clear
syms x n m
I1 = int(sin(m*pi*x)*sin(n*pi*x),x,0,1)
I2 = int(sin(m*pi*x)*sin(m*pi*x),x,0,1)
I3 = int(x*sin(n*pi*x),x,0,1)
```

I1 =

$$-\frac{m\cos(\pi m)\sin(\pi n) - n\cos(\pi n)\sin(\pi m)}{\pi\ (-n^2+m^2)}$$

I2 =

$$\frac{1}{2} - \frac{\sin(2\pi m)}{4 m \pi}$$

I3 =

$$\frac{\sin(\pi n) - \pi n\cos(\pi n)}{n^2\pi^2}$$

Since m, n are integers,

$$I_1 = 0, \quad I_2 = \frac{1}{2}, \quad I_3 = -\frac{\cos(n\pi)}{n\pi}$$

$$\therefore \int_0^1 x \sin(n\pi x)\,dx = \frac{-\cos(n\pi)}{n\pi} = \frac{(-1)^{n+1}}{n\pi}$$

$$\sqrt{2} \sum_{m=1}^{\infty} b_m \int_0^1 \sin(m\pi x)\sin(n\pi x)\,dx = \sqrt{2}\, b_n \left(\frac{1}{2}\right)$$

$$\therefore \frac{\sqrt{2}}{2} b_n = \frac{(-1)^{n+1}}{n\pi}, \quad \underline{b_n = \sqrt{2}\,\frac{(-1)^{n+1}}{n\pi}}$$

(b)

$$S_N(x) = \sqrt{2} \sum_{n=1}^{N} b_n \sin(n\pi x) = \sqrt{2} \sum_{n=1}^{N} \frac{\sqrt{2}}{n\pi}(-1)^{n+1} \sin(n\pi x)$$

$$= 2\sum_{n=1}^{N} \frac{(-1)^{n+1}}{n\pi} \sin(n\pi x)$$

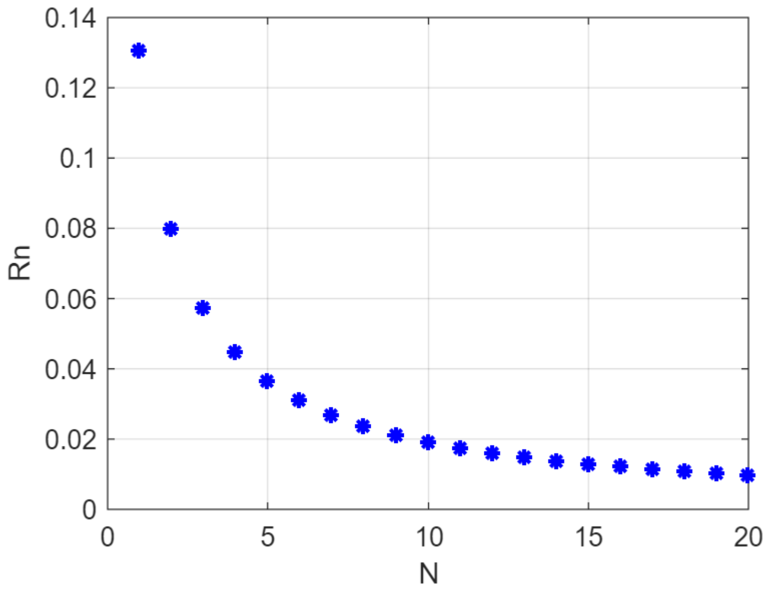
Need to compute:

$$R_N = \int_0^1 \left(x - S_N(x)\right)^2 dx$$

Using MATLAB,

```
clear
syms x
f = x;
Nmax = 20;      % number in series
N = [1 Nmax];   % array for x-axis
Rn = [1 Nmax]; % array for mean square error
Sn = 0;         % initialize for partial sums
for m = 1:Nmax  % compute partial sums, error
    N(m) = m;
    coef = 2*(-1)^(m+1)/(m*pi);
    Sn = Sn + coef*sin(m*pi*x);    % next partial sum
    Rn(m) = vpaintegral((f-Sn)^2,x,0,1);  % mean square error
end

plot(N,Rn,'b*','LineWidth',1.5)
grid on
xlabel('N');ylabel('Rn')
```



(c)

Modify above MATLAB code.

```
clear
syms x
f = x;
Nmax = 40;     % # in series, should be enough
Sn = 0;        % initialize for partial sums
for m = 1:Nmax  % compute partial sums, error
    coef = 2*(-1)^(m+1)/(m*pi);
    Sn = Sn + coef*sin(m*pi*x);        % next partial sum
    Rn = vpaintegral((f-Sn)^2,x,0,1);  % mean square error
    if Rn < 0.01
        sprintf('m = %u  %0.4f',m,Rn)
        break
    end
end
```

```
ans = 'm = 20  0.0099'
```

$\therefore$ At $\underline{n=20}$, $R_n = 0.0099 < 0.01$

3.

