3.1 Homogeneous Differential Equations with Constant Coefficients

Note Title 5/23/2018

1.

$r^2 + 2r - 3 = (r+3)(r-1) = 0, \quad r = 1, -3$

$\therefore \; y(t) = c_1 e^{t} + c_2 e^{-3t}$

2.

$r^2 + 3r + 2 = (r+2)(r+1) = 0, \quad r = -1, -2$

$\therefore \; y(t) = c_1 e^{-t} + c_2 e^{-2t}$

3.

$6r^2 - r - 1 = (3r+1)(2r-1) = 0, \quad r = \frac{1}{2}, -\frac{1}{3}$

$\therefore \; y(t) = c_1 e^{\frac{t}{2}} + c_2 e^{-\frac{t}{3}}$

4.

$r^2 + 5r + 0 = r(r+5) = 0, \quad r = 0, -5$

$\therefore \; y(t) = c_1 + c_2 e^{-5t}$

5.

$$4r^2 - 9 = 4\left(r^2 - \frac{9}{4}\right) = 4\left(r+\frac{3}{2}\right)\left(r-\frac{3}{2}\right) = 0, \quad r = \pm\frac{3}{2}$$

$$\therefore \underline{y(t) = c_1 e^{\frac{3}{2}t}} + c_2 e^{-\frac{3}{2}t}$$

6.

$$r^2 - 2r - 2 = 0, \quad r = \frac{2 \pm \sqrt{4 - 4(1)(-2)}}{2} = 1 \pm \sqrt{3}$$

$$\therefore \underline{y(t) = c_1 e^{(1+\sqrt{3})t} + c_2 e^{(1-\sqrt{3})t}}$$

In each of Problems 7 through 12, find the solution of the given initial value problem. Sketch the graph of the solution and describe its behavior as t increases.

7.

$$r^2 + r - 2 = (r+2)(r-1) = 0 \Rightarrow r = 1, -2$$

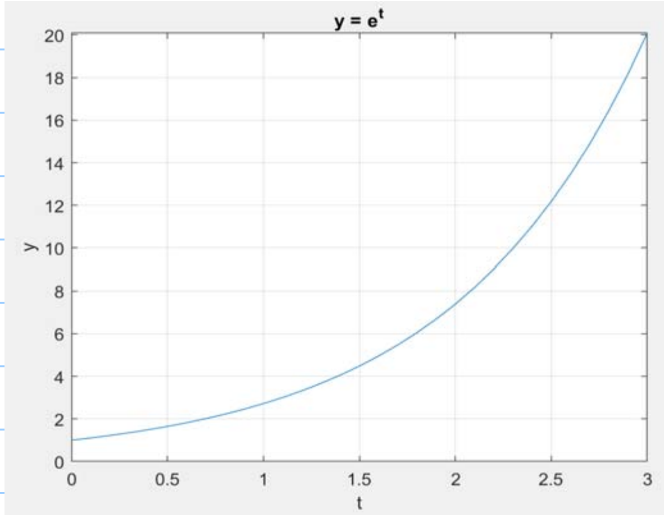
$$\therefore y = c_1 e^{t} + c_2 e^{-2t} \qquad y' = c_1 e^{t} - 2c_2 e^{-2t}$$

$$\left.\begin{aligned} y(0) = 1 &: \quad c_1 + c_2 = 1 \\ y'(0) = 1 &: \quad c_1 - 2c_2 = 1 \end{aligned}\right\} \quad 3c_2 = 0, \; c_2 = 0 \quad \therefore c_1 = 1$$

$$\therefore \underline{y(t) = e^{t}} \qquad \underline{\text{as } t \to \infty}, \; y \to \infty$$

MATLAB code

```
clear, clc
r1 = 1; r2 = -2;
c1 = 1; c2 = 0;
t = 0:0.1:3;
y = c1*exp(r1*t) + c2*exp(r2*t);
plot(t,y)
grid on
xlabel 't', ylabel 'y'
title 'y = e^t'
```



8.

$r^2 + 4r + 3 = (r+3)(r+1) = 0, \quad r = -1, -3$

$\therefore \; y = c_1 e^{-t} + c_2 e^{-3t} \qquad y' = -c_1 e^{-t} - 3c_2 e^{-3t}$

$y(0) = 2: \quad c_1 + c_2 = 2$

$y'(0) = -1: \quad -c_1 - 3c_2 = -1$

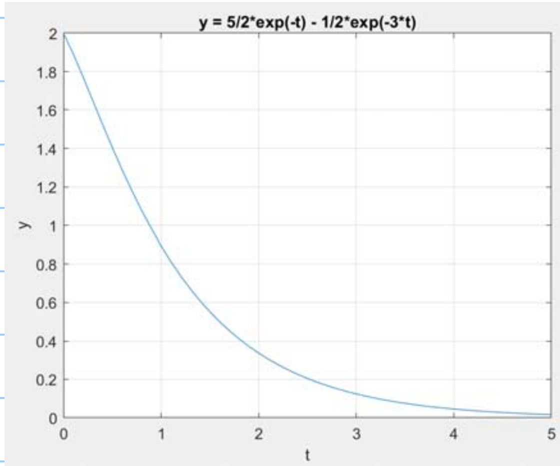
$-2c_2 = 1, \quad c_2 = -\frac{1}{2}$

$\therefore \; c_1 = \frac{5}{2}$

$\therefore \; y(t) = \frac{5}{2} e^{-t} - \frac{1}{2} e^{-3t}, \qquad y \to 0 \text{ as } t \to \infty$

MATLAB code

```
clear, clc
r1 = -1; r2 = -3;
c1 = 5/2; c2 = -1/2;
t = 0:0.1:5;
y = c1*exp(r1*t) + c2*exp(r2*t);
plot(t,y)
grid on
xlabel 't', ylabel 'y'
title 'y = 5/2*exp(-t) - 1/2*exp(-3*t)'
```



9.

$r^2 + 3r = r(r+3) = 0, \quad r = 0, -3$

$\therefore y = c_1 + c_2 e^{-3t} \qquad y' = -3c_2 e^{-3t}$

$y(0) = -2: \quad c_1 + c_2 = -2$

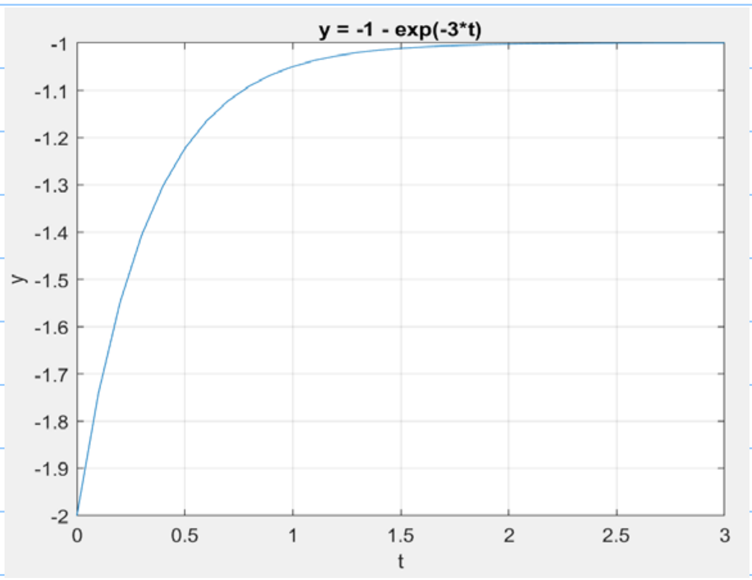
$y'(0) = 3: \quad -3c_2 = 3$

$c_2 = -1, \quad c_1 = -1$

$\therefore y(t) = \underline{-1 - e^{-3t}}, \quad y \to \underline{-1} \text{ as } t \to \infty$

MATLAB code

```
clear, clc
r1 = 0; r2 = -3;
c1 = -1; c2 = -1;
t = 0:0.1:3;
y = c1*exp(r1*t) + c2*exp(r2*t);
plot(t,y)
grid on
xlabel 't', ylabel 'y'
title 'y = -1 - exp(-3*t)'
```



10.

$2r^2 + r - 4 = 0, \quad r = \frac{-1 \pm \sqrt{1+32}}{4} = -\frac{1}{4} \pm \frac{\sqrt{33}}{4}$

$\therefore y = c_1 e^{\left(-\frac{1}{4} + \frac{\sqrt{33}}{4}\right)t} + c_2 e^{\left(-\frac{1}{4} - \frac{\sqrt{33}}{4}\right)t}$

$y(0) = 0: \quad c_1 + c_2 = 0$

$y'(0) = 1: \quad c_1\left(-\frac{1}{4} + \frac{\sqrt{33}}{4}\right) + c_2\left(-\frac{1}{4} - \frac{\sqrt{33}}{4}\right) = 1$

$\therefore \quad c_1\left[-\frac{1}{4} + \frac{\sqrt{33}}{4} - \left(-\frac{1}{4} - \frac{\sqrt{33}}{4}\right)\right] = 1, \quad c_1 \frac{\sqrt{33}}{2} = 1,$

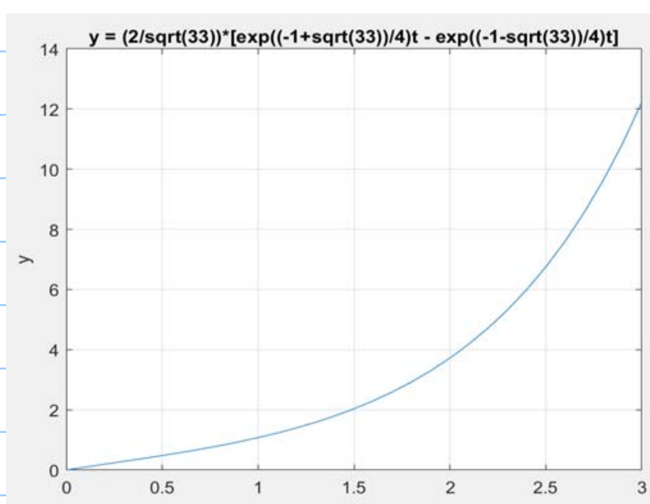
$c_1 = \frac{2}{\sqrt{33}}, \quad c_2 = -\frac{2}{\sqrt{33}}$

$$\therefore y(t) = \frac{2}{\sqrt{33}} e^{(-1+\sqrt{33})t/4} - \frac{2}{\sqrt{33}} e^{(-1-\sqrt{33})t/4}$$

Since $\frac{-1+\sqrt{33}}{4} > 0$, $y \to +\infty$ as $t \to \infty$

MATLAB code:

```
clear, clc
r1 = (-1 + sqrt(33))/4; r2 = (-1 - sqrt(33))/4;
c1 = 2/sqrt(33); c2 = -2/sqrt(33);
t = 0:0.1:3;
y = c1*exp(r1*t) + c2*exp(r2*t);
plot(t,y)
grid on
xlabel 't', ylabel 'y'
title 'y = (2/sqrt(33))*[exp((-1+sqrt(33))/4)t - exp((-1-sqrt(33))/4)t]'
```



11.

$r^2 + 8r - 9 = (r+9)(r-1) = 0, \quad r = 1, -9$

$\therefore y = c_1 e^{t} + c_2 e^{-9t} \qquad y' = c_1 e^{t} - 9c_2 e^{-9t}$

$y(1) = 1: \quad c_1 e + c_2 e^{-9} = 1$ $10c_2 e^{-9} = 1, \quad c_2 = \frac{e^9}{10}$

$y'(1) = 0: \quad c_1 e - 9c_2 e^{-9} = 0$ $10c_1 e = 9, \quad c_1 = \frac{9}{10e}$

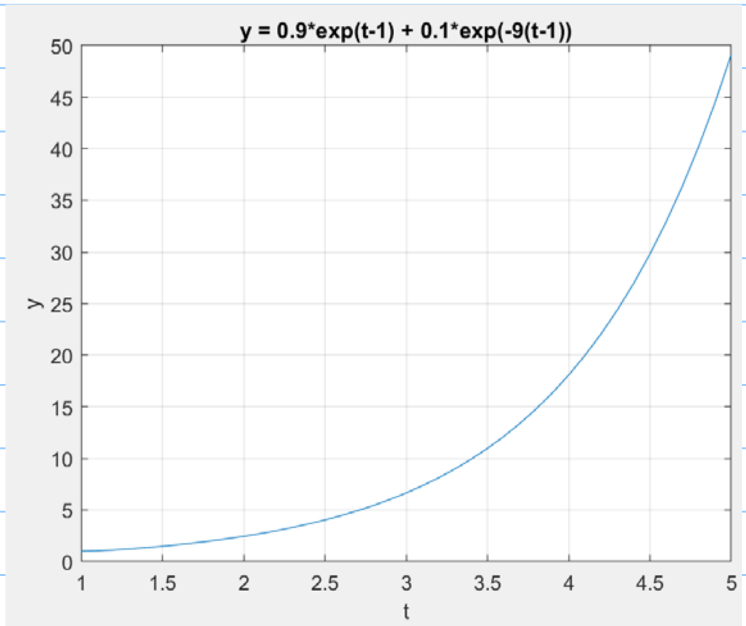
$$\therefore y(t) = \frac{9}{10e} e^{t} + \frac{e^9}{10} e^{-9t}$$

$$\text{Or,} \quad y(t) = \frac{9}{10} e^{t-1} + \frac{1}{10} e^{-9(t-1)}$$

$\therefore y \to +\infty$ as $t \to \infty$

MATLAB code

```
clear, clc
r1 = 1; r2 = -9;
c1 = 9/(10*exp(1)); c2 = exp(9)/10;
t = 1:0.1:5;
y = c1*exp(r1*t) + c2*exp(r2*t);
plot(t,y)
grid on
xlabel 't', ylabel 'y'
title 'y = 0.9*exp(t-1) + 0.1*exp(-9(t-1))'
```



12.

$$4r^2-1=(2r+1)(2r-1)=0,\quad r=\pm\frac{1}{2}$$

$$\therefore\ y=c_1e^{\frac{1}{2}t}+c_2e^{-\frac{1}{2}t},\quad y'=\frac{c_1}{2}e^{\frac{1}{2}t}-\frac{c_2}{2}e^{-\frac{1}{2}t}$$

$$y(-2)=1:\ \frac{c_1}{e}+c_2e=1$$

$$y'(-2)=-1:\ \frac{c_1}{2e}-\frac{c_2e}{2}=-1$$

$$\frac{c_1}{e}=-\frac{1}{2},\quad c_1=-\frac{e}{2}$$

$$c_2=\frac{3}{2e}$$

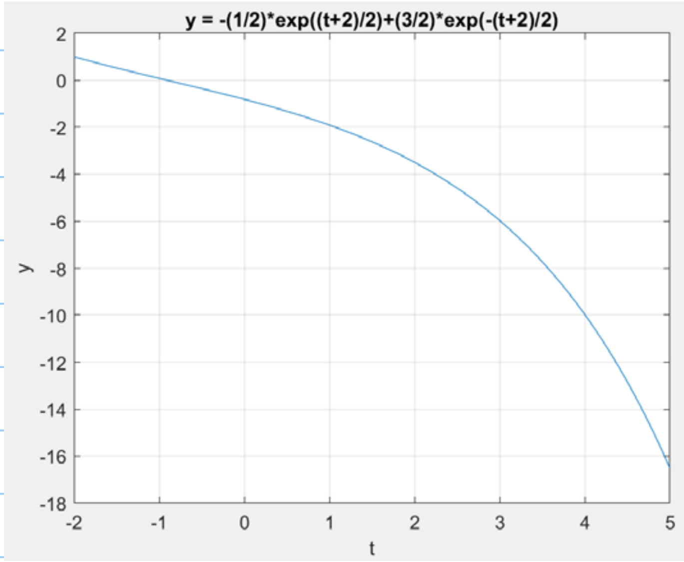
$$\therefore\ y(t)=-\frac{e}{2}e^{\frac{1}{2}t}+\frac{3}{2e}e^{-\frac{1}{2}t}=-\frac{1}{2}e^{\frac{t}{2}+1}+\frac{3}{2}e^{-\frac{t}{2}-1}$$

$$\text{Or, } y(t)=-\frac{1}{2}e^{(t+2)/2}+\frac{3}{2}e^{-(t+2)/2}$$

$\therefore\ y\to-\infty$ as $t\to\infty$

MATLAB code:

```
clear, clc
r1 = 1/2; r2 = -1/2;
c1 = -exp(1)/2; c2 = 3/(2*exp(1));
t = -2:0.1:5;
y = c1*exp(r1*t) + c2*exp(r2*t);
plot(t,y)
grid on
xlabel 't', ylabel 'y'
title 'y = -(1/2)*exp((t+2)/2)+(3/2)*exp(-(t+2)/2)'
```



13.

$r_1 = 2, \; r_2 = -3, \; \therefore \; (r-2)(r+3) = r^2 + r - 6$

$\therefore \; y'' + y' - 6y = 0$

14.

(a) $r^2 - 1 = (r+1)(r-1) = 0, \; r = 1, -1$

$\therefore \; y = c_1 e^{t} + c_2 e^{-t}, \quad y' = c_1 e^{t} - c_2 e^{-t}$

$y(0) = \frac{5}{4} : \; c_1 + c_2 = \frac{5}{4}$

$y'(0) = -\frac{3}{4} : \; c_1 - c_2 = -\frac{3}{4}$

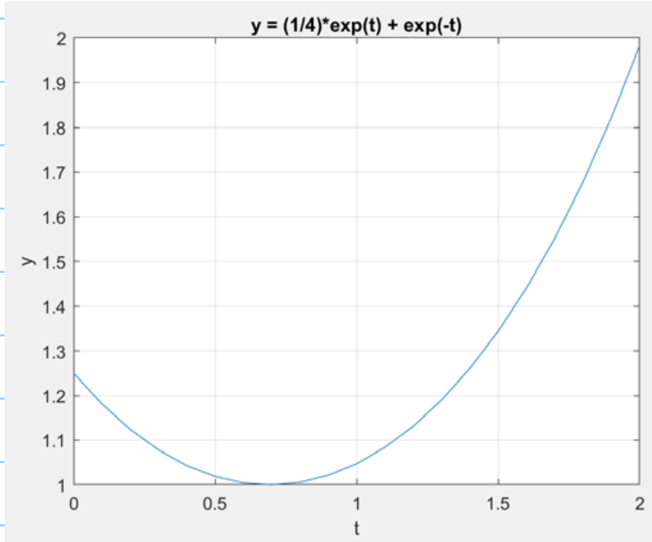
$2c_1 = \frac{1}{2}, \; c_1 = \frac{1}{4}$

$c_2 = 1$

$\therefore \; y(t) = \frac{1}{4} e^{t} + e^{-t}$

(b) MATLAB code:

```
clear, clc
r1 = 1; r2 = -1;
c1 = 1/4; c2 = 1;
t = 0:0.1:2;
y = c1*exp(r1*t) + c2*exp(r2*t);
plot(t,y)
grid on
xlabel 't', ylabel 'y'
title 'y = (1/4)*exp(t) + exp(-t)'
```



(c) $y'(t) = \frac{1}{4}e^{t} - e^{-t} = 0 \Rightarrow e^{2t} = 4, \ 2t = \ln(4)$

$\therefore t = \frac{1}{2}\ln(4) = \ln(2)$

$\therefore y(\ln(2)) = \frac{1}{4}e^{\ln(2)} + e^{-\ln(2)} = \frac{2}{4} + \frac{1}{2} = 1$

$\therefore$ Minimum value at $t = \ln(2)$ is $\underline{y = 1}$

15.

(a) $2r^2 - 3r + 1 = (2r-1)(r-1) = 0, \ r = \frac{1}{2}, 1$

$\therefore y = c_1 e^{t} + c_2 e^{\frac{1}{2}t}, \qquad y' = c_1 e^{t} + \frac{c_2}{2} e^{\frac{1}{2}t}$

$y(0) = 2: \quad c_1 + c_2 = 2$

$y'(0) = \frac{1}{2}: \quad c_1 + \frac{c_2}{2} = \frac{1}{2}$

$\frac{1}{2}c_2 = \frac{3}{2}, \ c_2 = 3$

$c_1 = -1$

$\therefore \underline{y(t) = -e^{t} + 3e^{\frac{1}{2}t}}$

(b) $y' = -e^{t} + \frac{3}{2}e^{\frac{1}{2}t} = 0 \Rightarrow \frac{3}{2} = e^{\frac{1}{2}t}$

$\therefore \ln\left(\frac{3}{2}\right) = \frac{1}{2}t, \ t = 2\ln\left(\frac{3}{2}\right) = \ln\left(\frac{9}{4}\right)$

$y\left(\ln\left(\frac{9}{4}\right)\right) = -e^{\ln\left(\frac{9}{4}\right)} + 3e^{\frac{1}{2}\left(2\ln\left(\frac{3}{2}\right)\right)} = -\frac{9}{4} + \frac{9}{2} = \frac{9}{4}$

Note $y''(\ln(\frac{9}{4})) = -e^{\ln(\frac{9}{4})} + \frac{3}{4}e^{\frac{1}{2}(2\ln(\frac{3}{2}))} = -\frac{9}{4} + \frac{9}{8} < 0$

$\therefore t = \ln(\frac{9}{4})$ is a local max.

$\therefore$ at $t = \ln(\frac{9}{4})$, $\underline{y = \frac{9}{4}}$

(c) $y(t) = -e^{t} + 3e^{\frac{1}{2}t} = 0 \Rightarrow 3 = e^{\frac{1}{2}t}$, $\ln(3) = \frac{1}{2}t$

$\therefore$ at $\underline{t = \ln(9)}$, $y = 0$

16.

(a) $r^2 - r - 2 = (r-2)(r+1) = 0$, $r = 2, -1$

$\therefore y = c_1 e^{2t} + c_2 e^{-t}$, $y' = 2c_1 e^{2t} - c_2 e^{-t}$

$y(0) = \alpha$: $c_1 + c_2 = \alpha$ } $3c_1 = \alpha + 2$, $c_1 = \frac{\alpha+2}{3}$

$y'(0) = 2$: $2c_1 - c_2 = 2$ } $3c_2 = 2\alpha + 2$, $c_2 = \frac{2\alpha+2}{3}$

$\therefore \underline{y(t) = \left(\frac{\alpha+2}{3}\right)e^{2t} + \left(\frac{2\alpha+2}{3}\right)e^{-t}}$

(b) For $y \to 0$ as $t \to \infty$, must get rid of e^{2t} term. $\therefore \frac{\alpha+2}{3} = 0 \Rightarrow \underline{\alpha = -2}$

17.

$$r^2 - (2\alpha - 1)r + \alpha(\alpha - 1) = [r - (\alpha - 1)][r - \alpha] = 0$$

$\therefore r = \alpha - 1, \alpha \quad \therefore y = c_1 e^{\alpha t} + c_2 e^{(\alpha-1)t}$

(a) $y \to 0$ as $t \to \infty$ if $\alpha < 0$ and $\alpha - 1 < 0$. $\therefore$ $y \to$ <u>0 if $\alpha < 0$</u>

(b) $y \to \pm\infty$ if ($\alpha > 0$ and $c_1 \neq 0$) or ($\alpha - 1 > 0$ and $c_2 \neq 0$).

$\therefore$ For all nonzero solutions, the solution that covers both cases is: $y \to \pm\infty$ if <u>$\alpha > 1$</u>

18.

$$r^2 + (3 - \alpha)r - 2(\alpha - 1) = (r + 2)(r + 1 - \alpha) = 0$$

$\therefore r = -2, \alpha - 1 \quad \therefore y = c_1 e^{-2t} + c_2 e^{(\alpha-1)t}$

(a) $y \to 0$ as $t \to \infty$ if $\alpha - 1 < 0$, or <u>$\alpha < 1$</u>

(b) To become unbounded, $\alpha - 1 > 0$ or $\alpha > 1$ <u>and</u>

$c_2 \neq 0$. If $c_2 = 0$, α unimportant.

$\therefore$ No value of α guarantees all solutions will become unbounded.

19.

(a)

$r^2 + 5r + 6 = (r+3)(r+2) = 0, \quad r = -2, -3$

$\therefore y = c_1 e^{-2t} + c_2 e^{-3t}, \quad y' = -2c_1 e^{-2t} - 3c_2 e^{-3t}$

$y(0) = 2: \quad c_1 + c_2 = 2$

$y'(0) = \beta: \quad -2c_1 - 3c_2 = \beta$

$-c_2 = 4 + \beta, \quad c_2 = -(4+\beta)$

$c_1 = 6 + \beta$

$\therefore y(t) = (6+\beta)e^{-2t} - (4+\beta)e^{-3t}$

(b)

$y'(t) = -2(6+\beta)e^{-2t} + 3(4+\beta)e^{-3t} = 0$

Or, $3(4+\beta)e^{-t} = 2(6+\beta)$,

Or, $e^{t} = \dfrac{12+3\beta}{12+2\beta}, \quad t_m = \ln\left[\dfrac{12+3\beta}{12+2\beta}\right], \beta \neq -4, -6$

$$\therefore y_m = (6+\beta)e^{-2\ln\left[\frac{12+3\beta}{12+2\beta}\right]} - (4+\beta)e^{-3\ln\left[\frac{12+3\beta}{12+2\beta}\right]}$$

$$= (6+\beta)\left[\frac{12+2\beta}{12+3\beta}\right]^2 - (4+\beta)\left[\frac{12+2\beta}{12+3\beta}\right]^3$$

$$= \frac{2^2}{3^2}(6+\beta)\frac{(6+\beta)^2}{(4+\beta)^2} - \frac{2^3}{3^3}(4+\beta)\frac{(6+\beta)^3}{(4+\beta)^3}$$

$$= \frac{4}{9}\frac{(6+\beta)^3}{(4+\beta)^2} - \frac{8}{27}\frac{(6+\beta)^3}{(4+\beta)^2}$$

$$\therefore y_m = \frac{4}{27}\frac{(6+\beta)^3}{(4+\beta)^2}$$

Note: without doing the 2nd derivative test, note that $y'(0) = \beta > 0$, $y(0) = 2$, and $\lim_{t\to\infty} y(t) = 0$.

$\therefore$ As $y(t)$ is continuous, there is a max on $(0, \infty)$, and there is only one critical point $\Rightarrow y_m$ a local max at t_m.

(c)

$$y_m \geq 4 \iff \frac{4}{27}\frac{(6+\beta)^3}{(4+\beta)^2} \geq 4 \iff \frac{(6+\beta)^3}{(4+\beta)^2} \geq 27$$

$$\iff (6+\beta)^3 - 27(4+\beta)^2 \geq 0$$

Using MATLAB to factor:

```
syms b
factor((6+b)^3 - 27*(4+b)^2)
```

```
ans =

[ b + 3, b^2 - 12*b - 72]
```

$$\therefore\ y_m \geq 4 \iff (\beta+3)(\beta^2-12\beta-72) \geq 0$$

$$\iff \beta^2 - 12\beta - 72 \geq 0 \quad \text{since } \beta > 0$$

Solving quadratic: $\beta = \dfrac{12 \pm \sqrt{144+4(72)}}{2}$

or, $\beta = 6 \pm 6\sqrt{3}$

$$\therefore\ y_m \geq 4 \iff \left[\beta - (6+6\sqrt{3})\right]\left[\beta - (6-6\sqrt{3})\right] \geq 0$$

$$\iff \beta \geq 6+6\sqrt{3} \text{ and } \beta \geq 6-6\sqrt{3}$$

Since $6-6\sqrt{3} < 0$ and $\beta > 0$,

$$\therefore\ \underline{y_m \geq 4 \iff \beta \geq 6+6\sqrt{3} \approx 16.3923}$$

(d)

$$\lim_{\beta\to\infty} t_m = \lim_{\beta\to\infty} \ln\left[\frac{12+3\beta}{12+2\beta}\right] = \lim_{\beta\to\infty}\left[\frac{\frac{12}{\beta}+3}{\frac{12}{\beta}+2}\right] = \underline{\ln\left[\tfrac{3}{2}\right]}$$

$$\lim_{\beta\to\infty} y_m = \lim_{\beta\to\infty} \frac{4}{27}\frac{(6+\beta)^3}{(4+\beta)^2} = \lim_{\beta\to\infty} \frac{4}{27}\frac{\beta^3}{\beta^2}\frac{\left(\frac{6}{\beta}+1\right)^3}{\left(\frac{4}{\beta}+1\right)^3}$$

$$= \underline{\infty}$$

20.

(a) if $\underline{y = \frac{d}{c}}$, Then $y' = 0$, $y'' = 0$, and $cy = c\left(\frac{d}{c}\right) = d$

(b)

$Y' = y'$, $Y'' = y''$

$\therefore ay'' + by' + cy = aY'' + bY' + c(Y + y_e) = d$

$\therefore aY'' + bY' + cY = d - cy_e = d - c\left(\frac{d}{c}\right) = 0$

$\therefore \underline{aY'' + bY' + cY = 0}$

21.

(a)

$ar^2 + br + c = 0$, $r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Real, different $\Rightarrow b^2 - 4ac > 0 \Rightarrow \frac{b^2}{4a} > c$

$b^2 - 4ac > 0$, $-b + \sqrt{b^2 - 4ac} < 0$, $-b - \sqrt{b^2 - 4ac} < 0$

$\therefore b^2 > 4ac$, $b > \sqrt{b^2 - 4ac}$, $b > -\sqrt{b^2 - 4ac}$

Since $\sqrt{b^2-4ac} > 0$, $b > 0$

$\therefore b > \sqrt{b^2-4ac} \Rightarrow b^2 > b^2 - 4ac \Rightarrow 4ac > 0 \Rightarrow c > 0$

$\therefore$ $a > 0$, $b > 0$ and $0 < c < \frac{b^2}{4a}$

(b)

$ar^2 + br + c = 0$, $\frac{-b \pm \sqrt{b^2-4ac}}{2a}$

Real w/ 2 solutions $\Rightarrow b^2 - 4ac > 0$

(1) Suppose $b > 0$. $\therefore -b < 0$, and $-b - \sqrt{b^2-4ac} < 0$

$\therefore -b + \sqrt{b^2-4ac} > 0 \Rightarrow \sqrt{b^2-4ac} > b \Rightarrow$

$b^2 - 4ac > b^2 \Rightarrow -4ac > 0 \Rightarrow c < 0$

$\therefore$ $b > 0$ and $c < 0$

(2) Suppose $b < 0$ $\therefore -b > 0$ and $\therefore -b + \sqrt{b^2-4ac} > 0$

$\therefore -b - \sqrt{b^2-4ac} < 0 \Rightarrow -b < \sqrt{b^2-4ac} \Rightarrow$

$(-b)^2 < b^2 - 4ac \Rightarrow 4ac < 0 \Rightarrow c < 0$

$\therefore$ $b < 0$ and $c < 0$

(3) Suppose $b = 0$ $\therefore r = \frac{\pm\sqrt{-4ac}}{2a} \Rightarrow -4ac > 0$

$\Rightarrow c < 0$

$\therefore$ (1), (2), (3) $\Rightarrow c < 0$ This automatically means $b^2 - 4ac > 0$ since $a > 0$

$\therefore$ Only need $\underline{\underline{c < 0}}$

(c)

$ar^2 + br + c = 0$, $r = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Real, different $\Rightarrow b^2 - 4ac > 0$

Positive $\Rightarrow b \neq 0$, and

$-b + \sqrt{b^2 - 4ac} > 0$ and $-b - \sqrt{b^2 - 4ac} > 0$

Adding, $-2b > 0$, $b < 0$

Also, $-b - \sqrt{b^2 - 4ac} > 0 \Rightarrow -b > \sqrt{b^2 - 4ac}$,

$\therefore b^2 > b^2 - 4ac \Rightarrow 4ac > 0 \Rightarrow c > 0$

$\therefore b < 0$, $c > 0$ and $b^2 - 4ac > 0 \Rightarrow \dfrac{b^2}{4a} > c$

$\therefore \underline{b < 0 \text{ and } 0 < c < \dfrac{b^2}{4a}}$

3.2 Solutions of Linear Homogeneous Equations; the Wronskian

Note Title 6/1/2018

1.

$$\begin{vmatrix} e^{2t} & e^{-3t/2} \\ 2e^{2t} & \left(-\frac{3}{2}\right)e^{-3t/2} \end{vmatrix} = \left(-\frac{3}{2}\right)e^{t/2} - 2e^{t/2} = -\frac{7}{2}e^{t/2}$$

2.

$$\begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = \cos^2 t + \sin^2 t = 1$$

3.

$$\begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & e^{-2t} - 2te^{-2t} \end{vmatrix} = e^{-4t} - 2te^{-4t} - \left(-2te^{-4t}\right)$$

$$= e^{-4t}$$

4.

$$\begin{vmatrix} e^{t}\sin t & e^{t}\cos t \\ e^{t}\sin t + e^{t}\cos t & e^{t}\cos t - e^{t}\sin t \end{vmatrix}$$

$$= e^{2t}\sin t\cos t - e^{2t}\sin^2 t - \left(e^{2t}\sin t\cos t + e^{2t}\cos^2 t\right)$$

$$= -e^{2t}\sin^2 t - e^{2t}\cos^2 t = -e^{2t}$$

5.

$$\begin{vmatrix} \cos^2\theta & 1+\cos(2\theta) \\ -2\cos\theta\sin\theta & -2\sin(2\theta) \end{vmatrix}$$

$$= -2\cos^2\theta\sin 2\theta - \left(-2\cos\theta\sin\theta - 2\cos\theta\sin\theta\cos(2\theta)\right)$$

using $\sin 2\theta = 2\sin\theta\cos\theta$

$$= -2\cos^2\theta\sin 2\theta + \sin 2\theta + \sin 2\theta\cos 2\theta$$

$$= \sin 2\theta(1-2\cos^2\theta) + \sin 2\theta\cos 2\theta$$

using $\cos 2\theta = \cos^2\theta - \sin^2\theta = \cos^2\theta - (1-\cos^2\theta) = 2\cos^2\theta - 1$

$$= \sin 2\theta(-\cos 2\theta) + \sin 2\theta\cos 2\theta = \underline{0}$$

6.

Standard form: $y'' + \frac{3}{t}y = 1 \quad \therefore t \neq 0$

As intial condition is at $t=1$, $\underline{0 < t < \infty}$

7.

Standard form: $y'' + \frac{3}{t-4}y' + \frac{4}{t(t-4)}y = \frac{2}{t(t-4)}$

$\therefore t \neq 0, 4$ Intial condition at $t=3$. $\therefore 0 < t < 4$

8.

$\ln|t| \Rightarrow t \neq 0$, continuous elsewhere

Initial condition at $t=2$. $\therefore$ $\underline{0 < t < \infty}$

9.

Standard form: $y'' + \frac{1}{x-2} y' + \tan x \, y = 0$

$\therefore x \neq 2$. For $\tan(x)$, $x \neq n\frac{\pi}{2}$, $n = \pm 1, \pm 3, \pm 5, \ldots$

Initial condition at $x = 3$. $\therefore$ $\underline{2 < x < \frac{3}{2}\pi}$

10.

(a) $y_1 = t^2$, $y_1' = 2t$, $y_1'' = 2$

$\therefore t^2(2) - 2(t^2) = \underline{0}$ ✓

$y_2 = \frac{1}{t}$, $y_2' = -\frac{1}{t^2}$, $y_2'' = \frac{2}{t^3}$

$\therefore t^2\left(\frac{2}{t^3}\right) - 2\left(\frac{1}{y}\right) = \underline{0}$ ✓

(b) $y = c_1 t^2 + c_2 t^{-1}$

$y' = 2c_1 t - c_2 t^{-2}$ $\quad$ $y'' = 2c_1 + 2c_2 t^{-3}$

$$\therefore t^2(2c_1 + 2c_2 t^{-3}) - 2(c_1 t^2 + c_2 t^{-1})$$

$$= 2c_1 t^2 + 2c_2 t^{-1} - 2c_1 t^2 - 2c_2 t^{-1} = \underline{0}$$

11.

(a) $y_1 = 1, \; y_1' = 0, \; y_1'' = 0$

$$\therefore 1(0) + (0)^2 = \underline{0}$$

$y_2 = t^{1/2}, \; y_2' = \frac{1}{2} t^{-\frac{1}{2}}, \; y_2'' = -\frac{1}{4} t^{-\frac{3}{2}}$

$$\therefore t^{1/2}\left(-\frac{1}{4} t^{-3/2}\right) + \left(\frac{1}{2} t^{-\frac{1}{2}}\right)^2 = -\frac{1}{4} t^{-1} + \frac{1}{4} t^{-1} = \underline{0}$$

(b) $y = c_1 + c_2 t^{\frac{1}{2}}, \; y' = \frac{c_2}{2} t^{-\frac{1}{2}}, \; y'' = -\frac{c_2}{4} t^{-3/2}$

$$\therefore \left(c_1 + c_2 t^{\frac{1}{2}}\right)\left(-\frac{c_2}{4} t^{-3/2}\right) + \left(\frac{c_2}{2} t^{-\frac{1}{2}}\right)^2 =$$

$$-\frac{c_1 c_2}{4} t^{-\frac{3}{2}} - \frac{c_2^2}{4} t^{-1} + \frac{c_2^2}{4} t^{-1} = \underline{-\frac{c_1 c_2}{4} t^{-3/2} \neq 0}$$

$\therefore y = c_1 + c_2 t^{\frac{1}{2}}$ not a solution when $c_1 c_2 \neq 0$

(c) Doesn't contradict Theorem 3.2.2 because $y y'' + (y')^2$ is <u>not linear</u>.

12.

(a) $y = c\phi(t)$, $y' = c\phi'(t)$, $y'' = c\phi''(t)$

$$\therefore y'' + py' + qy = c\phi'' + p(c\phi') + q(c\phi)$$
$$= c(\phi'' + p\phi' + q\phi)$$
$$= c\,g(t) \neq g(t)$$

(b) Does not contradict Theorem 3.2.2 because

$y'' + py' + qy = g$ is <u>not</u> homogeneous.

13.

$y = \sin(t^2)$, $y' = 2t\cos(t^2)$, $y'' = 2\cos(t^2) - 4t^2\sin(t^2)$

$\therefore y'' + p(t)y' + q(t)y =$

$$2\cos(t^2) - 4t^2\sin(t^2) + 2t\,p(t)\cos(t^2) + q(t)\sin(t^2)$$

If t is allowed to be 0, then the above becomes

$$2(1) - 4(0)(0) + 2(0)p(0)(1) + q(0)\,0 = 2 \neq 0$$

$\therefore$ **No**, $y = \sin(t^2)$ can't be a solution.

14.

$$W = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = \begin{vmatrix} e^{2t} & g \\ 2e^{2t} & g' \end{vmatrix} = 3e^{4t}$$

$$\therefore g'e^{2t} - 2g\,e^{2t} = 3e^{4t}$$

$$\therefore g' - 2g = 3e^{2t}, \text{ a linear 1st order equation.}$$

$$\therefore \text{Let } u = \exp \int -2 = e^{-2t}$$

$$\therefore \frac{d}{dt}\left(e^{-2t} g\right) = 3, \quad e^{-2t} g = 3t + c, \quad c \text{ a constant}$$

$$\therefore g(t) = 3t\,e^{2t} + c\,e^{2t}$$

$$\therefore g'(t) = 3e^{2t} + 6t\,e^{2t} + 2c\,e^{2t}$$

$$\therefore W = \begin{vmatrix} e^{2t} & (3t + c)e^{2t} \\ 2e^{2t} & (3 + 2c + 6t)e^{2t} \end{vmatrix}$$

$$= (3 + 2c + 6t)e^{4t} - (6t + 2c)e^{4t} = 3e^{4t}$$

$$\therefore g(t) = \underline{3t\,e^{2t} + c\,e^{2t}}, \quad c \text{ a constant}$$

15.

$$W = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = fg' - f'g = t\cos t - \sin t$$

$$u = f + 3g \Rightarrow u' = f' + 3g'$$

$$v = f - g \Rightarrow v' = f' - g'$$

$$\therefore W(u,v) = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = uv' - u'v$$

$$= (f+3g)(f'-g') - (f'+3g')(f-g)$$

$$= ff' - fg' + 3gf' - 3gg' - [ff' + 3fg' - f'g - 3gg']$$

$$= -4fg' + 4f'g = -4(fg' - f'g)$$

$$\therefore W(u,v) = -4W(f,g)$$

$$= -4(t\cos t - \sin t) = -4t\cos t + 4\sin t$$

16.

(a) $$W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$$

$$W[y_3, y_4] = \begin{vmatrix} a_1 y_1 + a_2 y_2 & b_1 y_1 + b_2 y_2 \\ a_1 y_1' + a_2 y_2' & b_1 y_1' + b_2 y_2' \end{vmatrix}$$

$$= \begin{vmatrix} a_1 y_1 & b_1 y_1 + b_2 y_2 \\ a_1 y_1' & b_1 y_1' + b_2 y_2' \end{vmatrix} + \begin{vmatrix} a_2 y_2 & b_1 y_1 + b_2 y_2 \\ a_2 y_2' & b_1 y_1' + b_2 y_2' \end{vmatrix}$$

$$= \begin{vmatrix} a_1 y_1 & b_1 y_1 \\ a_1 y_1' & b_1 y_1' \end{vmatrix} + \begin{vmatrix} a_1 y_1 & b_2 y_2 \\ a_1 y_1' & b_2 y_2' \end{vmatrix} + \begin{vmatrix} a_2 y_2 & b_1 y_1 \\ a_2 y_2' & b_1 y_1' \end{vmatrix} + \begin{vmatrix} a_2 y_2 & b_2 y_2 \\ a_2 y_2' & b_2 y_2' \end{vmatrix}$$

$$= a_1 b_1 \begin{vmatrix} y_1 & y_1 \\ y_1' & y_1' \end{vmatrix} + a_1 b_2 \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} + a_2 b_1 \begin{vmatrix} y_2 & y_1 \\ y_2' & y_1' \end{vmatrix} + a_2 b_2 \begin{vmatrix} y_2 & y_2 \\ y_2' & y_2' \end{vmatrix}$$

$$= a_1 b_1 (0) + a_1 b_2 W[y_1, y_2] - a_2 b_1 W[y_1, y_2] + a_2 b_2 (0)$$

$$\therefore W[y_3, y_4] = (a_1 b_2 - a_2 b_1) W[y_1, y_2]$$

using linear properties of determinants

(b) By definition of a fundamental set of solutions, $W[y_1, y_2] \neq 0$.

From (a), $W[y_3, y_4] \neq 0 \iff a_1 b_2 - a_2 b_1 \neq 0$

$\therefore$ y_3 and y_4 are a fundamental set of solutions $\iff a_1 b_2 - a_2 b_1 \neq 0$.

17.

$r^2 + r - 2 = (r+2)(r-1) = 0, \quad r = 1, -2$

$\therefore \; y = c_1 e^{t} + c_2 e^{-2t}, \quad y' = c_1 e^{t} - 2c_2 e^{-2t}$

$t_0 = 0$: Set $y_1(0) = 1 : \; c_1 + c_2 = 1$, $\quad y_1'(0) = 0 : \; c_1 - 2c_2 = 0$ $\quad 3c_2 = 1, \; c_2 = \frac{1}{3}$, $\; \therefore \; c_1 = \frac{2}{3}$

$\therefore \; \underline{y_1 = \frac{2}{3} e^{t} + \frac{1}{3} e^{-2t}}$

Set $y_2(0) = 0 : \; c_1 + c_2 = 0$, $\quad y_2'(0) = 1 : \; c_1 - 2c_2 = 1$ $\quad 3c_2 = -1, \; c_2 = -\frac{1}{3}$, $\; \therefore \; c_1 = \frac{1}{3}$

$\therefore \; \underline{y_2 = \frac{1}{3} e^{t} - \frac{1}{3} e^{-2t}}$

18.

$r^2 + 4r + 3 = (r+3)(r+1) = 0, \quad r = -1, -3$

$\therefore \; y = c_1 e^{-t} + c_2 e^{-3t}, \quad y' = -c_1 e^{-t} - 3c_2 e^{-3t}$

$t_0 = 1$: Set $y_1(1) = 1 : \; c_1 e^{-1} + c_2 e^{-3} = 1$

$$y_1'(1) = 0: \quad -c_1 e^{-1} - 3c_2 e^{-3} = 0$$

$$\therefore \quad -2c_2 e^{-3} = 1, \quad c_2 = -\tfrac{1}{2} e^{3}$$

$$-c_1 e^{-1} + \tfrac{3}{2} = 0, \quad c_1 = \tfrac{3}{2} e$$

$$\therefore \quad y_1 = \tfrac{3}{2} e^{1-t} - \tfrac{1}{2} e^{3-3t}$$

Set $y_2(1) = 0: \quad c_1 e^{-1} + c_2 e^{-3} = 0$

$y_2'(1) = 1: \quad -c_1 e^{-1} - 3c_2 e^{-3} = 1$

$\therefore$ $-2c_2 e^{-3} = 1$, $c_2 = -\tfrac{1}{2} e^{3}$, $c_1 = \tfrac{1}{2} e$

$$\therefore \quad y_2 = \tfrac{1}{2} e^{1-t} - \tfrac{1}{2} e^{3-3t}$$

Or, using standard of t_0 is the least most time,

$$y_1 = \tfrac{3}{2} e^{-(t-1)} - \tfrac{1}{2} e^{-3(t-1)}, \quad y_2 = \tfrac{1}{2} e^{-(t-1)} - \tfrac{1}{2} e^{-3(t-1)}$$

19.

(a) $y_1' = -2\sin(2t)$, $y_1'' = -4\cos(2t)$

$$\therefore \quad y_1'' + 4y_1 = [-4\cos(2t)] + 4[\cos(2t)] = 0$$

(b) $y_2' = 2\cos(2t)$, $y_2'' = -4\sin(2t)$

$\therefore y_2'' + 4y_2 = [-4\sin(2t)] + 4[\sin(2t)] = 0$

(c) $W = \begin{vmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{vmatrix} = 2\cos^2(2t) + 2\sin^2(2t) = 2 \neq 0$

$\therefore$ yes, y_1, y_2 form a fundamental set of solutions, by Theorem 3.2.4

20.

(a) $y_1' = e^t$, $y_1'' = e^t$

$\therefore y_1'' - 2y_1' + y = (e^t) - 2(e^t) + (e^t) = 0$

(b) $y_2' = e^t + te^t$, $y_2'' = 2e^t + te^t$

$\therefore y_2'' - 2y_2' + y_2 = [2e^t + te^t] - 2[e^t + te^t] + [te^t] = 0$

(c) $W = \begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix} = e^{2t} + te^{2t} - te^{2t} = e^{2t} \neq 0$

$\therefore$ yes, y_1, y_2 form a fundamental set of solutions

21.

(a) $y_1' = 1$, $y_1'' = 0$

$$\therefore x^2 y_1'' - x(x+2)y_1' + (x+2)y_1 = x^2(0) - x(x+2)(1) + (x+2)(x)$$
$$= \underline{0}$$

(b) $y_2' = e^x + xe^x$, $y_2'' = 2e^x + xe^x$

$$\therefore x^2 y_2'' - x(x+2)y_2' + (x+2)y_2 =$$

$$x^2(2e^x + xe^x) - x(x+2)(e^x + xe^x) + (x+2)(xe^x) =$$

$$2x^2e^x + x^3e^x - x^2e^x - 2xe^x - x^3e^x - 2x^2e^x + x^2e^x + 2xe^x$$

$= \underline{0}$ (all terms cancel)

(c) $W = \begin{vmatrix} x & xe^x \\ 1 & e^x + xe^x \end{vmatrix} = xe^x + x^2e^x - xe^x = x^2e^{2x} > 0$ for $x > 0$

$\therefore W \neq 0$ for $x > 0$

$\therefore$ $\underline{yes}$, y_1, y_2 form a fundamental set of solutions.

22.

(a) $y_1' = -e^{-t}$, $y_1'' = e^{-t}$ $\therefore y'' - y' - 2y = (e^{-t}) - (-e^{-t}) - 2(e^{-t})$
$$= 2e^{-t} - 2e^{-t} = \underline{0}$$

$y_2' = 2e^{2t}$, $y_2'' = 4e^{2t}$ $\therefore y'' - y' - 2y = (4e^{2t}) - (2e^{2t}) - 2(e^{2t})$
$$= 4e^{2t} - 4e^{2t} = \underline{0}$$

$\therefore y_1, y_2$ are solutions.

$$W[y_1, y_2] = \begin{vmatrix} e^{-t} & e^{2t} \\ -e^{-t} & 2e^{2t} \end{vmatrix} = 2e^{t} - (-e^{t}) = 3e^{t} \neq 0$$

$\therefore$ By Theorem 3.2.4, y_1, y_2 form a fundamental set of solutions.

(b)

yes, because they are linear combinations of y_1 and y_2, The $y'' - y' - 2y = 0$ is a linear homogeneous equation.

(c)

(1) $W[y_1, y_3] = \begin{vmatrix} y_1 & -2y_2 \\ y_1' & -2y_2' \end{vmatrix} = -2W[y_1, y_2] \neq 0$

$\therefore$ yes for $\{y_1, y_3\}$

(2) $W[y_2, y_3] = \begin{vmatrix} y_2 & -2y_2 \\ y_2' & -2y_2' \end{vmatrix} = -2\begin{vmatrix} y_2 & y_2 \\ y_2' & y_2' \end{vmatrix} = 0$

$\therefore$ No for $\{y_2, y_3\}$

(3) $W[y_1, y_4] = \begin{vmatrix} y_1 & y_1+2y_2 \\ y_1' & y_1'+2y_2' \end{vmatrix} = \begin{vmatrix} y_1 & y_1 \\ y_1' & y_1' \end{vmatrix} + 2\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$= 0 + 2W[y_1, y_2] \neq 0$

$\therefore$ <u>yes</u> for $\{y_1, y_4\}$

(4) $W[y_4, y_5] = \begin{vmatrix} y_1+2y_2 & 2y_1-2y_3 \\ y_1'+2y_2' & 2y_1'-2y_3' \end{vmatrix}$

$= \begin{vmatrix} y_1 & 2y_1-2y_3 \\ y_1' & 2y_1'-2y_3' \end{vmatrix} + 2\begin{vmatrix} y_2 & 2y_1-2y_3 \\ y_2' & 2y_1'-2y_3' \end{vmatrix}$

$= 2\begin{vmatrix} y_1 & y_1 \\ y_1' & y_1' \end{vmatrix} - 2\begin{vmatrix} y_1 & y_3 \\ y_1' & y_3' \end{vmatrix} + 4\begin{vmatrix} y_2 & y_1 \\ y_2' & y_1' \end{vmatrix} - 4\begin{vmatrix} y_2 & y_3 \\ y_2' & y_3' \end{vmatrix}$

$= 2(0) - 2\begin{vmatrix} y_1 & -2y_2 \\ y_1' & -2y_2' \end{vmatrix} - 4\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} - 4\begin{vmatrix} y_2 & -2y_2 \\ y_2' & -2y_2' \end{vmatrix}$

using $y_3 = -2y_2$, $\begin{vmatrix} a & a \\ b & b \end{vmatrix} = 0$, $\begin{vmatrix} a & c \\ b & d \end{vmatrix} = -\begin{vmatrix} c & a \\ d & b \end{vmatrix}$

$= 0 + 4\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} - 4\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} + 8\begin{vmatrix} y_2 & y_2 \\ y_2' & y_2' \end{vmatrix}$

$= 0 + 4W[y_1, y_2] - 4W[y_1, y_2] + 8(0) = \underline{0}$

$\therefore$ <u>No</u> for $\{y_4, y_5\}$

23.

Standard form: $y'' - \frac{(t+2)}{t} y' + \frac{(t+2)}{t^2} y = 0$

Using Abel's Theorem, $W(t) = c \exp\left(\int \frac{t+2}{t} dt\right)$

$$\int \frac{t+2}{t} dt = \int 1 + \frac{2}{t} dt = t + 2\ln(t)$$

$$\exp(t + 2\ln(t)) = t^2 e^t$$

$\therefore$ $\underline{W(t) = ct^2 e^t}$, c a constant

24.

Standard form: $y'' + \tan(t) y' - \frac{t}{\cos t} y = 0$

$\therefore$ $W(t) = c_1 \exp\left(-\int \tan(t) dt\right) = c_1 \exp(\ln|\cos(t)|)$

$= c_1 |\cos(t)| = c\cos(t)$, c_1 a constant, $c = \pm c_1$

25.

Standard form: $y'' - \frac{2x}{1-x^2} y' + \frac{\alpha(\alpha+1)}{1-x^2} y = 0$

$\therefore W(x) = c \exp\left(-\int \frac{-2x}{1-x^2} dx\right) = c \exp[-\ln(1-x^2)]$

$= \frac{c}{1-x^2}$, c a constant

26.

$(py')' + qy = py'' + p'y' + qy = 0$ [1]

$\therefore$ Since $p > 0$, [1] becomes: $y'' + \frac{p'}{p} y' + \frac{q}{p} y = 0$

$\therefore W(t) = c \exp\left(-\int \frac{p'(t)}{p(t)} dt\right) = c \exp[-\ln(p(t))]$

$= c \exp\left[\ln \frac{1}{p(t)}\right] = \frac{c}{p(t)}$, c a constant

27.

For $t > 0$, the equation in standard form is:

$y'' + \frac{2}{t} y' + e^t y = 0$

$\therefore W(t) = c\exp\left(-\int \frac{2}{t}\,dt\right) = c\exp(-2\ln t)$

$= c\exp\left(\ln\frac{1}{t^2}\right) = \frac{c}{t^2}$, c a constant

Since $W(1) = 2 = \frac{c}{(1)^2}$, $c = 2$

$\therefore W(t) = \frac{2}{t^2}$ $\therefore W(5) = \frac{2}{25}$

28.

$W(t) = c\exp\left(-\int p(t)\,dt\right) = c'$, a constant

$\Rightarrow p(t) = 0$. That is, $\int_{t_0}^{t} p(s)\,ds = K$, a constant

$\therefore \frac{d}{dt}\int_{t_0}^{t} p(s)\,ds = p(t) = \frac{d}{dt}(K) = 0$.

It says nothing about $g(t)$

$\therefore p(t) = 0$ for all t

29.

If y_1 and y_2 were a fundamental set of solutions, then $W[y_1, y_2](t) \neq 0$ for all $t \in I$, by definition.

But $W[y_1, y_2](t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} =$

$= \begin{vmatrix} 0 & 0 \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = 0$, a contradiction.

$\therefore$ y_1, y_2 can't form a fundamental set of solutions.

30.

Suppose y_1, y_2 are a fundamental set of solutions on I.

$\therefore$ $W[y_1, y_2](t) \neq 0$ for all $t \in I$.

$\therefore$ $y_1 y_2' - y_2 y_1' \neq 0$ for all $t \in I$.

$\therefore$ $W'(t) = y_1 y_2'' + y_1' y_2' - y_2' y_1' - y_2 y_1''$

$= y_1(t) y_2''(t) - y_2(t) y_1''(t)$ for all $t \in I$

$\therefore$ if t_0 is an inflection point, Then $y_1''(t_0) = y_2''(t_0) = 0$.

$\therefore$ $W'(t_0) = 0 \Rightarrow W(t_0) = c$, c a constant

By #28, $p(t) = 0$, so $p(t_0) = 0$, unless

y_1 and y_2 are not a fundamental set of solutions.

If they are a fundamental set, then $p(t_0)=0$ and

$\therefore\ y_1''(t_0) + p(t_0)y_1'(t_0) + q(t_0)y_1(t_0) = 0 = q(t_0)y_1(t_0)$

$y_2''(t_0) + p(t_0)y_2'(t_0) + q(t_0)y_2(t_0) = 0 = q(t_0)y_2(t_0)$

$y_1(t_0)$ and $y_2(t_0)$ can't both be 0 and be a fundamental set of solutions (by #29).

$\therefore\ q(t_0) = 0.$

$\therefore$ If y_1, y_2 have a common inflection point in I, then (1) if y_1, y_2 are a fundamental set, then

$p(t_0) = q(t_0) = 0$, $t_0 \in I$, the inflection point.

or (2) y_1, y_2 are not a fundamental set of solutions.

31.

If exact, then $(Py')' + (fy)' = 0$ implies

$P'y' + Py'' + f'y + fy' = 0$,

or $Py'' + (P' + f)y' + f'y = 0$

$\therefore$ $P' + f = Q$ and $f' = R$

Differentiating the first equation,

$\therefore$ $P'' + f' = Q'$. Sustituting $f' = R$,

$P'' + R = Q'$, or $\underline{P''(x) - Q'(x) + R(x) = 0}$

32.

(a) $P(x) = 1$, $Q(x) = x$, $R(x) = 1$. $\therefore$ $P'' - Q' + R = 0 - 1 + 1 = \underline{0}$.
$\therefore$ yes, exact.

(b) Let $f'(x) = R(x) = 1 \Rightarrow f(x) = x + c$, c a constant

$P' + f = Q \Rightarrow 0 + x + c = x \Rightarrow c = 0 \therefore f(x) = x$

$\therefore y'' + xy' + y = (P_y')' + (fy)' = (y')' + (xy)' = 0$

Integrating, $y' + xy = c_1$, a constant

Now use $\exp\left(\int x\,dx\right) = e^{\frac{x^2}{2}}$ as integrating factor

$\therefore \frac{d}{dx}\left(y e^{\frac{x^2}{2}}\right) = c_1 e^{\frac{x^2}{2}}$

$\therefore y e^{\frac{x^2}{2}} = c_1 \int_{x_0}^{x} e^{\frac{u^2}{2}} du + c_2$, c_2 a constant

since $\frac{d}{dx} \int_{x_0}^{x} e^{\frac{u^2}{2}} du = e^{\frac{x^2}{2}}$ by the

Fundamental Theorem of Calculus

$\therefore y(x) = c_1 e^{-\frac{x^2}{2}} \int_{x_0}^{x} e^{\frac{u^2}{2}} du + c_2 e^{-\frac{x^2}{2}}$, c_1, c_2 constants

33.

(a) $P(x) = x$, $Q(x) = -\cos x$, $R(x) = \sin x$

$\therefore P'' - Q' + R = 0 - \sin x + \sin x = 0. \therefore$ yes, exact

(6) Let $f'(x) = R(x) = \sin x \Rightarrow f(x) = -\cos x + c$,
c a constant

$\therefore P' + f = Q \Rightarrow 1 - \cos x + c = -\cos x \Rightarrow c = -1$

$\therefore f(x) = -\cos x - 1$

$$\therefore (Py')' + (fy)' = (xy')' + [(-\cos x - 1)y]'$$

$$= xy'' + y' - \cos x\, y' - y' + (\sin x)y$$

$$= xy'' - \cos x\, y' + (\sin x)y = 0$$

Integrating,

$$xy' + (-\cos x - 1)y = c_1, \quad c_1 \text{ a constant}$$

Or, $y' + \left(-\frac{\cos x}{x} - \frac{1}{x}\right)y = \frac{c_1}{x}$, as $x > 0$

Let $u(x) = \exp\left[\int_{x_0}^{x}\left(-\frac{\cos t}{t} - \frac{1}{t}\right)dt\right]$

$\therefore (yu)' = \frac{c_1 u(x)}{x}$, $yu = c_1 \int_{x_0}^{x} \frac{u(t)}{t}\,dt + c_2$

$\therefore y(x) = \frac{c_1}{u(x)} \int_{x_0}^{x} \frac{u(t)}{t}\,dt + \frac{c_2}{u(x)}$, c_1, c_2 constants

and $u(x) = \exp\left[\int_{x_0}^{x}\left(-\frac{\cos t}{t} - \frac{1}{t}\right)dt\right]$

34.

