4.1 General Theory of nth Order Linear Differential Equations

Note Title 12/17/2018

1.

4, 3, and t are all continuous on $\underline{-\infty < t < \infty}$

2.

Standard form: $y^{(4)} + \frac{e^t}{t(t-1)} y'' + \frac{4t^2}{t(t-1)} y = 0$

Coefficient functions are discontinuous at $t=0, 1$.

$\therefore$ $\underline{t<0}$, or $\underline{0<t<1}$, or $\underline{t>1}$

3.

Standard form: $y^{(4)} + \frac{x+1}{x-1} y'' + \frac{\tan(x)}{x-1} y = 0$

Coefficients discontinuous at $x=1$, $x = \frac{\pi}{2} \pm n\pi$, $n = 0, 1, 2, \ldots$

$\therefore$ $\underline{-\frac{\pi}{2} < x < 1}$, or $\underline{1 < x < \frac{\pi}{2}}$, or $\underline{\frac{2n-1}{2}\pi < x < \frac{2n+1}{2}\pi}$, $n = \pm 1, 2, 3, \ldots$

4.

Standard form: $y^{(6)} + \frac{x^2}{x^2-4} + \frac{9}{x^2-4}\, y = 0$

Coefficients discontinuous at $x = \pm 2$.

$\therefore$ $\underline{-\infty < x < -2}$, or $\underline{-2 < x < 2}$, or $\underline{2 < x < \infty}$

5.

Consider $K_1(2t-3) + K_2(t^2+1) + K_3(2t^2-t) = 0$

$\therefore$ $(K_2 + 2K_3)t^2 + (2K_1 - K_3)t + (-3K_1 + K_2) = 0$

Or, $\begin{array}{c} t^2: \\ t: \\ 1: \end{array} \begin{bmatrix} 0 & 1 & 2 \\ 2 & 0 & -1 \\ -3 & 1 & 0 \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Using MATLAB,

```
clear,clc
M = [0, 1, 2; 2, 0, -1; -3, 1, 0];
det(M)
```

```
ans = 7.0000
```

$\therefore$ $K_1 = K_2 = K_3 = 0$, so f_1, f_2, f_3 are <u>independent</u>

6.

Consider $K_1(2t-3) + K_2(2t^2+1) + K_3(3t^2+t) = 0$

Or,
$$\begin{array}{c} t^2: \\ t: \\ 1: \end{array} \begin{bmatrix} 0 & 2 & 3 \\ 2 & 0 & 1 \\ -3 & 1 & 0 \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Using MATLAB,

```
clear,clc
M = [0, 2, 3; 2, 0, 1; -3, 1, 0];
rref(M)
```

```
ans = 3×3
    1.0000         0    0.5000
         0    1.0000    1.5000
         0         0         0
```

$\therefore$ Determinant $= 0$, and letting $K_3 = -2$

then $K_1 = 1$, $K_2 = 3$

$\therefore$ Linearly dependent, and $(1) f_1 + (3) f_2 - (2) f_3 = 0$

7.

Consider $K_1(2t-3) + K_2(t^2+1) + K_3(2t^2-t) + K_4(t^2+t+1) = 0$

Or,
$$\begin{array}{c} t^2: \\ t: \\ 1: \end{array} \begin{bmatrix} 0 & 1 & 2 & 1 \\ 2 & 0 & -1 & 1 \\ -3 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \\ K_3 \\ K_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Using MATLAB,

```
clear,clc
M = [0, 1, 2, 1; 2, 0, -1, 1; -3, 1, 0, 1];
rats(rref(M))
```

```
ans = 3×56 char array
    '      1            0            0          2/7      '
    '      0            1            0         13/7      '
    '      0            0            1         -3/7      '
```

The column space of a 3x4 matrix is linearly dependent.

From reduced matrix, let $k_4 = -7$.

$\therefore\ k_1 = 2,\ k_2 = 13,\ k_3 = -3$

$\therefore\ 2f_1 + 13f_2 - 3f_3 - 7f_4 = 0$

8.

$y = 1 :\ y'' = 0,\ y^{(4)} = 0$

$y = t :\ y'' = 0,\ y^{(4)} = 0$

$y = \cos(t) :\ y'' = -\cos(t),\ y''' = \sin(t)$

$$y^{(4)} = \cos(t)$$

$$\therefore \; y^{(4)} + y'' = \cos(t) - \cos(t) = 0$$

$$y = \sin(t): \quad y'' = -\sin(t), \quad y''' = -\cos(t)$$

$$y^{(4)} = \sin(t)$$

$$\therefore \; y^{(4)} + y'' = \sin(t) - \sin(t) = 0$$

Wronskian:
$$\begin{vmatrix} 1 & t & \cos(t) & \sin(t) \\ 0 & 1 & -\sin(t) & \cos(t) \\ 0 & 0 & -\cos(t) & -\sin(t) \\ 0 & 0 & \sin(t) & -\cos(t) \end{vmatrix}$$

add row 3 to row 1

add row 4 to row 2

equivalent to:
$$\begin{vmatrix} 1 & t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\cos(t) & -\sin(t) \\ 0 & 0 & \sin(t) & -\cos(t) \end{vmatrix}$$

$$= (1)(1)\left[\cos^2(t) + \sin^2(t)\right] + t(0) + 0 + 0$$

$$= \underline{1}$$

9.

$y = e^t$: $y''' = e^t$, $2y'' = 2e^t$, $y' = e^t$

$\therefore (e^t) + (2e^t) - (e^t) - (2e^t) = \underline{0}$

$y = e^{-t}$: $y' = -e^{-t}$, $y'' = e^{-t}$, $y''' = -e^{-t}$

$\therefore (-e^{-t}) + (2e^{-t}) - (-e^{-t}) - (2e^{-t}) = \underline{0}$

$y = e^{-2t}$: $y' = -2e^{-2t}$, $y'' = 4e^{-2t}$, $y''' = -8e^{-2t}$

$\therefore (-8e^{-2t}) + (8e^{-2t}) - (-2e^{-2t}) - (2e^{-2t}) = \underline{0}$

Wronskian: $\begin{vmatrix} e^t & e^{-t} & e^{-2t} \\ e^t & -e^{-t} & -2e^{-2t} \\ e^t & e^{-t} & 4e^{-2t} \end{vmatrix} \Leftrightarrow$ subtract row 1 from row 2

$$\begin{vmatrix} e^t & e^{-t} & e^{-2t} \\ 0 & -2e^{-t} & -3e^{-2t} \\ 0 & 0 & 3e^{-2t} \end{vmatrix} = e^t(-6e^{-3t} - 0) + 0 + 0$$

$$= \underline{-6e^{-2t}}$$

10.

$y = 1$: $y' = 0$, $y'' = 0$, $y''' = 0$

$\therefore x(0) - (0) = \underline{0}$

$y = x$: $y' = 1$, $y'' = 0$, $y''' = 0$

$\therefore\ x(0) - (0) = \underline{0}$

$y = x^3$: $y' = 3x^2$, $y'' = 6x$, $y''' = 6$

$\therefore\ x(6) - (6x) = \underline{0}$

Wronskian : $\begin{vmatrix} 1 & x & x^3 \\ 0 & 1 & 3x^2 \\ 0 & 0 & 6x \end{vmatrix} = 1(6x - 0) + 0 + 0$

$= \underline{6x}$

11.

$y = x$: $y' = 1$, $y'' = 0$, $y''' = 0$

$\therefore\ x^3(0) + x^2(0) - 2x(1) + 2x = \underline{0}$

$y = x^2$: $y' = 2x$, $y'' = 2$, $y''' = 0$

$\therefore\ x^3(0) + x^2(2) - 2x(2x) + 2(x^2) = \underline{0}$

$y = \frac{1}{x}$: $y' = -\frac{1}{x^2}$, $y'' = 2x^{-3}$, $y''' = -6x^{-4}$

$\therefore\ x^3(-6x^{-4}) + x^2(2x^{-3}) - 2x(-x^{-2}) + 2(x^{-1})$

$= -6x^{-1} + 2x^{-1} + 2x^{-1} + 2x^{-1} = \underline{0}$

Wronskian : $\begin{vmatrix} x & x^2 & x^{-1} \\ 1 & 2x & -x^{-2} \\ 0 & 2 & 2x^{-3} \end{vmatrix} = x\left(4x^{-2} + 2x^{-2}\right) - 1\left(2x^{-1} - 2x^{-1}\right) + 0$

$$= 6x^{-1} - 0 + 0 = \underline{\frac{6}{x}}$$

12.

(a) $\begin{vmatrix} 5 & \sin^2(t) & \cos(2t) \\ 0 & \sin(2t) & -2\sin(2t) \\ 0 & 2\cos(2t) & -4\cos(2t) \end{vmatrix}$ $\quad \frac{d}{dt}\sin^2(t) = 2\sin(t)\cos(t) = \sin(2t)$

$$= 5\left[-4\sin(2t)\cos(2t) + 4\sin(2t)\cos(2t)\right] = \underline{0}$$

(b) Show the functions are linearly dependent.

Note $\cos(2t) = \cos^2(t) - \sin^2(t) = 1 - 2\sin^2(t)$

$\therefore$ Consider $K_1(5) + K_2(\sin^2 t) + K_3(1 - 2\sin^2 t)$

$\therefore$ If $K_2 = 2$, $K_3 = 1$, $K_1 = -\frac{1}{5}$, then

$$-\tfrac{1}{5}(5) + 2(\sin^2 t) + 1(1 - 2\sin^2 t) = 0$$

$\therefore$ 5, $\sin^2 t$, $\cos(2t)$ are <u>linearly dependent</u>

$\therefore$ They can't form a fundamental set of solutions to any 3rd order linear homogeneous differential equation, and so it is not true that $W[\ ]\neq 0$ (i.e., $W[\ ]=0$), using Theorem 4.1.3

13.

This all depends on the linearity of the derivative and the distributive law of multiplication over addition.

$$L[c_1y_1 + c_2y_2] = [c_1y_1 + c_2y_2]^{(n)} + p_1(t)[c_1y_1 + c_2y_2]^{(n-1)} + \dots +$$
$$p_n(t)[c_1y_1 + c_2y_2]$$
$$= c_1[y_1]^{(n)} + c_2[y_2]^{n} + p_1(t)\left[c_1[y_1]^{(n-1)} + c_2[y_2]^{(n-1)}\right] + \dots +$$
$$p_n(t)[c_1y_1 + c_2y_2]$$
$$= c_1[y_1]^{(n)} + p_1(t)c_1[y_1]^{(n-1)} + \dots + p_n(t)c_1y_1$$
$$+ c_2[y_2]^{(n)} + p_1(t)c_2[y_2]^{(n-1)} + \dots + p_n(t)c_2y_2$$

$$= c_1 L[y_1] + c_2 L[y_2]$$

14.

(a)

Using $(t^4)' = 4t^3$, $(4t^3)' = 4\cdot3t^2$, $(4\cdot3t^2)' = 4\cdot3\cdot2t$, and

$(4\cdot3\cdot2t)' = 4\cdot3\cdot2\cdot1$ as a pattern,

$$L[t^n] = a_0 (t^n)^{(n)} + a_1 (t^n)^{(n-1)} + \dots + a_n (t^n)$$

$$= a_0 n! + a_1 \frac{n!}{1} t + a_2 \frac{n!}{2!} t^2 + \dots + a_{n-1} n t^{n-1} + a_n t^n$$

$$= \underline{\underline{\sum_{i=0}^{n} a_i \frac{n!}{i!} t^i}}$$

(b)

$(e^{rt})' = re^{rt}$ $(e^{rt})'' = r^2 e^{rt}$, $(e^{rt})^{(3)} = r^3 e^{rt}$,

$$L[e^{rt}] = a_0 r^n e^{rt} + a_1 r^{n-1} e^{rt} + \dots + a_n e^{rt}$$

$$= \underline{\underline{e^{rt}(a_0 r^n + a_1 r^{n-1} + \dots + a_n)}}$$

(c)

Consider a "characteristic equation" of $r^4 - 5r^2 + 4 = 0$, using the $y = e^{rt}$ approach.

$\therefore (r^2-4)(r^2-1) = 0$, $\therefore r = \pm 2, \pm 1$.

$\therefore \underline{y = e^{2t}, e^{-2t}, e^{t}, e^{-t}}$

These solutions are independent as

$$W[e^t, e^{-t}, e^{2t}, e^{-2t}] = \begin{vmatrix} e^t & e^{-t} & e^{2t} & e^{-2t} \\ e^t & -e^{-t} & 2e^{2t} & -2e^{-2t} \\ e^t & e^{-t} & 4e^{2t} & 4e^{-2t} \\ e^t & -e^{-t} & 8e^{2t} & -8e^{-2t} \end{vmatrix}$$

$$= \begin{vmatrix} e^t & e^{-t} & e^{2t} & e^{-2t} \\ 0 & -2e^{-t} & e^{2t} & -3e^{-2t} \\ 0 & 0 & 3e^{2t} & 3e^{-2t} \\ 0 & -2e^{-t} & 7e^{2t} & -9e^{-2t} \end{vmatrix}$$

$$= \begin{vmatrix} e^t & e^{-t} & e^{2t} & e^{-2t} \\ 0 & -2e^{-t} & e^{2t} & -3e^{-2t} \\ 0 & 0 & 3e^{2t} & 3e^{-2t} \\ 0 & 0 & 6e^{2t} & -6e^{-2t} \end{vmatrix}$$

$$= e^t(-2e^{-t})[-18e^{2t}e^{-2t} - 18e^{2t}e^{-2t}] = 72 \neq 0$$

$\therefore$ They form a fundamental set over $-\infty < t < \infty$.

15.

(a)

$$W(t) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = y_1\left(y_2' y_3'' - y_2'' y_3'\right) - y_2\left(y_1' y_3'' - y_1'' y_3'\right) + y_3\left(y_1' y_2'' - y_1'' y_2'\right)$$

— cancel

$$\begin{aligned} \therefore W'(t) = {} & y_1'\left(y_2' y_3'' - y_2'' y_3'\right) + y_1\left(\underline{y_2'' y_3''} + y_2' y_3''' - y_2''' y_3' - \underline{y_2'' y_3''}\right) \\ & - y_2'\left(y_1' y_3'' - y_1'' y_3'\right) - y_2\left(\underline{y_1'' y_3''} + y_1' y_3''' - y_1''' y_3' - \underline{y_1'' y_3''}\right) \\ & + y_3'\left(y_1' y_2'' - y_1'' y_2'\right) + y_3\left(\underline{y_1'' y_2''} + y_1' y_2''' - y_1''' y_2' - \underline{y_1'' y_2''}\right) \\ = {} & \begin{vmatrix} y_1' & y_2' & y_3' \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} + \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1''' & y_2''' & y_3''' \end{vmatrix} \end{aligned}$$

rows 1, 2 identical $\Rightarrow = 0$

$$= \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1''' & y_2''' & y_3''' \end{vmatrix}$$

(b)

$$y_1''' = -p_1 y_1'' - p_2 y_1' - p_3 y_1$$

$$y_2''' = -p_1 y_2'' - p_2 y_2' - p_3 y_2$$

$$y_3''' = -p_1 y_3'' - p_2 y_3' - p_3 y_3$$

$$\therefore W' = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ -p_1 y_1'' - p_2 y_1' - p_3 y_1 & -p_1 y_2'' - p_2 y_2' - p_3 y_2 & -p_1 y_3'' - p_2 y_3' - p_3 y_3 \end{vmatrix} \quad [1]$$

But $W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$

Using properties of determinants,

$$-p_1 W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ -p_1 y_1'' & -p_1 y_2'' & -p_1 y_3'' \end{vmatrix}$$

using $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c-la & d-lb \end{vmatrix}$, l a number

$$-p_1 W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ -p_1 y_1'' - p_2 y_1' & -p_1 y_2'' - p_2 y_2' & -p_1 y_3'' - p_2 y_3' \end{vmatrix}$$

$$= \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ -p_1 y_1'' - p_2 y_1' - p_3 y_1 & -p_1 y_2'' - p_2 y_2' - p_3 y_2 & -p_1 y_3'' - p_2 y_3' - p_3 y_3 \end{vmatrix} \quad [2]$$

$\therefore$ From [1], [2], $\underline{\underline{W'(t) = -p_1(t) W(t)}}$

(c)

From (b), with $W[y_1, y_2, y_3](t) = W(t)$,

$W'(t) = -p_1(t) W(t)$, a first-order linear separable equation.

$\therefore$ $W(t) = c \exp\left(-\int p_1(t)\,dt\right)$, c a constant.

$\therefore$ $W(t)$ is either everywhere 0 $(c=0)$ or nowhere 0 $(c \neq 0)$.

(d)

(1) The derivative of an nxn determinant is the sum n nxn determinants obtained by differentiating, respectively, the first row, 2nd row, ..., nth row.

$\therefore$ If $W[y_1, y_2, \dots, y_n](t) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & & & \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$

then $W'(t) = \begin{vmatrix} y_1' & y_2' & \cdots & y_n' \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & & & \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} +$

$\begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1'' & y_2'' & \cdots & y_n'' \\ y_1'' & y_2'' & \cdots & y_n'' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} +$

$$\cdots + \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & & & \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} +$$

$$\begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & & & \\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_n^{(n-2)} \\ y_1^{(n)} & y_2^{(n)} & \cdots & y_n^{(n)} \end{vmatrix}$$

All the above determinands, except the very last one, have two adjacent rows that are equal, and so their value is 0.

$$\therefore \frac{d}{dt} W[y_1, y_2, \ldots, y_n](t) = \begin{vmatrix} y_1 & \cdots & y_2 & \cdots & y_n \\ y_1' & \cdots & y_2' & \cdots & y_n' \\ \vdots & & \vdots & & \vdots \\ y_1^{(n-2)} & \cdots & y_2^{(n-2)} & \cdots & y_n^{(n-2)} \\ y_1^{(n)} & & y_2^{(n)} & \cdots & y_n^{(n)} \end{vmatrix} \quad [3]$$

(2) From the differential equation, for each $i = 1, \ldots, n$,

$$y_i^{(n)} = -p_1 y_i^{(n-1)} - p_2 y_i^{(n-2)} - \cdots - p_{n-1} y_i' - p_n y_i$$

Now consider $-p_1(t)W[y_1,\dots,y_n](t)$

$$= \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ \vdots & \vdots & & \vdots \\ -p_1 y_1^{(n-1)} & -p_1 y_2^{(n-1)} & \cdots & -p_1 y_n^{(n-1)} \end{vmatrix}$$

$\therefore$ We can obtain [3] from $-p_1(t)W[y_1,\dots,y_n](t)$ by multiplying each element in row K $(k=1,2,\dots,n-1)$ of $-p_1(t)W[y_1,\dots,y_n](t)$ by $p_{n-K+1}(t)$, and subtracting that from row n.

$\therefore$ $W'(t) = -p_1(t)W(t)$

(3) The solution to $W' = -p_1 W$ is:

$$W[y_1,\dots,y_n](t) = c\exp\left(-\int p_1(t)\,dt\right), \quad c \text{ a constant}$$

16.

$p_1 = 2$. $\therefore$ $c\exp\left(-\int 2\,dt\right) = \underline{ce^{-2t}}$, c a constant

17.

Standard form: $y''' + \frac{2}{t}y'' - \frac{1}{t}y' + y = 0$

$p_1(t) = \frac{2}{t}$ $\therefore$ $c\exp\left(-\int \frac{2}{t}\,dt\right) = ce^{-2\ln t} = \underline{\underline{\frac{c}{t^2}}}$

18.

(a)

On $0 < t < 1$, $|t| = t$. $\therefore$ $f(t) = t^2 \cdot t = t^3 = g(t)$.

$\therefore$ Choose constants $c_1 = 1$, $c_2 = -1$.

Then $c_1 f(t) + c_2 g(t) = 0$ for all $t \in (0,1)$

(b)

On $-1 < t < 0$, $|t| = -t$. $\therefore$ $f(t) = t^2(-t) = -t^3 = -g(t)$

$\therefore$ Choose constants $c_1 = 1$, $c_2 = 1$.

$\therefore$ $c_1 f(t) + c_2 g(t) = 0$, for all $t \in (-1, 0)$

(c)

Suppose $f(t)$, $g(t)$ are dependent for all $t \in (-1,1)$.

$\therefore$ There are constants c_1, c_2, not both 0, s.t. $c_1 f(t) + c_2 g(t) = 0$ for all $t \in (-1,1)$.

$\therefore$ Let $t = \frac{1}{2}$

Then $c_1 f(\frac{1}{2}) + g(\frac{1}{2}) = c_1(\frac{1}{8}) + c_2(\frac{1}{8}) = 0$ [1]

Let $t = -\frac{1}{2}$.

Then $c_1 f(-\frac{1}{2}) + c_2 g(-\frac{1}{2}) = c_1(\frac{1}{8}) + c_2(-\frac{1}{8}) = 0$ [2]

Adding [1], [2], $2c_1(\frac{1}{8}) = 0 \Rightarrow c_1 = 0$. $\therefore c_2 = 0$.

This contradicts $c_1 \neq 0$ or $c_2 \neq 0$, so f, g are not linearly dependent $\Rightarrow$

$f(t)$ and $g(t)$ are <u>linearly independent</u>.

(d)

For $0 < t < 1$, $f(t) = g(t) = t^3$ $\therefore f'(t) = g'(t) = 3t^2$

$$\therefore W[f,g](t) = \begin{vmatrix} t^3 & t^3 \\ 3t^2 & 3t^2 \end{vmatrix} = 0$$

For $-1 < t < 0$, $f(t) = -t^3$, $g(t) = t^3$.

$\therefore f' = -3t^2$, $g(t) = 3t^2$.

$$\therefore W[f,g] = \begin{vmatrix} -t^3 & t^3 \\ -3t^2 & 3t^2 \end{vmatrix} = -3t^5 + 3t^5 = 0$$

For $t=0$, $f(t)=0$, $g(t)=0$. $g'(t) = 3t^2$

$\therefore g'(0) = 0$.

$$\lim_{h\to 0^+} \frac{f(h)-f(0)}{h} = \lim_{h\to 0^+} \frac{h^3 - 0}{h} = \lim_{h\to 0^+} h^2 = 0$$

$$\lim_{h\to 0^-} \frac{f(h)-f(0)}{h} = \lim_{h\to 0^-} \frac{-h^3 - 0}{h} = \lim_{h\to 0^-} -h^2 = 0$$

$\therefore f'(t)$ exists at $t=0$, and $f'(0) = 0$.

$$\therefore W[f,g](0) = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} = 0.$$

$\therefore$ For $-1<t<1$, $\underline{W[f,g](t) = 0}$

(e)

The definition of independence is the negative of dependence. Thus, linear independence doesn't mean for each $t \in I$, $c_1 f_1(t) + \dots + c_n f_n(t) = 0$ implies $c_1 = \dots = c_n = 0$. It means

$c_1 f_1(t) + \ldots + c_n f_n(t) = 0 \Rightarrow c_1 = \ldots = c_n = 0$

when all $t \in I$ are considered. That is, $[c_1, \ldots, c_n] = [0, \ldots, 0]$ is the only consistent solution to $c_1 f_1(t) + \ldots + c_n f_n(t) = 0$. The same set of constants $[c_1, \ldots, c_n]$ must be used for every $t \in I$. So, $f(t)$ and $g(t)$ are linearly independent if they are not a constant multiple of each other at least somewhere on I.

Regarding Theorem 4.1.3, $f(t)$ and $g(t)$ above can't be solutions to some ordinary 2nd order linear homogeneous equation on $(-1,1)$.

If they are solutions to some equation, $L[y]=0$, where $L[y] = y'' + py' + qy$, p, q continuous on $(-1,1)$, then let $\phi(t) = c_1 f(t) + c_2 g(t)$ on $(-1,1)$.

Since $W[f,g](t_0) = 0$, some $t_0 \in (-1,1)$, consider

$L[y]=0$, $y(t_0)=0$, $y'(t_0)=0$. There is a nontrivial solution $[c_1, c_2] \neq [0,0]$ to:

$$\begin{aligned} c_1 f(t_0) + c_2 g(t_0) &= 0 \\ c_1 f'(t_0) + c_2 g'(t_0) &= 0 \end{aligned} \Rightarrow \begin{aligned} c_1 f(t_0) g'(t_0) + c_2 g(t_0) g'(t_0) &= 0 \\ c_1 f'(t_0) g(t_0) + c_2 g(t_0) g'(t_0) &= 0 \end{aligned}$$

$$\Rightarrow c_1 [f(t_0) g'(t_0) - f'(t_0) g(t_0)] = 0 \Rightarrow c_1 W[f,g](t_0) = 0$$

Similarly, $c_2 W[f,g](t_0) = 0$

Since $W[f,g](t_0)=0$, any c_1, c_2 will do, so pick a non-zero solution such as $c_1 = c_2 = 1$.

