5.1 Review of Power Series

Note Title 2/11/2019

1.

Using the ratio test, $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-3)^{n+1}}{(x-3)^n}\right| = |x-3|$

$\therefore \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = |x-3|.$

$|x-3| < 1 \Rightarrow$ series converges

$|x-3| > 1 \Rightarrow$ series diverges

$\therefore$ radius of convergence around 3 = $\underline{1}$

2.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)\,x^{n+1}}{2^{n+1}} \cdot \frac{2^n}{n\,x^n}\right| = \left|\frac{(n+1)\,x}{2n}\right|$$

$\therefore \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x}{2}\right|$ $\therefore$ if $\left|\frac{x}{2}\right| < 1$ converges

$\therefore |x| < 2$, or radius of convergence = $\underline{2}$

3.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{2n+2}}{(n+1)!}\cdot\frac{n!}{x^{2n}}\right| = \left|\frac{x^2}{n+1}\right|$$

$$\therefore \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = x^2 \lim_{n\to\infty}\left|\frac{1}{n+1}\right| = 0, \text{ for all } x.$$

$\therefore$ radius of convergence $= \underline{\infty}$

4.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{2^{n+1}\,x^{n+1}}{2^n\,x^n}\right| = 2|x| < 1 \text{ , if } |x| < \frac{1}{2}$$

$\therefore$ radius of convergence $= \frac{1}{2}$

5.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-x_0)^{n+1}}{n+1}\cdot\frac{n}{(x-x_0)^n}\right| = \left|(x-x_0)\ \frac{n}{n+1}\right|$$

$$\therefore \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x-x_0|$$

$\therefore$ radius of convergence around $x_0 = \underline{1}$.

6.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1}(n+1)^2(x+2)^{n+1}}{3^{n+1}} \cdot \frac{3^n}{(-1)^n n^2 (x+2)^n}\right|$$

$$= \left| -\frac{(x+2)}{3}\left(\frac{n+1}{n}\right)^2 \right|$$

$$\therefore \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x+2}{3}\right| < 1 \iff |x+2| < 3$$

$\therefore$ radius of convergence around -2 = $\underline{3}$

7.

Let $f(x) = \sin(x)$

$f(0) = \sin(0) = 0$

$f'(0) = \cos(0) = 1$

$f''(0) = -\sin(0) = 0$

$f'''(0) = -\cos(0) = -1$

$f^{(4)}(0) = \sin(0) = 0$

$f^{(n)}(0) = 1$, $1 + 4n$, $n = 0, 1, 2, 3, \ldots$

$f^{(n)}(0) = -1$, $3 + 4n$, $n = 0, 1, 2, 3, \ldots$

$f^{(n)}(0) = 0$, $0 + 4n$, $n = 0, 1, 2, 3, \ldots$

$f^{n}(0) = 0$, $2 + 4n$, $n = 0, 1, 2, 3, \ldots$

Since $a_n = \dfrac{f^{(n)}(0)}{n!}$,

$a_0 = 0$ $\quad$ $a_4 = 0$

$a_1 = 1/1!$ $\quad$ $a_5 = 1/5!$

$a_2 = 0$ $\quad$ $a_6 = 0$

$a_3 = -1/3!$ $\quad$ $a_7 = -1/7!$

$$\therefore \sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

By the ratio test,

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{(-1)^n x^{2n+1}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x^2}{(2n+3)(2n+2)} \right| = 0 \text{ for all } x.$$

$$\therefore \sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad \rho = \infty$$

8.

(a) $f(0) = e^0 = 1$

$f^{(n)}(x) = e^x \quad \therefore f^{(n)}(0) = 1$

$$\therefore a_n (x-0)^n = \frac{f^{(n)}(0)}{n!} x^n = \frac{x^n}{n!}$$

$$\therefore e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

(b) $$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \left| \frac{x}{n+1} \right|$$

$\therefore$ For all x, $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 0$

$\therefore \underline{\rho = \infty}$

9.

(a) $f(1) = 1 \quad f'(1) = 1, \quad f^{(n)}(x) = 0$ for $n \geq 2$

$\therefore x = 1 + \frac{(1)}{1}(x-1) + 0(x-1)^2 + \ldots$

$\therefore x = 1 + (x-1)$

(b) Since $x = 1 + (x-1)$ for all x, $\rho = \infty$

10.

(a) $f(-1) = 1 \quad f'(-1) = -2 \quad f''(-1) = 2 \quad f^{(n)}(-1) = 0$ for $n \geq 3$

$\therefore x^2 = 1 + \frac{-2}{1}(x+1) + \frac{2}{2}(x+1)^2 + \frac{0}{3!} \cdot (x+1)^3 + 0 \ldots + 0 \ldots$

$\therefore \underline{x^2 = 1 - 2(x+1) + (x+1)^2}$

(b) Above is an identity for all x, $\therefore \underline{\rho = \infty}$

11.

(a) $f(1) = 0$ $\quad f'''(x) = 2x^{-3}$

$f'(1) = 1$ $\quad f^{(4)}(x) = -3(2)x^{-4}$

$f''(1) = -1$ $\quad f^{(5)}(x) = (-4)(-3)(2)x^{-5}$

$$\therefore f^{(n)}(x) = (-1)^{n+1}(n-1)!\, x^{-n}, \quad n \geq 1$$

$$\therefore f^{n}(1) = (-1)^{n+1}(n-1)! \quad \therefore \frac{f^{(n)}(1)}{n!} = \frac{(-1)^{n+1}}{n}$$

$$\therefore \ln(x) = 0 + \frac{1}{1}(x-1) + \frac{(-1)}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 + \ldots$$

$$\ln(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \ldots$$

$$= \underline{\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n}$$

(b) $$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+2}(x-1)^{n+1}}{(n+1)} \cdot \frac{n}{(-1)^{n+1}(x-1)^n}\right| = \left|\frac{n}{n+1}(x-1)\right|$$

$$\therefore \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x-1| \quad \therefore \text{converges if } |x-1| < 1$$

$$\therefore \underline{\rho = 1}$$

12.

(a) $f(0) = 1$ $\quad f(x) = (1-x)^{-1}$ $\quad f^n(x) = n!\,(1-x)^{-(n+1)}$

$f'(x) = (1-x)^{-2}$

$f''(x) = 2(1-x)^{-3}$ $\quad \therefore \dfrac{f^n(0)}{n!} = 1$

$$\therefore \frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots$$

$$= \underline{\sum_{n=0}^{\infty} x^n}$$

(b) $\left|\dfrac{a_{n+1}}{a_n}\right| = \left|\dfrac{x^{n+1}}{x^n}\right| = |x|$. $\therefore$ if $|x|<1$, converges

$\therefore \underline{\rho = 1}$

13.

(a) From #12, $f^n(x) = n!\,(1-x)^{-(n+1)}$, $n \geq 0$

$f(2) = -1$, $\quad \dfrac{f^n(2)}{n!} = (-1)^{-n-1} = \dfrac{1}{(-1)^{n+1}} = (-1)^{n+1}$

$$\therefore \frac{1}{1-x} = -1 + (x-2) - (x-2)^2 + (x-2)^3 - (x-2)^4 + \dots$$

$$= \underline{\sum_{n=0}^{\infty} (-1)^{n+1}(x-2)^n}$$

(b) $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+2}(x-2)^{n+1}}{(-1)^{n+1}(x-2)^n}\right| = |x-2|$

$\therefore$ if $|x-2|<1$, converges. $\therefore$ $\underline{\rho = 1}$

14.

(a) $y' = \sum_{n=1}^{\infty} n^2 x^{n-1} = 1 + 4x + 9x^2 + 16x^3 + \ldots$

$= \underline{\underline{\sum_{n=0}^{\infty} (n+1)^2 x^n}}$ (shifting index)

(b) $y'' = \sum_{n=1}^{\infty} (n+1)^2 n\, x^{n-1} = 4 + 18x + 48x^2 + 100x^3 + \ldots$

$= \underline{\underline{\sum_{n=0}^{\infty} (n+2)^2 (n+1) x^n}}$ (shifting index)

15.

(a)

(1) $\frac{d}{dx} a_n x^n = a_n n x^{n-1}$

$$\therefore y' = \sum_{n=1}^{\infty} a_n n x^{n-1} = \underline{\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n}$$

$$= a_1 x + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \dots$$

(2) $\frac{d}{dx} (n+1) a_{n+1} x^n = (n+1) n \, a_{n+1} x^{n-1}$

$$\therefore y'' = \sum_{n=1}^{\infty} (n+1)(n) a_{n+1} x^{n-1} = \underline{\underline{\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n}}$$

$$= 2a_2 + 6a_3 x + 12 a_4 x^2 + 20 a_5 x^3 + \dots$$

(b)

$$y'' = y \Rightarrow \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$\therefore$ By property #10 of text, p. 192,

$$a_n = (n+2)(n+1) a_{n+2}, \quad n = 0, 1, 2, \dots$$

$$\therefore \begin{array}{lcl} a_0 = 2a_2 & \text{or} & a_2 = a_0/2 \\ a_1 = 6a_3 & & a_3 = a_1/6 \\ a_2 = 12a_4 & & a_4 = a_2/12 \\ \vdots & & \vdots \end{array}$$

$\therefore$ Starting with arbitrary a_0 and a_1, all other coefficients are determined.

(c)

From (b), $a_n = (n+2)(n+1)a_{n+2}$, $n = 0, 1, 2, \ldots$

$$\therefore \underline{a_{n+2} = \frac{a_n}{(n+2)(n+1)}}, \quad n = 0, 1, 2, \ldots$$

16.

$$\sum_{n=0}^{\infty} a_n (x-1)^{n+1} = a_0 (x-1)^1 + a_1 (x-1)^2 + a_2 (x-1)^3 + \ldots$$

If start with $n=1$, then must make coefficient a_{n-1} to make $a_0, a_1, a_2 \ldots$, and must change $(x-1)^{n+1}$ to $(x-1)^n$ to make $(x-1)^1, (x-1)^2, (x-3)^3, \ldots$ for $n = 1, 2, 3, \ldots$

$$\therefore \sum_{n=0}^{\infty} a_n (x-1)^{n+1} = \sum_{n=1}^{\infty} a_{n-1} (x-1)^n$$

Another way to show this is by doing a change of variable. Let $m = n+1$. $\therefore$ $m-1 = n$, and when $n=0$, $m=1$

$$\therefore \sum_{n=0}^{\infty} a_n (x-1)^{n+1} = \sum_{m=1}^{\infty} a_{m-1}(x-1)^m = \sum_{n=1}^{\infty} a_{n-1}(x-1)^n$$

Since m is a dummy variable, can change the latter back to n.

17.

(shift index)

$$\sum_{K=0}^{\infty} a_{K+1} x^K = a_1 + \sum_{K=1}^{\infty} a_{K+1} x^K = a_1 + \sum_{K=0}^{\infty} a_{K+2} x^{K+1}$$

$$\therefore \sum_{K=0}^{\infty} a_{K+1} x^K + \sum_{K=0}^{\infty} a_K x^{K+1} = a_1 + \sum_{K=0}^{\infty} a_{K+2} x^{K+1} + \sum_{K=0}^{\infty} a_K x^{K+1}$$

$$= a_1 + \sum_{K=0}^{\infty} a_{K+2} x^{K+1} + a_K x^{K+1}$$

$$= a_1 + \sum_{K=0}^{\infty} (a_{K+2} + a_K) x^{K+1}$$

$$= a_1 + \sum_{K=1}^{\infty} (a_{K+1} + a_{K-1}) x^K$$ (shift index)

18.

Let $m = n-2$. $\therefore$ When $n=2$, $m=0$. $n = m+2$

$$\therefore \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{m=0}^{\infty} (m+2)(m+2-1)a_{m+2}x^m$$

$$= \underline{\underline{\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n}}$$ (switching back to n).

19.

$$x\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} n a_n x^n$$

$$\sum_{k=0}^{\infty} a_k x^k = a_0 + \sum_{k=1}^{\infty} a_k x^k = a_0 + \sum_{n=1}^{\infty} a_n x^n$$

$$\therefore x\sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{k=0}^{\infty} a_k x^k = \sum_{n=1}^{\infty} n a_n x^n + a_0 + \sum_{n=1}^{\infty} a_n x^n$$

$$= a_0 + \sum_{n=1}^{\infty} (n a_n + a_n)x^n$$

$$= a_0 + \sum_{n=1}^{\infty} (n+1)a_n x^n$$

$$= \sum_{n=0}^{\infty} (n+1) a_n x^n \qquad (a_0 = (0+1) a_0 x^0)$$

20.

$$\sum_{m=2}^{\infty} m(m-1) a_m x^{m-2} : \text{ Let } n = m-2. \quad m = 2 \Rightarrow n = 0$$

$$\therefore m = n+2$$

$$\therefore \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n \qquad [1]$$

$$x \sum_{k=1}^{\infty} k a_k x^{k-1} = \sum_{k=1}^{\infty} k a_k x^k = \sum_{k=0}^{\infty} k a_k x^k$$

as $(0) a_0 x^0 = 0$ and define $a_0 = 0$.

Now let $k = n$

$$\therefore \sum_{n=0}^{\infty} n a_n x^n \qquad [2]$$

Adding [1] and [2],

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} n a_n x^n$$

$$= \sum_{n=0}^{\infty} [(n+2)(n+1) a_{n+2} + n a_n] x^n$$

where a_0 can be any real number, but define it as $a_0 = 0$ for simplicity.

21.

$$\sum_{n=1}^{\infty} n a_n x^{n-1} \; : \; \text{Let } m = n-1 \; \therefore \; n = m+1$$
$$n = 1 \Rightarrow m = 0$$

$$\therefore \sum_{m=0}^{\infty} (m+1) a_{m+1} x^m, \text{ or } \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$= a_1 + \sum_{n=1}^{\infty} (n+1) a_{n+1} x^n \quad [1]$$

as $(0+1) a_{0+1} x^0 = a_1$

$$x \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+1} \; : \; \text{Let } m = n+1, \; \therefore \; n = m-1$$
$$n = 0 \Rightarrow m = 1$$

$$\therefore \sum_{m=1}^{\infty} a_{m-1} x^m, \text{ or } \sum_{n=1}^{\infty} a_{n-1} x^n \quad [2]$$

Adding [1] and [2],

$$a_1 + \sum_{n=1}^{\infty} (n+1) a_{n+1} x^n + \sum_{n=1}^{\infty} a_{n-1} x^n$$

$$= a_1 + \sum_{n=1}^{\infty} \left[(n+1) a_{n+1} + a_{n-1} \right] x^n$$

22.

$$x \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-1}$$

Let $m = n-1$, $\therefore n = m+1$. $n = 2 \Rightarrow m = 1$

$$\therefore \sum_{m=1}^{\infty} (m+1)(m) a_{m+1} x^m = \sum_{m=0}^{\infty} (m+1)(m) a_{m+1} x^m$$

since $(0+1)(0) a_{0+1} x^0 = 0$

$\therefore$ Switching back to n, $\sum_{n=0}^{\infty} (n+1)(n) a_{n+1} x^n$

Adding to $\sum_{n=0}^{\infty} a_n x^n$,

$$\sum_{n=0}^{\infty} (n+1)(n) a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=0}^{\infty} \left[(n+1)(n) a_{n+1} + a_n \right] x^n$$

23.

(a) $\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$

$\therefore \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + 2 \sum_{n=0}^{\infty} a_n x^n = 0$

$= \sum_{n=0}^{\infty} \left[(n+1) a_{n+1} + 2 a_n \right] x^n = 0$

For this to be true for all x, then

$(n+1) a_{n+1} + 2a_n = 0$,

or $a_{n+1} = -\frac{2}{n+1} a_n$

$\therefore$ $n=0$: $a_1 = -\frac{2}{1} a_0$

$n=1$: $a_2 = -\frac{2}{(2)} a_1 = -\frac{2}{(2)} \frac{(-2)}{1} a_0$

$n=2$: $a_3 = -\frac{2}{3} a_2 = -\frac{2}{3} \frac{(-2)}{2} \frac{(-2)}{1} a_0$

$n=3$: $a_4 = -\frac{2}{4} a_3 = -\frac{2}{4} \frac{(-2)}{3} \frac{(-2)}{2} \frac{(-2)}{1} a_0$

$\therefore \quad a_n = \underline{\dfrac{(-2)^n}{n!} a_0} \quad , n = 1, 2, 3, \ldots$

(b) $\sum_{n=0}^{\infty} a_n x^n$ is a Taylor series about 0.