(a) $P(x) = x^2$, $Q(x) = x$, $R(x) = -1$

$\therefore P'' - Q' + R = 2 - 1 - 1 = 0$. $\therefore$ yes, exact

(b) Let $f'(x) = R(x) = -1$, $\therefore f(x) = -x + c$, c a constant

$P' + f = Q \Rightarrow 2x - x + c = x \Rightarrow c = 0$, $f(x) = -x$

$\therefore (Py')' + (fy)' = (x^2y')' + (-xy)'$

$= x^2y'' + 2xy' - xy' - y = x^2y'' + xy' - y = 0$

Integrating,

$x^2y' - xy = K$, K a constant

Or, since $x > 0$, $y' - \frac{1}{x}y = \frac{K}{x^2}$

Let $u(x) = \exp\left(\int -\frac{dx}{x}\right) = \exp(-\ln x) = \frac{1}{x}$

$\therefore \left(\frac{1}{x}y\right)' = \frac{K}{x^3} \Rightarrow \frac{y}{x} = -\frac{1}{2}\frac{K}{x^2} + c_2$

Or, $y = \frac{c_1}{x} + c_2 x$, c_1, c_2 constants $\left(c_1 = -\frac{K}{2}\right)$

35.

$$(uPy')' + (fy)' = u'Py' + uP'y' + uPy'' + fy' + f'y$$

$$= uPy'' + (u'P + uP' + f)y' + f'y$$

$$= uPy'' + uQy' + uRy$$

$$\therefore \; u'P + uP' + f = uQ \qquad [1]$$

$$\text{and} \quad f' = uR \qquad [2]$$

Differentiating [1],

$$u''P + u'P' + u'P' + uP'' + f' = u'Q + uQ' \qquad [3]$$

and substituting [2] into [3],

$$u''P + 2u'P' + uP'' + uR = u'Q + uQ'$$

Or, $\underline{Pu'' + (2P' - Q)u' + (P'' - Q' + R)u = 0}$

36.

$P = x^2$, $Q = x$, $R = (x^2 - v^2)$

$\therefore P'' - Q' + R = 2 - 1 + x^2 - v^2 \neq$, so not exact

$\therefore$ Consider $u(x)$ s.t. $uPy'' + uQy' + uR = 0$

Let $f'(x) = uR = ux^2 - uv^2$

$u'P + uP' + f = uQ$ becomes

$x^2 u' + 2xu + f = xu$,

Or, $x^2 u' + xu + f = 0$

Differentiating,

$x^2 u'' + 2xu' + xu' + u + f' = 0$

Substituting $f' = ux^2 - uv^2$,

$x^2 u'' + 3xu' + u + x^2 u - v^2 u = 0$,

Or, $\underline{x^2 u'' + 3xu' + (1 + x^2 - v^2)u = 0}$

37.

$P = 1,\ Q = 0,\ R = -x.$

$P'' - Q' + R = 0 - 0 - x \neq 0.$ $\therefore$ Not exact

$\therefore$ Consider $u(x)$ s.t. $uPy'' + uQy' + uR = 0$

Let $f' = uR = -xu$

$u'P + uP' + f = uQ$ becomes

$u' + f = 0$

Differentiating, $u'' + f' = 0$

Substituting for f', $\underline{u'' - xu = 0}$

38.

(a) From problem # 35, $\mu(x)$ must satisfy,

$$P\mu'' + (2P' - Q)\mu' + (P'' - Q' + R)\mu = 0.$$

For This equation to be the same as,

$Py'' + Qy' + Ry = 0$, Then

So $2P' - Q = Q$, equating the coefficients for μ' and y'. $\therefore 2P' = 2Q$, or $\underline{P' = Q}$

It then follows, equating coefficients for μ and y, $P'' - Q' + R = R$, so $P'' = Q'$, which is true since $P' = Q$.

(6) For ,

$P(x) = x^2$, $Q(x) = x$, so $P' = 2x \neq x = Q$.

$\therefore$ #36 is <u>not</u> self-adjoint

For ,

$P(x) = 1$, $Q(x) = 0$, and $P' = 0 = Q$.

Also, from #37, the adjoint is

$\mu'' - x\mu = 0$, so the adjoint is the same the original, so it <u>is</u> self-adjoint.

3.3 Complex Roots of the Characteristic Equation

Note Title 7/16/2018

1.

$$\exp(2-3i) = e^2(\cos 3 - i\sin 3) = e^2\cos(3) - ie^2\sin(3)$$

$$\approx -7.3151 - i(1.0427)$$

2.

$$e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1 + i(0) = -1$$

3.

$$e^{2-\frac{\pi}{2}i} = e^2\left[\cos\left(\frac{\pi}{2}\right) - i\sin\left(\frac{\pi}{2}\right)\right] = e^2\left[0 - i(1)\right] = -ie^2$$

$$\approx -7.3891\, i$$

4.

$$2^{1-i} = 2\cdot 2^{-i}, \quad 2 = e^x, \therefore x = \ln(2)$$

$$\therefore 2^{1-i} = 2\cdot e^{\ln(2)(-i)} = 2\left[\cos(\ln(2)) - i\sin(\ln(2))\right]$$

$$\approx 1.5385 - 1.2779\, i$$

5.

$r^2 - 2r + 2 = 0$, $r = \dfrac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$

$\therefore y = e^{(1 \pm i)t} = e^t(\cos t \pm \sin t)$

Consider $u(t) = e^t \cos(t)$, $v(t) = e^t \sin(t)$

$$W[u,v](t) = \begin{vmatrix} e^t \cos t & e^t \sin t \\ e^t \cos t - e^t \sin t & e^t \sin t + e^t \cos t \end{vmatrix}$$

$$= e^{2t} \sin t \cos t + e^{2t} \cos^2 t - e^{2t} \sin t \cos t + e^{2t} \sin^2 t$$

$$= e^{2t} \neq 0 \quad \therefore u, v \text{ a fundamental set}$$

$\therefore y(t) = \underline{c_1 e^t \cos(t) + c_2 e^t \sin(t)}$, c_1, c_2 constants

6.

$r^2 - 2r + 6 = 0$, $r = \dfrac{2 \pm \sqrt{4-24}}{2} = 1 \pm \sqrt{5}\, i$

$\therefore y = e^{(1 \pm \sqrt{5} i)t} = e^t[\cos(\sqrt{5}t) \pm i \sin(\sqrt{5}t)]$

Let $u(t) = e^t \cos \sqrt{5} t$, $v(t) = e^t \sin(\sqrt{5} t)$

$$W[u,v](t) = \begin{vmatrix} e^{t}\cos(\sqrt{5}t) & e^{t}\sin\sqrt{5}t \\ e^{t}\cos\sqrt{5}t - \sqrt{5}e^{t}\sin\sqrt{5}t & e^{t}\sin\sqrt{5}t + \sqrt{5}e^{t}\cos\sqrt{5}t \end{vmatrix}$$

$$= e^{2t}\sin\sqrt{5}t\cos\sqrt{5}t + \sqrt{5}e^{2t}\cos^2\sqrt{5}t$$
$$- e^{2t}\sin\sqrt{5}t\cos\sqrt{5}t + \sqrt{5}e^{2t}\sin^2\sqrt{5}t$$

$$= \sqrt{5}e^{2t} \neq 0 \quad \therefore u, v \text{ a fundamental set}$$

$$\therefore y(t) = c_1 e^{t}\cos(\sqrt{5}t) + c_2 e^{t}\sin(\sqrt{5}t), \quad c_1, c_2 \text{ constants}$$

7.

$$r^2 + 2r + 2 = 0, \quad r = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$$

$$\therefore y = e^{(-1\pm i)t} = e^{-t}(\cos t \pm i\sin t)$$

Let $u(t) = e^{-t}\cos(t)$ $v(t) = e^{-t}\sin(t)$

Using MATLAB live script,

```
clear, clc
syms t
u = exp(-t)*cos(t);
v = exp(-t)*sin(t);
A = [u v; diff(u) diff(v)]
det(A)  % Compute Wronskian
```

$\therefore W[u,v](t) = e^{-2t} \neq 0$

$\therefore u, v$ a fundamental set

A =

$$\begin{pmatrix} e^{-t}\cos(t) & e^{-t}\sin(t) \\ -e^{-t}\cos(t) - e^{-t}\sin(t) & e^{-t}\cos(t) - e^{-t}\sin(t) \end{pmatrix}$$

ans = $e^{-2t}\,(\cos(t)^2 + \sin(t)^2)$

$\therefore\ y(t) = c_1 e^{-t}\cos(t) + c_2 e^{-t}\sin(t)$, c_1, c_2 constants

8.

$r^2 + 6r + 13 = 0$, $r = \dfrac{-6 \pm \sqrt{36-52}}{2} = -3 \pm 2i$

$y = e^{(-3\pm 2i)t} = e^{-3t}\left[\cos(2t) \pm i\sin(2t)\right]$

Let $u(t) = e^{-3t}\cos(2t)$, $v(t) = e^{-3t}\sin(2t)$

Using MATLAB live script

```
clear, clc
syms t
u = exp(-3*t)*cos(2*t);
v = exp(-3*t)*sin(2*t);
A = [u v; diff(u) diff(v)]
det(A)   % Compute Wronskian
```

A =

$$\begin{pmatrix} \cos(2t)\,e^{-3t} & \sin(2t)\,e^{-3t} \\ -3\cos(2t)\,e^{-3t} - 2\sin(2t)\,e^{-3t} & 2\cos(2t)\,e^{-3t} - 3\sin(2t)\,e^{-3t} \end{pmatrix}$$

ans = $2e^{-6t}\left(\cos(2t)^2 + \sin(2t)^2\right)$

$\therefore\ W[u,v](t) = 2e^{-6t} \neq 0$

$\therefore\ u, v$ form a fundamental set

$\therefore\ y(t) = c_1 e^{-3t}\cos(2t) + c_2 e^{-3t}\sin(2t)$, c_1, c_2 constants

9.

9. $y'' + 2y' + 1.25y = 0$

$r^2 + 2r + 1.25 = 0$, $r = \dfrac{-2 \pm \sqrt{4-5}}{2} = -1 \pm \frac{1}{2}i$

$$y = e^{(-1 \pm \frac{1}{2}i)t} = e^{-t}\left[\cos\frac{t}{2} \pm i\sin\frac{t}{2}\right]$$

Let $u(t) = e^{-t}\cos\left(\frac{t}{2}\right)$, $v(t) = e^{-t}\sin\left(\frac{t}{2}\right)$

Using MATLAB live script,

```
clear, clc
syms t
u = exp(-t)*cos(t/2);
v = exp(-t)*sin(t/2);
A = [u v; diff(u) diff(v)]
simplify(det(A))  % Compute Wronskian
```

A =

$$\begin{pmatrix} \cos\left(\frac{t}{2}\right) e^{-t} & \sin\left(\frac{t}{2}\right) e^{-t} \\ -\cos\left(\frac{t}{2}\right) e^{-t} - \frac{\sin\left(\frac{t}{2}\right) e^{-t}}{2} & \frac{\cos\left(\frac{t}{2}\right) e^{-t}}{2} - \sin\left(\frac{t}{2}\right) e^{-t} \end{pmatrix}$$

ans =

$$\frac{e^{-2t}}{2}$$

$\therefore W[u,v](t) = \frac{e^{-2t}}{2} \neq 0$

So, u,v form a fundamental set

$$\therefore y(t) = c_1 e^{-t}\cos\left(\frac{t}{2}\right) + c_2 e^{-t}\sin\left(\frac{t}{2}\right), \quad c_1, c_2 \text{ constants}$$

10.

$$9r^2 + 9r - 4 = 0, \quad r = \frac{-9 \pm \sqrt{81+144}}{18} = \frac{-9 \pm 15}{18} = \frac{1}{3}, -\frac{4}{3}$$

$$\therefore e^{\frac{t}{3}}, e^{-\frac{4}{3}t} \quad W\left[e^{\frac{t}{3}}, e^{-\frac{4}{3}t}\right] = \begin{vmatrix} e^{t/3} & e^{-\frac{4t}{3}} \\ \frac{1}{3}e^{t/3} & -\frac{4}{3}e^{-\frac{4t}{3}} \end{vmatrix}$$

$$= -\frac{4}{3}e^{-t} - \frac{1}{3}e^{-t} = -\frac{5}{3}e^{-t} \neq 0$$

$\therefore e^{t/3}, e^{-4t/3}$ form a fundamental set

$\therefore\ y(t) = c_1 e^{t/3} + c_2 e^{-4t/3}$, c_1, c_2 constants

11.

$r^2 + 4r + 6.25 = 0$, $r = \frac{-4 \pm \sqrt{16-25}}{2} = -2 \pm \frac{3}{2} i$

$\therefore\ y = e^{(-2 \pm \frac{3}{2} i)t} = e^{-2t}\left[\cos(\tfrac{3}{2}t) \pm i \sin(\tfrac{3}{2}t)\right]$

Let $u(t) = e^{-2t}\cos(\tfrac{3}{2}t)$, $v(t) = e^{-2t}\sin(\tfrac{3}{2}t)$

Using MATLAB live script,

```
clear, clc
syms t
u = exp(-2*t)*cos(3*t/2);
v = exp(-2*t)*sin(3*t/2);
A = [u v; diff(u) diff(v)]
simplify(det(A))  % Compute Wronskian
```

A =

$$\begin{pmatrix} \sigma_2 e^{-2t} & \sigma_1 e^{-2t} \\ -2\sigma_2 e^{-2t} - \frac{3\sigma_1 e^{-2t}}{2} & \frac{3\sigma_2 e^{-2t}}{2} - 2\sigma_1 e^{-2t} \end{pmatrix}$$

where

$$\sigma_1 = \sin\left(\frac{3t}{2}\right)$$

$$\sigma_2 = \cos\left(\frac{3t}{2}\right)$$

ans =

$$\frac{3 e^{-4t}}{2}$$

$\therefore\ W[u,v](t) = \frac{3}{2} e^{-4t} \neq 0$

$\therefore\ u, v$ form a fundamental set

$\therefore\ y(t) = c_1 e^{-2t}\cos(\tfrac{3}{2}t) + c_2 e^{-2t}\sin(\tfrac{3}{2}t)$

12.

$r^2 + 4 = 0$, $r = \pm 2i$, $y = e^{\pm 2it} = \cos(2t) \pm i\sin(2t)$

Let $u = \cos(2t)$, $v = \sin(2t)$

$$W[u,v](t) = \begin{vmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{vmatrix} = 2 \neq 0$$

$\therefore$ u, v a fundamental set

$\therefore$ $y(t) = c_1\cos(2t) + c_2\sin(2t)$, c_1, c_2 constants

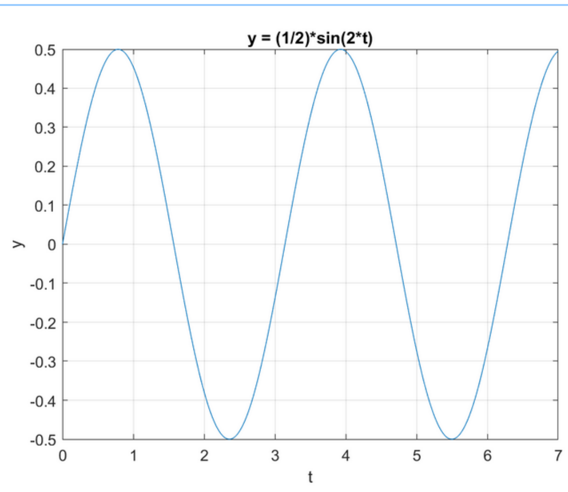
$y(0) = 0 \Rightarrow c_1(1) = 0 \Rightarrow c_1 = 0$

$y'(0) = 1 \Rightarrow y'(t) = 2c_2\cos(2t)\Big|_{t=0} = 2c_2 = 1$, $c_2 = \frac{1}{2}$

$\therefore$ $y(t) = \frac{1}{2}\sin(2t)$

Using MATLAB,

```
clear, clc
t = 0:0.01:7;
y = (1/2)*sin(2*t);
plot(t,y)
grid on
xlabel 't', ylabel 'y'
title 'y = (1/2)*sin(2*t)'
```



Behavior is oscillatory, no damping or unbound growth.

13.

$r^2 - 2r + 5 = 0$, $r = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i$

$\therefore y = e^t[\cos(2t) \pm i\sin(2t)]$

Let $u = e^t\cos(2t)$, $v = e^t\sin(2t)$

Using MATLAB live script,

```
clear, clc
syms t
u = exp(t)*cos(2*t);
v = exp(t)*sin(2*t);
A = [u v; diff(u) diff(v)]
simplify(det(A))  % Compute Wronskian
```

A =

$$\begin{pmatrix} \cos(2t)\,e^t & \sin(2t)\,e^t \\ \cos(2t)\,e^t - 2\sin(2t)\,e^t & 2\cos(2t)\,e^t + \sin(2t)\,e^t \end{pmatrix}$$

ans = $2e^{2t}$

$\therefore W[u,v](t) = 2e^{2t} \neq 0$

$\therefore u, v$ form a fundamental set

$\therefore y(t) = c_1 e^t\cos(2t) + c_2 e^t \sin(2t)$

$y(\frac{\pi}{2}) = 0 \Rightarrow c_1 e^{\pi/2}(-1) = 0 \Rightarrow c_1 = 0$

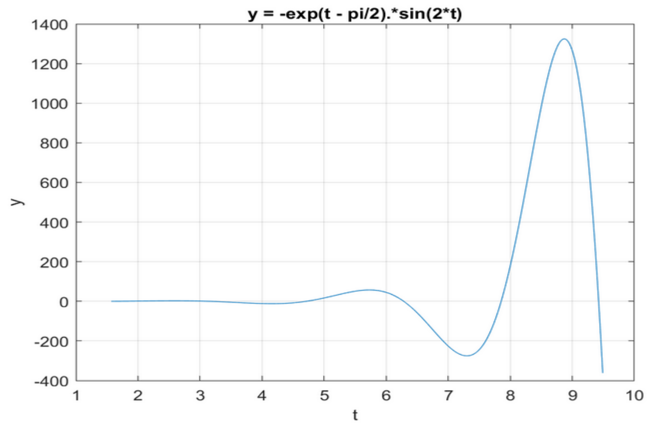
$\therefore y'(t) = c_2 e^t\sin(2t) + 2c_2 e^t\cos(2t)$

$y'(\frac{\pi}{2}) = 2 \Rightarrow 0 + 2c_2 e^{\pi/2}(-1) \Rightarrow c_2 = -e^{-\frac{\pi}{2}}$

$\therefore y(t) = -e^{t-\frac{\pi}{2}}\sin(2t)$

Using MATLAB,

```
clear, clc
t = pi/2:0.01:9.5;
y = -exp(t - pi/2).*sin(2*t);
plot(t,y)
grid on
xlabel 't', ylabel 'y'
title 'y = -exp(t - pi/2).*sin(2*t)'
```



Behavior is unbounded oscillation as $t \to \infty$

14.

$r^2 + 1 = 0$, $r = \pm i$, $y = \cos t \pm i \sin t$

Let $u = \cos t$, $v = \sin t$, $W[u,v] = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = 1 \neq 0$

$\therefore\ y(t) = c_1 \cos(t) + c_2 \sin(t)$

$y\left(\frac{\pi}{3}\right) = 2 \Rightarrow c_1\left(\frac{1}{2}\right) + c_2\left(\frac{\sqrt{3}}{2}\right) = 2$

$y'\left(\frac{\pi}{3}\right) = -4 \Rightarrow -c_1\left(\frac{\sqrt{3}}{2}\right) + c_2\left(\frac{1}{2}\right) = -4$

$\left(\frac{3}{2} + \frac{1}{2}\right)c_2 = 2\sqrt{3} - 4$

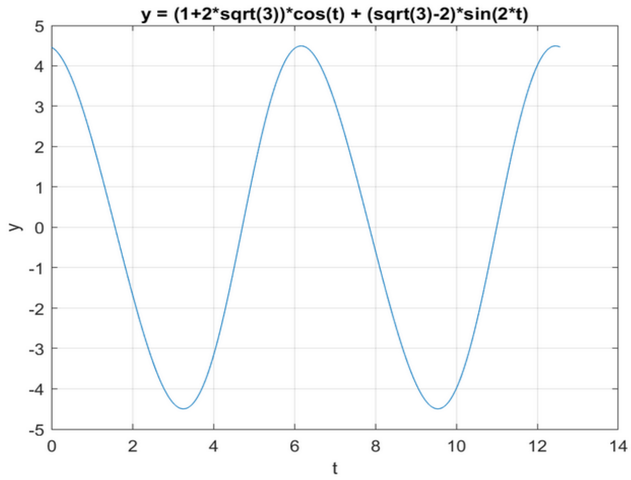
$c_2 = \sqrt{3} - 2$

$c_1\left(\frac{1}{2} + \frac{3}{2}\right) = 2 + 4\sqrt{3}$, $c_1 = 1 + 2\sqrt{3}$

$\therefore\ y(t) = (1 + 2\sqrt{3})\cos(t) + (\sqrt{3} - 2)\sin(t)$

Using MATLAB,

```
clear, clc
t = 0:0.01:4*pi;
y = (1+2*sqrt(3))*cos(t) + (sqrt(3)-2)*sin(2*t);
plot(t,y)
grid on
xlabel 't', ylabel 'y'
title 'y = (1+2*sqrt(3))*cos(t) + (sqrt(3)-2)*sin(2*t)'
```



Behavior is oscillatory

15.

$r^2 + 2r + 2 = 0$, $r = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$

$y = e^{-t}[\cos(t) \pm i \sin(t)]$

Let $u(t) = e^{-t}\cos t$, $v(t) = e^{-t}\sin(t)$

From #7, $W[u,v](t) = e^{-2t} \neq 0$, so a fundamental set.

$\therefore y(t) = c_1 e^{-t}\cos(t) + c_2 e^{-t}\sin(t)$

$y(\frac{\pi}{4}) = c_1 e^{-\frac{\pi}{4}} \frac{\sqrt{2}}{2} + c_2 e^{-\frac{\pi}{4}} \frac{\sqrt{2}}{2} = 2$

$y'(t) = c_1[-e^{-t}\cos t - e^{-t}\sin t] + c_2[-e^{-t}\sin t + e^{-t}\cos t]$

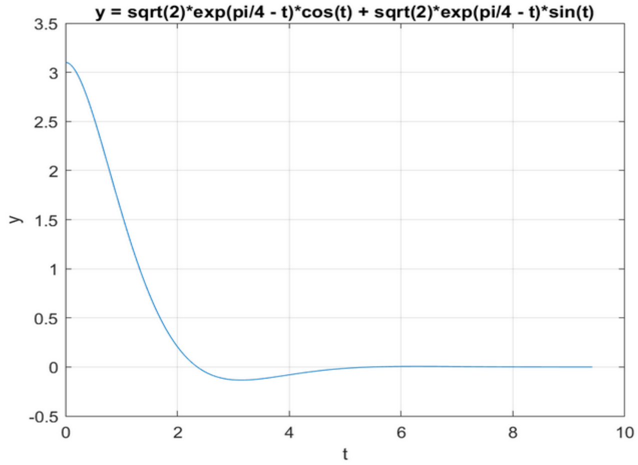
$y'(\frac{\pi}{4}) = -2c_1(e^{-\frac{\pi}{4}} \frac{\sqrt{2}}{2}) + 0 = -2 \Rightarrow c_1 = \sqrt{2}\, e^{\frac{\pi}{4}}$

$\therefore$ From $y\left(\frac{\pi}{4}\right)$, $1 + c_2 e^{-\frac{\pi}{4}} \frac{\sqrt{2}}{2} = 2$, $c_2 = \sqrt{2}\, e^{\frac{\pi}{4}}$

$$\therefore y(t) = \sqrt{2}\, e^{\frac{\pi}{4} - t} \cos(t) + \sqrt{2}\, e^{\frac{\pi}{4} - t} \sin(t)$$

Using MATLAB live script,

```
clear, clc
t = 0:0.01:3*pi;
y = sqrt(2)*exp(pi/4 - t).*cos(t) + sqrt(2)*exp(pi/4 - t).*sin(t);
plot(t,y)
grid on
xlabel 't', ylabel 'y'
title 'y = sqrt(2)*exp(pi/4 - t)*cos(t) + sqrt(2)*exp(pi/4 - t)*sin(t)'
```



Behavior is a decaying oscillation

16.

$$3r^2 - r + 2 = 0, \quad r = \frac{1 \pm \sqrt{1 - 24}}{6} = \frac{1}{6} \pm i \frac{\sqrt{23}}{6}$$

$$u = e^{t/6} \left[\cos\left(\frac{\sqrt{23}}{6} t\right) \pm i \sin\left(\frac{\sqrt{23}}{6} t\right) \right]$$

Let $x(t) = e^{t/6}\cos\left(\frac{\sqrt{23}}{6}t\right)$, $y(t) = e^{t/6}\sin\left(\frac{\sqrt{23}}{6}t\right)$

Using MATLAB live script.

```
clear, clc
syms t
x = exp(t/6)*cos(sqrt(23)*t/6);
y = exp(t/6)*sin(sqrt(23)*t/6);
A = [x y; diff(x) diff(y)]
simplify(det(A))  % Compute Wronskian
```

A =

$$\begin{pmatrix} \sigma_2 & \sigma_1 \\ \frac{\sigma_2}{6} - \frac{\sqrt{23}\, e^{t/6} \sin\left(\frac{\sqrt{23}\, t}{6}\right)}{6} & \frac{\sigma_1}{6} + \frac{\sqrt{23}\, e^{t/6} \cos\left(\frac{\sqrt{23}\, t}{6}\right)}{6} \end{pmatrix}$$

where

$$\sigma_1 = e^{t/6} \sin\left(\frac{\sqrt{23}\, t}{6}\right)$$

$$\sigma_2 = e^{t/6} \cos\left(\frac{\sqrt{23}\, t}{6}\right)$$

ans =

$$\frac{\sqrt{23}\, e^{t/3}}{6}$$

$\therefore W[x,y](t) = \frac{\sqrt{23}}{6} e^{t/3} \neq 0.$

$\therefore$ x, y form a fundamental set

$$\therefore u(t) = c_1 e^{t/6}\cos\left(\frac{\sqrt{23}}{6}t\right) + c_2 e^{t/6}\sin\left(\frac{\sqrt{23}}{6}t\right)$$

$$u(0) = 2 \Rightarrow c_1 = 2$$

$$u'(t) = \frac{2}{6}e^{t/6}\cos\left(\frac{\sqrt{23}}{6}t\right) - \frac{2\sqrt{23}}{6}e^{t/6}\sin\left(\frac{\sqrt{23}}{6}t\right)$$

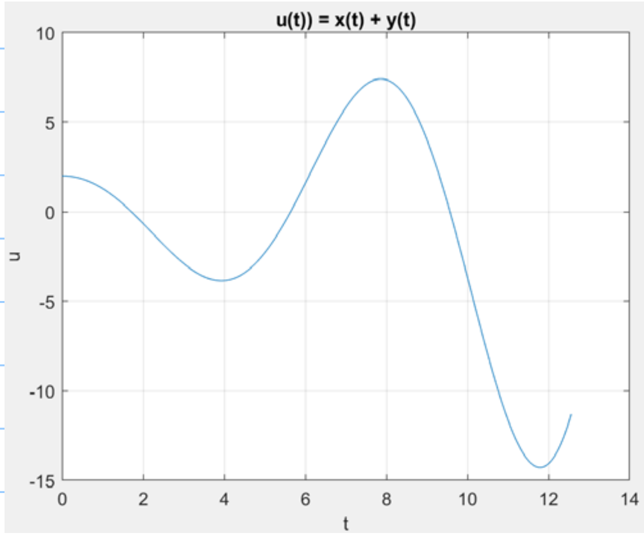
$$+ \frac{c_2}{6}e^{t/6}\sin\left(\frac{\sqrt{23}}{6}t\right) + c_2\frac{\sqrt{23}}{6}e^{t/6}\cos\left(\frac{\sqrt{23}}{6}t\right)$$

$$\therefore u'(0) = 0 \Rightarrow \frac{1}{3} - 0 + 0\ c_2\frac{\sqrt{23}}{6} = 0,\quad c_2 = \frac{-2}{\sqrt{23}}$$

$$\therefore u(t) = 2e^{t/6}\cos\left(\frac{\sqrt{23}}{6}t\right) - \frac{2}{\sqrt{23}}e^{t/6}\sin\left(\frac{\sqrt{23}}{6}t\right)$$

Using MATLAB to plot,

```
clear, clc
t = 0:0.01:4*pi;
x = 2*exp(t/6).*cos(sqrt(23)*t/6);
y = (-2/sqrt(23))*exp(t/6).*sin(sqrt(23)*t/6);
plot(t,x+y)
grid on
xlabel 't', ylabel 'u'
title 'u(t)) = x(t) + y(t)'
```



$\therefore$ at $t \approx 11$, $u(t) = -10$

Using MATLAB to be more precise,

```
clear, clc
syms t x y;
x = 2*exp(t/6)*cos(sqrt(23)*t/6);
y = (-2/sqrt(23))*exp(t/6)*sin(sqrt(23)*t/6);
vpasolve(abs(x+y)==10, t, 11)
```

ans = 10.759770554354898553146701430956

$\therefore$ at $t = 10.7598$, $|u(t)| = 10$

17.

(a) $5r^2 + 2r + 7 = 0$, $r = \dfrac{-2 \pm \sqrt{4-140}}{10} = -\dfrac{1}{5} \pm \dfrac{\sqrt{34}}{5} i$

$u = e^{-t/5}\left[\cos\left(\frac{\sqrt{34}}{5}t\right) \pm i \sin\left(\frac{\sqrt{34}}{5}t\right)\right]$

Let $x(t) = e^{-t/5}\cos\left(\frac{\sqrt{34}}{5}t\right)$, $y(t) = e^{-t/5}\sin\left(\frac{\sqrt{34}}{5}t\right)$

Using MATLAB live script,

```
clear, clc
syms t
x = exp(-t/5)*cos(sqrt(34)*t/5);
y = exp(-t/5)*sin(sqrt(34)*t/5);
A = [x y; diff(x) diff(y)]
simplify(det(A))  % Compute Wronskian
```

A =

$$\begin{pmatrix} \sigma_2 & \sigma_1 \\ -\frac{\sigma_2}{5} - \frac{\sqrt{34}\, e^{-\frac{t}{5}} \sin\left(\frac{\sqrt{34}\, t}{5}\right)}{5} & \frac{\sqrt{34}\, e^{-\frac{t}{5}} \cos\left(\frac{\sqrt{34}\, t}{5}\right)}{5} - \frac{\sigma_1}{5} \end{pmatrix}$$

where

$$\sigma_1 = e^{-\frac{t}{5}} \sin\left(\frac{\sqrt{34}\, t}{5}\right)$$

$$\sigma_2 = e^{-\frac{t}{5}} \cos\left(\frac{\sqrt{34}\, t}{5}\right)$$

ans =

$$\frac{\sqrt{34}\, e^{-\frac{2t}{5}}}{5}$$

$\therefore W[x, y] \neq 0 \Rightarrow x, y$ form a fundamental set

$$\therefore u(t) = c_1 e^{-t/5} \cos\left(\frac{\sqrt{34}}{5}t\right) + c_2 e^{-t/5} \sin\left(\frac{\sqrt{34}}{5}t\right)$$

$$u(0) = 2 \Rightarrow c_1 + 0 = 2, \quad c_1 = 2$$

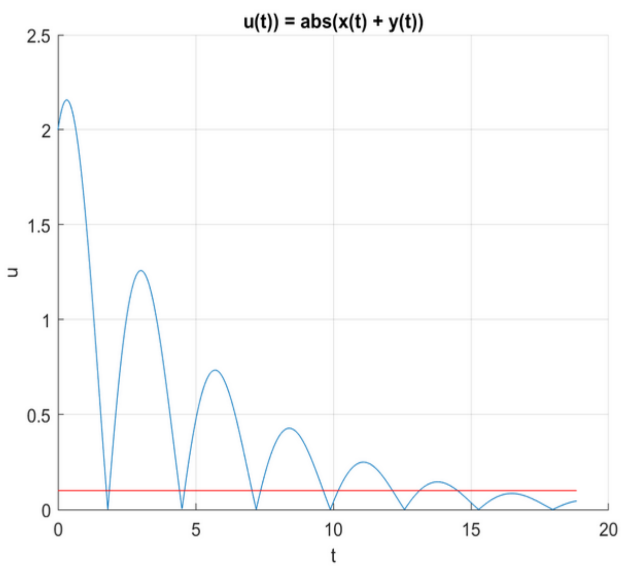
$$u'(t) = -\frac{2}{5} e^{-t/5} \cos\left(\frac{\sqrt{34}}{5}t\right) - \frac{2\sqrt{34}}{5} e^{-t/5} \sin\left(\frac{\sqrt{34}}{5}t\right)$$
$$- \frac{c_2}{5} e^{-t/5} \sin\left(\frac{\sqrt{34}}{5}t\right) + c_2 \frac{\sqrt{34}}{5} e^{-t/5} \cos\left(\frac{\sqrt{34}}{5}t\right)$$

$$u'(0) = 1 \Rightarrow -\frac{2}{5} - 0 + 0 + c_2 \frac{\sqrt{34}}{5} = 1 \Rightarrow c_2 = \frac{7}{\sqrt{34}}$$

$$\therefore u(t) = 2 e^{-t/5} \cos\left(\frac{\sqrt{34}}{5}t\right) + \frac{7}{\sqrt{34}} e^{-t/5} \sin\left(\frac{\sqrt{34}}{5}t\right)$$

(6) Use MATLAB to plot.

```
clear, clc
t = 0:0.01:6*pi;
x = 2*exp(-t/5).*cos(sqrt(34)*t/5);
y = (7/sqrt(34))*exp(-t/5).*sin(sqrt(34)*t/5);
hold on
plot(t,abs(x+y))
plot(t,0*t + 0.1, 'r')  %line at u = 0.1
grid on
xlabel 't', ylabel 'u'
title 'u(t)) = abs(x(t) + y(t))'
```



$|u(t)| = 0.1$ just before $t = 15$, and stays below 0.1 as $t \to \infty$

Use MATLAB to get precise value.

```
clear, clc
syms t x y;
x = 2*exp(-t/5)*cos(sqrt(34)*t/5);
y = (7/sqrt(34))*exp(-t/5).*sin(sqrt(34)*t/5);
vpasolve(abs(x+y)==0.1, t, 14.9)
```

ans = 14.511535629651747777356371881711

$\therefore\ T \approx 14.5115$

For $t > T$, $|u(t)| \le 0.1$

18.

(a)

$r^2 + 2r + 6 = 0$, $r = \dfrac{-2 \pm \sqrt{4-24}}{2} = -1 \pm i\sqrt{5}$

$\therefore\ y = e^{-t}\left[\cos(\sqrt{5}t) \pm i\sin(\sqrt{5}t)\right]$

Let $u(t) = e^{-t}\cos(\sqrt{5}t)$, $v(t) = e^{-t}\sin(\sqrt{5}t)$

```
clear, clc
syms t
u = exp(-t)*cos(sqrt(5)*t);
v = exp(-t)*sin(sqrt(5)*t);
A = [u v; diff(u) diff(v)]
simplify(det(A))  % Compute Wronskian
```

A =

$$\begin{pmatrix} e^{-t}\cos(\sqrt{5}\,t) & e^{-t}\sin(\sqrt{5}\,t) \\ -e^{-t}\cos(\sqrt{5}\,t) - \sqrt{5}\,e^{-t}\sin(\sqrt{5}\,t) & \sqrt{5}\,e^{-t}\cos(\sqrt{5}\,t) - e^{-t}\sin(\sqrt{5}\,t) \end{pmatrix}$$

ans = $\sqrt{5}\,e^{-2t}$

$\therefore\ W[x,y](t) \neq 0$,

$\therefore\ u, v$ form a fundamental set

MATLAB live script

$\therefore\ y(t) = c_1 e^{-t}\cos(\sqrt{5}t) + c_2 e^{-t}\sin(\sqrt{5}t)$, c_1, c_2 constants

$y(0) = 2 \Rightarrow C_1 = 2$

$y'(t) = -2e^{-t}\cos(\sqrt{5}t) - 2\sqrt{5}e^{-t}\sin(\sqrt{5}t)$

$\quad - C_2 e^{-t}\sin(\sqrt{5}t) + C_2\sqrt{5}e^{-t}\cos(\sqrt{5}t)$

$\therefore\ y'(0) = \alpha \Rightarrow -2 - 0 - 0 + C_2\sqrt{5} = \alpha$,

$$C_2 = \frac{\alpha + 2}{\sqrt{5}}$$

$$\therefore\ y(t) = \underline{2e^{-t}\cos(\sqrt{5}t) + \left(\frac{\alpha+2}{\sqrt{5}}\right)e^{-t}\sin(\sqrt{5}t)}$$

(b)

$$y(1) = \frac{2}{e}\cos(\sqrt{5}) + \left(\frac{\alpha+2}{e\sqrt{5}}\right)\sin(\sqrt{5}) = 0$$

$$\therefore\ \frac{\alpha+2}{e\sqrt{5}} = \frac{-2\cos(\sqrt{5})}{e\sin(\sqrt{5})}$$

$$\therefore\ \alpha = -2\sqrt{5}\cot(\sqrt{5}) - 2$$

$$\approx \underline{1.50878}$$

(c)

$$y(t) = 0 \Rightarrow 2e^{-t}\cos(\sqrt{5}t) + \left(\frac{\alpha+2}{\sqrt{5}}\right)e^{-t}\sin(\sqrt{5}t) = 0$$

$$\therefore\ -\left(\frac{\alpha+2}{\sqrt{5}}\right)\sin(\sqrt{5}t) = 2\cos(\sqrt{5}t),$$

$$\tan(\sqrt{5}t) = \frac{-2\sqrt{5}}{\alpha+2} < 0 \quad \text{since } \alpha \geq 0$$

Since $\tan(-\theta) = -\tan(\theta)$, then $\sqrt{5}t < 0$

Since $\tan(\theta \pm n\pi) = \tan\theta$, for $n = 0, 1, 2, \ldots$

Consider $\tan(\sqrt{5}t - \pi) = \tan\left[\sqrt{5}\left(t - \frac{\pi}{\sqrt{5}}\right)\right]$

since $-\frac{\pi}{2} < \sqrt{5}t - \pi < \frac{\pi}{2}$ yields a value of t for which $t > 0$: $\frac{1}{\sqrt{5}}\frac{\pi}{2} < t < \frac{1}{\sqrt{5}}\frac{3}{2}\pi$

$$\therefore\ \tan(\sqrt{5}t - \pi) = \frac{-2\sqrt{5}}{\alpha+2} \Rightarrow$$

$$\tan(\pi - \sqrt{5}t) = \frac{2\sqrt{5}}{\alpha+2}$$

$$\therefore\ \pi - \sqrt{5}t = \operatorname{Arctan}\left(\frac{2\sqrt{5}}{\alpha+2}\right)$$

$$\Rightarrow \sqrt{5}t = \pi - \operatorname{Arctan}\left(\frac{2\sqrt{5}}{\alpha+2}\right)$$

$$\therefore\ t = \frac{1}{\sqrt{5}}\left[\pi - \operatorname{Arctan}\left(\frac{2\sqrt{5}}{\alpha+2}\right)\right]$$

(d)

$$\lim_{\alpha\to\infty} \text{Arctan}\left(\frac{2\sqrt{5}}{\alpha+2}\right) = \text{Arctan}(0) = 0$$

$$\therefore \lim_{\alpha\to\infty} \frac{1}{\sqrt{5}}\left[\pi - \text{Arctan}\left(\frac{2\sqrt{5}}{\alpha+2}\right)\right] = \frac{\pi}{\sqrt{5}}$$

19.

$$\frac{d}{dt} e^{\lambda t}\cos(\mu t) = \lambda e^{\lambda t}\cos(\mu t) - \mu e^{\lambda t}\sin(\mu t)$$

$$\frac{d}{dt} e^{\lambda t}\sin(\mu t) = \lambda e^{\lambda t}\sin(\mu t) + \mu e^{\lambda t}\cos(\mu t)$$

$$\therefore W\{e^{\lambda t}\cos(\mu t), e^{\lambda t}\sin(\mu t)\} =$$

$$\begin{vmatrix} e^{\lambda t}\cos(\mu t) & e^{\lambda t}\sin(\mu t) \\ \lambda e^{\lambda t}\cos(\mu t) - \mu e^{\lambda t}\sin(\mu t) & \lambda e^{\lambda t}\sin(\mu t) + \mu e^{\lambda t}\cos(\mu t) \end{vmatrix}$$

$$= \lambda e^{2\lambda t}\cos(\mu t)\sin(\mu t) + \mu e^{2\lambda t}\cos^2(\mu t)$$

cancel

$$- \lambda e^{2\lambda t}\sin(\mu t)\cos(\mu t) + \mu e^{2\lambda t}\sin^2(\mu t)$$

$$= \mu e^{2\lambda t}\left[\cos^2(\mu t) + \sin^2(\mu t)\right] = \mu e^{2\lambda t}$$

20.

(a) $y_1''(t) = \frac{d}{dt}\left(\frac{d}{dt}\cos t\right) = \frac{d}{dt}(-\sin t) = -\cos t$

$\therefore y_1'' + y_1 = (-\cos t) + (\cos t) = 0$

$y_2''(t) = \frac{d}{dt}\left(\frac{d}{dt}\sin t\right) = \frac{d}{dt}(\cos t) = -\sin t$

$\therefore y_2'' + y_2 = (-\sin t) + (\sin t) = 0$

$W[\cos t, \sin t] = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = \cos^2 t + \sin^2 t = 1 \neq 0.$

(b)

$\frac{d}{dt}(e^{it}) = ie^{it} \quad \therefore \frac{d^2}{dt^2}(e^{it}) = \frac{d}{dt}(ie^{it}) = i^2 e^{it} = -e^{it}$

$\therefore y'' + y = (-e^{it}) + (e^{it}) = 0$

$\therefore$ By Theorem 3.2.4 on page 114 of text,

There must be constants c_1 and c_2 such that

$$e^{it} = c_1 \cos t + c_2 \sin t$$

(c)

For $t=0$, $e^{it} = e^{i(0)} = e^0 = 1$

$\therefore 1 = c_1 \cos(0) + c_2 \sin(0) = c_1$. $\therefore$ $\underline{c_1 = 1}$

(d)

$$\frac{d}{dt}(e^{it}) = ie^{it} = \frac{d}{dt}(c_1 \cos t + c_2 \sin t)$$

$$= -c_1 \sin t + c_2 \cos t$$

$\therefore$ For $t=0$, $ie^{i(0)} = ie^0 = i = -c_1 \sin(0) + c_2 \cos(0) = c_2$

$\therefore$ $c_2 = i$

$\therefore$ $\underline{e^{it} = \cos t + i \sin t}$

21.

$$e^{it} = \cos(t) + i \sin(t)$$
$$e^{-it} = \cos(t) - i \sin(t)$$

$\therefore$ $e^{it} + e^{-it} = 2\cos(t) \Rightarrow \dfrac{e^{it} + e^{-it}}{2} = \underline{\cos(t)}$

$$e^{it} - e^{-it} = 2i\sin(t) \Rightarrow \frac{e^{it} - e^{-it}}{2i} = \underline{\sin(t)}$$

22.

Let $r_1 = a + bi$, $r_2 = c + di$

$\therefore r_1 + r_2 = (a+c) + (b+d)i$

$$\therefore e^{r_1 t} e^{r_2 t} = e^{at}[\cos(bt) + i\sin(bt)]e^{ct}[\cos(dt) + i\sin(dt)]$$

$$= e^{at}e^{ct}[\cos(bt)\cos(dt) + i^2\sin(bt)\sin(dt)$$

$$+ i\sin(bt)\cos(dt) + i\cos(bt)\sin(dt)]$$

$$= e^{(a+c)t}[\cos(bt)\cos(dt) - \sin(bt)\sin(dt)$$

$$+ i(\sin(bt)\cos(dt) + \cos(bt)\sin(dt))]$$

$$= e^{(a+c)t}[\cos[(b+d)t] + i\sin[(b+d)t]]$$

$$= e^{[(a+c)+(b+d)i]t}$$

$$= \underline{e^{(r_1+r_2)t}}$$

23.

From $ay'' + by' + cy = 0$, we get $ar^2 + br + c = 0$,

where r is from $y = e^{rt}$.

$$\therefore\ r = \lambda \pm i\mu = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b}{2a} \pm i\frac{\sqrt{4ac - b^2}}{2a}$$

since $b^2 - 4ac < 0$.

$$\therefore\ \lambda = \frac{-b}{2a} \text{ and } \mu = \frac{\sqrt{4ac - b^2}}{2a}$$

$$u(t) = e^{\lambda t}\cos(\mu t)$$

$$\therefore\ u'(t) = \lambda e^{\lambda t}\cos(\mu t) - \mu e^{\lambda t}\sin(\mu t)$$

$$= \left(-\frac{b}{2a}\right)e^{\lambda t}\cos(\mu t) - \left(\frac{\sqrt{4ac - b^2}}{2a}\right)e^{\lambda t}\sin(\mu t)$$

$$u''(t) = \lambda^2 e^{\lambda t}\cos(\mu t) - \lambda\mu e^{\lambda t}\sin(\mu t)$$

$$- \lambda\mu e^{\lambda t}\sin(\mu t) - \mu^2 e^{\lambda t}\cos(\mu t)$$

$$= \lambda^2 e^{\lambda t}\cos(\mu t) - 2\lambda\mu e^{\lambda t}\sin(\mu t) - \mu^2 e^{\lambda t}\cos(\mu t)$$

$$= \left(\frac{b^2}{4a^2}\right) e^{\lambda t}\cos(\mu t) + \left(\frac{b\sqrt{4ac-b^2}}{2a^2}\right) e^{\lambda t}\sin(\mu t) - \left(\frac{4ac-b^2}{4a^2}\right) e^{\lambda t}\cos(\mu t)$$

$\therefore au'' + bu' + cu =$

$$\left(\frac{b^2}{4a}\right) e^{\lambda t}\cos(\mu t) + \left(\frac{b\sqrt{4ac-b^2}}{2a}\right) e^{\lambda t}\sin(\mu t) - \left(\frac{4ac-b^2}{4a}\right) e^{\lambda t}\cos(\mu t)$$

cancel

$$- \frac{b^2}{2a} e^{\lambda t}\cos(\mu t) - \frac{b\sqrt{4ac-b^2}}{2a} e^{\lambda t}\sin(\mu t)$$

$$+ c e^{\lambda t}\cos(\mu t)$$

$$= e^{\lambda t}\cos(\mu t)\left[\frac{b^2}{4a} + \frac{b^2-4ac}{4a} - \frac{b^2}{2a} + c\right]$$

$$= e^{\lambda t}\cos(\mu t)\left[\frac{b^2}{4a} + \frac{b^2-4ac}{4a} - \frac{2b^2}{4a} + \frac{4ac}{4a}\right]$$

$$= e^{\lambda t}\cos(\mu t)\,[0] = 0$$

$\therefore au'' + bu' + cu = 0 \Rightarrow$ <u>$u(t)$ is a solution</u>

$$v(t) = e^{\lambda t}\sin(\mu t)$$

$$v'(t) = \lambda e^{\lambda t}\sin(\mu t) + \mu e^{\lambda t}\cos(\mu t)$$

$$= \left(-\frac{b}{2a}\right) e^{\lambda t}\sin(\mu t) + \left(\frac{\sqrt{4ac-b^2}}{2a}\right) e^{\lambda t}\cos(\mu t)$$

$$V''(t) = \lambda^2 e^{\lambda t}\sin(\mu t) + \lambda\mu e^{\lambda t}\cos(\mu t)$$

$$+ \lambda\mu e^{\lambda t}\cos(\mu t) - \mu^2 e^{\lambda t}\sin(\mu t)$$

$$= \left(\frac{b^2}{4a^2}\right)e^{\lambda t}\sin(\mu t) - \left(\frac{b\sqrt{4ac-b^2}}{2a^2}\right)e^{\lambda t}\cos(\mu t) - \left(\frac{4ac-b^2}{4a^2}\right)e^{\lambda t}\sin(\mu t)$$

$\therefore\ aV'' + bV' + cV =$

$$\left(\frac{b^2}{4a}\right)e^{\lambda t}\sin(\mu t) - \left(\frac{b\sqrt{4ac-b^2}}{2a}\right)e^{\lambda t}\cos(\mu t) - \left(\frac{4ac-b^2}{4a}\right)e^{\lambda t}\sin(\mu t)$$

cancel

$$-\left(\frac{b^2}{2a}\right)e^{\lambda t}\sin(\mu t) + \left(\frac{b\sqrt{4ac-b^2}}{2a}\right)e^{\lambda t}\cos(\mu t)$$

$$+ c e^{\lambda t}\sin(\mu t)$$

$$= e^{\lambda t}\sin(\mu t)\left[\frac{b^2}{4a} - \frac{4ac-b^2}{4a} - \frac{b^2}{2a} + c\right]$$

$$= e^{\lambda t}\sin(\mu t)\left[\frac{b^2}{4a} + \frac{b^2-4ac}{4a} - \frac{2b^2}{4a} + \frac{4ac}{4a}\right]$$

$$= e^{\lambda t}\sin(\mu t)\left[\,0\,\right] = 0$$

$\therefore\ aV'' + bV' + cV = 0 \Rightarrow$ <u>$V(t)$ is a solution</u>

24.

Assume $p(t)$, $q(t)$ are everywhere continuous.

Let t_1 and t_2 be consecutive zeros of $y_1(t)$, such that $t_1 < t_2$. $\therefore$ $y_1(t_1) = 0$, $y_1(t_2) = 0$, and for $t_1 < t < t_2$, $y_1(t) \neq 0$.

From Purcell, "Calculus with Analytic Geometry", p. 173,

7.8.1 Rolle's theorem. *If a function f is continuous on the closed interval [a, b] and is differentiable on the open interval (a, b), and if $f(a) = f(b)$, then there exists a number z in (a, b) such that $f'(z) = 0$.*

Suppose there are no zeros of y_2 on $[t_1, t_2]$

$\therefore$ Consider $\frac{y_1(t)}{y_2(t)}$ on $[t_1, t_2]$. This function is continuous on $[t_1, t_2]$, and $\frac{y_1(t_1)}{y_2(t_1)} = 0$, $\frac{y_1(t_2)}{y_2(t_2)} = 0$

$\therefore$ By Rolle's Theorem, there is a z, $t_1 < z < t_2$, s.t. $\frac{y_2(z)y_1'(z) - y_1(z)y_2'(z)}{y_2(z)^2} = 0$, or

$y_2(z)\,y_1'(z) - y_1(z)\,y_2'(z) = 0$

$\therefore$ There is a z s.t. the Wronskian of y_1, y_2 is 0.

By Abel's Theorem (Theorem 3.2.7 of text), $W[y_1, y_2](t) = 0$ for all of t where $p(t)$, $q(t)$ are continuous, and this contradicts $y_1(t), y_2(t)$ being a fundamental set.

$\therefore$ This is at least one zero of $y_2(t)$ between t_1 and t_2.

Suppose there is more than one zero of $y_2(t)$ between t_1 and t_2. Call them s_1 and s_2.

By similar reasoning to the above, there must be a zero of $y_1(t)$ between s_1 and s_2, contradicting $y_1(t) \neq 0$ for $t_1 < t < t_2$.

$\therefore$ There is at most one zero of $y_2(t)$ between consecutive zeros of $y_1(t)$.

The above argument assumes that "consecutive zeros" is a valid concept for the functions $y_1(t), y_2(t)$, i.e., between 2 zeros, you are assured no other zero exists between the two zeros.

25.

(a) $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$ Think of $y[x(t)] = y(t)$ and use chain rule.

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dx} \cdot \frac{dx}{dt}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right) \cdot \frac{dx}{dt} + \frac{dy}{dx} \cdot \frac{d}{dt}\left(\frac{dx}{dt}\right)$$

$$= \left[\frac{d}{dx}\left(\frac{dy}{dx}\right) \cdot \frac{dx}{dt}\right] \cdot \frac{dx}{dt} + \frac{dy}{dx} \cdot \frac{d^2x}{dt^2}$$

$$\therefore \frac{d^2y}{dt^2} = \frac{d^2y}{dx^2} \cdot \left(\frac{dx}{dt}\right)^2 + \frac{dy}{dx} \cdot \frac{d^2x}{dt^2}$$

$$\therefore \text{ if } x = \ln(t), \quad \frac{dx}{dt} = \frac{1}{t}, \quad \frac{d^2x}{dt^2} = -\frac{1}{t^2}$$

$$\therefore \frac{dy}{dt} = \frac{dy}{dx}\left(\frac{1}{t}\right)$$

$$\frac{d^2y}{dt^2} = \frac{d^2y}{dx^2}\left(\frac{1}{t^2}\right) + \frac{dy}{dx}\left(-\frac{1}{t^2}\right)$$

(6)

$$t^2\frac{d^2y}{dt^2} + \alpha t\frac{dy}{dt} + \beta y =$$

$$t^2\left[\frac{d^2y}{dx^2}\cdot\left(\frac{1}{t^2}\right) + \frac{dy}{dx}\left(-\frac{1}{t^2}\right)\right] + \alpha t\left[\frac{dy}{dx}\left(\frac{1}{t}\right)\right] + \beta y$$

$$= \frac{d^2y}{dx^2} - \frac{dy}{dx} + \alpha\frac{dy}{dx} + \beta y$$

$$= \frac{d^2y}{dx^2} + (\alpha-1)\frac{dy}{dx} + \beta y = 0$$

Observe that differential equation (34) has constant coefficients. If $y_1(x)$ and $y_2(x)$ form a fundamental set of solutions of equation (34), then $y_1(\ln t)$ and $y_2(\ln t)$ form a fundamental set of solutions of equation (33).

Pf: If $W[y_1, y_2](x) = y_1(x)\cdot y_2'(x) - y_1'(x)\cdot y_2(x) \neq 0$

then
$$\begin{vmatrix} y_1[\ln(t)] & y_2[\ln(t)] \\ \frac{d}{dt}y_1[\ln(t)] & \frac{d}{dt}y_2[\ln(t)] \end{vmatrix}$$

$$= \begin{vmatrix} y_1[\ln(t)] & y_2[\ln(t)] \\ \frac{1}{t}y_1'[\ln(t)] & \frac{1}{t}y_2'[\ln(t)] \end{vmatrix}$$

$$= \frac{1}{t}\left[y_1[\ln(t)]\, y_2'[\ln(t)] - y_1'[\ln(t)]\, y_2[\ln(t)] \right]$$

$\neq 0$, for $t > 0$

26.

Using the above method, with $\alpha = 1$ and $x = \ln(t)$,

$y''(x) + y(x) = 0 \quad (\alpha - 1 = 0)$

$\therefore r^2 + 1 = 0$, $r = \pm i$, $y(x) = \cos(x) \pm i\sin(x)$,

$W[\cos x, \sin x] = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1 \neq 0.$

$\therefore \cos(x), \sin(x)$ a fundamental set.

$\therefore \cos(\ln(t)), \sin(\ln(t))$ a fundamental set.

$\therefore \underline{y(t) = c_1 \cos[\ln(t)] + c_2 \sin[\ln(t)]}$, c_1, c_2 constants

27.

$\alpha = 4$, $\therefore \alpha - 1 = 3$ Let $x = \ln(t)$

$\therefore y''(x) + 3y'(x) + 2y(x) = 0$

$\therefore\ r^2 + 3r + 2 = 0, \quad r = -1, -2$

$\therefore\ y(x) = e^{-x}, e^{-2x} \qquad W[e^{-x}, e^{-2x}] = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix} = -e^{-3x} \neq 0$

$\therefore\ e^{-x}, e^{-2x}$ a fundamental set.

$\therefore\ e^{-\ln t}, e^{-2\ln t}$ a fundamental set, or $\frac{1}{t}, \frac{1}{t^2}$

$\therefore\ y(t) = \dfrac{c_1}{t} + \dfrac{c_2}{t^2}$, c_1, c_2 constants

28.

$\alpha = -4, \quad \alpha - 1 = -5 \qquad$ Let $x = \ln(t)$

$\therefore\ y''(x) - 5y'(x) - 6y(x) = 0$

$\therefore\ r^2 - 5r - 6 = 0 = (r-6)(r+1). \quad \therefore\ r = 6, -1$

$\therefore\ y(x) = e^{-x}, e^{6x} \quad W[e^{-x}, e^{6x}] = \begin{vmatrix} e^{-x} & e^{6x} \\ -e^{-x} & 6e^{6x} \end{vmatrix} = 7e^{5x} \neq 0$

$\therefore\ e^{-x}, e^{6x}$ a fundamental set

$\therefore\ e^{-\ln(t)}, e^{6\ln(t)} = \frac{1}{t}, t^6$ a fundamental set

$\therefore\ y(t) = \dfrac{c_1}{t} + c_2 t^6$, c_1, c_2 constants

29.

$\alpha = -4,\ \alpha - 1 = -5$ Let $x = \ln(t)$

$\therefore\ y''(x) - 5y'(x) + 6y(x) = 0$

$\therefore\ r^2 - 5r + 6 = 0 = (r-2)(r-3),\ r = 2, 3$

$\therefore\ y(x) = e^{2x}, e^{3x}$ $\quad W[e^{2x}, e^{3x}] = \begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{vmatrix} = e^{5x} \neq 0$

$\therefore\ e^{2x}, e^{3x}$ a fundamental set

$\therefore\ e^{2\ln(t)}, e^{3\ln(t)} = t^2, t^3$ a fundamental set

$\therefore\ y(t) = c_1 t^2 + c_2 t^3$, c_1, c_2 constants

30.

$\alpha = 3,\ \alpha - 1 = 2$ Let $x = \ln(t)$,

$\therefore\ y''(x) + 2y'(x) - 3y(x) = 0$,

$\therefore\ r^2 + 2r - 3 = 0 = (r+3)(r-1),\ r = 1, -3$

$\therefore\ y(x) = e^{x}, e^{-3x}$ $\quad W[e^{x}, e^{-3x}] = \begin{vmatrix} e^{x} & e^{-3x} \\ e^{x} & -3e^{-3x} \end{vmatrix} = -4e^{-2x} \neq 0$

$\therefore\ e^{x}, e^{-3x}$ a fundamental set

$\therefore\ e^{\ln(t)}, e^{-3\ln(t)} = t, \frac{1}{t^3}$ a fundamental set

$\therefore\ y(t) = c_1 t + \dfrac{c_2}{t^3}$, c_1, c_2 constants

31.