$\therefore \phi(t) = f(t) + g(t)$ satisfies

$L[y]=0$, $y(t_0)=0$, $y'(t_0)=0$, since $\phi(t)$ is a linear combination of $f(t)$ and $g(t)$, and $\phi \neq 0$ for some $t \in (-1,1)$, such as $t=\frac{1}{2}$.

But the function $y=0$ on $(-1,1)$ satisfies $L[y]=0$, $y(t_0)=0$, $y'(t_0)=0$. $\therefore \phi$ contradicts the uniqueness of a solution, so the assumption that f, g are solutions to some $L[y]$ on $(-1,1)$ must be false.

$\therefore$ $f(t)$ and $g(t)$ don't contradict 4.1.3 because they don't fulfill the criteria of the converse part of the theorem on interval $(-1,1)$.

19.

$y' = y_1'v + y_1v'$

$y'' = y_1''v + y_1'v' + y_1'v' + y_1v'' = y_1''v + 2y_1'v' + y_1v''$

$y''' = y_1'''v + y_1''v' + 2y_1''v' + 2y_1'v'' + y_1'v'' + y_1v'''$

$= y_1'''v + 3y_1''v' + 3y_1'v'' + y_1v'''$

$\therefore y''' + p_1y'' + p_2y' + p_3y =$

$y_1'''v + 3y_1''v' + 3y_1'v'' + y_1v'''$

$+ p_1(y_1''v + 2y_1'v' + y_1v'')$

$+ p_2(y_1'v + y_1v') + p_3y_1v$

$$= (y_1''' + p_1 y_1'' + p_2 y_1' + p_3 y_1) v$$ $= 0$ since y_1 is a solution

$$+ (3y_1'' + 2p_1 y_1' + p_2 y_1) v'$$

$$+ (3y_1' + p_1 y_1) v''$$

$$+ y_1 v''' = 0$$

$$\therefore \underline{y_1 v''' + (3y_1' + p_1 y_1) v'' + (3y_1'' + 2p_1 y_1' + p_2 y_1) v' = 0}$$

20.

Standard form: $y''' - \frac{2t-3}{t-2} y'' + \frac{t}{t-2} y' - \frac{1}{t-2} y = 0, \; t \neq 2$

Checking for $y = e^t$, $y''' = y'' = y' = e^t$

$$\therefore e^t - \frac{2t-3}{t-2} e^t + \frac{t}{t-2} e^t - \frac{1}{t-2} e^t = 0$$

$$\therefore 1 - \frac{2t-3}{t-2} + \frac{t}{t-2} - \frac{1}{t-2} = \frac{t-2-2t+3+t-1}{t-2} = \underline{0}$$

$\therefore$ Let $y = ve^t$. Using the formula in #19,

$$e^t v''' + \left(3e^t + \frac{3-2t}{t-2} e^t\right) v'' + \left(3e^t + \frac{6-4t}{t-2} e^t + \frac{t}{t-2} e^t\right) v' = 0$$

Dividing by e^t and simplifying,

$v''' + \frac{t-3}{t-2} v'' = 0$ as terms for v' cancel.

$\therefore$ This becomes a first order equation in v''.

$\therefore \frac{v'''}{v''} = -\frac{t-3}{t-2}$, $\ln(v'') = -\int\left(1 - \frac{1}{t-2}\right)dt$

$\therefore \ln(v'') = -t + \ln(t-2) + c$, c a constant

$v'' = c_1(t-2)e^{-t} = c_1 te^{-t} - 2c_1 e^{-t}$, $c_1 = e^c$

$\therefore v' = \int\left[c_1 te^{-t} - 2c_1 e^{-t}\right]dt$

$= c_1(-t-1)e^{-t} + 2c_1 e^{-t} + c_2$, c_2 a constant

$= -c_1 te^{-t} + c_1 e^{-t} + c_2$

$\therefore v = -c_1 \int te^{-t} + c_1 \int e^{-t} + \int c_2$

$= -c_1(-t-1)e^{-t} - c_1 e^{-t} + c_2 t + c_3$, c_3 a constant

$= c_1 te^{-t} + c_2 t + c_3$

$\therefore y = ve^t = \underline{\underline{c_1 t + c_2 te^t + c_3 e^t}}$

Note: t, te^t, and e^t are linearly independent on $t < 2$, and so form a fundamental set.

21.

Check: $y_1' = 2t$, $y_1'' = 2$, $y_1''' = 0$

$$\therefore -3t(t+2)(2) + 6(1+t)(2t) - 6(t^2) =$$

$$-6t^2 - 12t + 12t + 12t^2 - 6t^2 = \underline{0}$$

$$y_2' = 3t^2, \quad y_2'' = 6t, \quad y_2''' = 6$$

$$\therefore t^2(t+3)(6) - 3t(t+2)(6t) + 6(1+t)(3t^2) - 6(t^3) =$$

$$6t^3 + 18t^2 - 18t^3 - 36t^2 + 18t^2 + 18t^3 - 6t^3 = \underline{0}$$

Let $y = vt^2$, $\therefore y' = v't^2 + 2tv$ (using the $y_1 = t^2$ solution)

$$y'' = v''t^2 + v'2t + 2v't + 2v = t^2v'' + 4tv' + 2v$$

$$y''' = v'''t^2 + 2tv'' + 4tv'' + 4v' + 2v'$$

$$= v'''t^2 + v''6t + 6v'$$

Standard form:

$$y''' - \frac{3(t+2)}{t(t+3)}y'' + \frac{6(t+1)}{t^2(t+3)}y' - \frac{6}{t^2(t+3)}y = 0$$

Using formula in #19,

$$t^2v''' + \left[6t - \frac{3t^2(t+2)}{t(t+3)}\right]v'' + \left[6 - \frac{12t(t+2)}{t(t+3)} + \frac{6t^2(t+1)}{t^2(t+3)}\right]v' = 0$$

$$\therefore\ t^2 v''' + \left[\frac{6t(t+3) - 3t(t+2)}{t+3}\right] v'' + \left[\frac{6(t+3) - 12(t+2) + 6(t+1)}{t+3}\right] v' = 0$$

$$\therefore\ t^2 v''' + \left[\frac{6t^2 + 18t - 3t^2 - 6t}{t+3}\right] v'' + \left[\frac{\overbrace{6t + 18 - 12t - 24 + 6t + 6}^{=0}}{t+3}\right] v' = 0$$

$$\therefore\ t^2 v''' + \frac{3t^2 + 12t}{t+3} v'' = t^2 v''' + \frac{3t(t+4)}{t+3} v'' = 0$$

$$\therefore\ v''' + \frac{3(t+4)}{t^2+3t} v'' = 0$$

→ multiply top + bottom by 2, since $(t^2+3t)' = 2t+3$

$$\frac{v'''}{v''} = -\frac{3t+12}{t^2+3t} = -\frac{3(2t+3)+15}{2(t^2+3t)} = -\frac{3}{2}\frac{(2t+3)}{t^2+3t} - \frac{15}{2(t^2+3t)}$$

$$\therefore\ \ln(v'') = -\frac{3}{2}\ln(t^2+3t) - \frac{15}{6}\ln\left(\frac{t}{t+3}\right) + c$$

using $\int \frac{du}{u(a+bu)} = \frac{1}{a}\ln\left|\frac{u}{a+bu}\right| + C$ and $t > 0$

$$\therefore\ v'' = K_1 (t^2+3t)^{-\frac{3}{2}} \left(\frac{t}{t+3}\right)^{-\frac{5}{2}}, \quad K_1 = e^c,\ c \text{ a constant}$$

$$= K_1 \frac{(t+3)^{5/2}}{t^{5/2}(t+3)^{3/2} t^{3/2}} = K_1 \frac{(t+3)}{t^4} = K_1 t^{-3} + 3K_1 t^{-4}$$

$$\therefore\ v' = \frac{K_1}{-2} t^{-2} + \frac{3K_1}{-3} t^{-3} + c_2$$

$$v = \frac{K_1}{2} t^{-1} + \frac{3}{6} K_1 t^{-2} + c_2 t + c_3$$

$$= c_1 (t^{-1} + t^{-2}) + c_2 t + c_3, \quad c_1 = \frac{K_1}{2}$$

$$\therefore\ y = vt^2 = \underline{c_1(t+1) + c_2 t^3 + c_3 t^2}$$

Check for $y = c_1(t+1)$:

$$y' = c_1, \quad y'' = 0, \quad y''' = 0$$

$$\therefore\ t^2(t+3)(0) - 3t(t+2)(0) + 6(1+t)(c_1) - 6(c_1(t+1)) = \underline{0}$$

And, $t+1$, t^2, t^3 are linearly independent.

1.

$$\sqrt{1^2+1^2} = \sqrt{2} \quad . \quad 1+i = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i\right)$$

$$= \sqrt{2}\left[\cos\left(\frac{\pi}{4} + 2n\pi\right) + i\sin\left(\frac{\pi}{4} + 2n\pi\right)\right]$$

$$= \underline{\sqrt{2}\, e^{i\left(\frac{\pi}{4} + 2n\pi\right)}}, \quad n = 0, \pm 1, \pm 2, \ldots$$

2.

$$\sqrt{(-1)^2 + (\sqrt{3})^2} = 2 \quad \therefore \quad -1 + \sqrt{3}\, i = 2\left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)$$

$$\theta = \arctan\left(\frac{\sqrt{3}/2}{-1/2}\right) = \arctan(-\sqrt{3}) = -\frac{\pi}{3} \equiv -\frac{\pi}{3} + \pi = \frac{2}{3}\pi$$

$$\therefore -1 + \sqrt{3}\, i = 2\left[\cos\left(\frac{2}{3}\pi + 2n\pi\right) + i\sin\left(\frac{2}{3}\pi + 2n\pi\right)\right]$$

$$= \underline{2\, e^{i\left(\frac{2}{3}\pi + 2n\pi\right)}}, \quad n = 0, \pm 1, \pm 2, \ldots$$

3.

$$-3 = 3(-1 + 0i) = 3\left[\cos(\pi + 2n\pi) + i\sin(\pi + 2n\pi)\right]$$

$$= \underline{3\, e^{i(\pi + 2n\pi)}}, \quad n = 0, \pm 1, \pm 2, \ldots$$

4.

$$\sqrt{(\sqrt{3})^2+(-1)^2} = 2 \quad \therefore \sqrt{3} - i = 2\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$$

$$\theta = \arctan\left(\frac{-1/2}{\sqrt{3}/2}\right) = \arctan\left(-\frac{\sqrt{3}}{3}\right) = -\frac{\pi}{6}$$

Note $\sin\theta = -\frac{1}{2}$, $\cos\theta = \frac{\sqrt{3}}{2} \Rightarrow$ Quadrant $\underline{IV}$.

$$\therefore \theta = -\frac{\pi}{6} + 2\pi = \frac{11}{6}\pi$$

$$\therefore \sqrt{3} - i = 2\left[\cos\left(\frac{11}{6}\pi + 2n\pi\right) + i\sin\left(\frac{11}{6}\pi + 2n\pi\right)\right]$$

$$= \underline{2e^{i\left(\frac{11}{6}\pi + 2n\pi\right)}}, \quad n = 0, \pm 1, \pm 2, \ldots$$

5.

$$1 = \cos(2n\pi) + i\sin(2n\pi) = e^{i(2n\pi)}, \quad n = 0, 1, 2$$

$$\therefore 1^{1/3} = e^{i(2n\pi)1/3} = e^{i\left(\frac{2}{3}n\pi\right)}$$

$\therefore$ For $n = 0, 1, 2$: $e^{i(0)}, e^{i\left(\frac{2}{3}\pi\right)}, e^{i\left(\frac{4}{3}\pi\right)}$

$$\therefore 1, \cos\left(\frac{2}{3}\pi\right) + i\sin\left(\frac{2}{3}\pi\right), \cos\left(\frac{4}{3}\pi\right) + i\sin\left(\frac{4}{3}\pi\right)$$

Or, $\underline{1, -\frac{1}{2} + i\frac{\sqrt{3}}{2}, -\frac{1}{2} - i\frac{\sqrt{3}}{2}}$

6.

$$\sqrt{1^2+(-1)^2}=\sqrt{2} \quad \therefore\ 1-i=\sqrt{2}\left(\frac{1}{\sqrt{2}}-i\frac{1}{\sqrt{2}}\right)$$

$$\therefore\ \cos\theta=\frac{1}{\sqrt{2}}\ ,\ \sin\theta=-\frac{1}{\sqrt{2}}\ .\ \therefore\ \text{Quadrant }\underline{IV}$$

$$\therefore\ \theta=\arctan\left(\frac{-1/\sqrt{2}}{1/\sqrt{2}}\right)=\arctan(-1)=\frac{3}{4}\pi+\pi=\frac{7}{4}\pi$$

$$\therefore\ 1-i=\sqrt{2}\left[\cos\left(\frac{7}{4}\pi+2n\pi\right)+i\sin\left(\frac{7}{4}\pi+2n\pi\right)\right]$$

$$\therefore\ 1-i=\sqrt{2}\,e^{i\left(\frac{7}{4}\pi+2n\pi\right)}\ ,\quad n=0,\pm1,\pm2,\ldots$$

$$\therefore\ (1-i)^{\frac{1}{2}}=2^{\frac{1}{4}}e^{i\left(\frac{7}{8}\pi+n\pi\right)}\ ,\quad n=0,\pm1,\pm2,\ldots$$

$$\therefore\ (1-1)^{\frac{1}{2}}=2^{\frac{1}{4}}e^{i\left(\frac{7}{8}\pi\right)}\ ,\ 2^{\frac{1}{4}}e^{i\left(\frac{15}{8}\pi\right)} \qquad e^{i\frac{15}{8}\pi}=e^{i\left(-\frac{\pi}{8}\right)}$$

7.

$$\cos\left(\frac{\pi}{3}+2n\pi\right)+i\sin\left(\frac{\pi}{3}+2n\pi\right)=e^{i\left(\frac{\pi}{3}+2n\pi\right)}$$

$$\therefore\ \left[2\left(\cos\left(\frac{\pi}{3}\right)+i\sin\left(\frac{\pi}{3}\right)\right)\right]^{1/2}=\left[2e^{i\left(\frac{\pi}{3}+2n\pi\right)}\right]^{\frac{1}{2}}$$

$$=2^{\frac{1}{2}}\,e^{i\left(\frac{\pi}{6}+n\pi\right)}\ ,\quad n=0,\pm1,\pm2,\ldots$$

$$=\sqrt{2}\,e^{i\frac{\pi}{6}}\ ,\ \sqrt{2}\,e^{i\left(\frac{7}{6}\pi\right)}$$

$$=\sqrt{2}\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)\ ,\ \sqrt{2}\left(\cos\frac{7}{6}\pi+i\sin\frac{7}{6}\pi\right)$$

$$=\sqrt{2}\left(\frac{\sqrt{3}}{2}+i\frac{1}{2}\right)\ ,\ \sqrt{2}\left(-\frac{\sqrt{3}}{2}+i\left(-\frac{1}{2}\right)\right)$$

$$= \frac{1}{\sqrt{2}}\left(\sqrt{3}+i\right), \quad -\frac{1}{\sqrt{2}}\left(\sqrt{3}+i\right)$$

8.

Characteristic equation: $r^3 - r^2 - r + 1 = 0$

From MATLAB

```
clear, clc
p = [1, -1, -1, 1];
rats(roots(p))
```

```
ans = 3×14 char array
    '      -1      '
    '       1      '
    '       1      '
```

$\therefore \; e^{-t}, e^{t}, te^{t}$ since $r=1$ is repeated.

$\therefore \; y(t) = \underline{c_1 e^{-t} + c_2 e^{t} + c_3 t e^{t}}$, c_i constants

9.

Characteristic equation: $r^3 - 3r^2 + 3r - 1 = 0$

Since $(r-1)^3 = r^3 - 3r^2 + 3r - 1$, $r = 1$, repeated $3\times$.

$\therefore \; y(t) = \underline{c_1 e^{t} + c_2 t e^{t} + c_3 t^2 e^{t}}$, c_i constants

10.

Characteristic equation: $r^4 - 4r^3 + 4r^2 = r^2(r-2)^2 = 0$

$\therefore\ r = 0, 0, 2, 2 \quad \therefore\ e^0, te^0, e^{2t}, te^{2t}$

$\therefore\ y(t) = \underline{c_1 + c_2 t + c_3 e^{2t} + c_4 t e^{2t}}$

11.

Characteristic equation: $r^6 + 1 = 0$, $r = (-1)^{1/6}$

$r = -1 = \cos(\pi + 2n\pi) + i\sin(\pi + 2n\pi)$

$= e^{i(\pi + 2n\pi)}, \quad n = 0, \pm 1, \pm 2, \ldots$

$\therefore\ (-1)^{1/6} = e^{i\left(\frac{\pi}{6} + \frac{n\pi}{3}\right)}, \quad n = 0, 1, 2, 3, 4, 5$

$\therefore\ \theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5}{6}\pi, \frac{7}{6}\pi, \frac{9}{6}\pi, \frac{11}{6}\pi$

$\therefore\ \cos\theta + i\sin\theta \Rightarrow \frac{\sqrt{3}}{2} + i\left(\frac{1}{2}\right), i, -\frac{\sqrt{3}}{2} + i\left(\frac{1}{2}\right)$

$-\frac{\sqrt{3}}{2} - i\left(\frac{1}{2}\right), -i, \frac{\sqrt{3}}{2} - i\left(\frac{1}{2}\right)$

$\therefore\ e^{\pm\frac{\sqrt{3}}{2}} e^{i\left(\frac{1}{2}\right)}, \ e^{\pm\frac{\sqrt{3}}{2}} e^{i\left(-\frac{1}{2}\right)}, \ e^{i(\pm 1)}$

$e^{i\left(\frac{1}{2}\right)} \equiv \cos\left(\frac{t}{2}\right), \sin\left(\frac{t}{2}\right)$

$e^{i\left(-\frac{1}{2}\right)} \equiv \cos\left(\frac{t}{2}\right), -\sin\left(\frac{t}{2}\right)$

$e^{i} \equiv \cos(t), \sin(t)$, $e^{-i} \equiv \cos(t), -\sin(t)$.

The constant coefficients will take care of the positive/negative signs.

$$\therefore \; y(t) = e^{\frac{\sqrt{3}}{2}t}\left[c_1 \cos\left(\frac{t}{2}\right) + c_2 \sin\left(\frac{t}{2}\right)\right] + e^{-\frac{\sqrt{3}}{2}t}\left[c_3 \cos\left(\frac{t}{2}\right) + c_4 \sin\left(\frac{t}{2}\right)\right] + c_5 \cos(t) + c_6 \sin(t) \quad , \; c_i \text{ constants}$$

12.

Characteristic equation: $r^6 - 3r^4 + 3r^2 - 1 = (r^2-1)^3 = 0$

$\therefore \; r = 1, 1, 1, -1, -1, -1$

$$\therefore \; y(t) = c_1 e^{t} + c_2 t e^{t} + c_3 t^2 e^{t} + c_4 e^{-t} + c_5 t e^{-t} + c_6 t^2 e^{-t} \quad , \; c_i \text{ constants}$$

13.

Characteristic equation: $r^6 - r^2 = r^2(r^4 - 1) = 0$

$\therefore \; r = 0, 0, 1, -1, i, -i$

$$\therefore y(t) = c_1 + c_2 t + c_3 e^{t} + c_4 e^{-t}$$
$$+ c_5 \cos(t) + c_6 \sin(t), \quad c_i \text{ constants}$$

14.

Characteristic equation: $r^5 - 3r^4 + 3r^3 - 3r^2 + 2r = 0$

$$\therefore r(r^4 - 3r^3 + 3r^2 - 3r + 2) = 0$$

Using MATLAB

```
clear, clc
p = [1, -3, 3, -3, 2];
rats(roots(p))
```

```
ans = 4×29 char array
    '        2       +       0i        '
    '        1       +       0i        '
    '        0       +       1i        '
    '        0       -       1i        '
```

$$\therefore r = 0, 1, 2, i, -i$$

$$\therefore e^{0t}, e^{t}, e^{2t}, e^{it}, e^{-it}$$

$$\therefore y(t) = c_1 + c_2 e^{t} + c_3 e^{2t} + c_4 \cos(t) + c_5 \sin(t)$$

15.

Characteristic equation: $r^8 + 8y^4 + 16 = (r^4 + 4)^2 = 0$

$$\therefore (r^2 + 2i)^2 (r^2 - 2i)^2 = 0, \quad 2i = 2\left[\cos\left(\tfrac{\pi}{2}\right) + i \sin\left(\tfrac{\pi}{2}\right)\right]$$

$$\therefore 2i = 2e^{i\left(\frac{\pi}{2} + 2n\pi\right)}, \quad n = 0, \pm 1, \pm 2, \ldots$$

$$-2i = 2(-i) = 2\left[\cos\left(\tfrac{3}{2}\pi\right) + i \sin\left(\tfrac{3}{2}\pi\right)\right]$$

$\therefore -2i = 2e^{i\left(\frac{3}{2}\pi + 2n\pi\right)}, \quad n = 0, \pm 1, \pm 2, \ldots$

$\therefore r^2 + 2i = 0 \Rightarrow r^2 = -2i \Rightarrow r = \sqrt{2}\, e^{i\left(\frac{3}{4}\pi + n\pi\right)}, \quad n = 0, 1$

$r^2 - 2i = 0 \Rightarrow r^2 = 2i \Rightarrow r = \sqrt{2}\, e^{i\left(\frac{\pi}{4} + n\pi\right)}, \quad n = 0, 1$

$\therefore r = \sqrt{2}\left[\cos\left(\frac{3}{4}\pi + n\pi\right) + i\sin\left(\frac{3}{4}\pi + n\pi\right)\right], \quad n = 0, 1$

$r = \sqrt{2}\left[\cos\left(\frac{\pi}{4} + n\pi\right) + i\sin\left(\frac{\pi}{4} + n\pi\right)\right], \quad n = 0, 1$

$\therefore r = \sqrt{2}\left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right), \ \sqrt{2}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)$

$r = \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right), \ \sqrt{2}\left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)$

$\therefore r = -1 + i, \ 1 - i, \ 1 + i, \ -1 - i$

Note each root is repeated as $(r^2 + 2i)^2 (r^2 - 2i)^2 = 0$

As $y(t) = e^{(-1+i)t} \equiv y(t) = e^{-t}(c_1\cos(t) + c_2\sin(t))$

and $y(t) = e^{(-1-i)t} \equiv y(t)\, e^{-t}(c_1\cos(t) + c_2\sin(t))$

as the constants c_i take care of the signs for stating solutions in generality, the roots $r = -1 + i$, $r = -1 - i$ and $r = 1 - i$, $r = 1 + i$ yield the same solution forms. These roots yield

4 independent solutions, and the repeated nature yields 4 additional independent solutions.

$$\therefore y(t) = e^{-t}\left[c_1\cos(t) + c_2\sin(t) + c_3 t\cos(t) + c_4 t\sin(t)\right]$$
$$+ e^{t}\left[c_5\cos(t) + c_6\sin(t) + c_7 t\cos(t) + c_8 t\sin(t)\right]$$

16.

Characteristic equation: $r^4 + 2r^2 + 1 = (r^2+1)^2 = 0$

$r^2 + 1 = 0 \Rightarrow r = \pm i$. $\therefore (r+i)^2(r-i)^2 = 0$

$\therefore r = i, i, -i, -i \quad \therefore y(t) = e^{it}, te^{it}, e^{-it}, te^{-it}$

Or, $y(t) = c_1\cos(t) + c_2\sin(t) + c_3 t\cos(t) + c_4 t\sin(t)$

17.

Characteristic equation: $r^3 + 5r^2 + 6r + 2 = 0$

$\therefore (r+1)(r^2+4r+2) = 0, \; r = -1, -2+\sqrt{2}, -2-\sqrt{2}$

$\therefore y(t) = c_1 e^{-t} + c_2 e^{(-2+\sqrt{2})t} + c_3 e^{(-2-\sqrt{2})t}$

18.

Characteristic equation: $r^4 - 7r^3 + 6r^2 + 30r - 36 = 0$

Using MATLAB:

```
clear, clc
syms r
factor(r^4 - 7*r^3 + 6*r^2 + 30*r - 36)
```

ans = $(r-3 \quad r+2 \quad r^2-6r+6)$

$\therefore r = 3, -2 \Rightarrow e^{3t}, e^{-2t}$

For $r^2 - 6r + 6 = 0$, $r = \frac{6 \pm \sqrt{36-24}}{2} = 3 \pm \sqrt{3}$

$$\therefore y(t) = c_1 e^{3t} + c_2 e^{-2t} + c_3 e^{(3+\sqrt{3})t} + c_4 e^{(3-\sqrt{3})t}$$

19.