$\therefore$ Find a function s.t. $\dfrac{f^n(0)}{n!} = \dfrac{(-2)^n}{n!} a_0$

or $f^{(n)}(0) = (-2)^n$ and $f(0) = a_0$.

Note that for $f(x) = e^x$, $f^n(0) = 1$

and for $f(x) = e^{ax}$, $f^{(n)}(x) = a^n e^{ax}$

so $f^{(n)}(0) = a^n$

$\therefore$ for $f(x) = e^{-2x}$, $f^{(n)}(0) = (-2)^n$

$\therefore$ for $f(x) = a_0 e^{-2x}$, $f^n(0) = (-2)^n a_0$

$\therefore \underline{f(x) = a_0 e^{-2x}}$

Alternatively,

let $y = \sum_{n=0}^{\infty} a_n x^n$. $\therefore y' = \sum_{n=1}^{\infty} n a_n x^n$

$\therefore$ The problem is equivalent to:

$$y' + 2y = 0$$

The solution to this is $y = ce^{-2x}$,

c a constant, and $y(0) = c$.

From $y = \sum_{n=0}^{\infty} a_n x^n$, $y(0) = a_0$.

$$\therefore \; y(x) = \underline{a_0 e^{-2x}}$$

5.2 Series Solutions Near an Ordinary Point, Part I

Note Title 2/21/2019

1.

(a) Let $y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$

$\therefore y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$

$\therefore \sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} - a_n \right] x^n = 0$

$\therefore (n+2)(n+1) a_{n+2} - a_n = 0$

$\therefore$ Recurrence relation: $\underline{a_{n+2} = \dfrac{a_n}{(n+2)(n+1)}}$

(b) $a_2 = \dfrac{a_0}{2 \cdot 1}$ $\qquad a_3 = \dfrac{a_1}{3 \cdot 2}$

$a_4 = \dfrac{a_2}{4 \cdot 3} = \dfrac{a_0}{4!}$ $\qquad a_5 = \dfrac{a_3}{5 \cdot 4} = \dfrac{a_1}{5!}$

$\therefore a_{2n} = \dfrac{a_0}{(2n)!}, \; n = 1, 2, 3, \ldots$ $\qquad a_{2n+1} = \dfrac{a_1}{(2n+1)!}, \; n = 1, 2, 3, \ldots$

$$\therefore\ a_0 \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = \underline{a_0}\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots\right)$$

$$a_1 \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = \underline{a_1}\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots\right)$$

(c)

$$y_1(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots\right) \qquad y_1(0) = 1$$

$$y_1'(x) = \sum_{n=1}^{\infty} \frac{(2n)\, x^{2n-1}}{(2n)!} = \sum_{n=1}^{\infty} \frac{x^{2n-1}}{(2n-1)!}$$

$$= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = y_2(x) \qquad y_1'(0) = 0$$

$$y_2(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \qquad y_2(0) = 0$$

$$y_2'(x) = \sum_{n=0}^{\infty} \frac{(2n+1)\, x^{2n}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = y_1(x)$$

$$y_2'(0) = 1$$

$$\therefore\ W[y_1, y_2](0) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$$

$\therefore$ $y_1(x)$, $y_2(x)$ form a fundamental set of solutions

(d)

From (c), $\underline{y_1(x)} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$

$\underline{y_2(x)} = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$

2.

(a)

Let $y = \sum_{n=0}^{\infty} a_n x^n \quad \therefore y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$

$y'' = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$

$\therefore y'' + 3y' = \sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} + 3(n+1) a_{n+1} \right] x^n = 0$

$\therefore (n+2)(n+1) a_{n+2} + 3(n+1) a_{n+1} = 0, \quad \underline{a_{n+2}} = -\frac{3a_{n+1}}{n+2}, \; n \geq 0$

(b)

$a_2 = -\frac{3a_1}{2} \qquad a_3 = -\frac{3a_2}{3} = \frac{(-3)^2 a_1}{3 \cdot 2} \qquad a_4 = -\frac{3a_3}{4} = \frac{(-3)^3 a_1}{4!}$

$$\therefore a_n = \frac{(-3)^{n-1}}{n!} a_1, \quad n \geq 1$$

$$\therefore y = a_0 + a_1 x + \frac{(-3)}{2} a_1 x^2 + \frac{(-3)^2}{3!} a_1 x^3 + \frac{(-3)^3}{4!} a_1 x^4 + \ldots$$

$$= a_0 (1) + a_1 \sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{n!} x^n$$

$$\therefore \underline{y_1 = 1} \qquad \underline{y_2 = \sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{n!} x^n}$$

(c)

$$y_1(0) = 1 \qquad y_1'(0) = 0$$

$$y_2(0) = 0 \qquad y_2' = \sum_{n=1}^{\infty} \frac{(-3)^{n-1} (n) x^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{(-3)^{n-1} x^{n-1}}{(n-1)!}$$

$$\therefore y_2'(0) = 1.$$

$$\therefore W[y_1, y_2](0) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = \underline{1} \neq 0$$

$\therefore$ y_1, y_2 form a fundamental set of solutions.

(d)

From (b), $\underline{y_1 = 1}$, $\underline{y_2 = \sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{n!} x^n}$

3.

(a)

Let $y = \sum_{n=0}^{\infty} a_n x^n$, $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$xy' = \sum_{n=1}^{\infty} n a_n x^n = \sum_{n=0}^{\infty} n a_n x^n$$

$$\therefore y'' - xy' - y = \sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} - n a_n - a_n\right] x^n$$

$$= \sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} - (n+1) a_n\right] x^n$$

$\therefore (n+2)(n+1) a_{n+2} - (n+1) a_n = 0$, $\underline{a_{n+2} = \frac{a_n}{n+2}}$, $n \geq 0$

(b)

$a_2 = \frac{a_0}{2}$ $\qquad$ $a_3 = \frac{a_1}{3}$

$a_4 = \frac{a_2}{4} = \frac{a_0}{4 \cdot 2}$ $\qquad$ $a_5 = \frac{a_3}{5} = \frac{a_1}{5 \cdot 3}$

$$a_6 = \frac{a_4}{6} = \frac{a_0}{6\cdot 4\cdot 2} \qquad a_7 = \frac{a_5}{7} = \frac{a_1}{7\cdot 5\cdot 3}$$

$$a_{2n} = \frac{a_0}{2\cdot 4\cdot 6\cdots(2n)},\ n \geq 1 \qquad a_{2n+1} = \frac{a_1}{1\cdot 3\cdot 5\cdot 7\cdots(2n+1)},\ n \geq 0$$

$$\therefore y = a_0\left(1 + \sum_{n=1}^{\infty} \frac{x^{2n}}{2\cdot 4\cdot 6\cdots(2n)}\right) + a_1 \sum_{n=0}^{\infty} \frac{x^{2n+1}}{1\cdot 3\cdot 5\cdot 7\cdots(2n+1)}$$

$$= a_0\left(1 + \frac{x^2}{2} + \frac{x^4}{2\cdot 4} + \frac{x^6}{2\cdot 4\cdot 6} + \ldots\right)$$

$$+ a_1\left(\frac{x}{1} + \frac{x^3}{1\cdot 3} + \frac{x^5}{1\cdot 3\cdot 5} + \frac{x^7}{1\cdot 3\cdot 5\cdot 7} + \ldots\right)$$

Note: $2\cdot 4\cdot 6\cdots(2n) = 2(1)\cdot 2(2)\cdot 2(3)\cdots 2(n) = 2^n n!$

$$1\cdot 3\cdot 5\cdot 7\cdots(2n+1) = \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdots(2n+1)}{2\cdot 4\cdot 6\cdots(2n)} = \frac{(2n+1)!}{2^n n!}$$

$$\therefore y_1(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n(n!)} \qquad y_2(x) = \sum_{n=0}^{\infty} \frac{2^n(n!)\,x^{2n+1}}{(2n+1)!}$$

(c)

$$y_1(0) = 1 \qquad y_1'(x) = \left(x + \frac{x^3}{2} + \frac{x^5}{2\cdot 4} + \ldots\right),\ y_1'(0) = 0$$

$$y_2(0) = 0 \qquad y_2'(x) = \left(1 + \frac{x^2}{1} + \frac{x^4}{1\cdot 3} + \ldots\right)\ y_2'(0) = 1$$

$$\therefore W[y_1, y_2](0) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$$

$\therefore y_1, y_2$ form a fundamental set of solutions

(d)

From (b), $\underline{y_1(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n (n!)}}$ $\quad \underline{y_2(x) = \sum_{n=0}^{\infty} \frac{2^n (n!) x^{2n+1}}{(2n+1)!}}$

4.

(a)

Let $y = \sum_{n=0}^{\infty} a_n (x-1)^n \quad \therefore y' = \sum_{n=1}^{\infty} n a_n (x-1)^{n-1}$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^n$$

Note $x = 1 + x - 1$

$$\therefore xy' = (1 + x - 1) \sum_{n=1}^{\infty} n a_n (x-1)^{n-1}$$

$$= \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} + \sum_{n=1}^{\infty} n a_n (x-1)^n$$

$$= \sum_{n=0}^{\infty} (n+1) a_{n+1} (x-1)^n + \sum_{n=0}^{\infty} n a_n (x-1)^n$$

$$= \sum_{n=0}^{\infty} \left[(n+1) a_{n+1} + n a_n \right] (x-1)^n$$

$$\therefore y'' - xy' - y = \sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} - (n+1) a_{n+1} - n a_n - a_n \right] (x-1)^n$$

$$\therefore (n+2)(n+1) a_{n+2} - (n+1) a_{n+1} - (n+1) a_n = 0, \quad n \geq 0$$

$$\therefore \underline{(n+2) a_{n+2} - a_{n+1} - a_n = 0}, \quad \text{or } a_{n+2} = \frac{a_{n+1} + a_n}{n+2}$$

(b)

$$a_2 = \frac{a_0}{2} + \frac{a_1}{2} \qquad a_3 = \frac{a_2}{3} + \frac{a_1}{3} = \frac{a_0}{6} + \frac{a_1}{6} + \frac{a_1}{3} = \underline{\frac{a_0}{6} + \frac{a_1}{2}}$$

$$a_4 = \frac{a_3}{4} + \frac{a_2}{4} = \frac{a_0}{24} + \frac{a_1}{8} + \frac{a_0}{8} + \frac{a_1}{8} = \underline{\frac{a_0}{6} + \frac{a_1}{4}}$$

$$\therefore y = \sum_{n=0}^{\infty} a_n (x-1)^n = a_0 + a_1 (x-1) +$$

$$\left(\frac{a_0}{2} + \frac{a_1}{2}\right)(x-1)^2 + \left(\frac{a_0}{6} + \frac{a_1}{2}\right)(x-1)^3 + \left(\frac{a_0}{6} + \frac{a_1}{4}\right)(x-1)^4 + \dots$$

$$= a_0 \left[1 + \tfrac{1}{2}(x-1)^2 + \tfrac{1}{6}(x-1)^3 + \tfrac{1}{6}(x-1)^4 + \dots \right]$$

$$+ a_1 \left[(x-1) + \tfrac{1}{2}(x-1)^2 + \tfrac{1}{2}(x-1)^3 + \tfrac{1}{4}(x-1)^4 + \dots \right]$$

$$\therefore \underline{y_1(x)} = 1 + \tfrac{1}{2}(x-1)^2 + \tfrac{1}{6}(x-1)^3 + \tfrac{1}{6}(x-1)^4 + \dots$$

$$\underline{y_2(x)} = (x-1) + \tfrac{1}{2}(x-1)^2 + \tfrac{1}{2}(x-1)^3 + \tfrac{1}{4}(x-1)^4 + \dots$$

Using MATLAB to compute the coefficients

```
clear, clc
syms a0 a1
n = 6;  %number of desired terms
%allocate space - array to hold coefficients
a = sym(zeros(n,2));
% initialize first entries for the for loop
a(1,:) = [0 a1];
a(2,:) = [a0/2 a1/2];
% i index gets coefficient for (x-1)^i
for i = 3:n
    a(i,:) = a(i-1,:)/i + a(i-2,:)/i;
end
a  % display coefficients
```

a =

$$\begin{pmatrix} 0 & a_1 \\ \frac{a_0}{2} & \frac{a_1}{2} \\ \frac{a_0}{6} & \frac{a_1}{2} \\ \frac{a_0}{6} & \frac{a_1}{4} \\ \frac{a_0}{15} & \frac{3\,a_1}{20} \\ \frac{7\,a_0}{180} & \frac{a_1}{15} \end{pmatrix}$$

$a(2,:) \to (x-1)^2$

$a(4,:) \to (x-1)^4$

$a(6,:) \to (x-1)^6$

(c)

From (b), $y_1(1) = 1 \qquad y_2(1) = 0$

$$y_1'(x) = (x-1) + \tfrac{1}{2}(x-1)^2 + \tfrac{2}{3}(x-1)^3 + \dots \qquad y_1'(1) = 0$$

$$y_2'(x) = 1 + (x-1) + \tfrac{3}{2}(x-1)^2 + (x-1)^3 + \dots \qquad y_2'(1) = 1$$

$$\therefore\ W[y_1, y_2](1) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$$

$\therefore\ y_1, y_2$ form a fundamental set of solutions

(d)

Pattern not apparent.

5.

(a)

Let $y = \sum_{n=0}^{\infty} a_n x^n \quad \therefore \quad y'' = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$

$\therefore \; K^2 x^2 y = K^2 \sum_{n=0}^{\infty} a_n x^{n+2} = K^2 \sum_{n=2}^{\infty} a_{n-2} x^n$

$y'' = 2a_2 + 6a_3 x + \sum_{n=2}^{\infty} (n+2)(n+1) a_{n+2} x^n$

$\therefore \; y'' + K^2 x^2 y = 2a_2 + 6a_3 x + \sum_{n=2}^{\infty} \left[(n+2)(n+1) a_{n+2} + K^2 a_{n-2}\right] x^n$