$\alpha = 7,\ \alpha - 1 = 6$ Let $x = \ln(t)$

$\therefore\ y''(x) + 6y'(x) + 10y(x) = 0$

$\therefore\ r^2 + 6r + 10 = 0,\quad r = \dfrac{-6 \pm \sqrt{36-40}}{2} = -3 \pm i$

$\therefore\ y(x) = e^{-3x}\left(\cos(x) \pm i\sin(x)\right)$

Consider $e^{-3x}\cos(x)$, $e^{-3x}\sin(x)$ Use MATLAB

```
clear, clc
syms x
u = exp(-3*x)*cos(x);
v = exp(-3*x)*sin(x);
A = [u v; diff(u) diff(v)]
simplify(det(A))  % Compute Wronskian
```

A =

$$\begin{pmatrix} e^{-3x}\cos(x) & e^{-3x}\sin(x) \\ -3e^{-3x}\cos(x) - e^{-3x}\sin(x) & e^{-3x}\cos(x) - 3e^{-3x}\sin(x) \end{pmatrix}$$

ans = e^{-6x}

$\therefore\ W[u, v] \neq 0$

$\therefore\ u, v$ a fundamental set

$\therefore\ y(t) = e^{-3\ln(t)}\cos[\ln(t)],\ e^{-3\ln(t)}\sin[\ln(t)]$

$= \dfrac{1}{t^3}\cos[\ln(t)],\ \dfrac{1}{t^3}\sin[\ln(t)]$

a fundamental set.

$$\therefore\ y(t) = C_1 t^{-3}\cos[\ln(t)] + C_2 t^{-3}\sin[\ln(t)]$$

C_1, C_2 constants

32.

(a)

Consider $y(x(t))$

By the chain rule, $\frac{d}{dt} y(x(t)) = \frac{dy}{dx}\cdot\frac{dx}{dt}$

$$\therefore\ \frac{d^2}{dt^2} y(x(t)) = \frac{d}{dt}\left[\frac{dy}{dx}\cdot\frac{dx}{dt}\right]$$

$$= \frac{d}{dt}\left(\frac{dy}{dx}\right)\cdot\frac{dx}{dt} + \frac{dy}{dx}\cdot\frac{d}{dt}\left(\frac{dx}{dt}\right)$$

$$= \left[\frac{d}{dx}\left(\frac{dy}{dx}\right)\cdot\frac{dx}{dt}\right]\cdot\frac{dx}{dt} + \frac{dy}{dx}\cdot\frac{d^2x}{dt^2}$$

$$= \left[\frac{d^2y}{dx^2}\cdot\frac{dx}{dt}\right]\cdot\frac{dx}{dt} + \frac{d^2x}{dt^2}\cdot\frac{dy}{dx}$$

$$= \left(\frac{dx}{dt}\right)^2\cdot\frac{d^2y}{dx^2} + \frac{d^2x}{dt^2}\cdot\frac{dy}{dx}$$

(b)

(32) is $\frac{d^2}{dt^2} y(t) + p(t) \frac{d}{dt} y(t) + q(t) y(t) = 0$

or, $\frac{d^2}{dt^2} y(x(t)) + p(t) \frac{d}{dt} y(x(t)) + q(t) y(x(t)) = 0$

Using (a) for substitution,

$$\left[\left(\frac{dx}{dt} \right)^2 \frac{d^2 y}{dx^2} + \frac{d^2 x}{dt^2} \frac{dy}{dx} \right] + p(t) \left[\frac{dy}{dx} \frac{dx}{dt} \right] + q(t) y(x(t)) = 0$$

Collecting terms in $\frac{d^2 y}{dx^2}$ and $\frac{dy}{dx}$,

$$\left(\frac{dx}{dt} \right)^2 \frac{d^2 y}{dx^2} + \left[\frac{d^2 x}{dt^2} + p(t) \frac{dx}{dt} \right] \frac{dy}{dx} + q(t) y(t) = 0$$

Note: The above assumes $y(x(t)) \equiv y(t)$; i.e., that there is an intermediate function $x(t)$ s.t. $y(t)$ can be expressed as $y(x(t))$. This can always be done using $x(t) = t$, but other $x(t)$ functions may be possible.

(c)

Since $q(t) > 0$, Then divide (35) by $q(t)$

$$\therefore \frac{\left(\frac{dx}{dt}\right)^2}{q(t)} \frac{d^2y}{dx^2} + \frac{\left(\frac{d^2x}{dt^2} + p(t)\frac{dx}{dt}\right)}{q(t)} \frac{dy}{dx} + y = 0$$

Choose $\frac{\left(\frac{dx}{dt}\right)^2}{q(t)} = 1$. Note $\frac{d\,x(t)}{dt} = \frac{d}{dt}\int_{t_0}^{t} q(s)^{\frac{1}{2}}ds = q(t)^{\frac{1}{2}}$

If $\frac{\left(\frac{dx}{dt}\right)^2}{q(t)} = K \neq 1$, then first divide equation (35) by K.

(d)

The coefficient of $\frac{dy}{dx}$ is: $\frac{d^2x}{dt^2} + p(t)\frac{dx}{dt}$

We want the ratio with $q(t)$ to be a constant

$$\therefore \frac{\frac{d^2x}{dt^2} + p(t)\frac{dx}{dt}}{q(t)} = \text{constant}$$

With $\frac{dx}{dt} = q(t)^{\frac{1}{2}}$ from (c), then $\frac{d^2x}{dt^2} = \frac{1}{2}q(t)^{-1/2}q'(t)$

$$\therefore \frac{\frac{d^2x}{dt^2} + p(t)\frac{dx}{dt}}{q(t)} = \frac{\left(\frac{q'(t)}{2q(t)^{1/2}}\right) + p(t)\left(q(t)^{\frac{1}{2}}\right)}{q(t)}$$

$$= \frac{\frac{q'(t) + 2p(t)q(t)}{2q(t)^{1/2}}}{q(t)} = \frac{q'(t) + 2p(t)q(t)}{2q(t)^{3/2}}$$

$$\therefore \frac{q' + 2pq}{2q^{3/2}} = \text{constant} \iff \frac{q' + 2pq}{q^{3/2}} = \text{constant}$$

(e)

If $q(t) < 0$, then choose

$$X(t) = \int_{t_0}^{t} (-q(s))^{\frac{1}{2}}\, ds, \quad \text{so } \frac{dx}{dt} = [-q(t)]^{1/2}$$

$$\therefore \frac{d^2x}{dt^2} = \frac{1}{2}[-q(t)]^{-\frac{1}{2}} \cdot -q'(t) = -\frac{q'(t)}{2[-q(t)]^{1/2}}$$

$\therefore$ Coefficient of $\frac{dy}{dx}$ to be a constant becomes

$$\frac{\frac{-q'}{2(-q)^{1/2}} + p(-q)^{\frac{1}{2}}}{q} = \frac{-q'(t) - 2p(t)q(t)}{2(-q(t))^{3/2}}$$

More simply, $\underline{\dfrac{q' + 2pq}{(-q)^{3/2}}}$ must be a constant

33.

Note $e^{-t^2} > 0$. $\therefore$ Choose $x(t) = \int (e^{-t^2})^{\frac{1}{2}} = \int e^{-\frac{t^2}{2}}$

$$\therefore \frac{\left(\frac{dx}{dt}\right)^2}{q(t)} = \frac{\left(e^{-\frac{t^2}{2}}\right)^2}{e^{-t^2}} = \frac{e^{-t^2}}{e^{-t^2}} = 1$$

And $$\frac{q' + 2pq}{2q^{3/2}} = \frac{-2te^{-t^2} + 2(t)(e^{-t^2})}{2(e^{-t^2})^{3/2}} = 0$$

$\therefore y''(x) + y(x) = 0$, so $r^2+1=0$, $r = \pm i$

$\therefore y(x) = \cos(x) \pm i\sin(x)$.

$W[\cos x, \sin x] = 1 \Rightarrow \cos(x), \sin(x)$ a fundamental set.

$\therefore \underline{y = C_1 \cos(x) + C_2 \sin(x)}$, $x = \int e^{-t^2/2}$

C_1, C_2 constants

34.

Since $t=0$ prevents $q(t)=t^2$ from being 0, the above methods can't be employed.

If you restrict t: $0<t<\infty$, $-\infty<t<0$,

choose $x(t) = \int (t^2)^{\frac{1}{2}} = \frac{t^2}{2}$, where $q(t)=t^2>0$

$\therefore \frac{dx}{dt} = t$, $\frac{\left(\frac{dx}{dt}\right)^2}{q(t)} = \frac{t^2}{t^2} = 1$

And $\frac{q' + 2pq}{2q^{3/2}} = \frac{2t + 2(3t)(t^2)}{2(t^2)^{3/2}} = \frac{2t+6t^3}{2t^3}$

This is <u>not</u> a constant

$\therefore$ Not possible with methods above to convert to equation with constant coefficients.

35.

Since $t\neq 0$, divide by t. $\therefore$ y is a solution $\Leftrightarrow$ it is a solution to $y'' + \frac{(t^2-1)}{t} y' + t^2 y = 0$

$\therefore q(t) = t^2 > 0$ on $0<t<\infty$

Let $x(t) = \int (t^2)^{\frac{1}{2}} = \frac{t^2}{2}$, $q'(t) = 2t$

Note $\frac{\left(\frac{dx}{dt}\right)^2}{q(t)} = \frac{(t)^2}{t^2} = 1$

$\therefore \frac{q' + 2pq}{2q^{3/2}} = \frac{2t + 2\left(\frac{t^2-1}{t}\right)(t^2)}{2(t^2)^{3/2}} = \frac{2t^2 + 2t^4 - 2t^2}{2t^4} = 1$

$\therefore$ This is a constant, so can convert to an equation with constant coefficients:

$y''(x) + y'(x) + y(x) = 0$

$\therefore r^2 + r + 1 = 0$, $r = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

$\therefore$ Let $u = e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}}{2} x\right)$, $v = e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}}{2} x\right)$

Using MATLAB,

```
clear, clc
syms x
u = exp(-x/2)*cos(sqrt(3)*x/2);
v = exp(-x/2)*sin(sqrt(3)*x/2);
A = [u v; diff(u) diff(v)]
simplify(det(A))  % Compute Wronskian
```

A =

$$\begin{pmatrix} \sigma_2 & \sigma_1 \\ -\frac{\sigma_2}{2} - \frac{\sqrt{3}\, e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\, x}{2}\right)}{2} & \frac{\sqrt{3}\, e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right)}{2} - \frac{\sigma_1}{2} \end{pmatrix}$$

where

$$\sigma_1 = e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\, x}{2}\right)$$

$$\sigma_2 = e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right)$$

ans =

$$\frac{\sqrt{3}\, e^{-x}}{2}$$

$\therefore W[u,v] \neq 0$, so u, v form a fundamental set.

$$\therefore\ y = C_1 e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}}{2}x\right) + C_2 e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}}{2}x\right)$$

Since $x = \frac{t^2}{2}$, then

$$\boxed{y = c_1 e^{-t^2/4}\cos\left(\frac{\sqrt{3}}{4}t^2\right) + c_2 e^{-t^2/4}\sin\left(\frac{\sqrt{3}}{4}t^2\right)}$$

C_1, C_2 are constants

36.

$$q(t) = -e^{-t^2} < 0, \quad q'(t) = 2te^{-t^2}$$

Let $x(t) = \int (-q)^{\frac{1}{2}} = \int_{t_0}^{t} \left(e^{-s^2}\right)^{\frac{1}{2}} ds$

$$\therefore\ \frac{dx}{dt} = e^{-t^2/2} = (-q(t))^{\frac{1}{2}}$$

From #32(e), look at coefficient of $y'(x)$:

$$\frac{-q' - 2pq}{2(-q)^{3/2}} = \frac{-\left(2te^{-t^2}\right) - 2(t)\left(-e^{-t^2}\right)}{2\left(e^{-t^2}\right)^{3/2}} = 0$$

This is a constant.

$$\therefore\ y''(x) + y(x) = 0, \quad r^2 + 1 = 0, \quad r = \pm i$$

$\therefore\ y = \cos(x), \sin(x)$ which are a fundamental set.

$\therefore\ y = c_1 \cos(x) + c_2 \sin(x)$

Since $x = \int (e^{-t^2})^{\frac{1}{2}} dt$,

$$\therefore\ \boxed{y(t) = c_1 \cos\left(\int (e^{-t^2})^{\frac{1}{2}} dt\right) + c_2 \sin\left(\int (e^{-t^2})^{\frac{1}{2}} dt\right)}$$

c_1, c_2 are constants

3.4 Repeated Roots; Reduction of Order

Note Title 7/30/2018

1. **1.** $y'' - 2y' + y = 0$

$r^2 - 2r + 1 = 0 = (r-1)^2, \quad r = 1, 1$

$\therefore \; y = c_1 e^t + c_2 t e^t$

2.

$9r^2 + 6r + r = 0 = (3r+1)^2, \quad r = -\frac{1}{3}, -\frac{1}{3}$

$\therefore \; y = c_1 e^{-t/3} + c_2 t e^{-t/3}$

3.

$4r^2 - 4r - 3 = 0 = (2r-3)(2r+1), \quad r = \frac{3}{2}, -\frac{1}{2}$

$\therefore \; y = c_1 e^{\frac{3t}{2}} + c_2 e^{-\frac{t}{2}}$

4.

$r^2 - 2r + 10 = 0, \quad r = \frac{2 \pm \sqrt{4-40}}{2} = 1 \pm 3i$

$\therefore\ y = c_1 e^{t}\cos(3t) + c_2 e^{t}\sin(3t)$

Note: $W[e^{t}\cos(3t),\ e^{t}\sin(3t)] = 3e^{2t} \neq 0$

5.

$r^2 - 6r + 9 = 0 = (r-3)^2,\quad r = 3, 3$

$\therefore\ y = c_1 e^{3t} + c_2 t e^{3t}$

6.

$4r^2 + 17r + 4 = 0 = (4r+1)(r+4),\quad r = -\frac{1}{4}, -4$

$\therefore\ y = c_1 e^{-t/4} + c_2 e^{-4t}$

7.

$16r^2 + 24r + 9 = 0,\quad r = \dfrac{-24 \pm \sqrt{576 - 4(144)}}{32} = -\frac{3}{4}, -\frac{3}{4}$

$\therefore\ y = c_1 e^{-3t/4} + c_2 t e^{-3t/4}$

8.

$2r^2 + 2r + 1 = 0,\quad r = \dfrac{-2 \pm \sqrt{4 - 8}}{4} = -\frac{1}{2} \pm \frac{1}{2} i$

$$\therefore\ y = c_1 e^{-t/2}\cos\left(\frac{t}{2}\right) + c_2 e^{-t/2}\sin\left(\frac{t}{2}\right)$$

Note: $W\left[e^{at}\cos(bt),\ e^{at}\sin(bt)\right] = b e^{2at} \neq 0$ for $b \neq 0$

9.

$$9r^2 - 12r + 4 = 0 = (3r-2)^2, \quad r = \frac{2}{3}, \frac{2}{3}$$

$$\therefore\ y = c_1 e^{2t/3} + c_2 t e^{2t/3}$$

$$y(0) = c_1 = 2$$

$$y'(t) = 2\left(\frac{2}{3}\right)e^{2t/3} + c_2 e^{2t/3} + c_2\left(\frac{2}{3}\right)t e^{2t/3}$$

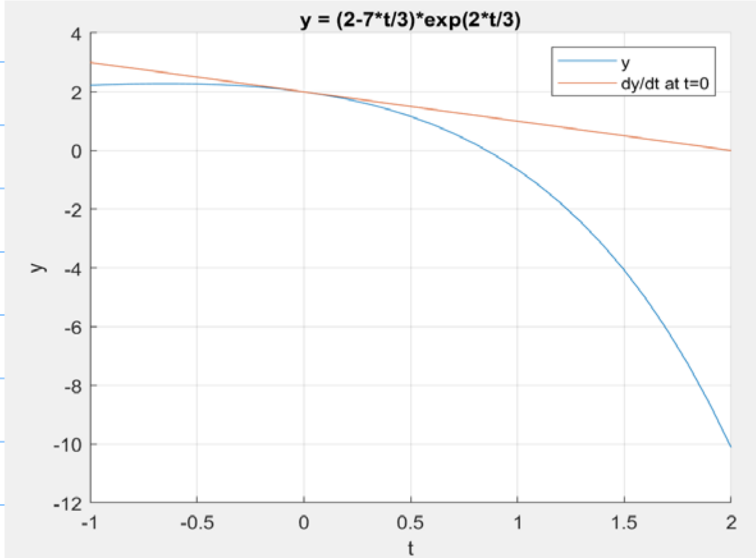
$$\therefore\ y'(0) = \frac{4}{3} + c_2 = -1 \Rightarrow c_2 = -\frac{7}{3}$$

$$\therefore\ y(t) = 2e^{2t/3} - \frac{7}{3}te^{2t/3}$$

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} e^{2t/3}\left(2 - \frac{7}{3}t\right) = -\infty$$

Using MATLAB to plot,

```
clear,clc;
t = -1:0.1:2;
eqn = (2-7*t/3).*exp(2*t/3);
hold on
plot(t, eqn)
plot(t, -t+2)
grid on
xlabel 't', ylabel 'y'
title 'y = (2-7*t/3)*exp(2*t/3)'
legend('y', 'dy/dt at t=0')
```



10.

$$r^2 - 6r + 9 = 0 = (r-3)^2, \quad r = 3, 3$$

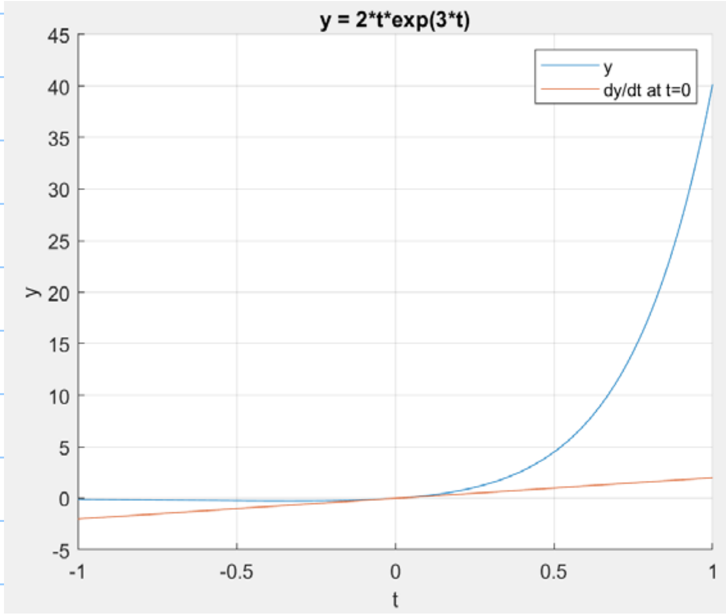
$$\therefore y = c_1 e^{3t} + c_2 t e^{3t}, \quad y' = 3c_1 e^{3t} + c_2 e^{3t} + 3c_2 t e^{3t}$$

$$y(0) = 0 \Rightarrow c_1 = 0 \qquad y'(0) = 2 \Rightarrow c_2 = 2$$

$$\therefore \underline{y(t) = 2te^{3t}} \qquad \therefore \lim_{t \to \infty} y(t) = \underline{+\infty}$$

Using MATLAB

```
clear,clc;
t = -1:0.01:1;
eqn = 2*t.*exp(3*t);
hold on
plot(t, eqn)
plot(t, 2*t)
grid on
xlabel 't', ylabel 'y'
title 'y = 2*t*exp(3*t)'
legend('y', 'dy/dt at t=0')
```



11.

$$r^2 + 4r + 4 = 0 = (r+2)^2, \quad r = -2, -2$$

$$\therefore y = c_1 e^{-2t} + c_2 t e^{-2t}, \quad y' = -2c_1 e^{-2t} + c_2 e^{-2t} - 2c_2 t e^{-2t}$$

$$y(-1) = c_1 e^2 - c_2 e^2 = 2, \qquad c_1 - c_2 = \frac{2}{e^2}$$

$$y'(-1) = -2c_1 e^2 + c_2 e^2 + 2c_2 e^2 = 1, \quad -2c_1 + 3c_2 = \frac{1}{e^2}$$

$$c_1 = \frac{7}{e^2}, \quad c_2 = \frac{5}{e^2}$$

$$\therefore y = \frac{7}{e^2} e^{-2t} + \frac{5}{e^2} t e^{-2t}$$

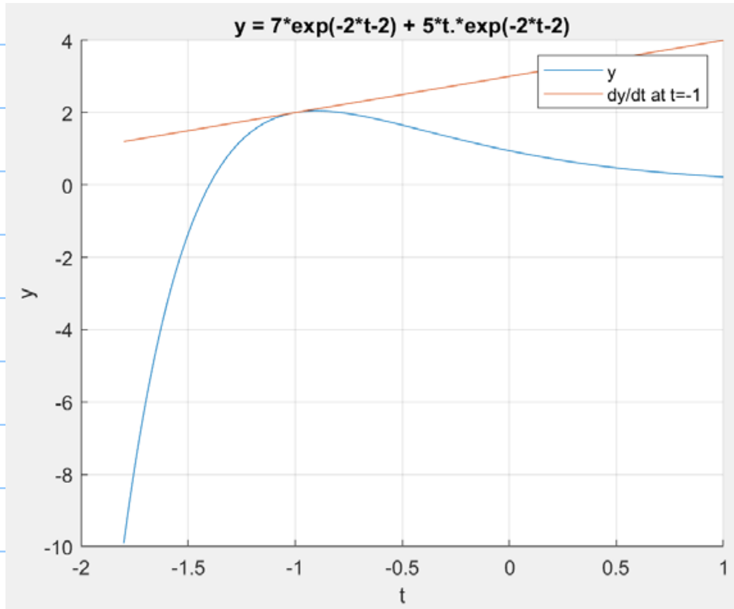
$$\text{Or, } y(t) = 7e^{-2t-2} + 5te^{-2t-2}$$

$$y(t) = 7e^{-2(t+1)} + 5te^{-2(t+1)}$$

$$\therefore \lim_{t\to\infty} y(t) = \lim_{t\to\infty} \frac{7+5t}{e^{2t+2}} = \lim_{t\to\infty} \frac{5}{2e^{2t+2}} = 0$$

Using MATLAB

```
clear,clc;
t = -1.8:0.01:1;
eqn = 7*exp(-2*t-2) + 5*t.*exp(-2*t-2);
hold on
plot(t, eqn)
plot(t, t+3)
grid on
xlabel 't', ylabel 'y'
title 'y = 7*exp(-2*t-2) + 5*t.*exp(-2*t-2)'
legend('y', 'dy/dt at t=-1')
```



12.

$r^2 - r + \frac{1}{4} = 0 = (r - \frac{1}{2})^2$, $r = \frac{1}{2}, \frac{1}{2}$

$\therefore y = c_1 e^{t/2} + c_2 t e^{t/2}$ $\quad y' = \frac{c_1}{2} e^{t/2} + c_2 e^{t/2} + \frac{c_2}{2} t e^{t/2}$

$y(0) = 2 \Rightarrow c_1 = 2$, $y'(0) = b = 1 + c_2$, $c_2 = b - 1$

$\therefore \underline{y(t) = 2e^{t/2} + (b-1)te^{t/2}}$

$y(t) = te^{t/2}\left[\frac{2}{t} + b - 1\right]$

As $t \to +\infty$, $\frac{2}{t} \to 0$, so that the term $(b-1)$

determines whether $te^{t/2} \to +\infty$ or $\to -\infty$.

If $b - 1 \geq 0$, then $y(t) \to +\infty$: i.e., $b \geq 1$
If $b - 1 < 0$, then $y(t) \to -\infty$: i.e., $b < 1$

$\therefore$ Critical value of b is at $\underline{b = 1}$.

13.

(a)

$4r^2 + 4r + 1 = 0 = (2r+1)$, $r = -\frac{1}{2}, -\frac{1}{2}$

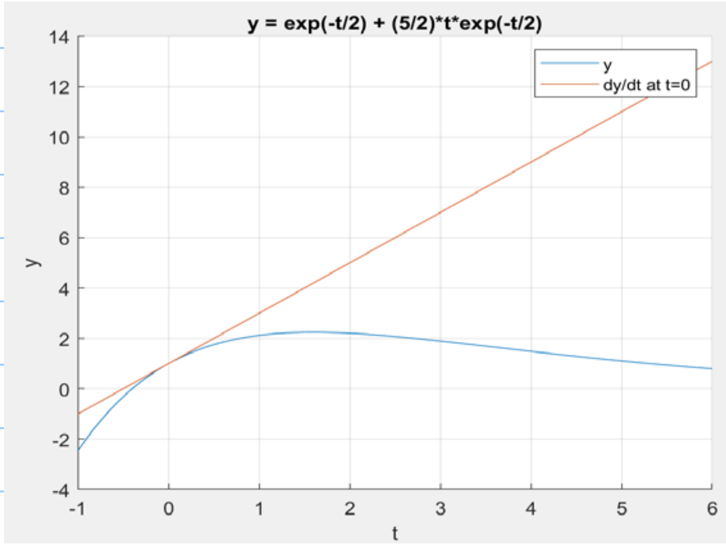
$\therefore y = c_1 e^{-t/2} + c_2 t e^{-t/2}$, $y' = -\frac{c_1}{2} e^{-t/2} + c_2 e^{-t/2} - \frac{c_2}{2} t e^{-t/2}$

$y(0) = 1 \Rightarrow c_1 = 1$, $y'(0) = 2 \Rightarrow -\frac{1}{2} + c_2 - 0 = 2$, $\therefore c_2 = \frac{5}{2}$

$\therefore y(t) = e^{-t/2} + \frac{5}{2} t e^{-t/2}$

Using MATLAB

```
clear,clc;
t = -1:0.01:6;
eqn = exp(-t/2) + (5/2)*t.*exp(-t/2);
hold on
plot(t, eqn)
plot(t, 2*t+1)
grid on
xlabel 't', ylabel 'y'
title 'y = exp(-t/2) + (5/2)*t*exp(-t/2)'
legend('y', 'dy/dt at t=0')
```



(b)

$y'(t) = e^{-t/2}\left(-\frac{1}{2} + \frac{5}{2} - \frac{5}{4}t\right) = e^{-t/2}\left(2 - \frac{5}{4}t\right)$

$\therefore y'(t) = 0$ at $t = \frac{8}{5}$

$$y\left(\frac{8}{5}\right) = e^{-\frac{8}{5}\cdot\frac{1}{2}} + \frac{5}{2}\left(\frac{8}{5}\right)e^{-\frac{8}{5}\cdot\frac{1}{2}} = e^{-\frac{4}{5}}(1+4)$$

$$\therefore \left(\frac{8}{5}, 5e^{-4/5}\right) \approx (1.6, 2.2466)$$

(c)

c_1 remains the same: $c_1 = 1$

From (a), $y'(t) = -\frac{1}{2}e^{-t/2} + c_2\left(1-\frac{t}{2}\right)e^{-t/2}$

$$\therefore y'(0) = 6 = -\frac{1}{2} + c_2, \quad c_2 = 6 + \frac{1}{2}$$

$$\therefore y(t) = e^{-t/2} + \left(6+\frac{1}{2}\right)t\,e^{-t/2}$$

(d)

(1) $y'(t) = -\frac{1}{2}e^{-t/2} + \left(6+\frac{1}{2}\right)e^{-t/2} - \frac{1}{2}\left(6+\frac{1}{2}\right)t\,e^{-t/2}$

$$y'(t) = 0 \Rightarrow \left[-\frac{1}{2} + \left(6+\frac{1}{2}\right) - \frac{1}{2}\left(6+\frac{1}{2}\right)t\right] = 0$$

$$\text{or, } \quad 6 - \frac{1}{2}\left(6+\frac{1}{2}\right)t = 0, \quad 26 = \left(6+\frac{1}{2}\right)t$$

$$\text{or, } \quad t = \frac{46}{26+1}$$

$$\therefore y\left(\frac{46}{26+1}\right) = e^{-\frac{26}{26+1}} + 26\,e^{-\frac{26}{26+1}}$$

$\therefore (t_m, y_m) = \left(\frac{4b}{2b+1}, (1+2b) e^{-\frac{2b}{2b+1}} \right)$

(2) $\lim_{b \to \infty} t_m = \lim_{b \to \infty} \frac{4b}{2b+1} = \lim_{b \to \infty} \frac{4}{2+\frac{1}{b}} = 2$

$\lim_{b \to \infty} (1+2b) e^{-\frac{2b}{2b+1}} = \lim_{b \to \infty} (1+2b) e^{-\frac{2}{2+1/b}} = +\infty$

$\therefore \lim_{b \to \infty} (t_m, y_m) = (2, +\infty)$

14.

Let r_1 and r_2 be the real roots. There are two possibilities, where c_1 and c_2 are constants:

(1) $y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$, $r_1 \neq r_2$

(2) $y(t) = c_1 e^{r_1 t} + c_2 t e^{r_1 t}$, $r_1 = r_2$

If $c_1 = c_2 = 0$, then $y(t) = 0$ for all t.

$\therefore$ Suppose at least one of c_1, c_2 is not zero

Case (1) $r_1 \neq r_2$

Consider $y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} = 0$. Suppose $c_1 \neq 0$.

$\therefore c_1 e^{(r_1 - r_2)t} = -c_2$.

(i) If $c_2 = 0$, then there is no value of t for which $c_1 e^{(r_1-r_2)t} = 0$ since $e^{(r_1-r_2)t} > 0$ for all t. $\therefore y(t) \neq 0$

(ii) If $c_2 \neq 0$, Then $e^{(r_1-r_2)t} = -\frac{c_2}{c_1}$. If $\frac{-c_2}{c_1} < 0$ There is no value of t to make This true, so $y(t) \neq 0$

(iii) If $c_2 \neq 0$ and $-\frac{c_2}{c_1} > 0$, Then

$(r_1 - r_2)t = \ln\left(\frac{-c_2}{c_1}\right)$, $t = \dfrac{\ln\left(-\frac{c_2}{c_1}\right)}{r_1 - r_2}$

so There is just one value of t for which $y(t) = 0$

Case (1) Summary: At most one value of t for which $y(t) = 0$.

Case (2) $r_1 = r_2$

Consider $y(t) = c_1 e^{r_1 t} + c_2 t e^{r_1 t} = 0$

Since $e^{r_1 t} \neq 0$ for all t, then $c_1 + c_2 t = 0$

$\therefore c_2 t = -c_1$

(i) if $c_1 = 0$, then since $c_2 \neq 0$ (c_1, c_2 not both 0),

$t = 0$

(ii) if $c_2 = 0$, then $c_1 \neq 0$, $y(t) = c_1 e^{r_1 t}$, so

there is no value of t for which $y(t) = 0$.

(iii) If $c_1 \neq 0$, $c_2 \neq 0$, then $c_1 + c_2 t = 0 \Rightarrow$

$t = -\frac{c_1}{c_2}$. $\therefore$ Just one value of t for

which $y(t) = 0$.

Case (2) summary: At most one value of t for

which $y(t) = 0$.

$\therefore$ Case (1), (2) $\Rightarrow$ $y(t)$ is either everywhere 0, never 0,

or at most one value of t for which $y(t) = 0$.

15.

(a) $r^2 + 2ar + a^2 = (r+a)^2, \quad r = -a, -a$

$\therefore y = e^{-at}$ is a solution

(b)

$$W[y_1, y_2](t) = c_1 \exp\left[-\int p(t)\,dt\right]$$

$$= c_1 \exp\left[-\int 2a\,dt\right]$$

$$= c_1 \exp\{-2at\} = c_1 e^{-2at}$$

(c)

For simplicity, choose $c_1 = 1$ to find a $y_2(t)$.

$\therefore y_1 y_2' - y_1' y_2 = (e^{-at})\, y_2'(t) - (-ae^{-at})\, y_2(t) = e^{-2at}$

$\therefore y_2'(t) + a y_2(t) = e^{-at}$, a first-order linear equation

Integrating factor = $\exp\left(\int a\,dt\right) = e^{at}$

$\therefore \frac{d}{dt}\left(e^{at} y_2(t)\right) = e^{at} e^{-at} = 1$

$\therefore e^{at} y_2(t) = t$, $y_2(t) = \underline{\underline{te^{-at}}}$

16.

(a) $\frac{e^{r_2 t} - e^{r_1 t}}{r_2 - r_1} = \left(\frac{1}{r_2 - r_1}\right) e^{r_2 t} + \left(\frac{1}{r_1 - r_2}\right) e^{r_1 t}$

which is a linear combination of $e^{r_1 t}$ and $e^{r_2 t}$

$\therefore$ By Theorem 3.2.2, p. 112 of the text,

$\frac{e^{r_2 t} - e^{r_1 t}}{r_2 - r_1}$ is a solution to $ay'' + by' + cy = 0$

(b) $\frac{d}{dr_2}(r_2 - r_1) = 1$, $\frac{d}{dr_2}\left(e^{r_2 t} - e^{r_1 t}\right) = te^{r_2 t}$

$\therefore \lim_{r_2 \to r_1} \frac{e^{r_2 t} - e^{r_1 t}}{r_2 - r_1} = \lim_{r_2 \to r_1} te^{r_2 t} = \underline{te^{r_1 t}}$

17.

(a) Since r_1 is a double root of $ar^2+br+c=0$, $a \neq 0$,

then $r^2 + \frac{b}{a}r + \frac{c}{a} = (r-r_1)^2$, from factorization

$\therefore ar^2 + br + c = a(r-r_1)^2$

$\therefore L[e^{rt}] = a(e^{rt})'' + b(e^{rt})' + c(e^{rt}) =$

$ar^2e^{rt} + bre^{rt} + ce^{rt} =$

$(ar^2+br+c)e^{rt} = \underline{a(r-r_1)^2 e^{rt}}$

$\therefore L[e^{r_1 t}] = a(r_1-r_1)^2 e^{r_1 t} = 0$

Since $L[y] = ay'' + by' + cy$, $e^{r_1 t}$ is a solution.

(b)

$\frac{\partial}{\partial r} L[e^{rt}] = \frac{\partial}{\partial r}\left[a(r-r_1)^2 e^{rt}\right]$, from (a)

$$= 2a(r-r_1)e^{rt} + at(r-r_1)^2 e^{rt}$$

But since $f(r,t) = e^{rt}$ is C^∞ (infinitely continuously differentiable), then mixed partials are equal: $f_{rtt} = f_{trt} = f_{ttr}$

So $\frac{\partial}{\partial r}\left[a\frac{d^2}{dt^2}(e^{rt}) + b\frac{d}{dt}(e^{rt}) + c(e^{rt})\right]$

$$= a\frac{d^2}{dt^2}\left[\frac{\partial}{\partial r}e^{rt}\right] + b\frac{d}{dt}\left[\frac{\partial}{\partial r}e^{rt}\right] + c\left[\frac{\partial}{\partial r}e^{rt}\right]$$

$\therefore \frac{\partial}{\partial r}L[e^{rt}] = L\left[\frac{\partial}{\partial r}e^{rt}\right] = L[te^{rt}]$

$\therefore L[te^{rt}] = 2a(r-r_1)e^{rt} + at(r-r_1)^2 e^{rt}$

$\therefore L[te^{r_1 t}] = 0 + 0 = 0$

$\therefore te^{r_1 t}$ is also a solution to $ay'' + by' + cy = 0$

18.

Let $y = v(t)t^2 \quad \therefore y' = v't^2 + 2tv$

$$y'' = t^2 v'' + 4tv' + 2v$$

$\therefore \; t^2(t^2v'' + 4tv' + 2v) - 4t(t^2v' + 2tv) + 6t^2v = 0$

Or, $t^4v'' + (4t^3 - 4t^3)v' + (2t^2 - 8t^2 + 6t^2)v = 0$

$\therefore \; t^4v'' = 0$, or $v''(t) = 0$

$\therefore \; v'(t) = K$, $v(t) = Kt + c$, K, c constants

$\therefore$ Let $v = t$ (choose $K = 1$, $c = 0$)

$\therefore \; y(t) = t^3$ is a solution.

$$W[t^2, t^3] = \begin{vmatrix} t^2 & t^3 \\ 2t & 3t^2 \end{vmatrix} = t^4 \neq 0 \text{ for } t > 0.$$

$\therefore \; t^2, t^3$ form a fundament set.

$\therefore \; \underline{y_2(t) = t^3}$

19.

Let $y(t) = tv(t)$ $\therefore \; y' = v + tv'$, $y'' = 2v' + tv''$

$\therefore \; t^2(2v' + tv'') + 2t(v + tv') - 2tv = 0$,

Or, $t^3v'' + (2t^2 + 2t^2)v' + (2t - 2t)v = 0$,

Or, $t^3v'' + 4t^2v' = 0$, or $v'' + \frac{4}{t}v' = 0$, $t > 0$

Let $u = \exp\left(\int \frac{4}{t}\right) = t^4$ be an integrating factor.

$\therefore\ t^4 v'' + 4t^3 v' = 0,\ \frac{d}{dt}(t^4 v') = 0,\ t^4 v' = c,$

$v' = ct^{-4},\ v(t) = \frac{c}{-3}t^{-3} + K,\ c, K$ constants

$\therefore$ Choose $v(t) = t^{-3}$ $(c=-3, K=0)$

$\therefore\ y_2(t) = (t^{-3})t = t^{-2}$

$W[t, t^{-2}] = \begin{vmatrix} t & t^{-2} \\ 1 & -2t^{-3} \end{vmatrix} = -3t^{-2} \neq 0$ for $t > 0$.

$\therefore\ t, t^{-2}$ a fundamental set

$\therefore\ \underline{y_2(t) = t^{-2}}$

20.

Let $y(t) = t^{-1}v(t)$ $\therefore\ y' = t^{-1}v' - t^{-2}v,\ y'' = t^{-1}v'' - 2t^{-2}v' + 2t^{-3}v$

$\therefore\ t^2(t^{-1}v'' - 2t^{-2}v' + 2t^{-3}v) + 3t(t^{-1}v' - t^{-2}v) + t^{-1}v = 0,$

Or, $tv'' + (-2+3)v' + (2t^{-1} - 3t^{-1} + t^{-1})v = 0$

Or, $tv'' + v' = 0$ $\therefore\ \frac{d}{dt}(tv') = 0,\ tv' = c,$

$v' = ct^{-1},\ v(t) = c\ln(t) + K,\ c, K$ constants

$\therefore$ Choose $v(t) = \ln(t)$ $(c=1, K=0)$

$\therefore\ y_2(t) = t^{-1}\ln(t)$

$$W[t^{-1}, t^{-1}\ln(t)] = \begin{vmatrix} t^{-1} & t^{-1}\ln(t) \\ -t^{-2} & -t^{-2}\ln(t) + t^{-2} \end{vmatrix}$$

$$= -t^{-3}\ln(t) + t^{-3} + t^{-3}\ln(t) = t^{-3} \neq 0 \text{ for } t > 0$$

$\therefore\ t^{-1}, t^{-1}\ln(t)$ form a fundamental set of solutions

$\therefore\ y_2(t) = \underline{t^{-1}\ln(t)}$

21.

Let $y(x) = v(x)\sin(x^2)$ $\quad\therefore\ y' = v'\sin(x^2) + 2xv\cos(x^2)$

$$y'' = v''\sin(x^2) + 4xv'\cos(x^2) + 2v\cos(x^2) - 4x^2v\sin(x^2)$$

$$\therefore\ x\left[v''\sin(x^2) + 4xv'\cos(x^2) + 2v\cos(x^2) - 4x^2v\sin(x^2)\right]$$

$$- \left[v'\sin(x^2) + 2xv\cos(x^2)\right] + 4x^3v\sin(x^2)$$

$$= x\sin(x^2)\,v'' + \left[4x^2\cos(x^2) - \sin(x^2)\right]v'$$

$$+ \left[2x\cos(x^2) - 4x^3\sin(x^2) - 2x\cos(x^2) + 4x^3\sin(x^2)\right]v = 0$$

$$\therefore\ x\sin(x^2)\,v'' + \left[4x^2\cos(x^2) - \sin(x^2)\right]v' = 0$$

Or, $v'' + \left[4x\cot(x^2) - \frac{1}{x}\right]v' = 0$

Integrating factor: $\exp\left(\int 4x\cot(x^2) - \frac{1}{x}\right)$

$$= \exp\left[2\ln(\sin(x^2)) - \ln(x)\right], \text{ for } x>0$$

$$= \exp\left[\ln \frac{\sin^2(x^2)}{x}\right] = \frac{\sin^2(x^2)}{x}$$

$$\therefore \frac{\sin^2(x^2)}{x} v'' + \left[4\cos(x^2)\sin(x^2) - \frac{\sin^2(x^2)}{x^2}\right] v' = 0$$

$$\therefore \frac{d}{dx}\left[\frac{\sin^2(x^2)}{x} v'\right] = 0, \quad \frac{\sin^2(x^2)}{x} v' = c$$

$$\therefore v' = cx\csc^2(x^2), \quad v = \frac{c}{2}\int 2x\csc^2(x^2) = -\frac{c}{2}\cot(x^2) + K$$

choose $c = -2$, $K = 0$.

$$\therefore v(x) = \cot(x^2), \quad \therefore y(x) = \cot(x^2)\sin(x^2) = \cos(x^2)$$

$$W[\sin(x^2), \cos(x^2)] = \begin{vmatrix} \sin(x^2) & \cos(x^2) \\ 2x\cos(x^2) & -2x\sin(x^2) \end{vmatrix} = -2x \neq 0 \text{ for } x>0$$

$\therefore$ $\sin(x^2)$, $\cos(x^2)$ form a fundamental set

$$\therefore y_2(x) = \cos(x^2)$$

22.

Let $y(x) = v(x)\, x^{-\frac{1}{2}} \sin(x)$

$$\therefore y' = v' x^{-1/2} \sin(x) - \tfrac{1}{2} v x^{-\frac{3}{2}} \sin(x) + v x^{-1/2} \cos(x)$$

$$\therefore y' = \left(x^{-\frac{1}{2}} \sin(x)\right) v' + \left(x^{-1/2} \cos(x) - \tfrac{1}{2} x^{-3/2} \sin(x)\right) v$$

$$y'' = v'' x^{-1/2} \sin(x) - \tfrac{1}{2} v' x^{-3/2} \sin(x) + v' x^{-1/2} \cos(x)$$

$$- \tfrac{1}{2} v' x^{-3/2} \sin(x) + \tfrac{3}{4} v x^{-5/2} \sin(x) - \tfrac{1}{2} v x^{-3/2} \cos(x)$$

$$+ v' x^{-1/2} \cos(x) - \tfrac{1}{2} v x^{-3/2} \cos(x) - v x^{-1/2} \sin(x)$$

$$\therefore y'' = \left(x^{-1/2} \sin(x)\right) v''$$

$$+ \left(- x^{-3/2} \sin(x) + 2 x^{-1/2} \cos(x)\right) v'$$

$$+ \left(\tfrac{3}{4} x^{-5/2} \sin(x) - x^{-3/2} \cos(x) - x^{-\frac{1}{2}} \sin(x)\right) v$$

$$\therefore x^2 y'' = x^{3/2} \sin(x)\, v'' \qquad [1]$$

$$+ \left(- x^{1/2} \sin(x) + 2 x^{3/2} \cos(x)\right) v'$$

$$+ \left(\tfrac{3}{4} x^{-\frac{1}{2}} \sin(x) - x^{1/2} \cos(x) - x^{3/2} \sin(x)\right) v$$

$$x y' = \left(x^{1/2} \sin(x)\right) v' + \left(x^{1/2} \cos(x) - \tfrac{1}{2} x^{-1/2} \sin(x)\right) v \qquad [2]$$

$$\left(x^2 - \tfrac{1}{4}\right) y = \left(x^2 - \tfrac{1}{4}\right)\left(x^{-1/2} \sin(x)\right) v = \left(x^{3/2} \sin(x) - \tfrac{1}{4} x^{-1/2} \sin(x)\right) v \qquad [3]$$

Adding [1], [2], [3],

$$x^{3/2}\sin(x)v'' + \left(x^{1/2}\sin(x) - x^{1/2}\sin(x) + 2x^{3/2}\cos(x)\right)v'$$

cancel

$$+ \Big[\tfrac{3}{4}x^{-1/2}\sin(x) - \tfrac{1}{2}x^{-1/2}\sin(x) - \tfrac{1}{4}x^{-1/2}\sin(x)$$

cancel

$$- x^{1/2}\cos(x) + x^{1/2}\cos(x) - x^{3/2}\sin(x) + x^{3/2}\sin(x)\Big]v$$

cancel cancel

$$= x^{3/2}\sin(x)\,v'' + \left[2x^{3/2}\cos(x)\right]v' = 0$$

Dividing by $x^{3/2}\sin(x)$,

$$v'' + 2\cot(x)\,v' = 0$$

Integrating factor: $\exp\left(\int 2\cot(x)\right) = \exp(2\ln|\sin(x)|)$

$$= \exp(\ln\sin^2(x)) = \sin^2(x)$$

$\therefore\ \sin^2(x)v'' + 2\sin(x)\cos(x)v' = 0,$

$$\frac{d}{dx}\left(\sin^2(x)v'(x)\right) = 0 \implies \sin^2(x)v'(x) = K$$

$$v'(x) = K\csc^2(x) \implies v(x) = -K\cot(x) + c$$

Choose $K=-1$, $c=0$. $\therefore\ v(x) = \cot(x)$

$\therefore\ y(x) = v(x)x^{-\frac{1}{2}}\sin(x) = \cot(x)\,x^{-\frac{1}{2}}\sin(x) = x^{-\frac{1}{2}}\cos(x)$

Use MATLAB to compute: $W[x^{-\frac{1}{2}}\sin(x),\ x^{-\frac{1}{2}}\cos(x)]$

```
clear, clc
syms x
u = (x^(-1/2))*sin(x);
v = (x^(-1/2))*cos(x);
A = [u v; diff(u) diff(v)]
simplify(det(A))  % Compute Wronskian
```

A =

$$\begin{pmatrix} \frac{\sin(x)}{\sqrt{x}} & \frac{\cos(x)}{\sqrt{x}} \\ \frac{\cos(x)}{\sqrt{x}} - \frac{\sin(x)}{2\,x^{3/2}} & -\frac{\cos(x)}{2\,x^{3/2}} - \frac{\sin(x)}{\sqrt{x}} \end{pmatrix}$$

ans =

$$-\frac{1}{x}$$

$\therefore W[x^{-\frac{1}{2}}\sin(x),\ x^{-\frac{1}{2}}\cos(x)] \neq 0$ for $x > 0$

$\therefore$ They form a fundamental set.

$\therefore\ y_2(x) = x^{-1/2}\cos(x)$

23.

(a) $y_1'(x) = (-\delta x)e^{-\delta x^2/2}$

$y_1''(x) = -\delta e^{-\delta x^2/2} + \delta^2 x^2 e^{-\delta x^2/2}$

$x y_1' = -\delta x^2 e^{-\delta x^2/2}$

$\delta(x y_1' + y_1) = \delta\left(-\delta x^2 e^{-\delta x^2/2} + e^{-\delta x^2/2}\right)$

$= -\delta^2 x^2 e^{-\delta x^2/2} + \delta e^{-\delta x^2/2}$

$$\therefore\ y_1'' + \delta(xy_1' + y_1) =$$

$$\left(-\delta e^{-\delta x^2/2} + \delta^2 x^2 e^{-\delta x^2/2}\right) + \left(-\delta^2 x^2 e^{-\delta x^2/2} + \delta e^{-\delta x^2/2}\right)$$

(cancel) (cancel)

$$= 0$$

(b) Let $y(x) = v(x)e^{-\delta x^2/2}$

$$y' = v'e^{-\delta x^2/2} - \delta x v e^{-\delta x^2/2}$$

$$y'' = v''e^{-\delta x^2/2} - \delta x v' e^{-\delta x^2/2} - \delta v e^{-\delta x^2/2} - \delta x v' e^{-\delta x^2/2} + \delta^2 x^2 v e^{-\delta x^2/2}$$

$$xy' = xv'e^{-\delta x^2/2} - \delta x^2 v e^{-\delta x^2/2}$$

$$\therefore\ \delta(xy' + y) = \delta x v' e^{-\delta x^2/2} - \delta^2 x^2 v e^{-\delta x^2/2} + \delta v e^{-\delta x^2/2}$$

$$\therefore\ y'' + \delta(xy' + y) = e^{-\delta x^2/2} v''$$

$$+ (-\delta x - \delta x + \delta x) e^{-\delta x^2/2} v'$$

$$+ (-\delta + \delta^2 x^2 - \delta^2 x^2 + \delta) e^{-\delta x^2/2} v$$

$$= e^{-\delta x^2/2} v'' + \delta x e^{-\delta x^2/2} v' = 0$$

Dividing by $e^{-\delta x^2/2}$ which is nonzero,

$$v'' + \delta x v' = 0$$

Integrating factor: $\exp\left(\int \delta x\right) = e^{\delta x^2/2}$

$$\therefore \frac{d}{dx}\left(e^{\delta x^2/2} v'\right) = 0, \quad e^{\delta x^2/2} v' = c$$

$$v' = c e^{-\delta x^2/2}, \quad v = c \int e^{-\delta x^2/2} + k$$

Choose $c = 1$, $k = 0$: $v(x) = \int_0^x e^{-\delta s^2/2} ds$

$$\therefore y_2(x) = e^{-\delta x^2/2} \int_0^x e^{-\delta s^2/2} ds$$

$$W[y_1(x), y_2(x)] = \begin{vmatrix} e^{-\delta x^2/2} & e^{-\delta x^2/2} \int_0^x e^{-\delta s^2/2} ds \\ -\delta x e^{-\delta x^2/2} & -\delta x e^{-\delta x^2/2} \int_0^x e^{-\delta s^2/2} ds + e^{-\delta x^2} \end{vmatrix}$$

$$= -\delta x e^{-\delta x^2} \int_0^x e^{-\delta s^2/2} ds + e^{-3\delta x^2/2} + \delta x e^{-\delta x^2} \int_0^x e^{-\delta s^2/2} ds$$

$$= e^{-3\delta x^2/2} \neq 0$$

$\therefore$ $y_1(x)$, $y_2(x)$ form a fundamental set.

$$\therefore y(x) = c_1 e^{-\delta x^2/2} + c_2 e^{-\delta x^2/2} \int_0^x e^{-\delta s^2/2} ds$$

c_1, c_2 constants

24.

$$W[y_1, y_2] = y_1 y_2' - y_1 y_2'$$

$$\left(\frac{y_2}{y_1}\right)' = \frac{y_1 y_2' - y_2 y_1'}{(y_1)^2} = \frac{W[y_1, y_2]}{(y_1)^2}$$

From Abel's Theorem, $W[y_1, y_2](t) = c_1 \exp\left[-\int_{t_0}^{t} p(r)\,dr\right]$

$$\therefore \frac{d}{dt}\left(\frac{y_2}{y_1}\right) = c_1 (y_1(t))^{-2} \exp\left[-\int_{t_0}^{t} p(r)\,dr\right]$$

$$\therefore \frac{y_2(t)}{y_1(t)} = c_1 \int_{t_0}^{t} \left(y_1(s)^{-2} \exp\left[-\int_{s_0}^{s} p(r)\,dr\right]\right) ds + c_2$$

$$\therefore y_2(t) = c_1 y_1(t) \int_{t_0}^{t} \left(y_1(s)^{-2} \exp\left[-\int_{s_0}^{s} p(r)\,dr\right]\right) ds + c_2 y_1(t)$$

Now choose $c_1 = 1$, $c_2 = 0$ to get a generic formula for an independent $y_2(t)$:

$$\underline{y_2(t) = y_1(t)} \int_{t_0}^{t} \left(y_1(s)^{-2} \exp\left[-\int_{s_0}^{s} p(r)\,dr\right]\right) ds$$

25.

Convert to: $y'' + \frac{3}{t} y' + \frac{1}{t^2} y = 0$

$-\int p(t)dt = -\int \frac{3}{t} dt = \ln(t^{-3})$

$\therefore \exp(\ln(t^{-3})) = t^{-3}$

$\therefore y_1(t)^{-2} (t^{-3}) = (t^{-1})^{-2} (t^{-3}) = t^{-1}$

$\therefore \int t^{-1} dt = \ln(t)$

$\therefore y_1(t) \ln(t) = t^{-1} \ln(t)$

$\therefore y_2(t) = t^{-1} \ln(t)$

26.

Convert to: $y'' - \frac{1}{t} y' + 4t^2 = 0$

$-\int p(t)dt = -\int -\frac{1}{t} dt = \ln(t)$

$\exp(\ln(t)) = t$

$\therefore \int (y_1^{-2})(t) = \int \frac{t}{\sin^2(t^2)} = \int t \csc^2(t^2)$

$$= -\frac{1}{2}\cot(t^2)$$

$$\therefore y_1(t)\left(-\frac{1}{2}\cot(t^2)\right) = \sin(t^2)\left(-\frac{1}{2}\frac{\cos(t^2)}{\sin(t^2)}\right)$$

$$= -\frac{1}{2}\cos(t^2)$$

$\therefore$ Choose $\underline{y_2(t) = \cos(t^2)}$

27.

Convert to: $y'' + \frac{1}{x}y' + \left(1 - \frac{1}{4x^2}\right)y = 0$

$$\therefore -\int p(x) = -\int \frac{1}{x} = \ln(x^{-1})$$

$$\therefore \exp\left(-\int p(x)\right) = \exp\left(\ln(x^{-1})\right) = x^{-1}$$

$$\therefore \int (y_1)^{-2}(x^{-1}) = \int x\sin^{-2}(x)\,x^{-1} = \int \csc^2(x)$$

$$= -\cot(x)$$

$$\therefore y_1(x)(-\cot(x)) = x^{-\frac{1}{2}}\sin(x)\left(-\frac{\cos(x)}{\sin(x)}\right) = -x^{-\frac{1}{2}}\cos(x)$$

$\therefore$ Choose $\underline{y_2(x) = x^{-1/2}\cos(x)}$

28.

From characteristic equation, $ar^2+br+c=0$,

$$r = -\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}, \text{ so that } -\frac{b}{2a} < 0$$

The 3 cases to consider are:

(1) $b^2-4ac > 0$ (2 different real roots)

$$\therefore b^2 > b^2-4ac, \quad \frac{b}{2a} > \frac{\sqrt{b^2-4ac}}{2a}$$

$$\therefore 0 > -\frac{b}{2a} + \frac{\sqrt{b^2-4ac}}{2a}, \text{ and also } -\frac{b}{2a} - \frac{\sqrt{b^2-4ac}}{2a} < 0$$

$\therefore$ both real roots are negative.

$$\therefore \lim_{t\to\infty} e^{rt} = 0 \text{ for } r = -\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$$

$$\therefore \lim_{t\to\infty} c_1 e^{r_1 t} + c_2 e^{r_2 t} = 0, \quad r_1 = \frac{-b+\sqrt{b^2-4ac}}{2a}$$

$$r_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

(2) $b^2 - 4ac = 0$ (repeated real roots)

$\therefore y(t) = c_1 e^{rt} + c_2 t e^{rt}$, $r = -\frac{b}{2a} < 0$

$\therefore \lim_{t \to \infty} c_1 e^{rt} = 0.$

Using L'Hopital's rule,

$$\lim_{t \to \infty} c_2 \frac{t}{e^{-rt}} = \lim_{t \to \infty} c_2 \frac{1}{-re^{-rt}} = 0$$

since $\lim_{t \to \infty} e^{-rt} = \infty.$

$$\therefore \lim_{t \to \infty} c_1 e^{rt} + c_2 t e^{rt} = 0$$

(3) $b^2 - 4ac < 0$ (roots are complex conjugates)

$$r = -\frac{b}{2a} \pm i\left(\frac{\sqrt{4ac - b^2}}{2a}\right)$$

$$\therefore y(t) = c_1 e^{(-b/2a)t} \cos\left(\frac{\sqrt{4ac-b^2}}{2a} t\right) + c_2 e^{(-b/2a)t} \sin\left(\frac{\sqrt{4ac-b^2}}{2a} t\right)$$

$$\text{But } \lim_{t \to \infty} \frac{c_1 \cos\left(\frac{\sqrt{4ac-b^2}}{2a} t\right)}{e^{\frac{b}{2a} t}} = 0$$

and $\lim_{t \to \infty} \dfrac{c_2 \sin\left(\frac{\sqrt{4ac-b^2}}{2a} t\right)}{e^{\frac{b}{2a}t}} = 0$

since $\max\left|\cos\left(\frac{\sqrt{4ac-b^2}}{2a} t\right)\right| = \max\left|\sin\left(\frac{\sqrt{4ac-b^2}}{2a} t\right)\right| = 1$

$\therefore \lim_{t \to \infty} y(t) = 0$

29.

(a) From the characteristic equation, $ar^2 + c = 0$,

$r = \pm\sqrt{-\frac{c}{a}} = \pm i\sqrt{\frac{c}{a}}$

$\therefore\ y(t) = c_1 \cos(\sqrt{\tfrac{c}{a}}t) + c_2 \sin(\sqrt{\tfrac{c}{a}}t)$

$\therefore\ |y(t)| \le |c_1 \cos(\sqrt{\tfrac{c}{a}}t)| + |c_2 \sin(\sqrt{\tfrac{c}{a}}t)| \le |c_1| + |c_2|$

$\therefore$ $y(t)$ is bounded for all t

(b) From the characteristic equation, $ar^2 + br = 0$,

$r = -\frac{b}{2a} \pm \frac{b}{2a} = 0, -\frac{b}{a}$ $\therefore$ two different roots.

$\therefore y(t) = c_1 e^{0t} + c_2 e^{-\frac{b}{a}t}$, or

$$y(t) = c_1 + c_2 e^{-\frac{b}{a}t}$$

$\therefore \lim_{t \to \infty} y(t) = c_1$ as $\lim_{t \to \infty} e^{-\frac{b}{a}t} = 0$

From $y(0) = y_0$, $y(0) = c_1 + c_2 e^{-\frac{b}{a}0} = c_1 + c_2$

$$\therefore c_1 + c_2 = y_0$$

From $y'(0) = y_0'$, $y'(t) = -\frac{b}{a} c_2 e^{-\frac{b}{a}t}$

$\therefore y'(0) = -\frac{b}{a} c_2 = y_0' \Rightarrow c_2 = -\frac{a y_0'}{b}$

$\therefore c_1 + -\frac{a y_0'}{b} = y_0$

$c_1 = y_0 + \frac{a}{b} y_0'$

$\therefore \lim_{t \to \infty} y(t) = y_0 + \frac{a}{b} y_0'$

30.