(a)

Let $f(x) = x(1-x) = \sum_{m=1}^{\infty} b_m \phi_m(x) = \sqrt{2} \sum_{m=1}^{\infty} b_m \sin(m\pi x)$

$\therefore \int_0^1 x(1-x)\sin(n\pi x)\,dx = \sqrt{2} \sum_{m=1}^{\infty} b_m \int_0^1 \sin(m\pi x)\sin(n\pi x)\,dx$

From (a), $\int_0^1 \sin(m\pi x)\sin(n\pi x)\,dx = 0 \ (m \neq n), \ \frac{1}{2} \ (m=n)$

$$\therefore \int_0^1 x(1-x)\sin(n\pi x)\,dx = \frac{\sqrt{2}}{2} b_n$$

Using MATLAB,

```
clear
syms x n
int(x*(1-x)*sin(n*pi*x),x,0,1)
```

ans =

$$\frac{4\sin\left(\frac{\pi n}{2}\right)^2}{n^3\pi^3} - \frac{\sin(\pi n)}{n^2\pi^2}$$

For integer n, $\sin(n\pi) = 0$, $\sin^2\left(\frac{\alpha}{2}\right) = \frac{1-\cos(\alpha)}{2}$

$$\therefore \sin^2\left(\frac{n\pi}{2}\right) = \frac{1-\cos(n\pi)}{2}$$

$$\therefore \int_0^1 x(1-x)\sin(n\pi x)\,dx = \frac{2(1-\cos(n\pi))}{n^3\pi^3} = \frac{\sqrt{2}}{2} b_n$$

$$\therefore b_n = \frac{2\sqrt{2}\left[1-\cos(n\pi)\right]}{n^3\pi^3}$$

(6)

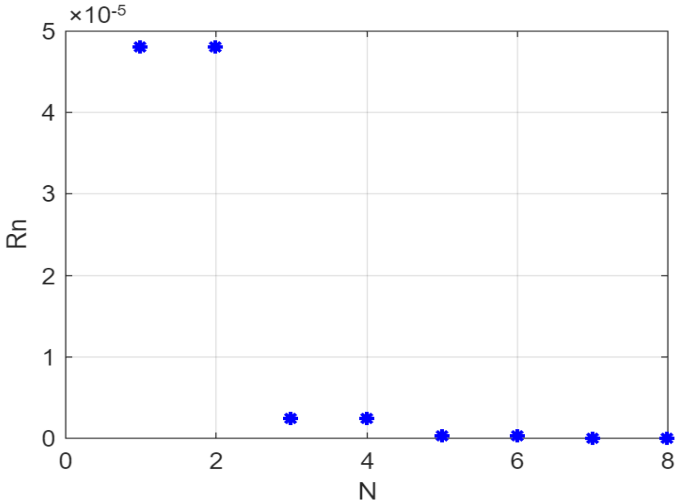
$$S_N(x) = \sqrt{2}\sum_{m=1}^{N} b_m \sin(m\pi x) = \frac{4}{\pi^3}\sum_{m=1}^{N}\frac{\left[1-\cos(m\pi)\right]}{m^3}\sin(m\pi x)$$

$$R_N = \int_0^1 \left(f(x) - S_N(x)\right)^2 dx = \int_0^1 \left[x(1-x) - S_N(x)\right]^2 dx$$

Using MATLAB, code & plot on the next page.

```
clear
syms x
f = x*(1-x);
Nmax = 8;         % number in series
N = [1 Nmax];    % array for x-axis
Rn = [1 Nmax];   % array for mean square error
Sn = 0;           % initialize for partial sums
for m = 1:Nmax   % compute partial sums, error
    N(m) = m;
    coef = 4*(1-cos(m*pi))/(m*pi)^3;
    Sn = Sn + coef*sin(m*pi*x);    % next partial sum
    Rn(m) = vpaintegral((f-Sn)^2,x,0,1);  % mean square error
end

plot(N,Rn,'b*','LineWidth',1.5)
grid on
xlabel('N');ylabel('Rn')
```



As shown in the plot, $S_N(x)$ is an excellent approximation to $f(x)$, even for $N=1$.

(c)

Using MATLAB,

```
clear
syms x
f = x*(1-x);
Nmax = 10;      % # in series, should be enough
Sn = 0;         % initialize
for m = 1:Nmax
    coef = 4*(1-cos(m*pi))/(m*pi)^3;
    Sn = Sn + coef*sin(m*pi*x);   % next partial sum
    Rn = vpaintegral((f-Sn)^2,x,0,1);  % mean square error
    if Rn < 0.01
        sprintf('m = %u  %0.6f',m,Rn)
        break
    end
end
```

```
ans = 'm = 1  0.000048'
```

$\therefore$ For $\underline{n=1}$, $R_n < 0.01$

4.

(a)

(1) $\lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} \frac{n\sqrt{x}}{e^{nx^2/2}} = \lim_{n\to\infty} \frac{\sqrt{x}}{\frac{x^2}{2}e^{nx^2/2}}$ using L'Hopital's rule

$= \frac{\sqrt{x}}{x^2/2} \lim_{n\to\infty} \frac{1}{e^{nx^2/2}} = 0, \quad 0 < x \le 1$

since $\lim_{n\to\infty} \frac{1}{e^n} = 0$.