$\therefore \; y'' + Kx^2 y = 0 \Rightarrow a_2 = 0, \; a_3 = 0,$

$(n+2)(n+1) a_{n+2} + K^2 a_{n-2} = 0 \quad , \; n \geq 2$

Or, $\quad a_{n+2} = \dfrac{-K^2 a_{n-2}}{(n+2)(n+1)} \quad , \; n \geq 2$

Or, shifing index, $\; a_{n+4} = \underline{\dfrac{-K^2 a_n}{(n+4)(n+3)}} \quad , \; n \geq 0$

with $a_2 = 0, \; a_3 = 0$

(b)

$$a_4 = \frac{-K^2 a_0}{4\cdot 3} \qquad a_5 = \frac{-K^2 a_1}{5\cdot 4}$$

$$a_6 = \frac{-K^2 a_2}{6\cdot 5} = 0 \qquad a_7 = \frac{-K^2 a_3}{7\cdot 6} = 0$$

$$a_8 = \frac{-K^2 a_4}{8\cdot 7} = \frac{K^4 a_0}{8\cdot 7\cdot 4\cdot 3} \qquad a_9 = \frac{-K^2 a_5}{9\cdot 8} = \frac{K^4 a_1}{9\cdot 8\cdot 5\cdot 4}$$

$$a_{10} = \frac{-K^2 a_6}{10\cdot 9} = 0 \qquad a_{11} = \frac{-K^2 a_7}{11\cdot 10} = 0$$

$$a_{12} = \frac{-K^2 a_8}{12\cdot 11} = \frac{-K^6 a_0}{12\cdot 11\cdot 8\cdot 7\cdot 4\cdot 3} \qquad a_{13} = \frac{-K^2 a_9}{13\cdot 12} = \frac{-K^6 a_1}{13\cdot 12\cdot 9\cdot 8\cdot 5\cdot 4}$$

$$\therefore y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + \left(\frac{-K^2 a_0}{4\cdot 3}\right)x^4 + \left(\frac{-K^2 a_1}{5\cdot 4}\right)x^5$$

$$+ \left(\frac{K^4 a_0}{8\cdot 7\cdot 4\cdot 3}\right)x^8 + \left(\frac{K^4 a_1}{9\cdot 8\cdot 5\cdot 4}\right)x^9$$

$$+ \left(\frac{-K^6 a_0}{12\cdot 11\cdot 8\cdot 7\cdot 4\cdot 3}\right)x^{12} + \left(\frac{-K^6 a_1}{13\cdot 12\cdot 9\cdot 8\cdot 5\cdot 4}\right)x^{13}$$

$$= a_0\left[1 - \frac{K^2}{4\cdot 3}x^4 + \frac{K^4}{8\cdot 7\cdot 4\cdot 3}x^8 - \frac{K^6}{12\cdot 11\cdot 8\cdot 7\cdot 4\cdot 3}x^{12} + \cdots\right]$$

$$+ a_1\left[x - \frac{K^2}{5\cdot 4}x^5 + \frac{K^4}{9\cdot 8\cdot 5\cdot 4}x^9 - \frac{K^6}{13\cdot 12\cdot 9\cdot 8\cdot 5\cdot 4}x^{13} + \cdots\right]$$

$$\therefore\ y_1(x) = 1 - \frac{K^2 x^4}{3\cdot 4} + \frac{K^4 x^8}{3\cdot4\cdot7\cdot8} - \frac{K^6 x^{12}}{3\cdot4\cdot7\cdot8\cdot11\cdot12} + \cdots$$

$$y_2(x) = x - \frac{K^2 x^5}{4\cdot5} + \frac{K^4 x^9}{4\cdot5\cdot8\cdot9} - \frac{K^6 x^{13}}{4\cdot5\cdot8\cdot9\cdot12\cdot13} + \cdots$$

(c)

From (b), $y_1(0) = 1 \quad y_2(0) = 0$

$y_1'(x) = 0 + \text{terms in } (x)\ ,\ y_1'(0) = 0$

$y_2'(x) = 1 + \text{terms in } (x),\ y_2'(0) = 1$

$$\therefore\ W[y_1, y_2](0) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$$

$\therefore$ $y_1(x)$, $y_2(x)$ form a fundamental set of solutions

(d)

$$y_1(x) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n K^{2n} x^{4n}}{3\cdot4\cdot7\cdot8\cdots(4n-1)(4n)}$$

$$y_2(x) = x + \sum_{n=1}^{\infty} \frac{(-1)^n K^{2n} x^{4n+1}}{4\cdot5\cdot8\cdot9\cdots(4n)(4n+1)}$$

$$= x\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n K^{2n} x^{4n}}{4\cdot5\cdot8\cdot9\cdots(4n)(4n+1)}\right]$$

6.

(a)

Let $y = \sum_{n=0}^{\infty} a_n x^n \quad \therefore \quad y'' = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$

$$\therefore (1-x)y'' + y = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^{n+1} + \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} + a_n \right] x^n - \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^{n+1}$$

first term

$$= 2a_2 + a_0 + \sum_{n=1}^{\infty} \left[(n+2)(n+1) a_{n+2} + a_n \right] x^n - \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^{n+1}$$

$$= 2a_2 + a_0 + \sum_{n=1}^{\infty} \left[(n+2)(n+1) a_{n+2} + a_n \right] x^n - \sum_{n=1}^{\infty} (n+1)(n) a_{n+1} x^n$$

shift index

$$= 2a_2 + a_0 + \sum_{n=1}^{\infty} \left[(n+2)(n+1) a_{n+2} + a_n - (n+1)(n) a_{n+1} \right] x^n$$

$\therefore$ Since $(1-x)y'' + y = 0$,

$\underline{2a_2 + a_0 = 0}$, and $\underline{(n+2)(n+1) a_{n+2} - n(n+1) a_{n+1} + a_n = 0}$ $\quad n \geq 1$

Or, $a_2 = \dfrac{-a_0}{2}$, $a_{n+2} = \dfrac{n}{n+2} a_{n+1} - \dfrac{a_n}{(n+2)(n+1)}$, $n \geq 1$

(b)

a_0 and a_1 are independent, and $a_2 = -\dfrac{a_0}{2}$

$\therefore$ Compute for $a_3, a_4 \ldots$

$\therefore$ Shift recurrence relation down for clarity, and to use MATLAB for computations

$$a_n = \left(\frac{n-2}{n}\right) a_{n-1} - \frac{a_{n-2}}{n(n-1)}, \quad n \geq 3$$

```
clear, clc
syms a0 a1
n = 6;  %number of desired terms
%allocate space - array to hold coefficients
a = sym(zeros(n,2));
% initialize first entries for the for loop
a(1,:) = [0 a1];
a(2,:) = [-a0/2 0];
% i index gets coefficient for x^i
for i = 3:n
    a(i,:) = ((i-2)/i)*a(i-1,:) - a(i-2,:)/(i*(i-1));
end
a  % display coefficients
```

a =

$$\begin{array}{cc} \begin{pmatrix} 0 & a_1 \\ -\frac{a_0}{2} & 0 \\ -\frac{a_0}{6} & -\frac{a_1}{6} \\ -\frac{a_0}{24} & -\frac{a_1}{12} \\ -\frac{a_0}{60} & -\frac{a_1}{24} \\ -\frac{7a_0}{720} & -\frac{a_1}{40} \end{pmatrix} & \begin{array}{l} a_1 \to x \\ a_2 \to x^2 \\ a_3 \to x^3 \\ a_4 \to x^4 \\ a_5 \to x^5 \\ a_6 \to x^6 \end{array} \end{array}$$

$$\therefore y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + \left(-\frac{a_0}{2}\right) x^2 + \left(-\frac{a_0}{6} - \frac{a_1}{6}\right) x^3$$

$$+ \left(-\frac{a_0}{24} - \frac{a_1}{12}\right) x^4 + \left(-\frac{a_0}{60} - \frac{a_1}{24}\right) x^5 + \ldots$$

$$= a_0\left[1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{24}x^4 - \frac{1}{60}x^5 + \dots\right]$$

$$+ a_1\left[x - \frac{1}{6}x^3 - \frac{1}{12}x^4 - \frac{1}{24}x^5 + \dots\right]$$

$$\therefore \boxed{\begin{aligned} y_1(x) &= 1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{24}x^4 + \dots \\ y_2(x) &= x - \frac{1}{6}x^3 - \frac{1}{12}x^4 - \frac{1}{24}x^5 + \dots \end{aligned}}$$

(c)

From (b), $y_1(0) = 1$, $y_2(0) = 0$

$y_1'(x) = -2x + (\text{other terms in } x) \therefore y_1'(0) = 0$

$y_2'(x) = 1 - \frac{1}{2}x^2 + (\text{other terms in } x) \therefore y_2'(0) = 1$

$\therefore W[y_1, y_2](0) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$

$\therefore y_1, y_2$ form a fundamental set of solutions.

(d)

Pattern for $y_1(x)$, $y_2(x)$ not apparent.

7.

(a)

Let $y = \sum_{n=0}^{\infty} a_n x^n \quad \therefore y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

$$xy' = \sum_{n=1}^{\infty} n a_n x^n \qquad y'' = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$y = a_0 + \sum_{n=1}^{\infty} a_n x^n \qquad = 2a_2 + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$\therefore y'' + xy' + 2y =$

$$\left[2a_2 + \sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2} x^n\right] + \left[\sum_{n=1}^{\infty} n a_n x^n\right] + \left[2a_0 + \sum_{n=1}^{\infty} 2a_n x^n\right]$$

$$= (2a_2 + 2a_0) + \sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2} + (n+2)a_n\right] x^n$$

$\therefore y'' + xy' + 2y = 0 \Rightarrow 2a_2 + 2a_0 = 0,$

$$(n+2)(n+1)a_{n+2} + (n+2)a_n = 0, \quad n \geq 1$$

$$\therefore a_2 = -a_0, \quad a_{n+2} = -\frac{a_n}{n+1}, \quad n \geq 1$$

$$\therefore \underline{a_{n+2} = \frac{-a_n}{n+1}}, \quad n \geq 0$$

(b)

$$a_2 = -a_0 \qquad a_3 = -\frac{a_1}{2}$$

$$a_4 = \frac{-a_2}{3} = \frac{a_0}{3} \qquad a_5 = \frac{-a_3}{4} = \frac{a_1}{2\cdot 4}$$

$$a_6 = -\frac{a_4}{5} = -\frac{a_0}{3\cdot 5} \qquad a_7 = -\frac{a_5}{6} = \frac{-a_1}{2\cdot 4\cdot 6}$$

$$= -\frac{2\cdot 4\cdot 6}{2\cdot 3\cdot 4\cdot 5\cdot 6}a_0 \qquad = \frac{-a_1}{2(1)\cdot 2(2)\cdot 2(3)}$$

$$\therefore a_{2n} = \frac{(-1)^n 2^n\cdot(n!)a_0}{(2n)!} \qquad a_{2n+1} = \frac{(-1)^n a_1}{2^n\cdot(n!)}$$

$$\therefore y = \underbrace{(a_0 + a_1 x)}_{n=0} + \underbrace{\left[(-a_0)x^2 + \left(-\frac{a_1}{2}\right)x^3\right]}_{n=1} + \underbrace{\left[\left(\frac{a_0}{3}\right)x^4 + \left(\frac{a_1}{2\cdot 4}\right)x^5\right]}_{n=2}$$

$$+ \underbrace{\left[\left(-\frac{a_0}{3\cdot 5}\right)x^6 + \left(\frac{-a_1}{2\cdot 4\cdot 6}\right)x^7\right]}_{n=3} + \underbrace{\left[\left(\frac{a_0}{3\cdot 5\cdot 7}\right)x^8 + \left(\frac{a_1}{2\cdot 4\cdot 6\cdot 8}\right)x^9\right]}_{n=4} + \dots$$

$$= a_0\left[1 - \frac{x^2}{1} + \frac{x^4}{1\cdot 3} - \frac{x^6}{1\cdot 3\cdot 5} + \frac{x^8}{1\cdot 3\cdot 5\cdot 7}\right]$$

$$+ a_1\left[x - \frac{x^3}{2} + \frac{x^5}{2\cdot 4} - \frac{x^7}{2\cdot 4\cdot 6} + \frac{x^9}{2\cdot 4\cdot 6\cdot 8}\right] + \dots$$

$$\therefore \underline{y_1(x)} = 1 - \frac{x^2}{1} + \frac{x^4}{1\cdot 3} - \frac{x^6}{1\cdot 3\cdot 5} + \dots$$

$$\underline{y_2(x)} = x - \frac{x^3}{2} + \frac{x^5}{2\cdot 4} - \frac{x^7}{2\cdot 4\cdot 6} + \dots$$

Check a_i with MATLAB:

```
clear, clc
syms a0 a1
n = 9;  %number of desired terms
%allocate space - array to hold coefficients
a = sym(zeros(n,2));
% initialize first entries for the for loop
a(1,:) = [0 a1];
a(2,:) = [-a0 0];
% i index gets coefficient for x^i
for i = 3:n
    a(i,:) = -a(i-2,:)/(i-1);
end
a  % display coefficients
```

a =

$$\begin{pmatrix} 0 & a_1 \\ -a_0 & 0 \\ 0 & -\frac{a_1}{2} \\ \frac{a_0}{3} & 0 \\ 0 & \frac{a_1}{8} \\ -\frac{a_0}{15} & 0 \\ 0 & -\frac{a_1}{48} \\ \frac{a_0}{105} & 0 \\ 0 & \frac{a_1}{384} \end{pmatrix} \quad \begin{array}{l} a_1 \to x \\ a_2 \to x^2 \\ a_3 \to x^3 \\ a_4 \to x^4 \\ a_5 \to x^5 \\ a_6 \to x^6 \\ a_7 \to x^7 \\ a_8 \to x^8 \\ a_9 \to x^9 \end{array}$$

(c)

From (b), $y_1(0) = 1$, $y_2(0) = 0$

$y_1'(x) = -2x + (\text{other terms in } x)$. $\therefore y_1'(0) = 0$

$y_2'(x) = 1 - \frac{3}{2}x^2 + (\text{other terms in } x)$. $\therefore y_2'(0) = 1$

$$\therefore W[y_1, y_2](0) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$$

$\therefore$ $y_1(x)$, $y_2(x)$ form a fundamental set of solutions.

(d)

From (b), $$\underline{\underline{y_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n (n!)}{(2n)!} x^{2n}}}, \quad \underline{\underline{y_2(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n (n!)} x^{2n+1}}}$$

8.

(a)

Let $y = \sum_{n=0}^{\infty} a_n (x-1)^n$

$$xy = (x-1+1)y = \sum_{n=0}^{\infty} a_n (x-1)^{n+1} + \sum_{n=0}^{\infty} a_n (x-1)^n$$

shift index

$$= \sum_{n=0}^{\infty} a_n (x-1)^{n+1} + a_0 + \sum_{n=0}^{\infty} a_{n+1} (x-1)^{n+1}$$

$$= \underline{a_0} + \sum_{n=0}^{\infty} (a_{n+1} + a_n)(x-1)^{n+1}$$

$$y' = \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} = a_1 + \sum_{n=2}^{\infty} n a_n (x-1)^{n-1}$$

$$= \underline{a_1} + \sum_{n=0}^{\infty} (n+2) a_{n+2} (x-1)^{n+1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2}$$

$$xy'' = (x-1+1)y'' =$$

$$\sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-1} + \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2}$$

shift index   write out 1st term

$$= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^{n+1} + 2a_2 + \sum_{n=3}^{\infty} n(n-1) a_n (x-1)^{n-2}$$

$$= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^{n+1} + 2a_2 + \sum_{n=0}^{\infty} (n+3)(n+2) a_{n+3} (x-1)^{n+1}$$

shift index

$$= 2a_2 + \sum_{n=0}^{\infty} \left[(n+3)(n+2) a_{n+3} + (n+2)(n+1) a_{n+2} \right] (x-1)^{n+1}$$

$$\therefore xy'' + y' + xy = (a_0 + a_1 + 2a_2) +$$

$$\sum_{n=0}^{\infty} \left[(n+3)(n+2) a_{n+3} + (n+2)(n+1) a_{n+2} + (n+2) a_{n+2} + a_{n+1} + a_n \right] (x-1)^{n+1}$$

combine

$$= (2a_2 + a_1 + a_0) + \sum_{n=0}^{\infty} \left[(n+3)(n+2) a_{n+3} + (n+2)^2 a_{n+2} + a_{n+1} + a_n \right] (x-1)^{n+1}$$

$$\therefore xy'' + y' + xy = 0 \Rightarrow$$

$$\boxed{\begin{array}{l} (1) \quad a_2 = -\dfrac{(a_1 + a_0)}{2}, \text{ and} \\[2ex] (2) \quad a_{n+3} = -\dfrac{(n+2)^2 a_{n+2} + a_{n+1} + a_n}{(n+3)(n+2)}, \quad n \geq 0 \end{array}}$$

(b)

Use MATLAB to do the computations.

a_0 and a_1 are independent. To initialize,

set $a_0 = [a_0 \ 0]$, $a_1 = [0 \ a_1]$, $a_2 = \left[-\frac{a_0}{2} \ \ -\frac{a_1}{2} \right]$