(a) $y' = \cos t$, $y'' = -\sin t$

$\therefore (-\sin t) + (K\sin^2 t)(\cos t) + (1 - K\cos t \sin t)\sin t$

cancel

$= -\sin t + K\sin^2 t\cos t + \sin t - K\cos t\sin^2 t = \underline{0}$

cancel

(b) If $0 < K < 2$, then $|K\cos t\sin t| < |2\cos t\sin t|$

$2\cos t\sin t = \sin 2t$

$\therefore |K\cos t\sin t| < |\sin 2t| < 1$

$\therefore 0 < 1 - |K\cos t\sin t|$

(1) If $K\cos t\sin t < 0$, $|K\cos t\sin t| = -K\cos t\sin t$

$\therefore 0 < -K\cos t\sin t \Rightarrow 1 < 1 - K\cos t\sin t$

$\therefore 0 < 1 < 1 - K\cos t\sin t$

(2) If $K\cos t\sin t \geq 0$, $|K\cos t\sin t| = K\cos t\sin t$

$\therefore\ 0 < 1 - |K\cos t \sin t| = 1 - K\cos t \sin t$

(c) If $0 < K < 2$, then $K > 0$. $\sin^2 t \geq 0$

$\therefore\ K\sin^2 t \geq 0$

(d) From Problem #28, all solutions $\to 0$ as $t \to \infty$

Here, $\lim_{t\to\infty} \sin t \neq 0$

The substitution is $x = \ln(t)$. From #25, Section 3.3,

$$y'(t) = \frac{1}{t} y'(x), \quad y''(t) = \frac{1}{t^2} y''(x) - \frac{1}{t^2} y'(x)$$

Then $t^2 y''(t) + \alpha t y'(t) + \beta y(t) = 0$, $t > 0$, becomes

$$y''(x) + (\alpha - 1) y'(x) + \beta y(x) = 0$$

31.

$\alpha = -3$, $\therefore\ \alpha - 1 = -4$

Convert to $y''(x) - 4y'(x) + 4y(x) = 0$

$\therefore\ r^2 - 4r - 4 = 0 = (r-2)^2$, $r = 2, 2$

$\therefore\ y(x) = c_1 e^{2x} + c_2 x e^{2x}$

$\therefore\ y(\ln(t)) = y(t) = c_1 e^{2\ln(t)} + c_2 \ln(t) e^{2\ln(t)}$

Or, $y(t) = c_1 t^2 + c_2 t^2 \ln(t)$

32.

$\alpha = 2,\ \therefore\ \alpha - 1 = 1$

Convert to $y''(x) + y'(x) + 0.25y(x) = 0$

$r^2 + r + 0.25 = 0,\quad r = \dfrac{-1 \pm \sqrt{1 - 4(0.25)}}{2} = -\frac{1}{2}, -\frac{1}{2}$

$\therefore\ y(x) = c_1 e^{-\frac{x}{2}} + c_2 x e^{-\frac{x}{2}}$

$\therefore\ y(\ln(t)) = y(t) = c_1 e^{-\frac{1}{2}\ln(t)} + c_2 \ln(t) e^{-\frac{1}{2}\ln(t)}$

Or, $y(t) = c_1 t^{-\frac{1}{2}} + c_2 t^{-\frac{1}{2}} \ln(t)$

33.

$\alpha = 3,\ \therefore\ \alpha - 1 = 2$

Convert to $y''(x) + 2y'(x) + y(x) = 0$

$\therefore\ r^2 + 2r + 1 = 0 = (r+1)^2,\quad r = -1, -1$

$\therefore\ y(x) = c_1 e^{-x} + c_2 x e^{-x}$

$\therefore\ y(\ln(t)) = y(t) = c_1 e^{-\ln(t)} + c_2 \ln(t) e^{-\ln(t)}$

Or, $y(t) = c_1 t^{-1} + c_2 t^{-1} \ln(t)$

34.

Convert to $t^2 y''(t) - 2t y'(t) + \frac{9}{4} y(t) = 0$

$\alpha = -2,\ \alpha - 1 = -3$

Convert to $y''(x) - 3y'(x) + \frac{9}{4} y(x) = 0$

$r^2 - 3r + \frac{9}{4} = (r - \frac{3}{2})^2 = 0,\ r = \frac{3}{2}, \frac{3}{2}$

$\therefore\ y(x) = c_1 e^{(3/2)x} + c_2 x e^{(3/2)x}$

$\therefore\ y(\ln(t)) = y(t) = c_1 e^{\frac{3}{2}\ln(t)} + c_2 \ln(t) e^{\frac{3}{2}\ln(t)}$

Or, $y(t) = c_1 t^{3/2} + c_2 t^{3/2} \ln(t)$

1.

Homogeneous: $r^2 - 2r - 3 = 0 = (r-3)(r+1)$, $r = 3, -1$.

$\therefore\ y_c(t) = c_1 e^{3t} + c_2 e^{-t}$

Particular: Let $y(t) = Ae^{2t}$, $y' = 2Ae^{2t}$, $y'' = 4Ae^{2t}$

$\therefore\ (4A - 4A - 3A)e^{2t} = 3e^{2t}$, $-3A = 3$, $A = -1$

$\therefore\ \underline{y(t) = -e^{2t}} + c_1 e^{3t} + c_2 e^{-t}$

2.

Homogeneous: $r^2 - r - 2 = 0 = (r-2)(r+1)$, $r = 2, -1$

$\therefore\ y_c(t) = c_1 e^{2t} + c_2 e^{-t}$

Particular: Let $y(t) = At^2 + Bt + C$, $y' = 2At + B$, $y'' = 2A$

$\therefore\ (2A) - (2At + B) - (2At^2 + 2Bt + 2C) = -2t + 4t^2$

Or, $(2A - B - 2C) + (-2A - 2B)t + (-2A)t^2 = -2t + 4t^2$

$\therefore\ -2A = 4$, $A = -2$

$$-2A-2B = 4-2B = -2, \quad B = 3$$

$$2A-B-2C = -4-3-2C = 0, \quad C = -\frac{7}{2}$$

$$\therefore y_p(t) = -2t^2 + 3t - \frac{7}{2}$$

$$\therefore y(t) = -2t^2 + 3t - \frac{7}{2} + c_1 e^{2t} + c_2 e^{-t}$$

3.

Homogeneous: $r^2 + r - 6 = (r+3)(r-2) = 0$, $r = -3, 2$

$$\therefore y_c(t) = c_1 e^{-3t} + c_2 e^{2t}$$

Particular: Let $y = Ae^{3t} + Be^{-2t}$

$$\therefore y' = 3Ae^{3t} + (-2)Be^{-2t}, \quad y'' = 9Ae^{3t} + 4Be^{-2t}$$

$$(9A + 3A - 6A)e^{3t} + (4B - 2B - 6B)e^{-2t} = 12e^{3t} + 12e^{-2t}$$

$$\therefore 6A = 12, \; A = 2 \qquad -4B = 12, \; B = -3$$

$$\therefore y(t) = c_1 e^{-3t} + c_2 e^{2t} + 2e^{3t} - 3e^{-2t}$$

4.

Homogeneous: $r^2 - 2r - 3 = 0 = (r-3)(r+1)$, $r = 3, -1$

$\therefore\ y_c(t) = c_1 e^{3t} + c_2 e^{-t}$

Particular: Let $y(t) = Ate^{-t}$,

Using MATLAB,

```
clear,clc
syms t A
c2 = 1; c1 = -2; c0 = -3;   %coeffs of diff eq
y = (A*t)*exp(-t);          %attempt
c2*diff(y,t,2) + c1*diff(y,t,1) + c0*y
```

ans = $-4\,A\,e^{-t}$

$\therefore$ No value of A yields $-3te^{-t}$

$\therefore$ try $y(t) = (At^2 + Bt)e^{-t}$. Don't need $At^2 + Bt + C$ since e^{-t} is part of $y_c(t)$.

Using MATLAB,

```
clear,clc
syms t A B
c2 = 1; c1 = -2; c0 = -3;   %coeffs of diff eq
y = (A*t^2 + B*t)*exp(-t);          %attempt
c2*diff(y,t,2) + c1*diff(y,t,1) + c0*y
```

ans = $2\,A\,e^{-t} - 4\,e^{-t}\,(B + 2\,A\,t)$

$\therefore\ (2A - 4B)e^{-t} - 8Ate^{-t} = -3te^{-t}$

$\therefore\ -8A = -3,\ A = \frac{3}{8},\ 2\left(\frac{3}{8}\right) - 4B = 0,\ B = \frac{3}{16}$

$\therefore\ y_p(t) = \left(\frac{3}{8}t^2 + \frac{3}{16}t\right)e^{-t}$

$\therefore\ y(t) = c_1 e^{3t} + c_2 e^{-t} + \left(\frac{3}{8}t^2 + \frac{3}{16}t\right)e^{-t}$

5.

Homogeneous: $r^2 + 2r = 0$, $r = 0, -2$

$\therefore y_c(t) = c_1 + c_2 e^{-2t}$

Particular:

(1) $y'' + 2y' = 3$ (c_1 already a solution, so try $y = At$)

Let $y(t) = At$, $y' = A$, $y'' = 0$

$\therefore y'' + 2y' = 0 + 2A = 3$, $A = \frac{3}{2}$

$\therefore y_{p_1}(t) = \frac{3}{2}t$ for $y'' + 2y' = 3$

(2) $y'' + 2y' = 4\sin(2t)$

Let $y(t) = A\cos(2t) + B\sin(2t)$

Using MATLAB,

```
clear,clc
syms t A B
c2 = 1; c1 = 2; c0 = 0;  %coeffs of diff eq
w =2;   %coeff for trig periods
y = A*cos(w*t) + B*sin(w*t);            %attempt
c2*diff(y,t,2) + c1*diff(y,t,1) + c0*y
```

ans = $4B\cos(2t) - 4A\cos(2t) - 4A\sin(2t) - 4B\sin(2t)$

$\therefore (-4A + 4B)\cos(2t) + (-4A - 4B)\sin(2t) = 4\sin(2t)$

$\therefore \left.\begin{aligned} -4A + 4B &= 0 \\ -4A - 4B &= 4 \end{aligned}\right\} -8A = 4$, $A = -\frac{1}{2}$, $B = -\frac{1}{2}$

$\therefore\ y_{p_2}(t) = -\frac{1}{2}\cos(2t) - \frac{1}{2}\sin(2t)$

$\therefore\ \underline{y(t) = c_1 + c_2 e^{-2t} + \frac{3}{2}t - \frac{1}{2}\cos(2t) - \frac{1}{2}\sin(2t)}$

6.

Homogeneous: $r^2 + 2r + 1 = 0 = (r+1)^2$, $r = -1, -1$

$\therefore\ y_c(t) = c_1 e^{-t} + c_2 t e^{-t}$

Particular: e^{-t} and te^{-t} are part of homogeneous solution. So try $y(t) = At^2 e^{-t}$

MATLAB:

```
clear,clc
syms t A B C
c2 = 1; c1 = 2; c0 = 1;  %coeffs of diff eq
y = A*(t^2)*exp(-t);          %attempt
c2*diff(y,t,2) + c1*diff(y,t,1) + c0*y
```

ans = $2\,A\,e^{-t}$

$\therefore A = 1$

$\therefore\ y_p(t) = t^2 e^{-t}$

$\therefore\ \underline{y(t) = c_1 e^{-t} + c_2 t e^{-t} + t^2 e^{-t}}$

7.

Homogeneous: $r^2 + 1 = 0$, $r = \pm i$

$\therefore\ y_c(t) = c_1\cos(t) + c_2\sin(t)$

Particular:

(1) Let $y(t) = A\cos(2t) + B\sin(2t)$

MATLAB:

```
clear,clc
syms t A B
c2 = 1; c1 = 0; c0 = 1;  %coeffs of diff eq
w =2;   %coeff for trig periods
y = A*cos(w*t) + B*sin(w*t);          %attempt
c2*diff(y,t,2) + c1*diff(y,t,1) + c0*y
```

ans = $-3\,A\cos(2\,t) - 3\,B\sin(2\,t)$

$\therefore -3A\cos(2t) - 3B\sin(2t) = 3\sin(2t)$

$\therefore A = 0,\ B = -1. \quad \therefore y_{p_1}(t) = -\sin(2t)$

(2) Let $y(t) = At\cos(2t) + Bt\sin(2t)$

MATLAB:

```
clear,clc
syms t A B
c2 = 1; c1 = 0; c0 = 1;  %coeffs of diff eq
w =2;   %coeff for trig periods
y = A*t*cos(w*t) + B*t*sin(w*t);          %attempt
c2*diff(y,t,2) + c1*diff(y,t,1) + c0*y
```

ans = $4\,B\cos(2\,t) - 4\,A\sin(2\,t) - 3\,A\,t\cos(2\,t) - 3\,B\,t\sin(2\,t)$

This yields $A=0, B=0$ for $t\cos(2t)$

Try $y(t) = (At+B)\cos(2t) + (Ct+D)\sin(2t)$

MATLAB:

```
clear,clc
syms t A B C D
c2 = 1; c1 = 0; c0 = 1;  %coeffs of diff eq
w =2;   %coeff for trig periods
y = (A*t+B)*cos(w*t) + (C*t+D)*sin(w*t);    %attempt
c2*diff(y,t,2) + c1*diff(y,t,1) + c0*y
```

ans = $4\,C\cos(2\,t) - 3\sin(2\,t)\,(D + C\,t) - 3\cos(2\,t)\,(B + A\,t) - 4\,A\sin(2\,t)$

$\therefore (4C - 3B)\cos(2t) + (-4A - 3D)\sin(2t) +$

$$(-3A)^{=-\frac{1}{3}}t\cos t+(-3c)^{=0}t\sin t = t\cos(2t)$$

$$\therefore\ C=0,\ A=-\tfrac{1}{3},\ 4c-3B=0 \Rightarrow B=0$$

$$-4A-3D=-4\left(-\tfrac{1}{3}\right)-3D=0,\ D=\tfrac{4}{9}$$

$$\therefore\ y_{p_2}(t)=-\tfrac{1}{3}t\cos(2t)+\tfrac{4}{9}\sin(2t)$$

$$\therefore\ y_{p_1}(t)+y_{p_2}(t)=-\sin(2t)-\tfrac{1}{3}t\cos(2t)+\tfrac{4}{9}\sin(2t)$$

$$=-\tfrac{1}{3}t\cos(2t)-\tfrac{5}{9}\sin(2t)$$

$$\therefore\ y(t)=c_1\cos(t)+c_2\sin(t)-\tfrac{1}{3}t\cos(2t)-\tfrac{5}{9}\sin(2t)$$

Lesson: (1) could have been avoided. When sin and cos have same period (ωt), go straight to method (2)

8.

Homogeneous: $r^2+\omega_0^2=0$, $r=\pm i\omega_0$

$$\therefore\ u_c(t)=C_1\cos(\omega_0 t)+C_2\sin(\omega_0 t)$$

Particular: Let $u(t)=A\cos(\omega t)+B\sin(\omega t)$

MATLAB:

```
clear,clc
syms t A B w w0
c2 = 1; c1 = 0; c0 = w0^2;  %coeffs of diff eq
y = (A)*cos(w*t) + (B)*sin(w*t);    %attempt
c2*diff(y,t,2) + c1*diff(y,t,1) + c0*y
```

ans =

$$w_0^2\ (A\cos(t\,w) + B\sin(t\,w)) - A\,w^2\cos(t\,w) - B\,w^2\sin(t\,w)$$

$$\therefore (Aw_0^2 - Aw^2)\cos(wt) + (Bw_0^2 - Bw^2)\sin(wt) = \cos(wt)$$

$$\therefore B = 0,\quad A(w_0^2 - w^2) = 1,\quad A = (w_0^2 - w^2)^{-1}$$

$$\therefore u(t) = c_1\cos(w_0 t) + c_2\sin(w_0 t) + \frac{1}{w_0^2 - w^2}\cos(wt)$$

9.

Homogeneous: As in #8 above, $u_c(t) = c_1\cos(w_0 t) + c_2\sin(w_0 t)$

Particular: Let $u(t) = At\cos(w_0 t) + Bt\sin(w_0 t)$

MATLAB:

```
clear,clc
syms t A B w0
c2 = 1; c1 = 0; c0 = w0^2;  %coeffs of diff eq
y = (A*t)*cos(w0*t) + (B*t)*sin(w0*t);    %attempt
c2*diff(y,t,2) + c1*diff(y,t,1) + c0*y
```

ans = $w_0^2\ (A\,t\cos(t\,w_0) + B\,t\sin(t\,w_0)) + 2\,B\,w_0\cos(t\,w_0) - 2\,A\,w_0\sin(t\,w_0) - A\,t\,w_0^2\cos(t\,w_0) - B\,t\,w_0^2\sin(t\,w_0)$

$$\therefore (Aw_0^2 - Aw_0^2)t\cos(w_0 t) + (Bw_0^2 - Bw_0^2)t\sin(w_0 t)$$

$$+ (2Bw_0)\cos(w_0 t) + (-2Aw_0)\sin(w_0 t) = \cos(w_0 t)$$

$$\therefore A = 0,\quad 2Bw_0 = 1,\quad B = \frac{1}{2w_0}$$

$$\therefore u_p(t) = \frac{1}{2w_0}t\sin(w_0 t)$$

$$\therefore u(t) = c_1\cos(w_0 t) + c_2\sin(w_0 t) + \left(\frac{1}{2w_0}\right)t\sin(w_0 t)$$

10.

Homogeneous: $r^2 + r + 4 = 0$, $r = \frac{-1 \pm i\sqrt{15}}{2}$

$\therefore y_c(t) = c_1 e^{-t/2} \cos\left(\frac{\sqrt{15}}{2} t\right) + c_2 e^{-t/2} \sin\left(\frac{\sqrt{15}}{2} t\right)$

Particular: Let $y(t) = Ae^{t} + Be^{-t}$

$\therefore y'' + y' + 4y = (A + A + 4A)e^{t} + (B - B + 4B)e^{-t} = e^{t} - e^{-t}$

$\therefore 6A = 1,\ A = \frac{1}{6} \qquad 4B = -1,\ B = -\frac{1}{4}$

$\therefore y_p(t) = \frac{1}{6} e^{t} - \frac{1}{4} e^{-t}$

$\therefore y(t) = c_1 e^{-t/2} \cos\left(\frac{\sqrt{15}}{2} t\right) + c_2 e^{-t/2} \sin\left(\frac{\sqrt{15}}{2} t\right)$

$+ \frac{1}{6} e^{t} - \frac{1}{4} e^{-t}$

11.

Homogeneous; $r^2 + r - 2 = 0 = (r+2)(r-1)$, $r = -2, 1$

$\therefore y_c(t) = C_1 e^{-2t} + C_2 e^{t}$

Particular: Let $y(t) = At + B$. $\therefore y' = A,\ y'' = 0$.

$\therefore y'' + y' - 2y = A - 2(At + B) = 2t$

$\therefore -2At + (A - 2B) = 2t, \; A = -1, \; B = -\frac{1}{2}$

$\therefore y_p(t) = -t - \frac{1}{2}$

$\therefore y(t) = c_1 e^{-2t} + c_2 e^{t} - t - \frac{1}{2}$

$y(0) = 0 : \quad c_1 + c_2 - \frac{1}{2} = 0$ } $3c_1 = -\frac{3}{2}, \; c_1 = -\frac{1}{2}$

$y'(0) = 1 : \quad -2c_1 + c_2 - 1 = 1$ } $c_2 = 1$

$\therefore y(t) = -\frac{1}{2} e^{-2t} + e^{t} - t - \frac{1}{2}$

12.

Homogeneous: $r^2 + 4 = 0, \; r = \pm 2i$

$\therefore y_c(t) = c_1 \cos(2t) + c_2 \sin(2t)$

Particular:

(1) $y'' + 4y = t^2 \quad$ Let $y(t) = At^2 + Bt + C$

$\therefore y'' + 4y = 2A + 4At^2 + 4Bt + 4C = t^2$

$\therefore 4A = 1, \; A = \frac{1}{4}. \; B = 0, \; 2(\frac{1}{4}) + 4C = 0, \; C = -\frac{1}{8}$

$\therefore y_{p_1}(t) = \frac{1}{4}t^2 - \frac{1}{8}$

(2) $y'' + 4y = 3e^{t} \quad$ Let $y(t) = Ae^{t}$

$\therefore Ae^t + 4Ae^t = 3e^t, \quad 5A = 3, \quad A = \frac{3}{5}$

$\therefore y_{p_2}(t) = \frac{3}{5}e^t$

$\therefore y(t) = c_1\cos(2t) + c_2\sin(2t) + \frac{3}{5}e^t + \frac{1}{4}t^2 - \frac{1}{8}$

$y(0) = 0: \quad c_1 + \frac{3}{5} - \frac{1}{8} = 0, \quad c_1 = -\frac{19}{40}$

$y'(0) = 2: \quad 2c_2 + \frac{3}{5} = 2, \quad c_2 = \frac{7}{10}$

$\therefore y(t) = \frac{-19}{40}\cos(2t) + \frac{7}{10}\sin(2t) + \frac{3}{5}e^t + \frac{1}{4}t^2 - \frac{1}{8}$

13.

Homogeneous: $r^2 - 2r + 1 = 0 = (r-1)^2, \quad r = 1, 1$

$\therefore y_c(t) = c_1 e^t + c_2 t e^t$

Particular: $y_{p_1}(t) = 4$ is a solution

$\therefore$ Look at $y'' - 2y' + y = te^t$, a form of $P_n(t)e^t$,

where $n = 1$. $\therefore$ Let $P_1(t) = At + B$

Since e^t, te^t are solutions for the homogeneous equation, start with $y(t) = t^2(At + B)e^t$, so that no term of $P_1(t)$ is a solution to the homogeneous equation.

MATLAB

```
clear,clc
syms t A B
c2 = 1; c1 = -2; c0 = 1;     %coeffs of diff eq
y = (A*t^3 + B*t^2)*exp(t);           %attempt
c2*diff(y,t,2) + c1*diff(y,t,1) + c0*y
```

ans = $e^t\,(2\,B + 6\,A\,t)$

$\therefore\ 2Be^t + 6Ate^t = te^t. \quad \therefore\ B = 0,\ A = \frac{1}{6}$

$\therefore\ y(t) = c_1e^t + c_2te^t + \frac{1}{6}t^3e^t + 4$

$y(0) = 1:\quad c_1 + 4 = 1,\ c_1 = -3$

$y'(0) = 1:\quad -3 + c_2 = 1,\ c_2 = 4$

$\therefore\ \underline{y(t) = -3e^t + 4te^t + \frac{1}{6}t^3e^t + 4}$

14.

Homogeneous: $r^2 + 4 = 0,\ r = \pm 2i$

$\therefore\ y_c(t) = c_1\cos(2t) + c_2\sin(2t)$

Particular: Let $y(t) = At\cos(2t) + Bt\sin(2t)$

Using MATLAB:

```
clear,clc
syms t A B
c2 = 1; c1 = 0; c0 = 4;  %coeffs of diff eq
w = 2;       %period of trig functions
y = (A*t)*cos(w*t) + (B*t)*sin(w*t);    %attempt
c2*diff(y,t,2) + c1*diff(y,t,1) + c0*y
```

ans = $4\,B\cos(2\,t) - 4\,A\sin(2\,t)$

$\therefore\ 4B\cos(2t) - 4A\sin(2t) = 3\sin(2t),\ B = 0,\ A = -\frac{3}{4}$

$\therefore\ y_p(t) = -\frac{3}{4}t\cos(2t)$

$\therefore\ y(t) = c_1\cos(2t) + c_2\sin(2t) - \frac{3}{4}t\cos(2t)$

$y(0) = 2:\quad c_1 = 2$

$y'(0) = -1:\quad y' = -4\sin(2t) + 2c_2\cos(2t) - \frac{3}{4}\cos(2t) + \frac{3}{2}t\sin(2t)$

$\therefore\ y'(0) = 2c_2 - \frac{3}{4} = -1,\ c_2 = -\frac{1}{8}$

$\therefore\ \underline{y(t) = 2\cos(2t) - \frac{1}{8}\sin(2t) - \frac{3}{4}t\cos(2t)}$

15.

Homogeneous: $r^2 + 2r + 5 = 0,\ r = \frac{-2 \pm \sqrt{4-20}}{2} = -1 \pm 2i$

$\therefore\ Y_c(t) = c_1e^{-t}\cos(2t) + c_2e^{-t}\sin(2t)$

Particular: Let $y(t) = Ate^{-t}\cos(2t) + Bte^{-t}\sin(2t)$

MATLAB:

```
clear,clc
syms t A B
c2 = 1; c1 = 2; c0 = 5;   %coeffs of diff eq
w = 2;        %period of trig functions
y = A*t*exp(-t)*cos(w*t) + B*t*exp(-t)*sin(w*t);  %attempt
c2*diff(y,t,2) + c1*diff(y,t,1) + c0*y
```

ans = $4\,B\cos(2\,t)\,e^{-t} - 4\,A\sin(2\,t)\,e^{-t}$

$\therefore\ 4B\,e^{-t}\cos(2t) - 4A\,e^{-t}\sin(2t) = 4e^{-t}\cos(2t)$

$\therefore\ B = 1,\ A = 0 \quad \therefore\ y_p(t) = t\,e^{-t}\sin(2t)$

$\therefore\ y(t) = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t) + t e^{-t}\sin(2t)$

$y(0) = 1:\ c_1 + 0 + 0 = 1,\ c_1 = 1$

$y'(0) = 0:\ y' = -e^{-t}\cos(2t) - 2e^{-t}\sin(2t)$
$\quad - c_2 e^{-t}\sin(2t) + 2c_2 e^{-t}\cos(2t)$
$\quad + e^{-t}\sin(2t) - t e^{-t}\sin(2t) + 2t e^{-t}\cos(2t)$

$\therefore\ y'(0) = -1 - 0 - 0 + 2c_2 + 0 - 0 + 0 = 0$

$\therefore\ 2c_2 = 1,\ c_2 = \frac{1}{2}$

$\therefore\ y(t) = e^{-t}\cos(2t) + \frac{1}{2}e^{-t}\sin(2t) + t e^{-t}\sin(2t)$

16.

Homogeneous: $r^2 + 3r = 0 = r(r+3)$, $r = 0, -3$

$\therefore y_c(t) = c_1 + c_2 e^{-3t}$

(a) For $P_n(t) = 2t^4$, $n = 4$, and since a constant solves the homogeneous equation,

$\therefore Y_1(t) = t\left(A_0 t^4 + A_1 t^3 + A_2 t^2 + A_3 t + A_4\right)$ *

For $P_n(t)e^{\alpha t} = t^2 e^{-3t}$, $n = 2$ and a constant solves the homogeneous equation.

$\therefore Y_2(t) = t\left(B_0 t^2 + B_1 t + B_2\right)e^{-3t}$ *

For $\sin(3t)$, $Y_3(t) = C\sin(3t) + D\cos(3t)$ *

(b) Using MATLAB,

$Y_1(t)$:

```
clear,clc
syms t A0 A1 A2 A3 A4
c2 = 1; c1 = 3; c0 = 0;  %coeffs of diff eq
y = t*(A0*t^4 + A1*t^3 + A2*t^2 + A3*t + A4);
P = c2*diff(y,t,2) + c1*diff(y,t,1) + c0*y;
collect(P,t)
```

ans = $(15\,A_0)\,t^4 + (20\,A_0 + 12\,A_1)\,t^3 + (12\,A_1 + 9\,A_2)\,t^2 + (6\,A_2 + 6\,A_3)\,t + 2\,A_3 + 3\,A_4$

Using the above answer, solve for coefficients:

```
format rat
A = [15 0 0 0 0; 20 12 0 0 0; 0 12 9 0 0; 0 0 6 6 0; 0 0 0 2 3];
B = [2 0 0 0 0]';
A\B
```

```
ans = 5×1
    2/15
   -2/9
    8/27
   -8/27
   16/81
```

B is from $2t^4$, so $B = [\overset{t^4}{2} \; \overset{t^3}{0} \; \overset{t^2}{0} \; \overset{t}{0} \; \overset{1}{0}]$

$\therefore A_0 = \frac{2}{15}, A_1 = -\frac{2}{9}, A_2 = \frac{8}{27}, A_3 = -\frac{8}{27}, A_4 = \frac{16}{81}$ *

$Y_2(t)$:

```
clear,clc
syms t B0 B1 B2
c2 = 1; c1 = 3; c0 = 0;  %coeffs of diff eq
y = t*(B0*t^2 + B1*t + B2)*exp(-3*t);
P = c2*diff(y,t,2) + c1*diff(y,t,1) + c0*y;
collect(P,t)
```

ans = $\left(-9\,B_0\,e^{-3\,t}\right)t^2 + \left(6\,B_0\,e^{-3\,t} - 6\,B_1\,e^{-3\,t}\right)t + 2\,B_1\,e^{-3\,t} - 3\,B_2\,e^{-3\,t}$

from t^2e^{-3t}

$\therefore -9B_0t^2 + (6B_0 - 6B_1)t + (2B_1 - 3B_2) = t^2$

Using the above answer, solve for coefficients:

```
format rat
A = [-9 0 0; 6 -6 0; 0 2 -3];
B = [1 0 0]';
A\B
```

ans = 3×1

	1
1	-1/9
2	-1/9
3	-2/27

$\therefore B_0 = -\frac{1}{9}, B_1 = -\frac{1}{9}, B_2 = -\frac{2}{27}$ *

$Y_3(t)$:

```
clear,clc
syms t C D
c2 = 1; c1 = 3; c0 = 0;  %coeffs of diff eq
y = C*sin(3*t) + D*cos(3*t);
P = c2*diff(y,t,2) + c1*diff(y,t,1) + c0*y;
```

ans =

$9\,C\cos(3\,t) - 9\,D\cos(3\,t) - 9\,C\sin(3\,t) - 9\,D\sin(3\,t)$ $= \sin(3t)$

$\therefore \ 9C - 9D = 0, \text{ or } C = D$

$-9C - 9D = 1, \ \therefore \ C = D = -\frac{1}{18}$ *

$$\therefore \ \underline{Y(t)} = C_1 + C_2 e^{-3t}$$
$$+ \frac{2}{15}t^5 - \frac{2}{9}t^4 + \frac{8}{27}t^3 - \frac{8}{27}t^2 + \frac{16}{81}t$$
$$+ \left(-\frac{1}{9}t^3 - \frac{1}{9}t^2 - \frac{2}{27}t\right)e^{-3t}$$
$$-\frac{1}{18}\sin(3t) - \frac{1}{18}\cos(3t)$$

17.

Homogeneous: $r^2 - 5r + 6 = (r-3)(r-2)$, $r = 3, 2$

$\therefore \ y_c(t) = c_1 e^{2t} + c_2 e^{3t}$

(a) For $e^t \cos(2t)$, let $Y_1(t) = Ae^t\cos(2t) + Be^t\sin(2t)$ *

For $e^{2t}(3t+4)\sin(t)$, although e^{2t} solves the homogeneous equation, $e^{(2\pm i)t}$ does not.

$Y_2(t) = (Ct + D)e^{2t}\cos(t) + (Et + F)e^{2t}\sin(t)$ *

$$\therefore \ \underline{Y(t)} = Ae^t\cos(2t) + Be^t\sin(2t)$$
$$+ (Ct + D)e^{2t}\cos(t) + (Et + F)e^{2t}\sin(t)$$

(6) Using MATLAB:

$Y_1(t)$:

```
clear,clc
syms t A B
c2 = 1; c1 = -5; c0 = 6;  %coeffs of diff eq
y = A*exp(t)*cos(2*t) + B*exp(t)*sin(2*t);
P = c2*diff(y,t,2) + c1*diff(y,t,1) + c0*y;
collect(P, [cos(2*t) sin(2*t)])
```

ans = $(-2\,A\,e^t - 6\,B\,e^t)\cos(2\,t) + (6\,A\,e^t - 2\,B\,e^t)\sin(2\,t)$ $= e^t\cos(2t)$

$\therefore \left.\begin{aligned} -2A - 6B &= 1 \\ 6A - 2B &= 0 \end{aligned}\right\} \quad -20B = 3,\ B = -3/20 \qquad A = \frac{1}{3}B,\ A = -1/20$

$\therefore Y_1(t) = -\frac{1}{20}e^t\cos(2t) - \frac{3}{20}e^t\sin(2t)$

$Y_2(t)$:

```
clear,clc
syms t C D E F
c2 = 1; c1 = -5; c0 = 6;  %coeffs of diff eq
y = exp(2*t)*((C*t+D)*cos(t) + (E*t+F)*sin(t));
P = c2*diff(y,t,2) + c1*diff(y,t,1) + c0*y;
collect(P, [cos(t) sin(t)])
```

ans = $(-e^{2t}\,(C + F + E\,t) - e^{2t}\,(D - 2\,E + C\,t))\cos(t) + (e^{2t}\,(D - E + C\,t) - e^{2t}\,(2\,C + F + E\,t))\sin(t)$

$\therefore -e^{2t}\left[(C+F+D-2E) + (E+C)t\right]\cos(t) +$

$e^{2t}\left[(D-E-2C-F) + (C-E)t\right]\sin(t) = e^{2t}(3t+4)\sin(t)$

$\therefore \left.\begin{aligned} E + C &= 0 \\ C - E &= 3 \end{aligned}\right\} \quad 2C = 3,\ C = \frac{3}{2},\ E = -\frac{3}{2}$

$\left.\begin{aligned} C + F + D - 2E &= 0 \\ D - E - 2C - F &= 4 \end{aligned}\right\} \quad \begin{aligned} F + D &= 2E - C = -\frac{9}{2} \\ D - F &= 4 + E + 2C = \frac{11}{2} \end{aligned}$

$\therefore 2D = 1,\ D = \frac{1}{2},\ F = -5$

$\therefore Y_2(t) = \left(\frac{3}{2}t + \frac{1}{2}\right)e^{2t}\cos(t) + \left(-\frac{3}{2}t - 5\right)e^{2t}\sin(t)$

$$\therefore \underline{Y(t)} = c_1 e^{2t} + c_2 e^{3t}$$
$$- \frac{1}{20} e^{t} \cos(2t) - \frac{3}{20} e^{t} \sin(2t)$$
$$+ \left(\frac{3}{2}t + \frac{1}{2}\right) e^{2t} \cos(t) + \left(-\frac{3}{2}t - 5\right) e^{2t} \sin(t)$$

18.

Homogeneous: $r^2 + 2r + 2 = 0$, $r = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$

$$\therefore Y_c(t) = c_1 e^{-t} \cos(t) + c_2 e^{-t} \sin(t)$$

(a) For $3e^{-t}$, let $Y_1(t) = A e^{-t}$ *

For $2e^{-t}\cos(t) + 4e^{-t}t^2 \sin(t)$, $e^{(-1 \pm i)t}$ is

already a solution to the homogeneous equation,

$$\therefore \text{Let } Y_2(t) = t(B_0 t^2 + B_1 t + B_2) e^{-t} \cos(t)$$
$$+ t(C_0 t^2 + C_1 t + C_2) e^{-t} \sin(t)$$

$$\therefore \underline{Y(t)} = A e^{-t}$$
$$+ t(B_0 t^2 + B_1 t + B_2) e^{-t} \cos(t)$$
$$+ t(C_0 t^2 + C_1 t + C_2) e^{-t} \sin(t)$$

(b) For $Y_1(t)$, $y_1'' + 2y_1' + 2y = Ae^{-t} + 2(-A)e^{-t} + 2Ae^{-t}$

$\therefore A - 2A + 2A = 3$ (from $3e^{-t}$)

$\therefore A = 3$, $Y_1(t) = 3e^{-t}$

$Y_2(t)$: Use MATLAB:

```
clear,clc
syms t B0 B1 B2 C0 C1 C2
c2 = 1; c1 = 2; c0 = 2;  %coeffs of diff eq
y = exp(-t)*(t*(B0*t^2 + B1*t + B2)*cos(t) ...
        + t*(C0*t^2 + C1*t + C2)*sin(t));
P = c2*diff(y,t,2) + c1*diff(y,t,1) + c0*y;
%Put coeffs of t terms in array T, terms in array D desc order
[T,D] = coeffs(P, t, 'All');
D  % display D
collect(T(1), [cos(t) sin(t)])  % t^2 coefficient
collect(T(2), [cos(t) sin(t)])  % t^1 coefficient
collect(T(3), [cos(t) sin(t)])  % t^0 coefficient
```

D = $(t^2 \quad t \quad 1)$

t^2 : ans = $\left(B_1 e^{-t} - e^{-t}\,(B_1 - 6\,C_0)\right)\cos(t) + \left(C_1 e^{-t} - e^{-t}\,(6\,B_0 + C_1)\right)\sin(t)$

cancel cancel

$\therefore [6C_0\, e^{-t}\cos(t) + (-6B_0)\, e^{-t}\sin(t)]\,t^2 = 4t^2 e^{-t}\sin(t)$

$\therefore 6C_0 = 0$, $C_0 = 0$ $-6B_0 = 4$, $B_0 = -2/3$

t^1 : ans = $\left(B_2 e^{-t} + e^{-t}\,(6\,B_0 - B_2 + 4\,C_1)\right)\cos(t) + \left(C_2 e^{-t} - e^{-t}\,(4\,B_1 - 6\,C_0 + C_2)\right)\sin(t)$

cancel cancel

$6B_0 + 4C_1 = 0$ $-4B_1 + 6C_0 = 0$ (no te^{-t} term)

$6(-\frac{2}{3}) + 4C_1 = 0$, $C_1 = 1$ As $C_0 = 0$, $\therefore B_1 = 0$

t^0 : ans = $\left(e^{-t}\,(2\,B_1 + 2\,C_2)\right)\cos(t) + \left(-e^{-t}\,(2\,B_2 - 2\,C_1)\right)\sin(t)$ $= 2e^{-t}\cos(t)$

$2B_1 + 2C_2 = 2$ $2B_2 - 2C_1 = 0$

As $B_1 = 0$, $C_2 = 1$ $B_2 = C_1 = 1$

Or, from matrices,

$$
\begin{matrix} t^2e^{-t}\cos(t) \\ t^2e^{-t}\sin(t) \\ t^1e^{-t}\cos(t) \\ t^1e^{-t}\sin(t) \\ t^0e^{-t}\cos(t) \\ t^0e^{-t}\sin(t) \end{matrix}
\begin{bmatrix} 0 & 0 & 0 & 6 & 0 & 0 \\ -6 & 0 & 0 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 & 4 & 0 \\ 0 & -4 & 0 & 6 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 & -2 & 0 \end{bmatrix}
\begin{bmatrix} B_0 \\ B_1 \\ B_2 \\ C_0 \\ C_1 \\ C_2 \end{bmatrix}
=
\begin{bmatrix} 0 \\ 4 \\ 0 \\ 0 \\ 2 \\ 0 \end{bmatrix}
\begin{matrix} t^2 e^{-t}\cos(t) \\ t^2 e^{-t}\sin(t) \\ t\, e^{-t}\cos(t) \\ t\, e^{-t}\sin(t) \\ 1\, e^{-t}\cos(t) \\ 1\, e^{-t}\sin(t) \end{matrix}
$$

↑ coefficients of above collected terms

↑ coefficients for $Y_2(t)$

↑ coefficients of terms from $2e^{-t}\cos t + 4e^{-t}t^2\sin t$

```
A = [0 0 0 6 0 0;...
    -6 0 0 0 0 0;...
     6 0 0 0 4 0;...
     0 -4 0 6 0 0;...
     0 2 0 0 0 2;...
     0 0 2 0 -2 0];
%X = [B0 B1 B2 C0 C1 C2]';
Z = [0 4 0 0 2 0]';
format rat
X = A\Z
```

```
X = 6×1

    -2/3
      0
      1
      0
      1
      1
```

$-2/3 \to B_0$, $0 \to B_1$, $1 \to B_2$, $0 \to C_0$, $1 \to C_1$, $1 \to C_2$

$$
\begin{aligned}
\therefore\ Y(t) &= c_1 e^{-t}\cos(t) + c_2 e^{-t}\sin(t) \\
&\quad + 3e^{-t} \\
&\quad + t\left(-\tfrac{2}{3}t^2 + 1\right)e^{-t}\cos(t) \\
&\quad + t(t+1)e^{-t}\sin(t)
\end{aligned}
$$

19.

Homogeneous: $r^2 + 4 = 0$, $r = \pm 2i$

$\therefore y_c(t) = c_1 \cos(2t) + c_2 \sin(2t)$

(a) Since e^{2i} is already a solution to homogeneous equation, use $t P_n(t)$, $n = 2$ from t^2.

$$\therefore \underline{y_p(t)} = t(A_0 t^2 + A_1 t + A_2)\cos(2t) + t(B_0 t^2 + B_1 t + B_2)\sin(2t)$$

(b) Use MATLAB:

```
clear,clc
syms t A0 A1 A2 B0 B1 B2
c2 = 1; c1 = 0; c0 = 4;  %coeffs of diff eq
y = t*(A0*t^2 + A1*t + A2)*cos(2*t) + ...
    t*(B0*t^2 + B1*t + B2)*sin(2*t);
P = c2*diff(y,t,2) + c1*diff(y,t,1) + c0*y;
%Put coeffs of t terms in array T, terms in array D desc order
[T,D] = coeffs(P, t, 'All');
D  % display degree of terms
collect(T(1), [cos(t) sin(t)])  % t^2 coefficient
collect(T(2), [cos(t) sin(t)])  % t^1 coefficient
collect(T(3), [cos(t) sin(t)])  % t^0 coefficient
```

Coefficients for: D = $(t^2 \quad t \quad 1)$

t^2: ans = $12 B_0 \cos(2t) - 12 A_0 \sin(2t)$

t^1: ans = $6 A_0 \cos(2t) + 8 B_1 \cos(2t) - 8 A_1 \sin(2t) + 6 B_0 \sin(2t)$

t^0: ans = $2 A_1 \cos(2t) + 4 B_2 \cos(2t) - 4 A_2 \sin(2t) + 2 B_1 \sin(2t)$

Use matrices to solve: $AX = Z$

$$\therefore A = \begin{bmatrix} 0 & 0 & 0 & 12 & 0 & 0 \\ -12 & 0 & 0 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 & 8 & 0 \\ 0 & -8 & 0 & 6 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 4 \\ 0 & 0 & -4 & 0 & 2 & 0 \end{bmatrix} \quad X = \begin{bmatrix} A_0 \\ A_1 \\ A_2 \\ B_0 \\ B_1 \\ B_2 \end{bmatrix} \quad Z = \begin{bmatrix} 0 \\ 1 \\ 6 \\ 0 \\ 7 \\ 0 \end{bmatrix} \begin{matrix} t^2 \cos \\ t^2 \sin \\ t \cos \\ t \sin \\ 1 \cos \\ 1 \sin \end{matrix}$$

from $t^2 \sin(2t) + (6t+7)\cos(2t)$ the right hand side of the diff. eq.

Using MATLAB:

```
A = [   0   0   0 12   0   0; ...
     -12   0   0   0   0   0; ...
       6   0   0   0   8   0; ...
       0  -8   0   6   0   0; ...
       0   2   0   0   0   4; ...
       0   0  -4   0   2   0];
%X = [A0 A1 A2 B0 B1 B2]'
Z = [0 1 6 0 7 0]';
format rat;   %answers in fractions
X = A\Z
```

```
X = 6×1

     -1/12    A0
      0       A1
     13/32    A2
      0          B0
     13/16       B1
      7/4        B2
```

$$\therefore \underline{y_p(t)} = t\left(-\frac{1}{12}t^2 + \frac{13}{32}\right)\cos(2t) + t\left(\frac{13}{16}t + \frac{7}{4}\right)\sin(2t)$$

20.

Homogeneous: $r^2 + 3r + 2 = (r+2)(r+1)$, $r = -2, -1$

$$\therefore y_c(t) = c_1 e^{-2t} + c_1 e^{-t}$$

(a) There are 3 groups:

$e^t(t^2+1)\sin(2t)$: $Y_{p_1}(t) = (A_0 t^2 + A_1 t + A_2)\cos(2t)e^t$
$+ (B_0 t^2 + B_1 t + B_2)\sin(2t)e^t$ (1)

$3e^{-t}\cos(t)$: e^{-t} is a homogeneous solution, but $e^{(-1\pm i)t}$ is not

$\therefore Y_{p_2}(t) = e^{-t}[D\cos(t) + E\sin(t)]$ (2)

$4e^t$: $Y_{p_3}(t) = Fe^t$ (3)

(b) $Y_{p_1}(t)$: Use MATLAB

```
clear,clc
syms t A0 A1 A2 B0 B1 B2
c2 = 1; c1 = 3; c0 = 2;  %coeffs of diff eq
y = (A0*t^2 + A1*t + A2)*exp(t)*cos(2*t) + ...
    (B0*t^2 + B1*t + B2)*exp(t)*sin(2*t);
P = c2*diff(y,t,2) + c1*diff(y,t,1) + c0*y;
%Put coeffs of t terms in array T, terms in array D desc order
[T,D] = coeffs(P, t, 'All');
D  % display D
collect(T(1), [cos(2*t) sin(2*t)])  % t^2 coefficient
collect(T(2), [cos(2*t) sin(2*t)])  % t coefficient
collect(T(3), [cos(2*t) sin(2*t)])  % 1 coefficient
```

Coefficients for: D = $(t^2 \quad t \quad 1)$

t^2: ans = $(2A_0 e^t + 10B_0 e^t)\cos(2t) + (2B_0 e^t - 10A_0 e^t)\sin(2t)$

t: ans = $(10A_0 e^t + 2A_1 e^t + 8B_0 e^t + 10B_1 e^t)\cos(2t) + (10B_0 e^t - 10A_1 e^t - 8A_0 e^t + 2B_1 e^t)\sin(2t)$

1: ans = $(2A_0 e^t + 5A_1 e^t + 2A_2 e^t + 4B_1 e^t + 10B_2 e^t)\cos(2t) +$
$(2B_0 e^t - 10A_2 e^t - 4A_1 e^t + 5B_1 e^t + 2B_2 e^t)\sin(2t)$

Use matrices to solve $AX = Z$

$$A = \begin{bmatrix} 2 & 0 & 0 & 10 & 0 & 0 \\ -10 & 0 & 0 & 2 & 0 & 0 \\ 10 & 2 & 0 & 8 & 10 & 0 \\ -8 & -10 & 0 & 10 & 2 & 0 \\ 2 & 5 & 2 & 0 & 4 & 10 \\ 0 & -4 & -10 & 2 & 5 & 2 \end{bmatrix} \quad X = \begin{bmatrix} A_0 \\ A_1 \\ A_2 \\ B_0 \\ B_1 \\ B_2 \end{bmatrix} \quad Z = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \begin{matrix} t^2\ e^t \cos \\ t^2\ e^t \sin \\ t\ e^t \cos \\ t\ e^t \sin \\ 1\ e^t \cos \\ 1\ e^t \sin \end{matrix}$$

↑ read numbers from above MATLAB answers

↖ From $e^t(t^2+1)\sin(2t)$

Using MATLAB:

```
A = [  2    0    0   10    0    0; ...
     -10    0    0    2    0    0; ...
      10    2    0    8   10    0; ...
      -8  -10    0   10    2    0; ...
       2    5    2    0    4   10; ...
       0   -4  -10    2    5    2];
%X = [A0 A1 A2 B0 B1 B2]'
Z = [0   1   0   0   0   1]';
format rat   %answers in fractions
X = A\Z
```

X = 6×1

	1	
1	-5/52	A_0
2	73/676	A_1
3	-87/745	A_2
4	1/52	B_0
5	10/169	B_1
6	-161/4590	B_2

$$\therefore\ Y_{p_1}(t) = \left(-\frac{5}{52}t^2 + \frac{73}{676}t - \frac{87}{745}\right)e^t\cos(2t)$$
$$+ \left(\frac{1}{52}t^2 + \frac{10}{169} - \frac{161}{4590}\right)e^t\sin(2t)$$

Note: MATLAB's "format rat" yields a rational "approximation" to the decimal value of a variable. So, $-\frac{87}{745} = -0.116778523\ldots$ and the text answer: $-\frac{4105}{35{,}152} = -0.116778561\ldots$ for A_2

To get fractions at back of text, use MATLAB's "rats"

```
A = [  2    0    0   10    0    0; ...
     -10    0    0    2    0    0; ...
      10    2    0    8   10    0; ...
      -8  -10    0   10    2    0; ...
       2    5    2    0    4   10; ...
       0   -4  -10    2    5    2];
%X = [A0 A1 A2 B0 B1 B2]'
Z = [0    1    0    0    0    1]';
format rat    %answers in fractions
X = rats(A\Z, 18)
```

```
X = 6×19 char array
   '      -5/52        '
   '      73/676       '
   '   -4105/35152     '
   '       1/52        '
   '      10/169       '
   '   -1233/35152     '
```

↑ increase accuracy by increasing this number

$Y_{p_2}(t)$: $D e^{-t}\cos(t) + E e^{-t}\sin(t)$

MATLAB:

```
clear,clc
syms t D E
c2 = 1; c1 = 3; c0 = 2;  %coeffs of diff eq
y = D*exp(-t)*cos(t) + E*exp(-t)*sin(t);
P = c2*diff(y,t,2) + c1*diff(y,t,1) + c0*y
```

$P = E\,e^{-t}\cos(t) - D\,e^{-t}\cos(t) - D\,e^{-t}\sin(t) - E\,e^{-t}\sin(t)$

$\therefore E - D = 3$ (from $3e^{-t}\cos(t)$)

$-E - D = 0$ (no $e^{-t}\sin(t)$ term)

$\therefore -2D = 3$, $D = -\frac{3}{2}$, $E = \frac{3}{2}$

$\therefore Y_{p_2}(t) = -\frac{3}{2}e^{-t}\cos(t) + \frac{3}{2}e^{-t}\sin(t)$

Y_{p_3}: $(Fe^{t})'' + 3(Fe^{t})' + 2(Fe^{t}) = 4e^{t}$

$\therefore F + 3F + 2F = 4$, $6F = 4$, $F = 2/3$

$\therefore Y_{p_3}(t) = \frac{2}{3}e^{t}$

21.

Homogeneous: $r^2+2r+5=0$, $r=\frac{-2\pm\sqrt{4-20}}{2}=-1\pm 2i$

$\therefore\ y_c(t)=c_1e^{-t}\cos(2t)+c_2e^{-t}\sin(2t)$

(a) (1) $3te^{-t}\cos(2t)$: Since $e^{(-1\pm 2i)t}$ a solution

to homogeneous, look at $te^{(-1\pm 2i)t}$, or,

$\underline{Y_{p_1}(t)}=t(A_0t+A_1)e^{-t}\cos(2t)+t(B_0t+B_1)e^{-t}\sin(2t)$

(2) $-2te^{-2t}\cos(t)$: $e^{(-2\pm i)t}$ is not a

solution to homogeneous,

$\therefore\ \underline{Y_{p_2}(t)}=(D_0t+D_1)e^{-2t}\cos(t)+(E_0t+E_1)e^{-2t}\sin(t)$

(b) Using MATLAB separately for (1) and (2)

(1) $Y_{p_1}(t)=t(A_0t+A_1)e^{-t}\cos(2t)+t(B_0t+B_1)e^{-t}\sin(2t)$

for: $3te^{-t}\cos(2t)$

```
clear,clc
syms t A0 A1 B0 B1
c2 = 1; c1 = 2; c0 = 5;  %coeffs of diff eq
y = t*(A0*t + A1)*exp(-t)*cos(2*t) + ...
    t*(B0*t + B1)*exp(-t)*sin(2*t);
P = c2*diff(y,t,2) + c1*diff(y,t,1) + c0*y;
%Put coeffs of t terms in array T, terms in array D desc order
[T,D] = coeffs(P, t, 'All');
D  % display D
collect(T(1), [cos(2*t) sin(2*t)])  % t^1 coefficient
collect(T(2), [cos(2*t) sin(2*t)])  % t^0 coefficient
```

Coefficients for: D = (t   1)

t : ans = $(8 B_0 e^{-t})\cos(2t) + (-8 A_0 e^{-t})\sin(2t)$

1 : ans = $(2 A_0 e^{-t} + 4 B_1 e^{-t})\cos(2t) + (2 B_0 e^{-t} - 4 A_1 e^{-t})\sin(2t)$

Use matrices to solve $AX = Z$

$$A = \begin{bmatrix} 0 & 0 & 8 & 0 \\ -8 & 0 & 0 & 0 \\ 2 & 0 & 0 & 4 \\ 0 & -4 & 2 & 0 \end{bmatrix} \quad X = \begin{bmatrix} A_0 \\ A_1 \\ B_0 \\ B_1 \end{bmatrix} \quad Z = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{matrix} t\,e^{-t}\cos(2t) \\ t\,e^{-t}\sin(2t) \\ 1\,e^{-t}\cos(2t) \\ 1\,e^{-t}\sin(2t) \end{matrix}$$

read numbers from MATLAB ans above

↑ from $3te^{-t}\cos(2t)$

```
A = [  0    0    8    0; ...
      -8    0    0    0; ...
       2    0    0    4; ...
       0   -4    2    0];
%X = [A0 A1 B0 B1]'
Z = [3    0    0    0]';
X = rats(A\Z)
```

```
X = 4×14 char array
    '       0      '      A0
    '     3/16     '      A1
    '      3/8     '      B0
    '       0      '      B1
```

$$\therefore\ Y_{p_1}(t) = \frac{3}{16} t e^{-t}\cos(2t) + \frac{3}{8} t^2 e^{-t}\sin(2t)$$

(2) $Y_{p_2}(t) = (D_0 t + D_1)e^{-2t}\cos(t) + (E_0 t + E_1)e^{-2t}\sin(t)$

for: $-2te^{-2t}\cos t$

Using MATLAB:

```
clear,clc
syms t D0 D1 E0 E1
c2 = 1; c1 = 2; c0 = 5;  %coeffs of diff eq
y = (D0*t + D1)*exp(-2*t)*cos(t) + ...
    (E0*t + E1)*exp(-2*t)*sin(t);
P = c2*diff(y,t,2) + c1*diff(y,t,1) + c0*y;
%Put coeffs of t terms in array T, terms in array D desc order
[T,D] = coeffs(P, t, 'All');
D  % display D
collect(T(1), [cos(t) sin(t)])  % t coefficient
collect(T(2), [cos(t) sin(t)])  % 1 coefficient
```

Coefficients for: D = $(t \quad 1)$

t : ans = $\left(4\,D_0\,e^{-2t} - 2\,E_0\,e^{-2t}\right)\cos(t) + \left(2\,D_0\,e^{-2t} + 4\,E_0\,e^{-2t}\right)\sin(t)$

1 : ans = $4\,D_1\,e^{-2t}\cos(t) - 2\,D_0\,e^{-2t}\cos(t) + 2\,E_0\,e^{-2t}\cos(t) - 2\,E_1\,e^{-2t}\cos(t)$

$- 2\,D_0\,e^{-2t}\sin(t) + 2\,D_1\,e^{-2t}\sin(t) - 2\,E_0\,e^{-2t}\sin(t) + 4\,E_1\,e^{-2t}\sin(t)$

Use matrices to solve $AX = Z$

$$A = \begin{bmatrix} 4 & 0 & -2 & 0 \\ 2 & 0 & 4 & 0 \\ -2 & 4 & 2 & -2 \\ -2 & 2 & -2 & 4 \end{bmatrix} \quad X = \begin{bmatrix} D_0 \\ D_1 \\ E_0 \\ E_1 \end{bmatrix} \quad Z = \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{matrix} t\,e^{-2t}\cos(t) \\ t\,e^{-2t}\sin(t) \\ 1\,e^{-2t}\cos(t) \\ 1\,e^{-2t}\sin(t) \end{matrix}$$

read numbers from above MATLAB ans

from $-2te^{-2t}\cos t$

```
A = [  4   0  -2   0; ...
       2   0   4   0; ...
      -2   4   2  -2; ...
      -2   2  -2   4];
%X = [D0 D1 E0 E1]'
Z = [-2   0   0   0]';
X = rats(A\Z)
```

```
X = 4×14 char array
    '     -2/5     '
    '     -7/25    '
    '      1/5     '
    '      1/25    '
```

D_0, D_1, E_0, E_1

$$\therefore\ y_{p_2}(t) = \left(-\frac{2}{5}t - \frac{7}{25}\right)e^{-2t}\cos(t) + \left(\frac{1}{5}t + \frac{1}{25}\right)e^{-2t}\sin(t)$$

22.

(a)

$y(t) = v(t)e^{-t}$, $y'(t) = v'e^{-t} - ve^{-t}$

$y''(t) = v''e^{-t} - v'e^{-t} - v'e^{-t} + ve^{-t}$

$= v''e^{-t} - 2v'e^{-t} + ve^{-t}$

$\therefore (v''e^{-t} - 2v'e^{-t} + ve^{-t}) - 3(v'e^{-t} - ve^{-t}) - 4(ve^{-t})$

$= v''e^{-t} - 5v'e^{-t} + (ve^{-t} + 3ve^{-t} - 4ve^{-t})$

$= v''e^{-t} - 5v'e^{-t} = 2e^{-t}$

$\therefore \underline{v'' - 5v' = 2}$

(b)

Since $w(t) = v'(t)$, $w'(t) = v''(t)$. Substituting

$v'' - 5v' = 2$, $\underline{w' - 5w = 2}$

Integrating factor is $\exp\left(\int -5\,dt\right) = e^{-5t}$

$\therefore\ w' e^{-5t} - 5e^{-5t} w = 2e^{-5t}$, or

$$\frac{d}{dt}\left(w e^{-5t}\right) = 2e^{-5t}$$

$\therefore\ w e^{-5t} = -\frac{2}{5} e^{-5t} + K$, K a constant

$$\therefore\ w = -\frac{2}{5} + K e^{5t}$$

(c)

From (b) $w = v' = -\frac{2}{5} + K e^{5t}$

Integrating, $v = -\frac{2}{5} t + \frac{K}{5} e^{5t} + c_2$

Letting $c_1 = \frac{K}{5}$, $v(t) = -\frac{2}{5} t + c_1 e^{5t} + c_2$

$$\therefore\ Y(t) = v(t) e^{-t} = -\frac{2}{5} t e^{-t} + c_1 e^{4t} + c_2 e^{-t}$$

23.

Homogeneous : $r^2 + \lambda^2 = 0$, $r = \pm \lambda i$

$$\therefore \underline{y_c(t)} = c_1 \cos(\lambda t) + c_2 \sin(\lambda t)$$

Since $\lambda \neq m\pi$ for $m = 1, \dots, N$,

$$\text{let } y_p(t) = \sum_{m=1}^{N} \left[A_m \cos(m\pi t) + B_m \sin(m\pi t) \right]$$

$$\therefore y_p' = \sum_{m=1}^{N} \left[-m\pi A_m \sin(m\pi t) + m\pi B_m \cos(m\pi t) \right]$$

$$y_p'' = \sum_{m=1}^{N} \left[-m^2\pi^2 A_m \cos(m\pi t) - m^2\pi^2 B_m \sin(m\pi t) \right]$$

$$\therefore y_p'' + \lambda^2 y_p = \sum_{m=1}^{N} (\lambda^2 - m^2\pi^2) \left[A_m \cos(m\pi t) + B_m \sin(m\pi t) \right]$$

Since $\sum_{m=1}^{N} a_m \sin(m\pi t)$ has no $\cos(m\pi t)$ term,

$$\therefore (\lambda^2 - m^2\pi^2) A_m = 0 \quad \text{and} \quad (\lambda^2 - m^2\pi^2) B_m = a_m$$

Or, $A_m = 0$, and $B_m = a_m / (\lambda^2 - m^2\pi^2)$

since $\lambda^2 \neq m^2\pi^2$

$$\therefore \underline{y_p(t)} = \sum_{m=1}^{N} \left(\frac{a_m}{\lambda^2 - m^2\pi^2} \right) \sin(m\pi t)$$

$$\therefore \underline{y(t) = c_1 \cos(\lambda t) + c_2 \sin(\lambda t) + \sum_{m=1}^{N} \left(\frac{a_m}{\lambda^2 - m^2\pi^2} \right) \sin(m\pi t)}$$

24.