For $x=0$, $S_n(x) = 0$ for all n, so $\lim_{n\to\infty} S_n(0) = 0$.

$\therefore \lim_{n\to\infty} S_n(x) = 0$ for $0 \le x \le 1$. $\therefore$ pointwise convergence.

(2) $R_n = \int_0^1 S_n^2(x)\,dx = \int_0^1 \left(n\sqrt{x}\, e^{-nx^2/2}\right)^2 dx = \int_0^1 n^2 x e^{-nx^2} dx$

$= -\frac{n}{2}\int_0^1 (-2nx)e^{-nx^2} = -\frac{n}{2}\left[e^{-nx^2}\right]_{x=0}^{x=1}$

$$= -\frac{n}{2}\left[e^{-n} - 1\right] = \underline{\frac{n}{2}(1-e^{-n})}$$

Since $\lim_{n\to\infty} \frac{n}{2}(1-e^{-n}) = +\infty$, $\lim_{n\to\infty} R_n = +\infty$

$\therefore$ no mean square convergence.

(6)

(1) $\{f(x) - S_n(x)\}^2 = [0 - x^n]^2 = x^{2n}$

$\therefore R_n = \int_0^1 x^{2n}\,dx = \left.\frac{x^{2n+1}}{2n+1}\right|_0^1 = \underline{\frac{1}{2n+1}}$, for all $x \in [0,1]$

$\therefore \lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{1}{2n+1} = 0 \Rightarrow$ mean square convergence.

(2) For $0 \le x < 1$, $\lim_{n\to\infty} x^n = 0$.

Pf: if $x=0$, then $x^n = 0$ for all n.

$\therefore$ Let x be any number $0 < x < 1$, and

let ϵ be s.t. $0 < \epsilon < 1$. $\therefore \log x < 0$, $\log \epsilon < 0$.

$\therefore$ Let N be s.t. $N > \dfrac{\log \epsilon}{\log x} > 0$.

$\therefore$ $N \log x < \log \epsilon \Rightarrow \log x^N < \log \epsilon \Rightarrow$

$x^N < \epsilon$. $\therefore$ If $n > N$, then

$x^n < x^N < \epsilon$ for any $0 < x < 1$.

$\therefore \lim\limits_{n \to \infty} x^n = 0$ for $0 \leq x < 1$.

But for $x = 1$, $x^n = 1$ for all n.

$\therefore \lim\limits_{n \to \infty} x^n = 1$ for $x = 1$.

$\therefore \lim\limits_{n \to \infty} x^n \neq 0$ for all $x \in [0,1]$

$\therefore$ No pointwise convergence for all $x \in [0,1]$.

5.

$$R_n = \int_0^1 r(x)\left[f(x) - S_n(x)\right]^2 dx = \int_0^1 r(x)\left[f(x) - \sum_{i=1}^{n} c_i \phi_i(x)\right]^2 dx$$

$$= \int_0^1 r(x)\left[f^2(x) - 2f(x)\sum_{i=1}^{n} c_i\phi_i(x) + \left(\sum_{i=1}^{n} c_i\phi_i(x)\right)^2\right] dx$$

$$= \int_0^1 r(x) f^2(x)dx - 2\int_0^1 r(x)f(x)\left[\sum_{i=1}^{n} c_i\phi_i(x)\right]dx + \int_0^1 r(x)\left(\sum_{i=1}^{n} c_i\phi_i(x)\right)^2 dx$$

$$= \int_0^1 r(x) f^2(x)dx - 2\int_0^1 \sum_{i=1}^{n} c_i\, r(x)f(x)\phi_i(x)\,dx + \int_0^1 r(x)\left(\sum_{i=1}^{n} c_i\phi_i(x)\right)^2 dx$$

$$= \int_0^1 r(x) f^2(x)dx - 2\sum_{i=1}^{n} c_i \underbrace{\int_0^1 r(x)f(x)\phi_i(x)\,dx}_{= a_i,\ \text{Fourier coefficient}} + \int_0^1 r(x)\left(\sum_{i=1}^{n} c_i\phi_i(x)\right)^2 dx$$

$$= \int_0^1 r(x) f^2(x)dx - 2\sum_{i=1}^{n} c_i a_i + \int_0^1 r(x)\left[\sum_{i=1}^{n} c_i\phi_i(x)\right]\left[\sum_{i=1}^{n} c_i\phi_i(x)\right]dx$$

$$= \int_0^1 r(x) f^2(x)dx - 2\sum_{i=1}^{n} c_i a_i + \int_0^1 \left[\sum_{i=1}^{n} c_i\, r(x)\phi_i(x)\right]\left[\sum_{i=1}^{n} c_i\phi_i(x)\right]dx$$

$$= \int_0^1 r(x) f^2(x)dx - 2\sum_{i=1}^{n} c_i a_i +$$