Shift the index down to write the recurrence relation

as $a_n = -\dfrac{(n-1)^2 a_{n-1} + a_{n-2} + a_{n-3}}{n(n-1)}, \quad n \geq 3$

Note that MATLAB vectors are 1-based, so $a_0 = [a_0 \ 0]$ doesn't "exist". $\therefore$ Compute a_3 outside loop, as a_3 is defined from a_0.

```
clear, clc
syms a0 a1
n = 6;  %number of desired terms
%allocate space - array to hold coefficients
a = sym(zeros(n,2));
% initialize first entries for the for loop
a(1,:) = [0 a1];
a(2,:) = [-a0/2 -a1/2];
i = 3; %use [a0 0] for a(0,:) outside loop
a(3,:) = -(((i-1)^2)*a(i-1,:) + a(i-2,:) + [a0 0])/(i*(i-1));
% i index gets coefficient for (x-1)^i
for i = 4:n
    a(i,:) = -(((i-1)^2)*a(i-1,:) + a(i-2,:) + a(i-3,:))/(i*(i-1));
end
a  % display coefficients
```

a =

$$\begin{pmatrix} 0 & a_1 \\ -\frac{a_0}{2} & -\frac{a_1}{2} \\ \frac{a_0}{6} & \frac{a_1}{6} \\ -\frac{a_0}{12} & -\frac{a_1}{6} \\ \frac{a_0}{12} & \frac{3\,a_1}{20} \\ -\frac{13\,a_0}{180} & -\frac{a_1}{8} \end{pmatrix} \begin{matrix} a_1 \to (x-1)^1 \\ a_2 \to (x-1)^2 \\ a_3 \to (x-1)^3 \\ a_4 \to (x-1)^4 \\ a_5 \to (x-1)^5 \\ a_6 \to (x-1)^6 \end{matrix}$$

$$\therefore y(x) = a_0 + a_1(x-1) + \left(-\frac{a_0}{2} - \frac{a_1}{2}\right)(x-1)^2 + \left(\frac{a_0}{6} + \frac{a_1}{6}\right)(x-1)^3$$

$$+\left(-\frac{a_0}{12}-\frac{a_1}{6}\right)(x-1)^4+\left(\frac{a_0}{12}+\frac{3a_1}{20}\right)(x-1)^5+\dots$$

$$=a_0\left[1-\frac{1}{2}(x-1)^2+\frac{1}{6}(x-1)^3-\frac{1}{12}(x-1)^4+\dots\right]$$

$$+a_1\left[(x-1)-\frac{1}{2}(x-1)^2+\frac{1}{6}(x-1)^3-\frac{1}{6}(x-1)^4+\dots\right]$$

$\therefore$

$$\boxed{\begin{aligned} y_1(x) &= 1-\frac{1}{2}(x-1)^2+\frac{1}{6}(x-1)^3-\frac{1}{12}(x-1)^4+\dots \\ y_2(x) &= (x-1)-\frac{1}{2}(x-1)^2+\frac{1}{6}(x-1)^3-\frac{1}{6}(x-1)^4+\dots \end{aligned}}$$

(c)

From (b), $y_1(1)=1$, $y_2(1)=0$

$y_1'(x) = -(x-1) + (\text{other terms in } (x-1))$ $\therefore y_1'(1)=0$

$y_2'(x) = 1-(x-1) + (\text{other terms in } (x-1))$ $\therefore y_2'(1)=1$

$\therefore W[y_1, y_2](1) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0.$

$\therefore$ $y_1(x)$, $y_2(x)$ form a fundamental set of solutions

(d)

Pattern for $y_1(x)$, $y_2(x)$ not apparent

9.

(a)

Let $y = \sum_{n=0}^{\infty} a_n x^n$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \therefore -3xy' = -\sum_{n=1}^{\infty} 3n a_n x^n = -\sum_{n=0}^{\infty} 3n a_n x^n$$

$n=0 \to 0$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$(3-x^2)y'' = 3y'' - x^2 y'' = \sum_{n=2}^{\infty} 3n(n-1) a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1) a_n x^n$$

shift index

shift index using $n=0 \to 0$, $n=1 \to 0$

$$= \sum_{n=0}^{\infty} 3(n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} n(n-1) a_n x^n$$

$$= \sum_{n=0}^{\infty} \left[3(n+2)(n+1) a_{n+2} - n(n-1) a_n \right] x^n$$

$$\therefore (3-x^2)y'' - 3xy' - y =$$

$$\sum_{n=0}^{\infty} \left[3(n+2)(n+1) a_{n+2} - n(n-1) a_n - 3n a_n - a_n \right] x^n$$

$$= \sum_{n=0}^{\infty} \left[3(n+2)(n+1) a_{n+2} - (\underbrace{n^2 - n + 3n + 1}_{=(n+1)^2}) a_n \right] x^n$$

$$= \sum_{n=0}^{\infty} \left[3(n+2)(n+1)a_{n+2} - (n+1)^2 a_n \right] x^n$$

$\therefore (3-x^2)y'' - 3xy' - y = 0 \Rightarrow$

$$3(n+2)(n+1)a_{n+2} - (n+1)^2 a_n = 0 \quad , \quad n \geq 0$$

Or, $a_{n+2} = \dfrac{(n+1)}{3(n+2)} a_n \quad , \quad n \geq 0$

(b)

$a_2 = \frac{1}{3 \cdot 2} a_0$ $\qquad$ $a_3 = \frac{2}{3 \cdot 3} a_1$

$a_4 = \dfrac{3}{3 \cdot 4} a_2 = \dfrac{1 \cdot 3}{3^2 \cdot 2 \cdot 4} a_0$ $\qquad$ $a_5 = \dfrac{4}{3 \cdot 5} a_3 = \dfrac{2 \cdot 4}{3^2 \cdot 3 \cdot 5} a_1$

$a_6 = \dfrac{5}{3 \cdot 6} a_4 = \dfrac{1 \cdot 3 \cdot 5}{3^3 \cdot 2 \cdot 4 \cdot 6} a_0$ $\qquad$ $a_7 = \dfrac{6}{3 \cdot 7} a_5 = \dfrac{2 \cdot 4 \cdot 6}{3^3 \cdot 3 \cdot 5 \cdot 7} a_1$

$\therefore a_{2n} = \dfrac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{3^n \cdot 2 \cdot 4 \cdot 6 \cdots (2n)} a_0$ $\qquad$ $a_{2n+1} = \dfrac{2 \cdot 4 \cdot 6 \cdots (2n)}{3^n \cdot 3 \cdot 5 \cdot 7 \cdots (2n+1)} a_1$

Note $2 \cdot 4 \cdot 6 \cdots (2n) = 2(1) \cdot 2(2) \cdot 2(3) \cdots (2)(n) = 2^n (n!)$

$\therefore a_{2n} = \dfrac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{3^n \cdot 2^n (n!)} a_0$ $\qquad$ $a_{2n+1} = \dfrac{2^n (n!)}{3^n [1 \cdot 3 \cdot 5 \cdots (2n+1)]} a_1$

$n \geq 1$ $\qquad$ $n \geq 1$

$$\therefore\ y(x) = a_0 + a_1 x + \left(\tfrac{1}{6}\right)a_0 x^2 + \left(\tfrac{2}{9}a_1\right)x^3 + \left(\tfrac{a_0}{24}\right)x^4 + \dots$$

$$= a_0\left[1 + \frac{1}{6}x^2 + \frac{1}{24}x^4 + \frac{5}{432}x^6 + \dots\right]$$

$$+ a_1\left[x + \frac{2}{9}x^3 + \frac{8}{135}x^5 + \frac{16}{945}x^7 + \dots\right]$$

$\therefore$

$$\boxed{\begin{aligned} y_1(x) &= 1 + \frac{1}{6}x^2 + \frac{1}{24}x^4 + \frac{5}{432}x^6 + \dots \\ y_2(x) &= x + \frac{2}{9}x^3 + \frac{8}{135}x^5 + \frac{16}{945}x^7 + \dots \end{aligned}}$$

Using MATLAB

```
clear, clc
syms a0 a1 x y(x) y1(x) y2(x)
Nterms = 8;  %number of desired terms
%allocate space - array to hold coefficients
a = sym(zeros(Nterms,2));
% initialize first entries for the for loop
a(1,:) = [0 a1];
a(2,:) = [a0/6 0];
% n index gets coefficient for x^n
for n = 3:Nterms
    a(n,:) = ((n-1)/(3*n))*a(n-2,:);
end
a  % display coefficients
y(x) = a0 + a1*x;
for n = 2:Nterms
    y(x) = y(x) + (a(n,1)+a(n,2))*x^n;
end
y(x)
y1(x) = subs(y(x),[a0 a1],[1 0])
y2(x) = subs(y(x),[a0 a1],[0 1])
```

a =

$$\begin{pmatrix} 0 & a_1 \\ \frac{a_0}{6} & 0 \\ 0 & \frac{2\,a_1}{9} \\ \frac{a_0}{24} & 0 \\ 0 & \frac{8\,a_1}{135} \\ \frac{5\,a_0}{432} & 0 \\ 0 & \frac{16\,a_1}{945} \\ \frac{35\,a_0}{10368} & 0 \end{pmatrix}
\begin{array}{l} a_1 \to x^1 \\ a_2 \to x^2 \\ a_3 \to x^3 \\ a_4 \to x^4 \\ a_5 \to x^5 \\ a_6 \to x^6 \\ a_7 \to x^7 \\ a_8 \to x^8 \end{array}$$

ans =

$$\frac{35\,a_0\,x^8}{10368} + \frac{16\,a_1\,x^7}{945} + \frac{5\,a_0\,x^6}{432} + \frac{8\,a_1\,x^5}{135} + \frac{a_0\,x^4}{24} + \frac{2\,a_1\,x^3}{9} + \frac{a_0\,x^2}{6} + a_1\,x + a_0$$

y1(x) =

$$\frac{35\,x^8}{10368} + \frac{5\,x^6}{432} + \frac{x^4}{24} + \frac{x^2}{6} + 1$$

y2(x) =

$$\frac{16\,x^7}{945} + \frac{8\,x^5}{135} + \frac{2\,x^3}{9} + x$$

(c)

From (b), $y_1(0) = 1$, $y_2(0) = 0$

$y_1'(x) = \frac{1}{3}x + (\text{other terms in } x) \quad \therefore y_1'(0) = 0$

$y_2'(x) = 1 + \frac{2}{3}x^2 + (\text{other terms in } x) \quad \therefore y_2'(0) = 1$

$\therefore W[y_1, y_2](0) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$

$\therefore y_1(x)$, $y_2(x)$ form a fundamental set of solutions.

(d)

From (b),

$$a_{2n} = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{3^n\cdot 2^n(n!)}a_0 \qquad a_{2n+1} = \frac{2^n(n!)}{3^n[1\cdot 3\cdot 5\cdots(2n+1)]}a_1$$

$n \geq 1$ $\qquad$ $n \geq 1$

$\therefore$

$$\boxed{\begin{aligned} y_1(x) &= 1 + \sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{3^n\cdot 2^n(n!)}x^{2n} \\ y_2(x) &= \sum_{n=0}^{\infty} \frac{2^n(n!)}{3^n[1\cdot 3\cdot 5\cdots(2n+1)]}x^{2n+1} \end{aligned}}$$

10.

(a)

Let $y = \sum_{n=0}^{\infty} a_n x^n = a_0 + \sum_{n=1}^{\infty} a_n x^n$

$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$, $xy' = \sum_{n=1}^{\infty} n a_n x^n$

$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$

$= 2a_2 + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} x^n$

$\therefore 2y'' + xy' + 3y =$

$(4a_2 + 3a_0) + \sum_{n=1}^{\infty} [2(n+2)(n+1)a_{n+2} + \underbrace{n a_n + 3a_n}_{=(n+3)a_n}] x^n$

$\therefore 2y'' + xy' + 3y = 0 \Rightarrow$

(1) $4a_2 + 3a_0 = 0$

(2) $2(n+2)(n+1)a_{n+2} + (n+3)a_n = 0, \quad n \geq 1$

But (1) is (2) with $n=0$

$\therefore \quad a_{n+2} = -\dfrac{(n+3)}{2(n+2)(n+1)} a_n, \quad n \geq 0$

(b)

$$a_2 = -\frac{3}{2\cdot 2\cdot 1} a_0 \qquad a_3 = -\frac{4}{2\cdot 3\cdot 2} a_1$$

$$a_4 = -\frac{5}{2\cdot 4\cdot 3} a_2 = \frac{(-1)^2\cdot 3\cdot 5}{2^2\, 1\cdot 2\cdot 3\cdot 4} a_0 \qquad a_5 = -\frac{6}{2\cdot 5\cdot 4} a_3 = \frac{(-1)^2\cdot 4\cdot 6}{2^2\cdot 2\cdot 3\cdot 4\cdot 5} a_1$$

$$\therefore\ a_{2n} = \frac{(-1)^n\, 1\cdot 3\cdot 5\cdots(2n+1)}{2^n\,(2n)!} a_0 \qquad a_{2n+1} = \frac{(-1)^n\, 2\cdot 4\cdot 6\cdots(2n+2)}{2^{n+1}\,(2n+1)!} a_1$$

Using MATLAB and using $a_n = -\frac{(n+1)}{2n(n-1)} a_{n-2},\ n \geq 3$

```
clear, clc
syms a0 a1 x y(x) y1(x) y2(x)
Nterms = 8;  %number of desired terms
%allocate space - array to hold coefficients
a = sym(zeros(Nterms,2));
% initialize first entries for the for loop
% note a(0,:) = [a0 0], but MATLAB 1-based
a(1,:) = [0 a1];
a(2,:) = [-3*a0/4 0];
% n index gets coefficient for x^n
for n = 3:Nterms
    a(n,:) = -((n+1)/(2*n*(n-1)))*a(n-2,:);
end
a  % display coefficients
y(x) = a0 + a1*x; % initialize
for n = 2:Nterms
    y(x) = y(x) + (a(n,1)+a(n,2))*x^n;
end
y(x)
y1(x) = subs(y(x),[a0 a1],[1 0])
y2(x) = subs(y(x),[a0 a1],[0 1])
```

a =

$$\begin{pmatrix} 0 & a_1 \\ -\frac{3a_0}{4} & 0 \\ 0 & -\frac{a_1}{3} \\ \frac{5a_0}{32} & 0 \\ 0 & \frac{a_1}{20} \\ -\frac{7a_0}{384} & 0 \\ 0 & -\frac{a_1}{210} \\ \frac{3a_0}{2048} & 0 \end{pmatrix} \quad \begin{array}{l} a_1 \to x^1 \\ a_2 \to x^2 \\ a_3 \to x^3 \\ a_4 \to x^4 \\ a_5 \to x^5 \\ a_6 \to x^6 \\ a_7 \to x^7 \\ a_8 \to x^8 \end{array}$$

ans =

$$\frac{3a_0x^8}{2048} - \frac{a_1x^7}{210} - \frac{7a_0x^6}{384} + \frac{a_1x^5}{20} + \frac{5a_0x^4}{32} - \frac{a_1x^3}{3} - \frac{3a_0x^2}{4} + a_1x + a_0$$

y1(x) =

$$\frac{3x^8}{2048} - \frac{7x^6}{384} + \frac{5x^4}{32} - \frac{3x^2}{4} + 1$$

y2(x) =

$$-\frac{x^7}{210} + \frac{x^5}{20} - \frac{x^3}{3} + x$$

$\therefore$

$$\boxed{\begin{aligned} y_1(x) &= 1 - \frac{3}{4}x^2 + \frac{5}{32}x^4 - \frac{7}{384}x^6 + \cdots \\ y_2(x) &= x - \frac{1}{3}x^3 + \frac{1}{20}x^5 - \frac{1}{210}x^7 \end{aligned}}$$

(c)

From (6), $y_1(0) = 1$, $y_2(0) = 0$

$y_1'(x) = -\frac{3}{2}x + (\text{other terms in } x)$. $\therefore y_1'(0) = 0$

$y_2'(x) = 1 - x^2 + (\text{other terms in } x)$. $\therefore y_2'(0) = 1$

$\therefore W[y_1, y_2](0) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$

$\therefore y_1(x), y_2(x)$ form a fundamental set of solutions.