Characteristic equation: $12r^4 + 31r^3 + 75r^2 + 37r + 5 = 0$

Using MATLAB:

```
clear, clc
syms r
factor(12*r^4 + 31*r^3 + 75*r^2 + 37*r + 5)
```

ans = $(3r+1 \quad 4r+1 \quad r^2+2r+5)$

$\therefore r = -\frac{1}{3}, -\frac{1}{4} \Rightarrow e^{-t/3}, e^{-t/4}$

For $r^2 + 2r + 5 = 0$, $r = \frac{-2 \pm \sqrt{4-20}}{2} = -1 \pm 2i$

$\therefore e^{-t}\cos(2t), e^{-t}\sin(2t).$

$\therefore\ \underline{y(t) = c_1 e^{-t/3} + c_2 e^{-t/4} + c_3 e^{-t}\cos(2t) + c_4 e^{-t}\sin(2t)}$

20.

Characteristic equation: $r^3 + r = 0$, $r = 0, \pm i$

$\therefore\ e^{0t}$, $\cos(t)$, $\sin(t)$

$\therefore\ y(t) = c_1 + c_2\cos(t) + c_3\sin(t)$

$y'(t) = -c_2\sin(t) + c_3\cos(t)$

$y''(t) = -c_2\cos(t) - c_3\sin(t)$

$y(0) = 0: \quad c_1 + c_2 = 0$

$y'(0) = 1: \quad c_3 = 1$

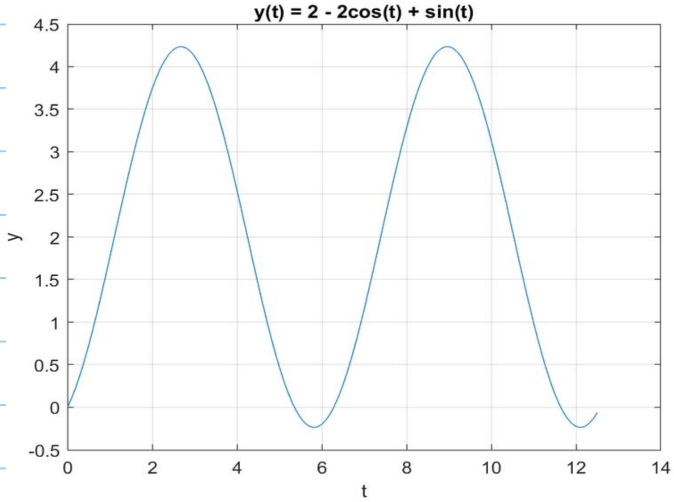
$y''(0) = 2: \quad -c_2 = 2$

$\therefore\ c_1 = 2,\ c_2 = -2,\ c_3 = 1$

$\therefore\ \underline{y(t) = 2 - 2\cos(t) + \sin(t)}$

Using MATLAB:

```
clear,clc
t = 0:0.1:4*pi;
eqn = 2 - 2*cos(t) + sin(t);
plot(t, eqn)
grid on
xlabel 't', ylabel 'y'
title 'y(t) = 2 - 2cos(t) + sin(t)'
```



As $t \to \infty$, $y(t)$ oscillates

21.

Characteristic equation: $r^4 + 1 = 0$, or $r^4 = -1$.

$-1 = \cos(\pi + 2n\pi) + i\sin(\pi + 2n\pi) = e^{i(\pi + 2n\pi)}$

$\therefore (-1)^{\frac{1}{4}} = e^{i\left(\frac{\pi}{4} + n\frac{\pi}{2}\right)}, \quad n = 0, 1, 2, 3$

$\therefore e^{i\frac{\pi}{4}}, e^{i\left(\frac{3}{4}\pi\right)}, e^{i\left(\frac{5}{4}\pi\right)}, e^{i\left(\frac{7}{4}\pi\right)}$

$\therefore r = \cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$

$r = \cos\left(\frac{3}{4}\pi\right) + i\sin\left(\frac{3}{4}\pi\right) = -\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$

$r = \cos\left(\frac{5}{4}\pi\right) + \sin\left(\frac{5}{4}\pi\right) = -\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$

$r = \cos\left(\frac{7}{4}\pi\right) + \sin\left(\frac{7}{4}\pi\right) = \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$

$\therefore$ Real parts $\Rightarrow e^{t/\sqrt{2}}, e^{-t/\sqrt{2}}$

$$\pm \frac{i}{\sqrt{2}} \Rightarrow \cos\left(\frac{t}{\sqrt{2}}\right), \sin\left(\frac{t}{\sqrt{2}}\right).$$

$$\therefore y(t) = e^{t/\sqrt{2}}\left[c_1 \cos(t/\sqrt{2}) + c_2 \sin(t/\sqrt{2})\right]$$
$$+ e^{-t/\sqrt{2}}\left[c_3 \cos(t/\sqrt{2}) + c_4 \sin(t/\sqrt{2})\right]$$

Using MATLAB:

```
clear, clc
syms t y(t) c1 c2 c3 c4
x = t/sqrt(2);
y(t) = exp(x)*(c1*cos(x) + c2*sin(x)) + ...
    exp(-x)*(c3*cos(x) + c4*sin(x));
subs(y, t, 0)
subs(diff(y,t,1), t, 0)
subs(diff(y,t,2), t, 0)
subs(diff(y,t,3), t, 0)
```

ans(t) = $c_1 + c_3$

ans(t) =

$$\frac{\sqrt{2}\,c_1}{2} + \frac{\sqrt{2}\,c_2}{2} - \frac{\sqrt{2}\,c_3}{2} + \frac{\sqrt{2}\,c_4}{2}$$

ans(t) = $c_2 - c_4$

ans(t) =

$$\frac{\sqrt{2}\,c_2}{2} - \frac{\sqrt{2}\,c_1}{2} + \frac{\sqrt{2}\,c_3}{2} + \frac{\sqrt{2}\,c_4}{2}$$

$$\therefore y(0) = 0 : \quad c_1 + c_3 = 0 \quad (0)$$

$$y'(0) = 0 : \quad \frac{c_1}{\sqrt{2}} + \frac{c_2}{\sqrt{2}} - \frac{c_3}{\sqrt{2}} + \frac{c_4}{\sqrt{2}} = 0 \quad (1)$$

$$y''(0) = -1 : \quad c_2 - c_4 = -1 \quad (2)$$

$$y'''(0) = 0 : \quad -\frac{c_1}{\sqrt{2}} + \frac{c_2}{\sqrt{2}} + \frac{c_3}{\sqrt{2}} + \frac{c_4}{\sqrt{2}} = 0 \quad (3)$$

$$(1) + (3) : \quad \sqrt{2}\, c_2 + \sqrt{2}\, c_4 = 0 \quad (4)$$

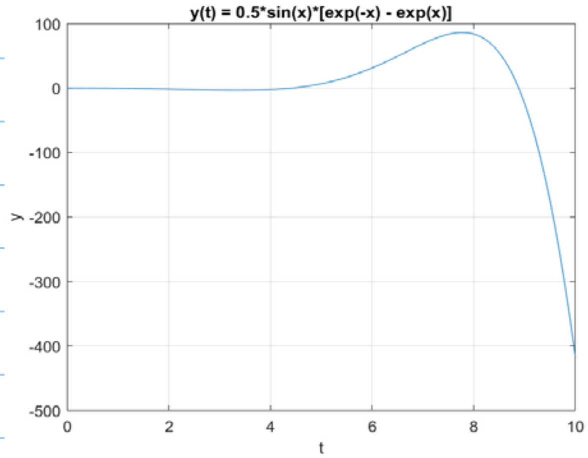
$$\sqrt{2} * (2) + (4) : \quad 2\sqrt{2}\, c_2 = -\sqrt{2} \Rightarrow \underline{c_2 = -\frac{1}{2}}$$

$$\therefore \underline{c_4 = \frac{1}{2}}$$

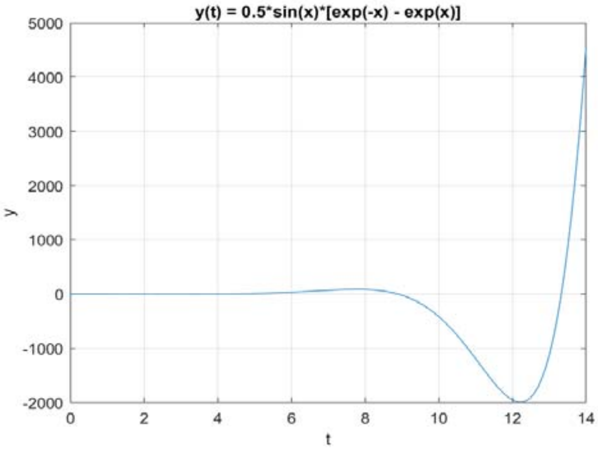
$$\therefore (1) : \frac{c_1}{\sqrt{2}} - \frac{c_3}{\sqrt{2}} = 0 \Rightarrow c_1 = c_3. \therefore (0) \Rightarrow c_1 = c_3 = 0$$

$$\therefore\ y(t) = -\frac{1}{2}e^{t/\sqrt{2}}\sin(t/\sqrt{2}) + \frac{1}{2}e^{-t/\sqrt{2}}\sin(t/\sqrt{2})$$

```
clear,clc
t = 0:0.1:10;
x = t/sqrt(2);
eqn = 0.5*sin(x).*(exp(-x) - exp(x));
plot(t, eqn)
grid on
xlabel 't', ylabel 'y'
title 'y(t) = 0.5*sin(x)*[exp(-x) - exp(x)]'
```




```
clear,clc
t = 0:0.1:14;
x = t/sqrt(2);
eqn = 0.5*sin(x).*(exp(-x) - exp(x));
plot(t, eqn)
grid on
xlabel 't', ylabel 'y'
title 'y(t) = 0.5*sin(x)*[exp(-x) - exp(x)]'
```



The term $\frac{1}{2}e^{-t/\sqrt{2}}\sin(t/\sqrt{2}) \to 0$ as $t \to \infty$

The term $-\frac{1}{2}e^{t/\sqrt{2}}\sin(t/\sqrt{2})$ oscillates as $t \to \infty$, as sin() oscillates positive/negative, and $e^{t/\sqrt{2}}$ grows unbounded.

$\therefore$ $y(t)$ oscillates in unbounded fashion as $t \to \infty$.

22.

(a) Characteristic equation: $r^4 - 4r^3 + 4r^2 = 0 = r^2(r^2 - 4r + 4)$

$\therefore r^2(r-2)^2 = 0, \quad r = 0, 0, 2, 2$

$\therefore e^{0t}, te^{0t}, e^{2t}, te^{2t}$

$\therefore y(t) = c_1 + c_2 t + c_3 e^{2t} + c_4 t e^{2t}$

$y'(t) = c_2 + 2c_3 e^{2t} + (c_4 + 2c_4 t)e^{2t}$

$y''(t) = 4c_3 e^{2t} + (4c_4 + 4c_4 t)e^{2t}$

$y'''(t) = 8c_3 e^{2t} + (12c_4 + 8c_4 t)e^{2t}$

$y(1) = c_1 + c_2 + c_3 e^2 + c_4 e^2 = -1 \quad (0)$

$y'(1) = c_2 + 2c_3 e^2 + 3c_4 e^2 = 2 \quad (1)$

$y''(1) = 4c_3 e^2 + 8c_4 e^2 = 0 \quad (2)$

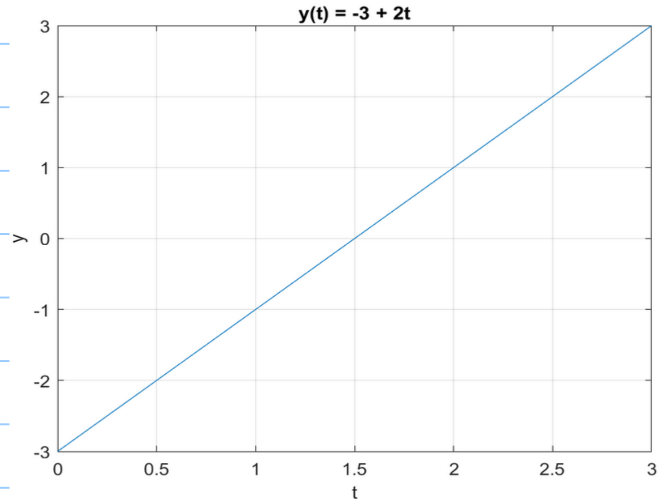
$y'''(1) = 8c_3 e^2 + 20c_4 e^2 = 0 \quad (3)$

$(2), (3) \Rightarrow c_3 = c_4 = 0 \quad \therefore c_2 = 2, \; c_1 = -3$

$\therefore \underline{y(t) = -3 + 2t}$

(b) Using MATLAB,

```
clear,clc;
t = 0:0.1:3;
eqn = -3 + 2*t;
plot(t, eqn)
grid on
xlabel 't', ylabel 'y'
title 'y(t) = -3 + 2t'
```



(c) As $t \to \infty$, $y(t) \to \infty$.

23.

(a) Characteristic equation: $2r^4 - r^3 - 9r^2 + 4r + 4 = 0$

From MATLAB,

```
clear, clc
syms r
factor(2*r^4 - r^3 -9*r^2 + 4*r + 4)
```

ans = $(r-1 \quad r-2 \quad 2r+1 \quad r+2)$

$\therefore r = 1, 2, -\frac{1}{2}, -2$. $\therefore e^{t}, e^{2t}, e^{-t/2}, e^{-2t}$

$\therefore y(t) = C_1 e^{t} + C_2 e^{2t} + C_3 e^{-t/2} + C_4 e^{-2t}$

$$y(0) = C_1 + C_2 + C_3 + C_4 = -2$$
$$y'(0) = C_1 + 2C_2 - C_3/2 - 2C_4 = 0$$
$$y''(0) = C_1 + 4C_2 + C_3/4 + 4C_4 = -2$$
$$y'''(0) = C_1 + 8C_2 - C_3/8 - 8C_4 = 0$$

Using MATLAB,

```
clear, clc
A = [1 1    1    1;...
     1 2 -1/2 -2;...
     1 4  1/4  4;...
     1 8 -1/8 -8];
B = [-2 0 -2 0]';
rats(A\B)     %From AX = B
```

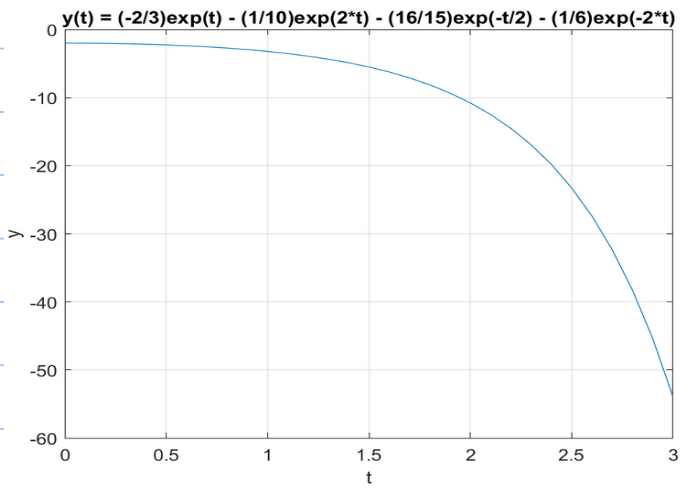
```
ans = 4×14 char array
    '     -2/3     '
    '     -1/10    '
    '    -16/15    '
    '     -1/6     '
```

$\therefore\ c_1 = -\frac{2}{3},\quad c_2 = -\frac{1}{10},\quad c_3 = -\frac{16}{15},\quad c_4 = -\frac{1}{6}$

$$\therefore\ y(t) = -\frac{2}{3}e^{t} - \frac{1}{10}e^{2t} - \frac{16}{15}e^{-t/2} - \frac{1}{6}e^{-2t}$$

(b) Using MATLAB,

```
clear,clc;
t = 0:0.1:3;
c1 = -2/3; c2 = -1/10; c3 = -16/15; c4 = -1/6;
eqn = c1*exp(t) + c2*exp(2*t) + c3*exp(-t/2) + c4*exp(-2*t);
plot(t, eqn)
grid on
xlabel 't', ylabel 'y'
title 'y(t) = (-2/3)exp(t) - (1/10)exp(2*t) - (16/15)exp(-t/2) - (1/6)exp(-2*t)'
```



(c) Dominant term is $-\frac{1}{10}e^{2t}$. $\therefore$ as $t \to \infty$, $y(t) \to -\infty$

24.

(a) Characteristic equation: $4r^3 + r + 5 = 0$

Using MATLAB,

```
clear, clc
syms r
factor(4*r^3 + r + 5)
p = [4 0 1 5];
rats(roots(p))
```

ans = $\left(r+1 \quad 4\,r^2-4\,r+5\right)$

ans = *3×29 char array*

```
'      1/2      +      1i      '
'      1/2      -      1i      '
'      -1       +      0i      '
```

$\therefore\ e^{-t},\ e^{t/2}\cos(t),\ e^{t/2}\sin(t)$

$\therefore\ y(t) = c_1 e^{-t} + c_2 e^{t/2}\cos(t) + c_3 e^{t/2}\sin(t)$

Using MATLAB,

```
clear, clc
syms t y(t) c1 c2 c3
y(t) = c1*exp(-t) + c2*exp(t/2)*cos(t) + c3*exp(t/2)*sin(t);
subs(y, t, 0)
subs(diff(y,t,1), t, 0)
subs(diff(y,t,2), t, 0)
```

ans(t) = $c_1 + c_2$

ans(t) =

$$\frac{c_2}{2} - c_1 + c_3$$

ans(t) =

$$c_1 - \frac{3\,c_2}{4} + c_3$$

And from equations in c_1, c_2, c_3,

```
A = [ 1     1    0;...
     -1   1/2    1;...
      1  -3/4    1];
B = [2 1 -1]';
rats(A\B)    %Solve AX = B
```

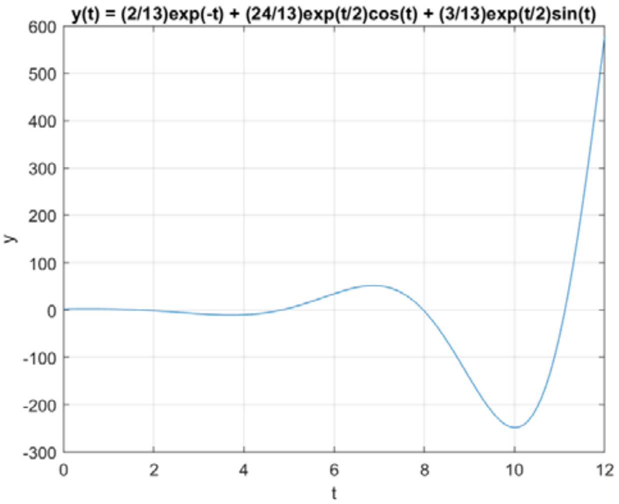
ans = *3×14 char array*

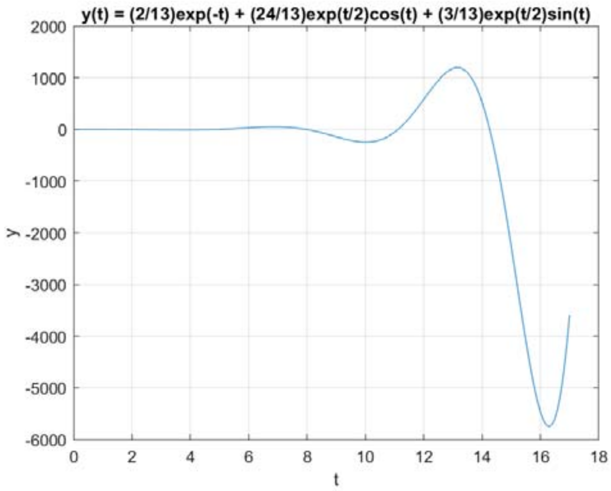
```
'     2/13     '
'    24/13     '
'     3/13     '
```

$$\therefore\ y(t) = \frac{2}{13} e^{-t} + \frac{24}{13} e^{t/2}\cos(t) + \frac{3}{13} e^{t/2}\sin(t)$$

(b)

```
clear,clc;
t = 0:0.1:12;
c1 = 2/13; c2 = 24/13; c3 = 3/13;
eqn = c1*exp(-t) + c2*exp(t/2).*cos(t) + c3*exp(t/2).*sin(t);
plot(t, eqn)
grid on
xlabel 't', ylabel 'y'
title 'y(t) = (2/13)exp(-t) + (24/13)exp(t/2)cos(t) + (3/13)exp(t/2)sin(t)'
```





$t_{max} = 12$ $\qquad$ $t_{max} = 17$

(c) As $t \to \infty$, $y(t)$ oscillates in unbounded fashion.

25.

(a) Characteristic equation: $6r^3 + 5r^2 + r = 0$

$\therefore r(6r^2 + 5r + 1) = r(3r+1)(2r+1) = 0, \; r = 0, -\frac{1}{3}, -\frac{1}{2}$

$\therefore e^{0t}, e^{-t/3}, e^{-t/2}$

$\therefore y(t) = c_1 + c_2 e^{-t/3} + c_3 e^{-t/2}$

$$y(0) = c_1 + c_2 + c_3 = -2$$

$$y'(0) = -c_2/3 - c_3/2 = 2$$

$$y''(0) = c_2/9 + c_3/4 = 0$$

Using MATLAB,

```
clear, clc
A = [ 1    1     1;...
      0  -1/3  -1/2;...
      0   1/9   1/4];
B = [-2 2 0]';
rats(A\B)   %Solve AX = B
```

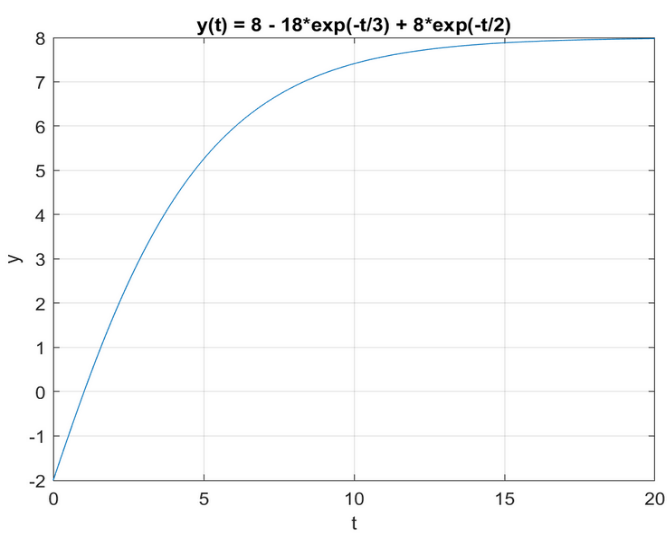
```
ans = 3×14 char array
    '      8       '
    '     -18      '
    '      8       '
```

$\therefore$ $c_1 = 8$, $c_2 = -18$, $c_3 = 8$

$$\therefore \; \underline{y(t) = 8 - 18e^{-t/3} + 8e^{-t/2}}$$

(3)

```
clear,clc;
t = 0:0.1:20;
c1 = 8; c2 = -18; c3 = 8;
eqn = c1 + c2*exp(-t/3) + c3*exp(-t/2);
plot(t, eqn)
grid on
xlabel 't', ylabel 'y'
title 'y(t) = 8 - 18*exp(-t/3) + 8*exp(-t/2)'
```



(c) $\lim\limits_{t\to\infty}\left(8-18e^{-t/3}+8e^{-t/2}\right)=8-0+0=\underline{8}$

26.

(a) $y'(t)=-3e^{-t}-\frac{1}{2}\sin(t)-\cos(t)$ $\quad y'(0)=-3-0-1=\underline{-4}$ ✓

$y''(t)=3e^{-t}-\frac{1}{2}\cos(t)+\sin(t)$ $\quad y''(0)=3-\frac{1}{2}+0=\underline{\frac{5}{2}}$ ✓

$y'''(t)=-3e^{-t}+\frac{1}{2}\sin(t)+\cos(t)$ $\quad y'''(0)=-3+0+1=\underline{-2}$ ✓

$y^{(4)}(t)=3e^{-t}+\frac{1}{2}\cos(t)-\sin(t)$

$\therefore\ y^{(4)}(t)=y(t)\implies \underline{y^{(4)}-y=0}$ ✓

and $y(0)=3+\frac{1}{2}-0=\underline{\frac{7}{2}}$ ✓

(b)

From $r^4-1=0$, $r=1,-1,i,-i.$

$\therefore\ y(t)=c_1e^{t}+c_2e^{-t}+c_3\cos(t)+c_4\sin(t)$

$$\begin{aligned}
y(0)&=c_1+c_2+c_3 &&=\tfrac{7}{2}\\
y'(0)&=c_1-c_2+c_4 &&=-4\\
y''(0)&=c_1+c_2-c_3 &&=5/2\\
y'''(0)&=c_1-c_2-c_4 &&=-15/8
\end{aligned}$$

Using MATLAB,

```
clear, clc
A = [ 1     1     1     0;...
      1    -1     0     1;...
      1     1    -1     0;...
      1    -1     0    -1];
B = [7/2  -4    5/2  -15/8]';
rats(A\B)    %Solve AX = B
```

```
ans = 4×14 char array
    '     1/32     '
    '    95/32     '
    '     1/2      '
    '   -17/16     '
```

$$\therefore\ y(t) = \frac{1}{32}e^{t} + \frac{95}{32}e^{-t} + \frac{1}{2}\cos(t) - \frac{17}{16}\sin(t)$$

27.