Homogeneous: $r^2+1=0$, $r=\pm i$,

$\therefore y_c(t) = c_1\cos(t) + c_2\sin(t)$

Particular:

(1) t: Let $y_{p_1}(t) = At+B$

$\therefore y_{p_1}'' + y_{p_1} = At+B = t$, $A=1$, $B=0$

$\therefore \underline{y_{p_1}(t) = t}$

(2) $\pi e^{\pi-t}$: Let $y_{p_2}(t) = Ae^{-t}$

$\therefore y_{p_2}'' + y = Ae^{-t} + Ae^{-t} = 2Ae^{-t} = \pi e^{\pi-t}$

$\therefore A = \frac{\pi}{2}e^{\pi}$ $\therefore y_{p_2}(t) = \frac{\pi}{2}e^{\pi-t}$

$$\therefore y(t) = \begin{cases} c_1\cos(t) + c_2\sin(t) + t, & 0 \le t \le \pi \\ c_1\cos(t) + c_2\sin(t) + \frac{\pi}{2}e^{\pi-t}, & t > \pi \end{cases}$$

Using $y(0) = 0$, $y'(0) = 1$,

(1) $c_1 + 0 + 0 = 0 \Rightarrow c_1 = 0$ for $0 \le t \le \pi$

$y'(t) = c_2 \cos(t) + 1 = 1 \Rightarrow c_2 = 0$, $0 \le t \le \pi$

$\therefore y(t) = t$, $0 \le t \le \pi$

(2) Since $y(t)$ is continuous at $t = \pi$,

$y(\pi) = \pi$ from (1), and $y'(t) = 1$ as $t \to \pi^-$

$c_1 \cos(\pi) + c_2 \sin(\pi) + \frac{\pi}{2} e^{\pi - \pi} = -c_1 + \frac{\pi}{2}$

$\therefore -c_1 + \frac{\pi}{2} = \pi \Rightarrow c_1 = -\frac{\pi}{2}$, $t > \pi$

$\therefore y'(t) = \frac{\pi}{2} \sin(t) + c_2 \cos(t) - \frac{\pi}{2} e^{\pi - t}$

$\therefore$ as $t \to \pi^+$, $\lim_{t \to \pi^+} y'(t) = -c_2 - \frac{\pi}{2}$

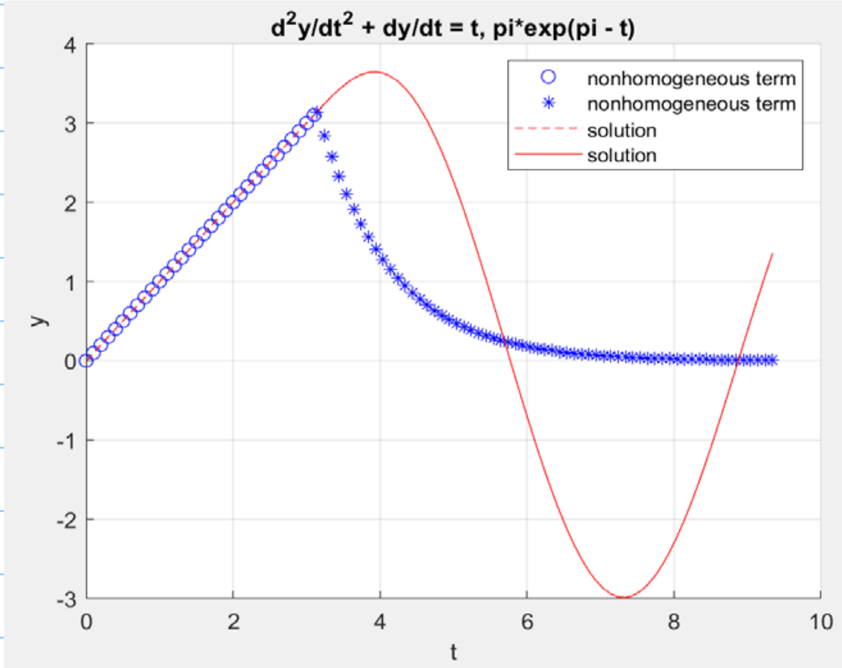
$\therefore -c_2 - \frac{\pi}{2} = 1$, $c_2 = -1 - \frac{\pi}{2}$, $t > \pi$

$\therefore y(t) = -\frac{\pi}{2} \cos(t) - \left(1 + \frac{\pi}{2}\right) \sin(t) + \frac{\pi}{2} e^{\pi - t}$, $t > \pi$

$$\therefore y(t) = \begin{cases} t, & 0 \le t \le \pi \\ -\frac{\pi}{2} \cos(t) - \left(1 + \frac{\pi}{2}\right) \sin(t) + \frac{\pi}{2} e^{\pi - t}, & t > \pi \end{cases}$$

Using MATLAB to plot

```
clear, clc
t = 0:0.1:pi;
s = pi:0.1:3*pi;
y1 = t;
y2 = (-pi/2)*cos(s) - (1+pi/2)*sin(s) + (pi/2)*exp(pi-s);
hold on
plot(t, t, 'ob')                % nonhomogeneous term
plot(s, pi*exp(pi-s), '*b')  % nonhomogeneous term
plot(t, y1, '--r')
plot(s, y2, 'r')
grid on
xlabel 't', ylabel 'y'
legend('nonhomogeneous term', 'nonhomogeneous term', ...
       'solution', 'solution')
title 'd^2y/dt^2 + dy/dt = t, pi*exp(pi - t)'
```



25.

Since $Y_1(t) - Y_2(t)$ is a solution to $ay'' + by' + cy = 0$, then by problem #28, Section 3.4, $Y_1(t) - Y_2(t) \to 0$ as $t \to \infty$

If $b = 0$, result <u>not</u> true, by problem #29, Sec. 3.4

26.

(a) Let $Y(t)$ be any solution to $ay'' + by' + cy = d$, $c \neq 0$. $\therefore$ $Y(t) - \frac{d}{c}$ is a solution to $ay'' + by' + cy = 0$, since:

$$a\left(y - \frac{d}{c}\right)'' + b\left(y - \frac{d}{c}\right)' + c\left(y - \frac{d}{c}\right) =$$

$$(ay'' + by' + cy) - d = d - d = 0$$

$\therefore$ By #25 above, $\lim_{t \to \infty} \left[Y(t) - \frac{d}{c} \right] = 0$, or

equivalently, $\lim_{t \to \infty} Y(t) = \frac{d}{c}$

(b) If $c = 0$, then if $Y_1(t)$ and $Y_2(t)$ are solutions to $ay'' + by' = d$, $a, b > 0$, then by #29.(b) of Section 3.4, $\lim_{t \to \infty} Y_1(t) - Y_2(t) = K$, K some constant.

$\therefore$ If $Y(t)$ is any solution to $ay'' + by' = d$, then $y - \frac{d}{b}t$ is a solution to $ay'' + by' = 0$ as $a\left(y - \frac{d}{b}t\right)'' + b\left(y - \frac{d}{b}t\right)' =$

$$ay'' + by' - d = d - d = 0$$

$$\therefore \lim_{t \to \infty} \left[Y(t) - \frac{d}{b}t \right] = K \iff \lim_{t \to \infty} \left[Y(t) - \left(\frac{d}{b}t + K \right) \right] = 0$$

$\therefore \lim_{t \to \infty} |Y(t)| = \infty$ since $\lim_{t \to \infty} \left| \frac{d}{b}t + K \right| = \infty$

for if $|Y(t)|$ were bounded, then since

$|Y(t) - (\frac{d}{b}t + k)|$ is bounded, $|\frac{d}{b}t + k|$ would be bounded, a contradiction.

(c) If $b = 0$ and $c = 0$, then $ay'' = d$, $y'' = \frac{d}{a}$,

$Y(t) = \frac{d}{2a}t^2 + c_1 t + c_2$, so $\lim_{t\to\infty} |Y(t)| = \infty$

27.

(a)

$(D - r_1)(D - r_2)y = (D - r_1)(Dy - r_2 y) = DDy - r_2 Dy - r_1 Dy + r_1 r_2 y$

$= (D^2 - (r_1 + r_2)D + r_1 r_2)y = (D^2 + bD + c)y$

(b)

Simple substitution yields $(D - r_1)(D - r_2)y = g(t)$, and by (a), this yields (36)

28.

$$r^2-3r-4=(r-4)(r+1). \quad \therefore (D-4)(D+1)y=3e^{2t}$$

$\therefore$ Let $u=(D+1)y$

(1) $\therefore (D-4)u = u'-4u = 3e^{2t}$

Integrating factor : e^{-4t}

$$\therefore \frac{d}{dt}\left(e^{-4t}u\right) = 3e^{2t}e^{-4t} = 3e^{-2t}$$

$$\therefore e^{-4t}u = -\frac{3}{2}e^{-2t} + c_1', \quad c_1' \text{ a constant}$$

$$\therefore u = -\frac{3}{2}e^{2t} + c_1'e^{4t}$$

(2) $\therefore (D+1)y = y'+y = -\frac{3}{2}e^{2t} + c_1'e^{4t}$

Integrating factor: e^{t}

$$\therefore \frac{d}{dt}\left(e^{t}y\right) = -\frac{3}{2}e^{3t} + c_1'e^{5t}$$

$$\therefore e^{t}y = -\frac{1}{2}e^{3t} + \frac{c_1'}{5}e^{5t} + c_2$$

$\therefore\ y = -\frac{1}{2}e^{2t} + c_1 e^{4t} + c_2 e^{-t}$

29.

$r^2 + 2r + 1 = (r+1)^2$. $\therefore\ (D+1)(D+1)y = 2e^{-t}$

Let $u = (D+1)y$

(1) $\therefore\ (D+1)u = 2e^{-t}$, or $u' + u = 2e^{-t}$

$\therefore\ \frac{d}{dt}(e^{t}u) = 2e^{-t}e^{t} = 2$, $\therefore\ e^{t}u = 2t + c_1$

$\therefore\ u = 2te^{-t} + c_1 e^{-t}$

(2) $\therefore\ (D+1)y = y' + y = u = 2te^{-t} + c_1 e^{-t}$

$\therefore\ \frac{d}{dt}(e^{t}y) = 2t + c_1$, $e^{t}y = t^2 + c_1 t + c_2$

$\therefore\ y = t^2 e^{-t} + c_1 t e^{-t} + c_2 e^{-t}$

30.

$r^2 + 2r = (r+0)(r+2)$. $\therefore\ (D+0)(D+2)y = 3 + 4\sin(2t)$

Let $u = (D+2)y$

(1) $\therefore Du = u' = 3 + 4\sin(2t)$

$\therefore u = 3t - 2\cos(2t) + c_1'$

(2) $\therefore (D+2)y = u = 3t - 2\cos(2t) + c_1'$

Integrating factor: e^{2t}

$\therefore \frac{d}{dt}(e^{2t}y) = 3te^{2t} - 2e^{2t}\cos(2t) + c_1' e^{2t}$

From $\int ue^{au}\,du = \frac{1}{a^2}(au-1)e^{au} + C$

and $\int e^{au}\cos bu\,du = \frac{e^{au}}{a^2+b^2}(a\cos bu + b\sin bu) + C$

$$e^{2t}y = \frac{3}{2}te^{2t} - \frac{3}{4}e^{2t} - 2\left[\frac{e^{2t}}{8}\left(2\cos(2t) + 2\sin(2t)\right)\right] + \frac{c_1'}{2}e^{2t} + c_2$$

$$\therefore y = \frac{3}{2}t - \frac{3}{4} - \frac{1}{2}\cos(2t) - \frac{1}{2}\sin(2t) + \frac{c_1'}{2} + c_2e^{-2t}$$

Letting $c_1 = -\frac{3}{4} + \frac{c_1'}{2}$,

$$y = \frac{3}{2}t - \frac{1}{2}\cos(2t) - \frac{1}{2}\sin(2t) + c_1 + c_2e^{-2t}$$

3.6 Variation of Parameters

Note Title 8/24/2018

1.

(a) $r^2 - 5r + 6 = (r-3)(r-2)$. $\therefore y_c(t) = c_1 e^{3t} + c_2 e^{2t}$

$\therefore$ Consider $y = u_1 e^{3t} + u_2 e^{2t}$, $y_1 = e^{3t}$, $y_2 = e^{2t}$

$\therefore W[y_1, y_2] = \begin{vmatrix} e^{3t} & e^{2t} \\ 3e^{3t} & 2e^{2t} \end{vmatrix} = -e^{5t}$

$$\therefore u_1 = -\int \frac{(e^{2t})(2e^{t})}{-e^{5t}} = \int 2e^{-2t} = -e^{-2t}$$

$$u_2 = \int \frac{(e^{3t})(2e^{t})}{-e^{5t}} = \int -2e^{-t} = 2e^{-t}$$

$\therefore y(t) = (-e^{-2t})e^{3t} + (2e^{-t})e^{2t} = -e^{t} + 2e^{t}$

$\therefore \underline{y(t) = e^{t}}$

(b) Let $y = Ae^{t}$ $\therefore y' = Ae^{t}$, $y'' = Ae^{t}$

$\therefore Ae^{t} - 5(Ae^{t}) + 6(Ae^{t}) = 2e^{t}$

$\therefore\ 2A = 2,\ A = 1,\quad \therefore\ y(t) = e^t$

2.

(a) $r^2 - r - 2 = 0 = (r-2)(r+1)$, $y_c(t) = c_1 e^{-t} + c_2 e^{2t}$

$\therefore$ Let $y = u_1 e^{-t} + u_2 e^{2t}$, $y_1 = e^{-t}$, $y_2 = e^{2t}$

$$W[y_1, y_2] = \begin{vmatrix} e^{-t} & e^{2t} \\ -e^{-t} & 2e^{2t} \end{vmatrix} = 3e^t$$

$$\therefore\ u_1 = -\int \frac{(e^{2t})(2e^{-t})}{3e^t} = -\frac{2}{3}t$$

$$u_2 = \int \frac{(e^{-t})(2e^{-t})}{3e^t} = \frac{2}{3}e^{-3t}$$

$$\therefore\ y(t) = \left(-\frac{2}{3}t\right)(e^{-t}) + \left(\frac{2}{3}e^{-3t}\right)(e^{2t})$$

$$= -\frac{2}{3}te^{-t} + \frac{2}{3}e^{-t}$$

But e^{-t} is among $y_c(t) = c_1 e^{-t} + c_2 e^{2t}$

$\therefore\ y_p(t) = -\frac{2}{3}te^{-t}$

(b) Since e^{-t} is among $y_c(t)$,

let $y(t) = (At)e^{-t}$

Using MATLAB,

```
clear,clc
syms t A
c2 = 1; c1 = -1; c0 = -2;  %coeffs of diff eq
y = (A*t)*exp(-t);         %attempt
c2*diff(y,t,2) + c1*diff(y,t,1) + c0*y
```

ans = $-3\,A\,e^{-t}$

$\therefore -3Ae^{-t} = 2e^{-t} \Rightarrow A = -\frac{2}{3}$

$\therefore \underline{y(t) = -\frac{2}{3}te^{-t}}$

3.

$4r^2 - 4r + 1 = 0 = (2r-1)^2, \quad \therefore y_c(t) = c_1 e^{t/2} + c_2 t e^{t/2}$

Convert to: $y'' - y' + \frac{1}{4}y = 4e^{t/2}$

(a) Let $y = u_1 e^{t/2} + u_2 t e^{t/2}, \quad y_1 = e^{t/2}, \quad y_2 = te^{t/2}$

$$W[y_1, y_2] = \begin{vmatrix} e^{t/2} & te^{t/2} \\ \frac{1}{2}e^{t/2} & e^{t/2} + \frac{t}{2}e^{t/2} \end{vmatrix} = e^t + \frac{t}{2}e^t - \frac{t}{2}e^t = e^t$$

$$\therefore u_1 = -\int \frac{(te^{t/2})(4e^{t/2})}{e^t} = -2t^2$$

$$u_2 = \int \frac{(e^{t/2})(4e^{t/2})}{e^t} = 4t$$

$\therefore\ y = (-2t^2)e^{t/2} + (4t)te^{t/2} = 2t^2e^{t/2}$

$\therefore\ y_p(t) = 2t^2e^{t/2}$

(b) Since $e^{t/2}$ and $te^{t/2}$ are among $c_1e^{t/2}$ and $c_2te^{t/2}$

Let $y = (At^2)e^{t/2}$

Using MATLAB,

```
clear,clc
syms t A
c2 = 1; c1 = -1; c0 = 1/4;  %coeffs of diff eq
y = (A*t^2)*exp(t/2);          %attempt
c2*diff(y,t,2) + c1*diff(y,t,1) + c0*y
```

ans = $2\,A\,e^{t/2}$

$\therefore\ 2Ae^{t/2} = 4e^{t/2} \Rightarrow A = 2$

$\therefore\ y_p(t) = 2t^2e^{t/2}$

4.

Homogeneous: $r^2+1=0$, $r=\pm i$

$\therefore\ y_c(t) = c_1\cos(t) + c_2\sin(t)$, c_1, c_2 constants

$\therefore$ Let $y_p = u_1(t)\cos(t) + u_2(t)\sin(t)$, $y_1 = \cos(t)$, $y_2 = \sin(t)$

$$\therefore W[y_1, y_2] = \begin{vmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{vmatrix} = 1$$

$$u_1 = -\int \sin(t)\tan(t) = \int -\frac{\sin^2(t)}{\cos(t)} = \int \frac{\cos^2(t) - 1}{\cos(t)}$$

$$= \sin(t) - \ln|\sec(t) + \tan(t)|$$

$$= \sin(t) - \ln[\sec(t) + \tan(t)] \quad \text{for } 0 < t < \frac{\pi}{2}$$

$$u_2 = \int \cos(t)\tan(t) = \int \sin(t) = -\cos(t)$$

$$\therefore y_p(t) = \cos(t)\sin(t) - \cos(t)\ln[\sec(t) + \tan(t)] - \cos(t)\sin(t)$$

$$\therefore \underline{y(t) = -\cos(t)\ln[\sec(t) + \tan(t)] + c_1\cos(t) + c_2\sin(t)}$$

5.

Homogeneous: $r^2 + 9 = 0$, $r = \pm 3i$,

$\therefore y_c(t) = c_1\cos(3t) + c_2\sin(3t)$, c_1, c_2 constants

Let $y_p(t) = u_1(t)\cos(3t) + u_2(t)\sin(3t)$, $y_1 = \cos(3t)$, $y_2 = \sin(3t)$

$\therefore W[y_1, y_2] = 3$

$$\therefore \quad u_1 = -\int \frac{\sin(3t)\left[9\sec^2(3t)\right]}{3} = \int \frac{-3\sin(3t)}{\cos^2(3t)} = \frac{-1}{\cos(3t)}$$

$$u_2 = \int \frac{\cos(3t)\left[9\sec^2(3t)\right]}{3} = 3\int \sec(3t)$$

$$= \ln\left[\sec(3t) + \tan(3t)\right] \quad \text{for } 0 < t < \frac{\pi}{6}$$

$$\therefore y_p(t) = \left(-\frac{1}{\cos(3t)}\right)\cos(3t) + \sin(3t)\ln\left[\sec(3t) + \tan(3t)\right]$$

$$= -1 + \sin(3t)\ln\left[\sec(3t) + \tan(3t)\right]$$

$$\therefore \underline{y(t)} = -1 + \sin(3t)\ln\left[\sec(3t) + \tan(3t)\right]$$

$$+ c_1\cos(3t) + c_2\sin(3t)$$

6.

Homogeneous: $r^2 + 4r + 4 = 0 = (r+2)^2$, $r = -2, -2$

$$\therefore y_c(t) = c_1 e^{-2t} + c_2 t e^{-2t}$$

Let $y_p(t) = u_1(t)e^{-2t} + u_2(t)te^{-2t}$, $y_1 = e^{-2t}$, $y_2 = te^{-2t}$

From MATLAB,

```
% Compute Wronskian
clear, clc
syms t
y1 = exp(-2*t);
y2 = t*exp(-2*t);
W = [y1 y2; diff(y1) diff(y2)];
det(W)
```

ans = e^{-4t}

$$\therefore \quad u_1 = -\int \frac{(t e^{-2t})(t^{-2} e^{-2t})}{e^{-4t}} = -\int \frac{1}{t} = -\ln(t), \quad t > 0$$

$$u_2 = \int \frac{(e^{-2t})(t^{-2} e^{-2t})}{e^{-4t}} = -\frac{1}{t}$$

$$\therefore \quad y_p(t) = (-\ln(t))\, e^{-2t} + \left(-\frac{1}{t}\right) t e^{-2t}$$

$$= -e^{-2t} \ln(t) - e^{-2t}$$

Since e^{-2t} is already part of $y_c(t)$,

$$\underline{y(t)} = -e^{-2t} \ln(t) + c_1 e^{-2t} + c_2 t e^{-2t}$$

7.

$$y'' + \frac{1}{4} y = \frac{1}{2} \sec(t/2)$$

Homogeneous: $r^2 + \frac{1}{4} = 0$, $r = \pm \frac{1}{2} i$

$$\therefore \quad y_c(t) = c_1 \cos(t/2) + c_2 \sin(t/2)$$

Let $y_p(t) = u_1(t)\cos(t/2) + u_2(t)\sin(t/2)$

$$y_1 = \cos(t/2) \ , \ y_2 = \sin(t/2)$$

$W[y_1, y_2] = \frac{1}{2}$

$\therefore$

$$u_1 = -\int \frac{[\sin(t/2)]\frac{1}{2}\sec(t/2)}{1/2} = -\int \tan(t/2)$$

$$= -2\ln(\sec(t/2)) = 2\ln[\cos(t/2)] \ , \text{ for } -\pi < t < \pi$$

$$u_2 = \int \frac{[\cos(t/2)]\frac{1}{2}\sec(t/2)}{1/2} = \int 1 = t$$

$\therefore y_p = 2\ln[\cos(t/2)]\cos(t/2) + t\sin(t/2)$

$\therefore \underline{y(t)} = 2\cos(t/2)\ln[\cos(t/2)] + t\sin(t/2)$

$$+ c_1\cos(t/2) + c_2\sin(t/2)$$

8.

Homogeneous: $r^2 - 2r + 1 = 0 = (r-1)^2$, $r = 1, 1$

$\therefore y_c(t) = c_1 e^t + c_2 t e^t$

Let $y_p(t) = u_1(t)e^t + u_2(t)te^t$, $y_1 = e^t$, $y_2 = te^t$

From MATLAB,

```
% Compute Wronskian
clear, clc
syms t
y1 = exp(t);  % edit y1, y2
y2 = t*exp(t);
W = [y1 y2; diff(y1) diff(y2)];
det(W)
```

ans = e^{2t}

$$\therefore\ u_1 = -\int \frac{(te^t)}{e^{2t}}\left(\frac{e^t}{1+t^2}\right) = -\int \frac{t}{1+t^2} = -\frac{1}{2}\ln(1+t^2)$$

$$u_2 = \int \frac{(e^t)}{e^{2t}}\left(\frac{e^t}{1+t^2}\right) = \int \frac{1}{1+t^2} = \text{Arctan}(t)$$

$$\therefore\ y_p(t) = -\frac{e^t}{2}\ln(1+t^2) + te^t\,\text{Arctan}(t)$$

$$\therefore\ y(t) = -\frac{e^t}{2}\ln(1+t^2) + te^t\,\text{Arctan}(t) + c_1e^t + c_2te^t$$

9.

Homogeneous : $r^2 - 5r + 6 = (r-3)(r-2)$, $r = 2, 3$

$$\therefore\ y_c(t) = c_1e^{2t} + c_2e^{3t}$$

Let $y_p(t) = u_1(t)e^{2t} + u_2(t)e^{3t}$, $y_1 = e^{2t}$, $y_2 = e^{3t}$

$$W[y_1, y_2] = \begin{vmatrix} e^{2t} & e^{3t} \\ 2e^{2t} & 3e^{3t} \end{vmatrix} = e^{5t}$$

$$\therefore \quad u_1(t) = -\int \frac{(e^{3t})\, g(t)}{e^{5t}} = -\int_{t_0}^{t} e^{-2s} g(s)\,ds$$

$$u_2(t) = \int \frac{(e^{2t})\, g(t)}{e^{5t}} = \int_{t_0}^{t} e^{-3s} g(s)\,ds$$

$$\therefore \quad y_p(t) = -e^{2t} \int_{t_0}^{t} e^{-2s} g(s)\,ds + e^{3t} \int_{t_0}^{t} e^{-3s} g(s)\,ds$$

$y_p(t)$ can be simplified by pulling the e^{2t} and e^{3t} terms into the integrand, since they act as constants relative to variable s. That is,

$$F(t) = f(t) \int_{t_0}^{t} g(s)\,ds = \int_{t_0}^{t} f(t)\, g(s)\,ds$$

$$\therefore \quad \underline{y(t)} = c_1 e^{2t} + c_2 e^{3t} + \int_{t_0}^{t} \left[e^{3(t-s)} - e^{2(t-s)} \right] g(s)\,ds$$

10.

$y_1 = t^2$: $y_1' = 2t$, $y_1'' = 2$

$\therefore t^2(2) - 2(t^2) = 0$ ✓

$y_2 = t^{-1}$: $y_2' = -t^{-2}$, $y_2'' = 2t^{-3}$

$\therefore t^2(2t^{-3}) - 2(t^{-1}) = 2t^{-1} - 2t^{-1} = 0$ ✓

$\therefore y_c(t) = c_1 t^2 + c_2 t^{-1}$

Standard form: $y'' - \frac{2}{t^2} y = 3 - t^{-2}$, $t > 0$

Let $y_p(t) = u_1(t) t^2 + u_2(t) t^{-1}$

$W[y_1, y_2] = \begin{vmatrix} t^2 & t^{-1} \\ 2t & -t^{-2} \end{vmatrix} = -1 - 2 = -3$

$\therefore u_1 = -\int \frac{(t^{-1})(3 - t^{-2})}{-3} = \int t^{-1} - \frac{t^{-3}}{3} = \ln(t) + \frac{t^{-2}}{6}$

$\therefore u_1 y_1 = t^2 u_1 = t^2 \ln(t) + \frac{1}{6}$

$$u_2 = \int \frac{(t^2)(3-t^{-2})}{-3} = \int -t^2 + \frac{1}{3} = -\frac{t^3}{3} + \frac{t}{3}$$

$$\therefore u_2 y_2 = t^{-1} u_2 = -\frac{t^2}{3} + \frac{1}{3}$$

Since $-\frac{t^2}{3} \in \{c_1 t^2 + c_2 t^{-1}\}$, drop $-\frac{t^2}{3}$ term

$$\therefore y_p(t) = t^2 \ln(t) + \frac{1}{6} + \frac{1}{3}$$

$$\therefore y_p(t) = t^2 \ln(t) + \frac{1}{2}$$

11.

Use MATLAB to check:

```
clear, clc
syms t
y1 = t;          %edit y1, y2, coefficients
y2 = t*exp(t);
a = t^2; b = -t*(t+2); c = t+2;
y1_check = a*diff(y1,2) + b*diff(y1,1) + c*y1;
y2_check = a*diff(y2,2) + b*diff(y2,1) + c*y2;
simplify(y1_check), simplify(y2_check)
```

ans = 0

ans = 0

Standard form: $y'' - \frac{t(t+2)}{t^2} y' + \frac{(t+2)}{t^2} y = 2t$, $t > 0$

Let $y_p(t) = u_1 y_1 + u_2 y_2 = u_1(t) + u_2(te^t)$

```
% Compute Wronskian
clear, clc
syms t
y1 = t;  % edit y1, y2
y2 = t*exp(t);
W = [y1 y2; diff(y1) diff(y2)];
det(W)
```

ans = $t^2 e^t$

$$\therefore u_1(t) = -\int \frac{(te^t)(2t)}{t^2e^t} = \int -2 = -2t$$

$$\therefore u_1 y_1 = (-2t)(t) = -2t^2$$

$$u_2(t) = \int \frac{(t)(2t)}{t^2e^t} = \int 2e^{-t} = -2e^{-t}$$

$$\therefore u_2 y_2 = (-2e^{-t})(te^t) = -2t$$

But $-2t \in \{y_c(t)\}$, so ignore in $y_p(t)$

$$\therefore y_p(t) = -2t^2$$

12.

Use MATLAB to check:

```
clear, clc
syms t
y1 = 1 + t;          %edit y1, y2, coefficients
y2 = exp(t);
a = t; b = -(1+t); c = 1;
y1_check = a*diff(y1,2) + b*diff(y1,1) + c*y1;
y2_check = a*diff(y2,2) + b*diff(y2,1) + c*y2;
simplify(y1_check), simplify(y2_check)
```

```
ans = 0

ans = 0
```

Standard form: $y'' - \frac{(1+t)}{t} y' + \frac{1}{t} y = te^{2t}, \; t > 0$

Let $y_p(t) = u_1 y_1 + u_2 y_2 = u_1(1+t) + u_2(e^t)$

```
% Compute Wronskian
clear, clc
syms t
y1 = 1 + t;  % edit y1, y2
y2 = exp(t);
W = [y1 y2; diff(y1) diff(y2)];
det(W)
```

ans = $t\,e^t$

$$\therefore\ u_1(t) = -\int \frac{(e^t)(te^{2t})}{te^t} = -\int e^{2t} = -\frac{1}{2}e^{2t}$$

$$\therefore\ u_1 y_1 = \left(-\frac{1}{2}e^{2t}\right)(1+t) = -\frac{1}{2}e^{2t} - \frac{1}{2}te^{2t}$$

$$u_2(t) = \int \frac{(1+t)(te^{2t})}{te^t} = \int (1+t)e^t = e^t + te^t - e^t = te^t$$

$$\therefore\ u_2 y_2 = (te^t)(e^t) = te^{2t}$$

$$\therefore\ y_p(t) = -\frac{1}{2}e^{2t} - \frac{1}{2}te^{2t} + te^{2t}$$

$$\therefore\ \underline{y_p(t)} = -\frac{1}{2}e^{2t} + \frac{1}{2}te^{2t} = \frac{1}{2}(t-1)e^{2t}$$

13.

Use MATLAB to check:

```
clear, clc
syms x
y1 = x^2;          %edit y1, y2, coefficients
y2 = x^2*log(x);
a = x^2; b = -3*x; c = 4;
y1_check = a*diff(y1,2) + b*diff(y1,1) + c*y1;
y2_check = a*diff(y2,2) + b*diff(y2,1) + c*y2;
simplify(y1_check), simplify(y2_check)
```

ans = 0

ans = 0

Standard form: $y'' - \frac{3}{x}y' + \frac{4}{x^2}y = \ln(x)$, $x > 0$

Let $y_p(x) = u_1 y_1 + u_2 y_2 = u_1(x^2) + u_2(x^2\ln(x))$

```
% Compute Wronskian
clear, clc
syms x
y1 = x^2;        % edit y1, y2
y2 = x^2*log(x);
W = [y1 y2; diff(y1) diff(y2)];
det(W)
```

ans = x^3

$$\therefore\ u_1(x) = -\int \frac{(x^2\ln(x))(\ln(x))}{x^3} = -\int \frac{[\ln(x)]^2}{x} = \frac{-[\ln(x)]^3}{3}$$

$$\therefore\ u_1 y_1 = \frac{-[\ln(x)]^3}{3} x^2$$

$$u_2(x) = \int \frac{(x^2)(\ln(x))}{x^3} = \int \frac{\ln(x)}{x} = \frac{[\ln(x)]^2}{2}$$

$$\therefore\ u_2 y_2 = \frac{[\ln(x)]^2}{2}(x^2\ln(x)) = \frac{x^2[\ln(x)]^3}{2}$$

$$\therefore\ \underline{y_p(x)} = \left(\frac{1}{2} - \frac{1}{3}\right)x^2[\ln(x)]^3 = \underline{\frac{1}{6}x^2[\ln(x)]^3}$$

14.

Use MATLAB to check:

```
clear, clc
syms x
y1 = x^(-1/2)*sin(x);          %edit y1, y2, coefficients
y2 = x^(-1/2)*cos(x);
a = x^2; b = x; c = x^2 - 1/4;
y1_check = a*diff(y1,2) + b*diff(y1,1) + c*y1;
y2_check = a*diff(y2,2) + b*diff(y2,1) + c*y2;
simplify(y1_check), simplify(y2_check)
```

ans = 0

ans = 0

Standard form: $y'' + \frac{1}{x} y' + \left(1 - \frac{1}{4x^2}\right) y = 3x^{-\frac{1}{2}} \sin(x), \quad x > 0$

```
% Compute Wronskian
clear, clc
syms x
y1 = x^(-1/2)*sin(x);          %edit y1, y2
y2 = x^(-1/2)*cos(x);
W = [y1 y2; diff(y1) diff(y2)];
det(W)
```

ans =

$$-\frac{\cos(x)^2 + \sin(x)^2}{x}$$

$$= -\frac{1}{x}$$

$$\therefore \quad u_1(x) = -\int \frac{\left[x^{-\frac{1}{2}} \cos(x)\right] 3x^{-\frac{1}{2}} \sin(x)}{-1/x} = \int 3 \cos(x) \sin(x)$$

$$= \frac{3}{2} \sin^2(x)$$

$$\therefore \quad u_1 y_1 = \frac{3}{2} \sin^2(x)\, x^{-\frac{1}{2}} \sin(x) = \frac{3}{2} x^{-\frac{1}{2}} \sin^3(x)$$

$$u_2(x) = \int \frac{\left[x^{-\frac{1}{2}} \sin(x)\right] 3x^{-\frac{1}{2}} \sin(x)}{-1/x} = -3 \int \sin^2(x)$$

$$= -3\left[\frac{1}{2} x - \frac{1}{4} \sin(2x)\right] = -\frac{3}{2} x + \frac{3}{4} \sin(2x)$$

$$\therefore u_2 y_2 = \left[-\frac{3}{2}x + \frac{3}{4}\sin(2x)\right] x^{-\frac{1}{2}}\cos(x)$$

$$= -\frac{3}{2}x^{\frac{1}{2}}\cos(x) + \frac{3}{4}x^{-\frac{1}{2}}\sin(2x)\cos(x)$$

$$= -\frac{3}{2}x^{\frac{1}{2}}\cos(x) + \frac{3}{2}x^{-\frac{1}{2}}\sin(x)\cos^2(x)$$

$$\therefore y_p(x) = u_1 y_1 + u_2 y_2 = \frac{3}{2}x^{-\frac{1}{2}}\sin^3(x)$$

$$+ \frac{3}{2}x^{-\frac{1}{2}}\sin(x)\cos^2(x) - \frac{3}{2}x^{\frac{1}{2}}\cos(x)$$

$$= \frac{3}{2}x^{-\frac{1}{2}}\sin(x)\left[\sin^2(x) + \cos^2(x)\right] - \frac{3}{2}x^{\frac{1}{2}}\cos(x)$$

$$= \frac{3}{2}x^{-\frac{1}{2}}\sin(x) - \frac{3}{2}x^{\frac{1}{2}}\cos(x)$$

But $\frac{3}{2}x^{-\frac{1}{2}}\sin(x) \in \{y_c(x)\}$, $\therefore$ drop term in y_p

$$\therefore y_p(x) = -\frac{3}{2}x^{\frac{1}{2}}\cos(x)$$

15.

Use MATLAB to check:

```
clear, clc
syms x
y1 = x^(-1/2)*sin(x);           %edit y1, y2, coefficients
y2 = x^(-1/2)*cos(x);
a = x^2; b = x; c = x^2 - 0.25;
y1_check = a*diff(y1,2) + b*diff(y1,1) + c*y1;
y2_check = a*diff(y2,2) + b*diff(y2,1) + c*y2;
simplify(y1_check), simplify(y2_check)
```

```
ans = 0

ans = 0
```

Standard form: $y'' + \frac{1}{x}y' + \left(1 - \frac{1}{4x^2}\right)y = \frac{g(x)}{x^2}, \quad x > 0$

```
% Compute Wronskian
clear, clc
syms x
y1 = x^(-1/2)*sin(x);          %edit y1, y2
y2 = x^(-1/2)*cos(x);
W = [y1 y2; diff(y1) diff(y2)];
det(W)
```

ans =

$$-\frac{\cos(x)^2+\sin(x)^2}{x}$$

$$= -\frac{1}{x}$$

$$\therefore \quad u_1(x) = -\int \frac{\left[x^{-\frac{1}{2}}\cos(x)\right]}{-\frac{1}{x}}\frac{g(x)}{x^2} = \int x^{-\frac{3}{2}}\cos(x)\,g(x)$$

$$\therefore u_1 y_1 = x^{-\frac{1}{2}}\sin(x)\int_{x_0}^{x} t^{-\frac{3}{2}}\cos(t)\,g(t)\,dt$$

$$u_2(x) = \int \frac{\left[x^{-\frac{1}{2}}\sin(x)\right]}{-1/x}\frac{g(x)}{x^2} = -\int x^{-\frac{3}{2}}\sin(x)\,g(x)$$

$$\therefore u_2 y_2 = -x^{-\frac{1}{2}}\cos(x)\int_{x_0}^{x} t^{-\frac{3}{2}}\sin(t)\,g(t)\,dt$$

$$\therefore y_p(x) = x^{-\frac{1}{2}}\sin(x)\int_{x_0}^{x} t^{-\frac{3}{2}}\cos(t)\,g(t)\,dt$$

$$- x^{-\frac{1}{2}}\cos(x)\int_{x_0}^{x} t^{-\frac{3}{2}}\sin(t)\,g(t)\,dt$$

$$= x^{-\frac{1}{2}}\int_{x_0}^{x} t^{-\frac{3}{2}} g(t)\left[\sin(x)\cos(t) - \cos(x)\sin(t)\right]dt$$

$$\therefore y_p(x) = x^{-\frac{1}{2}}\int_{x_0}^{x} t^{-\frac{3}{2}} g(t)\sin(x-t)\,dt$$

16.

(a) Equation (30) is:

$$Y(t) = -y_1(t)\int_{t_0}^{t} \frac{y_2(s)g(s)}{W[y_1, y_2](s)}\,ds + y_2(t)\int_{t_0}^{t} \frac{y_1(s)g(s)}{W[y_1, y_2](s)}\,ds, \qquad (30)$$

Note that the variable t acts as a constant relative to the variable s, so that $y_1(t)$ and $y_2(t)$ can be brought under the integral sign.

$$\therefore \quad Y(t) = -\int_{t_0}^{t} \frac{y_1(t)\,y_2(s)\,g(s)}{W[y_1,y_2](s)}\,ds + \int_{t_0}^{t} \frac{y_2(t)\,y_1(s)\,g(s)}{W[y_1,y_2](s)}\,ds$$

$$= \int_{t_0}^{t} \frac{y_2(t)\,y_1(s) - y_1(t)\,y_2(s)}{W[y_1,y_2](s)}\,g(s)\,ds$$

(b) Using the Fundamental Theorem of Calculus, and derivative of a product, using (30),

$$Y'(t) = -y_1(t)\frac{y_2(t)g(t)}{W[y_1,y_2](t)} + y_2(t)\frac{y_1(t)g(t)}{W[y_1,y_2](t)}$$

(cancel)

$$-y_1'(t)\int_{t_0}^{t}\frac{y_2(s)g(s)}{W[y_1,y_2](s)}ds + y_2'(t)\int_{t_0}^{t}\frac{y_1(s)g(s)}{W[y_1,y_2](s)}ds$$

$$\therefore Y'(t) = -y_1'(t)\int_{t_0}^{t}\frac{y_2(s)g(s)}{W[y_1,y_2](s)}ds + y_2'(t)\int_{t_0}^{t}\frac{y_1(s)g(s)}{W[y_1,y_2](s)}ds \quad [1]$$

$$= \int_{t_0}^{t}\frac{y_2'(t)y_1(s) - y_1'(t)y_2(s)}{W[y_1,y_2](s)}g(s)\,ds \quad [2]$$

Using [1], and using $W[y_1,y_2](t) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$

$$Y''(t) = \frac{\overbrace{[y_2'(t)y_1(t) - y_1'(t)y_2(t)]}^{=W[y_1,y_2](t)}g(t)}{W[y_1,y_2](t)}$$

$$-y_1''(t)\int_{t_0}^{t}\frac{y_2(s)g(s)}{W[y_1,y_2](s)}ds + y_2''(t)\int_{t_0}^{t}\frac{y_1(s)g(s)}{W[y_1,y_2](s)}ds$$

$$\therefore Y''(t) = g(t) + \int_{t_0}^{t}\frac{y_2''(t)y_1(s) - y_1''(t)y_2(s)}{W[y_1,y_2](s)}g(s)\,ds \quad [3]$$

$$\therefore L[Y(t)] = Y''(t) + p(t)Y'(t) + q(t)Y(t)$$

$$= g(t) + \int_{t_0}^{t} \frac{y_2''(t)y_1(s) - y_1''(t)y_2(s)}{W[y_1, y_2](s)} g(s)\,ds \quad (\text{from } [3])$$

$$+ p(t)\int_{t_0}^{t} \frac{y_2'(t)y_1(s) - y_1'(t)y_2(s)}{W[y_1, y_2](s)} g(s)\,ds \quad (\text{from } [2])$$

$$+ q(t)\int_{t_0}^{t} \frac{y_2(t)y_1(s) - y_1(t)y_2(s)}{W[y_1, y_2](s)} g(s)\,ds \quad (\text{from } (a))$$

$$= g(t)$$

(rearranging above integrals)

$$+ y_2''(t)\int_{t_0}^{t} \frac{y_1(s)g(s)}{W[y_1,y_2](s)}ds + p(t)y_2'(t)\int_{t_0}^{t} \frac{y_1(s)g(s)}{W[y_1,y_2](s)}ds + q(t)y_2(t)\int_{t_0}^{t} \frac{y_1(s)g(s)}{W[y_1,y_2](s)}ds$$

$$- y_1''(t)\int_{t_0}^{t} \frac{y_2(s)g(s)}{W[y_1,y_2](s)}ds - p(t)y_1'(t)\int_{t_0}^{t} \frac{y_2(s)g(s)}{W[y_1,y_2](s)}ds - q(t)y_1(t)\int_{t_0}^{t} \frac{y_2(s)g(s)}{W[y_1,y_2](s)}ds$$

$$= g(t) + \left[y_2''(t) + p(t)y_2'(t) + q(t)y_2(t)\right]\int_{t_0}^{t} \frac{y_1(s)g(s)}{W[y_1,y_2](s)}ds$$

$$- \left[y_1''(t) + p(t)y_1'(t) + q(t)y_1(t)\right]\int_{t_0}^{t} \frac{y_2(s)g(s)}{W[y_1,y_2](s)}ds$$

$$= g(t)$$

as $L[y_1(t)] = 0$ and $L[y_2(t)] = 0$

so $y_2'' + p y_2' + q y_2 = 0$, $y_1'' + p y_1'' + q y_1 = 0$

$\therefore$ $L[Y(t)] = g(t)$

Also, from (a),

$$Y(t_0) = \int_{t_0}^{t_0} \frac{y_2(t_0) y_1(s) - y_1(t_0) y_2(s)}{W[y_1, y_2](s)} g(s)\, ds = 0$$

and from [2],

$$Y'(t_0) = \int_{t_0}^{t_0} \frac{y_2'(t_0) y_1(s) - y_1'(t_0) y_2(s)}{W[y_1, y_2](s)} g(s)\, ds = 0$$

17.

$$L[y] = L[u(t) + v(t)] = L[u(t)] + L[v(t)] = 0 + g(t) = g(t)$$

$$y(t_0) = u(t_0) + v(t_0) = y_0 + 0 = y_0$$

$$y'(t_0) = u'(t_0) + v'(t_0) = y_0' + 0 = y_0'$$

18.

(a) From $r^2 + 1 = 0$, $r = \pm i$ $\therefore$ $\cos(t)$, $\sin(t)$ form a fundamental set. $\therefore$ Let $y_1(t) = \cos(t)$ and $y_2(t) = \sin(t)$ solve $L[y_1(t)] = 0$, $L[y_2(t)] = 0$, where $L[y] = y'' + y$

$\therefore$ By #16,

$$y(t) = \int_{t_0}^{t} \frac{\cos(s)\sin(t) - \cos(s)\sin(t)}{W[y_1, y_2](s)} g(s)\, ds$$

$$= \int_{t_0}^{t} \frac{\sin(t-s)}{W[y_1, y_2](s)} g(s)\,ds$$

But $W[y_1, y_2](s) = \begin{vmatrix} \cos(s) & \sin(s) \\ -\sin(s) & \cos(s) \end{vmatrix} = 1$

$$\therefore\ y(t) = \int_{t_0}^{t} \sin(t-s)\, g(s)\,ds$$

$\therefore$ By # 16, $L[y(t)] = g(t)$, and $y(t_0) = 0$, $y'(t_0) = 0$

(b)

(1) First find solution to $y'' + y = 0$, $y(0) = y_0$, $y'(t_0) = y_0'$

From $r^2 + 1 = 0 \Rightarrow r = \pm i \Rightarrow y(t) = C_1 \cos(t) + C_2 \sin(t)$

$\therefore\ y'(t) = -C_1 \sin(t) + C_2 \cos(t)$

$y(0) = C_1 \cos(0) + C_2 \sin(0) = C_1 \Rightarrow C_1 = y_0$

$y'(0) = -C_1 \sin(0) + C_2 \cos(0) = C_2 \Rightarrow C_2 = y_0'$

$\therefore\ y(t) = y_0 \cos(t) + y_0' \sin(t)$

(2) Using (a) and (1) above, using $t_0 = 0$,

$$y(t) = y_0\cos(t) + y_0'\sin(t) + \int_0^t \sin(t-s)\,g(s)\,ds$$

19.

In #16, $y_1(t)$ and $y_2(t)$ are solutions to $L[y]=0$

Since $L[y] = (D-a)(D-b)y$, $y_1 = e^{at}$, $y_2 = e^{bt}$

since $r^2-(a+b)r+ab=0=(r-a)(r-b)$

$$y_1(s)\,y_2'(s) - y_1'(s)\,y_2(s) = W[y_1,y_2](s)$$

$$\therefore W[e^{as},e^{bs}] = \begin{vmatrix} e^{as} & e^{bs} \\ ae^{as} & be^{bs} \end{vmatrix} = (b-a)e^{(a+b)s}$$

$\therefore$ From #16,

$$Y(t) = \int_{t_0}^{t} \frac{y_1(s)\,y_2(t) - y_1(t)\,y_2(s)}{W[y_1,y_2](s)}\,g(s)\,ds$$

$$= \int_{t_0}^{t} \frac{e^{as}e^{bt} - e^{at}e^{bs}}{(b-a)e^{(a+b)s}}\,g(s)\,ds$$

$$= \frac{1}{b-a}\int_{t_0}^{t} \left[e^{-bs}e^{bt} - e^{at}e^{-as}\right]g(s)\,ds$$

$$\therefore\ Y(t) = \frac{1}{b-a}\int_{t_0}^{t}\left[e^{b(t-s)} - e^{a(t-s)}\right] g(s)\, ds$$

20.

Because $r = \lambda \pm i\mu$ is a solution to the characteristic equation, $y_1(t) = e^{\lambda t}\cos(\mu t)$ and $y_2(t) = e^{\lambda t}\sin(\mu t)$ are a fundamental set of solutions for $L[y] = 0$.

Using MATLAB,

```
% Compute Wronskian
clear, clc
syms x u t
y1 = exp(x*t)*cos(u*t);        %edit y1, y2
y2 = exp(x*t)*sin(u*t);
W = [y1 y2; diff(y1,t) diff(y2,t)];
simplify(det(W))
```

ans = $u\, e^{2tx}$

$$\therefore\ W[y_1, y_2](s) = \mu e^{2\lambda s}$$

$\therefore$ Using #16,

$$Y(t) = \int_{t_0}^{t} \frac{\left[e^{\lambda s}\cos(\mu s)\right] e^{\lambda t}\sin(\mu t) - \left[e^{\lambda t}\cos(\mu t)\right] e^{\lambda s}\sin(\mu s)}{\mu e^{2\lambda s}}\, g(s)\, ds$$

$$= \int_{t_0}^{t} \frac{e^{\lambda s + \lambda t} \sin[\mu(t-s)]\, g(s)}{\mu e^{2\lambda s}}\, ds$$

$$\therefore\ y(t) = \frac{1}{\mu} \int_{t_0}^{t} e^{\lambda(t-s)} \sin[\mu(t-s)]\, g(s)\, ds$$

21.

Because $r = a$ is a double root to the characteristic equation, $y_1(t) = e^{at}$ and $y_2(t) = te^{at}$ form a fundamental set to $L[y] = (D-a)^2 y = 0$.

Using MATLAB,

```
% Compute Wronskian
clear, clc
syms a t
y1 = exp(a*t);          %edit y1, y2
y2 = t*exp(a*t);
W = [y1 y2; diff(y1,t) diff(y2,t)];
simplify(det(W))
```

ans = e^{2at}

$\therefore\ W[y_1, y_2](s) = e^{2as}$

$\therefore$ By #16,

$$Y(t) = \int_{t_0}^{t} \frac{(e^{as})te^{at} - (e^{at})(se^{as})}{e^{2as}}\, g(s)\, ds$$

$$= \int_{t_0}^{t} \left[t\, e^{a(t-s)} - s\, e^{a(t-s)} \right] g(s)\, ds$$

$$\therefore Y(t) = \int_{t_0}^{t} (t-s)\, e^{a(t-s)}\, g(s)\, ds$$

22.

The solution to $(D^2 + bD + c)y = g(t)$ depends on the roots to the characteristic equation $r^2 + br + c = 0$.

(1) For real unequal roots (problem #19),

$(D^2 + bD + c)y$ becomes $(D-u)(D-v)y$, $u \neq v$,

$$Y(t) = \int_{t_0}^{t} \left[\frac{e^{v(t-s)} - e^{u(t-s)}}{v-u} \right] g(s)\, ds$$

Here, $K(t-s) = \dfrac{e^{v(t-s)} - e^{u(t-s)}}{v-u}$,

where u, v are the real roots of $r^2 + br + c = 0$.

(2) For real repeated roots (problem # 21),

$(D^2 + bD + c)y$ becomes $(D-u)^2 y$,

$$y(t) = \int_{t_0}^{t} (t-s)\, e^{u(t-s)} g(s)\, ds$$

Here, $K(t-s) = (t-s)\, e^{u(t-s)}$,

where u is the double root of $r^2 + br + c = 0$

(3) For complex roots (problem # 20),

$(D^2 + bD + c)y$ becomes $[D - (\lambda + iu)][D - (\lambda - iu)]y$,

$$\therefore\ y(t) = \int_{t_0}^{t} \frac{e^{\lambda(t-s)}}{u} \sin[u(t-s)]\, g(s)\, ds$$

Here, $K(t-s) = \dfrac{e^{\lambda(t-s)} \sin[u(t-s)]}{u}$, where

$\lambda \pm iu$ represent the complex roots to $r^2 + br + c = 0$.

Note that in each case, $K(t-s)$ depends on u, v (1) or u (2) or λ, u (3), each from solutions to the characteristic equation, and so K only depends on y_1 and y_2, the solutions to the homogeneous equation, and not on $g(t)$.

23.

$$y'(t) = v'y_1 + vy_1'$$

$$y''(t) = v''y_1 + 2v'y_1' + vy_1''$$

$$\therefore\ y'' + py' + qy =$$

$$(v''y_1 + 2v'y_1' + vy_1'') + (pv'y_1 + pvy_1') + qvy_1$$

$$= y_1 v'' + (2v' + py_1)v' + \underbrace{(y_1'' + py_1' + qy_1)}_{=0}v$$

$$= y_1 v'' + (2y_1' + py_1)v'$$

$\therefore$ If $y_1(t)$ is a homogeneous equation solution, then for $y(t) = v(t)\, y_1(t)$,

$$y'' + py' + qy = g(t) \iff y_1 v'' + (2y_1' + py_1)v' = g(t)$$

24.

Standard form: $y'' - \frac{2}{t}y' + \frac{2}{t^2}y = 4$

Let $y = vy_1 = vt$, $y_1 = t$ a homogeneous solution.

$\therefore y_1' = 1$, $p(t) = -\frac{2}{t}$, $g(t) = 4$

$\therefore y_1 v'' + (2y_1' + py_1)v' =$

$tv'' + \left[2 + \left(-\frac{2}{t}\right)t\right]v' = tv'' = 4$

$\therefore v'' = \frac{4}{t}$, $v' = 4\ln(t) + c_1'$

$\therefore v(t) = 4t\ln(t) - 4t + c_1't + c_2$

using $\int \ln u \, du = u \ln u - u + C$

combine $-4t + c_1't = c_1 t$

$$\therefore y = vt = 4t^2\ln(t) + c_1 t^2 + c_2 t$$

25.

Standard form: $y'' + \frac{7}{t}y' + \frac{5}{t^2} = \frac{1}{t}$

Let $y = v\left(\frac{1}{t}\right)$, $y_1 = \frac{1}{t}$ a homogeneous solution

$\therefore y_1' = -\frac{1}{t^2}$, $p(t) = \frac{7}{t}$, $g(t) = \frac{1}{t}$

$\therefore y_1 v'' + (2y_1' + p y_1) v' =$

$\left(\frac{1}{t}\right)v'' + \left(-\frac{2}{t^2} + \frac{7}{t}\cdot\frac{1}{t}\right)v' = \left(\frac{1}{t}\right)v'' + \frac{5}{t^2}v' = \frac{1}{t}$

$\therefore v'' + \frac{5}{t}v' = 1$

Integrating factor: $\exp\left(\int \frac{5}{t}\right) = t^5$

$\therefore \frac{d}{dt}\left(t^5 v'\right) = t^5$, $t^5 v' = \frac{1}{6}t^6 + c_1'$

$\therefore v' = \frac{t}{6} + c_1' t^{-5}$, $v = \frac{t^2}{12} + \frac{c_1'}{-4}t^{-4} + c_2$

Make $c_1'/-4 = c_1$

$\therefore y = y_1 v = \left(\frac{1}{t}\right)v = \frac{t}{12} + c_1 t^{-5} + c_2 t^{-1}$

26.

Standard form: $y'' - \left(\frac{1}{t}+1\right)y' + \left(\frac{1}{t}\right)y = te^{2t}$

Let $y = (1+t)v$, $y_1 = 1+t$, a homogeneous solution.

$\therefore\ y_1' = 1,\ p(t) = -\left(\frac{1}{t}+1\right),\ g(t) = te^{2t}$

$\therefore\ y_1 v'' + (2y_1' + py_1)v' =$

$(1+t)v'' + \left[2 - \left(\frac{1}{t}+1\right)(1+t)\right]v' =$

$(1+t)v'' + \left[2 - \frac{1}{t} - 1 - 1 - t\right]v' =$

$(1+t)v'' - \left[\frac{1+t^2}{t}\right]v' = te^{2t}$

$\therefore\ v'' - \frac{t^2+1}{t^2+t}v' = \frac{t}{1+t}e^{2t}$

Integrating factor: $\exp\left(-\int \frac{t^2+1}{t(t+1)}\right) = \exp\left(-\int \frac{1}{t} + \frac{t-1}{t+1}\right)$

$= \exp\left(-\int \frac{1}{t} + \frac{t}{t+1} - \frac{1}{t+1}\right) = \exp\left(-\int \frac{1}{t} + 1 + \frac{-1}{t+1} - \frac{1}{t+1}\right)$

$= \exp\left(-\int \frac{1}{t} + 1 - \frac{2}{t+1}\right) = \exp\left[-\ln(t) - t + 2\ln(t+1)\right]$

$= \frac{(t+1)^2}{t}e^{-t}$

$$\therefore \frac{d}{dt}\left(\frac{(t+1)^2}{t} e^{-t} v'\right) = \frac{(t+1)^2}{t} e^{-t} \frac{t}{1+t} e^{2t} = (t+1) e^t$$

$$\therefore \frac{(t+1)^2}{t} e^{-t} v' = \int (te^t + e^t)$$

$$= (t-1)e^t + e^t + c_1 = te^t + c_1$$

$$\therefore v' = \frac{t^2 e^{2t}}{(t+1)^2} + c_1 \frac{t}{(t+1)^2} e^t$$

Using MATLAB,

```
clear, clc
syms t
int((t^2/(t+1)^2)*exp(2*t))
int(t*exp(t)/(t+1)^2)
```

ans =

$$\frac{e^{2t}\,(t-1)}{2\,(t+1)}$$

ans =

$$\frac{e^t}{t+1}$$

So, $\displaystyle\int \frac{t^2 e^{2t}}{(t+1)^2} + c_1 \frac{te^t}{(t+1)^2} = \frac{e^{2t}(t-1)}{2(t+1)} + c_1 \frac{e^t}{t+1} + c_2$

c_1, c_2 constants

$$\therefore v(t) = \frac{e^{2t}(t-1)}{2(t+1)} + c_1 \frac{e^t}{t+1} + c_2$$

$$\therefore y = y_1 v = (1+t)v = \frac{e^{2t}(t-1)}{2} + c_1 e^t + c_2 (1+t)$$

3.7 Mechanical and Electrical Vibrations

Note Title 9/18/2018

1.

$u = R\cos(\delta)\cos(2t) + R\sin(\delta)\sin(2t)$

$\therefore R\cos(\delta) = 3, \quad R\sin(\delta) = 4$

$\therefore R^2\cos^2(\delta) = 9, \quad R^2\sin^2(\delta) = 16$

$\therefore R^2[\cos^2(\delta) + \sin^2(\delta)] = 25, \quad \underline{R = 5}$

$\therefore \cos(\delta) = \frac{3}{5}, \quad \delta = \text{Arccos}(\frac{3}{5}) = \underline{0.927} \text{ rad}$

From $2t \Rightarrow \underline{\omega_0 = 2}$

$\therefore \underline{u = 5\cos(2t - 0.927)}$

2.

From $\pi t \Rightarrow \underline{\omega_0 = \pi}$

$u = R\cos(\delta)\cos(\pi t) + R\sin(\delta)\sin(\pi t)$

$\therefore R\cos(\delta) = -2, \quad R\sin(\delta) = -3$

$\therefore R^2 = (-2)^2 + (-3)^2 = 13, \quad \underline{R = \sqrt{13}}$

$$\frac{R\sin(\delta)}{R\cos(\delta)} = \frac{-3}{-2} = \frac{3}{2} = \tan(\delta)$$

$\sin(\delta) < 0$, $\cos(\delta) < 0 \Rightarrow$ Quadrant III

$\therefore \delta = \text{Arctan}\left(\frac{3}{2}\right) + \pi = \underline{4.124}$

$$\therefore u = \sqrt{13}\cos(\pi t - 4.124)$$

3.