(d)

From (6),

$$a_{2n} = \frac{(-1)^n \, 1\cdot 3\cdot 5\cdots(2n+1)}{2^n\,(2n)!}a_0 \qquad a_{2n+1} = \frac{(-1)^n \, 2\cdot 4\cdot 6\cdots(2n+2)}{2^{n+1}\,(2n+1)!}a_1$$

Note: $1\cdot 3\cdot 5\cdots(2n+1) = \dfrac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdots(2n)(2n+1)}{2\cdot 4\cdots(2n)} = \dfrac{(2n+1)!}{2^n(n!)}$

$$\therefore a_{2n} = \frac{(-1)^n(2n+1)!}{2^n(2n)!\,2^n(n!)} = \frac{(-1)^n(2n+1)!}{4^n(n!)(2n)!} \quad (\text{set } a_0 = 1)$$

Also, $2\cdot 4\cdot 6\cdots(2n+2) = 2(1)\cdot 2(2)\cdot 2(3)\cdots 2(n+1) = 2^{n+1}(n+1)!$

$$\therefore\ a_{2n+1} = \frac{(-1)^n 2^{n+1}(n+1)!}{2^{n+1}(2n+1)!} = \frac{(-1)^n (n+1)!}{(2n+1)!} \quad (\text{set } a_1 = 1)$$

$$\therefore\ \boxed{\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)!}{4^n (n!)(2n)!} x^{2n} \\ y_2(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)!}{(2n+1)!} x^{2n+1} \end{aligned}}$$

11.

(a)

$$\text{Let } y = \sum_{n=0}^{\infty} a_n (x-2)^n = a_0 + \sum_{n=1}^{\infty} a_n (x-2)^n \quad [1]$$

$$y' = \sum_{n=1}^{\infty} n a_n (x-2)^{n-1} \qquad x+1 = (x-2)+3$$

$$\therefore\ (x+1)y' = (x-2)y' + 3y'$$

$$= \sum_{n=1}^{\infty} n a_n (x-2)^n + \sum_{n=1}^{\infty} 3n a_n (x-2)^{n-1}$$

shift index

$$= \sum_{n=1}^{\infty} n a_n (x-2)^n + \sum_{n=0}^{\infty} 3(n+1) a_{n+1} (x-2)^n$$

$$= \sum_{n=1}^{\infty} n a_n (x-2)^n + 3a_1 + \sum_{n=1}^{\infty} 3(n+1) a_{n+1} (x-2)^n$$

write out 1st term

$$= 3a_1 + \sum_{n=1}^{\infty} [3(n+1) a_{n+1} + n a_n] (x-2)^n \qquad [2]$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x-2)^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-2)^n$$

$$= 2a_2 + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} (x-2)^n \qquad [3]$$

write out 1st term

$\therefore 2y'' + (x+1)y' + 3y =$ From [1], [2], [3]

$$(4a_2 + 3a_1 + 3a_0) + \sum_{n=1}^{\infty} [2(n+2)(n+1) a_{n+2} + 3(n+1) a_{n+1} + n a_n + 3a_n] (x-2)^n$$

$$= (4a_2 + 3a_1 + 3a_0) + \sum_{n=1}^{\infty} [2(n+2)(n+1) a_{n+2} + 3(n+1) a_{n+1} + (n+3) a_n] (x-2)^n$$

$\therefore 2y'' + (x+1)y' + 3y = 0 \Rightarrow$

(1) $4a_2 + 3a_1 + 3a_0 = 0$

(2) $2(n+2)(n+1) a_{n+2} + 3(n+1) a_{n+1} + (n+3) a_n = 0, \; n \geq 1$

Note that for $n=2$, (2) becomes (1).

$\therefore$

$$\boxed{2(n+2)(n+1) a_{n+2} + 3(n+1) a_{n+1} + (n+3) a_n = 0, \; n \geq 0}$$

(b)

a_0 and a_1 are independent.

$$a_{n+2} = -\frac{3(n+1)a_{n+1} + (n+3)a_n}{2(n+2)(n+1)} \qquad a_2 = -\frac{3}{4}a_1 - \frac{3}{4}a_0$$

Using MATLAB, shift index down

$$\therefore \quad a_n = \frac{-3(n-1)a_{n-1} - (n+1)a_{n-2}}{2n(n-1)}, \quad n \geq 3$$

```
clear, clc
syms a0 a1 x y(x) y1(x) y2(x)
Nterms = 5;  %number of desired terms
%allocate space - array to hold coefficients
a = sym(zeros(Nterms,2));
% initialize first entries for the for loop
% note a(0,:) = [a0 0], but MATLAB 1-based
a(1,:) = [0 a1];
a(2,:) = [-3*a0/4 -3*a1/4];
% n index gets coefficient for (x-2)^n
for n = 3:Nterms
    a(n,:) = (-3*(n-1)*a(n-1,:) - (n+1)*a(n-2,:))/(2*n*(n-1));
end
a  % display coefficients
y(x) = a0 + a1*(x-2); % initialize
for n = 2:Nterms
    y(x) = y(x) + (a(n,1)+a(n,2))*(x-2)^n;
end
y(x)
y1(x) = subs(y(x),[a0 a1],[1 0])
y2(x) = subs(y(x),[a0 a1],[0 1])
```

a =

$$\begin{pmatrix} 0 & a_1 \\ -\frac{3\,a_0}{4} & -\frac{3\,a_1}{4} \\ \frac{3\,a_0}{8} & \frac{a_1}{24} \\ \frac{a_0}{64} & \frac{9\,a_1}{64} \\ -\frac{39\,a_0}{640} & -\frac{31\,a_1}{640} \end{pmatrix} \begin{array}{l} a_1 \to (x-2)^1 \\ a_2 \to (x-2)^2 \\ a_3 \to (x-2)^3 \\ a_4 \to (x-2)^4 \\ a_5 \to (x-2)^5 \end{array}$$

ans =

$$a_0 - (x-2)^2\left(\frac{3\,a_0}{4} + \frac{3\,a_1}{4}\right) + (x-2)^3\left(\frac{3\,a_0}{8} + \frac{a_1}{24}\right) + (x-2)^4\left(\frac{a_0}{64} + \frac{9\,a_1}{64}\right) - (x-2)^5\left(\frac{39\,a_0}{640} + \frac{31\,a_1}{640}\right) + a_1\,(x-2)$$

y1(x) =

$$\frac{3\,(x-2)^3}{8} - \frac{3\,(x-2)^2}{4} + \frac{(x-2)^4}{64} - \frac{39\,(x-2)^5}{640} + 1$$

y2(x) =

$$x - \frac{3\,(x-2)^2}{4} + \frac{(x-2)^3}{24} + \frac{9\,(x-2)^4}{64} - \frac{31\,(x-2)^5}{640} - 2$$

$\therefore$

$$\boxed{\begin{aligned} y_1(x) &= 1 - \frac{3}{4}(x-2)^2 + \frac{3}{8}(x-2)^3 + \frac{1}{64}(x-2)^4 + \dots \\ y_2(x) &= (x-2) - \frac{3}{4}(x-2)^2 + \frac{1}{24}(x-2)^3 + \frac{9}{64}(x-2)^4 + \dots \end{aligned}}$$

(c)

From (b), $y_1(2) = 1$, $y_2(2) = 0$

$y_1'(x) = -\frac{3}{2}(x-2) +$ (other terms in $(x-2)$). $\therefore y_1'(2) = 0$

$y_2'(x) = 1 - \frac{3}{2}(x-2) +$ (other terms in $(x-2)$). $\therefore y_2'(2) = 1$

$\therefore W[y_1, y_2](2) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$

$\therefore y_1(x), y_2(x)$ form a fundamental set of solutions.

(d)

Pattern not apparent.

12.

(a) From #3, the recurrence relation is:

$$a_{n+2} = \frac{a_n}{n+2}, \; n \geq 0, \quad \text{or} \quad a_n = \frac{a_{n-2}}{n}, \; n \geq 2$$

$\therefore$ Using MATLAB, and using $a_2 = \frac{a_0}{2}$

```
clear, clc
syms a0 a1 x y(x) y1(x) y2(x)
Nterms = 6;  %number of desired terms
x0 = 0;  %the ordinary point
%allocate space - array to hold coefficients
a = sym(zeros(Nterms,2));
% initialize first entries for the for loop
% note a(0,:) = [a0 0], but MATLAB 1-based
a(1,:) = [0 a1];
a(2,:) = [a0/2 0];
% n index gets coefficient for x^n
for n = 3:Nterms
    a(n,:) = a(n-2,:)/n;
end
a;  % display coefficients
y(x) = a0 + a1*(x-x0); % initialize
for n = 2:Nterms       % build y(x) from coeffs
    y(x) = y(x) + (a(n,1)+a(n,2))*(x-x0)^n;
end
y(x)
```

ans =

$$\frac{a_0\,x^6}{48} + \frac{a_1\,x^5}{15} + \frac{a_0\,x^4}{8} + \frac{a_1\,x^3}{3} + \frac{a_0\,x^2}{2} + a_1\,x + a_0$$

$$\therefore \; y(x) = a_0 + a_1 x + \frac{a_0}{2}x^2 + \frac{a_1}{3}x^3 + \frac{a_0}{8}x^4 + \frac{a_1}{15}x^5 + \ldots$$

$$\therefore \; y(0) = 2 \Rightarrow a_0 = 2 \qquad y'(0) = 1 \Rightarrow a_1 = 1$$

$$\therefore\ y(x) = 2 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{4} + \dots$$

(b)

Using MATLAB – first part of code similar to (a), but now substitute using $a_0 = y(0)$, $a_1 = y'(0)$

```
clear, clc
syms a0 a1 x y(x) p(x) y4(x) y5(x)
Nterms = 6;  %number of desired terms
x0 = 0;  %the ordinary point
%allocate space - array to hold coefficients
a = sym(zeros(Nterms,2));
% initialize first entries for the for loop
% note a(0,:) = [a0 0], but MATLAB 1-based
a(1,:) = [0 a1];
a(2,:) = [a0/2 0];
% n index gets coefficient for x^n
for n = 3:Nterms
    a(n,:) = a(n-2,:)/n;
end
a;  % display coefficients
y(x) = a0 + a1*(x-x0); % initialize
for n = 2:Nterms      % build y(x) from coeffs
    y(x) = y(x) + (a(n,1)+a(n,2))*(x-x0)^n;
end
p(x) = subs(y(x),[x-x0 a0 a1],[x 2 1])
% MATLAB coeffs display in descending order
% So make in ascending order
c = fliplr(coeffs(p(x), 'All'))
y4(x) = 0; y5(x) = 0;
for n = 1:4
    y4(x) = y4(x) + c(n)*x^(n-1);
end
y4(x)
y5(x) = y4(x) + c(5)*x^(4)
figure
hold on
grid on
% x, y limits obtained by trial-and-error
axis([-1.1 1.1 0 6]);
fplot(y4(x))
fplot(y5(x))
xlabel 'x', ylabel 'y'
title 'Solution to y" - xy'' - y = 0'
legend('y4(x)', 'y5(x)')
```

p(x) =

$$\frac{x^6}{24} + \frac{x^5}{15} + \frac{x^4}{4} + \frac{x^3}{3} + x^2 + x + 2$$

c =

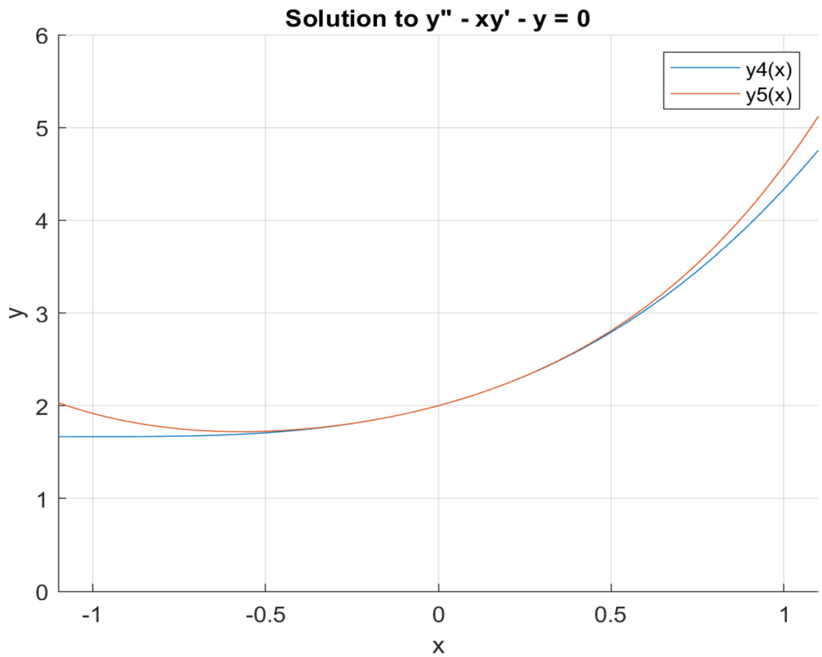
$$\begin{pmatrix} 2 & 1 & 1 & \frac{1}{3} & \frac{1}{4} & \frac{1}{15} & \frac{1}{24} \end{pmatrix}$$

ans =

$$\frac{x^3}{3} + x^2 + x + 2$$

y5(x) =

$$\frac{x^4}{4} + \frac{x^3}{3} + x^2 + x + 2$$



(c)

Plots start to depart after $|x| > 0.75$

$\therefore$ Use $|x| < 0.7$

13.

From #7, recurrence relation is: $a_{n+2} = \dfrac{-a_n}{n+1}$

or $a_n = -\dfrac{a_{n-2}}{n-1}$, $n \geq 2$. $\therefore a_2 = -a_0$

$\therefore$ Using MATLAB, and noting $y(0) = a_0$, $y'(0) = a_1$

```
clear, clc
syms a0 a1 x y(x) p(x) y4(x) y5(x)
Nterms = 5;  %number of desired terms
x0 = 0;  %the ordinary point
%allocate space - array to hold coefficients
a = sym(zeros(Nterms,2));
% initialize first entries for the for loop
% note a(0,:) = [a0 0], but MATLAB 1-based
a(1,:) = [0 a1];
a(2,:) = [-a0 0];
% n index gets coefficient for x^n
for n = 3:Nterms
    a(n,:) = -a(n-2,:)/(n-1);
end
a;  % display coefficients
y(x) = a0 + a1*(x-x0); % initialize
for n = 2:Nterms      % build y(x) from coeffs
    y(x) = y(x) + (a(n,1)+a(n,2))*(x-x0)^n;
end
y(x);
y0 = 4;  % initial conditions
dy0 = -1;
p(x) = subs(y(x),[x-x0 a0 a1],[x y0 dy0])
% MATLAB coeffs display in descending order
% So make in ascending order
c = fliplr(coeffs(p(x), 'All'))
y4(x) = 0; y5(x) = 0; % build y4(x), y5(x)
for n = 1:4
    y4(x) = y4(x) + c(n)*x^(n-1);
end
y4(x)
y5(x) = y4(x) + c(5)*x^(4)
figure
hold on
grid on
% x, y limits obtained by trial-and-error
axis([-1.1 1.1 0 6]);
fplot(y4(x))
fplot(y5(x))
xlabel 'x', ylabel 'y'
title 'Solution to y" - xy'' - y = 0'
legend('y4(x)', 'y5(x)')
```

```
p(x) =
```

$$-\frac{x^5}{8}+\frac{4\,x^4}{3}+\frac{x^3}{2}-4\,x^2-x+4$$

```
c =
```

$$\left(4 \quad -1 \quad -4 \quad \frac{1}{2} \quad \frac{4}{3} \quad -\frac{1}{8}\right)$$

```
ans =
```

$$\frac{x^3}{2}-4\,x^2-x+4$$

```
y5(x) =
```

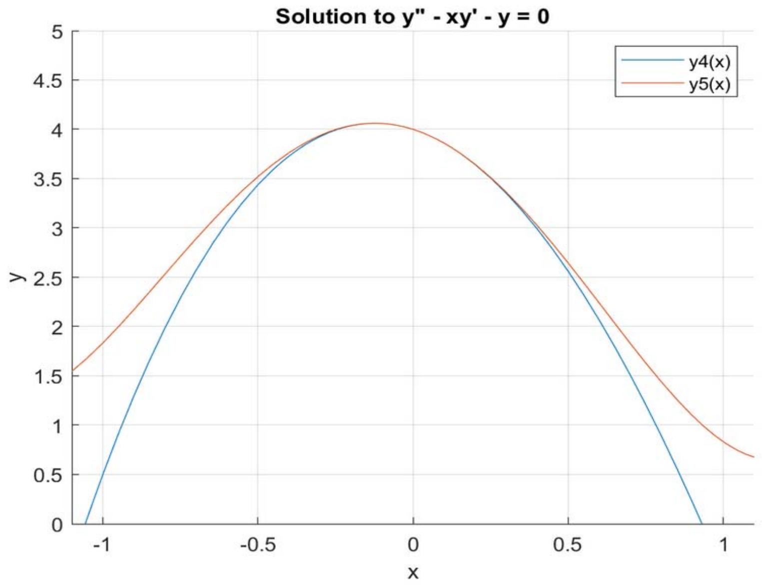
$$\frac{4\,x^4}{3}+\frac{x^3}{2}-4\,x^2-x+4$$

(a) $\therefore \underline{y(x)} = 4 - x - 4x^2 + \frac{1}{2}x^3 + \frac{4}{3}x^4 + \ldots$

(b) $y4(x) = 4 - x - 4x^2 + \frac{1}{2}x^3$

$y5(x) = 4 - x - 4x^2 + \frac{1}{2}x^3 + \frac{4}{3}x^4$

Plot on next page, from above code.