(a) For, $y(t) = \cos(t)$, $y^{(4)}(t) = \cos(t)$ $\therefore y^{(4)} - y = 0$

$y(t) = \sin(t)$, $y^{(4)}(t) = \sin(t)$ $\therefore y^{(4)} - y = 0$

For $y(t) = \cosh(t)$, $y' = \sinh(t)$, $y'' = \cosh(t)$, $y''' = \sinh(t)$, $y^{(4)} = \cosh(t)$

$\therefore y^{(4)} - y = 0$

For $y(t) = \sinh(t)$, $y' = \cosh(t)$, $y'' = \sinh(t)$, $y''' = \cosh(t)$, $y^{(4)} = \sinh(t)$

$\therefore y^{(4)} - y = 0$.

$\therefore$ For $y(t) = c_1\cos(t) + c_2\sin(t) + c_3\cosh(t) + c_4\sinh(t)$,

$y^{(4)} - y = 0$

For the Wronskian,

$$W[\cos, \sin, \cosh, \sinh](t) =$$

$$\begin{vmatrix} \cos & \sin & \cosh & \sinh \\ -\sin & \cos & \sinh & \cosh \\ -\cos & -\sin & \cosh & \sinh \\ \sin & -\cos & \sinh & \cosh \end{vmatrix}$$ add row 3 to row 1

$$= \begin{vmatrix} 0 & 0 & 2\cosh & 2\sinh \\ -\sin & \cos & \sinh & \cosh \\ -\cos & -\sin & \cosh & \sinh \\ \sin & -\cos & \sinh & \cosh \end{vmatrix}$$ add row 4 to row 2

$$= \begin{vmatrix} 0 & 0 & 2\cosh & 2\sinh \\ 0 & 0 & 2\sinh & 2\cosh \\ -\cos & -\sin & \cosh & \sinh \\ \sin & -\cos & \sinh & \cosh \end{vmatrix}$$ add $-\frac{1}{2} \times$ row 1 to row 3; add $-\frac{1}{2} \times$ row 2 to row 4

$$= \begin{vmatrix} 0 & 0 & 2\cosh & 2\sinh \\ 0 & 0 & 2\sinh & 2\cosh \\ -\cos & -\sin & 0 & 0 \\ \sin & -\cos & 0 & 0 \end{vmatrix}$$

$$= 2\cosh\left[0 - 0 + 2\cosh(\cos^2 + \sin^2)\right]$$

$$- 2\sinh\left[0 - 0 + 2\sinh(\cos^2 + \sin^2)\right]$$

$$= 4\cosh^2(t) - 4\sinh^2(t) = 4 \neq 0$$

$\therefore$ By Theorem 4.1.2,

$$y(t) = c_1 \cos(t) + c_2 \sin(t) + c_3 \cosh(t) + c_4 \sinh(t)$$

is a general solution to $y^{(4)} - y = 0$.

(b)
$$\begin{aligned}
y(0) &= c_1 + c_3 = 0 \quad (0)\\
y'(0) &= c_2 + c_4 = 0 \quad (1)\\
y''(0) &= -c_1 + c_3 = 1 \quad (2)\\
y'''(0) &= -c_2 + c_4 = 1 \quad (3)
\end{aligned}$$

$(0) + (2) = 2c_3 = 1$, $c_3 = \frac{1}{2}$, $\therefore c_1 = -\frac{1}{2}$
$(1) + (3) = 2c_4 = 1$, $c_4 = \frac{1}{2}$, $\therefore c_2 = -\frac{1}{2}$

$$\therefore \underline{y(t) = -\frac{1}{2}\cos(t) - \frac{1}{2}\sin(t) + \frac{1}{2}\cosh(t) + \frac{1}{2}\sinh(t)}$$

(c) Convenience is in the eye of the beholder.

I suppose since $\cos(0) = \cosh(0) = 1$ and $\sin(0) = \sinh(0) = 0$, it is easier to determine the constants c_i, compared to using e^t, e^{-t}.

28.

(a)

In this case, $p_1(t)$, the coefficient for $y'''(t)$, is 0. $\therefore W[y_1, y_2, y_3, y_4](t) = c\exp\left(-\int p_1(t)\,dt\right)$

$= c\exp(0) = c$

$\therefore$ The Wronskian is a constant

(b)

$$W = \begin{vmatrix} e^t & e^{-t} & \cos(t) & \sin(t) \\ e^t & -e^{-t} & -\sin(t) & \cos(t) \\ e^t & e^{-t} & -\cos(t) & -\sin(t) \\ e^t & -e^{-t} & \sin(t) & -\cos(t) \end{vmatrix}$$

Since $e^t, e^{-t}, \cos(t), \sin(t)$ all satisfy $y^{(4)} - y = 0$, the Wronskian doesn't depend on t, as shown in (a).

$\therefore$ Choose $t=0$ to simplify: $W = \begin{vmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & -1 \end{vmatrix}$ add row 3 to row 1 add row 4 to row 2

$$= \begin{vmatrix} 2 & 2 & 0 & 0 \\ 2 & -2 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & -1 \end{vmatrix}$$

add $-\frac{1}{2} \times$ row 1 to row 3
add $-\frac{1}{2} \times$ row 2 to row 4

$$= \begin{vmatrix} 2 & 2 & 0 & 0 \\ 2 & -2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix} = 2\left[-2(1-0) - 0 + 0\right] - 2\left[2(1-0) - 0 + 0\right.$$

$$= -8$$

$$\therefore \underline{W = -8}$$

(c)

This was done in 27(a). $\underline{W = 4}$

One could again evaluate at $t = 0$ to simplify.

$$\therefore W[\cosh(0), \sinh(0), \cos(0), \sin(0)] =$$

$$\begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{vmatrix}$$

add row 3 to row 1

add row 4 to row 2

$$= \begin{vmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{vmatrix} = 2\left[2(1-0) - 0 + 0\right] - 0 + 0 - 0$$

$$= \underline{4}$$

29.

(a)

Let L_1, L_2 be the equilibrium positions for m_1 and m_2, respectively.

Then $m_2 g - K_2 L_2 = 0$, or $m_2 g = K_2 L_2$ [a]

and $(m_1 + m_2) g - K_1 L_1 = 0$

which also yields $m_1 g - K_1 L_1 + K_2 L_2 = 0$ [b]

At equilibrium, distance between m_1 and m_2 is L_2.

From Newton's 2nd law, for m_2:

$m_2 u_2'' = m_2 g - K_2 [L_2 + (u_2 - u_1)]$ [1]

where $u_2 - u_1$ is the additional stretch of spring K_2 beyond L_2. Thus, just two forces act on m_2: gravity and spring K_2

For m_1, 3 forces act: gravity (downward) $= m_1 g$,

spring K_1 (upwards) $= -K_1(L_1 + u_1)$, and

spring K_2 (downwards) $= K_2[L_2 + (u_2 - u_1)]$

$$\therefore \quad m_1 u_1'' = m_1 g - K_1(L_1 + u_1) + K_2[L_2 + (u_2 - u_1)] \qquad [2]$$

[1] can be simplified as,

$$m_2 u_2'' = -K_2(u_2 - u_1) = -K_2 u_2 + K_2 u_1 \qquad [1']$$

since $m_2 g = K_2 L_2$ from [a]

or $$m_2 u_2'' + K_2 u_2 = K_2 u_1 \qquad [1'']$$

[2] can be simplified as

$$m_1 u_1'' = -K_1 u_1 + K_2 u_2 - K_2 u_1 \qquad [2']$$

as $m_1 g - K_1 L_1 + K_2 L_2 = 0$ from [b]

or, $$m_1 u_1'' + (K_1 + K_2) u_1 = K_2 u_2 \qquad [2'']$$

Using $m_1 = 1$, $m_2 = 1$, $K_1 = 3$, $K_2 = 2$, [1''], [2''] become:

$$\underline{u_2'' + 2u_2 = 2u_1} \quad \text{and} \quad \underline{u_1'' + 5u_1 = 2u_2}$$

(b)

(1) From $u_1'' + 5u_1 = 2u_2$, $u_2 = \frac{1}{2}u_1'' + \frac{5}{2}u_1$

$\therefore \left(\frac{1}{2}u_1'' + \frac{5}{2}u_1\right)'' + 2\left(\frac{1}{2}u_1'' + \frac{5}{2}u_1\right) = 2u_1$

or, $\frac{1}{2}u_1^{(4)} + \frac{5}{2}u_1'' + u_1'' + 5u_1 = 2u_1$

Or, $\underline{u_1^{(4)} + 7u_1'' + 6u_1 = 0}$

(2) Characteristic equation: $r^4 + 7r^2 + 6 = 0$

or $(r^2+1)(r^2+6) = 0$, $r = i, -i, i\sqrt{6}, -i\sqrt{6}$

$\therefore \underline{u_1(t) = c_1\cos(t) + c_2\sin(t) + c_3\cos(\sqrt{6}t) + c_4\sin(\sqrt{6}t)}$

(c)

(1) From $u_1'' + 5u_1 = 2u_2$, $u_1''(0) + 5u_1(0) = 2u_2(0)$

$\therefore u_1''(0) + 5(1) = 2(2) \Rightarrow \underline{u_1''(0) = -1}$

Also, $u_1''' + 5u_1' = 2u_2'$, so $u_1'''(0) + 5u_1'(0) = 2u_2'(0)$

$\therefore u_1'''(0) + 5(0) = 2(0) \Rightarrow \underline{u_1'''(0) = 0}$

$\therefore \underline{u_1''(0) = -1 \ , \ u_1'''(0) = 0}$

(2) $\therefore u_1(0) = c_1 + c_3 = 1$ (0)

$u_1'(0) = c_2 + \sqrt{6}c_4 = 0$ (1)

$u_1''(0) = -c_1 - 6c_3 = -1$ (2)

$u_1'''(0) = -c_2 - 6\sqrt{6}c_4 = 0$ (3)

(0) + (2) $\Rightarrow -5c_3 = 0 \Rightarrow c_3 = 0, \ c_1 = 1$

(1) + (3) $\Rightarrow (\sqrt{6} - 6\sqrt{6})c_4 = 0 \Rightarrow c_4 = 0, \ c_2 = 0$

$\therefore \underline{u_1(t) = \cos(t)}$

(3) From $u_2'' + 2u_2 = 2u_1$,

$u_2''(t) + 2u_2(t) = 2\cos(t), \ u_2(0) = 2, \ u_2'(0) = 0$

Homogeneous solution: $r^2 + 2 = 0 \Rightarrow r = \pm\sqrt{2}i$

$\therefore u_c(t) = c_1\cos(\sqrt{2}t) + c_2\sin(\sqrt{2}t)$

Particular: Let $u_p(t) = A\cos(t) + B\sin(t)$

$$\therefore u_p''(t) = -A\cos(t) - B\sin(t)$$

$$\therefore \left[-A\cos(t) - B\sin(t)\right] + 2\left[A\cos(t) + B\sin(t)\right]$$

$$= A\cos(t) + B\sin(t) = 2\cos(t)$$

$$\therefore A = 2, \ B = 0$$

$$\therefore u_p(t) = 2\cos(t)$$

$$\therefore u_2(t) = c_1\cos(\sqrt{2}t) + c_2\sin(\sqrt{2}t) + 2\cos(t)$$

$$u_2(0) = c_1 + 2 = 2 \Rightarrow c_1 = 0$$

$$\therefore u_2'(t) = \sqrt{2}c_2\cos(\sqrt{2}t) - 2\sin(t)$$

$$\therefore u_2'(0) = \sqrt{2}c_2 = 0 \Rightarrow c_2 = 0$$

$$\therefore \underline{\underline{u_2(t) = 2\cos(t)}}$$

(d)

We have: $u_1'' + 5u_1 = 2u_2$ [1]

$u_2'' + 2u_2 = 2u_1$ [2]

$u_1^{(4)} + 7u_1'' + 6u_1 = 0$ [3]

$$u_1(t) = c_1\cos(t) + c_2\sin(t) + c_3\cos(\sqrt{6}t) + c_4\sin(\sqrt{6}t) \quad [4]$$

where [4] comes from [3]

From [1], $u_1''(0) + 5(-2) = 2(1) \quad \therefore \underline{u_1''(0) = 12}$

$$u_1'''(t) + 5u_1'(t) = 2u_2'(t)$$

$$\therefore u_1'''(0) + 5(0) = 2(0) \quad \therefore \underline{u_1'''(0) = 0}$$

$$\therefore u_1(0) = -2,\ u_1'(0) = 0,\ u_1''(0) = 12,\ u_1'''(0) = 0 \quad [5]$$

From [4], [5]:

$$u_1(0) = c_1 + c_3 = -2 \quad (0)$$

$$u_1'(0) = c_2 \quad \sqrt{6}\,c_4 = 0 \quad (1)$$

$$u_1''(0) = -c_1 - 6c_3 = 12 \quad (2)$$

$$u_1'''(0) = -c_2 - 6\sqrt{6}\,c_4 = 0 \quad (3)$$

$$(0) + (2) \Rightarrow -5c_3 = 10,\quad c_3 = -2,\quad c_1 = 0$$

$$(1) + (3) \Rightarrow (\sqrt{6} - 6\sqrt{6})c_4 = 0,\quad c_4 = 0,\quad c_2 = 0$$

$$\therefore \underline{\underline{u_1(t) = -2\cos(\sqrt{6}t)}}$$

From [2], $u_2''(t) + 2u_2(t) = -4\cos(\sqrt{6}t)$

From (c) above, homogeneous solution is:

$$u_c(t) = c_1 \cos(\sqrt{2}t) + c_2 \sin(\sqrt{2}t)$$

For the particular solution, let

$$u_p(t) = A\cos(\sqrt{6}t) + B\sin(\sqrt{6}t)$$

$$\therefore u_p''(t) = -6A\cos(\sqrt{6}t) - 6B\sin(\sqrt{6}t)$$

$$\therefore u_2'' + 2u_2 = -4A\cos(\sqrt{6}t) - 4B\sin(\sqrt{6}t) = -4\cos(\sqrt{6}t)$$

$$\therefore A = 1,\ B = 0$$

$$\therefore u_2(t) = c_1\cos(\sqrt{2}t) + c_2\sin(\sqrt{2}t) + \cos(\sqrt{6}t)$$

From $u_2(0) = 1$, $c_1 + 0 + 1 = 1$, $c_1 = 0$

$u_2'(0) = 0$, $0 + \sqrt{2}c_2 + 0 = 0$, $c_2 = 0$

$$\therefore \underline{u_2(t)} = \cos(\sqrt{6}t)$$

$$\therefore \underline{u_1(t) = -2\cos(\sqrt{6}t)},\quad \underline{u_2(t) = \cos(\sqrt{6}t)}$$

30.

(a)

$$c_1 e^{r_1 t}\left(e^{-r_1 t}\right) + \dots + c_n e^{r_n t}\left(e^{-r_1 t}\right) =$$

$$c_1 + c_2 e^{(r_2 - r_1)t} + \dots + c_n e^{(r_n - r_1)t}$$

$$\frac{d}{dt}\left[c_1 + c_2 e^{(r_2-r_1)t} + \dots + c_n e^{(r_n - r_1)t}\right]$$

$$= c_2(r_2 - r_1)e^{(r_2-r_1)t} + \dots + c_n(r_n - r_1)e^{(r_n - r_1)t} \quad [a]$$

(b)

Multiplying [a] by $e^{-(r_2-r_1)t}$,

$$c_2(r_2-r_1)e^{(r_2-r_1)t}\left(e^{-(r_2-r_1)t}\right) + c_3(r_3-r_1)e^{(r_3-r_1)t}\left(e^{-(r_2-r_1)t}\right)$$

$$+ \dots + c_n(r_n - r_1)e^{(r_n-r_1)t}\left(e^{-(r_2-r_1)t}\right) = 0$$

$$= c_2(r_2-r_1)e^{(r_2-r_1)t-(r_2-r_1)t} \textcolor{red}{= e^0} + c_3(r_3-r_1)e^{(r_3-r_1)t-(r_2-r_1)t}$$
$$+ \dots + c_n(r_n-r_1)e^{(r_n-r_1)t-(r_2-r_1)t}$$

$$= c_2(r_2-r_1) + c_3(r_3-r_1)e^{(r_3-r_2)t} + \dots + c_n(r_n-r_1)e^{(r_n-r_2)t} = 0$$

Taking $\frac{d}{dt}$ of the last equation,

$$c_3(r_3-r_2)(r_3-r_1)e^{(r_3-r_2)t} + c_4(r_4-r_2)(r_4-r_1)e^{(r_4-r_2)t}$$
$$+ \dots + c_n(r_n-r_2)(r_n-r_1)e^{(r_n-r_2)t} = 0$$

(c)

The Kth iteration would multiply the last equation by the inverse of the exponential in the first term. So the third iteration multiplies by $e^{-(r_3-r_2)t}$, leaving a constant for the first term, which disappears after differentiation. After $n-1$ iterations, you are left with just one term with an exponential component:

$$c_n(r_n - r_{n-1})\cdots(r_n - r_2)(r_n - r_1)e^{(r_n - r_{n-1})t} = 0$$

Since $r_i \neq r_j$ for $i \neq j$, you can divide by $(r_n - r_{n-1})\cdots(r_n - 1)$ and by $e^{(r_n - r_{n-1})t}$ to get

$\underline{c_n = 0}$ and $\therefore$ $c_1 e^{r_1 t} + \ldots + c_{n-1} e^{r_{n-1} t} = 0$

(d)

First multiply by $e^{-r_1 t}$ to get:

$$c_1 + c_2 e^{(r_2 - r_1)t} + \ldots + c_n e^{(r_{n-1} - r_1)t} = 0$$

Differentiate to get:

$$c_2(r_2 - r_1)e^{(r_2 - r_1)t} + \ldots + c_n(r_n - r_1)e^{(r_{n-1} - r_1)t} = 0$$

For the "kth" iteration, up to $k = n-2$, multiply by $e^{-(r_k - r_{k-1})t}$, then differentiate, to finally get:

$$c_{n-1}(r_{n-1}-r_{n-2})\cdots(r_{n-1}-r_1)\,e^{(r_{n-1}-r_{n-2})t}=0$$

As $r_i \neq r_j$ for $i \neq j$, divide by

$(r_{n-1}-r_{n-2})\cdots(r_{n-1}-r_1)$ and $e^{(r_{n-1}-r_{n-2})t}$

to get $\underline{\underline{c_{n-1}=0}}$.

31.

(a)

For the term $(r-r_1)^s$, for $1 \leq k \leq s-1$, the

First derivative is: $s(r-r_1)^{s-1}$
2nd derivative is: $s(s-1)(r-r_1)^{s-2}$

$\vdots$

kth derivative is: $s(s-1)\cdots(s-(k-1))(r-r_1)^{s-k}$

$\vdots$

$s-1$ th derivative is: $s(s-1)\cdots(3)(2)(r-r_1)$

Thus, for $1 \le k \le s-1$, the kth derivative of $(r-r_1)^s$ evaluated at $r=r_1$ is zero.

Using the derivative of a product,

$$z'(r) = g(r)\frac{d}{dr}(r-r_1)^s + (r-r_1)^s g'(r)$$

$$z''(r) = g(r)\frac{d^2}{dr^2}(r-r_1)^s + \frac{d}{dr}(r-r_1)^s g''(r) + \frac{d}{dr}\left[(r-r_1)^s g'(r)\right]$$

$= g(r)\dfrac{d^2}{dr^2}(r-r_1)^s +$ other terms each of which has a derivative of $(r-r_1)^s$ of order less than 2.

$\vdots$

$z^{(k)}(r) = g(r)\dfrac{d^k}{dr^k}(r-r_1)^s +$ other terms each of which has a derivative of $(r-r_1)^s$ of order less than k.

$\vdots$

$z^{(s-1)}(r) = g(r)\dfrac{d^{s-1}}{dr^{s-1}}(r-r_1)^s +$ other terms each of which has a derivative of $(r-r_1)^s$ of order less than $(s-1)$

$= g(r)(s)(s-1)\cdots(3)(2)(r-r_1) +$ other terms each of which has a factor of $(r-r_1)$ of order ≥ 2

$\therefore$ For $1 \le K \le s-1$, $\underline{\underline{z^{(K)}(r_1) = 0}}$ as each term of $z^{(K)}(r)$ has a factor of $(r-r_1)$

$\therefore$ $z^{(s)}(r) = g(r)(s)(s-1)\cdots(3)(2)$ + other terms each of which has a factor of $(r-r_1)$ of order ≥ 1

$\therefore$ $z^{(s)}(r_1) = g(r_1)(s)(s-1)\cdots(3)(2) + 0$

and since $g(r_1) \neq 0$ by assumption,

$\underline{\underline{z^{(s)}(r_1) \neq 0}}$.

(b)

Note the function $f(r,t) = e^{rt}$ is C^{∞} (i.e., continuosly differentiable innumerable times).

$\therefore$ Mixed partial derivative are equal.

For example: $f_{rtt} = f_{trt} = f_{ttr}$

This is the justification for

$$\frac{\partial}{\partial r} L[e^{rt}] = L\left[\frac{\partial}{\partial r} e^{rt}\right].$$

Since $\frac{\partial}{\partial r} e^{rt} = t e^{rt}$, $L\left[\frac{\partial}{\partial r} e^{rt}\right] = L\left[t e^{rt}\right]$

$$\therefore \frac{\partial}{\partial r} L[e^{rt}] = L\left[\frac{\partial}{\partial r} e^{rt}\right] = L\left[t e^{rt}\right]$$

Also, $\frac{\partial^{s-1}}{\partial r^{s-1}} L[e^{rt}] = L\left[\frac{\partial^{s-1}}{\partial r^{s-1}} e^{rt}\right]$

and $\frac{\partial^{s-1}}{\partial r^{s-1}} e^{rt} = t^{s-1} e^{rt}$

$$\therefore \underline{\frac{\partial^{s-1}}{\partial r^{s-1}} L[e^{rt}] = L\left[t^{s-1} e^{rt}\right]}$$

(c)

Let $1 \leq k \leq s-1$.

$$\therefore L\left[t^k e^{rt}\right] = \frac{\partial^k}{\partial r^k} L[e^{rt}] = \frac{\partial^k}{\partial r^k}\left[e^{rt} Z(r)\right]$$

$$= \frac{\partial^k}{\partial r^k}\left[e^{rt}(r-r_1)^s g(r)\right]$$

Since $k < s$, every term has a factor of

$(r-r_1)$ of order at least $s-k$.

$\therefore L[t^k e^{r_1 t}] = 0$ since every term has $(r_1 - r_1)$ as a factor.

4.3 The Method of Undetermined Coefficients

Note Title 1/21/2019

1. **1.** $y''' - y'' - y' + y = 2e^{-t} + 3$

(a) Homogeneous: $r^3 - r^2 - 1 + 1 = 0$

Using MATLAB

```
clear, clc
p = [1, -1, -1, 1];
rats(roots(p))
```

```
ans = 3×14 char array
    '      -1      '
    '       1      '
    '       1      '
```

$\therefore y_c = c_1 e^{-t} + c_2 e^{t} + c_3 t e^{t}$

(b) Particular: $y_p = 3$ is one obvious solution

For $2e^{-t}$, since e^{-t} is a homogeneous solution,

let $y = (At)e^{-t}$, using MATLAB,

```
clear,clc
syms t A
c3 = 1; c2 = -1; c1 = -1; c0 = 1; %coeffs of diff eq
y = (A*t)*exp(-t); %attempt
c3*diff(y,t,3) + c2*diff(y,t,2) + c1*diff(y,t,1) + c0*y
```

ans = $4\,A\,e^{-t}$

$\therefore 4Ae^{-t} = 2e^{-t}, \quad A = \frac{1}{2}$

$\therefore y_p = \frac{1}{2} t e^{-t} + 3$

$\therefore \underline{y(t) = 3 + \frac{1}{2} t e^{-t} + c_1 e^{-t} + c_2 e^{t} + c_3 t e^{t}}$

2.

(a) Homogeneous: $r^4 - 1 = 0 = (r^2+1)(r^2-1)$, $r = \pm 1, \pm i$

$\therefore\ y_c = c_1 e^{t} + c_2 e^{-t} + c_3 \cos(t) + c_4 \sin(t)$

(b) Particular: Let $y_p = (At+B) + Ct\cos(t) + Dt\sin(t)$

Using MATLAB,

```
clear,clc
syms t A B C D
%coeffs of diff eq
c4 = 1; c3 = 0; c2 = 0; c1 = 0; c0 = -1;
%attempt
y = (A*t + B) + C*t*cos(t) + D*t*sin(t);
P = c4*diff(y,t,4) + c3*diff(y,t,3) + c2*diff(y,t,2) + c1*diff(y,t,1) + c0*y;
collect(P, [cos(t) sin(t)])
```

ans = $(-4\,D)\cos(t) + (4\,C)\sin(t) - B - A\,t$

$\therefore\ -At - B = 3t$, $A = -3$, $B = 0$

$-4D = 1$, $D = -\frac{1}{4}$ $\quad 4C = 0$, $C = 0$

$\therefore\ y_p = -3t - \frac{1}{4} t \sin(t)$

$\therefore\ \underline{y(t) = c_1 e^{t} + c_2 e^{-t} + c_3\cos(t) + c_4 \sin(t) - 3t - \frac{1}{4} t\sin(t)}$

3.