(a) $K = \frac{mg}{L} = \frac{(100\,g)(980\,cm/s^2)}{5\,cm} = 19{,}600\ g/s^2$

$\therefore 100u'' + 19{,}600u = 0$, or $u'' + 196u = 0$

Characteristic equation: $r^2 + 196 = 0$, $r = \pm 14i$

$\therefore u(t) = A\cos(14t) + B\sin(14t)$

$u(0) = 0$, $u'(0) = 10$

$u(0) = 0 \Rightarrow A = 0$

$u'(0) = 10 \Rightarrow 14B\cos(14(0)) = 14B = 10$, $B = \frac{5}{7}$

$\therefore\ u(t) = \frac{5}{7}\sin(14t)$, in cm, t in secs.

(b) $\sin(14t)$ is zero at $14t = 0$ and $14t = \pi$

$\therefore$ First return to equilibrium at $t = \frac{\pi}{14} = 0.224$ sec

4.

(a) $3\text{ in} = \frac{1}{4}\text{ ft}$. $\therefore\ K = \frac{F}{L} = \frac{3\text{ lb}}{\frac{1}{4}\text{ ft}} = 12\text{ lb/ft}$

$$\text{Mass} = \frac{F}{g} = \frac{3\text{ lb}}{32\text{ ft/s}^2} = \frac{3}{32}$$

$\therefore\ mu'' + Ku = \frac{3}{32}u'' + 12u = 0$, or

$u'' + 128u = 0 \qquad \sqrt{128} = 8\sqrt{2}$

$\therefore\ u(t) = A\cos(8\sqrt{2}\,t) + B\sin(8\sqrt{2}\,t)$

$u(0) = -1$ inch $= -\frac{1}{12}$ ft, interpreting "contracting" to mean 1 inch above the 3 inch equilibrium level.

$\therefore A = -\frac{1}{12}$

$u'(0) = 2 \text{ ft/s} \Rightarrow 8\sqrt{2}\, B = 2, \quad B = \frac{1}{4\sqrt{2}} = \frac{\sqrt{2}}{8}$

$\therefore u(t) = -\frac{1}{12}\cos(8\sqrt{2}\, t) + \frac{\sqrt{2}}{8}\sin(8\sqrt{2}\, t)$, in ft, t in secs

(b) From $8\sqrt{2}\, t$, $\omega = 8\sqrt{2}$ rad/sec

$T = \frac{2\pi}{\omega} = \frac{2\pi}{8\sqrt{2}} = \frac{\sqrt{2}}{8}\pi$ sec

$R\cos(\delta) = -\frac{1}{12} \quad R\sin(\delta) = \frac{\sqrt{2}}{8}$

$R = \sqrt{\left(-\frac{1}{12}\right)^2 + \left(\frac{\sqrt{2}}{8}\right)^2} = \sqrt{\frac{1}{144} + \frac{2}{64}} = \sqrt{\frac{4}{576} + \frac{18}{576}}$

$= \sqrt{\frac{22}{576}} = \sqrt{\frac{11}{288}} \approx 0.1954$

$\therefore$ Amplitude ≈ 0.1954 feet

$\tan\delta = \frac{\sqrt{2}/8}{-\frac{1}{12}} = -\frac{3\sqrt{2}}{2}$, $\sin\delta > 0$, $\cos\delta < 0$

$\therefore$ Quadrant II

$\therefore \delta = \text{Arctan}\left(-\frac{3\sqrt{2}}{2}\right) + \pi \approx 2.0113$ rad

$\therefore u(t) = 0.1954\cos(8\sqrt{2}\, t - 2.0113)$

5.

(a) K: $mg = Kx$, $K = \frac{mg}{x} = \frac{(20g)(980\ cm/s^2)}{5\ cm} = 3{,}920\ g/s^2$

$\gamma = 400$ dyn-s/cm

$\therefore mu'' + \gamma u' + Ku = 20u'' + 400u' + 3920u = 0$,

or, $u'' + 20u' + 196u = 0$

Characteristic equation: $r^2 + 20r + 196 = 0$,

$r = \frac{-20 \pm \sqrt{400 - 4(196)}}{2} = -10 \pm \frac{\sqrt{384}}{2}\, i = -10 \pm 4\sqrt{6}\, i$

$\therefore u(t) = e^{-10t}\left[A\cos(4\sqrt{6}t) + B\sin(4\sqrt{6}t)\right]$

Initial conditions: $u(0) = 2$, $u'(0) = 0$

$\therefore u(0) = 2 \Rightarrow A = 2$

$u'(t) = -10e^{-10t}\left[2\cos(4\sqrt{6}t) + B\sin(4\sqrt{6}t)\right]$

$+ e^{-10t}\left[-8\sqrt{6}\sin(8\sqrt{6}t) + 4B\sqrt{6}\cos(4\sqrt{6}t)\right]$

$\therefore u'(0) = 0 \Rightarrow -20 + 4B\sqrt{6} = 0$, $B = \frac{5}{\sqrt{6}}$

$\therefore\ u(t) = e^{-10t}\left[2\cos(4\sqrt{6}\,t) + \frac{5}{\sqrt{6}}\sin(4\sqrt{6}\,t)\right]$, in cm, t secs

Using $R = \sqrt{A^2+B^2} = \sqrt{2^2 + \frac{25}{6}} = \sqrt{\frac{49}{6}} = \frac{7}{\sqrt{6}}$

$A = R\cos\delta,\ B = R\sin\delta,\ \tan\delta = \frac{B}{A},$

$\therefore\ \tan\delta = \frac{5/\sqrt{6}}{2},\ \delta = \operatorname{Arctan}\left(\frac{5}{2\sqrt{6}}\right) \approx 1.0206$

$\therefore\ u(t) = e^{-10t}\left[\frac{7}{\sqrt{6}}\cos(4\sqrt{6}\,t - \delta)\right],$

$\delta = \operatorname{Arctan}\left(\frac{5}{2\sqrt{6}}\right) \approx 1.0206$

(b) Quasi-frequency: $4\sqrt{6}$ rad/s

Quasi-period: $\frac{2\pi}{4\sqrt{6}} = \frac{\pi}{2\sqrt{6}}$ sec

Undamped ($\gamma = 0$): $u'' + 196u = 0$,

characteristic equation: $r^2 + 196 = 0$, $r = \pm 14i$

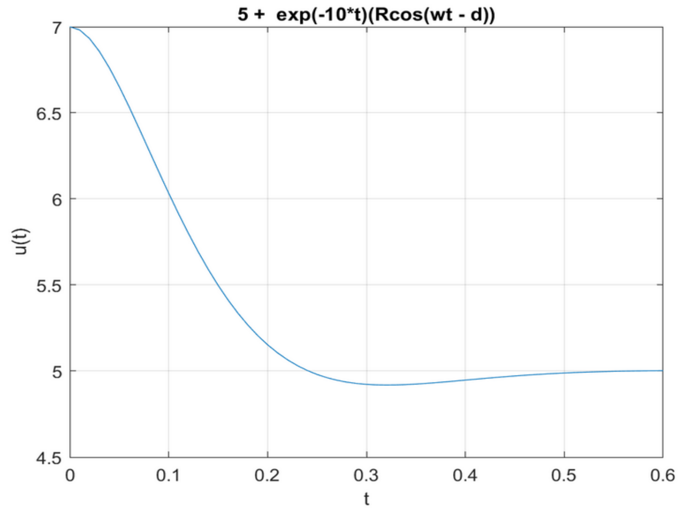
$\therefore\ u(t) = A\cos(14t) + B\sin(14t)$, $\omega = 14$

$\therefore\ T = \frac{2\pi}{14}$

$\therefore\ \frac{\text{Quasi-period}}{\text{Undamped period}} = \frac{\pi/2\sqrt{6}}{2\pi/14} = \frac{7}{2\sqrt{6}} \approx 1.4289$

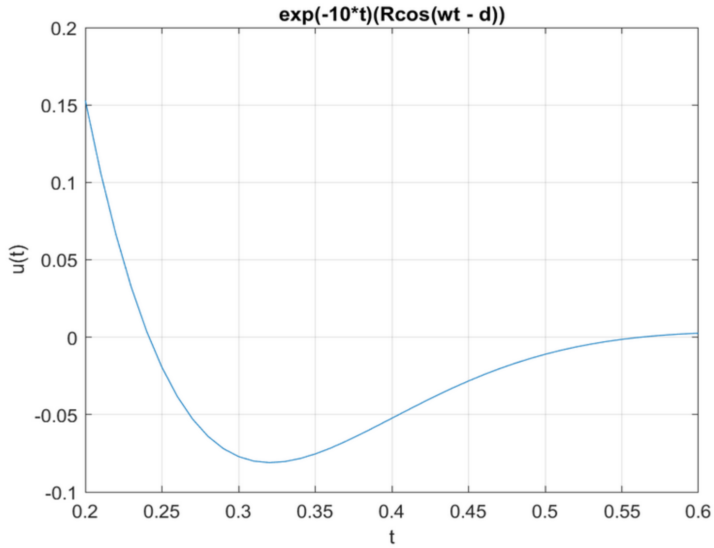
(c) Using MATLAB:

```
clear, clc
R = 7/sqrt(6);
w = 4*sqrt(6);
d = atan(5/(2*sqrt(6)));
t = 0:0.01:0.6;
u = exp(-10*t).*(R*cos(w*t - d));
plot(t, 5 + u)
grid on
xlabel 't', ylabel 'u(t)'
title '5 +  exp(-10*t)(Rcos(wt - d))'
```



(d) Using MATLAB, zoom in on equilibrium position, and only plot $u(t)$:

```
clear, clc
R = 7/sqrt(6);
w = 4*sqrt(6);
d = atan(5/(2*sqrt(6)));
t = 0.2:0.01:0.6;
u = exp(-10*t).*(R*cos(w*t - d));
plot(t, u)
grid on
xlabel 't', ylabel 'u(t)'
title 'exp(-10*t)(Rcos(wt - d))'
```



By inspection, for

$\tau \simeq 0.41$, $|u(t)| < 0.05$

Use MATLAB to solve $u(t) = -0.05$ precisely,

or $u(t) + 0.05 = 0$, using $t = 0.41$ as initial guess.

```
clear, clc
R = 7/sqrt(6);
w = 4*sqrt(6);
d = atan(5/(2*sqrt(6)));
syms t
u = exp(-10*t)*(R*cos(w*t - d));
vpasolve(u + 0.05, t, 0.41)
```

ans = 0.40454119866669580923467094660677

$\therefore \tilde{T} \approx 0.4045$ secs

6.

(a) $k: \frac{F}{x} = \frac{3\ N}{0.1\ \text{meter}} = 30\ N/m$

$\gamma: \frac{F}{V} = \frac{3\ N}{5\ m/s} = 0.6\ \frac{N \cdot s}{m}$

$m = 2\ kg$. Let $u(t)$ = position from equilibrium

$\therefore 2u'' + 0.6u' + 30u = 0$, or

$u'' + 0.3u' + 15u = 0$, $u(0) = 0.05\ m$, $u'(0) = 0.1\ m/s$

Characteristic equation: $r^2 + 0.3r + 15 = 0$,

$$r = \frac{-0.3 \pm \sqrt{0.09 - 60}}{2} = -0.15 \pm 3.87008$$

$$\therefore u(t) = e^{-0.15t}\left[A\cos(3.87008t) + B\sin(3.87008t)\right]$$

$$u(0) = 0.05 \Rightarrow \underline{A = 0.05}$$

$$u'(t) = (-0.15)e^{-0.15t}\left[0.05\cos(3.87008t) + B\sin(3.87008t)\right]$$

$$+ e^{-0.15t}\left[-(0.05)(3.87008)\sin(3.87008t) + (3.87008)B\cos(3.87008t)\right]$$

$$\therefore u'(0) = 0.1 \Rightarrow (-0.15)(0.05) + (3.87008)B = 0.1$$

$$\therefore \underline{B = 0.02778}$$

$$\therefore u(t) = e^{-0.15t}[(0.05)\cos(3.87008t) + (0.02778)\sin(3.87008t)]$$

Or, using $R = \sqrt{A^2 + B^2} = \sqrt{0.05^2 + 0.02778^2} = 0.057198$

$A = R\cos\delta$, $B = R\sin\delta$, $\delta = \text{Arctan}\left(\frac{B}{A}\right)$

$$\therefore \delta = \text{Arctan}\left(\frac{0.02778}{0.05}\right) = 0.50709 \text{ radians}$$

$$\therefore u(t) = e^{-0.15t}\left[0.057198\cos(3.87008t - 0.50709)\right]$$

in meters, t secs

(b) Quasi-frequency: $\mu = \underline{3.87008 \text{ rad/s}}$

(c) Natural frequency of undamped motion: $\gamma = 0$

$$\therefore u'' + 15u = 0, \quad r^2 + 15 = 0, \quad r = \pm\sqrt{15}\,i$$

$\therefore$ Natural frequency: $\omega = \sqrt{15}$ rad/s

$$\therefore \frac{\mu}{\omega} = \frac{3.87008}{\sqrt{15}} \approx \underline{0.99925}$$

7.

From $LQ'' + RQ' + \frac{1}{C}Q = 0$, (no current $\Rightarrow E(t) = 0$)

$(0.2)Q'' + (3\times10^2)Q' + (10^5)Q = 0$, or

$Q'' + (1.5\times10^3)Q' + (5\times10^5)Q = 0$

Characteristic equation: $r^2 + (1.5\times10^3)r + (5\times10^5) = 0$,

Using MATLAB,

```
clear, clc
p = [1 1.5e3 5e5];
roots(p)
```

```
ans = 2×1
   -1000
    -500
```

$\therefore Q(t) = C_1 e^{-1000t} + C_2 e^{-500t}$

$\therefore Q'(t) = -1000 C_1 e^{-1000t} - 500 C_2 e^{-500t}$

Initial conditions: $Q(0) = 10^{-6}$, $Q'(0) = 0$

$\therefore Q(0) = 10^{-6} = C_1 + C_2$

$Q'(0) = 0 = -1000 C_1 - 500 C_2$

From MATLAB,

```
A = [1 1 ; -1000 -500];
B = [1e-6 ; 0];
C = linsolve(A,B)  %solve A*C = B
```

```
C = 2×1
10^-5 ×
   -0.1000
    0.2000
```

$\therefore C_1 = -1.0\times10^{-6}$, $C_2 = 2.0\times10^{-6}$

$\therefore\ Q(t) = 10^{-6}\left(2e^{-500t} - e^{-1000t}\right)$ Coulombs, t in secs

8.

Undamped motion: $u'' + u = 0$, $r^2 + 1 = 0$, $r = \pm i$

$\therefore\ u(t) = c_1 \cos(t) + c_2 \sin(t)$

$\therefore$ Period $= 2\pi$

Damped motion: $r^2 + \gamma r + 1 = 0$, $r = \dfrac{-\gamma \pm \sqrt{\gamma^2 - 4}}{2}$

Assume $\gamma^2 - 4 < 0$ to get sinusoidal solution

$\therefore\ u(t) = e^{-\gamma/2}\left[A\cos\left(\frac{\sqrt{4-\gamma^2}}{2}t\right) + B\sin\left(\frac{\sqrt{4-\gamma^2}}{2}t\right)\right]$

$\therefore$ Quasi-period $= \dfrac{2\pi}{\frac{\sqrt{4-\gamma^2}}{2}} = \dfrac{4\pi}{\sqrt{4-\gamma^2}}$

$\therefore\ \dfrac{\text{Quasi-period}}{\text{Undamped period}} = \dfrac{4\pi/\sqrt{4-\gamma^2}}{2\pi} = 1.5$ (50% greater)

$\therefore\ \sqrt{4-\gamma^2} = \frac{4}{3}$, $\gamma^2 = 4 - \frac{16}{9} = \frac{20}{9}$, $\gamma = \sqrt{\frac{20}{9}} \simeq 1.4907$

Check: $\gamma^2 - 4 = \frac{20}{9} - 4 = -\frac{16}{9} < 0$,

validating assumption $\gamma^2 - 4 < 0$

9.

Spring constant: $K = \frac{mg}{2}$

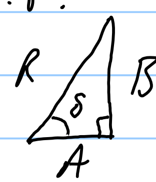
$\therefore mu''(t) + \left(\frac{mg}{2}\right)u(t) = 0$, or $u'' + \frac{g}{2}u = 0$

Characteristic equation: $r^2 + \frac{g}{2} = 0$, $r = \pm\sqrt{\frac{g}{2}}\, i$

$\therefore u(t) = A\cos\left(\sqrt{\frac{g}{2}}\, t\right) + B\sin\left(\sqrt{\frac{g}{2}}\, t\right)$, A, B constants

(1) If $A \neq$, $B \neq 0$, let $R = \sqrt{A^2 + B^2}$, and δ s.t.

$A = R\cos(\delta)$, $B = R\sin(\delta)$



$\therefore \delta = \operatorname{Arctan}\left(\frac{B}{A}\right)$

$\therefore u(t) = R\cos\left(\sqrt{\frac{g}{2}}\, t - \delta\right)$, which has period $\frac{2\pi}{\sqrt{g/2}}$

(2) If $A = 0$, $u(t) = B\sin\left(\sqrt{\frac{g}{2}}\, t\right)$, period $\frac{2\pi}{\sqrt{g/2}}$

(3) If $B = 0$, $u(t) = A\cos\left(\sqrt{\frac{g}{2}}\, t\right)$, period $\frac{2\pi}{\sqrt{g/2}}$

$\therefore$ In all cases, period $= \frac{2\pi}{\sqrt{g/2}} = \underline{2\pi\sqrt{\frac{2}{g}}}$

10.

By Theorem 3.2.1,

$$mu'' + \gamma u' + ku = 0, \quad u(t_0) = u_0, \quad u'(t_0) = 0$$

has a unique solution, call it $v(t)$, so

$$mv'' + \gamma v' + kv = 0, \quad v(t_0) = u_0, \quad v'(t_0) = 0 \quad [1]$$

and $mu'' + \gamma u' + ku = 0, \quad u(t_0) = 0, \quad u'(t_0) = u_0'$

has a unique solution, call it $w(t)$, so

$$mw'' + \gamma w' + kw = 0, \quad w(t_0) = 0, \quad w'(t_0) = u_0' \quad [2]$$

Let $u(t) = v(t) + w(t)$

$$\therefore mu'' + \gamma u' + ku = m(v+w)'' + \gamma(v+w)' + k(v+w)$$

$$= (mv'' + \gamma v' + kv) + (mw'' + \gamma w' + kw)$$

$$= 0 + 0 = 0,$$ since v satisfies [1] and w satisfies [2]

In addition, $u(t_0) = v(t_0) + w(t_0) = u_0 + 0 = u_0$

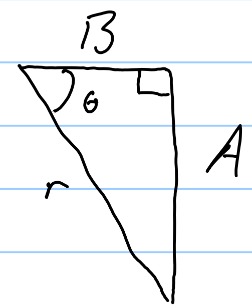
and $u'(t_0) = v'(t_0) + w'(t_0) = 0 + u_0' = u_0'$

$\therefore$ By Theorem 3.2.1, $u(t) = v(t) + w(t)$ is the unique solution to:

$mu'' + \gamma u' + ku = 0, \ u(t_0) = u_0, \ u'(t_0) = u_0'$

11.

(a) Let $r = \sqrt{A^2 + B^2}$, and θ s.t.



$A = r\sin(-\theta), \ B = r\cos(-\theta)$

or $-A = r\sin(\theta), \ B = r\cos(\theta)$, since $\sin(-\theta) = -\sin(\theta)$
$\cos(-\theta) = \cos(\theta)$

$\therefore -\frac{A}{B} = \tan(\theta)$, or $\theta = \text{Arctan}\left(-\frac{A}{B}\right)$

$\therefore A\cos(\omega_0 t) + B\sin(\omega_0 t) =$

$r\sin(-\theta)\cos(\omega_0 t) + r\cos(-\theta)\sin(\omega_0 t) =$

$r\sin(\omega_0 t - \theta)$ using $\sin(x+y) = \sin x \cos y + \cos x \sin y$

Alternatively, since $\cos(x) = \sin\left(x + \frac{\pi}{2} \pm 2n\pi\right), \ n = 0, 1, 2, \ldots$

$\therefore R\cos(\omega_0 t - \delta) = R\sin\left(\omega_0 t - \delta + \frac{\pi}{2} \pm 2n\pi\right)$

$$= R\sin\left[\omega_0 t - \left(\delta - \frac{\pi}{2} \pm 2n\pi\right)\right]$$

$$= r\sin(\omega_0 t - \theta), \text{ where}$$

$$r = R, \quad \theta = \delta - \frac{\pi}{2} \pm 2n\pi = \text{Arctan}\left(\frac{B}{A}\right) - \frac{\pi}{2} \pm 2n\pi$$

(b) From (a), $\underline{r = R} = \sqrt{A^2 + B^2}$

and $\theta = \delta - \frac{\pi}{2} \pm 2n\pi = \delta - \frac{\pi}{2} \pm \frac{4n\pi}{2}$

or, $\underline{\theta = \delta - \frac{\pi}{2}(1 \pm 4n)}$, $n = 0, 1, 2, \ldots$

and $\delta = \text{Arctan}\left(\frac{B}{A}\right)$

12.

From $LQ'' + RQ' + \frac{1}{C}Q = 0$, the characteristic equation is: $Lr^2 + Rr + \frac{r}{C} = 0$,

$$\therefore r = \frac{-R \pm \sqrt{R^2 - 4L/C}}{2L}$$

Critically damped means $R^2 - 4L/C = 0$

$$\therefore R = \sqrt{\frac{4(0.2)}{0.8 \times 10^{-6}}} = \underline{10^3 \text{ Ohms}}$$

13.

Characteristic equation: $mr^2 + \gamma r + ku = 0$,

$$r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$$

Critically damped: $\gamma^2 - 4mk = 0$

$\therefore u(t) = (A + Bt)e^{-\frac{\gamma}{2m}t}$

Suppose the mass crosses the equilibrium point for the first time at t_1: $u(t_1) = (A + Bt_1)e^{-\frac{\gamma}{2m}t_1} = 0$

$\therefore A + Bt_1 = 0$, as $e^{-\frac{\gamma}{2m}t_1} \neq 0$. $\therefore A = -Bt_1$

$\therefore u(t) = B(t - t_1)e^{-\frac{\gamma}{2m}t}$

For $t > t_1$, $t - t_1 > 0$ and $e^{-\frac{\gamma}{2m}t} > 0$

$\therefore (t - t_1)e^{-\frac{\gamma}{2m}t} > 0$ for all $t > t_1$

If $B \neq 0$, then $u(t) = B(t - t_1)e^{-\frac{\gamma}{2m}t}$ is never 0 for $t > t_1$, so $u(t)$ does <u>not cross equilibrium</u> again.

If $B=0$, then from $A=-Bt_1$, $A=0$, and $u(t)=0$, so the mass is motionless at the equilibrium spot, and so technically doesn't cross that point.

Overdamped: $\gamma^2-4mK>0$

$\therefore u(t)=Ae^{r_1t}+Be^{r_2t}$, where

$$r_1=\frac{-\gamma}{2m}+\frac{\sqrt{\gamma^2-4mK}}{2m},\quad r_2=\frac{-\gamma}{2m}-\frac{\sqrt{\gamma^2-4mK}}{2m}$$

Note $r_2<r_1<0$, as $\gamma>0$ and $m>0$.

Suppose the mass crosses the equilibrium point for the first time at time t_1: $u(t_1)=0=Ae^{r_1t_1}+Be^{r_2t_1}$

$\therefore Ae^{(r_1-r_2)t_1}+B=0$, $B=-Ae^{(r_1-r_2)t_1}$

$$\therefore u(t)=A\left(e^{r_1t}-e^{(r_1-r_2)t_1}e^{r_2t}\right)$$

$$=A\left(e^{r_1t}-e^{r_1t_1}e^{(t-t_1)r_2}\right)$$

$$=Ae^{r_1t_1}\left[e^{(t-t_1)r_1}-e^{(t-t_1)r_2}\right]$$

$\therefore$ For $t>t_1$, $t-t_1>0$, so $r_1>r_2\Rightarrow$

$$(t-t_1)r_1 > (t-t_1)r_2 \implies$$

$$e^{(t-t_1)r_1} > e^{(t-t_1)r_2} \implies$$

$$e^{(t-t_1)r_1} - e^{(t-t_1)r_2} > 0 \implies$$

$$e^{r_1 t_1}\left[e^{(t-t_1)r_1} - e^{(t-t_1)r_2}\right] > 0$$

$\therefore$ For $A \neq 0$, $u(t) \neq 0$ for $t > t_1$

$\therefore$ $u(t)$ does <u>not cross</u> equilibrium <u>again</u>

For $A=0$, $B = -Ae^{(r_1-r_2)t_1} \Rightarrow B=0$, so

$u(t)=0$ and mass is motionless at equilibrium point, and so technically doesn't cross that point.

14.

Characteristic equation: $mr^2 + \gamma r + K = 0$,

$$\therefore \quad r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4Km}}{2m}$$

Critically damped $\Rightarrow \gamma^2 - 4km = 0$, so $r = -\frac{\gamma}{2m}$

$\therefore u(t) = (A + Bt)e^{-\frac{\gamma}{2m}t}$, A, B constants

$u(0) = A \Rightarrow A = u_0$

$u'(t) = -\frac{\gamma}{2m}(u_0 + Bt)e^{-\frac{\gamma}{2m}t} + Be^{-\frac{\gamma}{2m}t}$ [1]

$\therefore u'(0) = -\frac{\gamma u_0}{2m} + B = v_0$, $B = v_0 + \frac{\gamma u_0}{2m}$

$\therefore u(t) = u_0 e^{-\frac{\gamma}{2m}t} + \left(v_0 + \frac{\gamma u_0}{2m}\right) t e^{-\frac{\gamma}{2m}t}$ [2]

(a) If $v_0 = 0$, $u(t) = u_0 e^{-\frac{\gamma}{2m}t} + \frac{\gamma u_0}{2m} t e^{-\frac{\gamma}{2m}t}$

(1) If $u(t) = 0$, then $u_0 + \frac{\gamma u_0}{2m} t = 0$

Assuming $u_0 \neq 0$, $1 + \frac{\gamma t}{2m} = 0$, $t = -\frac{2m}{\gamma} < 0$, since $m > 0, \gamma > 0$. $\therefore$ There is no $t > 0$ for which $u(t) = 0$

(2) Since $\lim_{t \to \infty} e^{-\frac{\gamma}{2m}t} = 0$,

$$\lim_{t \to \infty} u(t) = 0 + \frac{\gamma u_0}{2m} \lim_{t \to \infty} \frac{t}{e^{\frac{\gamma}{2m}t}}$$

$$= \frac{\gamma u_0}{2m} \lim_{t \to \infty} \frac{1}{\frac{\gamma}{2m} e^{\frac{\gamma}{2m}t}} = 0, \text{ using L'Hopital's rule}$$

$$\therefore \lim_{t \to \infty} u(t) = 0$$

(b) If $u_0 > 0$, setting $u(t) = 0$ from [2], and dividing by $e^{-\frac{\gamma}{2m}t}$, $u_0 + \left(v_0 + \frac{\gamma u_0}{2m}\right)t = 0$

$$\therefore t = \frac{-2m u_0}{2m v_0 + \gamma u_0}$$

t must be >0, and since $2m u_0 > 0$, then

$$2m v_0 + \gamma u_0 < 0$$

$$\therefore v_0 < -\frac{\gamma u_0}{2m}$$

15.

(a) Equation (26) is: $u = Re^{-\gamma t/(2m)} \cos(\mu t - \delta)$

Looking at critical points of $u(t)$ is messy.

$\therefore$ Note $|Re^{-\frac{\gamma}{2m}t} \cos(\mu t - \delta)| \le Re^{-\frac{\gamma}{2m}t}$ for all t, since $|\cos(\mu t-\delta)| \le 1$. $\therefore$ The relative maxima of $u(t)$ fall on the curve of $Re^{-\frac{\gamma}{2m}t}$, and this occurs when $\cos(\mu t-\delta)=1$.

$\cos(\mu t-\delta)$ has a period of $\frac{2\pi}{\mu}$, so successive points of times when $\cos(\mu t-\delta)=1$ have a period of $\frac{2\pi}{\mu}$. $\therefore$ Time between successive maxima is $\underline{\frac{2\pi}{\mu}}$.

(b)

From (a), let T_d and $T_d+\frac{2\pi}{\mu}$ be any two successive maxima.

$$\therefore \quad \frac{u(T_d)}{u(T_d+\frac{2\pi}{\mu})} = \frac{Re^{-\frac{\gamma}{2m}T_d}\cos(\mu T_d-\delta)}{Re^{-\frac{\gamma}{2m}(T_d+\frac{2\pi}{\mu})}\cos(\mu T_d+2\pi-\delta)} = \frac{1}{e^{-\frac{2\pi\gamma}{2m\mu}}} = e^{\frac{\pi\gamma}{m\mu}}$$

Since $\cos(\mu T_d-\delta)=\cos(\mu T_d+2\pi-\delta)$ and the factors R and $e^{-\frac{\gamma}{2m}T_d}$ cancel.

$$\therefore \frac{u(T_d)}{u\left(T_d + \frac{2\pi}{\mu}\right)} = e^{\frac{\pi\gamma}{m\mu}} = e^{\frac{\gamma}{2m}\frac{2\pi}{\mu}} = e^{\frac{\gamma T_d}{2m}}$$

(c)

From (b), $\frac{u(T_d)}{u\left(T_d + \frac{2\pi}{\mu}\right)} = e^{\frac{\pi\gamma}{m\mu}}$

$$\therefore \Delta = \log\left(e^{\frac{\pi\gamma}{m\mu}}\right) = \frac{\pi\gamma}{m\mu}$$

16.

From Problem #5, $m = 20g$, $\gamma = 400$ dyn-s/cm

$u(t) = e^{-10t}\left[\frac{7}{\sqrt{6}}\cos(4\sqrt{6}\,t - \delta)\right]$. $\therefore \mu = 4\sqrt{6}$

$$\therefore \Delta = \frac{\pi\gamma}{m\mu} = \frac{\pi(400)}{(20)(4\sqrt{6})} = \frac{5\pi}{\sqrt{6}}$$

17.

The general form of the solution is:

$$u(t) = Re^{-\gamma t/2m}\cos(\mu t - \delta)$$

Here, $\gamma = 0$ as there is no damping term.

$R = 3$, and period $= \frac{2\pi}{\mu} = \pi$, so $\mu = 2$

Also, $\mu^2 = \frac{k}{m}$ (see p. 150 of text).

$\therefore 4 = \frac{k}{3/2}$, so $\underline{k = 6}$

$\therefore u(t) = 3\cos(2t - \delta)$, so $u(0) = 3\cos(\delta) = 2$

$\therefore \cos(\delta) = \frac{2}{3}$

$u'(t) = -6\sin(2t - \delta)$, $\therefore u'(0) = 6\sin(\delta) = v$

$\therefore \sin(\delta) = \frac{v}{6}$

$\therefore \cos^2(\delta) + \sin^2(\delta) = 1 = \frac{4}{9} + \frac{v^2}{36}$, $v^2 = 36\left(\frac{5}{9}\right)$,

$\therefore \underline{v = \pm 2\sqrt{5}}$

18.

(a)

From equation (25) of text, p. 152,

$$u(t) = e^{-\gamma t/2m}\left[A\cos(\mu t) + B\sin(\mu t)\right], \quad \mu = \frac{1}{2m}(4km-\gamma^2)^{1/2}$$

$$\therefore u(0) = A = u_0$$

$$\therefore u(t) = e^{-\gamma t/2m}\left[u_0\cos(\mu t) + B\sin(\mu t)\right]$$

$$u'(t) = -\frac{\gamma}{2m}e^{-\gamma t/2m}\left[u_0\cos(\mu t) + B\sin(\mu t)\right] + e^{-\gamma t/2m}\left[-\mu u_0\sin(\mu t) + \mu B\cos(\mu t)\right]$$

$$\therefore u'(0) = -\frac{\gamma u_0}{2m} + \left[\mu B\right] = V_0, \quad \mu B = V_0 + \frac{u_0\gamma}{2m}$$

$$\therefore B = \frac{2mV_0 + \gamma u_0}{2m\mu} = \frac{2mV_0 + \gamma u_0}{(4km-\gamma^2)^{1/2}}$$

$$\therefore u(t) = e^{-\gamma t/2m}\left[u_0\cos(\mu t) + \frac{2mv_0 + \gamma u_0}{(4km-\gamma^2)^{1/2}}\sin(\mu t)\right]$$

$$\text{Or, } u(t) = \frac{e^{-\gamma t/2m}}{(4km-\gamma^2)^{\frac{1}{2}}}\left[(4km-\gamma^2)^{1/2}\cos(\mu t) + (2mv_0 + \gamma u_0)\sin(\mu t)\right]$$

(b)

From (a), $A = u_0$, $B = \dfrac{2mv_0 + \gamma u_0}{(4Km - \gamma^2)^{1/2}}$

$$\therefore R = \sqrt{A^2 + B^2} = \sqrt{u_0^2 + \frac{(2mv_0 + \gamma u_0)^2}{4Km - \gamma^2}}$$

$$= \sqrt{\frac{4Km\, u_0^2 - \gamma^2 u_0^2 + 4m^2 v_0^2 + 4m\gamma u_0 v_0 + \gamma^2 u_0^2}{4Km - \gamma^2}}$$

$$\therefore R = \sqrt{\frac{4m(Ku_0^2 + mv_0^2 + \gamma u_0 v_0)}{4Km - \gamma^2}}$$

$$\delta = \text{Arctan}\left(\frac{B}{A}\right) = \text{Arctan}\left[\frac{2mv_0 + \gamma u_0}{u_0 (4Km - \gamma^2)^{1/2}}\right]$$

and $\mu = \frac{1}{2m}(4Km - \gamma^2)^{1/2}$

(c)

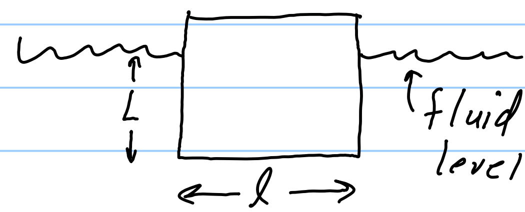
Note: $0 \le \gamma < \sqrt{4Km}$

As $\gamma \to 0$, $R \to \sqrt{\dfrac{4m(Ku_0^2 + mv_0^2)}{4Km}} = \sqrt{u_0^2 + \dfrac{m}{K} v_0^2}$

so R approaches a fixed value, like damped free vibrations.

As $\gamma \to \sqrt{4km}$, $4km - \gamma^2 \to 0$, so $\mu = \frac{1}{2m}(4km-\gamma^2)^{\frac{1}{2}} \to 0$, meaning the frequency approaches 0; i.e., no oscillatory motion, so $R \to \infty$ and becomes meaningless (critical damping), as $u(t) \to (A + Bt)\,e^{-\gamma t/2m}$

19.



(a) In equilibrium, the block is partially submerged by an amount L = depth of inferior edge of block below the fluid level. By Archimedes' principle, the upward buoyant force on the block is $\rho_0 l^2 L g$ (density × volume × g) and this equals the downward force of gravity. $\therefore\ mg = \rho_0 l^2 L g$

Assume downward is positive, and let $u(t) =$

position of inferior edge relative to equilibrium depth L.

By Newton's 2nd law, $mu''(t) = mg - F_b(t)$, where $F_b(t)$ = buoyant force at time t = $\rho_0 [L + u(t)] \ell^2 g$, wher $[L + u(t)] \ell^2$ = volume of block submerged, and the buoyant force always acts upward.

By definition, $m = \rho \ell^3$

$$\therefore \rho \ell^3 u''(t) - mg + \rho_0 [L + u(t)] \ell^2 g = 0$$

Using $mg = \rho_0 \ell^2 L g$ to cancel terms,

$$\rho \ell^3 u'' + \rho_0 \ell^2 g u = 0, \text{ or } \underline{\rho \ell u'' + \rho_0 g u = 0}$$

(b) From $u'' + \frac{\rho_0 g}{\rho \ell} u = 0$, $\frac{\rho_0 g}{\rho \ell}$ is a positive term,

$\therefore$ General solution is $A\cos(\omega t) + B\sin(\omega t)$,

where $\omega = \sqrt{\frac{\rho_0 g}{\rho \ell}}$, where the characteristic

equation is $r^2 + \left(\frac{\rho_0 g}{\rho \ell}\right) = 0$

$$\text{Period} = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\frac{\rho_0 g}{\rho l}}} = 2\pi\sqrt{\frac{\rho l}{\rho_0 g}}$$

20.

(a)

Characteristic equation: $r^2 + 2 = 0$, $r = \pm\sqrt{2}\,i$

$\therefore u(t) = A\cos(\sqrt{2}t) + B\sin(\sqrt{2}t)$

$u(0) = A = 0$. $\therefore u(t) = B\sin(\sqrt{2}t)$

$u'(t) = \sqrt{2}B\cos(\sqrt{2}t)$ $\quad u'(0) = \sqrt{2}B = 2 \Rightarrow B = \sqrt{2}$

$\therefore u(t) = \sqrt{2}\sin(\sqrt{2}t)$

(b)

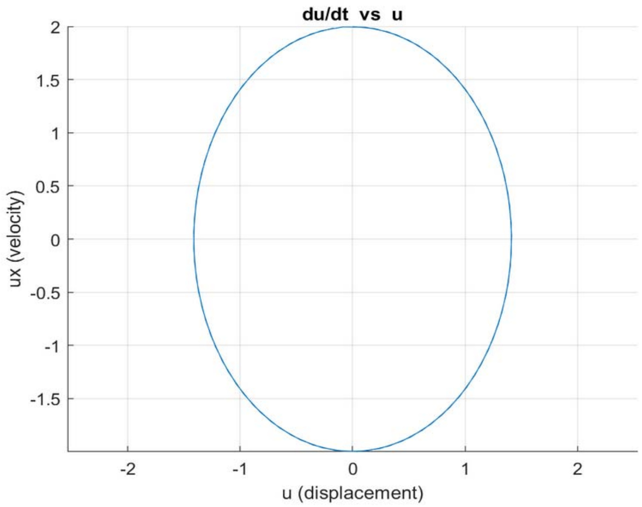
Using MATLAB:

```
clear,clc;
t = 0:0.1:4*pi/sqrt(2);  %two periods
u = sqrt(2)*sin(sqrt(2)*t);
ux= 2*cos(sqrt(2)*t);
hold on
plot(t,u)
plot(t,ux)
grid on
xlabel 't', ylabel 'displacement/velocity'
title 'u = sqrt(2)sin(sqrt(2)t)   ux = 2cos(sqrt(2)t)'
```

(c)

```
clear,clc;
t = 0:0.1:4*pi/sqrt(2);  %two periods
u = sqrt(2)*sin(sqrt(2)*t);
ux= 2*cos(sqrt(2)*t);
hold on
plot(u,ux)  % ux on y-axis, u on x-axis
axis equal
grid on
xlabel 'u (displacement)', ylabel 'ux (velocity)'
title 'du/dt  vs  u'
```

Using MATLAB



$u(0) = 0$ and $u'(0) = 2$
As t increases from 0, displacement $u(t)$ increases and $u'(t)$ decreases.

$\therefore$ Clockwise motion of parametric plot.

21.

(a)

Characteristic equation: $r^2 + \frac{1}{4}r + 2 = 0$

$$r = \frac{-\frac{1}{4} \pm \sqrt{\frac{1}{16} - 8}}{2} = -\frac{1}{8} \pm \frac{\sqrt{127}}{8} i$$

$$\therefore u(t) = e^{-\frac{t}{8}}\left[A\cos\left(\frac{\sqrt{127}}{8}t\right) + B\sin\left(\frac{\sqrt{127}}{8}t\right)\right]$$

$$u(0) = A = 0 \quad \therefore u(t) = Be^{-t/8}\sin\left(\frac{\sqrt{127}}{8}t\right)$$

$$u'(t) = -\frac{B}{8}e^{-t/8}\sin\left(\frac{\sqrt{127}}{8}t\right) + \frac{\sqrt{127}}{8}Be^{-t/8}\cos\left(\frac{\sqrt{127}}{8}t\right)$$

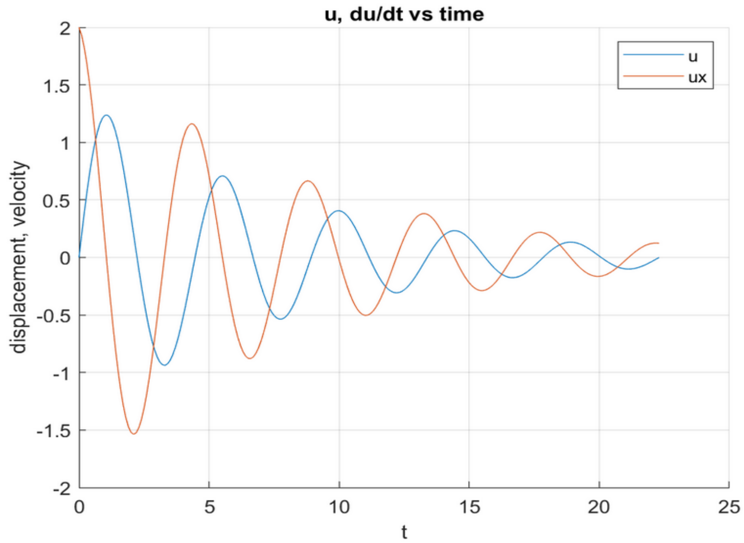
$$u'(0) = \frac{\sqrt{127}}{8}B = 2 \Rightarrow B = \frac{16}{\sqrt{127}}$$

$$\therefore u(t) = \frac{16}{\sqrt{127}} e^{-t/8} \sin\left(\frac{\sqrt{127}}{8}t\right)$$

(b)

Using MATLAB,

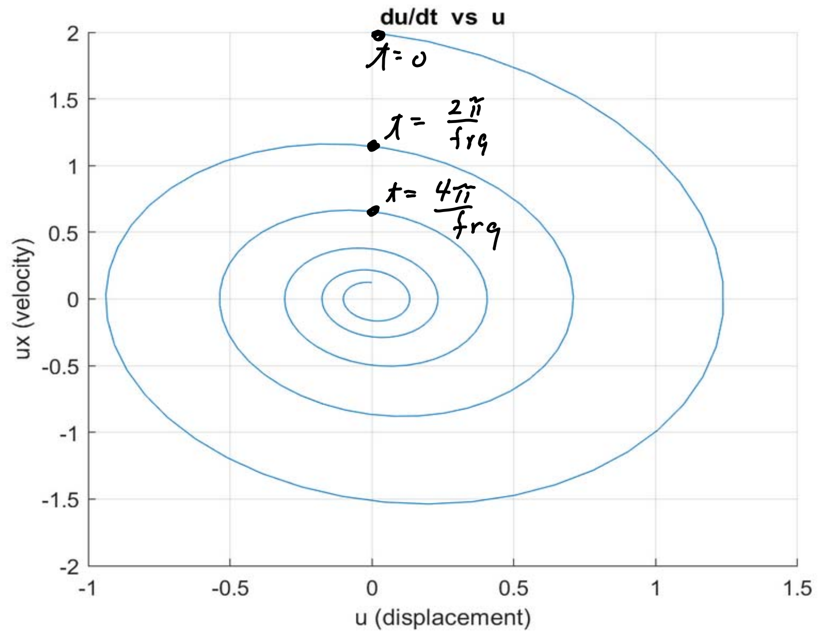
```
clear,clc;
frq = sqrt(127)/8;
B = 2/frq;
t = 0:0.1:10*pi/frq;  % five periods
u = B*exp(-t/8).*sin(frq*t);
ux= (-1/8)*u + frq*B*exp(-t/8).*cos(frq*t);
hold on
plot(t,u)
plot(t,ux)
grid on
legend('u', 'ux')
xlabel 't', ylabel 'displacement, velocity'
title 'u, du/dt vs time'
```



(c)

Using MATLAB

```
clear,clc;
frq = sqrt(127)/8;
B = 2/frq;
t = 0:0.1:10*pi/frq;  % five periods
u = B*exp(-t/8).*sin(frq*t);
ux= (-1/8)*u + frq*B*exp(-t/8).*cos(frq*t);
hold on
plot(u,ux)
grid on
xlabel 'u (displacement)'
ylabel 'ux (velocity)'
title 'du/dt  vs  u'
```



Points on the curve where $u=0$ represent successive quasi-periods in time

As t initially increases from $t=0$, $u(t)$ increases and $u'(t)$ decreases in value.

$\therefore$ Direction of motion in phase plane is <u>clockwise</u> as t increases.

22.

(a)

(1) The velocity imparted to the mass is 6 from $u'(0)=6$. $\therefore$ From Kinetic energy $= \frac{1}{2}mv^2$,

the K.E. imparted to the mass is $\frac{1}{2}mb^2$

(2) The potential energy of a spring is $\frac{1}{2}Kx^2$, K = spring constant, x = distance of mass from unstretched position. From $u(0)=a$, $x=a$, so potential energy of spring at $t=0$ is $\frac{1}{2}Ka^2$

(1) & (2) ⇒ total energy of system at $t=0$ is

$$\frac{1}{2}mb^2 + \frac{1}{2}Ka^2 = \frac{1}{2}(Ka^2 + mb^2)$$

(b)

Assume $K, m \neq 0$ $\therefore$ $u''(t) + \frac{K}{m}u(t) = 0$

Characteristic equation: $r^2 + \frac{K}{m} = 0$, $r = \pm\sqrt{\frac{K}{m}}\, i$

$\therefore$ $u(t) = A\cos\left(\sqrt{\frac{K}{m}}\,t\right) + B\sin\left(\sqrt{\frac{K}{m}}\,t\right)$

$u(0) = a \Rightarrow A = a$

$u'(t) = -a\sqrt{\frac{K}{m}}\sin\left(\sqrt{\frac{K}{m}}\,t\right) + B\sqrt{\frac{K}{m}}\cos\left(\sqrt{\frac{K}{m}}\,t\right)$

$u'(0) = b \Rightarrow B\sqrt{\frac{K}{m}} = b$, $B = b\sqrt{\frac{m}{K}}$

$$\therefore\ u(t) = a\cos\left(\sqrt{\tfrac{K}{m}}\,t\right) + b\sqrt{\tfrac{m}{K}}\sin\left(\sqrt{\tfrac{K}{m}}\,t\right)$$

(c)

Look at $\frac{1}{2}Ku(t)^2$ and $\frac{1}{2}mu'(t)^2$

$$u(t)^2 = a^2\cos^2\left(\sqrt{\tfrac{K}{m}}\,t\right) + 2ab\sqrt{\tfrac{m}{K}}\cos\left(\sqrt{\tfrac{K}{m}}\,t\right)\sin\left(\sqrt{\tfrac{K}{m}}\,t\right) + b^2\frac{m}{K}\sin^2\left(\sqrt{\tfrac{K}{m}}\,t\right)$$

$$u'(t) = -a\sqrt{\tfrac{K}{m}}\sin\left(\sqrt{\tfrac{K}{m}}\,t\right) + b\cos\left(\sqrt{\tfrac{K}{m}}\,t\right)$$

$$\therefore\ u'(t)^2 = a^2\frac{K}{m}\sin^2\left(\sqrt{\tfrac{K}{m}}\,t\right) - 2ab\sqrt{\tfrac{K}{m}}\cos\left(\sqrt{\tfrac{K}{m}}\,t\right)\sin\left(\sqrt{\tfrac{K}{m}}\,t\right) + b^2\cos^2\left(\sqrt{\tfrac{K}{m}}\,t\right)$$

For simplicity, let $x = \sqrt{\frac{K}{m}}\,t$

$$\therefore\ \frac{1}{2}Ku(t)^2 = \frac{1}{2}Ka^2\cos^2(x) + ab\sqrt{mK}\cos(x)\sin(x) + \frac{1}{2}mb^2\sin^2(x)$$

Cancel

$$\frac{1}{2}mu'(t)^2 = \frac{1}{2}Ka^2\sin^2(x) - ab\sqrt{mK}\cos(x)\sin(x) + \frac{1}{2}mb^2\cos^2(x)$$

$$\therefore\ \frac{1}{2}Ku(t)^2 + \frac{1}{2}mu'(t)^2 = \frac{1}{2}Ka^2\left[\cos^2(x) + \sin^2(x)\right] + \frac{1}{2}mb^2\left[\sin^2(x) + \cos^2(x)\right]$$

$\therefore \frac{1}{2} K u(t)^2 + \frac{1}{2} m u'(t)^2 = \frac{1}{2} K a^2 + \frac{1}{2} m b^2$

i.e., total energy at any time t = total system energy at time $t=0$.

23.

(a) Equation (21) is: $mu'' + \gamma u' + ku = 0.$ (21)

From Newton's 2nd law, Σ Forces = $mu''(t)$, where $u(t)$ = position of mass at time t.

The forces acting on the mass are from the spring ($F = -Ku(t)$) and air resistance ($F = -\gamma u'(t)$). Let the unstretched spring position be 0, positive to right.

$\therefore \; mu''(t) = -Ku(t) - \gamma u'(t)$,

or, $mu''(t) + \gamma u'(t) + Ku(t) = 0$

This is (21)

(b) The diffence is no gravity involved, so no initial stretching of spring. Thus, equilibrium for spring is unstretched position. With gravity, equilibrium position is the stretch position from mg.

24.

If $\epsilon > 0$, then ϵu^3 represents an additional amount, so that the spring force number is larger than just ku. This acts much like a stiffer spring (one with a larger K value).

If $\epsilon < 0$, then $|ku + \epsilon u^3| < |ku|$, so that the force value is smaller than just ku. This acts like a softer spring (one with a smaller K value).

(a)

From Newton's 2nd law, $\sum$ Forces $= mu''(t)$

The forces are from the spring $[F = -(Ku + \epsilon u^3)]$ and air resistance $(F = -\gamma u')$.

$\therefore \; -(Ku + \epsilon u^3) - \gamma u' = mu''$

$\therefore \; mu''(t) + \gamma u'(t) + Ku(t) + \epsilon u^3(t) = 0$

(b)

With $m=1$, $K=1$, $\gamma=0$, $\epsilon=0$,

$u''(t) + u(t) = 0$.

Characteristic equation: $r^2 + 1 = 0$, $r = \pm i$

$\therefore \; u(t) = A\cos(t) + B\sin(t)$

$u(0) = 0 \Rightarrow A = 0 \quad \therefore \; u'(t) = B\cos(t)$

$u'(0) = 1 \Rightarrow B = 1 \quad \therefore \; \underline{u(t) = \sin(t)}$

Amplitude $= \underline{1}$, period $= \frac{2\pi}{1} = \underline{2\pi}$

(c)

$u'' + u + \epsilon u^3 = 0$, or $u'' = -u - \epsilon u^3$

To generate numerical approximations of $u(t)$ over time, use the linear approximation using $u'(t)$:

$u(a+h) = u(a) + u'(a) \cdot h$, where $u(a)$ is the already known value and $u(a+h)$ is the next value (i.e., creating a sequence or array of values of $u(t)$ from $t=0$ to $t=$ end time).

This depends u on knowledge of $u(a)$ and $u'(a)$. Starting out a $t=0$, we know $u(0)$ and $u'(0)$. However, the next step in the sequence is:

$$u(a+2h) = u(a+h) + u'(a+h) \cdot h$$

We know $u(a+h)$ from the prior step.

To get $u'(a+h)$, use the linear approximation:

$$u'(a+h) = u'(a) + u''(a) \cdot h$$

We know $u'(a)$ from the prior step ($u'(0) = 1$ at the start), and $u''(a) = -u(a) - \epsilon u^3(a)$.

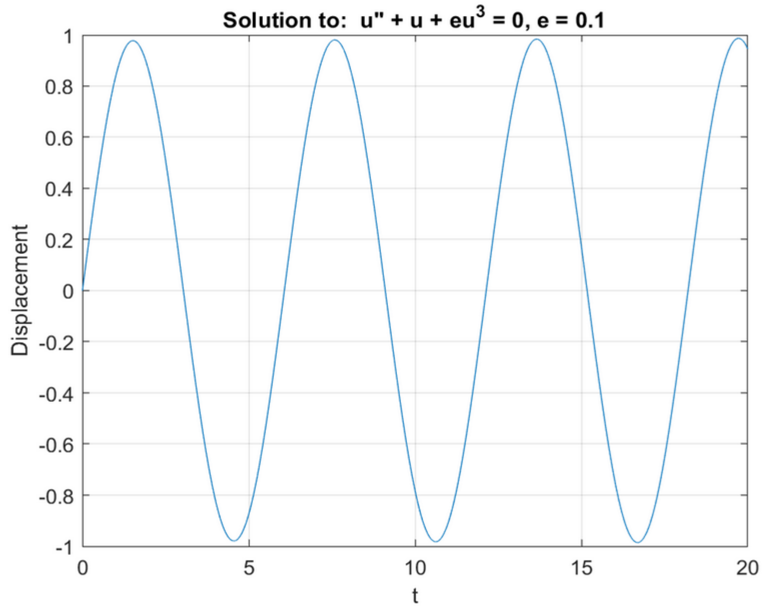
$\therefore$ Since $u(a)$ is known from a prior step, $u''(a)$ can be calculated, so $u''(a+h)$ can be calculated.

Use MATLAB to generate the arrays, and then plot $u(t)$ vs. time for a given ϵ.

Use trial-and-error to guess how long to plot for the time axis to investigate periodicity.

```
clear, clc;
e = 0.1;
h=0.001; T_end=20.0;  % try 20 secs
t=0:h:T_end;
u(1)=0;  % starting initial values
ux(1)=1;
for i = 2:length(t)
        %get next value of u(t) from prior u(t), u'(t)
    u(i) = u(i-1) + ux(i-1)*h;
        %calculate prior u"(t) from prior u(t) in order
        % to calculate next u'(t) to prepare for next
        % calculation of u(t) when enter loop again
    uxx(i-1) = -u(i-1) - e*(u(i-1))^3;
    ux(i) = ux(i-1) + uxx(i-1)*h;
end
plot(t,u)
grid on
xlabel 't', ylabel 'Displacement'
title 'Solution to:  u" + u + eu^3 = 0, e = 0.1'
```

Use a fine mesh, like $h = 0.001$ to improve accuracy.



The motion appears <u>periodic</u>.

Estimate amplitude: 0.97 period: 6

Modify code above to more accurately measure.

```
clear, clc;
e = 0.1;
h=0.001; T_end=20.0;  % try 20 secs
t=0:h:T_end;
u(1)=0;  % starting initial values
ux(1)=1;
Max = 0; Tp = 0;  % for calculating amplitude, period
for i = 2:length(t)
        %get next value of u(t) from prior u(t), u'(t)
    u(i) = u(i-1) + ux(i-1)*h;
        %calculate prior u"(t) from prior u(t) in order
        % to calculate next u'(t) to prepare for next
        % calculation of u(t) when enter loop again
    uxx(i-1) = -u(i-1) - e*(u(i-1))^3;
    ux(i) = ux(i-1) + uxx(i-1)*h;

    if u(i) > Max
        Max = u(i);
    end
    if (abs(u(i)) < 0.001) & ((t(i)>5) & (t(i)<7.5))
        Tp = t(i);
    end
end
plot(t,u)
grid on
xlabel 't', ylabel 'Displacement'
title 'Solution to:  u" + u + eu^3 = 0, e = 0.1'
Max, Tp
```

```
Max = 0.9871
Tp = 6.0700
```

∴ Amplitude = <u>0.9871</u>

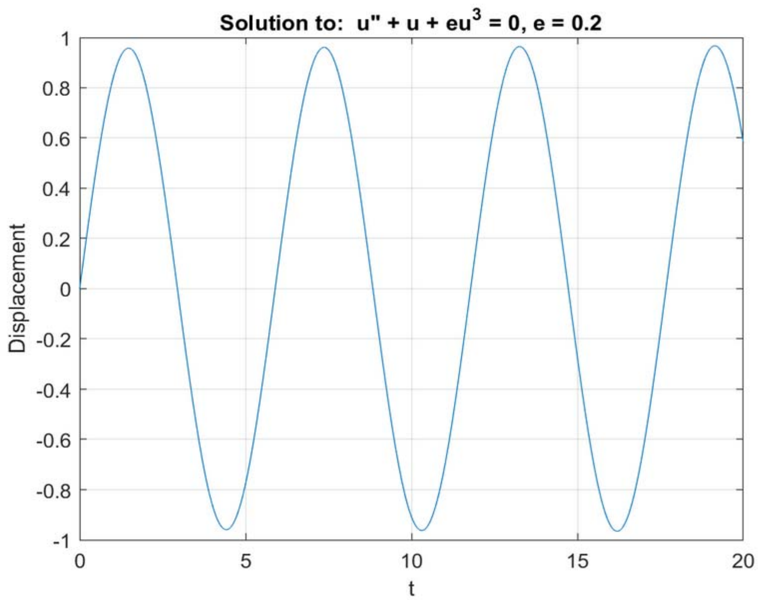
Period = <u>6.07</u>

(d)

For $\epsilon = 0.2$ Using MATLAB

```
clear, clc;
e = 0.2;
h=0.001; T_end=20.0;  % try 20 secs
t=0:h:T_end;
u(1)=0;  % starting initial values
ux(1)=1;
Max = 0; Tp = 0;  % for calculating amplitude, period
for i = 2:length(t)
        %get next value of u(t) from prior u(t), u'(t)
    u(i) = u(i-1) + ux(i-1)*h;
        %calculate prior u"(t) from prior u(t) in order
        % to calculate next u'(t) to prepare for next
        % calculation of u(t) when enter loop again
    uxx(i-1) = -u(i-1) - e*(u(i-1))^3;
    ux(i) = ux(i-1) + uxx(i-1)*h;

    if u(i) > Max
        Max = u(i);
    end
    if (abs(u(i)) < 0.001) & ((t(i)>5) & (t(i)<7.5))
        Tp = t(i);
    end
end
plot(t,u)
grid on
xlabel 't', ylabel 'Displacement'
title 'Solution to:  u" + u + eu^3 = 0, e = 0.2'
Max, Tp
```

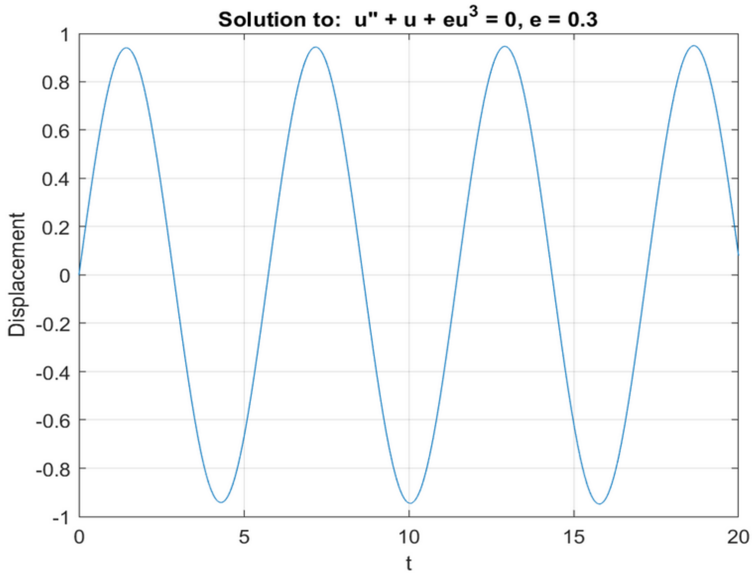



```
Max = 0.9671
Tp = 5.8930
```

For $\epsilon = 0.3$

```
clear, clc;
e = 0.3;
h=0.001; T_end=20.0;  % try 20 secs
t=0:h:T_end;
u(1)=0;  % starting initial values
ux(1)=1;
Max = 0; Tp = 0;  % for calculating amplitude, period
for i = 2:length(t)
        %get next value of u(t) from prior u(t), u'(t)
    u(i) = u(i-1) + ux(i-1)*h;
        %calculate prior u"(t) from prior u(t) in order
        % to calculate next u'(t) to prepare for next
        % calculation of u(t) when enter loop again
    uxx(i-1) = -u(i-1) - e*(u(i-1))^3;
    ux(i) = ux(i-1) + uxx(i-1)*h;

    if u(i) > Max
        Max = u(i);
    end
    if (abs(u(i)) < 0.001) & ((t(i)>5) & (t(i)<7.5))
        Tp = t(i);
    end
end
plot(t,u)
grid on
xlabel 't', ylabel 'Displacement'
title 'Solution to:  u" + u + eu^3 = 0, e = 0.3'
Max, Tp
```




```
Max = 0.9496
Tp = 5.7410
```

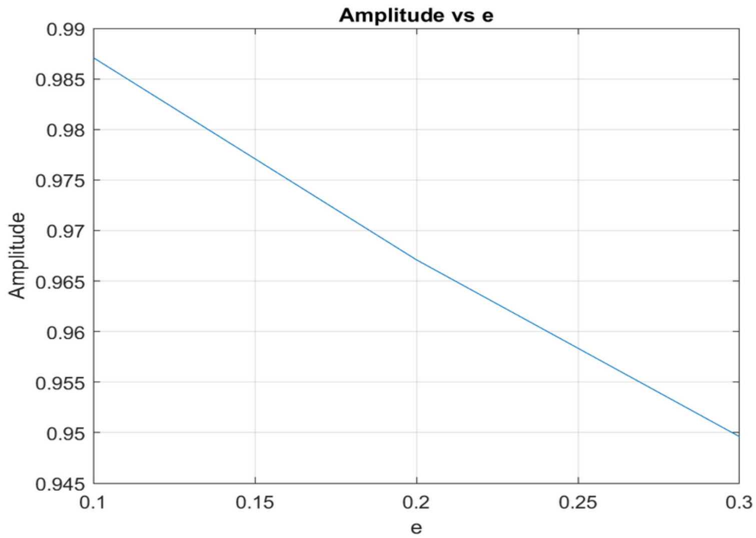
(e)

Using MATLAB and above values in (d):

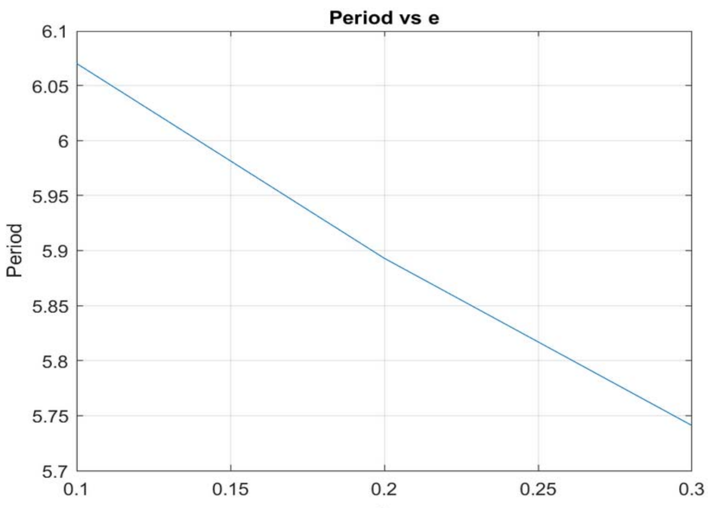
```
clear, clc;
e = [0.1 0.2 0.3];
Max = [0.9871 0.9671 0.9496];
Tp = [6.070 5.893 5.741];

plot(e, Max)
grid on
xlabel 'e', ylabel 'Amplitude'
title 'Amplitude vs e'

plot(e, Tp)
grid on
xlabel 'e', ylabel 'Period'
title 'Period vs e'
```



As ϵ increases, both Amplitude and Period decrease.