$$\int_0^1 \left[\sum_{i=1}^{n} c_i^2 r(x) \phi_i^2(x) + \sum_{\substack{i,j=1 \\ i \neq j}}^{n} r(x) c_i c_j \phi_i(x) \phi_j(x) \right] dx$$

$$= \int_0^1 r(x) f^2(x)\,dx - 2 \sum_{i=1}^{n} c_i a_i +$$

$$\sum_{i=1}^{n} c_i^2 \underbrace{\int_0^1 r(x) \phi_i^2(x)\,dx}_{=1} + \sum_{\substack{i,j=1 \\ i \neq j}}^{n} c_i c_j \underbrace{\int_0^1 r(x) \phi_i(x) \phi_j(x)\,dx}_{=0}$$

using orthogonality

$$= \int_0^1 r(x) f^2(x)\,dx - 2 \sum_{i=1}^{n} c_i a_i + \sum_{i=1}^{n} c_i^2$$

$$= \int_0^1 r(x) f^2(x)\,dx - \sum_{i=1}^{n} a_i^2 + \sum_{i=1}^{n} a_i^2 - \sum_{i=1}^{n} 2 a_i c_i + \sum_{i=1}^{n} c_i^2$$

$$= \int_0^1 r(x) f^2(x)\,dx - \sum_{i=1}^{n} a_i^2 + \sum_{i=1}^{n} \left(c_i^2 - 2 a_i c_i + a_i^2 \right)$$

$$= \int_0^1 r(x) f^2(x)\,dx - \sum_{i=1}^{n} a_i^2 + \sum_{i=1}^{n} (c_i - a_i)^2$$

The $\sum_{i=1}^{n} (c_i - a_i)^2$ has all nonnegative terms, and $\therefore$ is smallest if each term is zero; i.e., if $c_i = a_i$.

$\therefore$ R_n is minimized if $c_i = a_i$ for all i.

6.

(a)

(1) $\int_0^1 x^{-1/2}\,dx = \lim_{\alpha\to 0^+} \int_\alpha^1 x^{-\frac{1}{2}}\,dx = \lim_{\alpha\to 0^+} 2x^{1/2}\Big|_\alpha^1$

$= \lim_{\alpha\to 0^+} \left[2 - 2\alpha^{1/2}\right] = 2 - 0 = 2$

$\therefore \int_0^1 x^{-1/2}\,dx = 2$

(2) $f^2(x) = \left(x^{-1/2}\right)^2 = x^{-1}$

$\therefore \int_0^1 f^2(x)\,dx = \int_0^1 x^{-1}\,dx = \lim_{\alpha\to 0^+} \int_\alpha^1 x^{-1}\,dx$

$= \lim_{\alpha\to 0^+} \ln\Big|_\alpha^1 = \lim_{\alpha\to 0^+} \left(0 - \ln(\alpha)\right) = -\lim_{\alpha\to 0^+} \ln(\alpha)$

$= +\infty$ $\quad\therefore \int_0^1 f^2(x)\,dx$ does not exist.

(b)

(1) $f^2(x) = 1$, for all x, rational or irrational.

$\therefore \int_0^1 f^2(x)\,dx = \int_0^1 1\,dx = x\Big|_0^1 = 1.$

(2) Let P be a partition of $[0,1]$ consisting of $n+1$ distinct points s.t.

$P = \{x_i \in [0,1],\ x_0 = 0,\ x_n = 1,\ x_{i-1} < x_i \text{ for } i = 1, \dots, n\}$

$\therefore$ For any $i = 1, \dots, n$, $[x_{i-1}, x_i]$ contains both rational and irrational values.

$\therefore \inf\{f(x) : x_{i-1} \le x \le x_i\} = -1 = m_i$

$\sup\{f(x) : x_{i-1} \le x \le x_i\} = 1 = M_i$

where inf = greatest lower bound, sup = least upper bound.

$\therefore$ The lower sum of $f(x)$ for P is

$$L(f,P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) = -1(1-0) = -1$$

The upper sum of $f(x)$ for P is

$$U(f,P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) = 1(1-0) = 1.$$

This is true for every partition P of $[0,1]$ as

$n \to \infty$. $\therefore$ $\sup\{L(f,P)\} \neq \inf\{U(f,P)\}$

since $-1 \neq 1$. $\therefore$ By definition, $f(x)$ is

not Riemann integrable.

7.

(a)

Since $\int_0^1 1 \cdot 1 \, dx = 1$, $\underline{f_0(x) = 1}$ or $\underline{f_0(x) = -1}$, a polynomial of degree 0.