(a) Homogeneous: $r^3 + r^2 + r + 1 = 0$

Using MATLAB,

```
clear, clc
p = [1, 1, 1, 1];
rats(roots(p))
```

```
ans = 3×29 char array
    '      -1       +       0i     '
    '       0       +       1i     '
    '       0       -       1i     '
```

$$\therefore y_c = c_1 e^{-t} + c_2 \cos(t) + c_3 \sin(t)$$

(b) Particular: $y = (At)e^{-t} + (Bt + C)$

```
clear,clc
syms t A B C
%coeffs of diff eq
c3 = 1; c2 = 1; c1 = 1; c0 = 1;
%attempt
y = (A*t)*exp(-t) + (B*t + C);
P = c3*diff(y,t,3) + c2*diff(y,t,2) + c1*diff(y,t,1) + c0*y
```

P = $B + C + B\,t + 2\,A\,e^{-t}$

$$\therefore Bt = 4t \Rightarrow B = 4, \quad B + C = 0 \Rightarrow C = -4$$

$$2Ae^{-t} = e^{-t} \Rightarrow A = \frac{1}{2}$$

$$\therefore y_p = \frac{1}{2} t e^{-t} + 4t - 4$$

$$\therefore \underline{y(t) = c_1 e^{-t} + c_2 \cos(t) + c_3 \sin(t) + \frac{1}{2} t e^{-t} + 4t - 4}$$

4.

(a) Homogeneous: $r^4 - 4r^2 = r^2(r^2 - 4) = 0, \quad r = 0, 0, 2, -2$

$$\therefore y_c = c_1 + c_2 t + c_3 e^{2t} + c_4 e^{-2t}$$

(b) Particular: $y = At^4 + Bt^3 + Ct^2 + De^t$

Using MATLAB,

```
clear,clc
syms t A B C D
%coeffs of diff eq
c4 = 1; c3 = 0; c2 = -4; c1 = 0; c0 = 0;
%attempt
y = A*t^4 + B*t^3 + C*t^2 + D*exp(t);
P = c4*diff(y,t,4) + c3*diff(y,t,3) + c2*diff(y,t,2) + c1*diff(y,t,1) + c0*y;
collect(P, t)
```

ans = $(-48\,A)\,t^2 + (-24\,B)\,t + 24\,A - 8\,C - 3\,\mathrm{D}\,e^t$

$\therefore -3De^t = e^t, \quad D = -\frac{1}{3}$

$-48At^2 = t^2, \quad A = -\frac{1}{48} \qquad -24Bt = 0, \; B = 0$

$24A - 8C = 0, \quad -\frac{24}{48} - 8C = 0, \quad -8C = \frac{1}{2}, \quad C = -\frac{1}{16}$

$\therefore y_p = -\frac{1}{48}t^4 - \frac{1}{16}t^2 - \frac{1}{3}e^t$

$\therefore \underline{y(t) = c_1 + c_2 t + c_3 e^{2t} + c_4 e^{-2t} - \frac{1}{48}t^4 - \frac{1}{16}t^2 - \frac{1}{3}e^t}$

5.

(a) Homogeneous: $r^4 + 2r^2 + 1 = (r^2+1)^2 = 0, \; r = i, i, -i, -i$

$\therefore e^{it}, \; te^{it}, \; e^{-it}, \; te^{-it}$, or

$y_c = c_1\cos(t) + c_2\sin(t) + c_3 t\cos(t) + c_4 t\sin(t)$

(b) Particular: $y = 3$ is an obvious solution

Let $y_p = A\cos(2t) + B\sin(2t)$

Using MATLAB,

```
clear,clc
syms t A B
%coeffs of diff eq
c4 = 1; c3 = 0; c2 = 2; c1 = 0; c0 = 1;
%attempt
y = A*cos(2*t) + B*sin(2*t);
P = c4*diff(y,t,4) + c3*diff(y,t,3) + c2*diff(y,t,2) + c1*diff(y,t,1) + c0*y;
collect(P, [cos(2*t), sin(2*t)])
```

ans = $(9\,A)\cos(2\,t) + (9\,B)\sin(2\,t)$

$\therefore\ 9A\cos(2t) = \cos(2t),\ A = \frac{1}{9} \quad B = 0$

$\therefore\ y_p = 3 + \frac{1}{9}\cos(2t)$

$\therefore\ \underline{y(t)} = c_1\cos(t) + c_2\sin(t) + c_3 t\cos(t) + c_4 t\sin(t)$

$\underline{+ 3 + \frac{1}{9}\cos(2t)}$

6.

(a) Homogeneous: $r^6 + r^3 = r^3(r^3+1) = 0,\ r = 0,0,0,-1$

Using MATLAB,

```
clear, clc
p = [1, 0, 0, 1, 0, 0, 0];
rats(roots(p))
syms r
factor(r^3+1)
```

```
ans = 6×29 char array
    '      0      +      0i      '
    '      0      +      0i      '
    '      0      +      0i      '
    '     -1      +      0i      '
    '     1/2     +  1170/1351i  '
    '     1/2     -  1170/1351i  '
```

ans = $\left(r+1 \quad r^2 - r + 1\right)$

$\therefore r^3+1 = (r+1)(r^2-r+1) = 0$

For r^2-r+1, $r = \dfrac{1 \pm \sqrt{1-4}}{2} = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

$\therefore r = 0, 0, 0, -1, \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

$\therefore e^{0t}, te^{0t}, t^2 e^{0t}, e^{-t}, e^{t/2}\cos(\frac{\sqrt{3}}{2}t), e^{t/2}\sin(\frac{\sqrt{3}}{2}t)$

$\therefore y_c = C_1 + C_2 t + C_3 t^2 + C_4 e^{-t} + e^{t/2}\left[C_5\cos(\frac{\sqrt{3}}{2}t) + C_6\sin(\frac{\sqrt{3}}{2}t)\right]$

(b) Particular: Let $y_p = At^4$

$\therefore y_p' = 4At^3, \; y_p'' = 12At^2, \; y_p''' = 24At, \; y_p^{(6)} = 0$

$\therefore y_p^{(6)} + y_p''' = 24At = t, \; A = \frac{1}{24}$

$\therefore y_p = \frac{1}{24}t^4$

$\therefore \underline{y(t)} = C_1 + C_2 t + C_3 t^2 + C_4 e^{-t} + e^{t/2}\left[C_5\cos(\frac{\sqrt{3}}{2}t) + C_6\sin(\frac{\sqrt{3}}{2}t)\right]$

$\underline{+ \frac{1}{24}t^4}$

7.

(a) Homogeneous: $r^3 + 4r = r(r^2+4) = 0$, $r = 0, \pm 2i$

$\therefore y_c = c_1 + c_2\cos(2t) + c_3\sin(2t)$

(b) Particular: Let $y_p = t(At) = At^2$

$\therefore y_p' = 2At$, $y_p'' = 2A$, $y_p''' = 0$

$\therefore y_p''' + 4y_p' = 8At = t$, $A = \frac{1}{8}$

$\therefore y(t) = \frac{1}{8}t^2 + c_1 + c_2\cos(2t) + c_3\sin(2t)$

(c) $y(0) = c_1 + c_2 = 0$

$y'(0) = +2c_3 = 0 \Rightarrow c_3 = 0$

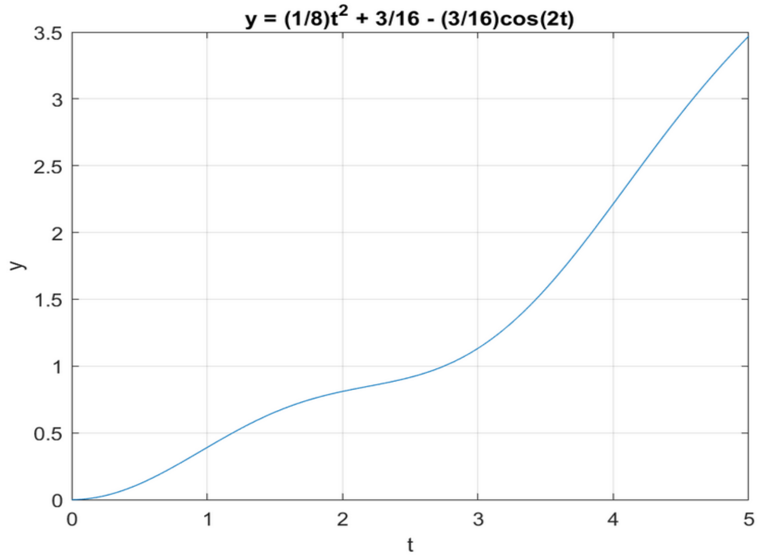
$y''(0) = \frac{1}{4} - 4c_2 = 1 \Rightarrow c_2 = -\frac{3}{16}$

$\therefore c_1 = \frac{3}{16}$

$\therefore \underline{y(t) = \frac{1}{8}t^2 + \frac{3}{16} - \frac{3}{16}\cos(2t)}$

(d) Using MATLAB

```
clear, clc;
t = 0:0.01:5;
y = (1/8)*t.^2 + 3/16 - (3/16)*cos(2*t);
plot(t,y)
grid on
xlabel 't', ylabel 'y'
title 'y = (1/8)t^2 + 3/16 - (3/16)cos(2t)'
```



8.

(a) Homogeneous: as in #5 above.

$$y_c = c_1\cos(t) + c_2\sin(t) + c_3 t\cos(t) + c_4 t\sin(t)$$

(b) Particular: Let $y = At + B$, $\therefore y'' = 0,\ y^{(4)} = 0$

$\therefore A = 3,\ B = 4$

$$\therefore y(t) = 3t + 4 + c_1\cos(t) + c_2\sin(t) + c_3 t\cos(t) + c_4 t\sin(t)$$

(c) Use MATLAB to do all the differentiation:

```
clear, clc
syms t c1 c2 c3 c4
y(t) = 3*t+4 + c1*cos(t) + c2*sin(t) + c3*t*cos(t) + c4*t*sin(t);
subs(y(t),t,0)
subs(diff(y(t),t,1),t,0)
subs(diff(y(t),t,2),t,0)
subs(diff(y(t),t,3),t,0)
```

ans = $c_1 + 4$

ans = $c_2 + c_3 + 3$

ans = $2\,c_4 - c_1$

ans = $-c_2 - 3\,c_3$

$\therefore\ y(0) = c_1 + 4 = 0 \Rightarrow \underline{c_1 = -4}$

$y'(0) = c_2 + c_3 + 3 = 0$

$y''(0) = 2c_4 - c_1 = 2c_4 + 4 = 1 \Rightarrow \underline{c_4 = -\frac{3}{2}}$

$y'''(0) = -c_2 - 3c_3 = 1$

Add $y'(0) + y'''(0)$: $-2c_3 = -2 \Rightarrow \underline{c_3 = 1}$

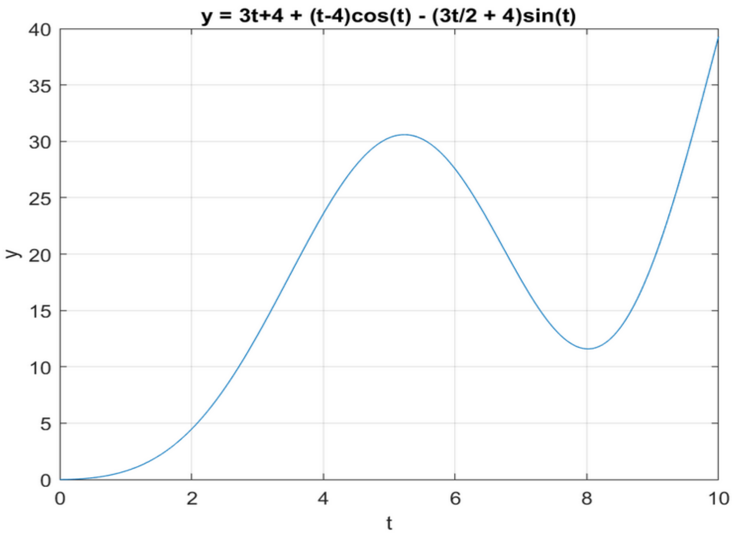
$\therefore\ \underline{c_2 = -4}$

$\therefore\ y(t) = 3t + 4 - 4\cos(t) - 4\sin(t) + t\cos(t) - \frac{3}{2}t\sin(t)$

or, $\underline{y(t) = 3t + 4 + (t-4)\cos(t) - (\frac{3}{2}t + 4)\sin(t)}$

(d) From MATLAB,

```
clear, clc;
t = 0:0.01:10;
y = 3*t+4 + (t-4).*cos(t) - (3*t/2 + 4).*sin(t);
plot(t,y)
grid on
xlabel 't', ylabel 'y'
title 'y = 3t+4 + (t-4)cos(t) - (3t/2 + 4)sin(t)'
```



9.

(a) Homogeneous: $r^4 + 2r^3 + r^2 + 8r - 12 = 0$

Use MATLAB,

```
clear, clc
syms r
factor(r^4 + 2*r^3 + r^2 + 8*r - 12)
solve(r^4 + 2*r^3 + r^2 + 8*r - 12 == 0)
```

ans = $\left(r+3 \quad r-1 \quad r^2+4\right)$

ans =

$$\begin{pmatrix} -3 \\ 1 \\ -2\,\mathrm{i} \\ 2\,\mathrm{i} \end{pmatrix}$$

$$\therefore\ y_c = c_1 e^{t} + c_2 e^{-3t} + c_3 \cos(2t) + c_4 \sin(2t)$$

(b) Particular: Let $y = Ae^{-t} + B\cos(t) + C\sin(t)$

```
clear,clc
syms t A B C
%coeffs of diff eq
c4 = 1; c3 = 2; c2 = 1; c1 = 8; c0 = -12;
%attempt
y = A*exp(-t) + B*cos(t) + C*sin(t);
P = c4*diff(y,t,4) + c3*diff(y,t,3) + c2*diff(y,t,2) + c1*diff(y,t,1) + c0*y;
collect(P, [cos(t) sin(t)])
```

ans = $(6\,C - 12\,B)\cos(t) + (-6\,B - 12\,C)\sin(t) - 20\,A\,e^{-t}$

$\therefore -20Ae^{-t} = -e^{-t}, \quad A = \frac{1}{20}$

$$\left.\begin{aligned} -12B + 6C &= 0 \\ -6B - 12C &= 12 \end{aligned}\right\} \quad -30B = 12, \quad B = -\frac{2}{5}, \quad C = -4/5$$

$\therefore \underline{y_p = \frac{1}{20}} e^{-t} - \frac{2}{5}\cos(t) - \frac{4}{5}\sin(t)$

(c) $\therefore y(t) = y_c(t) + y_p(t)$

Use MATLAB to solve for c_1, c_2, c_3, c_4

```
clear, clc
syms t c1 c2 c3 c4
yc(t) = c1*exp(t) + c2*exp(-3*t) + c3*cos(2*t) + c4*sin(2*t);
A = 1/20; B = -2/5; C = -4/5;
yp(t) = A*exp(-t) + B*cos(t) + C*sin(t);
y(t) = yc(t) + yp(t);
subs(y(t),t,0)
subs(diff(y(t),t,1),t,0)
subs(diff(y(t),t,2),t,0)
subs(diff(y(t),t,3),t,0)
```

$y(0)$ ans =

$c_1 + c_2 + c_3 - \frac{7}{20}$

$y'(0)$ ans =

$c_1 - 3c_2 + 2c_4 - \frac{17}{20}$

$y''(0)$ ans =

$c_1 + 9c_2 - 4c_3 + \frac{9}{20}$

$y'''(0)$ ans =

$c_1 - 27c_2 - 8c_4 + \frac{3}{4}$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & -3 & 0 & 2 \\ 1 & 9 & -4 & 0 \\ 1 & -27 & 0 & -8 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} \frac{7}{20} + 3 \\ \frac{17}{20} + 0 \\ -\frac{9}{20} - 1 \\ -\frac{3}{4} + 2 \end{bmatrix}$$

The right matrix comes from $y(0) = 3$,

$y'(0) = 0, \; y''(0) = -1, \; y'''(0) = 2$

Using MATLAB to solve the above simultaneous equations for c_1, c_2, c_3, c_4:

```
clear, clc
A = [1,   1,   1,   0;...
     1,  -3,   0,   2;...
     1,   9,  -4,   0;...
     1, -27,   0,  -8];
B = [  7/20 + 3;...
      17/20 + 0;...
      -9/20 - 1;...
      -3/4  + 2];
%solve AX = B
rats(A\B)
```

```
ans = 4×14 char array
    '     81/40    '
    '     73/520   '
    '     77/65    '
    '    -49/130   '
```

$c_1 = 81/40$

$c_2 = 73/520$

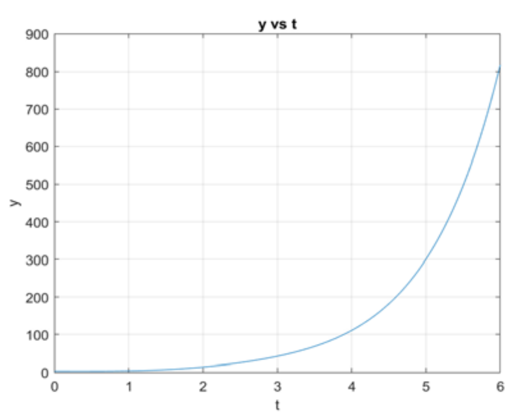
$c_3 = 77/65$

$c_4 = -49/130$

$$\therefore\ y(t) = \frac{1}{20}e^{-t} - \frac{2}{5}\cos(t) - \frac{4}{5}\sin(t) + \frac{81}{40}e^{t} + \frac{73}{520}e^{-3t} + \frac{77}{65}\cos(2t) - \frac{49}{130}\sin(2t)$$

(d) Using MATLAB to plot:

```
clear, clc
t = 0:0.01:6;
c1 = 81/40; c2 = 73/520; c3 = 77/65; c4 = -49/130;
yc = c1*exp(t) + c2*exp(-3*t) + c3*cos(2*t) + c4*sin(2*t);
A = 1/20; B = -2/5; C = -4/5;
yp = A*exp(-t) + B*cos(t) + C*sin(t);
y = yc + yp;
plot(t,y)
grid on
xlabel 't', ylabel 'y'
title 'y vs t'
```



Basically a plot of

$y = 3 + 2e^{t}$

10.

(a) Homogeneous: $r^3 - 2r^2 + r = r(r^2 - 2r + 1) = r(r-1)^2 = 0$

$\therefore e^{0t}, e^{t}, te^{t}$. $\therefore$ $\underline{y_c = c_1 + c_2 e^{t} + c_3 t e^{t}}$

(b) Particular: $r = 0$ is a root of multiplicity 1

$\therefore$ need a factor of $t\,(t^3)$. e^{t}, te^{t} already in homogeneous solution. $\therefore$ need factor of $t^2(e^{t})$

$\therefore$ $\underline{y_p = At^4 + Bt^3 + Ct^2 + Dt + Et^2 e^{t}}$

11.

(a) Homogeneous: $r^3 - r = r(r^2 - 1) = r(r+1)(r-1) = 0$

$\therefore e^{0t}, e^{t}, e^{-t}$ $\therefore$ $\underline{y_c = c_1 + c_2 e^{t} + c_3 e^{-t}}$

(b) Particular: e^{-t} is part of y_c.

$\therefore$ $\underline{y_p = t(At + B)e^{-t} + C\cos(t) + D\sin(t)}$

12.

(a) Homogeneous : $r^4 - r^3 - r^2 + r = 0$

MATLAB:
```
clear, clc
syms r
factor(r^4 - r^3 - r^2 + r)
```
ans = $(r \quad r+1 \quad r-1 \quad r-1)$

$\therefore e^{0t}, e^{t}, e^{t}, e^{-t}$. $\underline{y_c = c_1 + c_2 e^{t} + c_3 t e^{t} + c_4 e^{-t}}$

(b) Particular : $r = 0$ is a root. $\therefore$ for $y^2 + 4$,

use $t(At^2 + Bt + C)$

$\therefore y_p = t(At^2 + Bt + C) +$

$(Dt + E)\cos(t) + (Ft + G)\sin(t)$

13.

(a) Homogeneous : $r^4 + 2r^3 + 2r^2 = r^2(r^2 + 2r + 2) = 0$

$\therefore r = 0, 0, \dfrac{-1 \pm \sqrt{4-8}}{2}$, or $-1 \pm i$

$\therefore e^{0t}, te^{0t}, e^{-t}\cos(t), e^{-t}\sin(t)$

$\therefore y_c = C_1 + C_2 t + C_3 e^{-t}\cos(t) + C_4 e^{-t}\sin(t)$

(b) Particular : For te^{-t}, e^{-t} is not part of y_c.

$\therefore$ try $(Bt + C)e^{-t}$

For $e^{-t}\sin(t)$, already a part of y_c,

try $te^{-t}(D\cos(t) + E\sin(t))$

$$\therefore y_p = Ae^{t} + (Bt + C)e^{-t} + te^{-t}[D\cos(t) + E\sin(t)]$$

14.

Lemma: $[e^{\alpha t}u]^{(n)} = e^{\alpha t}\left[\sum_{k=0}^{n} \binom{n}{k} \alpha^k u^{(n-k)}\right]$

$$= e^{\alpha t}\left[u^{(n)} + n\alpha u^{(n-1)} + \dots + \binom{n}{i}\alpha^i u^{(n-i)} + \dots \alpha^n u\right]$$

where $\binom{n}{k} = \dfrac{n!}{(n-k)!\,k!}$, the binomial coefficient.

and $u^{(n)}$ = nth derivative of $u(t)$, $u^{(0)} = u$.

Proof: (a) For $n=1$, $[e^{\alpha t}u]' = e^{\alpha t}u' + \alpha e^{\alpha t}u$

$$= e^{\alpha t}[u' + \alpha u]$$

and $\binom{1}{0} = 1$, $\binom{1}{1} = 1$

(b) For $n=2$, $[e^{\alpha t}u]'' = [e^{\alpha t}(u'+\alpha u)]'$

$$= e^{\alpha t}(u'+\alpha u)' + \alpha e^{\alpha t}(u'+\alpha u)$$

$$= e^{\alpha t}(u''+\alpha u') + e^{\alpha t}(\alpha u' + \alpha^2 u)$$

$$= e^{\alpha t}[u'' + 2\alpha u' + \alpha^2 u]$$

and $\binom{2}{0} = 1$, $\binom{2}{1} = 2$, $\binom{2}{2} = 1$

(c) Assume $[e^{\alpha t}u]^{(p)} = e^{\alpha t}\left[\sum_{K=0}^{p}\binom{p}{K}\alpha^K u^{(p-K)}\right]$

Consider $[e^{\alpha t}u]^{(p+1)}$

$$= [(e^{\alpha t}u)']^{(p)}$$

$$= [e^{\alpha t}u' + \alpha e^{\alpha t}u]^{(p)}$$

$$= [e^{\alpha t}u']^{(p)} + \alpha[e^{\alpha t}u]^{(p)}$$

$$= e^{\alpha t}\left[\sum_{K=0}^{p}\binom{p}{K}\alpha^K (u')^{(p-K)}\right] + \alpha e^{\alpha t}\left[\sum_{K=0}^{p}\binom{p}{K}\alpha^K u^{(p-K)}\right]$$

$$= e^{\alpha t}\left[\sum_{K=0}^{p}\binom{p}{K}\alpha^K u^{(p-K+1)}\right] + e^{\alpha t}\left[\sum_{K=0}^{p}\binom{p}{K}\alpha^{K+1} u^{(p-K)}\right]$$

$$= e^{\alpha t}\left[\binom{p}{0}\alpha^0 u^{(p+1)} + \sum_{K=1}^{p}\binom{p}{K}\alpha^K u^{(p-K+1)} + \sum_{K=0}^{p}\binom{p}{K}\alpha^{K+1}u^{(p-K)}\right]$$

$$= e^{\alpha t}\left[u^{(p+1)} + \sum_{k=0}^{p-1} \binom{p}{k+1} \alpha^{k+1} u^{(p-k)} + \right.$$

shift k index

$$\left. \sum_{k=0}^{p} \binom{p}{k} \alpha^{k+1} u^{(p-k)} \right]$$

$$= e^{\alpha t}\left[u^{(p+1)} + \sum_{k=0}^{p-1} \binom{p}{k+1} \alpha^{k+1} u^{(p-k)} + \right.$$

$$\left. \sum_{k=0}^{p-1} \binom{p}{k} \alpha^{k+1} u^{(p-k)} + \binom{p}{p} \alpha^{p+1} u^{(p-p)} \right]$$

$$= e^{\alpha t}\left[u^{(p+1)} + \sum_{k=0}^{p-1} \binom{p+1}{k+1} \alpha^{k+1} u^{(p-k)} + \alpha^{p+1} u \right]$$

using Pascal's rule: $\binom{p}{k} + \binom{p}{k-1} = \binom{p+1}{k}$, $1 \le k \le p$

$\therefore \binom{p}{k+1} + \binom{p}{k} = \binom{p+1}{k+1}$, $1 \le k+1 \le p$ or $0 \le k \le p-1$

$$\therefore \left[e^{\alpha t} u\right]^{(p+1)} = e^{\alpha t}\left[u^{(p+1)} + \sum_{k=0}^{p-1} \binom{p+1}{k+1} \alpha^{k+1} u^{(p-k)} + \alpha^{p+1} u \right]$$

But for $k = p$, $\binom{p+1}{k+1} \alpha^{k+1} u^{(p-k)} = \binom{p+1}{p+1} \alpha^{p+1} u^{(0)} = \alpha^{p+1} u$

$$\therefore \left[e^{\alpha t} u\right]^{(p+1)} = e^{\alpha t}\left[u^{(p+1)} + \sum_{k=0}^{p} \binom{p+1}{k+1} \alpha^{k+1} u^{(p-k)} \right]$$

$$= e^{\alpha t}\left[u^{(p+1)} + \sum_{k=1}^{p+1} \binom{p+1}{k} \alpha^{k} u^{(p-k+1)} \right]$$

And for $K=0$, $\binom{p+1}{K}\alpha^K u^{(p-K+1)} = \binom{p+1}{0}\alpha^0 u^{(p+1)}$

$$= u^{(p+1)}$$

$$\therefore \left[e^{\alpha t} u\right]^{(p+1)} = e^{\alpha t}\left[\sum_{K=0}^{p+1}\binom{p+1}{K}\alpha^K u^{(p+1-K)}\right]$$

$\therefore$ True for $p \Rightarrow$ true for $p+1$

$\therefore$ Lemma true for all $n \geq 1$

Back to the main proposition:

If $a_0 y^{(n)} + \ldots + a_n y = e^{\alpha t}(b_0 t^m + \ldots + b_m)$ and

$y = e^{\alpha t} u$, then

$K_0 u^{(n)} + \ldots + K_n u = b_0 t^m + \ldots + b_m$, K_i constants

Use induction as in above Lemma.