(c) Plots start to separate for $|x| > 0.5$

$\therefore \underline{|x| < 0.5}$

14.

(a)

Let $y = \sum_{n=0}^{\infty} a_n x^n$ [1]

$\therefore y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$, $xy' = \sum_{n=1}^{\infty} n a_n x^n = \sum_{n=0}^{\infty} n a_n x^n$ [2]

$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$,

$\therefore (1-x)y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1) a_n x^{n-1}$

$$= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} (n+1)(n) a_{n+1} x^n$$

shifting indices

$$= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} (n+1)(n) a_{n+1} x^n$$

works for $n=0$

$$= \sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} - (n+1)(n) a_{n+1}\right] x^n \quad [3]$$

$\therefore (1-x)y'' + xy' - y =$ From [1], [2], [3]

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} - (n+1)(n) a_{n+1} + n a_n - a_n\right] x^n$$

$$= \sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} - (n+1)(n) a_{n+1} + (n-1) a_n\right] x^n$$

$\therefore (1-x)y'' + xy' - y = 0 \Rightarrow$

$$(n+2)(n+1) a_{n+2} - (n+1)(n) a_{n+1} + (n-1) a_n = 0, \quad n \geq 0$$

Or, $a_{n+2} = \dfrac{n}{n+2} a_{n+1} - \dfrac{(n-1)}{(n+2)(n+1)} a_n, \quad n \geq 0$

Or, $a_n = \dfrac{n-2}{n} a_{n-1} - \dfrac{(n-3)}{n(n-1)} a_{n-2}, \quad n \geq 2$

$\therefore a_2 = \frac{1}{2} a_0$

Using MATLAB,

```
clear, clc
syms a0 a1 x y(x) p(x) y4(x) y5(x)
Nterms = 5;  %number of desired terms
x0 = 0;  %the ordinary point
%allocate space - array to hold coefficients
a = sym(zeros(Nterms,2));
% initialize first entries for the for loop
% note a(0,:) = [a0 0], but MATLAB 1-based
a(1,:) = [0 a1];
a(2,:) = [a0/2 0];
% n index gets coefficient for x^n
for n = 3:Nterms
    a(n,:) = ((n-2)/n)*a(n-1,:) - ((n-3)/(n*(n-1)))*a(n-2,:);
end
a;  % display coefficients
y(x) = a0 + a1*(x-x0); % initialize
for n = 2:Nterms      % build y(x) from coeffs
    y(x) = y(x) + (a(n,1)+a(n,2))*(x-x0)^n;
end
y(x);
y0 = -3;  % initial conditions
dy0 = 2;
p(x) = subs(y(x),[x-x0 a0 a1],[x y0 dy0])
% MATLAB coeffs display in descending order
% So make in ascending order
c = fliplr(coeffs(p(x), 'All'))
y4(x) = 0; y5(x) = 0; % build y4(x), y5(x)
for n = 1:4
    y4(x) = y4(x) + c(n)*x^(n-1);
end
y4(x)
y5(x) = y4(x) + c(5)*x^(4)
figure
hold on
grid on
% x, y limits obtained by trial-and-error
axis([-1.1 1.1 -7 -2]);
fplot(y4(x))
fplot(y5(x))
xlabel 'x', ylabel 'y'
title 'Solution to (1-x)y" + xy'' - y = 0'
legend('y4(x)', 'y5(x)', 'Location', 'southeast')
```

p(x) =

$$-\frac{x^5}{40}-\frac{x^4}{8}-\frac{x^3}{2}-\frac{3\,x^2}{2}+2\,x-3$$

$$\therefore\ \underline{y(x) = -3 + 2x - \frac{3}{2}x^2 - \frac{1}{2}x^3 - \frac{1}{8}x^4 + \ldots}$$

c =

$$\left(-3 \quad 2 \quad -\frac{3}{2} \quad -\frac{1}{2} \quad -\frac{1}{8} \quad -\frac{1}{40}\right)$$

ans =

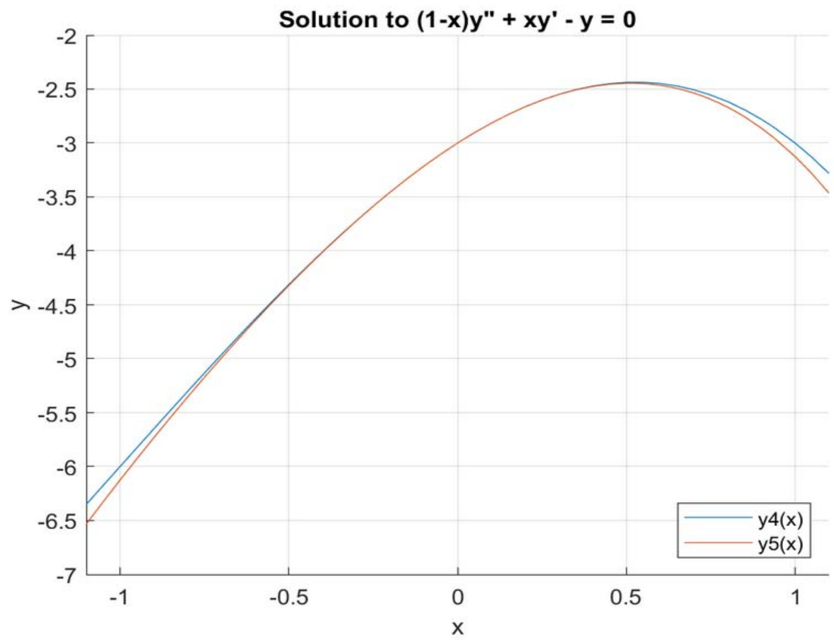
$$-\frac{x^3}{2}-\frac{3\,x^2}{2}+2\,x-3$$

y5(x) =

$$-\frac{x^4}{8}-\frac{x^3}{2}-\frac{3\,x^2}{2}+2\,x-3$$

(b)

From output of code above,



(c)

Plots start to separate for $|x| > 0.8$

$\therefore$ Use $\underline{|x| < 0.8}$

15.

(a) Using $t = x-1$, $(x^2-1) = (x-1)(x+1) = t(t+2) = t^2+2t$

$y(x) = y(t+1)$. $\therefore \frac{dy}{dx} = \frac{dy}{dt}$, and $y''(x) = y''(t)$

$$\therefore y'' + t^2 y' + (t^2+2t)y = 0$$

Let $y = \sum_{n=0}^{\infty} a_n t^n$

$$\therefore (t^2+2t)y = \sum_{n=0}^{\infty} a_n t^{n+2} + \sum_{n=0}^{\infty} 2a_n t^{n+1}$$

$$= \sum_{n=2}^{\infty} a_{n-2} t^n + \sum_{n=1}^{\infty} 2a_{n-1} t^n$$

write out 1st term, combine

$$= 2a_0 t + \sum_{n=2}^{\infty} (a_{n-2} + 2a_{n-1}) t^n \qquad [1]$$

$$y' = \sum_{n=1}^{\infty} n a_n t^{n-1}, \quad t^2 y' = \sum_{n=1}^{\infty} n a_n t^{n+1} = \sum_{n=2}^{\infty} (n-1) a_{n-1} t^n \qquad [2]$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} t^n$$

write out first 2 terms

$$= 2a_2 + 6a_3 t + \sum_{n=2}^{\infty} (n+2)(n+1) a_{n+2} t^n \qquad [3]$$

$$\therefore\ y'' + t^2 y' + (t^2 + 2t)y =$$ From [1], [2], [3]

$$[2a_2 + (2a_0 + 6a_3)t] +$$

$$\sum_{n=2}^{\infty} \left[(n+2)(n+1)a_{n+2} + (n-1)a_{n-1} + a_{n-2} + 2a_{n-1}\right] t^n$$

$$= [2a_2 + (2a_0 + 6a_3)t] + \sum_{n=2}^{\infty} \left[(n+2)(n+1)a_{n+2} + (n+1)a_{n-1} + a_{n-2}\right] t^n$$

$$\therefore\ y'' + t^2 y' + (t^2 + 2t)y = 0 \implies$$

(1) $2a_2 = 0$, or $a_2 = 0$

(2) $2a_0 + 6a_3 = 0$, or $a_3 = -\frac{1}{3}a_0$

(3) $(n+2)(n+1)a_{n+2} + (n+1)a_{n-1} + a_{n-2} = 0$, $n \geq 2$

$$\therefore\ a_{n+2} = -\frac{a_{n-1}}{n+2} - \frac{a_{n-2}}{(n+2)(n+1)}, \quad n \geq 2$$

Or, $$a_n = -\frac{a_{n-3}}{n} - \frac{a_{n-4}}{n(n-1)}, \quad n \geq 5$$ [4]

with $a_2 = 0$, $a_3 = -\frac{1}{3}a_0$, $a_4 = -\frac{a_1}{4} - \frac{a_0}{12}$

Now use MATLAB using [4] as recurrence relation.

```
clear, clc
syms a0 a1 t y(t) y1(t) y2(t)
Nterms = 8;  %number of desired terms
t0 = 0;  % the ordinary point
%allocate space - array to hold coefficients
a = sym(zeros(Nterms,2));
% initialize first entries for the for loop
% note a(0,:) = [a0 0], but MATLAB 1-based
a(1,:) = [0 a1];
a(2,:) = [0 0];
a(3,:) = [-a0/3 0];
a(4,:) = [-a0/12 -a1/4];
% n index gets coefficient for (t-t0)^n
for n = 5:Nterms
    a(n,:) = -a(n-3,:)/n - a(n-4,:)/(n*(n-1));
end
a  % display coefficients
y(t) = a0 + a1*(t-t0); % initialize
for n = 2:Nterms
    y(t) = y(t) + (a(n,1)+a(n,2))*(t-t0)^n;
end
y(t)
y1(t) = subs(y(t),[a0 a1],[1 0])
y2(t) = subs(y(t),[a0 a1],[0 1])
```

a =

$$\begin{pmatrix} 0 & a_1 \\ 0 & 0 \\ -\frac{a_0}{3} & 0 \\ -\frac{a_0}{12} & -\frac{a_1}{4} \\ 0 & -\frac{a_1}{20} \\ \frac{a_0}{18} & 0 \\ \frac{5\,a_0}{252} & \frac{a_1}{28} \\ \frac{a_0}{672} & \frac{3\,a_1}{280} \end{pmatrix}
\begin{array}{l} a_1 \to t^1 \\ a_2 \to t^2 \\ a_3 \to t^3 \\ a_4 \to t^4 \\ a_5 \to t^5 \\ a_6 \to t^6 \\ a_7 \to t^7 \\ a_8 \to t^8 \end{array}$$

ans =

$$\left(\frac{a_0}{672}+\frac{3\,a_1}{280}\right)t^8+\left(\frac{5\,a_0}{252}+\frac{a_1}{28}\right)t^7+\frac{a_0\,t^6}{18}-\frac{a_1\,t^5}{20}+\left(-\frac{a_0}{12}-\frac{a_1}{4}\right)t^4-\frac{a_0\,t^3}{3}+a_1\,t+a_0$$

y1(t) =

$$\frac{t^8}{672}+\frac{5\,t^7}{252}+\frac{t^6}{18}-\frac{t^4}{12}-\frac{t^3}{3}+1$$

y2(t) =

$$\frac{3\,t^8}{280}+\frac{t^7}{28}-\frac{t^5}{20}-\frac{t^4}{4}+t$$

Now substituting $x-1$ for t in $y_1(t)$ and $y_2(t)$,

$$\boxed{\begin{aligned} y_1(x) &= 1-\frac{1}{3}(x-1)^3-\frac{1}{12}(x-1)^4+\frac{1}{18}(x-1)^6+\dots \\ y_2(x) &= (x-1)-\frac{1}{4}(x-1)^4-\frac{1}{20}(x-1)^5+\frac{1}{28}(x-1)^7+\dots \end{aligned}}$$

(b)

$$x^2-1=(x+1)(x-1)=(x-1+2)(x-1)=(x-1)^2+2(x-1)$$

$$\therefore\ y''+(x-1)^2y'+(x^2-1)y=$$

$$y''+(x-1)^2y'+\left[(x-1)^2+2(x-1)\right]y$$

Let $y=\sum_{n=0}^{\infty}a_n(x-1)^n$

$$\therefore\ \left[(x-1)^2+2(x-1)\right]y=\sum_{n=0}^{\infty}a_n(x-1)^{n+2}+\sum_{n=0}^{\infty}2a_n(x-1)^{n+1}$$

shift index

write out 1st term

$$=\sum_{n=1}^{\infty}a_{n-1}(x-1)^{n+1}+2a_0(x-1)+\sum_{n=1}^{\infty}2a_n(x-1)^{n+1}$$

$$=2a_0(x-1)+\sum_{n=1}^{\infty}\left[2a_n+a_{n-1}\right](x-1)^{n+1}\quad [1]$$

$$y'=\sum_{n=1}^{\infty}na_n(x-1)^{n-1},\ \therefore (x-1)^2y'=\sum_{n=1}^{\infty}na_n(x-1)^{n+1}\quad [2]$$

$$y''=\sum_{n=2}^{\infty}n(n-1)a_n(x-1)^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-1)^n$$

writing out 1st term

$$=2a_2+\sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2}(x-1)^n$$

$$=2a_2+\sum_{n=0}^{\infty}(n+3)(n+2)a_{n+3}(x-1)^{n+1}$$

write out 1st term

$$= 2a_2 + 6a_3(x-1) + \sum_{n=1}^{\infty} (n+3)(n+2)a_{n+3}(x-1)^{n+1} \quad [3]$$

$$\therefore\ y'' + (x-1)^2 y' + (x^2-1)y = \qquad \text{From } [1], [2], [3]$$

$$2a_0(x-1) + 2a_2 + 6a_3(x-1) +$$

$$\sum_{n=1}^{\infty} \left[(n+3)(n+2)a_{n+3} + na_n + 2a_n + a_{n-1}\right](x-1)^{n+1}$$

$$= 2a_2 + (2a_0 + 6a_3)(x-1) + \sum_{n=1}^{\infty} \left[(n+3)(n+2)a_{n+3} + (n+2)a_n + a_{n-1}\right](x-1)^{n+1}$$

$$\therefore\ y'' + (x-1)^2 y' + (x^2-1)y = 0 \implies$$

(1) $2a_2 = 0$

(2) $2a_0 + 6a_3 = 0$

(3) $(n+3)(n+2)a_{n+3} + (n+2)a_n + a_{n-1} = 0, \quad n \geq 1$

Or, $a_2 = 0, \quad a_3 = -\frac{1}{3}a_0, \quad a_{n+3} = \dfrac{-a_n}{n+3} - \dfrac{a_{n-1}}{(n+3)(n+2)}, \quad n \geq 1$

Or, $a_2 = 0, \quad a_3 = -\frac{1}{3}a_0, \quad a_n = -\dfrac{a_{n-3}}{n} - \dfrac{a_{n-4}}{n(n-1)}, \quad n \geq 4$