(b)

$f_1(x) = a_1 x + a_0$

$\int_0^1 (1) \cdot (a_1 x + a_0) \, dx = 0 \Rightarrow \frac{a_1 x^2}{2} + a_0 x \Big|_0^1 = \frac{a_1}{2} + a_0 = 0$ [1]

$\int_0^1 (a_1 x + a_0)^2 dx = 1 \Rightarrow \int_0^1 a_1^2 x^2 + 2a_1 a_0 x + a_0^2 \, dx = 1$

$\Rightarrow \frac{a_1^2 x^3}{3} + a_1 a_0 x^2 + a_0^2 x \Big|_0^1 = \frac{a_1^2}{3} + a_1 a_0 + a_0^2 = 1$ [2]

Substituting [1] into [2],

$$\frac{a_1^2}{3} + a_1\left(-\frac{a_1}{2}\right) + \frac{a_1^2}{4} = 1 \Rightarrow 4a_1^2 - 6a_1^2 + 3a_1^2 = 12,$$

$\therefore a_1^2 = 12,\ a_1 = \pm 2\sqrt{3},\ \therefore$ from [1], $a_0 = \mp\sqrt{3}$

$$\therefore f_1(x) = \underline{2\sqrt{3}x - \sqrt{3}} \text{ or } f_1(x) = \underline{-2\sqrt{3}x + \sqrt{3}}$$

(c)

(i) Using $f_1(x) = 2\sqrt{3}x - \sqrt{3}$,

Let $f_2(x) = a_2x^2 + a_1x + a_0$

$$\int_0^1 1\cdot f_2(x)\,dx = \frac{a_2x^3}{3} + \frac{a_1x^2}{2} + a_0x\Big|_0^1 = \frac{a_2}{3} + \frac{a_1}{2} + a_0 = 0 \quad [1]$$

Using MATLAB to do the computations,

```
clear
syms x a2 a1 a0
f1 = 2*sqrt(3)*x - sqrt(3);
f2 = a2*x^2 + a1*x + a0;
i20 = int(f2,x,0,1)
i21 = int(f2*f1,x,0,1)
i22 = int(f2*f2,x,0,1)
subs(i20,a1,-a2)
subs(i22,[a1,a0],[-a2,a2/6])
```

i20 =

$$\frac{a_2}{3} + \frac{a_1}{2} + a_0$$

i21 =

$$\frac{\sqrt{3}\,(a_2 + a_1)}{6}$$

i22 =

$$\frac{a_2^2}{5} + \frac{a_1 a_2}{2} + \frac{a_1^2}{3} + \frac{2\,a_0 a_2}{3} + a_0 a_1 + a_0^2$$

ans =

$$-\frac{a_2}{6} + a_0$$

ans =

$$\frac{a_2^2}{180}$$

$$\therefore \int_0^1 f_2(x) f_1(x)\,dx = \overbrace{\frac{\sqrt{3}}{6}(a_2 + a_1)}^{= \text{ i21 in MATLAB}} = 0$$

$$\therefore a_1 = -a_2 \quad [2]$$

Substituting [2] into [1],

we get $-\frac{a_2}{6} + a_0 = 0$, or, $a_0 = \frac{a_2}{6}$ [3]

$$\int_0^1 f_2(x)\overset{=\,i22 \text{ in MATLAB}}{f_2(x)}\,dx = \frac{a_2^2}{5} + \frac{a_1 a_2}{2} + \frac{a_1^2}{3} + \frac{2a_0 a_0}{3} + a_0 a_1 + a_0^2 = 1 \quad [4]$$

Substituting [2] and [3] into [4],

$\frac{a_2^2}{180} = 1$, or $a_2 = 6\sqrt{5}$

From [2], $a_1 = -6\sqrt{5}$

From [3], $a_0 = \sqrt{5}$

$\therefore f_2(x) = \underline{6\sqrt{5}x^2 - 6\sqrt{5}x + \sqrt{5}}$

(2) If $p_1(x) = -f_1(x)$, then $\int_0^1 p_1 f_2\,dx = -\int_0^1 f_1 f_2\,dx = 0$

and if $p_2(x) = -f_2(x)$, $\int_0^1 p_2 p_2\,dx = \int_0^1 f_2 f_2\,dx = 1$

and $\int p_2 f_1 = -\int f_2 f_1 = 0$, $\int p_2 f_0 = -\int f_2 f_0 = 0$

and $\int p_2 p_1 = \int f_2 f_1 = 0$.

$\therefore$ Can choose $\underline{\underline{f_2(x) = \pm\sqrt{5}(6x^2 - 6x + 1)}}$

(d)

Since $\int_0^1 (c_i f_i)(c_j f_j)\,dx = c_i c_j \int_0^1 (f_i)(f_j)\,dx = 0$

for $i \neq j$, where c_i, c_j are constants,

we just need to multiply f_i by a constant

so that $g_i(1) = c_i f_i(1) = 1$.

$\therefore\ g_0(x) = 1$

$f_1(x) = \pm\sqrt{3}(2x-1)$, $f_1(1) = \pm\sqrt{3}$

$\therefore\ g_1(x) = 2x-1$

$f_2(x) = \pm\sqrt{5}(6x^2-6x+1)$, $f_2(1) = \pm\sqrt{5}$

$\therefore\ g_2(x) = 6x^2-6x+1$

$\therefore\ \boxed{g_0(x) = 1,\ g_1(x) = 2x-1,\ g_2(x) = 6x^2-6x+1}$

8.