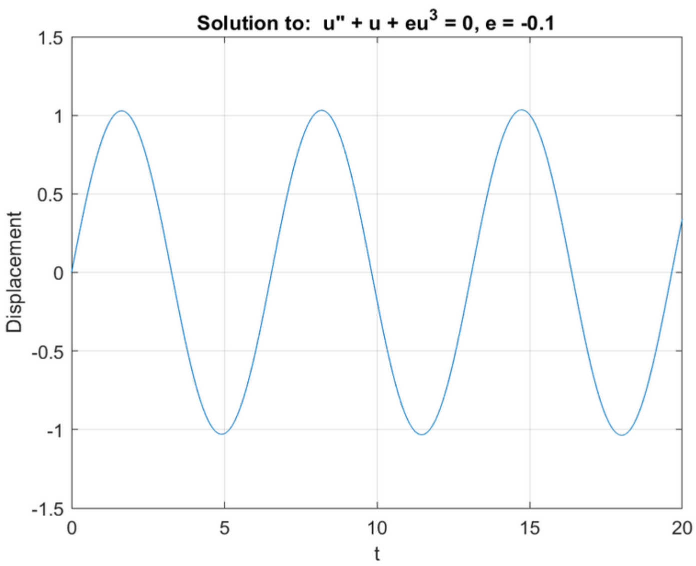


(f)

$\epsilon = -0.1$

```
clear, clc;
e = -0.1;
h=0.001; T_end=20.0;  % try 20 secs
t=0:h:T_end;
u(1)=0;  % starting initial values
ux(1)=1;
Max = 0; Tp = 0;  % for calculating amplitude, period
for i = 2:length(t)
        %get next value of u(t) from prior u(t), u'(t)
    u(i) = u(i-1) + ux(i-1)*h;
        %calculate prior u"(t) from prior u(t) in order
        % to calculate next u'(t) to prepare for next
        % calculation of u(t) when enter loop again
    uxx(i-1) = -u(i-1) - e*(u(i-1))^3;
    ux(i) = ux(i-1) + uxx(i-1)*h;

    if u(i) > Max
        Max = u(i);
    end
    if (abs(u(i)) < 0.001) & ((t(i)>5) & (t(i)<7.5))
        Tp = t(i);
    end
end
plot(t,u)
grid on
xlabel 't', ylabel 'Displacement'
title 'Solution to:  u" + u + eu^3 = 0, e = -0.1'
Max, Tp
```

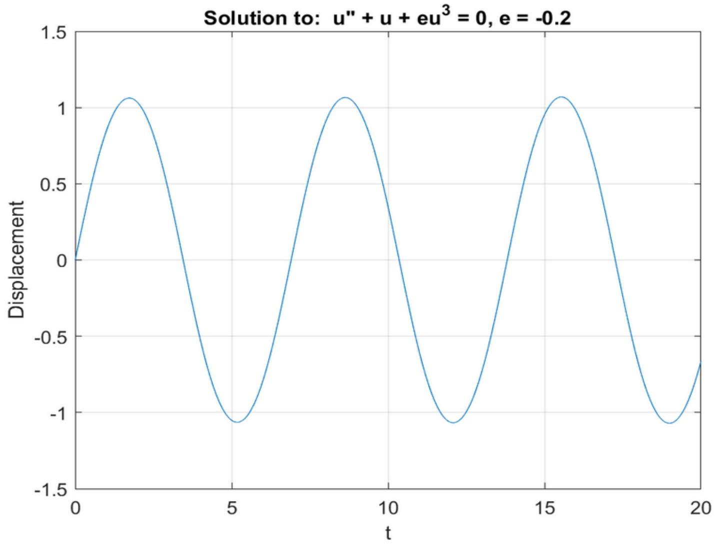



```
Max = 1.0347
Tp = 6.5500
```

$\epsilon = -0.2$

```
clear, clc;
e = -0.2;
h=0.001; T_end=20.0;  % try 20 secs
t=0:h:T_end;
u(1)=0;  % starting initial values
ux(1)=1;
Max = 0; Tp = 0;  % for calculating amplitude, period
for i = 2:length(t)
        %get next value of u(t) from prior u(t), u'(t)
    u(i) = u(i-1) + ux(i-1)*h;
        %calculate prior u"(t) from prior u(t) in order
        % to calculate next u'(t) to prepare for next
        % calculation of u(t) when enter loop again
    uxx(i-1) = -u(i-1) - e*(u(i-1))^3;
    ux(i) = ux(i-1) + uxx(i-1)*h;

    if u(i) > Max
        Max = u(i);
    end
    if (abs(u(i)) < 0.001) & ((t(i)>5) & (t(i)<7.5))
        Tp = t(i);
    end
end
plot(t,u)
grid on
xlabel 't', ylabel 'Displacement'
title 'Solution to:  u" + u + eu^3 = 0, e = -0.2'
Max, Tp
```

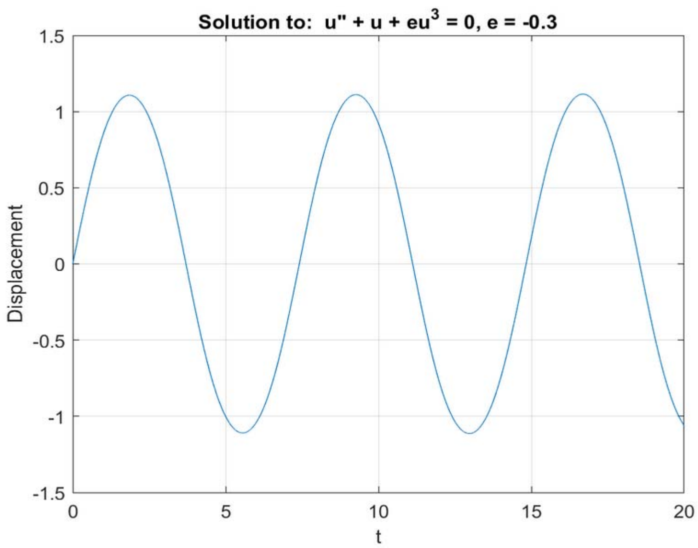



```
Max = 1.0691
Tp = 6.9010
```

$e = -0.3$

```
clear, clc;
e = -0.3;
h=0.001; T_end=20.0;  % try 20 secs
t=0:h:T_end;
u(1)=0;  % starting initial values
ux(1)=1;
Max = 0; Tp = 0;  % for calculating amplitude, period
for i = 2:length(t)
        %get next value of u(t) from prior u(t), u'(t)
    u(i) = u(i-1) + ux(i-1)*h;
        %calculate prior u"(t) from prior u(t) in order
        % to calculate next u'(t) to prepare for next
        % calculation of u(t) when enter loop again
    uxx(i-1) = -u(i-1) - e*(u(i-1))^3;
    ux(i) = ux(i-1) + uxx(i-1)*h;

    if u(i) > Max
        Max = u(i);
    end
    if (abs(u(i)) < 0.001) & ((t(i)>5) & (t(i)<7.5))
        Tp = t(i);
    end
end
plot(t,u)
grid on
xlabel 't', ylabel 'Displacement'
title 'Solution to:  u" + u + eu^3 = 0, e = -0.3'
Max, Tp
```

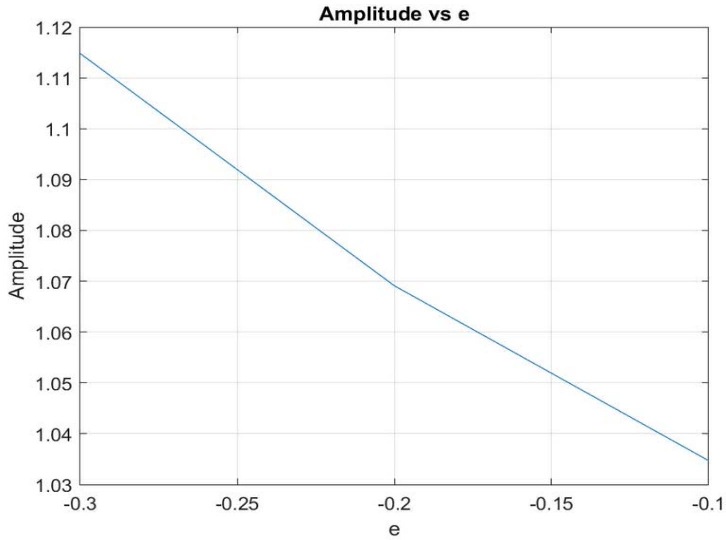



```
Max = 1.1148
Tp = 7.4100
```

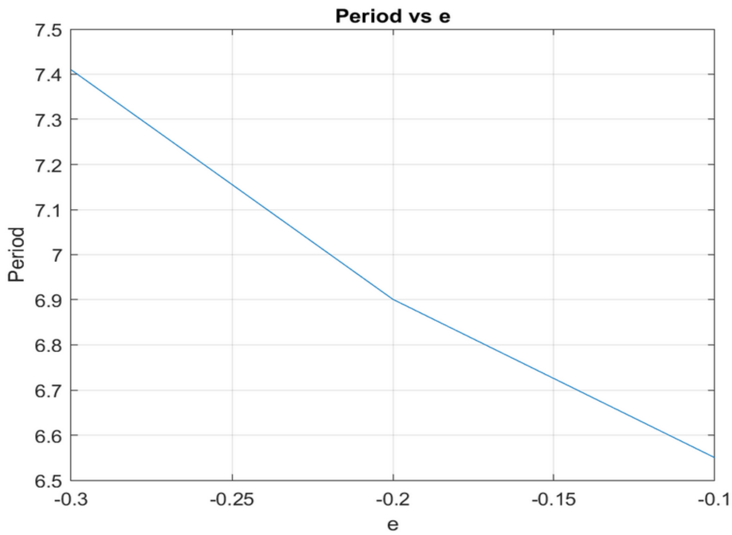
```
clear, clc;
e = [-0.3 -0.2 -0.1];
Max = [1.1148 1.0691 1.0347];
Tp = [7.410 6.901 6.550];

plot(e, Max)
grid on
xlabel 'e', ylabel 'Amplitude'
title 'Amplitude vs e'

plot(e, Tp)
grid on
xlabel 'e', ylabel 'Period'
title 'Period vs e'
```



As ϵ increases, Amplitude and Period decrease, as with positive ϵ's Amplitudes and Periods are greater for $\epsilon < 0$ compared to $\epsilon > 0$.



3.8 Forced Periodic Vibrations

Note Title 11/28/2018

Let $A = \frac{1}{2}(a+b)$, $B = \frac{1}{2}(a-b)$

1.

$$\sin(a) = \sin(A+B) = \sin A \cos B + \cos A \sin B$$
$$\sin(b) = \sin(A-B) = \sin A \cos B - \cos A \sin B$$

$$\therefore \sin(a) - \sin(b) = 2\cos A \sin B$$

$$\therefore \sin(7t) - \sin(6t) = \underline{2\cos\left(\tfrac{13}{2}t\right)\sin\left(\tfrac{1}{2}t\right)}$$

2.

$$\cos(a) = \cos(A+B) = \cos A \cos B - \sin A \sin B$$
$$\cos(b) = \cos(A-B) = \cos A \cos B + \sin A \sin B$$

$$\therefore \cos(a) + \cos(b) = 2\cos A \cos B$$

$$\therefore \cos(\pi t) + \cos(2\pi t) = \underline{2\cos\left(\tfrac{3}{2}\pi t\right)\cos\left(\tfrac{\pi}{2}t\right)}$$

3.

As in #1, $\sin(a) + \sin(b) = 2\sin A \cos B$

$\therefore \sin(3t) + \sin(4t) = 2\sin(\frac{7}{2}t)\cos(\frac{1}{2}t)$

using $\cos(x) = \cos(-x)$

4.

$k = \frac{F}{x} = \frac{(5\,kg)(9.8\,m/s^2)}{0.1\,m} = 490 = 5(98)$

$\gamma = \frac{F}{v} = \frac{2\,N}{0.04\,m/s} = 50$

Let $u(t)$ = position (in meters) from equilibrium.

$\therefore \quad 5u'' + 50u' + 490u = 10\sin(t/2)$

Or, $u''(t) + 10u'(t) + 98u(t) = 2\sin(\frac{t}{2})$

$u(0) = 0, \quad u'(0) = 0.03$

5.

(a) Homogeneous: characteristic equation is

$r^2 + 10r + 98 = 0, \quad r = \frac{-10 \pm \sqrt{100-392}}{2},$

$r = -5 \pm i\sqrt{73}$ as $292 = 2^2 \cdot 73$

$\therefore u_c(t) = e^{-5t}\left[c_1\cos(\sqrt{73}\,t) + c_2\sin(\sqrt{73}\,t)\right]$

Particular: let $u_p(t) = A\cos\left(\frac{t}{2}\right) + B\sin\left(\frac{t}{2}\right)$

Using MATLAB,

```
clear,clc
syms t A B w0
c2 = 1; c1 = 10; c0 = 98;  %coeffs of diff eq
w0 = 1/2;
y = A*cos(w0*t) + B*sin(w0*t);     %attempt
u = c2*diff(y,t,2) + c1*diff(y,t,1) + c0*y;
collect(u, [cos(w0*t), sin(w0*t)])
```

ans =

$$\left(\frac{391\,A}{4} + 5\,B\right)\cos\left(\frac{t}{2}\right) + \left(\frac{391\,B}{4} - 5\,A\right)\sin\left(\frac{t}{2}\right)$$

$$\therefore \frac{391A}{4} + 5B = 0$$

$$\frac{391B}{4} - 5A = 2$$

```
M=[391/4, 5; -5, 391/4];
N=[0; 2];
X = rats(M\N, 18)     % solve M*X = N, X = [A; B]
```

```
X = 2×19 char array
    '    -160/153281   '
    '    3128/153281   '
```

The rats() function produces answers in fractional form. The "18" integer is arbitrary, used to adjust accuracy of the fractional form, a larger number giving greater accuracy. 18 was chosen as the fractions agree with the answer in the back of the book.

$$\therefore u_p(t) = \frac{-160}{153,281}\cos\left(\frac{t}{2}\right) + \frac{3128}{153,281}\sin\left(\frac{t}{2}\right)$$

$$\therefore u(t) = e^{-5t}\left[c_1\cos(\sqrt{73}\,t) + c_2\sin(\sqrt{73}\,t)\right] - \frac{160}{153,281}\cos\left(\frac{t}{2}\right) + \frac{3128}{153,281}\sin\left(\frac{t}{2}\right)$$

$$u(0) = c_1 - \frac{160}{153,281} = 0, \quad c_1 = \frac{160}{153,281}$$

$$\therefore u(t) = e^{-5t}\left[\frac{160}{153,281}\cos(\sqrt{73}\,t) + c_2\sin(\sqrt{73}\,t)\right] - \frac{160}{153,281}\cos(t/2) + \frac{3128}{153,281}\sin(t/2)$$

$$\therefore u'(0) = -5\left[\frac{160}{153,281}\right] + \left[\sqrt{73}\,c_2\right] + \frac{1}{2}\left(\frac{3128}{153,281}\right) = \frac{3}{100}$$

$$\therefore \sqrt{73}\,c_2 = \frac{3}{100} - \frac{1564}{153,281} + \frac{800}{153,281} = \frac{3}{100} - \frac{764}{153,281}$$

$$\therefore \underline{c_2} = \frac{3}{100\sqrt{73}} - \frac{764}{153,281\sqrt{73}} = \underline{0.002927865\,1}$$

and $153,281\, c_2 = 448.7861 \simeq \dfrac{258,052}{575}$

To check this computation for c_2,

from MATLAB:

```
clear,clc
syms t u(t) c2
%Define constants to simplify typing coefficients
x1 = 160/153281; x3 = 3128/153281; sq = sqrt(73);
%Enter u(t) to determine c2 from u'(0) = 0.03
u(t) = exp(-5*t)*(x1*cos(sq*t) + c2*sin(sq*t)) - ...
    x1*cos(t/2) + x3*sin(t/2);
%Calculate u'(t)
Du = diff(u(t), t);
v = subs(Du,t,0)
c2 = solve(v == 3/100, c2)
c2 = vpa(c2)
c2rat = rats(double(153281*c2), 18)
```

v =

$$\sqrt{73}\, c_2 + \frac{11493036051691091}{2305843009213693952}$$

c2 =

$$\frac{1442056355617993189\ \sqrt{73}}{42081634918149914624 00}$$

c2 = 0.0029278650716217856181447697932 58

c2rat = '   258052/575   '

$$\therefore\ u(t) = \frac{1}{153{,}281}\left[160\, e^{-5t}\cos(\sqrt{73}\,t) + \frac{258{,}052}{575}\, e^{-5t}\sin(\sqrt{73}\,t) - 160\cos(t/2) + 3128\sin(t/2)\right]$$

$153{,}281\, c_2$ was calculated to factor out the common 153,281.

(b) **b.** Identify the transient and steady-state parts of the solution.

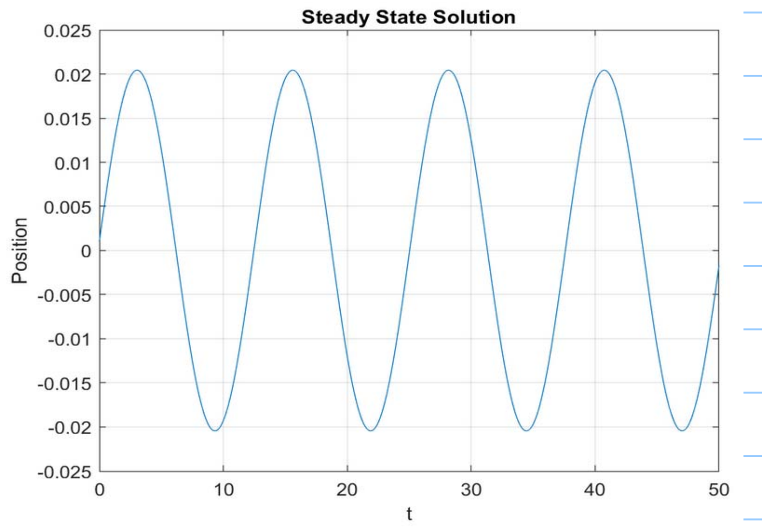
Transient: $\frac{1}{153{,}281}\left[160\, e^{-5t}\cos(\sqrt{73}\,t) + \frac{258{,}052}{575}\, e^{-5t}\sin(\sqrt{73}\,t)\right]$

Steady state: $\frac{1}{153{,}281}\left[-160\cos(t/2) + 3128\sin(t/2)\right]$

(c)

Using MATLAB,

```
t = 0:0.01:50;
ss = x1*cos(t/2) + x3*sin(t/2);
plot(t, ss)
grid on
xlabel 't', ylabel 'Position'
title 'Steady State Solution'
```



(d)

The problem becomes, using #4,

$$5u'' + 50u' + 490u = 2\cos(\omega t)$$

$$u(0) = 0, \quad u'(0) = 0.03$$

From the text, p.161, equation (14),

$$\omega_{max}^2 = \frac{k}{m}\left(1 - \frac{\gamma^2}{2mk}\right) = \frac{490}{5}\left(1 - \frac{50^2}{2(5)(490)}\right) = 48$$

$$\therefore \omega_{max} = \sqrt{48} = \underline{4\sqrt{3} \text{ radians/sec}}$$

6.

$m = \frac{8 \text{ lbs}}{32 \text{ ft/s}^2} = \frac{1}{4} \frac{\text{lbs-s}^2}{\text{ft}}$ $\quad K = \frac{F}{x} = \frac{8}{0.5 \text{ ft}} = 16 \text{ lbs/ft}$

Note no damping.

$\therefore \frac{1}{4} u''(t) + 16 u(t) = 8 \sin(8t)$, or $u'' + 64u = 32 \sin(8t)$

$u(0) = \frac{1}{4}$ ft, $\quad u'(0) = 0$

(a) Characteristic equation: $r^2 + 64 = 0$, $r = \pm 8i$

$\therefore u_c(t) = c_1 \cos(8t) + c_2 \sin(8t)$

Particular: let $u_p(t) = At\cos(8t) + Bt\sin(8t)$

Using MATLAB,

```
clear,clc
syms t A B
c2 = 1; c1 = 0; c0 = 64;  %coeffs of diff eq
y = A*t*cos(8*t) + B*t*sin(8*t);
P = c2*diff(y,t,2) + c1*diff(y,t,1) + c0*y
```

P = $16\,B\cos(8\,t) - 16\,A\sin(8\,t)$

$\therefore 16B\cos(8t) - 16A\sin(8t) = 32\sin(8t)$

$\therefore B = 0, \quad A = -2$

$\therefore\ u_p(t) = -2t\cos(8t)$

$\therefore\ u(t) = c_1\cos(8t) + c_2\sin(8t) - 2t\cos(8t)$

$u(0) = \frac{1}{4} \Rightarrow c_1 = \frac{1}{4}$

$u'(t) = -8c_1\sin(8t) + 8c_2\cos(8t) - 2\cos(8t) + 16t\sin(8t)$

$\therefore\ u'(0) = 0 \Rightarrow 8c_2 - 2 = 0,\ c_2 = \frac{1}{4}$

$\therefore\ u(t) = \frac{1}{4}\cos(8t) + \frac{1}{4}\sin(8t) - 2t\cos(8t)$ ft.

(b) $u'(t) = -2\sin(8t) + 2\cos(8t) - 2\cos(8t) + 16t\sin(8t)$

$= -2\sin(8t) + 16t\sin(8t)$

$\therefore\ u'(t) = 0 \Rightarrow 2\sin(8t) = 16t\sin(8t)$

if $\sin(8t) \neq 0$, $2 = 16t$, $t = \frac{1}{8}$

For $\sin(8t) = 0$, $8t = n\pi$, $n = 0, 1, 2, 3, \ldots$

$\therefore\ t = 0, \frac{\pi}{8}, \frac{\pi}{4}, \frac{3}{8}\pi$

Since $0 < \frac{1}{8} < \frac{\pi}{8}$, first 4 times at which $u'(t) = 0$: $0, \frac{1}{8}, \frac{\pi}{8}, \frac{\pi}{4}$

7.

$m = \frac{8 \text{ lbs}}{32 \text{ ft/s}^2} = \frac{1}{4} \frac{\text{lbs}-\text{s}^2}{\text{ft}}$ $K = \frac{8 \text{ lbs}}{0.5 \text{ ft}} = 16 \text{ lbs/ft}$

(a)

Apparently, $u(0) = 0$, $u'(0) = 0$

$\therefore \frac{1}{4}u'' + \frac{1}{4}u' + 16u = 4\cos(2t)$,

Or, $u'' + u' + 64u = 16\cos(2t)$

Characteristic equation: $r^2 + r + 64 = 0$,

$r = \frac{-1 \pm \sqrt{1-256}}{2}$

$\therefore$ Transient solution:

$$e^{-\frac{1}{2}}\left[c_1 \cos\left(\frac{\sqrt{255}}{2}t\right) + c_2 \sin\left(\frac{\sqrt{255}}{2}t\right)\right.$$

Steady state:

Let $u_p(t) = A\cos(2t) + B\sin(2t)$

MATLAB:

```
clear,clc
syms t A B
c2 = 1; c1 = 1; c0 = 64;  %coeffs of diff eq
y = A*cos(2*t) + B*sin(2*t);
P = c2*diff(y,t,2) + c1*diff(y,t,1) + c0*y;
collect(P,[cos(2*t), sin(2*t)])
```

ans = $(60\,A + 2\,B)\cos(2\,t) + (60\,B - 2\,A)\sin(2\,t)$

Continuing the above code:

```
M = [60, 2; -2, 60];
N = [16; 0];
X = rats(M\N)    % solve M*X = N, in rational form
```

```
X = 2×14 char array
    '    240/901    '
    '      8/901    '
```

$\therefore A = \frac{240}{901}, \quad B = \frac{8}{901}$

$\therefore$ Steady state solution:

$$u(t) = \frac{8}{901}\left[30\cos(2t) + \sin(2t)\right] \text{ ft}$$

(b)

From equation (13) in text on p. 161,

$$\frac{Rk}{F_0} = \left(\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \Gamma\left(\frac{\omega}{\omega_0}\right)^2\right)^{-1/2} \quad \text{where } \Gamma = \frac{\gamma^2}{mk}. \qquad (13)$$

Here, $F_0 = 4$, $\gamma = \frac{1}{4}$, $k = 16$, $\omega = 2$, $\omega_0^2 = \frac{16}{m}$

$$\therefore R = \frac{4}{16}\left[\left(1 - \frac{4}{16/m}\right)^2 + \frac{1/16}{m(16)} \cdot \frac{4}{16/m}\right]^{-\frac{1}{2}}$$

$$= \frac{1}{4}\left[\left(1 - \frac{m}{4}\right)^2 + \frac{1}{256m} \cdot \frac{m}{4}\right]^{-\frac{1}{2}}$$

$\therefore R = \frac{1}{4}\left[\left(1-\frac{m}{4}\right)^2 + \frac{1}{1024}\right]^{-\frac{1}{2}}$

$\therefore \frac{dR}{dm} = -\frac{1}{8}\left[\quad\right]^{-\frac{3}{2}}\left[2\left(1-\frac{m}{4}\right)\left(-\frac{1}{4}\right)\right] = 0$

$\therefore \frac{m}{4} = 1, \ \underline{m = 4 \text{ slugs}}$

This is a relative max, as $m \neq 4$ makes the term $\left(1-\frac{m}{4}\right)^2$ in the denominator positive, so R will be smaller.

8.

Let $u(t)$ = position of mass at time t.

$\therefore 2u'' + u' + 3u = 3\cos(3t) - 2\sin(3t)$, here $\gamma = 1$

From $2r^2 + r + 3 = 0$, $r = \frac{-1 \pm \sqrt{1-24}}{4}$, so homogeneous solution is $u_c(t) = e^{-t/4}(c_1\cos(\mu t) + c_2\sin(\mu t))$

and $u_c(t) \to 0$ as $t \to \infty$

$\therefore$ Steady state solution is the particular solution.

Let $u_p(t) = A\cos(3t) + B\sin(3t)$

Using MATLAB:

```
clear,clc
syms t A B
c2 = 2; c1 = 1; c0 = 3;  %coeffs of diff eq
w = 3;
y = A*cos(w*t) + B*sin(w*t);
P = c2*diff(y,t,2) + c1*diff(y,t,1) + c0*y;
collect(P,[cos(w*t), sin(w*t)])
```

ans = $(3B - 15A)\cos(3t) + (-3A - 15B)\sin(3t)$

$\therefore \left.\begin{array}{l} -15A + 3B = 3 \\ -3A - 15B = -2 \end{array}\right\} -78A = 13,\ A = -\frac{1}{6},\ B = \frac{1}{6}$

$\therefore u_p(t) = -\frac{1}{6}\cos(3t) + \frac{1}{6}\sin(3t)$

$= R\cos(3t - \delta)$

$R\cos\delta = -\frac{1}{6},\quad R\sin\delta = \frac{1}{6},\quad \delta = \operatorname{Arctan}(-1)$

$\therefore \delta = -\frac{\pi}{4} + \pi = \frac{3}{4}\pi$

$R = \sqrt{(-\frac{1}{6})^2 + (\frac{1}{6})^2} = \frac{\sqrt{2}}{6}$

$\therefore$ Steady state: $\frac{\sqrt{2}}{6}\cos(3t - \frac{3}{4}\pi)$ meters

9.

(a)

Let $U(t)$ be the particular solution expressed as

$$U(t) = R\cos(\omega t - \delta). \qquad (10)$$

$\therefore U(t) = R\cos(\omega t)\cos(\delta) + R\sin(\omega t)\sin(\delta)$

Substituting into: $mu'' + \gamma u' + ku = F_0\cos(\omega t). \qquad (8)$

$U' = -R\omega\sin(\omega t - \delta) \qquad U'' = -R\omega^2\cos(\omega t - \delta)$,

$-Rm\omega^2\cos(\omega t-\delta) - R\gamma\omega\sin(\omega t-\delta) + RK\cos(\omega t-\delta) =$

$R(K - m\omega^2)\cos(\omega t-\delta) - R\gamma\omega\sin(\omega t-\delta) =$

$R(K-m\omega^2)[\cos(\omega t)\cos(\delta) + \sin(\omega t)\sin(\delta)$

$-R\gamma\omega[\sin(\omega t)\cos(\delta) - \cos(\omega t)\sin(\delta)] =$

$[R(K-m\omega^2)\cos(\delta) + R\gamma\omega\sin(\delta)]\cos(\omega t)$

$+[R(K-m\omega^2)\sin(\delta) - R\gamma\omega\cos(\delta)]\sin(\omega t) = F_0\cos(\omega t)$

$\therefore R[(K-m\omega^2)\cos(\delta) + \gamma\omega\sin(\delta)] = F_0 \qquad [1]$

$R[(K-m\omega^2)\sin(\delta) - \gamma\omega\cos(\delta)] = 0 \qquad [2]$

Letting $\underline{\omega_0^2 = \frac{k}{m}}$ or $k = m\omega_0^2$, the above equations become

$$\begin{bmatrix} m(\omega_0^2-\omega^2) & \gamma\omega \\ -\gamma\omega & m(\omega_0^2-\omega^2) \end{bmatrix} \begin{bmatrix} \cos(\delta) \\ \sin(\delta) \end{bmatrix} = \begin{bmatrix} F_0/R \\ 0 \end{bmatrix}$$

Let $d = m^2(\omega_0^2-\omega^2)^2 + \gamma^2\omega^2$ [3]

(the determinant of the left matrix, and always ≥ 0)

$\therefore$ From Cramer's Rule,

$$\cos(\delta) = \frac{(F_0/R)\, m(\omega_0^2-\omega^2)}{d}, \quad \sin(\delta) = \frac{(F_0/R)\gamma\omega}{d} \quad [4]$$

$$\therefore \cos^2(\delta) + \sin^2(\delta) = 1 = \frac{(F_0/R)^2\left[m^2(\omega_0^2-\omega^2) + \gamma^2\omega^2\right]}{d^2},$$

Or, using [3], $1 = \dfrac{(F_0/R)^2 d}{d^2}$, $\therefore \dfrac{F_0}{R} = \sqrt{d}$

$$\underline{\text{Let } \Delta = \sqrt{d} = \sqrt{m^2(\omega_0^2-\omega^2)^2 + \gamma^2\omega^2}}$$

$$\therefore \underline{R = \frac{F_0}{\Delta}}$$

Using this to substitute into [4], and $d = \Delta^2$,

$$\underline{\cos(\delta) = \frac{m(\omega_0^2-\omega^2)}{\Delta}}, \quad \underline{\sin(\delta) = \frac{\gamma\omega}{\Delta}}$$

(b)

From $R = \frac{F_0}{\Delta}$, $\frac{R}{F_0} = \frac{1}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2}}$

$$= \frac{1}{\sqrt{m^2 \omega_0^4 \left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right)^2 + m^2 \omega_0^4 \frac{\gamma^2 \omega^2}{m^2 \omega_0^4}}}$$

now use $\omega_0^2 = \frac{K}{m}$

$$= \frac{1}{K\sqrt{\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \frac{\gamma^2}{m^2 \omega_0^2}\left(\frac{\omega}{\omega_0}\right)^2}}$$

$$\therefore \frac{RK}{F_0} = \frac{1}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_0}\right)^2\right]^2 + \frac{\gamma^2}{Km}\left(\frac{\omega}{\omega_0}\right)^2}} = \frac{1}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_0}\right)^2\right]^2 + \Gamma\left(\frac{\omega}{\omega_0}\right)^2}}$$

where $\Gamma = \frac{\gamma^2}{Km}$

(c)

(1) From (b),

$$R = \frac{F_0}{K}\left[\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \frac{\gamma^2}{Km}\left(\frac{\omega}{\omega_0}\right)^2\right]^{-\frac{1}{2}} \quad [R]$$

$$\therefore \frac{dR}{d\omega} = -\frac{F_0}{2K}\left[\quad\right]^{-\frac{3}{2}}\left[2\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right)\left(\frac{-2\omega}{\omega_0^2}\right) + \frac{2\gamma^2 \omega}{Km\omega_0^2}\right]$$

Setting to zero, $\dfrac{\gamma^2 \omega}{K m \omega_0^2} = \dfrac{2\omega}{\omega_0^2}\left(1 - \dfrac{\omega^2}{\omega_0^2}\right)$

or, $\dfrac{\gamma^2}{Km} = \dfrac{2(\omega_0^2 - \omega^2)}{\omega_0^2}$, $\omega^2 = \omega_0^2 - \dfrac{\omega_0^2 \gamma^2}{2Km}$

$\therefore \omega_{max}^2 = \omega_0^2\left(1 - \dfrac{\gamma^2}{2mK}\right)$, or $\dfrac{\omega_{max}^2}{\omega_0^2} = 1 - \dfrac{\gamma^2}{2mK}$

Substituting into [R],

$$R_{max} = \frac{F_0}{K}\left[\left(\frac{\gamma^2}{2mK}\right)^2 + \frac{\gamma^2}{mK}\left(1 - \frac{\gamma^2}{2mK}\right)\right]^{-\frac{1}{2}} \qquad [R_{max}]$$

$$= \frac{F_0}{K}\left[\frac{\gamma^4}{4m^2K^2} + \frac{\gamma^2}{mK} - \frac{\gamma^4}{2m^2K^2}\right]^{-\frac{1}{2}}$$

$$= \frac{F_0}{K}\left[\frac{\gamma^2}{mK} - \frac{\gamma^4}{4m^2K^2}\right]^{-\frac{1}{2}} = \frac{F_0}{\gamma K}\left[\frac{1}{mK} - \frac{\gamma^2}{4m^2K^2}\right]^{-\frac{1}{2}}$$

Now using $K = m\omega_0^2$

$$= \frac{F_0}{\gamma K}\left[\frac{1}{m^2\omega_0^2} - \frac{\gamma^2}{4m^3K\omega_0^2}\right]^{-\frac{1}{2}}$$

$$= \frac{F_0 m \omega_0}{\gamma K}\left[1 - \frac{\gamma^2}{4mK}\right]^{-\frac{1}{2}} = \frac{F_0 \omega_0}{\gamma \omega_0^2}\left(1 - \frac{\gamma^2}{4mK}\right)^{-\frac{1}{2}}$$

$$\therefore R_{max} = \frac{F_0}{\gamma \omega_0 \sqrt{1 - \gamma^2/4mk}} = \frac{F_0}{\gamma \omega_0}\left(1 - \frac{\gamma^2}{4mk}\right)^{-\frac{1}{2}}$$

This can be shown to be a relative max and not a relative minimum without doing the second derivative test. Just evaluate R at $\frac{\omega_{max}^2}{\omega_0^2} + \epsilon$, $\epsilon \neq 0$. You eventually get

$$R_\epsilon = \frac{F_0/k}{\sqrt{\left(\frac{\gamma^2}{2mk}\right)^2 + \frac{\gamma^2}{mk}\left(1 - \frac{\gamma^2}{2mk}\right) + \epsilon^2}} < \frac{F_0/k}{\sqrt{\left(\frac{\gamma^2}{2mk}\right)^2 + \frac{\gamma^2}{mk}\left(1 - \frac{\gamma^2}{2mk}\right)}} = R_{max}$$

(2) Note, for $|x| < 1$, $(1+x)^k = 1 + kx + \frac{k(k-1)}{2}x^2 + \ldots$

For any real number k (Taylor expansion).

$$\therefore R_{max} = \frac{F_0}{\gamma \omega_0}\left(1 - \frac{\gamma^2}{4mk}\right)^{-\frac{1}{2}} \simeq \frac{F_0}{\gamma \omega_0}\left(1 + \frac{\gamma^2}{8mk}\right)$$

as long as $\frac{\gamma^2}{4mk} < 1$. For small γ, this is true.

(d)

$\frac{RK}{F_0}$: R from $R\cos(\omega t-\delta)$, the response from $u(t) = u_c(t) + u_p(t)$, is in meters.

K, from $F = -Kx$, is in Newtons/meter

F_0, from $F_0\cos(\omega t)$, the driving force, is in meters.

$$\therefore \frac{RK}{F_0} \equiv \frac{(\text{meters})(\text{Newtons}/\text{meter})}{\text{Newtons}} = \text{dimensionless}$$

$\frac{\omega}{\omega_0}$: both are of $\sec^{-1}$, $\therefore$ $\frac{\sec^{-1}}{\sec^{-1}} \Rightarrow$ dimensionless

$\frac{\gamma^2}{mK}$: γ, from $F = -\gamma v$, is in $\frac{\text{Newtons}}{\text{meters}/\text{sec}}$

K, from $F = -Kx$, is in $\frac{\text{Newtons}}{\text{meter}}$

$$\therefore \frac{\gamma^2}{mK} \equiv \frac{\left(\frac{\text{Newtons}}{\text{meters}/\text{sec}}\right)^2}{(\text{Kg})\left(\frac{\text{Newtons}}{\text{meter}}\right)} = \frac{\text{Newtons}-\sec^2}{\text{Kg}-\text{meters}}$$

$$= \frac{\left(\text{Kg}\cdot\frac{\text{meters}}{\sec^2}\right)\cdot\sec^2}{\text{Kg}\cdot\text{meters}} = \text{dimensionless}$$

10.

Equation (10) is: $U(t) = R\cos(\omega t - \delta)$. (10)

$\therefore\ U'(t) = -\omega R\sin(\omega t - \delta)$ Assume $\gamma \neq 0$.

For any given ω, the speed is a max when $|\sin(\omega t - \delta)| = 1$, so $|U'(t)| = \omega R$,

or, $$V(\omega) = \omega R = \frac{\omega F_0}{\Delta} = \frac{\omega F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}}$$

To find the max speed, set $\frac{dV(\omega)}{d\omega} = 0$.

For simplicity, note that $V(\omega)$ is a max when $\frac{\gamma}{F_0}V(\omega)$ is a max.

$$\frac{\gamma}{F_0}V(\omega) = \gamma\omega\left[m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2\right]^{-\frac{1}{2}}$$

$$= \gamma\frac{\omega_0}{\omega_0}\omega\left[m^2\omega_0^4\left(1 - \frac{\omega^2}{\omega_0^2}\right)^2 + \gamma^2\omega_0^2\left(\frac{\omega^2}{\omega_0^2}\right)\right]^{-\frac{1}{2}}$$

$$= \gamma\omega_0\frac{\omega}{\omega_0}(\gamma\omega_0)^{-1}\left[\frac{m^2}{\gamma^2}\omega_0^2\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \left(\frac{\omega}{\omega_0}\right)^2\right]^{-\frac{1}{2}}$$

$$= \frac{\omega}{\omega_0}\left[\frac{m^2\omega_0^2}{\gamma^2}\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \left(\frac{\omega}{\omega_0}\right)^2\right]^{-\frac{1}{2}}$$

$\therefore$ Let $x = \frac{\omega}{\omega_0}$, and let $c = \frac{m^2 \omega_0^2}{\gamma^2}$, $s(x) = \frac{\gamma}{F_0} V(\omega)$

$$\therefore s(x) = x\left[c(1-x^2)^2 + x^2\right]^{-1/2}$$

$$\therefore s'(x) = \left[c(1-x^2)^2 + x^2\right]^{-\frac{1}{2}}$$

$$- \frac{x}{2}\left[c(1-x^2)^2 + x^2\right]^{-3/2}\left[2c(1-x^2)(-2x) + 2x\right]$$

Setting to 0, and rearranging,

$$c(1-x^2)^2 + x^2 = -2x^2 c(1-x^2) + x$$

$$\therefore cx^4 - 2cx^2 + c + x^2 = -2cx^2 + 2cx^4 + x$$

$$cx^4 - x^2 + x - c = 0$$

$x = 1$ is a solution as $c - 1 + 1 - c = 0$.

$$cx^4 - x^2 + x - c = (x-1)(cx^3 + cx^2 + (c-1)x + c)$$

Note $c = \frac{m^2 \omega_0^2}{\gamma^2} > 0$.

$\therefore$ Consider $x^3 + x^2 + \frac{c-1}{c} x + 1$

(1) If $c \geq 1$, Then since $x = \frac{\omega}{\omega_0} > 0$, there is no other x s.t. $s'(x) = 0$ since all terms are positive.

(2) If $0 < c < 1$, then for $x > 1$,

$x^2 > \left|\frac{c-1}{c}\right| x$, so $x^3 + x^2 + \frac{c-1}{c}x + 1 > 0$

If $0 < c < 1$ and $0 < x < 1$, then

$\left|\frac{c-1}{c}\right| x < 1$, so $x^3 + x^2 + \frac{c-1}{c}x + 1 > 0$

$\therefore$ $x = 1$ is the only root for $s'(x) = 0$.

Use MATLAB to do second derivative test, and also check above work:

```
clear,clc
syms x c s(x)
s(x) = x/sqrt(c*(1-x^2)^2 + x^2);
d1 = diff(s(x),x,1)
solve(d1==0,x)
d2 = diff(s(x),x,2);
subs(d2,x,1)
```

d1 =

$$\frac{1}{\sqrt{c\,(x^2-1)^2 + x^2}} - \frac{x\,(2\,x + 4\,c\,x\,(x^2-1))}{2\,\left(c\,(x^2-1)^2 + x^2\right)^{3/2}}$$

ans =

$$\begin{pmatrix} -1 \\ 1 \\ -i \\ i \end{pmatrix}$$

ans = $-4\,c$

Thus, $x = 1$ is a critical point, and other solutions are $x = -1, \pm i$, which are all not valid for $x = \frac{\omega}{\omega_0}$

Second derivative shows $s''(x) = -4c$ at $x = 1$, and $-4c < 0$ since $c > 0$.

$\therefore$ $x = 1$ is a relative max for $s(x)$.

$\therefore$ $x = \frac{\omega}{\omega_0} = 1$ is a max for $V(\omega)$, and so maximum velocity is reached when $\omega = \omega_0$

The above assumed $\gamma \neq 0$. If $\gamma = 0$, there is no max velocity as Example 4, p. 166 of text showed.

11.

Homogeneous solution: $r^2 + 1 = 0 \Rightarrow r = \pm i$

$\therefore u_c(t) = C_1 \cos(t) + C_2 \sin(t)$

(a) $F(t) = F_0 t$, F_0 a constant

Let $u_p(t) = F_0 t$. $\therefore u_p'' = 0$.

$\therefore u(t) = C_1 \cos(t) + C_2 \sin(t) + F_0 t$

$u(0) = 0 \Rightarrow C_1 = 0$

$u'(0) \Rightarrow C_2 + F_0 = 0 \Rightarrow C_2 = -F_0$

$$\therefore \; u(t) = -F_0 \sin(t) + F_0 t, \quad 0 \le t \le \pi$$

Note: $u(\pi) = F_0 \pi$

$u'(\pi) = F_0 + F_0 = 2F_0$

(b) $F_0(2\pi - t)$

Note $\frac{d^2}{dt^2} F_0(2\pi - t) = 0 \quad \therefore \; u_p(t) = F_0(2\pi - t)$

$$\therefore \; u(t) = c_1 \cos(t) + c_2 \sin(t) + F_0(2\pi - t)$$

From (a), $u(\pi) = F_0 \pi$, $u'(\pi) = 2F_0$

$\therefore \; -c_1 + F_0 \pi = F_0 \pi \Rightarrow c_1 = 0$

$u'(t) = c_2 \cos(t) - F_0 . \quad \therefore \; u'(\pi) = -c_2 - F_0 = 2F_0$

$\therefore \; c_2 = -3F_0$

$$\therefore \; u(t) = -3F_0 \sin(t) + F_0(2\pi - t), \quad \pi < t \le 2\pi$$

Note: $u(2\pi) = 0$, $u'(2\pi) = -4F_0$

(c) $0 \quad \therefore \; u_p(t) = 0$

$$\therefore \; u(t) = c_1 \cos(t) + c_2 \sin(t)$$

From (a), $u(2\pi) = 0 \Rightarrow c_1 = 0$

$u'(2\pi) = -4F_0 \Rightarrow c_2 = -4F_0$

$\therefore\ u(t) = -4F_0 \sin(t),\quad 2\pi < t$

$$\therefore\quad u(t) = \begin{cases} F_0\,(t - \sin(t)) & 0 \le t \le \pi \\ -3F_0 \sin(t) + F_0\,(2\pi - t) & \pi < t \le 2\pi \\ -4F_0 \sin(t) & 2\pi < t \end{cases}$$

12.

Let $Q(t)$ be the charge on the capacitor at any t.

$\therefore\ LQ'' + RQ' + \frac{1}{C}Q = 12$

Or, $Q'' + (5\times10^3)Q' + \frac{1}{0.25\times10^{-6}}Q = 12$

Or, $Q'' + (5\times10^3)Q' + (4\times10^6)Q = 12$

Characteristic equation: $r^2 + (5\times10^3)r + 4\times10^6 = 0,$

$r = 1\times10^3,\ 4\times10^3$

$\therefore$ Homogeneous solution: $C_1 e^{-1\times10^3 t} + C_2 e^{-4\times10^3 t}$

Particular solution: $4\times10^6 Q_p = 12$, $Q_p = 3\times10^{-6}$

$\therefore Q(t) = C_1 e^{-1\times10^3 t} + C_2 e^{-4\times10^3 t} + 3\times10^{-6}$

$Q(0)=0 \Rightarrow C_1 + C_2 + 3\times10^{-6} = 0$ [1]

$Q'(0)=0 \Rightarrow -1\times10^3 C_1 - 4\times10^3 C_2 = 0$ [2]

$\therefore 10^3 C_1 + 10^3 C_2 = -3\times10^{-3}$ [1']

$\therefore -3\times10^3 C_2 = -3\times10^{-3}$, $C_2 = 10^{-6}$

$\therefore C_1 = -4\times10^{-6}$

$\therefore Q(t) = 10^{-6}\left(e^{-4000t} - 4e^{-1000t} + 3\right)$ Coulombs

$\lim_{t\to\infty} Q(t) = 3\times10^{-6}$ Coulomb

$Q(0.001) = 10^{-6}(0.0183 - 1.4715 + 3) \simeq 1.547\times10^{-6}$ C

$Q(0.01) = 10^{-6}(4.25\times10^{-18} - 1.82\times10^{-4} + 3) \simeq 3\times10^{-6}$ C

13.

(a)

From text, p. 165, with $m=1$, $k=1$, $\omega_0=1$, $F_0=3$

$$u(t) = c_1\cos(t) + c_2\sin(t) + \frac{3}{1-\omega^2}\cos(\omega t)$$

$$u(0) = c_1 + \frac{3}{1-\omega^2} = 0 \Rightarrow c_1 = \frac{-3}{1-\omega^2}$$

$$u'(t) = \frac{3}{1-\omega^2}\sin(t) + c_2\cos(t) - \frac{3\omega}{1-\omega^2}\sin(\omega t)$$

$$\therefore u'(0) = c_2 = 0.$$

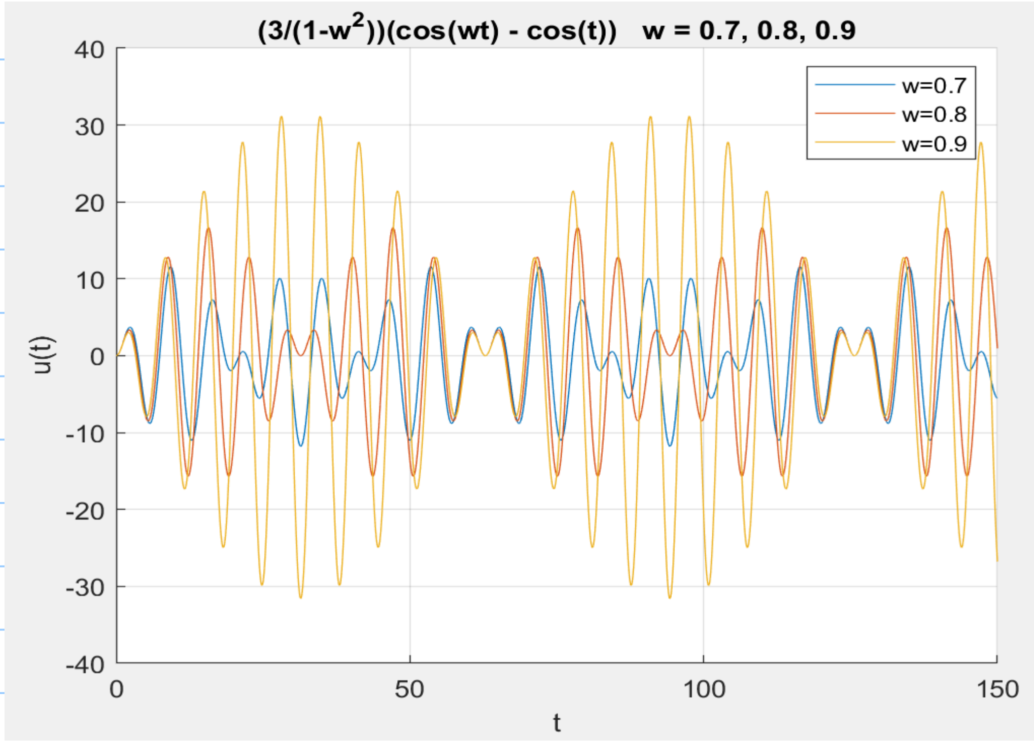
$$\therefore u(t) = \frac{3}{1-\omega^2}\left[\cos(\omega t) - \cos(t)\right]$$

(b)

Use MATLAB

As $\omega \to 1$, amplitude of beat gets larger,
and the beat itself becomes more obvious.

```
clear,clc;
t = 0:0.01:150;
hold on
grid on
for w = [0.7, 0.8, 0.9]
    eqn = (3/(1-w^2))*(cos(w*t) - cos(t));
    plot(t, eqn)
end
xlabel 't', ylabel 'u(t)'
title '(3/(1-w^2))(cos(wt) - cos(t))   w = 0.7, 0.8, 0.9'
legend('w=0.7', 'w=0.8', 'w=0.9')
```



14.

(a)

As in #13 above,

$$u(t) = c_1 \cos(t) + c_2 \sin(t) + \frac{3}{1-\omega^2}\cos(\omega t)$$

$$u(0) = 1 \Rightarrow C_1 + \frac{3}{1-\omega^2} = 1, \quad C_1 = \frac{-\omega^2 - 2}{1-\omega^2} = \frac{\omega^2+2}{\omega^2-1}$$

$$u'(t) = -\frac{w^2+2}{w^2-1}\sin(t) + C_2\cos(t) + \frac{3w}{w^2-1}\sin(wt)$$

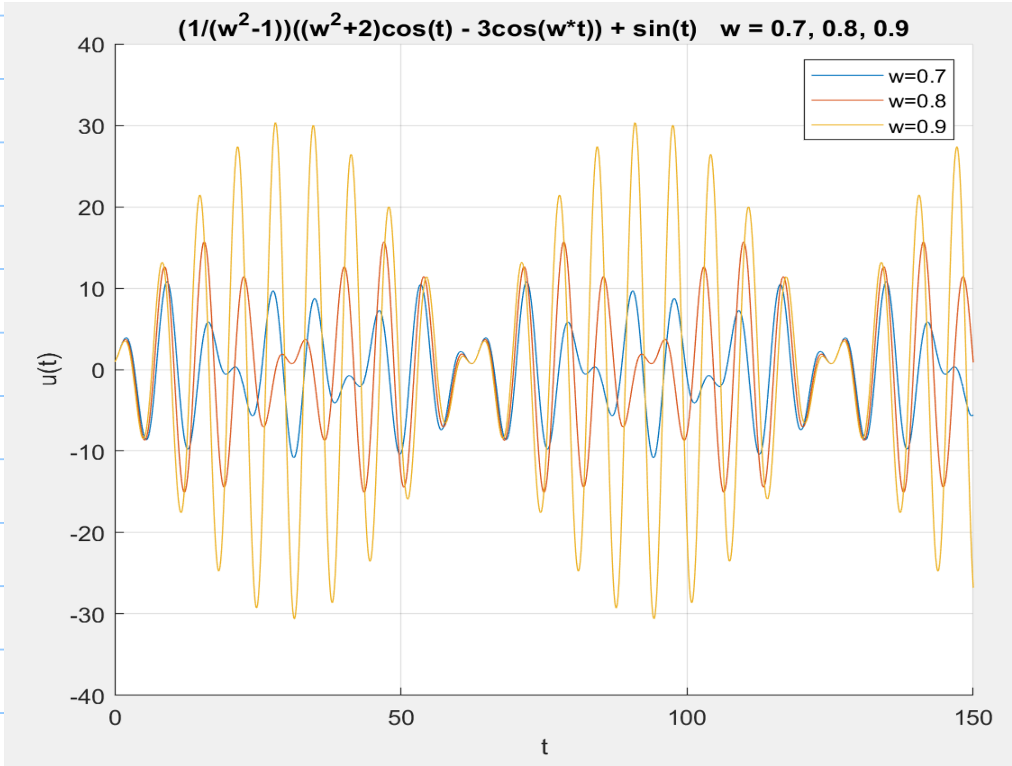
$$\therefore u'(0) = 1 \Rightarrow C_2 = 1$$

$$\therefore u(t) = \frac{1}{w^2-1}\left[(w^2+2)\cos(t) - 3\cos(wt)\right] + \sin(t)$$

(b)

MATLAB:

```
clear,clc;
t = 0:0.01:150;
hold on
grid on
for w = [0.7, 0.8, 0.9]
    eqn = (1/(w^2-1))*((w^2+2)*cos(t) - 3*cos(w*t)) + sin(t);
    plot(t, eqn)
end
xlabel 't', ylabel 'u(t)'
title '(1/(w^2-1))((w^2+2)cos(t) - 3cos(w*t)) + sin(t)   w = 0.7, 0.8, 0.9'
legend('w=0.7', 'w=0.8', 'w=0.9')
```



There is very little difference between this non-zero initial conditions problem and #13. The amplitudes of the beats appear very similar. Perhaps the similarity is due to the relatively small initial conditions.

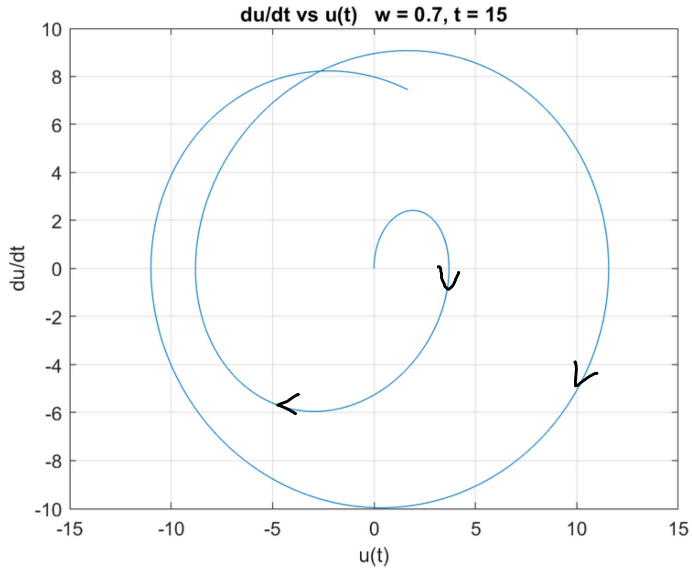
15.