(a) For $n=1$, $a_0 y' + a_1 y = a_0(e^{\alpha t}u(t))' + a_1(e^{\alpha t}u(t))$

$$= a_0(\alpha e^{\alpha t} u + e^{\alpha t} u') + a_1 e^{\alpha t} u$$

$$= e^{\alpha t}\left[a_0 u'(t) + (a_0\alpha + a_1)u(t)\right] = e^{\alpha t}(b_0 t^m + \ldots + b_m)$$

$$\therefore a_0 u' + (a_0\alpha + a_1)u = (b_0 t^m + \ldots + b_m)$$

$\therefore\; K_0 = a_0 \;,\; K_1 = a_0\alpha + a_1 \;,\;$ all constants

$\therefore$ True for $n=1$

(b) Suppose true for $p>1$:

$$a_0 y^{(p)} + a_1 y^{(p-1)} + \ldots + a_p y = e^{\alpha t}(b_0 t^m + \ldots + b_m)$$

If $y = e^{\alpha t}u$, then there are constants K_i

$$\text{s.t. } e^{\alpha t}(K_0 u^{(p)} + \ldots + K_p u) = e^{\alpha t}(b_0 t^m + \ldots + b_m)$$

Now consider $p+1$:

$$a_x y^{(p+1)} + a_0 y^{(p)} + \ldots + a_p y = e^{\alpha t}(b_0 t^m + \ldots + b_m)$$

Let $y = e^{\alpha t}u$, then there are constants K_i s.t.

$$a_x (e^{\alpha t}u)^{(p+1)} + e^{\alpha t}(K_0 u^{(p)} + \ldots + K_p u) = e^{\alpha t}(b_0 t^m + \ldots + b_m) \quad [1]$$

Using the Lemma,

$$a_x (e^{\alpha t}u)^{(p+1)} = a_x e^{\alpha t}\left[\sum_{K=0}^{p+1} \binom{p+1}{K} \alpha^K u^{(p+1-K)}\right]$$

$\therefore$ [1] becomes:

$$a_x e^{\alpha t}\left[\sum_{K=0}^{p+1} \binom{p+1}{K} \alpha^K u^{(p+1-K)} + e^{\alpha t}(K_0 u^{(p)} + \ldots + K_p u)\right.$$

$$= e^{\alpha t}(b_0 t^m + \ldots + b_m)$$

Or, upon dividing by $e^{\alpha t}$,

$$a_x\binom{p+1}{0}\alpha^0 u^{(p+1)} + \dots + a_x\binom{p+1}{p+1}\alpha^{p+1} u + K_0 u^{(p)} + \dots + K_p u$$

$$= a_x u^{(p+1)} + \left[a_x\binom{p+1}{1}\alpha + K_0\right]u^{(p)} + \dots + \left[a_x \alpha^{p+1} + K_p\right]u$$

All of the coefficients $\left[a_x\binom{p+1}{i}\alpha^i + K_{i-1}\right]$, $1 \le i \le p+1$

are constants as they are the sum and

products of constants. Call these constants r_i.

$$\therefore a_x u^{(p+1)} + r_1 u^{(p)} + \dots + r^{p+1} u = (b_0 t^m + b_m)$$

$\therefore$ whenever true for $p > 1$, true for $p+1$.

$\therefore$ For all $n \ge 1$,

$$a_0 y^{(n)} + \dots + a_n y = e^{\alpha t}(b_0 t^m + \dots + b_m)$$

reduces to $K_0 u^{(n)} + \dots + K_n u = b_0 t^m + \dots + b_m$,

where K_i are constants, when $y = e^{\alpha t} u(t)$.

From the above, $\underline{K_0 = a_0}$

For K_n, from the Lemma,

$$\left[e^{\alpha t} u\right]^{(n)} = e^{\alpha t}\left[\sum_{K=0}^{n}\binom{n}{K}\alpha^K u^{(n-K)}\right]$$

$\therefore$ The $u = u^{(0)}$ term has contributions from each $\sum_{K=0}^{n} \binom{n}{K} \alpha^{K} u^{(n-K)}$ term when $K =$ upper limit of Σ each time $y = e^{\alpha t} u$ is used in $a_0 y^{(n)} + \ldots + a_n y$.

$$\therefore \quad a_0 \binom{n}{n} \alpha^n + a_1 \binom{n-1}{n-1} \alpha^{n-1} + \ldots + a_{n-1} \binom{1}{1} \alpha^1 + a_n \binom{0}{0} \alpha^0$$

$$= a_0 \alpha^n + a_1 \alpha^{n-1} + \ldots + a_{n-1} \alpha + a_n$$

$$\therefore K_n = \underline{\underline{\sum_{K=0}^{n} a_K \alpha^{n-K}}}$$

15.

$$(D-a)(D-b)f = (D-a)\left[Df - bf\right]$$

$$= D^2 f - D(bf) - aDf + abf$$

$$= D^2 f - bDf - aDf + abf$$

$$= D^2 f - (b+a)Df + abf$$

$$(D-b)(D-a)f = (D-b)\left[Df - af\right]$$

$$= D^2 f - D(af) - bDf + baf$$

$$= D^2 f - a D f - b D f + b a f$$

$$= D^2 f - (a+b) D f + b a f$$

$\therefore$ Since $(b+a) = (a+b)$ and $ab = ba$,

$$\underline{(D-a)(D-b) f = (D-b)(D-a)}$$

16.

So the problem was originally from $(r-2)^3(r+1) =$
$r^4 - 5r^3 + 6r^2 + 4r - 8$, or $y^{(4)} - 5y''' + 6y'' + 4y' - 8y$
and the homogeneous solution is

$$y_c = c_1 e^{2t} + c_2 t e^{2t} + c_3 t^2 e^{2t} + c_4 e^{-t}$$

(a)

$$(D-2)\, 3e^{2t} = D(3e^{2t}) - 2(3e^{2t})$$

$$= 6e^{2t} - 6e^{2t} = \underline{0}$$

$$(D+1)^2(-te^{-t}) = (D+1)\left[(D+1)(-te^{-t})\right]$$

$$= (D+1)\left[D(-te^{-t}) - te^{-t}\right]$$

$$= (D+1)\left[-e^{-t} + te^{-t} - te^{-t}\right]$$

$$= (D+1)\left[-e^{-t}\right] = D\left[-e^{-t}\right] - e^{-t}$$

$$= e^{-t} - e^{-t} = \underline{0}$$

$$\therefore (D-2)(D+1)^2(3e^{2t}) = (D+1)^2(D-2)(3e^{2t}) \quad \text{(by \#15)}$$

$$= (D+1)^2(0) \quad \text{(from above)}$$

$$= \underline{0}$$

$$(D-2)(D+1)^2(-te^{-t}) = (D-2)(0) \quad \text{(from above)}$$

$$= \underline{0}$$

(b)

$$(D-2)(D+1)^2\left[(D-2)^3(D+1)\right]Y = (D-2)(D+1)^2(3e^{2t} - te^{-t})$$

From (a), the right hand side $= 0$.

$$\therefore (D-2)(D+1)^2\left[(D-2)^3(D+1)\right]Y = 0$$

$$\therefore (D-2)\left[(D+1)^2(D-2)^3(D+1)\right]y=0$$

$$\therefore (D-2)\left[(D-2)^3(D+1)^2(D+1)\right]y=0 \quad \text{(by \#15)}$$

$$\therefore (D-2)(D-2)^3(D+1)^3\,y=0$$

$$\therefore (D-2)^4(D+1)^3\,y=0$$

The corresponding characteristic equation is:

$$(r-2)^4(r+1)^3=0, \quad r=2,2,2,2,-1,-1,-1$$

$$\therefore y(t)=c_1e^{2t}+c_2te^{2t}+c_3t^2e^{2t}+c_4t^3e^{2t}$$
$$+c_5e^{-t}+c_6te^{-t}+c_7t^2e^{-t}$$

Using #31 from Section 4.2, page 181.

(c)

Note that since e^{2t}, te^{2t}, t^2e^{2t}, and e^{-t} solve

$(D-2)^3(D+1)y=0$, then they solve

$$(D-2)(D+1)^2\left[(D-2)^3(D+1)\right]y=0$$

or, $(D-2)^4(D+1)^3 y = 0$

$\therefore$ Finding y_p form by finding an equation for which y_p is the homogeneous solution.

17.

(10)

$D(t^3) = 3t^2$, $D^2(t^3) = 6t$, $D^3(t^3) = 6$, $D^4(t^3) = 0$

$(D-1)(2e^t) = 0$

$\therefore$ $D^4(D-1)$ annihilates $t^3 + 2e^t$

$y''' - 2y'' + y' = (y'' - 2y' + y)' = D(D-1)^2 y$

$\therefore$ $D^4(D-1)D(D-1)^2 y = D^5(D-1)^3 y$

Solution to $D^5(D-1)^3 y = 0$ is equivalent to $r^5(r-1)^3 = 0$ as characteristic equation.

$\therefore$ $e^{0t}, te^{0t}, t^2e^{0t}, t^3e^{0t}, t^4e^{0t}, e^t, te^t, t^2e^t$

Or, $Y(t) = c_1 + c_2 t + c_3 t^2 + c_4 t^3 + c_5 t^4 + c_6 e^t + c_7 te^t + c_8 t^2 e^t$

Solutions to $D(D-1)^2 y = 0$ include e^{0t}, e^{t}, te^{t}

$\therefore$ Set $c_1 = 0$, $c_6 = 0$, $c_7 = 0$

$\therefore Y(t) = c_2 t + c_3 t^2 + c_4 t^3 + c_5 t^4 + c_8 t^2 e^t$

(11)

As shown in #16 above, $(D+1)^2$ annihilates te^{-t}

$(D^2+1)(2\cos t) = D^2(2\cos(t)) + 2\cos(t)$

$= -2\cos(t) + 2\cos(t) = 0$

$\therefore (D^2+1)(D+1)^2$ annihilates $te^{-t} + 2\cos(t)$

$y''' - y' = D(D^2-1)y = D(D+1)(D-1)y$

$\therefore (D^2+1)(D+1)^2 D(D+1)(D-1)y = 0$

Or, $(D^2+1)(D+1)^3(D-1)D\, y = 0$

$\therefore e^{0t}, e^{t}, e^{-t}, te^{-t}, t^2 e^{-t}, e^{it}, e^{-it}$

$\therefore Y(t) = c_1 + c_2 e^{t} + c_3 e^{-t} + c_4 te^{-t} + c_5 t^2 e^{-t} +$

$c_6 \cos(t) + c_7 \sin(t)$

For $D(D+1)(D-1)y = 0$, e^{0t}, e^{-t}, e^{t} are solutions

$\therefore$ set $c_1 = 0$, $c_2 = 0$, $c_3 = 0$

$\therefore\ y(t) = c_4 t e^{-t} + c_5 t^2 e^{-t} + c_6 \cos(t) + c_7 \sin(t)$

(12)

$D^3(t^2+4) = 0$ so D^3 annihilates t^2+4

$D[t\sin(t)] = \sin(t) + t\cos(t)$
$D^2[t\sin(t)] = 2\cos(t) - t\sin(t)$
$D^3[t\sin(t)] = -3\sin(t) - t\cos(t)$
$D^4[t\sin(t)] = -4\cos(t) + t\sin(t)$

$$\therefore\ D^4 + 2D^2 = -4\cos(t) + t\sin(t) + 4\cos(t) - 2t\sin(t)$$
$$= -t\sin(t)$$

$\therefore\ D^4 + 2D^2 + 1 = -t\sin(t) + t\sin(t) = 0$

$\therefore\ D^4 + 2D^2 + 1 = (D^2+1)^2$ annihilates $t\sin(t)$

$\therefore\ \underline{D^3(D^2+1)^2}$ annihilates $t^2 + 4 + t\sin(t)$

$$y^{(4)} - y''' - y'' + y' = D^4 - D^3 - D^2 + D = D(D^3 - D^2 - D + 1)$$
$$= D[D^3 - D - (D^2-1)]$$
$$= D[D(D^2-1) - (D^2-1)]$$
$$= D(D-1)(D^2-1) = \underline{D(D+1)(D-1)^2}$$

$\therefore$ Solve $D^3(D^2+1)^2 \, D(D+1)(D-1)^2 y = 0$

Or, $D^4(D^2+1)^2(D-1)^2(D+1)y = 0$

$\therefore e^{0t}, te^{0t}, t^2e^{0t}, t^3e^{0t}, e^{it}, e^{-it}, te^{it}, te^{-it},$

e^{t}, te^{t}, e^{-t} [1]

Solution set to $D(D+1)(D-1)^2 y = 0$ is

$e^{0t}, e^{-t}, e^{t}, te^{t}$ [2]

$\therefore$ Solution set [1] minus solution set [2] gives

$t, t^2, t^3, e^{\pm it}, te^{\pm it}$

$\therefore y(t) = c_1 t + c_2 t^2 + c_3 t^3 + c_4 \cos(t) + c_5 \sin(t) +$

$c_6 t\cos(t) + c_7 t\sin(t)$

(13)

$\underline{(D-1)}(3e^{t}) = 0$

$\underline{(D+1)^2}(2te^{-t}) = 0$ from #16 above

$$\sin(t) = \frac{\cos(t) + i\sin(t) - [\cos(-t) + i\sin(-t)]}{2i}$$

$$= \frac{e^{it} - e^{-it}}{2i}$$

$$\therefore e^{-t}\sin(t) = \frac{e^{(-1+i)t} - e^{(-1-i)t}}{2i}$$

$(D-\lambda)e^{\lambda t} = 0$ for real and complex λ.

$\therefore [D-(-1+i)][D-(-1-i)]$ annihilates $e^{-t}\sin(t)$.

Or, $(D+1-i)(D+1+i) = (D+1)^2 - i^2 = (D+1)^2 + 1$

$(D+1)^2 + 1 = D^2 + 2D + 2$

$\therefore (\underline{D^2 + 2D + 2})\, e^{-t}\sin(t) = 0$

$\therefore (D^2+2D+2)(D+1)^2(D-1)$ annihilates $3e^{t} + 2te^{-t} + e^{-t}\sin(t)$

$y^{(4)} + 2y''' + 2y'' = D^2(D^2+2D+2)y$

$\therefore$ Solve $D^2(D^2+2D+2)^2(D+1)^2(D-1)y = 0$

$\therefore e^{0t}, te^{0t}, e^{(-1\pm i)t}, te^{(-1\pm i)t}, e^{-t}, te^{-t}, e^{t}$ [1]

Solution set to $D^2(D^2+2D+2)y = 0$ is

$e^{0t}, te^{0t}, e^{(-1\pm i)t}$ [2]

Solution set [1] minus set [2] is: $te^{(-1\pm i)t}, e^{-t}, te^{-t}, e^{t}$

$$\therefore \underline{y(t) = te^{-t}[c_1\cos(t) + c_2\sin(t)] + c_3e^{-t} + c_4te^{-t} + c_5e^{t}}$$

4.4 The Method of Variation of Parameters

Note Title 2/1/2019

1.

(a) Homogeneous: $r^3+r = r(r^2+1)=0$, $r=0, \pm i$

$\therefore y_c(t) = c_1 + c_2\cos(t) + c_3\sin(t)$ [1]

(b) Particular: Let $y_p = u_1 + u_2\cos(t) + u_3\sin(t)$

$$\therefore W = \begin{vmatrix} 1 & \cos(t) & \sin(t) \\ 0 & -\sin(t) & \cos(t) \\ 0 & -\cos(t) & -\sin(t) \end{vmatrix} \quad 1\left[\sin^2(t)+\cos^2(t)\right] - 0 + 0$$

$$\therefore W\left[1, \cos(t), \sin(t)\right] = 1$$

$$u_1' = \begin{vmatrix} 0 & \cos(t) & \sin(t) \\ 0 & -\sin(t) & \cos(t) \\ \tan(t) & -\cos(t) & -\sin(t) \end{vmatrix} = \tan(t)\left[\cos^2(t)+\sin^2(t)\right] = \tan(t)$$

$$u_2' = \begin{vmatrix} 1 & 0 & \sin(t) \\ 0 & 0 & \cos(t) \\ 0 & \tan(t) & -\sin(t) \end{vmatrix} = -\tan(t)\cos(t) = -\sin(t)$$

$$u_3' = \begin{vmatrix} 1 & \cos(t) & 0 \\ 0 & -\sin(t) & 0 \\ 0 & -\cos(t) & \tan(t) \end{vmatrix} = -\sin(t)\tan(t) = -\frac{\sin^2(t)}{\cos(t)}$$

$$\therefore u_1 = \int \tan(t) = -\ln|\cos(t)| = -\underline{\ln(\cos(t))}, \quad -\frac{\pi}{2} < t < \frac{\pi}{2}$$

$$u_2 = \int -\sin(t) = \underline{\cos(t)}$$

$$u_3 = \int \frac{-\sin^2(t)}{\cos(t)} = \int \frac{\cos^2(t)-1}{\cos(t)} = \int \cos(t) - \int \sec(t)$$

$$= \sin(t) - \ln|\sec(t) + \tan(t)|$$

$$= \underline{\sin(t) - \ln(\sec(t) + \tan(t))}, \quad -\frac{\pi}{2} < t < \frac{\pi}{2}$$

$$\therefore y_p(t) = -\ln[\cos(t)] + \cos^2(t) + \sin^2(t) - \sin(t)\ln[\sec(t) + \tan(t)]$$

$$= -\ln[\cos(t)] - \sin(t)\ln[\sec(t) + \tan(t)] \quad [2]$$

(c) $$\underline{y(t)} = c_1 + c_2\cos(t) + c_3\sin(t) - \ln[\cos(t)] - \sin(t)\ln[\sec(t) + \tan(t)]$$

2.

(a) Homogeneous: $r^3 - r = r(r^2-1) = r(r+1)(r-1)$, $r = 0, 1, -1$

$$\therefore y_c(t) = c_1 + c_2 e^t + c_3 e^{-t} \quad [1]$$

(b) Particular: Let $y_p = u_1 + u_2 e^t + u_3 e^{-t}$

$$W = \begin{vmatrix} 1 & e^{t} & e^{-t} \\ 0 & e^{t} & -e^{-t} \\ 0 & e^{t} & e^{-t} \end{vmatrix} = 1\left(e^{t}e^{-t} + e^{t}e^{-t}\right) = 2$$

$$u_1' = \frac{1}{2}\begin{vmatrix} 0 & e^{t} & e^{-t} \\ 0 & e^{t} & -e^{-t} \\ t & e^{t} & e^{-t} \end{vmatrix} = \frac{1}{2}t\left(-e^{t}e^{-t} - e^{t}e^{-t}\right) = -t$$

$$u_2' = \frac{1}{2}\begin{vmatrix} 1 & 0 & e^{-t} \\ 0 & 0 & -e^{-t} \\ 0 & t & e^{-t} \end{vmatrix} = \frac{1}{2}\left(0 + te^{-t}\right) = \frac{1}{2}te^{-t}$$

$$u_3' = \frac{1}{2}\begin{vmatrix} 1 & e^{t} & 0 \\ 0 & e^{t} & 0 \\ 0 & e^{t} & t \end{vmatrix} = \frac{1}{2}\left(te^{t} - 0\right) = \frac{1}{2}te^{t}$$

$$\therefore u_1 = \int -t = -\frac{t^2}{2}$$

$$u_2 = \int \frac{1}{2}te^{-t} = \frac{1}{2}(-t-1)e^{-t}$$

using $\int ue^{au}\,du = \frac{1}{a^2}(au-1)e^{au} + C$

$$u_3 = \int \frac{1}{2}te^{t} = \frac{1}{2}(t-1)e^{t}$$

$$\therefore y_p(t) = -\frac{t^2}{2} + \frac{1}{2}(-t-1)e^{-t}e^{t} + \frac{1}{2}(t-1)e^{t}e^{-t}$$

$$= -\frac{t^2}{2} - \frac{t}{2} - \frac{1}{2} + \frac{t}{2} - \frac{1}{2} = -\frac{t^2}{2} - 1 \qquad [2]$$

(c) $y(t) = c_1 + c_2e^{t} + c_3e^{-t} - \frac{t^2}{2}$ (the -1 of y_p is in c_1)

3.

(a) Homogeneous: $r^3 - 2r^2 - r + 2 = 0$

Using MATLAB:
```
clear, clc
syms r
factor(r^3 - 2*r^2 - r + 2)
```

ans = $(r-1 \quad r-2 \quad r+1)$

$$\therefore \; y_c(t) = c_1 e^{t} + c_2 e^{2t} + c_3 e^{-t}$$

(b) Particular: Let $y_p(t) = u_1 e^{t} + u_2 e^{2t} + u_3 e^{-t}$

$$W = \begin{vmatrix} e^{t} & e^{2t} & e^{-t} \\ e^{t} & 2e^{2t} & -e^{-t} \\ e^{t} & 4e^{2t} & e^{-t} \end{vmatrix} = \begin{vmatrix} e^{t} & e^{2t} & e^{-t} \\ 0 & e^{2t} & -2e^{-t} \\ 0 & 3e^{2t} & 0 \end{vmatrix}$$

subtract row 1 from row 2, row 3

$$= e^{t}\left[0 - (3e^{2t})(-2e^{-t})\right] = 6e^{2t}$$

$$u_1' = \frac{1}{6e^{2t}} \begin{vmatrix} 0 & e^{2t} & e^{-t} \\ 0 & 2e^{2t} & -e^{-t} \\ e^{4t} & 4e^{2t} & e^{-t} \end{vmatrix} = \frac{1}{6e^{2t}}\left[e^{4t}\left(-e^{2t}e^{-t} - 2e^{2t}e^{-t}\right)\right]$$

$$= \frac{1}{6e^{2t}} e^{4t}(-3e^{t}) = -\frac{1}{2}e^{3t}$$

$$u_2' = \frac{1}{6e^{2t}} \begin{vmatrix} e^{t} & 0 & e^{-t} \\ e^{t} & 0 & -e^{-t} \\ e^{t} & e^{4t} & e^{-t} \end{vmatrix} = \frac{1}{6e^{2t}}\left[-e^{4t}\left(-e^{t}e^{-t} - e^{t}e^{-t}\right)\right]$$

$$= \frac{1}{3}e^{2t}$$

$$u_3' = \frac{1}{6e^{2t}} \begin{vmatrix} e^t & e^{2t} & 0 \\ e^t & 2e^{2t} & 0 \\ e^t & 4e^{2t} & e^{4t} \end{vmatrix} = \frac{1}{6e^{2t}}\left[e^{4t}\left(2e^{3t} - e^{3t}\right)\right]$$

$$= \frac{1}{6} e^{5t}$$

$\therefore$

$$u_1 = \int -\frac{1}{2} e^{3t} = -\frac{1}{6} e^{3t}$$

$$u_2 = \int \frac{1}{3} e^{2t} = \frac{1}{6} e^{2t}$$

$$u_3 = \int \frac{1}{6} e^{5t} = \frac{1}{30} e^{5t}$$

$$\therefore y_p(t) = \left(-\frac{1}{6} e^{3t}\right) e^{t} + \left(\frac{1}{6} e^{2t}\right) e^{2t} + \left(\frac{1}{30} e^{5t}\right) e^{-t}$$

$$= \frac{1}{30} e^{4t}$$

(c) $\therefore \underline{y(t) = c_1 e^{t} + c_2 e^{2t} + c_3 e^{-t} + \frac{1}{30} e^{4t}}$

4.