$P_0(x) = 1.$

$P_1(x)$: Let $P_1(x) = a_1 x + a_0$

$$\int_{-1}^{1} P_0 P_1 \, dx = \int_{-1}^{1} a_1 x + a_0 \, dx = \frac{a_1}{2}x^2 + a_0 x \Big|_{-1}^{1} = 2a_0 = 0$$

$\therefore a_0 = 0. \quad \therefore P_1(1) = a_1 = 1.$

$\therefore \underline{P_1(x) = x}$

$P_2(x)$: Let $P_2(x) = a_2 x^2 + a_1 x + a_0$

$$\int_{-1}^{1} P_0 P_2 = \int_{-1}^{1} (a_2 x^2 + a_1 x + a_0) dx = \frac{a_2}{3}x^3 + \frac{a_1}{2}x^2 + a_0 x \Big|_{-1}^{1}$$

$$= \left(\frac{a_2}{3} + \frac{a_1}{2} + a_0\right) - \left(-\frac{a_2}{3} + \frac{a_1}{2} - a_0\right) = \frac{2a_2}{3} + 2a_0 = 0$$

$$\therefore a_0 = -\frac{a_2}{3}$$

$$\int_{-1}^{1} P_1 P_2 = \int_{-1}^{1} (a_2 x^3 + a_1 x^2 + a_0 x) dx = \frac{a_2}{4}x^4 + \frac{a_1}{3}x^3 + \frac{a_0}{2}x^2 \Big|_{-1}^{1}$$

$$= \left(\frac{a_2}{4} + \frac{a_1}{3} + \frac{a_0}{2}\right) - \left(\frac{a_2}{4} - \frac{a_1}{3} + \frac{a_2}{2}\right) = \frac{2a_1}{3} = 0$$

$\therefore a_1 = 0$

$\therefore P_2(x) = a_2 x^2 - \frac{a_2}{3}. \quad P_2(1) = a_2 - \frac{a_2}{3} = 1 \Rightarrow$

$$a_2 = \frac{3}{2} \quad \therefore a_0 = -\frac{1}{2}$$

$$\therefore P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$$

$P_3(x)$: Let $P_3(x) = a_3x^3 + a_2x^2 + a_1x + a_0$

$$\int_{-1}^{1} P_0 P_3 = \int_{-1}^{1}(a_3x^3 + a_2x^2 + a_1x + a_0)dx = \frac{a_3}{4}x^4 + \frac{a_2}{3}x^3 + \frac{a_1}{2}x^2 + a_0x\Big|_{-1}^{1}$$

$$= \left(\frac{a_3}{4} + \frac{a_2}{3} + \frac{a_1}{2} + a_0\right) - \left(\frac{a_3}{4} - \frac{a_2}{3} + \frac{a_1}{2} - a_0\right) = \frac{2a_2}{3} + 2a_0 = 0$$

$$\therefore a_0 = -\frac{a_2}{3}$$

$$\int_{-1}^{1} P_1 P_3 = \int_{-1}^{1}(a_3x^4 + a_2x^3 + a_1x^2 + a_0x)dx = \frac{a_3}{5}x^5 + \frac{a_2}{4}x^4 + \frac{a_1}{3}x^3 + \frac{a_0}{2}x^2\Big|_{-1}^{1}$$

$$= \left(\frac{a_3}{5} + \frac{a_2}{4} + \frac{a_1}{3} + \frac{a_0}{2}\right) - \left(-\frac{a_3}{5} + \frac{a_2}{4} - \frac{a_1}{3} + \frac{a_0}{2}\right) = \frac{2a_3}{5} + \frac{2a_1}{3} = 0$$

$$\therefore a_1 = -\frac{3}{5}a_3$$

using above substitutions

$$\int_{-1}^{1} P_2 P_3 = \int_{-1}^{1}\left(\frac{3}{2}x^2 - \frac{1}{2}\right)\left(a_3x^3 + a_2x^2 - \frac{3}{5}a_3x - \frac{a_2}{3}\right)dx$$

Use MATLAB to compute

```
clear
syms x a2 a3
p2 = (3/2)*x^2 - 1/2;
p3 = a3*x^3 + a2*x^2 -(3/5)*a3*x - a2/3;
i23 = int(p2*p3,x,-1,1)
```

i23 =

$$\frac{4\,a_2}{15}$$

$$\therefore \int_{-1}^{1} P_2 P_3 = \frac{4}{15}a_2 = 0 \Rightarrow a_2 = 0. \quad \therefore a_0 = \frac{-a_2}{3} = 0$$

$$\therefore P_3(x) = a_3x^3 - \frac{3}{5}a_3x \qquad \therefore P_3(1) = a_3 - \frac{3}{5}a_3 = 1$$

$\therefore \frac{2}{5} a_3 = 1, \ a_3 = \frac{5}{2}$

$$\therefore \underline{P_3(x) = \frac{5}{2}x^3} - \frac{3}{2}x$$

Summary:

$$\boxed{\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= \frac{3}{2}x^2 - \frac{1}{2} \\ P_3(x) &= \frac{5}{2}x^3 - \frac{3}{2}x \end{aligned}}$$

9.