From #13, $u(t) = \frac{3}{1-\omega^2}\left[\cos(\omega t) - \cos(t)\right]$

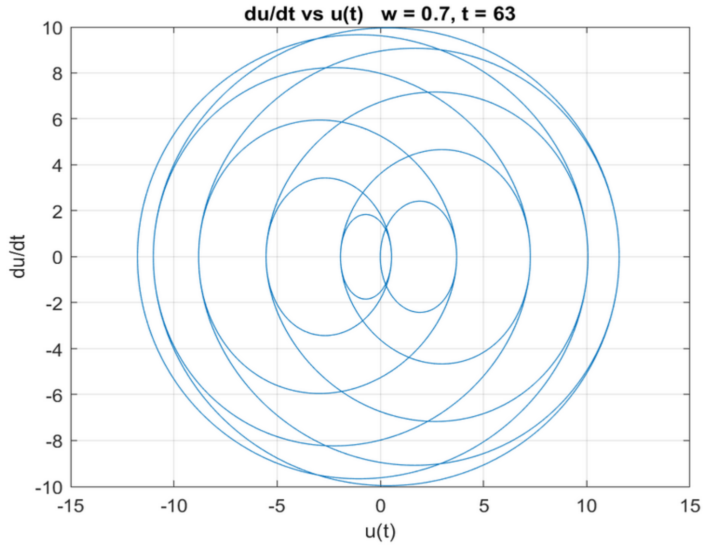
$$\therefore\ u'(t) = \frac{3}{1-\omega^2}\left[\sin(t) - \omega \sin(\omega t)\right]$$

Using MATLAB

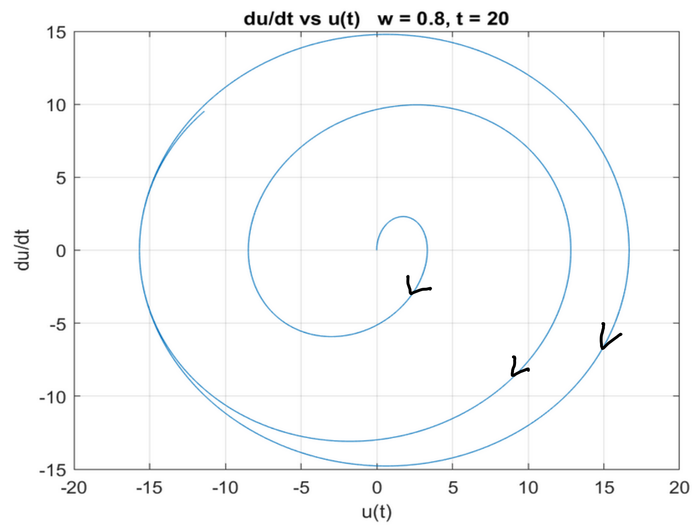
```
clear,clc;
t = 0:0.01:15;
w = 0.7;
u = (3/(1-w^2)).*(cos(w*t) - cos(t));
du = (3/(1-w^2)).*(sin(t) -w*sin(w*t));
plot(u, du)
grid on
xlabel 'u(t)', ylabel 'du/dt'
title 'du/dt vs u(t)   w = 0.7, t = 15'
```



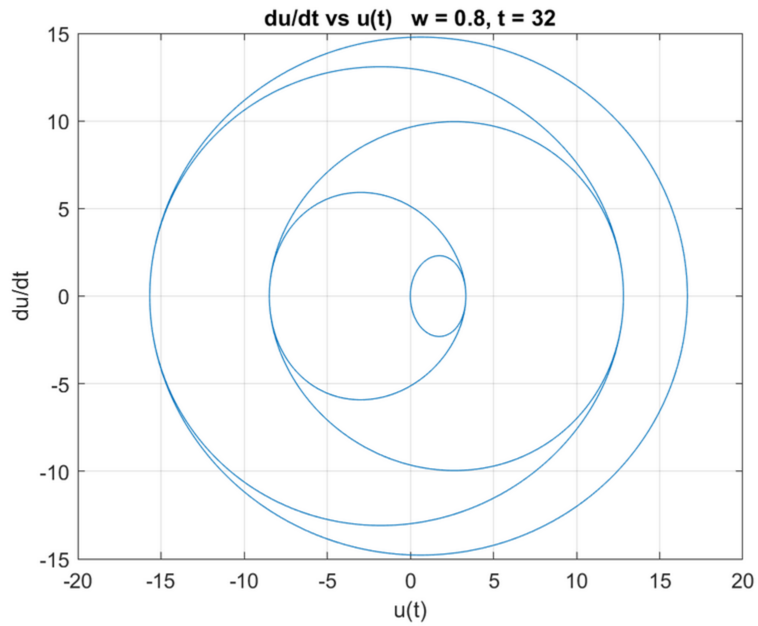
At $t = 15$, two ends are "loose". Need around $t = 63$ to tie ends (by trial and error).



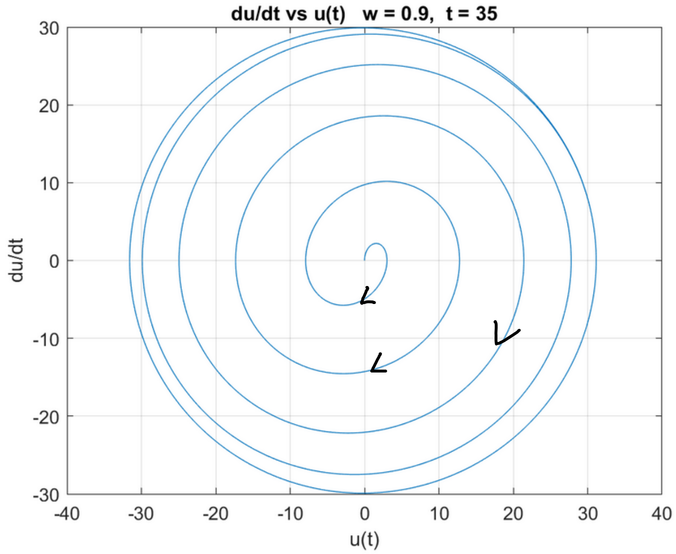

```
t = 0:0.01:20;
w = 0.8;
u = (3/(1-w^2)).*(cos(w*t) - cos(t));
du = (3/(1-w^2)).*(sin(t) -w*sin(w*t));
plot(u, du)
grid on
xlabel 'u(t)', ylabel 'du/dt'
title 'du/dt vs u(t)    w = 0.8, t = 20'
```



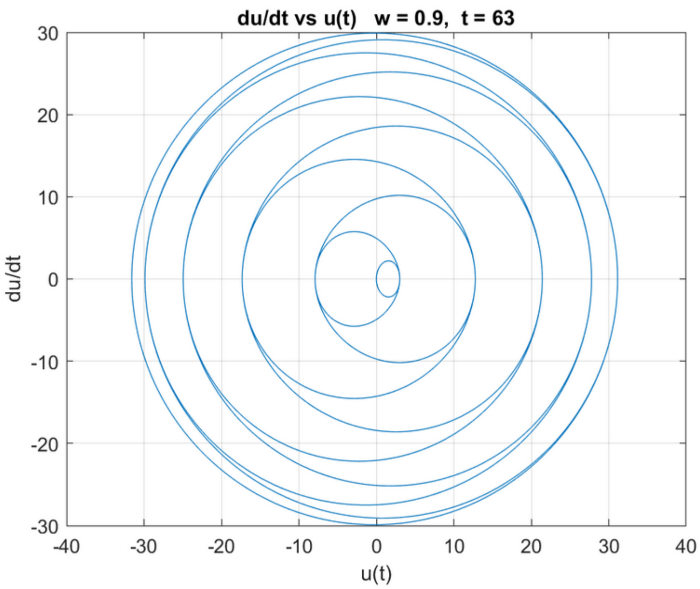
t = 32 gives a closed loop



```
t = 0:0.01:35;
w = 0.9;
u = (3/(1-w^2)).*(cos(w*t) - cos(t));
du = (3/(1-w^2)).*(sin(t) -w*sin(w*t));
plot(u, du)
grid on
xlabel 'u(t)', ylabel 'du/dt'
title 'du/dt vs u(t)   w = 0.9,  t = 35'
```



$t = 63$ gives a closed loop



Homogeneous: $8r^2 + r + 32 = 0$, $r = \frac{-1 \pm \sqrt{1023}}{16}$

$$\therefore u_c(t) = e^{-t/16}\left(c_1 \cos\left(\frac{\sqrt{1023}}{16}t\right) + c_2 \sin\left(\frac{\sqrt{1023}}{16}t\right)\right)$$

16.

(a) From text, p. 161, with $F_0 = 3$, $\omega = \frac{1}{4}$,

$\omega_0 = 2$, $m = 1$, $\gamma = \frac{1}{8}$, $K = 4$

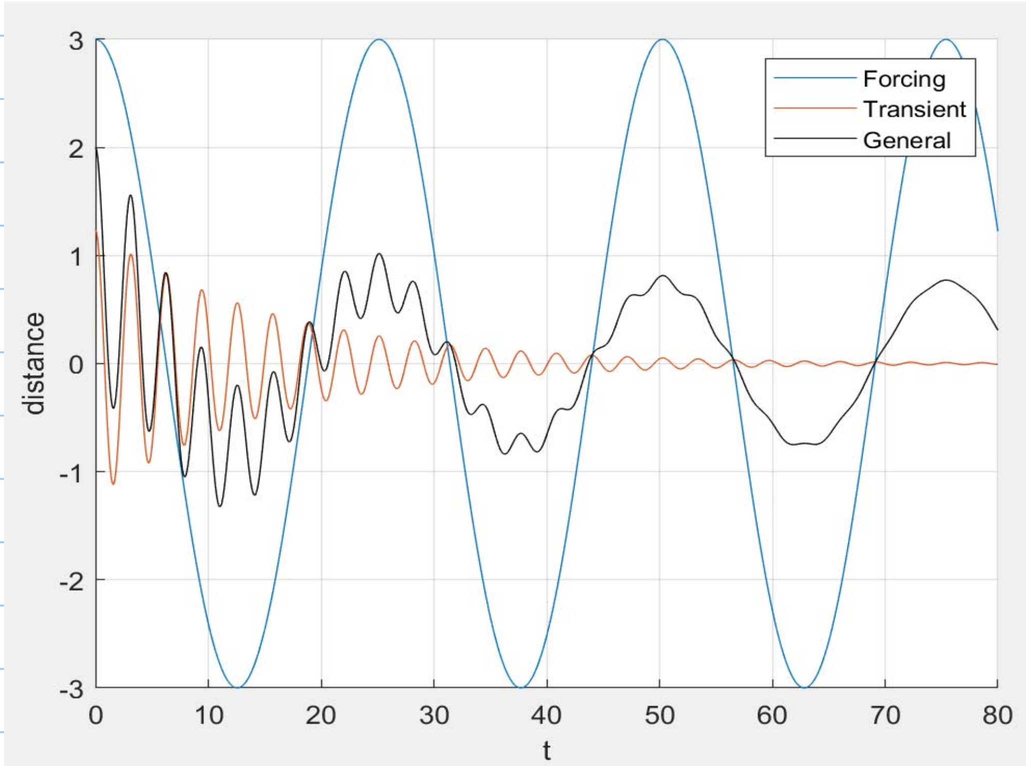
$$U_p(t) = R\cos(\omega t - \delta)$$

Use MATLAB to do the calculations of

$\Delta = \sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}$, $R = F_0/\Delta$,

and $\delta = \arctan\left[\frac{\gamma\omega}{m(\omega_0^2 - \omega^2)}\right]$, taking care to possibly adjust for proper quadrant for δ.

```
clear, clc
syms t c1 c2
w0 = 2;   %from sqrt(k/m)
g = 1/8;  %damping coefficient
F0 = 3; w = 1/4;  %forcing function amplitude, frequency
F(t) = F0*cos(w*t);
wc = sqrt(1023)/16;  %Homogeneous solution frequency
%Homogeneous solution
Uc(t) = exp(-t/16)*(c1*cos(wc*t) + c2*sin(wc*t));
delta = sqrt((w0^2 - w^2)^2 + (g^2)*(w^2));
R = F0/delta;
d = atan(g*w/(w0^2 - w^2)); %check for quadrant
%Particular solution
Up(t) = R*cos(w*t - d);
%General solution
U(t) = Uc(t) + Up(t);
dU(t) = diff(U,t);
%Determine c1 from u(0)=2, store in k1
k1 = solve(subs(U(t),t,0)==2);
%Get c2 from u'(0)=0, using k1 for c1, store in k2
k2 = solve(subs(dU(t),[t,c1],[0,k1])==0,c2);
%Create transient, general solutions  for plotting
T(t) = subs(Uc(t), [c1,c2], [k1,k2]);
G(t) = subs(U(t), [c1,c2], [k1,k2]);
%Create u'(t) for plotting phase map
DU(t) = subs(dU(t), [c1,c2], [k1,k2]);
%time for elimination of transient from trial and error
t = 0:0.1:80;
hold on
plot(t,F(t))        %Forcing function
plot(t,T(t))        %Transient solution
plot(t,G(t),'k')    %General solution
grid on
xlabel 't', ylabel 'distance'
legend('Forcing', 'Transient', 'General')
hold off
```



Note: $\frac{\gamma^2}{mk} = \frac{1/64}{4} = 0.0039$, so resonance is very possible, but ω $(=\frac{1}{4})$ is not near ω_0 $(=2)$.

so $\frac{\omega}{\omega_0} = 0.125$, so is far to the left on the graph of text, p.162, figure 3.8.2

Note that $\delta = \arctan\left(\frac{\gamma\omega}{\omega_0^2 - \omega^2}\right) = 0.0079$, close to 0.

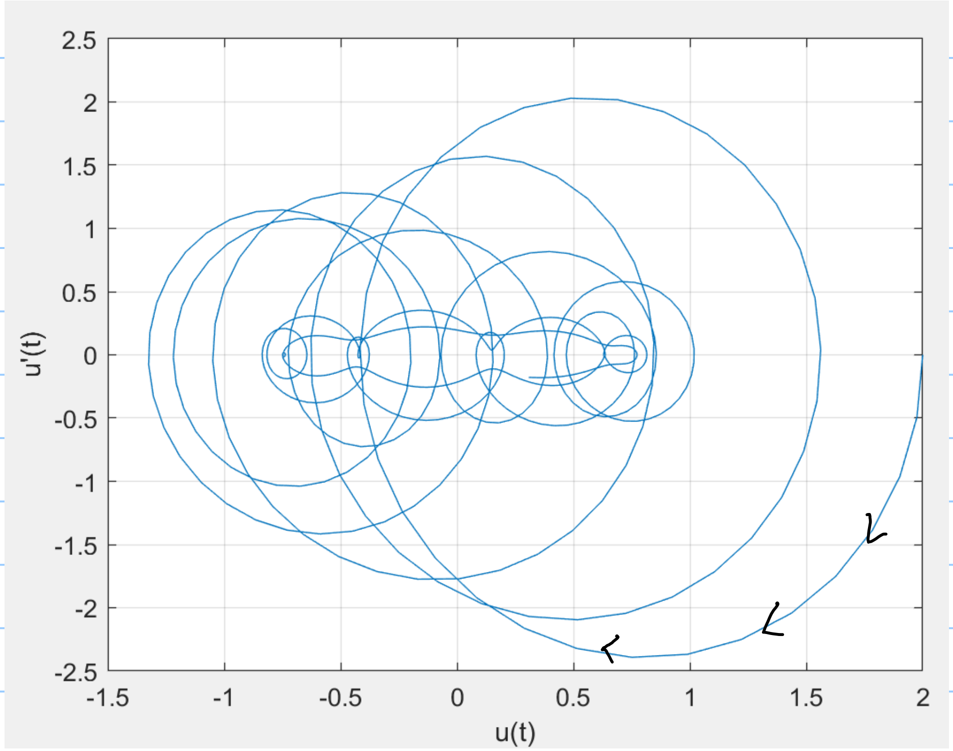
So the phase offset of the steady state is essentially non-existant, as the plot confirms: forcing function and resultant solution are in phase.

Comparing amplitudes: resultant amplitude (≈ 0.8) is about 1/4 of forcing function. To check,

$R = \frac{F_0}{\Delta} = \frac{3}{\Delta} = 0.7619$

(b) Phase plot - continuation of above code:

```
plot(G(t),DU(t))
grid on
xlabel 'u(t)', ylabel 'u''(t)'
```



Amplitude (u(t)) eventually bounces between ± 0.7619, and the velocity (u'(t)) bounces between around ± 0.3

17.

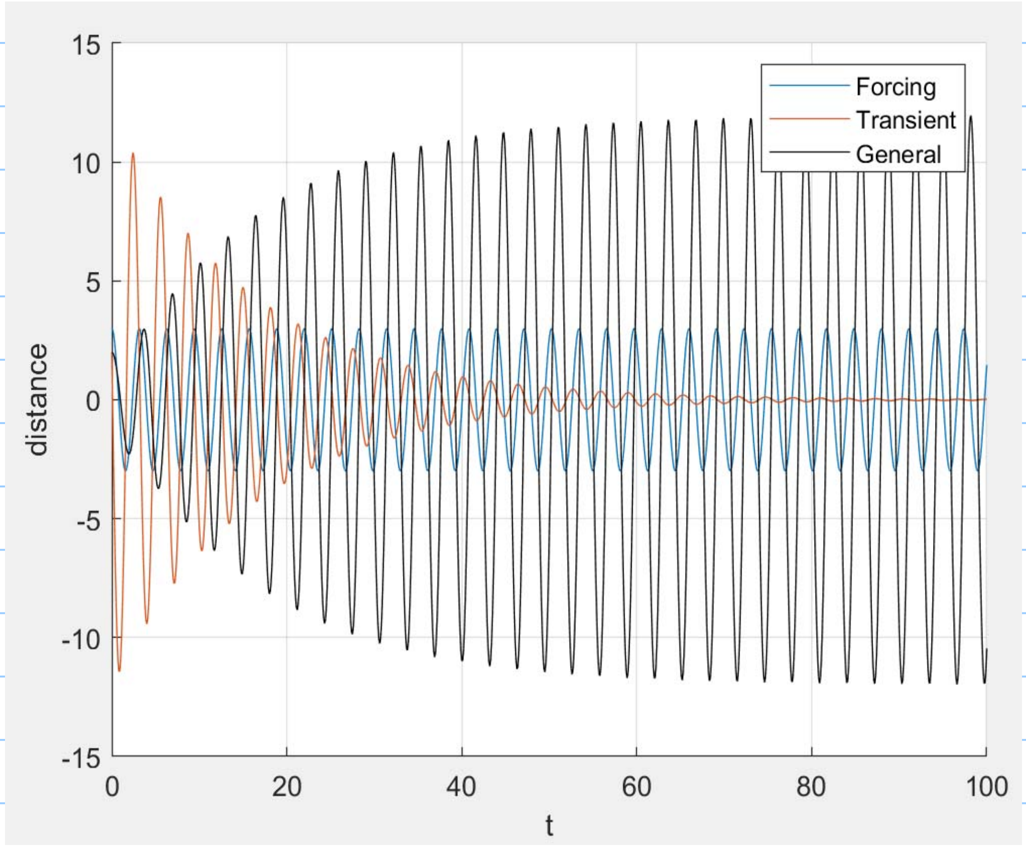
$F_0 = 3, \ \omega = 2$ (so $\omega = \omega_0 = 2$: look for resonance)

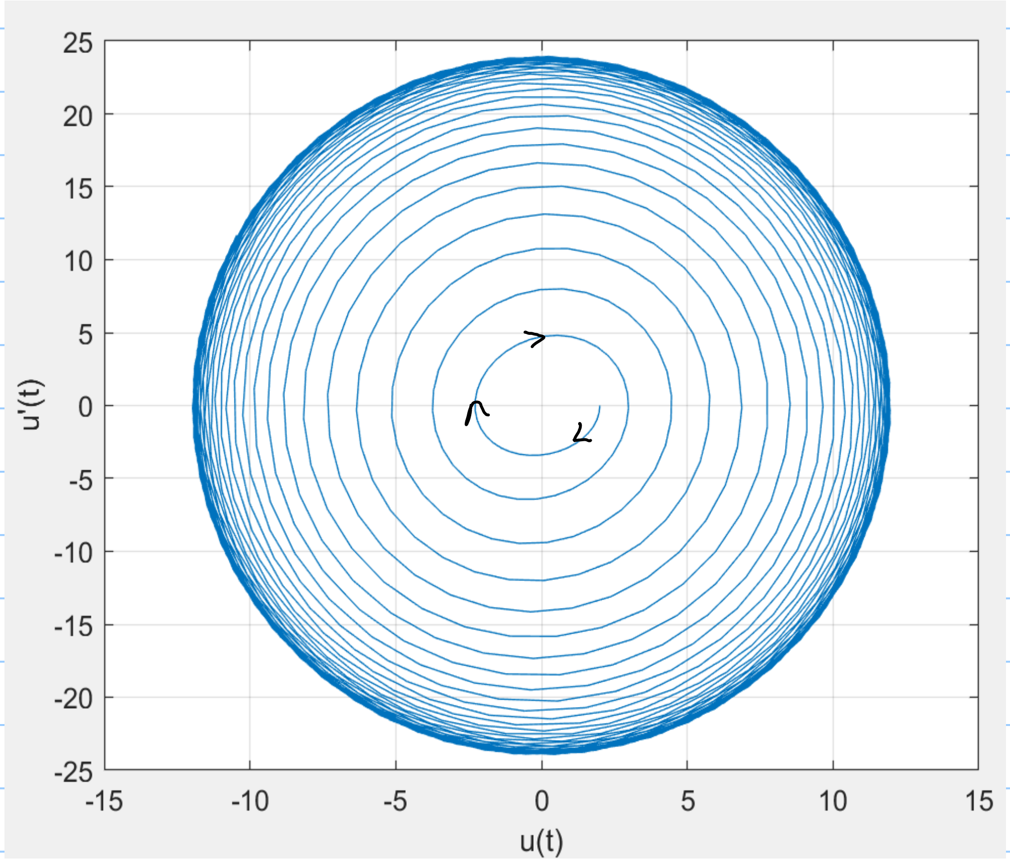
$m = 1, \ \gamma = \frac{1}{8}, \ k = 4$ Here, $\Delta = \sqrt{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2} = \gamma\omega$

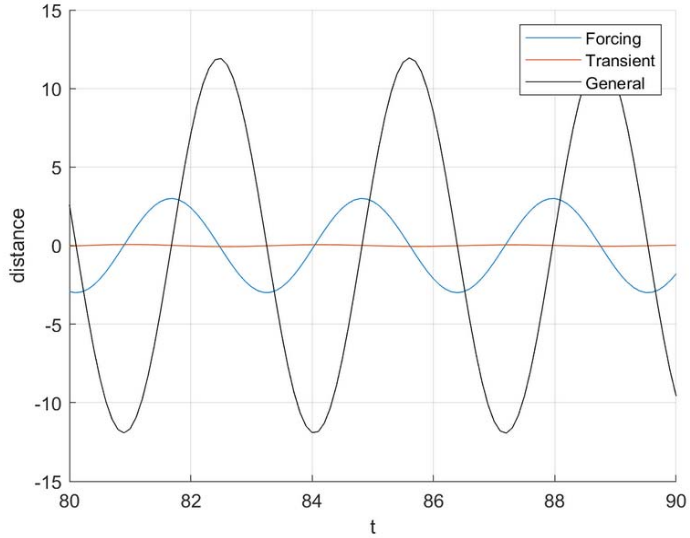
Note $\cos(\delta) = \frac{\omega_0^2 - \omega^2}{\Delta} = 0, \ \sin(\delta) = \frac{\gamma\omega}{\Delta}$

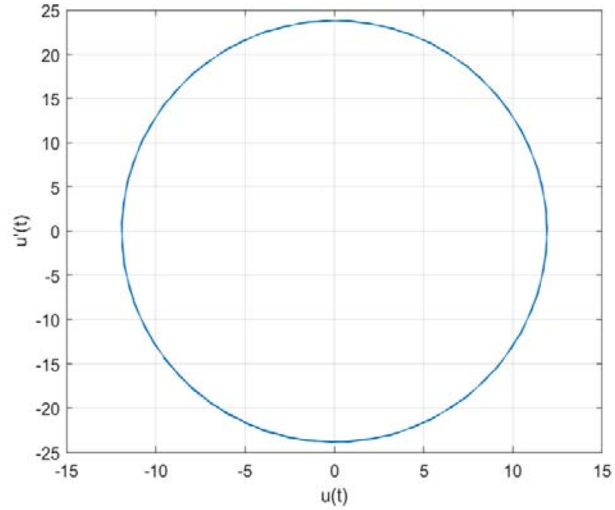
So use $\delta = \arcsin\left(\frac{\gamma\omega}{\Delta}\right) = \arcsin(1) = \frac{\pi}{2}$

```
clear, clc
syms t c1 c2
w0 = 2;   %from sqrt(k/m)
g = 1/8;  %damping coefficient
F0 = 3; w = 2;  %forcing function amplitude, frequency
F(t) = F0*cos(w*t);
wc = sqrt(1023)/16;  %Homogeneous solution frequency
%Homogeneous solution
Uc(t) = exp(-t/16)*(c1*cos(wc*t) + c2*sin(wc*t));
delta = sqrt((w0^2 - w^2)^2 + (g^2)*(w^2));
R = F0/delta;
d = asin(g*w/delta); %check for quadrant
%Particular solution
Up(t) = R*cos(w*t - d);
%General solution
U(t) = Uc(t) + Up(t);
dU(t) = diff(U,t);
%Determine c1 from u(0)=2, store in k1
k1 = solve(subs(U(t),t,0)==2);
%Get c2 from u'(0)=0, using k1 for c1, store in k2
k2 = solve(subs(dU(t),[t,c1],[0,k1])==0,c2);
%Create transient, general solutions  for plotting
T(t) = subs(Uc(t), [c1,c2], [k1,k2]);
G(t) = subs(U(t), [c1,c2], [k1,k2]);
%Create u'(t) for plotting phase map
DU(t) = subs(dU(t), [c1,c2], [k1,k2]);
%time for elimination of transient from trial and error
t = 0:0.1:100;
hold on
plot(t,F(t))       %Forcing function
plot(t,T(t))       %Transient solution
plot(t,G(t),'k')   %General solution
grid on
xlabel 't', ylabel 'distance'
legend('Forcing', 'Transient', 'General')
hold off
plot(G(t),DU(t))
grid on
xlabel 'u(t)', ylabel 'u''(t)'
```


Forcing
Transient
General
distance
t
15
10
5
0
-5
-10
-15
0
20
40
60
80
100


u'(t)
u(t)
25
20
15
10
5
0
-5
-10
-15
-20
-25
-15
-10
-5
0
5
10
15





Plots from $80 \leq t \leq 90$ shown above (zoom in).

This better shows the resultant solution $\frac{\pi}{2}$ out of phase with the forcing function: the peak of the resultant is when the forcing function is zero.

The resultant amplitude is larger than the forcing function (resonance) by a factor of

$$\frac{R}{F_0} = \frac{1}{\Delta} = \frac{1}{\gamma w} = \frac{1}{\frac{1}{8}(2)} = 4$$

18.

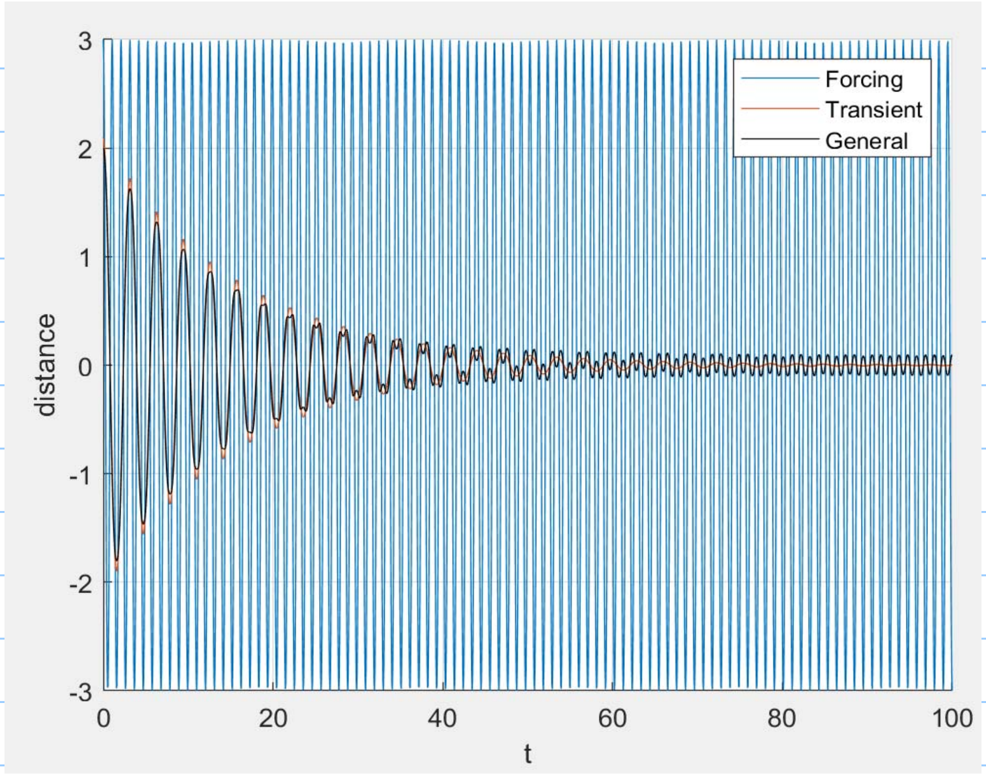
$F_0 = 3$, $\omega = 6$ (so no resonance as $\omega >> \omega_0 = 2$)

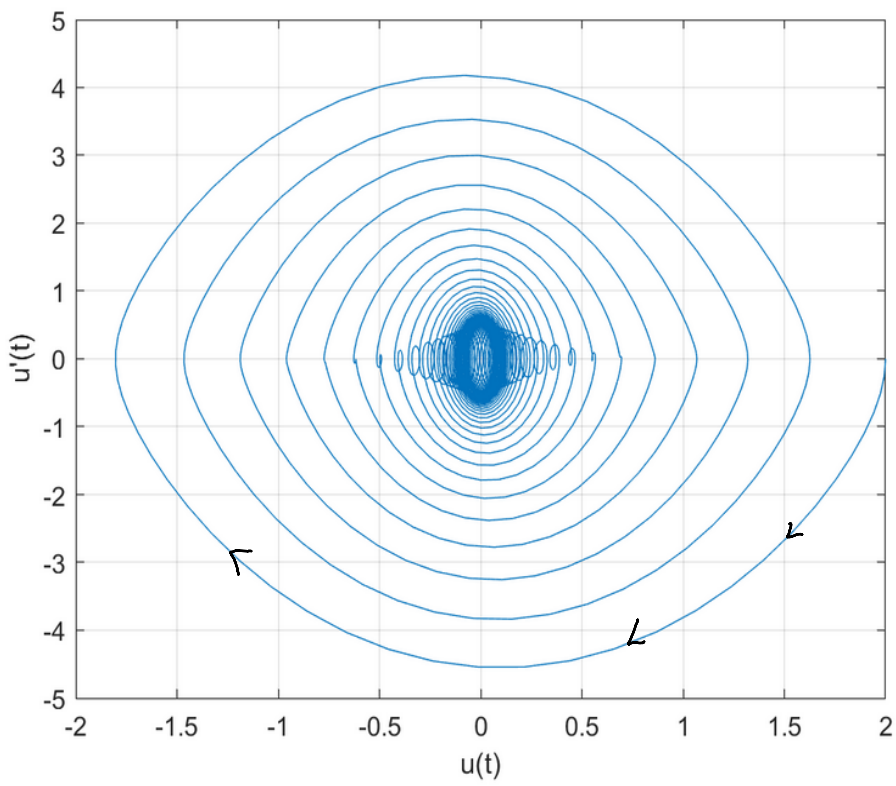
Note $\cos(\delta) = \frac{\omega_0^2 - \omega^2}{\Delta} < 0$, $\sin(\delta) = \frac{\gamma\omega}{\Delta} > 0$.

$\therefore$ δ is in quadrant 2, $\delta = \arctan\left(\frac{\gamma\omega}{\omega_0^2 - \omega^2}\right) + \pi$

$\therefore$ Modify code to reflect that

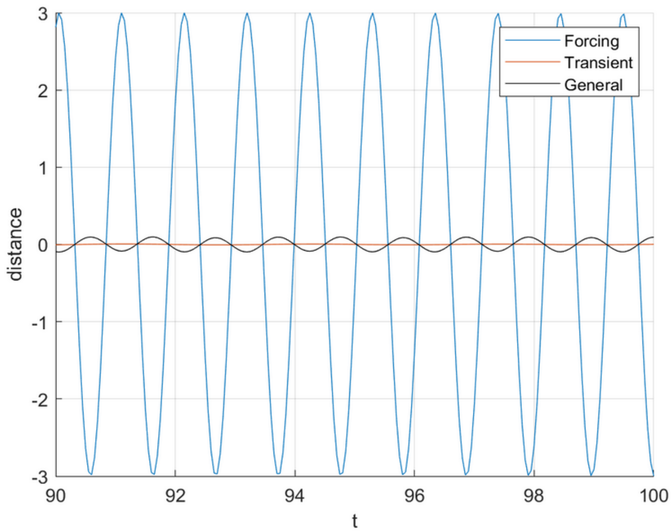
```
clear, clc
syms t c1 c2
w0 = 2;   %from sqrt(k/m)
g = 1/8;  %damping coefficient
F0 = 3; w = 6;  %forcing function amplitude, frequency
F(t) = F0*cos(w*t);
wc = sqrt(1023)/16;  %Homogeneous solution frequency
%Homogeneous solution
Uc(t) = exp(-t/16)*(c1*cos(wc*t) + c2*sin(wc*t));
delta = sqrt((w0^2 - w^2)^2 + (g^2)*(w^2));
R = F0/delta;
d = atan(g*w/(w0^2 - w^2)) + pi; %check for quadrant
%Particular solution
Up(t) = R*cos(w*t - d);
%General solution
U(t) = Uc(t) + Up(t);
dU(t) = diff(U,t);
%Determine c1 from u(0)=2, store in k1
k1 = solve(subs(U(t),t,0)==2);
%Get c2 from u'(0)=0, using k1 for c1, store in k2
k2 = solve(subs(dU(t),[t,c1],[0,k1])==0,c2);
%Create transient, general solutions  for plotting
T(t) = subs(Uc(t), [c1,c2], [k1,k2]);
G(t) = subs(U(t), [c1,c2], [k1,k2]);
%Create u'(t) for plotting phase map
DU(t) = subs(dU(t), [c1,c2], [k1,k2]);
%time for elimination of transient from trial and error
t = 0:0.05:100;
hold on
plot(t,F(t))       %Forcing function
plot(t,T(t))       %Transient solution
plot(t,G(t),'k')   %General solution
grid on
xlabel 't', ylabel 'distance'
legend('Forcing', 'Transient', 'General')
hold off
plot(G(t),DU(t))
grid on
xlabel 'u(t)', ylabel 'u''(t)'
```

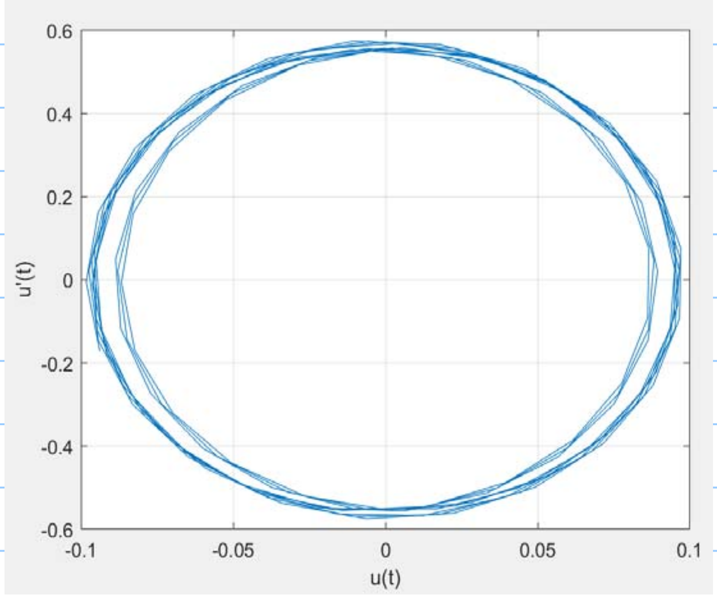




Zoom in to better visualize the steady state.

Choose $90 \le t \le 100$





The resultant solution is out of phase, by about π, with the forcing function: the amplitude of one is at the trough of the other. And the calculated value of $\delta = \arctan\left(-\frac{6}{8(32}\right) + \pi \approx \pi$

The amplitude of the resultant solution is much smaller than the forcing function.

$R = \frac{F_0}{\Delta} \approx 0.09$, or about $\frac{1}{10}$.

19.

(a)

As in #24 of Section 3.7, use MATLAB to generate an array u(t) of values at several time points: u(0), u(0+h), u(0+2h), ..., etc., h = increment of time, such as h = 0.001 sec. Use Taylor series approximations:

$$u(0+h) = u(0) + u'(0)\cdot h$$
$$u(0+2h) = u(0+h) + u'(0+h)\cdot h$$
$$u(0+3h) = u(0+2h) + u'(0+2h)\cdot h$$
$$\vdots$$

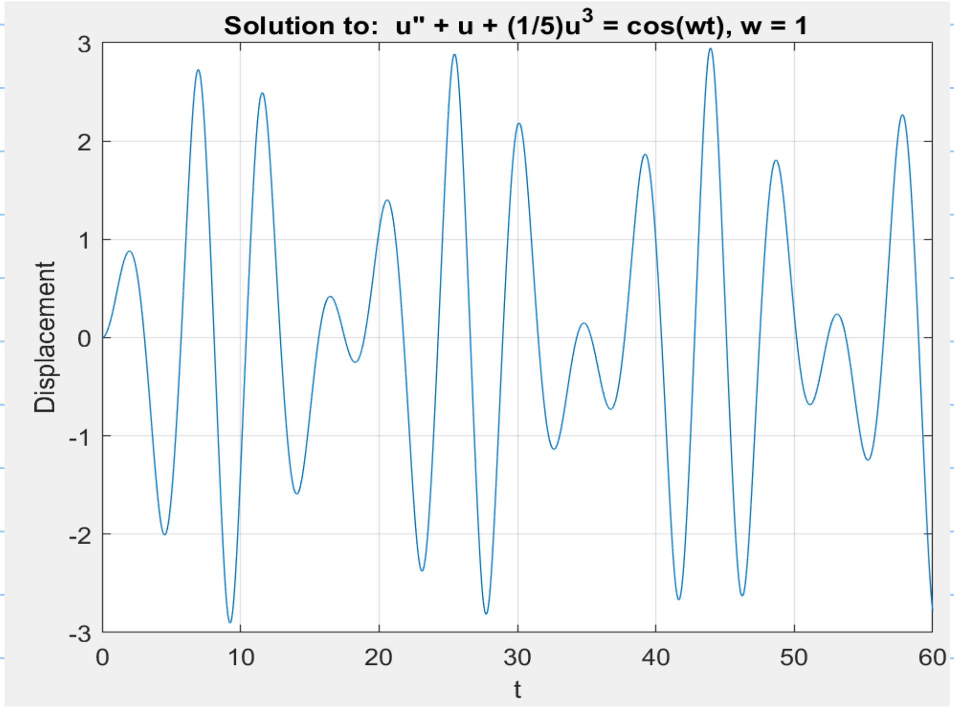
and

$$u'(0+h) = u'(0) + u''(0)\cdot h$$
$$u'(0+2h) = u'(0+h) + u''(0+h)\cdot h$$
$$u'(0+3h) = u'(0+2h) + u''(0+2h)\cdot h$$

using $u''(x) = -u(x) - \frac{1}{5}u^3(x) + \cos(\omega x)$

u(0), u'(0), and u''(x) are known, so the rest can be calculated and stored in u(t) for plotting.

```
clear, clc;
%hardening coefficient, forcing frequency
e = 1/5; w = 1;
h = 0.001; T_end = 60.0;  % try 60 secs long
t=0:h:T_end;           %time points
n = T_end/h + 1;      %size of array
%pre-allocate memory for arrays, initialize
u = zeros(1, n);
ux = zeros(1, n);   % ux() is u'(), uxx() is u"()
uxx = zeros(1, n);
% starting initial conditions, MATLAB not zero-based
u(1)=0;
ux(1)=0;
for i = 2:length(t)
        %get next value of u(t) from prior u(t), u'(t)
    u(i) = u(i-1) + ux(i-1)*h;
        %calculate prior u"(t) from prior u(t) in order
        % to calculate next u'(t) to prepare for next
        % calculation of u(t) when enter loop again
    uxx(i-1) = -u(i-1) - e*(u(i-1))^3 + cos(w*(i-1)*h);
    ux(i) = ux(i-1) + uxx(i-1)*h;
end
plot(t,u)
grid on
xlabel 't', ylabel 'Displacement'
title 'Solution to:  u" + u + (1/5)u^3 = cos(wt), w = 1'
```



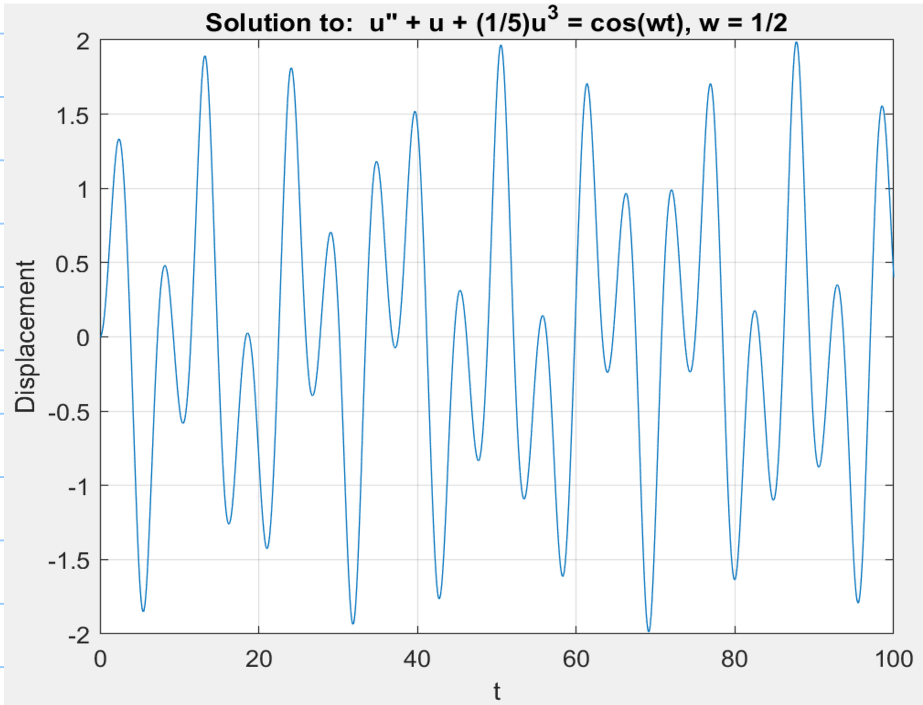
The solution does appear to exhibit a beat, with a period ≈ 20 secs between maxima.

(b)

Plot for $w = \frac{1}{2}, \frac{3}{4}, \frac{5}{4}, \frac{3}{2}, 2$

$w = \frac{1}{2}$

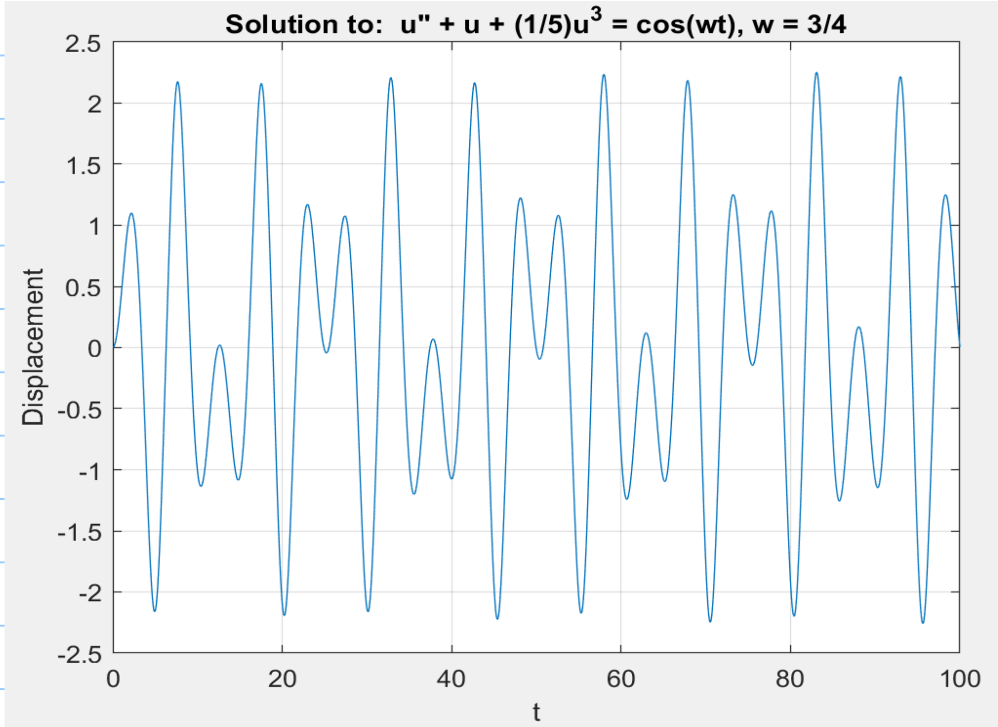
```
clear, clc;
%hardening coefficient, forcing frequency
e = 1/5; w = 1/2;
h = 0.001; T_end = 100.0;  % try 100 secs long
t=0:h:T_end;           %time points
n = T_end/h + 1;      %size of array
%pre-allocate memory for arrays, initialize
u = zeros(1, n);
ux = zeros(1, n);   % ux() is u'(), uxx() is u"()
uxx = zeros(1, n);
% starting initial conditions, MATLAB not zero-based
u(1)=0;
ux(1)=0;
for i = 2:length(t)
        %get next value of u(t) from prior u(t), u'(t)
    u(i) = u(i-1) + ux(i-1)*h;
        %calculate prior u"(t) from prior u(t) in order
        % to calculate next u'(t) to prepare for next
        % calculation of u(t) when enter loop again
    uxx(i-1) = -u(i-1) - e*(u(i-1))^3 + cos(w*(i-1)*h);
    ux(i) = ux(i-1) + uxx(i-1)*h;
end
plot(t,u)
grid on
xlabel 't', ylabel 'Displacement'
title 'Solution to:  u" + u + (1/5)u^3 = cos(wt), w = 1/2'
```



Not much of a beat.

$\omega = \frac{3}{4}$

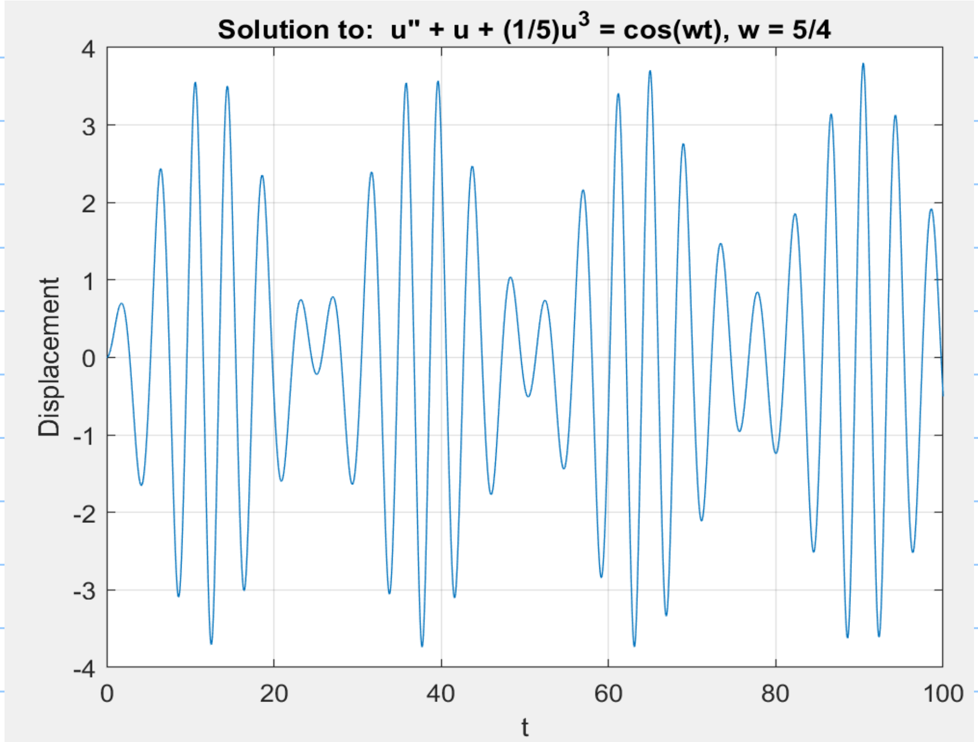
```
clear, clc;
%hardening coefficient, forcing frequency
e = 1/5; w = 3/4;
h = 0.001; T_end = 100.0;  % try 100 secs long
t=0:h:T_end;          %time points
n = T_end/h + 1;      %size of array
%pre-allocate memory for arrays, initialize
u = zeros(1, n);
ux = zeros(1, n);   % ux() is u'(), uxx() is u"()
uxx = zeros(1, n);
% starting initial conditions, MATLAB not zero-based
u(1)=0;
ux(1)=0;
for i = 2:length(t)
        %get next value of u(t) from prior u(t), u'(t)
    u(i) = u(i-1) + ux(i-1)*h;
        %calculate prior u"(t) from prior u(t) in order
        % to calculate next u'(t) to prepare for next
        % calculation of u(t) when enter loop again
    uxx(i-1) = -u(i-1) - e*(u(i-1))^3 + cos(w*(i-1)*h);
    ux(i) = ux(i-1) + uxx(i-1)*h;
end
plot(t,u)
grid on
xlabel 't', ylabel 'Displacement'
title 'Solution to:  u" + u + (1/5)u^3 = cos(wt), w = 3/4'
```



Not much of a beat

$w = \frac{5}{4}$

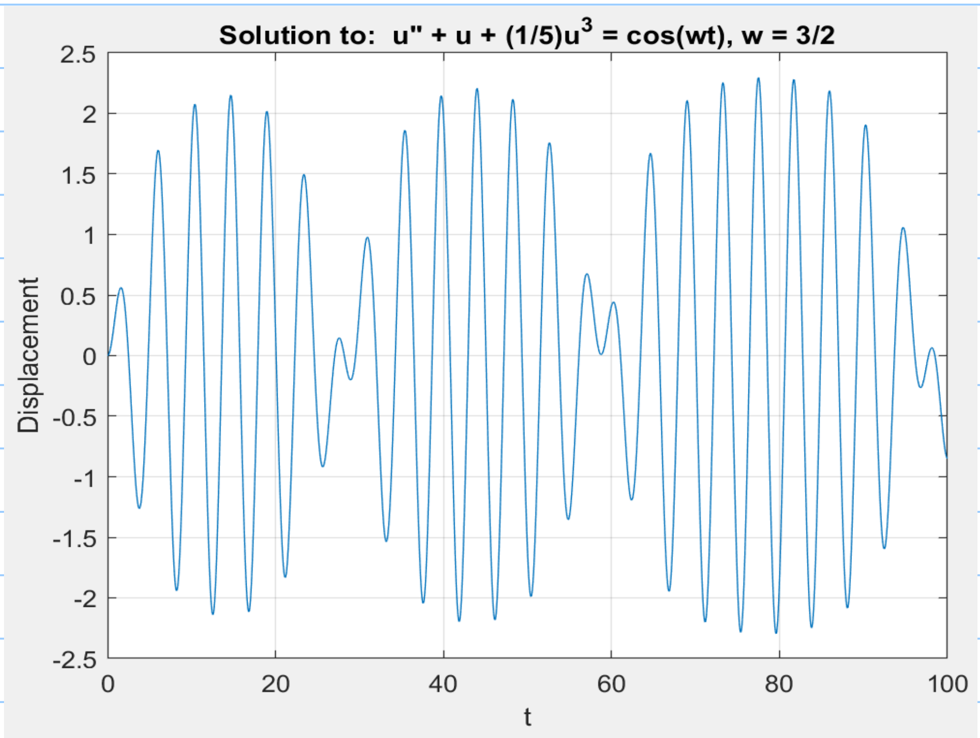
```
clear, clc;
%hardening coefficient, forcing frequency
e = 1/5; w = 5/4;
h = 0.001; T_end = 100.0;  % try 100 secs long
t=0:h:T_end;           %time points
n = T_end/h + 1;     %size of array
%pre-allocate memory for arrays, initialize
u = zeros(1, n);
ux = zeros(1, n);    % ux() is u'(), uxx() is u"()
uxx = zeros(1, n);
% starting initial conditions, MATLAB not zero-based
u(1)=0;
ux(1)=0;
for i = 2:length(t)
        %get next value of u(t) from prior u(t), u'(t)
    u(i) = u(i-1) + ux(i-1)*h;
        %calculate prior u"(t) from prior u(t) in order
        % to calculate next u'(t) to prepare for next
        % calculation of u(t) when enter loop again
    uxx(i-1) = -u(i-1) - e*(u(i-1))^3 + cos(w*(i-1)*h);
    ux(i) = ux(i-1) + uxx(i-1)*h;
end
plot(t,u)
grid on
xlabel 't', ylabel 'Displacement'
title 'Solution to:  u" + u + (1/5)u^3 = cos(wt), w = 5/4'
```



There is a definite beat, with a period of $\approx$ 25 secs.

$\omega = \frac{3}{2}$

```
clear, clc;
%hardening coefficient, forcing frequency
e = 1/5; w = 3/2;
h = 0.001; T_end = 100.0;  % try 100 secs long
t=0:h:T_end;           %time points
n = T_end/h + 1;      %size of array
%pre-allocate memory for arrays, initialize
u = zeros(1, n);
ux = zeros(1, n);   % ux() is u'(), uxx() is u"()
uxx = zeros(1, n);
% starting initial conditions, MATLAB not zero-based
u(1)=0;
ux(1)=0;
for i = 2:length(t)
        %get next value of u(t) from prior u(t), u'(t)
    u(i) = u(i-1) + ux(i-1)*h;
        %calculate prior u"(t) from prior u(t) in order
        % to calculate next u'(t) to prepare for next
        % calculation of u(t) when enter loop again
    uxx(i-1) = -u(i-1) - e*(u(i-1))^3 + cos(w*(i-1)*h);
    ux(i) = ux(i-1) + uxx(i-1)*h;
end
plot(t,u)
grid on
xlabel 't', ylabel 'Displacement'
title 'Solution to:  u" + u + (1/5)u^3 = cos(wt), w = 3/2'
```

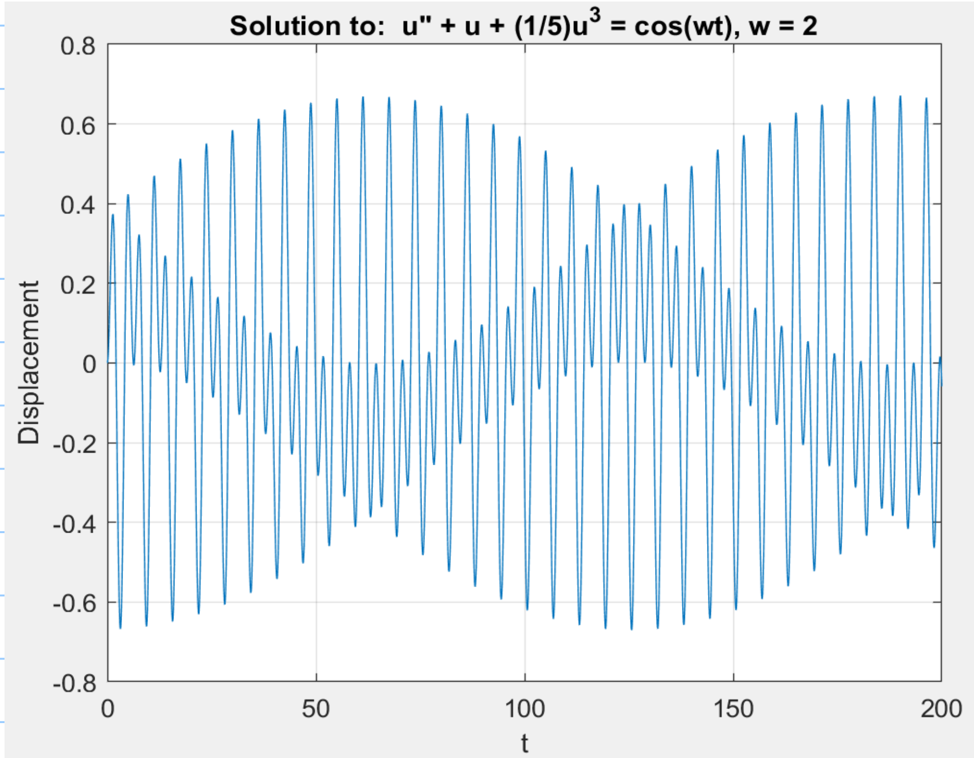


There is a definite beat, period ≈ 40 secs.

w = 2,

200 secs, and use finer mesh of time points: h = 0.0001

```
clear, clc;
%hardening coefficient, forcing frequency
e = 1/5; w = 2;
h = 0.0001; T_end = 200.0;  % try 200 secs long
t=0:h:T_end;               %time points
n = T_end/h + 1;          %size of array
%pre-allocate memory for arrays, initialize
u = zeros(1, n);
ux = zeros(1, n);   % ux() is u'(), uxx() is u"()
uxx = zeros(1, n);
% starting initial conditions, MATLAB not zero-based
u(1)=0;
ux(1)=0;
for i = 2:length(t)
        %get next value of u(t) from prior u(t), u'(t)
    u(i) = u(i-1) + ux(i-1)*h;
        %calculate prior u"(t) from prior u(t) in order
        % to calculate next u'(t) to prepare for next
        % calculation of u(t) when enter loop again
    uxx(i-1) = -u(i-1) - e*(u(i-1))^3 + cos(w*(i-1)*h);
    ux(i) = ux(i-1) + uxx(i-1)*h;
end
plot(t,u)
grid on
xlabel 't', ylabel 'Displacement'
title 'Solution to:  u" + u + (1/5)u^3 = cos(wt), w = 2'
```



No discernable beat when examine |Amplitude| over time.