(a) Homogeneous: $r^3 - r^2 + r - 1 = 0$

Using MATLAB:

```
clear, clc
syms r
factor(r^3 - r^2 + r - 1)
```

ans = $(r^2 + 1 \quad r - 1)$

$\therefore r = 1, i, -i$

$$\therefore\ y_c(t) = c_1 e^{t} + c_2\cos(t) + c_3\sin(t)$$

(b) Particular: Let $y_p(t) = u_1 e^{t} + u_2\cos(t) + u_3\sin(t)$

Use MATLAB to do the computations:

```
clear, clc
syms t
y1 = exp(t); y2 = cos(t); y3 = sin(t);
B = [0; 0 ; exp(-t)*sin(t)];
w1 = [y1; diff(y1,t,1); diff(y1,t,2)];
w2 = [y2; diff(y2,t,1); diff(y2,t,2)];
w3 = [y3; diff(y3,t,1); diff(y3,t,2)];
A = [w1, w2, w3];
W = simplify(det(A))
u1_d = det([B, w2, w3])/W
u2_d = det([w1, B, w3])/W
u3_d = det([w1, w2, B])/W
```

W = $2\,e^{t}$

u1_d =

$$\frac{e^{-2t}\,\left(\cos(t)^2\sin(t)+\sin(t)^3\right)}{2}$$

u2_d =

$$-\frac{e^{-t}\,\left(\cos(t)\sin(t)-\sin(t)^2\right)}{2}$$

u3_d =

$$-\frac{e^{-t}\,\left(\sin(t)^2+\cos(t)\sin(t)\right)}{2}$$

$$\therefore\ u_1' = \frac{e^{-2t}}{2}\left[\cos^2(t)\sin(t) + \sin^2(t)\sin(t)\right]$$

$$= \frac{e^{-2t}}{2}\sin(t)$$

$$u_2' = \frac{e^{-t}}{2}\sin^2(t) - \frac{e^{-t}}{4}\sin(2t)$$

$$u_3' = -\frac{e^{-t}}{2}\sin^2(t) - \frac{e^{-t}}{4}\sin(2t)$$

Using $\cos(2t) = \cos^2(t) - \sin^2(t) = 1 - 2\sin^2(t)$,

$$\sin^2(t) = \frac{1-\cos(2t)}{2} = \frac{1}{2} - \frac{\cos(2t)}{2}$$

$\therefore$ The above becomes:

$$u_1' = \frac{e^{-2t}}{2}\sin(t)$$

$$u_2' = \frac{e^{-t}}{4} - \frac{e^{-t}\cos(2t)}{4} - \frac{e^{-t}}{4}\sin(2t)$$

$$u_3' = -\frac{e^{-t}}{4} + \frac{e^{-t}\cos(2t)}{4} - \frac{e^{-t}}{4}\sin(2t)$$

From a table of integrals,

$$\int e^{au}\sin bu\,du = \frac{e^{au}}{a^2+b^2}(a\sin bu - b\cos bu) + C$$

$$\int e^{au}\cos bu\,du = \frac{e^{au}}{a^2+b^2}(a\cos bu + b\sin bu) + C$$

$$\therefore u_1 = \frac{1}{2}\int e^{-2t}\sin(t) = \frac{e^{-2t}}{10}\left[-2\sin(t) - \cos(t)\right]$$

$$u_2 = \int \frac{e^{-t}}{4} - \frac{1}{4}\left[\int e^{-t}\cos(2t) + e^{-t}\sin(2t)\right]$$

$$= -\frac{e^{-t}}{4} - \frac{1}{4}\left[\frac{e^{-t}}{5}\left(-\cos(2t) + 2\sin(2t) - \sin(2t) - 2\cos(2t)\right)\right]$$

$$= -\frac{e^{-t}}{4} - \frac{e^{-t}}{20}\left[-3\cos(2t) + \sin(2t)\right]$$

$$u_3 = \int -\frac{e^{-t}}{4} + \frac{1}{4}\left[\int e^{-t}\cos(2t) - e^{-t}\sin(2t)\right]$$

$$= \frac{e^{-t}}{4} + \frac{1}{4}\left[\frac{e^{-t}}{5}\left(-\cos(2t) + 2\sin(2t) + \sin(2t) + 2\cos(2t)\right)\right]$$

$$= \frac{e^{-t}}{4} + \frac{e^{-t}}{20}\left[\cos(2t) + 3\sin(2t)\right]$$

$$\therefore y_p = u_1 e^{t} + u_2\cos(t) + u_3\sin(t)$$

$$= \frac{e^{-t}}{10}\left[-2\sin(t) - \cos(t)\right]$$

$$- \frac{e^{-t}}{4}\cos(t) - \frac{e^{-t}}{20}\left[-3\cos(2t)\cos(t) + \sin(2t)\cos(t)\right]$$

use $\cos(\alpha-\beta)$ use $\sin(\alpha-\beta)$

$$+ \frac{e^{-t}}{4}\sin(t) + \frac{e^{-t}}{20}\left[\cos(2t)\sin(t) + 3\sin(2t)\sin(t)\right]$$

$$= \frac{e^{-t}}{20}\left[-4\sin(t) - 2\cos(t) - 5\cos(t) + 5\sin(t)\right]$$

$$+ \frac{e^{-t}}{20}\left[3\cos(2t - t) - \sin(2t - t)\right]$$

$$= \frac{e^{-t}}{20}\left[-7\cos(t) + \sin(t)\right] + e^{-t}\left[3\cos(t) - \sin(t)\right]$$

$$= \frac{e^{-t}}{20}\left[-4\cos(t)\right] = \underline{-\frac{1}{5}e^{-t}\cos(t)}$$

(c) $\therefore \underline{y(t) = c_1 e^{t} + c_2\cos(t) + c_3\sin(t) - \frac{1}{5}e^{-t}\cos(t)}$

Just out of curiosity:

```
clear, clc
syms t
y1 = exp(t); y2 = cos(t); y3 = sin(t);
B = [0; 0 ; exp(-t)*sin(t)];
w1 = [y1; diff(y1,t,1); diff(y1,t,2)];
w2 = [y2; diff(y2,t,1); diff(y2,t,2)];
w3 = [y3; diff(y3,t,1); diff(y3,t,2)];
A = [w1, w2, w3];
W = simplify(det(A));
u1_d = det([B, w2, w3])/W;
u2_d = det([w1, B, w3])/W;
u3_d = det([w1, w2, B])/W;
u1 = int(u1_d);
u2 = int(u2_d);
u3 = int(u3_d);
y = y1*u1 + y2*u2 + y3*u3;
simplify(y)
```

ans =

$$-\frac{e^{-t}\cos(t)}{5}$$

Wow!

5.

(a) Homogeneous: As shown in #4 above,

$$y_c(t) = c_1 e^t + c_2 \cos(t) + c_3 \sin(t)$$

(b) Particular: Using MATLAB as in #4,

```
clear, clc
syms t
%homogeneous solutions
y1 = exp(t); y2 = cos(t); y3 = sin(t);
g = sec(t);    %nonhomogeneous function
B = [0; 0 ; g];
w1 = [y1; diff(y1,t,1); diff(y1,t,2)];
w2 = [y2; diff(y2,t,1); diff(y2,t,2)];
w3 = [y3; diff(y3,t,1); diff(y3,t,2)];
A = [w1, w2, w3];
W = simplify(det(A))  %compute Wronskian
u1_d = det([B, w2, w3])/W
u2_d = det([w1, B, w3])/W
u3_d = det([w1, w2, B])/W
```

W = $2e^t$

u1_d =

$$\frac{e^{-t}\left(\cos(t)^2 + \sin(t)^2\right)}{2\cos(t)}$$

u2_d =

$$-\frac{\cos(t) - \sin(t)}{2\cos(t)}$$

u3_d =

$$-\frac{\cos(t) + \sin(t)}{2\cos(t)}$$

$$\therefore\ u_1' = \frac{e^{-t}}{2\cos(t)} \qquad u_2' = -\frac{1}{2} + \frac{\sin(t)}{2\cos(t)}$$

$$u_3' = -\frac{1}{2} - \frac{\sin(t)}{2\cos(t)}$$

$$\therefore\ u_1 = \int_{t_0}^{t} \frac{e^{-s}}{2\cos(s)}\,ds \qquad u_2 = \int -\frac{1}{2} + \frac{\sin(t)}{2\cos(t)} = -\frac{t}{2} - \frac{1}{2}\ln[\cos(t)]$$

$$u_3 = \int -\frac{1}{2} - \frac{\sin(t)}{2\cos(t)} = -\frac{t}{2} + \frac{1}{2}\ln[\cos(t)]$$

$$\therefore\ y_p = \frac{e^{t}}{2}\int_{t_0}^{t} \frac{e^{-s}}{\cos(s)}\,ds - \frac{t\cos(t)}{2} - \frac{\cos(t)}{2}\ln[\cos(t)]$$

$$- \frac{t\sin(t)}{2} + \frac{\sin(t)}{2}\ln[\cos(t)]$$

$$= \frac{e^{t}}{2}\int_{t_0}^{t} \frac{e^{-s}}{\cos(s)}\,ds - \frac{t}{2}\left[\cos(t) + \sin(t)\right]$$

$$- \frac{1}{2}\left[\cos(t) - \sin(t)\right]\ln[\cos(t)]$$

(c) $$y(t) = c_1 e^{t} + c_2\cos(t) + c_3\sin(t) + \frac{e^{t}}{2}\int_{t_0}^{t} \frac{e^{-s}}{\cos(s)}\,ds$$

$$- \frac{t}{2}\left[\cos(t) + \sin(t)\right] - \frac{1}{2}\left[\cos(t) - \sin(t)\right]\ln[\cos(t)]$$

where $t_0 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

6.

(a) Homogeneous: As in #2 above,

$$y_c(t) = C_1 + C_2 e^{t} + C_3 e^{-t}$$

(b) Particular: Using MATLAB:

```
clear, clc
syms t
%homogeneous solutions
y1 = 1; y2 = exp(t); y3 = exp(-t);
g = csc(t);    %nonhomogeneous function
B = [0; 0 ; g];
w1 = [y1; diff(y1,t,1); diff(y1,t,2)];
w2 = [y2; diff(y2,t,1); diff(y2,t,2)];
w3 = [y3; diff(y3,t,1); diff(y3,t,2)];
A = [w1, w2, w3]
W = simplify(det(A))  %compute Wronskian
u1_d = det([B, w2, w3])/W
u2_d = det([w1, B, w3])/W
u3_d = det([w1, w2, B])/W
```

A =

$$\begin{pmatrix} 1 & e^{t} & e^{-t} \\ 0 & e^{t} & -e^{-t} \\ 0 & e^{t} & e^{-t} \end{pmatrix}$$

W = 2

u1_d =

$$-\frac{1}{\sin(t)}$$

u2_d =

$$\frac{e^{-t}}{2\sin(t)}$$

u3_d =

$$\frac{e^{t}}{2\sin(t)}$$

$$\therefore \; u_1 = \int -\csc(t) = -\ln\left|\csc(t) - \cot(t)\right|$$

(using a table of integrals)

$$= -\ln\left[\csc(t) - \cot(t)\right], \quad \text{for } 0 < t < \pi$$

$$u_2 = \frac{1}{2}\int_{t_0}^{t} \frac{e^{-s}}{\sin(s)}\,ds, \quad 0 < t_0 < \pi$$

$$u_3 = \frac{1}{2}\int_{t_0}^{t} \frac{e^{s}}{\sin(s)}\,ds, \quad 0 < t_0 < \pi$$

$$\therefore\ y_p(t) = -\ln[\csc(t)-\cot(t)] + \frac{e^{t}}{2}\int_{t_0}^{t}\frac{e^{-s}}{\sin(s)}\,ds + \frac{e^{-t}}{2}\int_{t_0}^{t}\frac{e^{s}}{\sin(s)}\,ds \quad, 0<t_0<\pi$$

(c) $$\therefore\ \underline{y(t)} = C_1 + C_2e^{t} + C_3e^{-t} - \ln[\csc(t)-\cot(t)] + \underline{\frac{e^{t}}{2}\int_{t_0}^{t}\frac{e^{-s}}{\sin(s)}\,ds} + \frac{e^{-t}}{2}\int_{t_0}^{t}\frac{e^{s}}{\sin(s)}\,ds \quad, 0<t_0<\pi$$

Note: $-\ln[\csc(t)-\cot(t)] = \ln[\csc(t)+\cot(t)]$

$\iff 0 = \ln[\csc(t)+\cot(t)] + \ln[\csc(t)-\cot(t)]$

$\iff 0 = \ln\left([\csc(t)+\cot(t)][\csc(t)-\cot(t)]\right)$

$\iff 0 = \ln\left(\csc^2(t)-\cot^2(t)\right)$

$\iff 1 = \csc^2(t)-\cot^2(t)$

$\iff 1+\cot^2(t) = \csc^2(t)$

$\iff 1 + \frac{\cos^2(t)}{\sin^2(t)} = \frac{1}{\sin^2(t)}$

$\iff \frac{\sin^2(t)+\cos^2(t)}{\sin^2(t)} = \frac{1}{\sin^2(t)}$ ✓

$\therefore$ Answer could use $\ln[\csc(t)+\cot(t)]$ instead of $-\ln[\csc(t)-\cot(t)]$, and as part of c_1 could use $\ln\left[\frac{\csc(t)+\cot(t)}{\csc(t_0)+\cot(t_0)}\right]$ as the divisor $\csc(t_0)+\cot(t_0)$ is a constant. The answer in this form (as in back of the book) keeps this term consistent with the other terms in the particular solution.

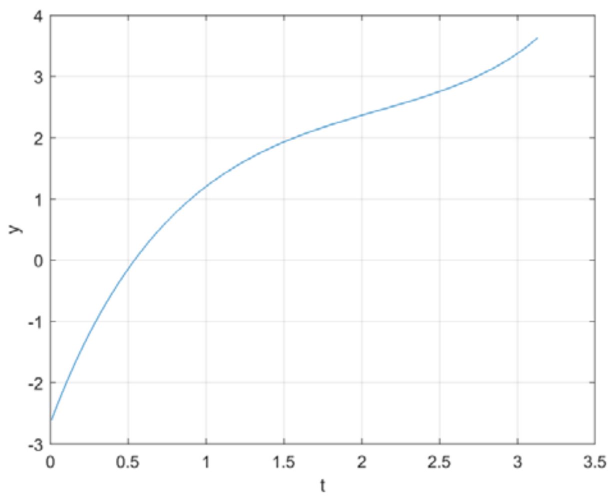
Note: to plot a function involving an integral, such as $y(t) = \int_{t_0}^{t} \cosh(t-s)\csc(s)\,ds$, over an interval (a, b), there are three commands in MATLAB to accomplish this: subs( ), trapz(), and integral( ). In each case, a vector for the x-axis interval (a, b) is created, such as

```
a = -pi/2; b = pi/2;
d = 0.01;  %keep interval open
t = a + d : 0.01 : b - d;
```

A vector of the same length must be created for y(t). If the vector for (a,b) has many elements (smoother, more accurate plot), the subs() method is prohibitively long time-wise. trapz() and integral() are very fast. An example of each method is given below plotting $y(t) = 3 - \exp(-t + \pi/2) + \ln(\csc(t) + \cot(t)) + \int_{t_0}^{t} \cosh(t-s)\csc(s)\,ds$ and using $t_0 = \pi/2$.

subs():

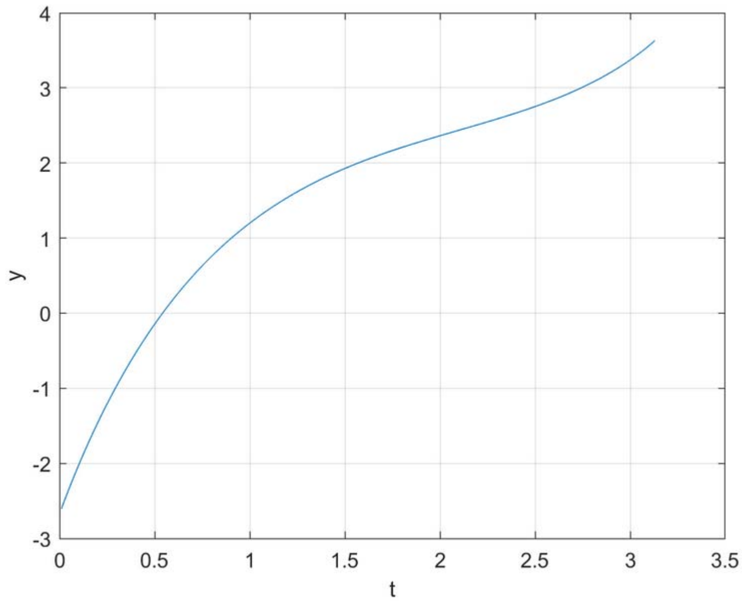
```
clear, clc
a = 0; b = pi;
d = 0.01;  %keep interval open
t = a + d : 0.01 : b - d;
% yc is an example homogeneous solution
yc = 3 - exp(-t + pi/2) + log(csc(t) + cot(t));
syms s x
t0 = pi/2;  % t0 is in (a,b)
intp(x) = int((cosh(x-s)*csc(s)),s,t0,x); %function to plot
yp = subs(intp(x),x,t);  % create the vector for yp
plot(t, yc + yp)
xlabel('t'), ylabel('y')
grid on
```



this code takes forever to run

trapz():

```
clear, clc
a = 0; b = pi;
d = 0.01;  %keep interval open
t = a + d : 0.01 : b - d;
% yc is an example homogeneous solution
yc = 3 - exp(-t + pi/2) + log(csc(t) + cot(t));
t0 = pi/2;  % t0 is in (a,b)
yp = zeros(1, length(t)); % preallocate memory
for i = 1:length(t)  %create vector for yp
  upperbound = t(i);
  h = (upperbound - t0)/1000;
  x = t0:h:upperbound;
  y = cosh(upperbound-x).*csc(x);
  yp(i) = trapz(x,y); %integrate y from t0 to t(i)
end
plot(t, yc + yp)
xlabel('t'), ylabel('y')
grid on
```

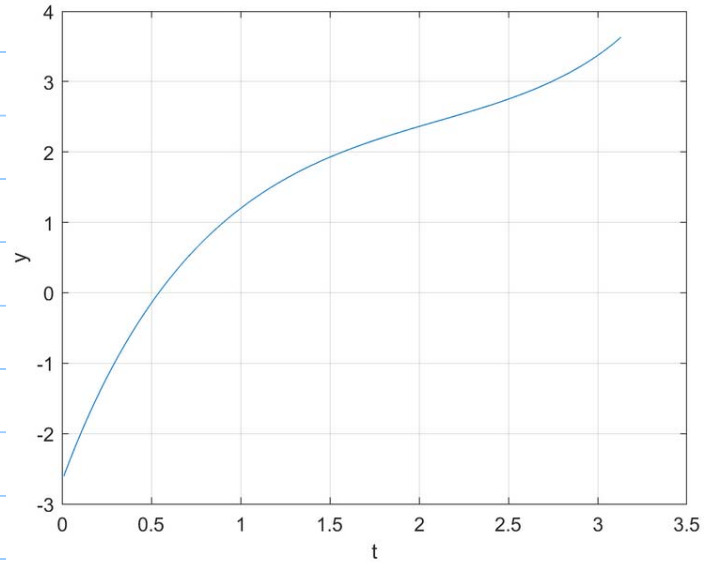


this code runs almost instantaneously

same plot as from subs()

integral()

```
clear, clc
a = 0; b = pi;
d = 0.01;  %keep interval open
t = a + d : 0.01 : b - d;
% yc is an example homogeneous solution
yc = 3 - exp(-t + pi/2) + log(csc(t) + cot(t));
t0 = pi/2;  % t0 is in (a,b)
func = @(x,c) cosh(c - x).*csc(x);
yp = zeros(1, length(t)); %preallocate memory for speed
for i = 1:length(t)
  %create yp vector integrating func from t0 to t(i)
  yp(i) = integral(@(x)func(x, t(i)), t0, t(i));
end
plot(t, yc + yp)
xlabel('t'), ylabel('y')
grid on
```



This code also runs almost instantaneously

Note all plots are identical. The integral() method has slightly less code, but involves the awkward @ function declaration. The function $\int_{t_0}^{t} \cosh(t-s)\csc(s)\,ds$ was specifically chosen as the upperbound t is also in the integrand. If a simple function like $y(t) = \int_{t_0}^{t} \tan(s)\,ds$ were to be plotted, then just `func = @(x) tan(x);` and `integral(func, t0, t(i))` could be used.

Note that for both trapz() and integral(), sometimes the upperbound $< t_0$, so that integration is performed "backwards" or right-to-left. Both MATLAB functions seem to handle this correctly. The subs() method avoids use of for loops, but is not recommended because of speed factors. The integral method will be used for problems #7, #8 below.

7.

From #5 above, and using $t_0 = 0$

$$y(t) = c_1 e^t + c_2 \cos(t) + c_3 \sin(t) + \frac{e^t}{2} \int_0^t \frac{e^{-s}}{\cos(s)}\, ds$$

$$- \frac{t}{2}\left[\cos(t) + \sin(t)\right] - \frac{1}{2}\left[\cos(t) - \sin(t)\right] \ln\left[\cos(t)\right]$$

$$\therefore \; y(0) = c_1 + c_2$$

Use MATLAB to do the rest of the differentiation.

```
clear, clc
syms t c1 c2 c3 s
yc = c1*exp(t) + c2*cos(t) + c3*sin(t);
yp1 = (exp(t)/2)*int((exp(-s)/cos(s)),s,0,t);
yp2 = -(t/2)*(cos(t) + sin(t));
yp3 = -0.5*(cos(t) - sin(t))*log(cos(t));
y = yc + yp1 + yp2 + yp3;
subs(y,t,0)
subs(diff(y,t,1),t,0)
subs(diff(y,t,2),t,0)
```

ans = $c_1 + c_2$

ans = $c_1 + c_3$

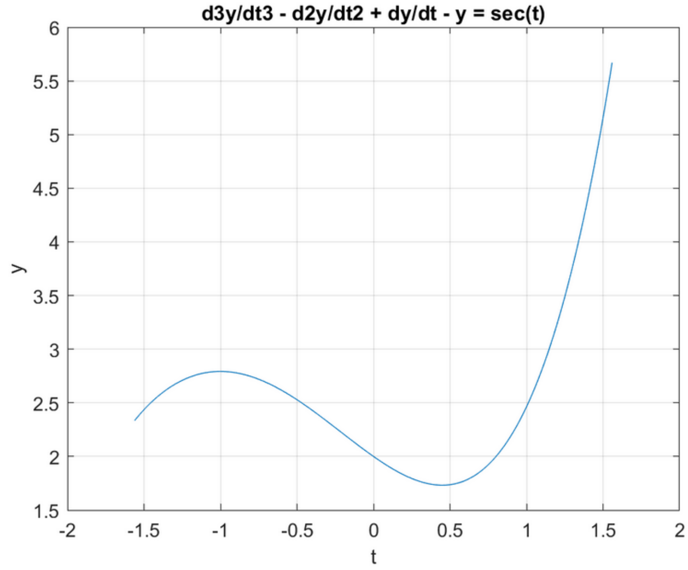
ans = $c_1 - c_2$

$$\therefore \left.\begin{aligned} y(0) &= c_1 + c_2 = 2 \\ y'(0) &= c_1 + c_3 = -1 \\ y''(0) &= c_1 - c_2 = 1 \end{aligned}\right\} \therefore 2c_1 = 3, \; c_1 = \frac{3}{2}, \quad c_2 = \frac{1}{2}, \quad c_3 = -\frac{5}{2}$$

$$\therefore y(t) = \frac{3}{2}e^t + \frac{1}{2}\cos(t) - \frac{5}{2}\sin(t) + \frac{e^t}{2}\int_0^t \frac{e^{-s}}{\cos(s)}\,ds - \frac{t}{2}\left[\cos(t) + \sin(t)\right] - \frac{1}{2}\left[\cos(t) - \sin(t)\right]\ln\left[\cos(t)\right]$$

MATLAB:

```
clear, clc
a = -pi/2; b = pi/2;
d = 0.01;  %keep interval open
t = a + d : 0.01 : b - d;
c1 = 3/2; c2 = 1/2; c3 = -5/2;
yc = c1*exp(t) + c2*cos(t) + c3*sin(t);
t0 = 0;  % t0 is in (a,b)
%evaluate integral from t0 to x, where x is in (a,b)
%thus int1 is a vector of t as it varies from a to b
func = @(x) exp(-x)./cos(x);
int1 = zeros(1, length(t)); %preallocate memory for speed
for i = 1:length(t)
  %create yp vector integrating func from t0 to t(i)
  int1(i) = integral(func, t0, t(i));
end
yp1 = (exp(t)/2).*int1;
yp2 = -(t/2).*(cos(t) + sin(t));
yp3 = -0.5*(cos(t) - sin(t)).*log(cos(t));
y = yc + yp1 + yp2 + yp3;
plot(t, y)
grid on
xlabel 't', ylabel 'y'
title 'd3y/dt3 - d2y/dt2 + dy/dt - y = sec(t)'
```



Plot from

$-\frac{\pi}{2} < t < \frac{\pi}{2}$

8.