(a)

By definition, $R_n = \int_0^1 r(x)\left[f(x) - \sum_{i=1}^{n} a_i \phi_i(x)\right]^2 dx$

$$= \int_0^1 r(x)\left[f^2(x) - 2f(x)\sum_{i=1}^{n} a_i \phi_i(x) + \left(\sum_{i=1}^{n} a_i \phi_i(x)\right)^2\right] dx$$

$$= \int_0^1 r(x) f^2(x)\,dx - 2\int_0^1 r(x) f(x) \sum_{i=1}^{n} a_i \phi_i(x)\,dx$$

$$+ \int_0^1 r(x)\left(\sum_{i=1}^{n} a_i \phi_i(x)\right)^2 dx$$

$$= \int_0^1 r(x) f^2(x)\,dx - 2\sum_{i=1}^{n} a_i \underbrace{\int_0^1 r(x) f(x) \phi_i(x)\,dx}_{= a_i \text{ by equation (9)}}$$

$$+ \int_0^1 r(x) \left(\sum_{i=1}^{n} a_i^2 \phi_i^2(x) + \sum_{\substack{i,j=1 \\ i \neq j}}^{n} a_i a_j \phi_i \phi_j \right) dx$$

$$= \int_0^1 r(x) f^2(x)\,dx - 2\sum_{i=1}^{n} a_i^2$$

$$+ \sum_{i=1}^{n} a_i^2 \overbrace{\int_0^1 r(x) \phi_i^2(x)\,dx}^{=1 \text{ by orthonormality}} + \sum_{\substack{i,j=1 \\ i \neq j}}^{n} a_i a_j \underbrace{\int_0^1 r(x) \phi_i(x) \phi_j(x)\,dx}_{=0 \text{ by orthogonality}}$$

$$= \int_0^1 r(x) f^2(x)\,dx - 2\sum_{i=1}^{n} a_i^2 + \sum_{i=1}^{n} a_i^2 + 0$$

$$\therefore R_n = \int_0^1 r(x) f^2(x)\,dx - \sum_{i=1}^{n} a_i^2$$

(b)

Since $R_n = \int_0^1 r(x)[f(x) - S_n(x)]^2\,dx$ and $r(x) \geq 0$ for all $x \in [0,1]$, then $r(x)[f(x) - S_n(x)]^2 \geq 0$ for all $x \in [0,1]$. $\therefore R_n \geq 0$

$\therefore$ From (a), $R_n = \int_0^1 r(x) f^2(x)\,dx - \sum_{i=1}^{n} a_i^2 \geq 0$

$$\therefore \int_0^1 r(x) f^2(x)\,dx \geq \sum_{i=1}^{n} a_i^2 \quad \text{for all } n.$$

(c)

Since $a_i^2 \geq 0$, $\sum_{i=1}^{\infty} a_i^2$ is a monotonic increasing series and is bounded above by $\int_0^1 r(x) f^2(x)\,dx$ by (b). $\therefore$ It converges.

(d)

Let $\epsilon > 0$. From (c), let $L = \lim_{n\to\infty} \sum_{i=1}^{n} a_i^2 = \sum_{i=1}^{\infty} a_i^2$

$\therefore \exists N > 0$ s.t. if $n > N$, then

$$\left| \sum_{i=1}^{n} a_i^2 - \sum_{i=1}^{\infty} a_i^2 \right| < \epsilon$$

$$\therefore |-1| \left| \sum_{i=1}^{n} a_i^2 - \int_0^1 r(x) f^2(x)\,dx + \int_0^1 r(x) f^2(x)\,dx - \sum_{i=1}^{\infty} a_i^2 \right| < \epsilon$$

$$\Rightarrow \left| \int_0^1 r(x) f^2(x)\,dx - \sum_{i=1}^{n} a_i^2 - \left(\int_0^1 r(x) f^2(x)\,dx - \sum_{i=1}^{\infty} a_i^2 \right) \right| < \epsilon$$

$$\therefore \lim_{n\to\infty} R_n = \lim_{n\to\infty} \left(\int_0^1 r(x) f^2(x)\,dx - \sum_{i=1}^{n} a_i^2 \right) = \int_0^1 r(x) f^2(x)\,dx - \sum_{i=1}^{\infty} a_i^2$$

(e)

" $\sum_{i=1}^{\infty} a_i \phi_i(x)$ converges to $f(x)$ in the mean" means

$$\lim_{n\to\infty} \int_0^1 r(x)\left[f(x) - \sum_{i=1}^{n} a_i \phi_i(x)\right]^2 dx = 0$$

$$\Longleftrightarrow \lim_{n\to\infty} \int_0^1 r(x)\left[f(x) - S_n(x)\right]^2 dx = 0$$ since $S_n(x) = \sum_{i=1}^{n} a_i \phi_i(x)$

$$\Longleftrightarrow \lim_{n\to\infty} R_n = 0$$ by definition of R_n

(1) $\therefore$ If $\lim_{n\to\infty} R_n = 0$, then by (d)

$$\lim_{n\to\infty} R_n = \int_0^1 r(x) f^2(x)dx - \sum_{i=1}^{\infty} a_i^2 = 0$$

$$\Rightarrow \int_0^1 r(x) f^2(x)dx = \sum_{i=1}^{\infty} a_i^2$$

(2) If $\int_0^1 r(x) f^2(x)\,dx = \sum_{i=1}^{\infty} a_i^2$, then

$$\int_0^1 r(x) f^2(x)dx - \sum_{i=1}^{\infty} a_i^2 = 0 \implies$$

$$\lim_{n\to\infty} R_n = 0 \quad \text{by (d)}$$

10.