Or,

$$\boxed{\begin{array}{l} a_n = -\dfrac{a_{n-3}}{n} - \dfrac{a_{n-4}}{n(n-1)}, \quad n \geq 5 \\[2ex] \text{with } a_2 = 0, \quad a_3 = -\frac{1}{3}a_0, \quad a_4 = -\dfrac{a_1}{4} - \dfrac{a_0}{12} \end{array}}$$

This is the exact $\underline{\text{same recurrence relation as in (a)}}$

$\therefore$ result is the same

Using MATLAB to corroborate,

```
clear, clc
syms a0 a1 x y(x) y1(x) y2(x)
Nterms = 8;  %number of desired terms
x0 = 1;  % the ordinary point
%allocate space - array to hold coefficients
a = sym(zeros(Nterms,2));
% initialize first entries for the for loop
% note a(0,:) = [a0 0], but MATLAB 1-based
a(1,:) = [0 a1];
a(2,:) = [0 0];
a(3,:) = [-a0/3 0];
a(4,:) = [-a0/12 -a1/4];
% n index gets coefficient for (x-x0)^n
for n = 5:Nterms
    a(n,:) = -a(n-3,:)/n - a(n-4,:)/(n*(n-1));
end
a  % display coefficients
y(x) = a0 + a1*(x-x0); % initialize
for n = 2:Nterms
    y(x) = y(x) + (a(n,1)+a(n,2))*(x-x0)^n;
end
y(x)
y1(x) = subs(y(x),[a0 a1],[1 0])
y2(x) = subs(y(x),[a0 a1],[0 1])
```

a =

$$\begin{pmatrix} 0 & a_1 \\ 0 & 0 \\ -\frac{a_0}{3} & 0 \\ -\frac{a_0}{12} & -\frac{a_1}{4} \\ 0 & -\frac{a_1}{20} \\ \frac{a_0}{18} & 0 \\ \frac{5\,a_0}{252} & \frac{a_1}{28} \\ \frac{a_0}{672} & \frac{3\,a_1}{280} \end{pmatrix} \quad \begin{array}{l} a_1 \to (x-1)^1 \\ a_2 \to (x-1)^2 \\ a_3 \to (x-1)^3 \\ a_4 \to (x-1)^4 \\ a_5 \to (x-1)^5 \\ a_6 \to (x-1)^6 \\ a_7 \to (x-1)^7 \\ a_8 \to (x-1)^8 \end{array}$$

ans =

$$a_0 - (x-1)^4\left(\frac{a_0}{12}+\frac{a_1}{4}\right) + (x-1)^7\left(\frac{5\,a_0}{252}+\frac{a_1}{28}\right) + (x-1)^8\left(\frac{a_0}{672}+\frac{3\,a_1}{280}\right) + a_1\,(x-1) - \frac{a_0\,(x-1)^3}{3} + \frac{a_0\,(x-1)^6}{18} - \frac{a_1\,(x-1)^5}{20}$$

y1(x) =

$$\frac{(x-1)^6}{18} - \frac{(x-1)^4}{12} - \frac{(x-1)^3}{3} + \frac{5\,(x-1)^7}{252} + \frac{(x-1)^8}{672} + 1$$

y2(x) =

$$x - \frac{(x-1)^4}{4} - \frac{(x-1)^5}{20} + \frac{(x-1)^7}{28} + \frac{3\,(x-1)^8}{280} - 1$$

$$\therefore \boxed{\begin{aligned} y_1(x) &= 1 - \tfrac{1}{3}(x-1)^3 - \tfrac{1}{12}(x-1)^4 + \tfrac{1}{18}(x-1)^6 + \dots \\ y_2(x) &= (x-1) - \tfrac{1}{4}(x-1)^4 - \tfrac{1}{20}(x-1)^5 + \tfrac{1}{28}(x-1)^7 + \dots \end{aligned}}$$

16.

Equation 10 is: $a_n = a_{2k+1} = \frac{(-1)^k}{(2k+1)!} a_1, \quad k = 1, 2, 3, \ldots .$ (10)

given: $(n+2)(n+1)a_{n+2} + a_n = 0, \quad n = 0, 1, 2, 3, \ldots .$ (8)

(1) For $K=1$, $a_{2k+1} = a_3 = \frac{(-1)^1}{3!} a_1 = -\frac{a_1}{6}$

Using (8), when $n=1$, $(3)(2)a_3 + a_1 = 0 \Rightarrow a_3 = -\frac{a_1}{6}$

$\therefore$ (10) is true for $K=1$

(2) Suppose (10) is true for $K > 1$.

Consider $K+1$: $a_{2(K+1)+1} = a_{2K+3}$

By (8), when $n = 2K+1$, $n+2 = 2K+3$.

$\therefore (2K+1+2)(2K+1+1)a_{2K+3} + a_{2K+1} = 0$,

or, $a_{2K+3} = \frac{-a_{2K+1}}{(2K+3)(2K+2)}$ [1]

By assumption, (10) is true for $K>1$ so that $a_{2K+1} = \frac{(-1)^K}{(2K+1)!} a_1$

Substituting this into [1],

$$a_{2k+3} = \frac{-(-1)^k}{(2k+3)(2k+2)(2k+1)!} a_1 = \frac{(-1)^{k+1}}{(2k+3)!} a_1$$

This is (10) for $k+1$.

$\therefore$ True for $k>1 \Rightarrow$ true for $k+1$

(1) + (2) $\Rightarrow$ (10) true for all $k \geq 1$

17.

(1) For $y_1(x)$, $$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{3n+3}}{2\cdot 3\cdots(3n+2)(3n+3)} \cdot \frac{2\cdot 3\cdots(3n-1)(3n)}{x^{3n}}\right|$$

$$= |x^3| \left|\frac{2\cdot 3\cdots(3n-1)(3n)}{2\cdot 3\cdots(3n-1)(3n)(3n+1)(3n+2)(3n+3)}\right|$$

$$= |x^3| \left|\frac{1}{(3n+1)(3n+2)(3n+3)}\right|$$

$$\therefore \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = |x^3| \lim_{n\to\infty} \left|\frac{1}{(3n+1)(3n+2)(3n+3)}\right| = 0$$

This is true for all x.

$\therefore$ $y_1(x)$ converges for all x.

(2) For $y_2(x)$, $\left|\dfrac{a_{n+1}}{a_n}\right| = \left|\dfrac{x^{3n+4}}{3\cdot 4\cdots(3n+3)(3n+4)} \cdot \dfrac{3\cdot 4\cdots(3n)(3n+1)}{x^{3n+1}}\right|$

$$= |x^3| \left|\frac{3\cdot 4\cdots(3n)(3n+1)}{3\cdot 4\cdots(3n)(3n+1)(3n+2)(3n+3)(3n+4)}\right|$$

$$= |x^3| \left|\frac{1}{(3n+2)(3n+3)(3n+4)}\right|$$

$$\therefore \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x^3| \lim_{n\to\infty}\left|\frac{1}{(3n+2)(3n+3)(3n+4)}\right| = 0$$

This is true for all x.

$\therefore$ $y_2(x)$ converges for all x

18.

(a)

Let $y = \sum_{n=0}^{\infty} a_n x^n \quad \therefore \lambda y = \sum_{n=0}^{\infty} \lambda a_n x^n$

write out 1st term

$$= \lambda a_0 + \sum_{n=1}^{\infty} \lambda a_n x^n \qquad [1]$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad 2xy' = \sum_{n=1}^{\infty} 2n a_n x^n \qquad [2]$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^n = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

write out 1st term

$$= 2a_2 + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} x^n \quad [3]$$

$\therefore y'' - 2xy' + \lambda y =$

$$(\lambda a_0 + 2a_2) + \sum_{n=1}^{\infty} \left[(n+2)(n+1) a_{n+2} - 2n a_n + \lambda a_n \right] x^n$$

$$= (\lambda a_0 + 2a_2) + \sum_{n=1}^{\infty} \left[(n+2)(n+1) a_{n+2} + (\lambda - 2n) a_n \right] x^n$$

$\therefore y'' - 2xy' - \lambda y = 0 \implies$

(1) $\lambda a_0 + 2a_2 = 0$

$$(2)\quad (n+2)(n+1)a_{n+2} + (\lambda - 2n)a_n = 0, \quad n \geq 1$$

Or, $a_2 = -\frac{\lambda}{2} a_0$

$$a_{n+2} = -\frac{(\lambda - 2n)}{(n+2)(n+1)} a_n, \quad n \geq 1$$

But this also works for $n=0$. Shift index

to get: $$a_n = -\frac{(\lambda - 2n + 4)}{n(n-1)} a_{n-2}, \quad n \geq 2$$

Use MATLAB (use v for λ as can't type λ into MATLAB)

```
clear, clc
syms a0 a1 x v y(x) y1(x) y2(x)  % v = lambda
Nterms = 8;  %number of desired terms
x0 = 0;  % the ordinary point
%allocate space - array to hold coefficients
a = sym(zeros(Nterms,2));
% initialize first entries for the for loop
% note a(0,:) = [a0 0], but MATLAB 1-based
a(1,:) = [0 a1];
a(2,:) = [-v*a0/2 0];
% n index gets coefficient for (x-x0)^n
for n = 3:Nterms
    a(n,:) = -((v-2*n+4)/(n*(n-1)))*a(n-2,:);
end
a;  % display of coefficients
y(x) = a0 + a1*(x-x0); % initialize
for n = 2:Nterms
    y(x) = y(x) + (a(n,1)+a(n,2))*(x-x0)^n;
end
y(x);
y1(x) = subs(y(x),[a0 a1],[1 0])
y2(x) = subs(y(x),[a0 a1],[0 1])
```

y1(x) =

$$\frac{v\left(\frac{v}{12}-\frac{1}{3}\right)\left(\frac{v}{30}-\frac{4}{15}\right)\left(\frac{v}{56}-\frac{3}{14}\right)x^8}{2} - \frac{v\left(\frac{v}{12}-\frac{1}{3}\right)\left(\frac{v}{30}-\frac{4}{15}\right)x^6}{2} + \frac{v\left(\frac{v}{12}-\frac{1}{3}\right)x^4}{2} - \frac{v\,x^2}{2} + 1$$

y2(x) =

$$-\left(\frac{v}{6}-\frac{1}{3}\right)\left(\frac{v}{20}-\frac{3}{10}\right)\left(\frac{v}{42}-\frac{5}{21}\right)x^7 + \left(\frac{v}{6}-\frac{1}{3}\right)\left(\frac{v}{20}-\frac{3}{10}\right)x^5 + \left(\frac{1}{3}-\frac{v}{6}\right)x^3 + x$$

The terms $v\left(\frac{v}{12}-\frac{1}{3}\right)$, $\left(\frac{v}{30}-\frac{4}{15}\right)$ in $y_1(x)$ and $\left(\frac{v}{6}-\frac{1}{3}\right)$, $\left(\frac{v}{20}-\frac{3}{10}\right)$ in $y_2(x)$ repeat.

$$\frac{v\left(\frac{v}{12}-\frac{1}{3}\right)}{2} = \frac{v(v-4)}{24} = \frac{v(v-4)}{4!}$$

$$\left(\frac{v}{30}-\frac{4}{15}\right) = \frac{v-8}{6\cdot 5}.$$

$$\therefore y_1(x) = 1 - \frac{vx^2}{2!} + \frac{v(v-4)}{4!}x^4 - \frac{v(v-4)(v-8)}{6!}x^6$$

$$\frac{v}{6}-\frac{1}{3} = \frac{v-2}{6} = \frac{v-2}{3\cdot 2}, \quad \frac{v}{20}-\frac{3}{10} = \frac{v-6}{20} = \frac{v-6}{5\cdot 4}$$

$$\frac{v}{42}-\frac{5}{21} = \frac{v-10}{42} = \frac{v-10}{7\cdot 6}$$

$$\therefore y_2(x) = x - \frac{(v-2)}{3!}x^3 + \frac{(v-2)(v-6)}{5!}x^5 - \frac{(v-2)(v-6)(v-10)}{7!}x^7$$

Reverting back to λ

$$\boxed{\begin{aligned} y_1(x) &= 1 - \frac{\lambda x^2}{2!} + \frac{\lambda(\lambda-4)}{4!}x^4 - \frac{\lambda(\lambda-4)(\lambda-8)}{6!}x^6 + \cdots \\ y_2(x) &= x - \frac{(\lambda-2)}{3!}x^3 + \frac{(\lambda-2)(\lambda-6)}{5!}x^5 - \frac{(\lambda-2)(\lambda-4)(\lambda-10)}{7!}x^7 + \cdots \end{aligned}}$$

$$\text{Or,}\quad y_1(x) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n \lambda(\lambda-4)\cdots[\lambda-(n-1)4]\, x^{2n}}{(2n)!}$$

$$y_2(x) = x + \sum_{n=1}^{\infty} \frac{(-1)^n (\lambda-2)(\lambda-6)\cdots(\lambda-[2+(n-1)4])\, x^{2n+1}}{(2n+1)!}$$

Note $y_1(0) = 1$, $y_2(0) = 0$

$y_1'(x) = -\lambda x +$ (other terms in x). $\therefore y_1'(0) = 0$

$y_2'(x) = 1 + \frac{(\lambda-2)x^2}{2} +$ (other terms in x). $\therefore y_2'(0) = 1$

$$\therefore W[y_1, y_2](0) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$$

$\therefore y_1(x), y_2(x)$ form a fundamental set of solutions.

(b)

Note $y_1(x)$ has $\lambda(\lambda-4)(\lambda-8)\cdots$ in its summation series. $\therefore$ For $\lambda = 0, 4, 8, \ldots$, the summation series will terminate and become finite.

Similarly for $y_2(x)$ which has $(\lambda-2)(\lambda-6)(\lambda-10)\cdots$,

for $\lambda = 2, 6, 10, \ldots$

Using $y_1(x)$ and $y_2(x)$ from (a),

$\lambda = 0: \quad y_1(x) = \underline{1}$

$\lambda = 4: \quad y_1(x) = 1 - \frac{4}{2!}x^2 = \underline{1 - 2x^2}$

$\lambda = 8: \quad y_1(x) = 1 - \frac{8}{2!}x^2 + \frac{8(4)}{4!} = \underline{1 - 4x^2 + \frac{4}{3}x^4}$

$\lambda = 2: \quad y_2(x) = \underline{x}$

$\lambda = 6: \quad y_2(x) = x - \frac{4}{3!}x^3 = \underline{x - \frac{2}{3}x^3}$

$\lambda = 10: \quad y_2(x) = x - \frac{8}{3!}x^3 + \frac{8(4)}{5!}x^5 = \underline{x - \frac{4}{3}x^3 + \frac{4}{15}x^5}$

Note since $y(x) = a_0 y_1(x) + a_1 y_2(x)$, each polynomial is determined up up a multiplicative constant: a_0 or a_1.

(c)

For $n = 0, 1, 2, \ldots, 5$: $\lambda = 0, 2, 4, 6, 8, 10$

$\therefore$ From (b), the polynomials, either $y_1(x)$ or $y_2(x)$, are obtained. To get the coefficient of x^n, the highest power of the polynomial, to be 2^n, multiply $y_1(x)$ or $y_2(x)$ from (b) by an appropriate constant.

$\therefore$ $H_0(x)$: $n = 0$, so $\lambda = 0$ $\therefore$ $y_1(x) = 1 = x^0$

coefficient of x^0 is $2^0 = 1$.