Assume $-\frac{\pi}{2} < t < \frac{\pi}{2}$
from $\tan(t)$

(a) Homogeneous: From # 2 above,

$$y_c(t) = C_1 + C_2 e^{t} + C_3 e^{-t}$$

(b) Particular: Using MATLAB

```
clear, clc
syms t
%homogeneous solutions
y1 = 1; y2 = exp(t); y3 = exp(-t);
g = tan(t);    %nonhomogeneous function
B = [0; 0 ; g];
w1 = [y1; diff(y1,t,1); diff(y1,t,2)];
w2 = [y2; diff(y2,t,1); diff(y2,t,2)];
w3 = [y3; diff(y3,t,1); diff(y3,t,2)];
A = [w1, w2, w3]
W = simplify(det(A))  %compute Wronskian
u1_d = det([B, w2, w3])/W
u2_d = det([w1, B, w3])/W
u3_d = det([w1, w2, B])/W
```

A =

$$\begin{pmatrix} 1 & e^{t} & e^{-t} \\ 0 & e^{t} & -e^{-t} \\ 0 & e^{t} & e^{-t} \end{pmatrix}$$

W = 2

u1_d = $-\tan(t)$

u2_d =

$$\frac{e^{-t}\tan(t)}{2}$$

u3_d =

$$\frac{e^{t}\tan(t)}{2}$$

$$\therefore\ u_1(t) = \int -\tan(t) = \ln[\cos(t)]$$

$$u_2(t) = -\frac{1}{2}\int_{t_0}^{t} e^{-s}\tan(s)\,ds$$

$$u_3(t) = \frac{1}{2}\int_{t_0}^{t} e^{s}\tan(s)\,ds$$

$$\therefore\ y_p(t) = y_1u_1 + y_2u_2 + y_3u_3$$

$$= \ln[\cos(t)] + \frac{e^{t}}{2}\int_{t_0}^{t} e^{-s}\tan(s)\,ds + \frac{e^{-t}}{2}\int_{t_0}^{t} e^{s}\tan(s)\,ds$$

(c) $\therefore\ y(t) = c_1 + c_2e^{t} + c_3e^{-t} + \ln[\cos(t)] + \frac{e^{t}}{2}\int_{t_0}^{t} e^{-s}\tan(s)\,ds + \frac{e^{-t}}{2}\int_{t_0}^{t} e^{s}\tan(s)\,ds$

Use MATLAB to do the differentiation to determine c_1, c_2, c_3.

```
clear, clc
syms t c1 c2 c3 s
yc = c1 + c2*exp(t) + c3*exp(-t);
yp1 = log(cos(t));
yp2 = (exp(t)/2)*int((exp(-s)*tan(s)),s,pi/4,t);
yp3 = (exp(-t)/2)*int((exp(s)*tan(s)),s,pi/4,t);
y = yc + yp1 + yp2 + yp3;
subs(y,t,pi/4)
subs(diff(y,t,1),t,pi/4)
subs(diff(y,t,2),t,pi/4)
```

ans =

$$c_1 + \log\left(\frac{\sqrt{2}}{2}\right) + c_2\, e^{\pi/4} + c_3\, e^{-\frac{\pi}{4}}$$

ans =

$$c_2\, e^{\pi/4} - c_3\, e^{-\frac{\pi}{4}}$$

ans =

$$c_2\, e^{\pi/4} + c_3\, e^{-\frac{\pi}{4}}$$

$$\therefore \begin{bmatrix} 1 & e^{\pi/4} & e^{-\pi/4} \\ 0 & e^{\pi/4} & -e^{-\pi/4} \\ 0 & e^{\pi/4} & e^{-\pi/4} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2 - \ln\left(\frac{\sqrt{2}}{2}\right) \\ 1 \\ -1 \end{bmatrix}$$

$$\therefore 2c_2 e^{\pi/4} = 0, \quad \underline{c_2 = 0}$$

$$\therefore c_3 e^{-\pi/4} = -1, \quad \underline{c_3 = -e^{\pi/4}}, \quad c_1 - 1 = 2 - \ln\frac{\sqrt{2}}{2}$$

$$\therefore c_1 = 3 - \ln\left(\frac{\sqrt{2}}{2}\right)$$

$$\therefore \underline{y(t)} = 3 - \ln\left(\frac{\sqrt{2}}{2}\right) - e^{-t+\pi/4} + \ln[\cos(t)]$$

$$\underline{+ \frac{e^{t}}{2}\int_{\pi/4}^{t} e^{-s}\tan(s)\,ds + \frac{e^{-t}}{2}\int_{\pi/4}^{t} e^{s}\tan(s)\,ds}$$

Note: $\ln[\cos(t)] = -\ln\left[\frac{1}{\cos(t)}\right] = -\ln[\sec(t)]$

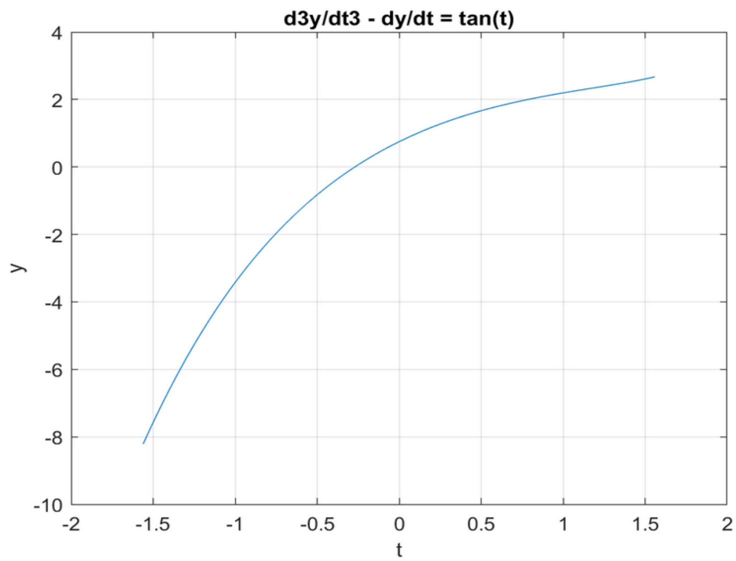
and $\cos(t) > 0$ for $-\frac{\pi}{2} < t < \frac{\pi}{2}$

Also, $-\ln\left(\frac{\sqrt{2}}{2}\right) = -\ln\left(\frac{1}{\sqrt{2}}\right) = \ln(\sqrt{2}) =$

$$\ln 2^{1/2} = \frac{1}{2}\ln(2)$$

(d) From MATLAB:

```
clear, clc
a = -pi/2; b = pi/2;
d = 0.01;  %keep interval open
t = a + d : 0.01 : b - d;
yc = 3 + 0.5*log(2) - exp(-t + pi/4);
t0 = pi/4;  % t0 is in (a,b)
%evaluate integrals from t0 to x, where x is in (a,b)
func2 = @(x) exp(-x).*tan(x);
func3 = @(x) exp(x).*tan(x);
int2 = zeros(1, length(t)); %preallocate memory for speed
int3 = zeros(1, length(t));
for i = 1:length(t)
  %create vector integrating func from t0 to t(i)
  int2(i) = integral(func2, t0, t(i));
  int3(i) = integral(func3, t0, t(i));
end
yp1 = log(cos(t));
yp2 = (exp(t)/2).*int2;
yp3 = (exp(-t)/2).*int3;
y = yc + yp1 + yp2 + yp3;
plot(t, y)
grid on
xlabel 't', ylabel 'y'
title 'd3y/dt3 - dy/dt = tan(t)'
```



plot over $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

9.

Note $W[x, x^2, \frac{1}{x}] = \frac{6}{x}$

MATLAB:

```
% Compute Wronskian
clear, clc
syms x
y1 = x;
y2 = x^2;
y3 = 1/x;
W = [y1 y2 y3; ...
     diff(y1,x) diff(y2,x) diff(y3,x); ...
     diff(y1,x,2) diff(y2,x,2) diff(y3,x,2)];
simplify(det(W))
```

ans =

$$\frac{6}{x}$$

$\therefore$ $x, x^2, \frac{1}{x}$ form a fundamental set of solutions.

$$\therefore \quad y_c(x) = c_1 x + c_2 x^2 + c_3 \frac{1}{x}$$

Since $x > 0$, equation becomes:

$$y''' + \frac{1}{x} y'' - \frac{2}{x^2} y' + \frac{2}{x^3} y = 2x$$

Let $y_p = u_1 x + u_2 x^2 + u_3 \left(\frac{1}{x}\right)$

Use MATLAB for the computations:

```
clear, clc
syms x
%homogeneous solutions
y1 = x; y2 = x^2; y3 = 1/x;
g = 2*x;     %nonhomogeneous function
B = [0; 0 ; g];
w1 = [y1; diff(y1,x,1); diff(y1,x,2)];
w2 = [y2; diff(y2,x,1); diff(y2,x,2)];
w3 = [y3; diff(y3,x,1); diff(y3,x,2)];
A = [w1, w2, w3];
W = simplify(det(A));  %compute Wronskian
u1_d = det([B, w2, w3])/W
u2_d = det([w1, B, w3])/W
u3_d = det([w1, w2, B])/W
u1 = int(u1_d)
u2 = int(u2_d)
u3 = int(u3_d)
y = y1*u1 + y2*u2 + y3*u3;
simplify(y)
```

u1_d = $-x^2$

u2_d =

$$\frac{2x}{3}$$

u3_d =

$$\frac{x^4}{3}$$

u1 =

$$-\frac{x^3}{3}$$

u2 =

$$\frac{x^2}{3}$$

u3 =

$$\frac{x^5}{15}$$

ans =

$$\frac{x^4}{15}$$

$$\therefore y_p(x) = \frac{x^4}{15}$$

10.

From #4, $y_c(t) = c_1 e^t + c_2 \cos(t) + c_3 \sin(t)$

Let $y_p(t) = u_1 e^t + u_2 \cos(t) + u_3 \sin(t)$

Using MATLAB:

```
clear, clc
syms t g(t)
%homogeneous solutions
y1 = exp(t); y2 = cos(t); y3 = sin(t);
B = [0; 0 ; g];
w1 = [y1; diff(y1,t,1); diff(y1,t,2)];
w2 = [y2; diff(y2,t,1); diff(y2,t,2)];
w3 = [y3; diff(y3,t,1); diff(y3,t,2)];
A = [w1, w2, w3];
W = simplify(det(A))  %compute Wronskian
u1_d = det([B, w2, w3])/W
u2_d = det([w1, B, w3])/W
u3_d = det([w1, w2, B])/W
```

W = $2\,e^t$

u1_d(t) =

$$\frac{e^{-t}\,\left(g(t)\cos(t)^2 + g(t)\sin(t)^2\right)}{2}$$

u2_d(t) =

$$-\frac{e^{-t}\,\left(e^t\cos(t)\,g(t) - e^t\,g(t)\sin(t)\right)}{2}$$

u3_d(t) =

$$-\frac{e^{-t}\,\left(e^t\cos(t)\,g(t) + e^t\,g(t)\sin(t)\right)}{2}$$

$$\therefore\ u_1' = \frac{e^{-t}}{2} g(t)$$

$$u_2' = \frac{1}{2}\left[-\cos(t) + \sin(t)\right] g(t)$$

$$u_3' = \frac{1}{2}\left[-\cos(t) - \sin(t)\right] g(t)$$

$$u_1 = \frac{1}{2}\int_{t_0}^{t} e^{-s} g(s)\,ds$$

$$u_2 = \frac{1}{2}\int_{t_0}^{t} \left[-\cos(s) + \sin(s)\right] g(s)\,ds$$

$$u_3 = \frac{1}{2}\int_{t_0}^{t} \left[-\cos(s) - \sin(s)\right] g(s)\,ds$$

$$\therefore\ y_p(t) = u_1 e^{t} + u_2 \cos(t) + u_3 \sin(t)$$

$$= \frac{1}{2}\int_{t_0}^{t} e^{(t-s)} g(s)\,ds$$

$= -\sin(t-s)$

$$+ \frac{1}{2}\int_{t_0}^{t} \left[-\cos(t)\cos(s) + \cos(t)\sin(s)\right] g(s)\,ds$$

$$+ \frac{1}{2}\int_{t_0}^{t} \left[-\sin(t)\cos(s) - \sin(t)\sin(s)\right] g(s)\,ds$$

$= -\cos(t-s)$

$$= \frac{1}{2} \int_{t_0}^{t} \left[e^{(t-s)} - \sin(t-s) - \cos(t-s) \right] g(s)\, ds$$

where t_0 is in the interval over which $g(t)$ is continuous.

11.

(a) Homogeneous: $r^4 - 1 = (r^2+1)(r^2-1) = 0$, $r = i, -i, 1, -1$

$\therefore y_c = c_1 \cos(t) + c_2 \sin(t) + c_3 e^{t} + c_4 e^{-t}$

Since $\cosh(t) = \frac{e^t + e^{-t}}{2}$, $\sinh(t) = \frac{e^t - e^{-t}}{2}$

$\therefore \cosh(t) + \sinh(t) = e^t$, $\cosh(t) - \sinh(t) = e^{-t}$

$$\therefore y_c = c_1 \cos(t) + c_2 \sin(t) + c_3 [\cosh(t) + \sinh(t)] + c_4 [\cosh(t) - \sinh(t)]$$

$$= c_1 \cos(t) + c_2 \sin(t) + [c_3 + c_4] \cosh(t) + [c_3 - c_4] \sinh(t)$$

Or, $y_c(t) = c_1 \cos(t) + c_2 \sin(t) + c_3' \cosh(t) + c_4' \sinh(t)$

(b) Particular: Let $y_p(t) = u_1(t)\cos(t) + u_2(t)\sin(t) + u_3(t)\cosh(t) + u_4(t)\sinh(t)$

Using MATLAB:

```
clear, clc
syms t g(t)
%homogeneous solutions
y1 = cos(t); y2 = sin(t); y3 = cosh(t); y4 = sinh(t);
B = [0; 0; 0; g];
w1 = [y1; diff(y1,t,1); diff(y1,t,2); diff(y1,t,3)];
w2 = [y2; diff(y2,t,1); diff(y2,t,2); diff(y2,t,3)];
w3 = [y3; diff(y3,t,1); diff(y3,t,2); diff(y3,t,3)];
w4 = [y4; diff(y4,t,1); diff(y4,t,2); diff(y4,t,3)];
A = [w1, w2, w3, w4]
W = simplify(det(A))  %compute Wronskian
u1_d = simplify(det([B, w2, w3, w4])/W)
u2_d = simplify(det([w1, B, w3, w4])/W)
u3_d = simplify(det([w1, w2, B, w4])/W)
u4_d = simplify(det([w1, w2, w3, B])/W)
u1 = int(u1_d)  %integrate above derivatives
u2 = int(u2_d)
u3 = int(u3_d)
u4 = int(u4_d)
y = y1*u1 + y2*u2 + y3*u3 + y4*u4;
simplify(y)
```

A =

$$\begin{pmatrix} \cos(t) & \sin(t) & \cosh(t) & \sinh(t) \\ -\sin(t) & \cos(t) & \sinh(t) & \cosh(t) \\ -\cos(t) & -\sin(t) & \cosh(t) & \sinh(t) \\ \sin(t) & -\cos(t) & \sinh(t) & \cosh(t) \end{pmatrix}$$

W = 4

u1_d(t) =

$$\frac{g(t)\sin(t)}{2}$$

u2_d(t) =

$$-\frac{\cos(t)\,g(t)}{2}$$

u3_d(t) =

$$-\frac{g(t)\sinh(t)}{2}$$

u4_d(t) =

$$\frac{\cosh(t)\,g(t)}{2}$$

u1(t) =

$$\int \frac{g(t)\sin(t)}{2}\,dt$$

u2(t) =

$$-\int \frac{\cos(t)\,g(t)}{2}\,dt$$

u3(t) =

$$-\int \frac{g(t)\sinh(t)}{2}\,dt$$

u4(t) =

$$\int \frac{\cosh(t)\,g(t)}{2}\,dt$$

ans(t) =

$$\frac{\cos(t)\int g(t)\sin(t)dt}{2} - \frac{\sin(t)\int \cos(t)\,g(t)dt}{2} - \frac{\cosh(t)\int g(t)\sinh(t)dt}{2} + \frac{\sinh(t)\int \cosh(t)\,g(t)dt}{2}$$

$$\therefore y_p(t) = \frac{\cos(t)}{2}\int_{t_0}^{t} \sin(s)g(s)\,ds - \frac{\sin(t)}{2}\int_{t_0}^{t}\cos(s)g(s)\,ds - \frac{\cosh(t)}{2}\int_{t_0}^{t}\sinh(s)g(s)\,ds + \frac{\sinh(t)}{2}\int_{t_0}^{t}\cosh(s)g(s)\,ds$$

$$= \frac{1}{2}\int_{t_0}^{t}\Big[\underbrace{\cos(t)\sin(s) - \sin(t)\cos(s)}_{= -\sin(t-s)} - \underbrace{\cosh(t)\sinh(s) + \sinh(t)\cosh(s)}_{= \sinh(t-s)}\Big]g(s)\,ds$$

$$\therefore\ y_p(t) = \frac{1}{2}\int_{t_0}^{t}\left[\sinh(t-s) - \sin(t-s)\right]g(s)\,ds$$

where t_0 is in the interval over which $g(t)$ is continuous.

Note: if $y_c = c_1\cos(t) + c_2\sin(t) + c_3e^{t} + c_4e^{-t}$

```
clear, clc
syms t g(t)
%homogeneous solutions
y1 = cos(t); y2 = sin(t); y3 = exp(t); y4 = exp(-t);
B = [0; 0; 0; g];
w1 = [y1; diff(y1,t,1); diff(y1,t,2); diff(y1,t,3)];
w2 = [y2; diff(y2,t,1); diff(y2,t,2); diff(y2,t,3)];
w3 = [y3; diff(y3,t,1); diff(y3,t,2); diff(y3,t,3)];
w4 = [y4; diff(y4,t,1); diff(y4,t,2); diff(y4,t,3)];
A = [w1, w2, w3, w4]
W = simplify(det(A))  %compute Wronskian
u1_d = simplify(det([B, w2, w3, w4])/W)
u2_d = simplify(det([w1, B, w3, w4])/W)
u3_d = simplify(det([w1, w2, B, w4])/W)
u4_d = simplify(det([w1, w2, w3, B])/W)
u1 = int(u1_d)
u2 = int(u2_d)
u3 = int(u3_d)
u4 = int(u4_d)
y = y1*u1 + y2*u2 + y3*u3 + y4*u4;
simplify(y)
```

ans(t) =

$$\frac{\cos(t)\int g(t)\sin(t)\,dt}{2} - \frac{\sin(t)\int \cos(t)\,g(t)\,dt}{2} - \frac{e^{-t}\int e^{t}\,g(t)\,dt}{4} + \frac{e^{t}\int e^{-t}\,g(t)\,dt}{4}$$

Or,

$$y_p(t) = \frac{1}{2}\int_{t_0}^{t}\left[\frac{e^{(t-s)} - e^{-(t-s)}}{2} - \sin(t-s)\right]g(s)\,ds$$

$$= \frac{1}{2}\int_{t_0}^{t}\left[\sinh(t-s) - \sin(t-s)\right] g(s)\,ds$$

answer in same form.

12.

(a) Homogeneous: $r^3 - 3r^2 + 3r - 1 = (r-1)^3 = 0, \quad r = 1,1,1$

$\therefore\ y_c = c_1 e^t + c_2 t e^t + c_3 t^2 e^t$

(b) Particular: let $y_p = u_1(t)e^t + u_2(t)te^t + u_3(t)t^2e^t$

Using MATLAB:

```
clear, clc
syms t g(t)
%homogeneous solutions
y1 = exp(t); y2 = t*exp(t); y3 = t^2*exp(t);
%g = tan(t);    %nonhomogeneous function
B = [0; 0 ; g];
w1 = [y1; diff(y1,t,1); diff(y1,t,2)];
w2 = [y2; diff(y2,t,1); diff(y2,t,2)];
w3 = [y3; diff(y3,t,1); diff(y3,t,2)];
A = [w1, w2, w3]
W = simplify(det(A))  %compute Wronskian
u1_d = det([B, w2, w3])/W
u2_d = det([w1, B, w3])/W
u3_d = det([w1, w2, B])/W
u1 = int(u1_d)
u2 = int(u2_d)
u3 = int(u3_d)
y = y1*u1 + y2*u2 + y3*u3;
simplify(y)
```

A =

$$\begin{pmatrix} e^t & t\,e^t & t^2\,e^t \\ e^t & e^t + t\,e^t & t^2\,e^t + 2\,t\,e^t \\ e^t & 2\,e^t + t\,e^t & 2\,e^t + t^2\,e^t + 4\,t\,e^t \end{pmatrix}$$

W = $2\,e^{3\,t}$

u1_d(t) =

$$\frac{t^2\,e^{-t}\,g(t)}{2}$$

u2_d(t) = $-t\,e^{-t}\,g(t)$

u3_d(t) =

$$\frac{e^{-t}\,g(t)}{2}$$

u1(t) =

$$\int \frac{t^2\,e^{-t}\,g(t)}{2}\,dt$$

u2(t) =

$$-\int t\,e^{-t}\,g(t)\,dt$$

u3(t) =

$$\int \frac{e^{-t}\,g(t)}{2}\,dt$$

ans(t) =

$$\frac{e^t \int t^2\,e^{-t}\,g(t)\,dt}{2} + \frac{t^2\,e^t \int e^{-t}\,g(t)\,dt}{2} - t\,e^t \int t\,e^{-t}\,g(t)\,dt$$

$$\therefore\ y_p(t) = \frac{e^t}{2}\int_{t_0}^{t} s^2 e^{-s} g(s)\,ds + \frac{t^2 e^t}{2}\int_{t_0}^{t} e^{-s} g(s)\,ds - te^t\int_{t_0}^{t} s e^{-s} g(s)\,ds$$

$$= \frac{1}{2}\int_{t_0}^{t}\left[s^2 e^{(t-s)} + t^2 e^{(t-s)} - 2ts\,e^{(t-s)}\right] g(s)\,ds$$

$$= \frac{1}{2}\int_{t_0}^{t} (t-s)^2 e^{(t-s)} g(s)\,ds$$

(c) With $g(t) = t^{-2}e^t$, then $g(s) = s^{-2}e^s$

$$\therefore\ (t-s)^2 e^{(t-s)} s^{-2} e^s = \left(\frac{t-s}{s}\right)^2 e^t$$

$$= \left(\frac{t^2}{s^2} - \frac{2t}{s} - 1\right) e^t$$

$$\therefore\ y_p(t) = \frac{1}{2}\int_{t_0}^{t}\left(\frac{t^2}{s^2} - \frac{2t}{s} - 1\right) e^t\,ds$$

$$= \frac{e^t}{2}\int_{t_0}^{t}\left(\frac{t^2}{s^2} - \frac{2t}{s} - 1\right) ds$$

$$= \frac{e^t}{2}\left[-\frac{t^2}{s} - 2t\ln|s| - s\right]_{t_0}^{t}$$

$$= \frac{e^t}{2}\left[-t - 2t\ln|t| - t + \frac{t^2}{t_0} + 2t\ln|t_0| + t_0\right]$$

$$= -te^{t} - te^{t}\ln|t| + \frac{t^2e^{t}}{2t_0} + \ln|t_0|\,te^{t} + \left(\frac{t_0}{2}\right)e^{t}$$

$$= \underbrace{\left(\frac{t_0}{2}\right)e^{t} + \left(\ln|t_0| - 1\right)te^{t} + \left(\frac{1}{2t_0}\right)t^2e^{t}}_{\text{all contained in } y_c(t)} - te^{t}\ln|t|$$

$$\therefore\ y_p(t) = -te^{t}\ln|t|$$