a_n as the nth Fourier coefficient of a square integrable function f means $f(x) = \sum_{i=1}^{\infty} a_i \phi_i(x)$, where $a_i = \int_0^1 r(x) f(x) \phi_i(x) dx$, in the Sturm-Liouville problem. Problem # 9(b) showed $\sum_{i=1}^{\infty} a_i^2$ is bounded, and so it converges since it is a monotonically increasing series. Let $S_n = \sum_{i=1}^{n} a_i^2$ and $S = \sum_{i=1}^{\infty} a_i^2$. Since $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_i^2 = S = \lim_{n\to\infty} S_{n-1}$, then $\lim_{n\to\infty} a_n^2 =$

$$\lim_{n\to\infty} (S_n - S_{n-1}) = S - S = 0, \text{ so } \lim_{n\to\infty} a_n^2 = 0.$$

$$\therefore \lim_{n\to\infty} a_n = 0.$$

Pf: Let $\epsilon > 0$. Since $\lim_{n\to\infty} a_n^2 = 0$, then for ϵ^2 there exists an N s.t. if $n > N$, then

$|a_n^2| < \epsilon^2 \Rightarrow |a_n|^2 < \epsilon^2 \Rightarrow |a_n| < \epsilon$, or

$|a_n - 0| < \epsilon$ when $n > N$. By definition,

$\lim_{n\to\infty} a_n = 0$.

11.

If, for some function $f(x)$, $f(x) = \sum_{i=1}^{\infty} \phi_i(x)$, then $a_i = 1$ for $i = 1, 2, 3, \ldots$. $\therefore$ $\sum_{i=1}^{\infty} a_i^2$ wouldn't converge and $\lim_{n\to\infty} a_n = 1 \neq 0$, contradicting the result of #10.

12.

If $f(x) = \sum_{i=1}^{\infty} \frac{\phi_i(x)}{\sqrt{i}}$ for some square integrable function $f(x)$, then here, $a_i = \frac{1}{\sqrt{i}}$, so $a_i^2 = \frac{1}{i}$. By

Problem # 9(6), $\sum_{i=1}^{\infty} a_i^2$ is bounded and thus converges since a_i^2 is a monotonic increasing sequence. Since $a_i^2 = \frac{1}{i}$, then $\sum_{i=1}^{\infty} a_i^2 = \sum_{i=1}^{\infty} \frac{1}{i}$.
But this is the harmonic series, which does not converge. $\therefore \sum_{i=1}^{\infty} a_i^2$ does not converge, so $f(x)$, defined as $\sum_{i=1}^{\infty} \frac{1}{\sqrt{i}} \phi_i(x)$, cannot exist as a square integrable function.

13.

Given $f(x) = \sum_{i=1}^{\infty} a_i \phi_i(x)$, where $\phi_i(x)$ are orthonormal eigenfunctions defined on $0 \le x \le 1$, then

$$f^2(x) = \left(\sum_{i=1}^{\infty} a_i \phi_i(x) \right)^2 = \sum_{i=1}^{\infty} a_i^2 \phi_i^2(x) + \sum_{\substack{i,j=1 \\ i \neq j}} a_i a_j \phi_i(x) \phi_j(x)$$

$$\therefore r(x) f^2(x) = \sum_{i=1}^{\infty} a_i^2 r(x) \phi_i^2(x) + \sum_{\substack{i,j=1 \\ i \neq j}}^{\infty} a_i a_j r(x) \phi_i(x) \phi_j(x)$$

$$\therefore \int_0^1 r(x) f^2(x) dx = \int_0^1 \left(\sum_{i=1}^{\infty} a_i^2 r(x) \phi_i^2(x) \right) dx \quad +$$

$$\int_0^1 \left(\sum_{\substack{i,j=1 \\ i \neq j}}^{\infty} a_i a_j r(x) \phi_i(x) \phi_j(x) \right) dx$$

$$= \sum_{i=1}^{\infty} a_i^2 \underbrace{\int_0^1 r(x) \phi_i^2(x) dx}_{=1 \text{ by definition of orthonormality}} +$$

$$\sum_{\substack{i,j=1 \\ i \neq j}}^{\infty} a_i a_j \underbrace{\int_0^1 r(x) \phi_i(x) \phi_j(x) dx}_{=0 \text{ by orthonormality}}$$

$$= \sum_{i=1}^{\infty} a_i^2$$

$$\therefore \underline{\int_0^1 r(x) f^2(x) dx = \sum_{i=1}^{\infty} a_i^2}$$