$\therefore$ $\underline{H_0(x) = 1}$

$H_1(x)$: $n = 1 \Rightarrow \lambda = 2$. $\therefore$ $y_2(x) = x$

x is the highest power, $\therefore$ want $2^1 = 2$ as coefficient. $\therefore$ $2y_2(x) = \underline{H_1(x) = 2x}$

$H_2(x)$: $n = 2 \Rightarrow \lambda = 4$. $\therefore$ $y_1(x) = 1 - 2x^2$

$n = 2 \Rightarrow 2^n = 4$, so 4 as coefficient of x^2

$$\therefore (-2)\,y_1(x) = H_2(x) = -2 + 4x^2$$

$H_3(x)$: $n = 3 \Rightarrow \lambda = 6$. $\therefore Y_2(x) = x - \frac{2}{3}x^3$

$n = 3 \Rightarrow 2^n = 8$, $\therefore$ 8 as coefficient of x^3

$$\therefore 12\,y_2(x) = H_3(x) = 12x - 8x^3$$

$H_4(x)$: $n = 4 \Rightarrow \lambda = 8$. $\therefore y_1(x) = 1 - 4x^2 + \frac{4}{3}x^4$

$n = 4 \Rightarrow 2^n = 16$, $\therefore$ 16 as coefficient for x^4

$$\therefore 12\,y_1(x) = H_4(x) = 12 - 48x^2 + 16x^4$$

$H_5(x)$: $n = 5 \Rightarrow \lambda = 10$. $\therefore Y_2(x) = x - \frac{4}{3}x^3 + \frac{4}{15}x^5$

$n = 5 \Rightarrow 2^n = 32$, $\therefore$ 32 as coefficient for x^5

$$\therefore 32\left(\tfrac{15}{4}\right) Y_2(x) = H_5(x) = 120x - 160x^3 + 32x^5$$

19.

(a)

$\sin(0) = 0$, so $\sin(x)$ solves the initial value.

$\sqrt{1-y^2} = \sqrt{1-\sin^2(x)} = \sqrt{\cos^2(x)} = \cos(x)$, $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

$y'(x) = \cos(x)$, $\therefore y' = \sqrt{1-y^2}$ for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

(b)

By squaring, consider $(y')^2 = 1-y^2$, or

$(y')^2 + y^2 - 1 = 0$

Let $y = \sum_{n=0}^{\infty} a_n x^n$

$$\therefore y^2 = \left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} a_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n,$$

where $c_n = a_n a_0 + a_{n-1} a_1 + \dots + a_0 a_n$

So, $c_0 = a_0^2$,

$c_1 = a_1 a_0 + a_0 a_1 = 2 a_0 a_1$

$c_2 = a_2 a_0 + a_1 a_1 + a_0 a_2 = a_1^2 + 2 a_0 a_1$

$c_3 = a_3 a_0 + a_2 a_1 + a_1 a_2 + a_0 a_3 = 2 a_0 a_3 + 2 a_1 a_2$

$c_4 = a_4 a_0 + a_3 a_1 + a_2 a_2 + a_1 a_3 + a_0 a_4$

$= a_2^2 + 2 a_0 a_4 + 2 a_1 a_3$

$$\therefore\ y^2(x) = a_0^2 + (2a_0a_1)x + (a_1^2 + 2a_0a_1)x^2 + (2a_0a_3 + 2a_1a_2)x^3$$
$$+ (a_2^2 + 2a_0a_4 + 2a_1a_3)x^4 + \dots$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1} x^n$$

$$\therefore\ (y')^2 = \left(\sum_{n=0}^{\infty} (n+1)a_{n+1} x^n\right)\left(\sum_{n=0}^{\infty} (n+1)a_{n+1} x^n\right) = \sum_{n=0}^{\infty} d_n x^n$$

where $d_n = \underset{n}{[(n+1)a_{n+1}]}\underset{0}{[a_1]} + \underset{n-1}{[na_n]}\underset{1}{[2a_2]} + \dots + \underset{0}{[a_1]}\underset{n}{[(n+1)a_{n+1}]}$

so, $d_0 = a_1^2$

$$d_1 = 2a_2a_1 + a_1(2a_2) = 4a_1a_2$$

$$d_2 = (3a_3)(a_1) + (a_1)(2a_2) + (a_1)(3a_3) = 2a_1a_2 + 6a_1a_3$$

$$d_3 = (4a_4)(a_1) + (3a_3)(2a_2) + (2a_2)(3a_3) + (a_1)(4a_4)$$
$$= 8a_1a_4 + 12a_2a_3$$

$$d_4 = (5a_5)(a_1) + (4a_4)(2a_2) + (3a_3)(3a_3) + (2a_2)(4a_4) + (a_1)(5a_5)$$
$$= 10a_1a_5 + 16a_2a_4 + 9a_3^2$$

$$\therefore\ (y')^2 = a_1^2 + (4a_1a_2)x + (2a_1a_2 + 6a_1a_3)x^2 + (8a_1a_4 + 12a_2a_3)x^3$$
$$+ (10a_1a_5 + 16a_2a_4 + 9a_3^2)x^4 + \dots$$

$\therefore (y')^2 + y^2 - 1 =$

$$(a_1^2 + a_0^2 - 1) + (4a_1a_2 + 2a_0a_1)x + (2a_1a_2 + 6a_1a_3 + a_1^2 + 2a_0a_1)x^2$$
$$+ (8a_1a_4 + 12a_2a_3 + 2a_0a_3 + 2a_1a_2)x^3 + \dots$$

$\therefore (y')^2 + y^2 - 1 = 0$, and $y(0) = 0$ (initial condition)

and $y'(0) = 1$, since $y' = \sqrt{1-y^2}$

(0) $a_0 = 0$ (from $y(0) = 0$)

(1) $a_1 = 1$ (from $y'(0) = 1$) $\therefore a_0^2 + a_1^2 - 1 = 0$

(2) $4\overset{=1 \text{ from (1)}}{a_1} a_2 + 2\overset{=0 \text{ from (0)}}{a_0} a_1 = 0 \quad \therefore 4a_2 = 0, \; a_2 = 0$

(3) $2a_1\overset{=0}{a_2} + 6\overset{=1}{a_1}a_3 + \overset{=1}{a_1^2} + 2\overset{=0}{a_0}a_1 = 0$, or $6a_3 + 1 = 0$, $a_3 = -\frac{1}{6}$

(4) $8\overset{=1}{a_1}a_4 + 12\overset{=0}{a_2}a_3 + 2\overset{=0}{a_0}a_3 + 2a_1\overset{=0}{a_2} = 0$, or $8a_4 = 0$, $a_4 = 0$

Summary:

$a_0 = 0 \qquad a_1 = 1$

$a_2 = 0 \qquad a_3 = -\frac{1}{6}$

$a_4 = 0$

$\therefore \underline{y(x) = x - \frac{1}{6}x^3 + \dots}$

Use MATLAB to confirm above calculations.

```
syms n a(n) x
f = a(n)*x^n;               % n-th term of f(x)
df = (n+1)*a(n+1)*x^n;      % n-th term of f'(x)
sf = symsum(f,n,0,4)        % summation terms of f(x) to x^4
sdf =symsum(df,n,0,3)       % summation terms of f'(x) to x^3
s = sdf^2 + sf^2 - 1;       % (y')^2 + y^2 - 1:  add the terms
p = collect(s,x)            % collect terms in polynomial form
c = coeffs(p,x,'all');      % 'All' gives coeffs in descending order
l = length(c);              % number of terms

c(l)                        % constant term
c(l-1)                      % coeff of x
c(l-2)                      % coeff of x^2
c(l-3)                      % coeff of x^3
subs(c(l), [a(0),a(1)], [0,1])    % use a0 = 0, a1 =1 to
subs(c(l-1), [a(0),a(1)], [0,1])  % simplify above expressions
subs(c(l-2), [a(0),a(1)], [0,1])
subs(c(l-3), [a(0),a(1)], [0,1])
```

sf = $a(4)\,x^4 + a(3)\,x^3 + a(2)\,x^2 + a(1)\,x + a(0)$

sdf = $4\,a(4)\,x^3 + 3\,a(3)\,x^2 + 2\,a(2)\,x + a(1)$

constant ans = $a(0)^2 + a(1)^2 - 1$

x^1 ans = $2\,a(0)\,a(1) + 4\,a(1)\,a(2)$

x^2 ans = $2\,a(0)\,a(2) + 6\,a(1)\,a(3) + a(1)^2 + 4\,a(2)^2$

x^3 ans = $2\,a(0)\,a(3) + 2\,a(1)\,a(2) + 8\,a(1)\,a(4) + 12\,a(2)\,a(3)$

constant ans = 0

x^1 ans = $4\,a(2)$

x^2 ans = $6\,a(3) + 4\,a(2)^2 + 1$

x^3 ans = $2\,a(2) + 8\,a(4) + 12\,a(2)\,a(3)$

$\therefore\ 4a_2 = 0,\quad a_2 = 0$

$6a_3 + 1 = 0,\quad a_3 = -\frac{1}{6}$

$8a_4 = 0,\quad a_4 = 0$

$\therefore\ y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots = x - \frac{1}{6}x^3 + \ldots$

20.

From #7, $y(x) = a_0\left[1 - \frac{x^2}{1} + \frac{x^4}{1\cdot 3} - \frac{x^6}{1\cdot 3\cdot 5} + \frac{x^8}{1\cdot 3\cdot 5\cdot 7}\right]$

$+ a_1\left[x - \frac{x^3}{2} + \frac{x^5}{2\cdot 4} - \frac{x^7}{2\cdot 4\cdot 6} + \frac{x^9}{2\cdot 4\cdot 6\cdot 8}\right] + \ldots$

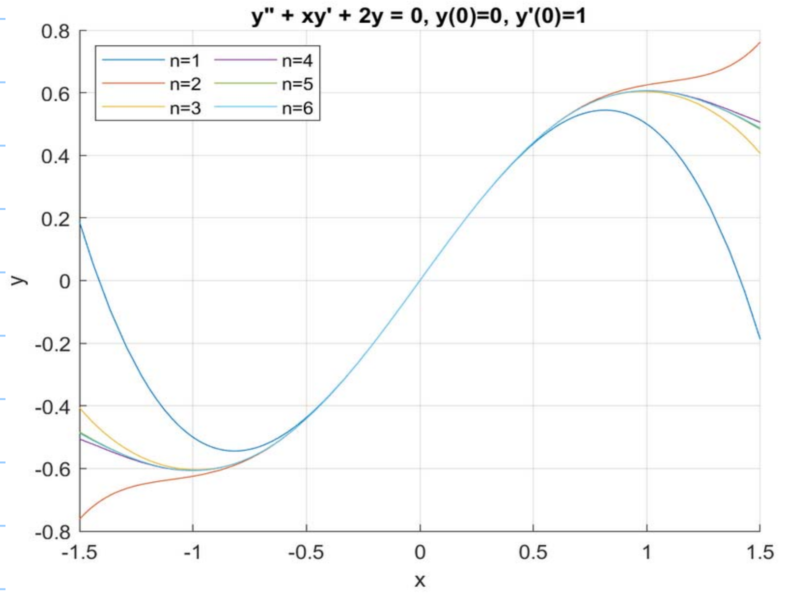
$y(0) = 0 \Rightarrow a_0 = 0$

$y'(0) = 1 \Rightarrow a_1 = 1$

$\therefore\ y(x) = x - \frac{x^3}{2} + \frac{x^5}{2\cdot 4} - \frac{x^7}{2\cdot 4\cdot 6} + \frac{x^9}{2\cdot 4\cdot 6\cdot 8} + \ldots + \frac{(-1)^n}{2^n(n!)}x^{2n+1} + \ldots$

Using MATLAB,

```
clear, clc
syms x y(x)
y(x) = x;  % Don't plot this
figure
hold on
grid on
for n = 1:6
    y(x) = y(x) + ((-1)^n)*(x^(2*n+1))/((2^n)*factorial(n));
    fplot(y(x), [-1.5,1.5])
end
xlabel 'x', ylabel 'y'
title 'y" + xy'' + 2y = 0, y(0)=0, y''(0)=1'
legend('n=1', 'n=2', 'n=3', 'n=4', 'n=5', 'n=6', ...
       'Location', 'northwest', 'NumColumns', 2)
```



21.

Let $y = \sum_{n=0}^{\infty} a_n x^n \qquad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

$$\therefore (4-x^2)y'' = \sum_{n=2}^{\infty} 4n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^n$$

$n(n-1) = 0$ for $n = 0, 1$

$$= \sum_{n=0}^{\infty} 4(n+2)(n+1)a_{n+2} x^n - \sum_{n=0}^{\infty} n(n-1)a_n x^n$$

$$= \sum_{n=0}^{\infty} \left[4(n+2)(n+1)a_{n+2} - n(n-1)a_n\right] x^n$$

$$\therefore (4-x^2)y'' + 2y = \sum_{n=0}^{\infty} \left[4(n+2)(n+1)a_{n+2} - n(n-1)a_n + 2a_n\right] x^n$$

$-n^2 + n + 2 = -(n-2)(n+1)$

$$= \sum_{n=0}^{\infty} \left[4(n+2)(n+1)a_{n+2} - (n-2)(n+1)a_n\right] x^n$$

$$\therefore (4-x^2)y'' + 2y = 0 \Rightarrow 4(n+2)(n+1)a_{n+2} - (n-2)(n+1)a_n = 0,$$

$$\text{Or,} \quad a_{n+2} = \frac{(n-2)}{4(n+2)} a_n, \quad n \geq 0$$

$$a_2 = \frac{-2}{4\cdot 2} a_0 \qquad a_3 = \frac{-1}{4\cdot 3} a_1$$

$$a_4 = 0 \qquad a_5 = \frac{1}{4\cdot 5} a_3 = \frac{-1}{4^2\cdot 3\cdot 5} a_1$$

$$\therefore a_6, a_8, a_{10}, \ldots = 0 \qquad a_7 = \frac{3}{4\cdot 7} a_5 = \frac{-3}{4^3\cdot 3\cdot 5\cdot 7} a_1$$

$$a_9 = \frac{5}{4\cdot 9} a_7 = \frac{-3\cdot 5}{4^4\cdot 3\cdot 5\cdot 7\cdot 9} a_1$$

$$\therefore a_{2n+1} = -\frac{1}{4^n (2n-1)(2n+1)} a_1, \quad n \geq 2$$

$$\therefore y(x) = a_0 + a_1 x - \frac{a_0}{4} x^2 - \frac{a_1}{4\cdot 3} x^3 - a_1 \left(\frac{x^5}{4^2\cdot 3\cdot 5} + \frac{x^7}{4^3\cdot 5\cdot 7} + \ldots \right)$$

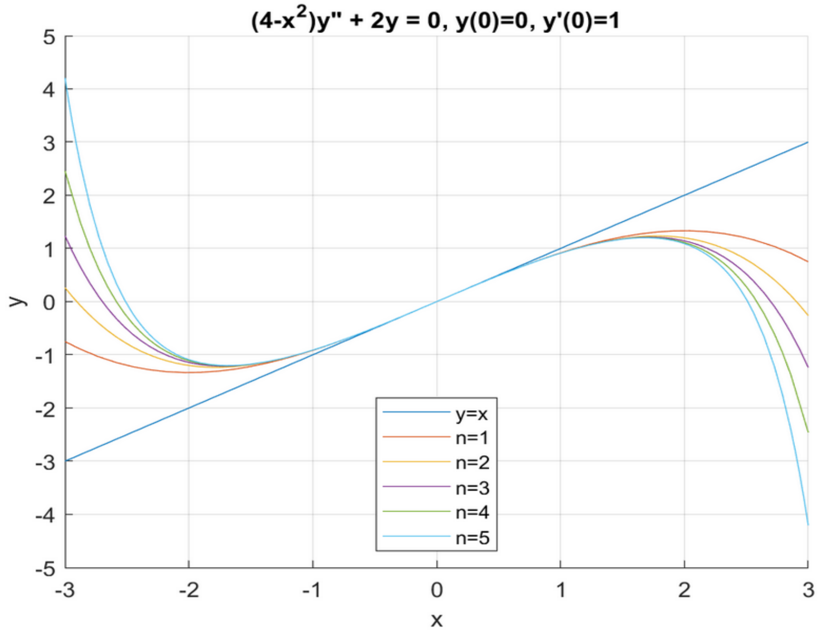
$$y(0) = 0 \Rightarrow a_0 = 0 \qquad y'(0) = 1 \Rightarrow a_1 = 1$$

$$\therefore y(x) = x - \frac{x^3}{4\cdot 1\cdot 3} - \frac{x^5}{4^2\cdot 3\cdot 5} - \frac{x^7}{4^3\cdot 5\cdot 7} - \ldots - \frac{x^{2n+1}}{4^n (2n-1)(2n+1)} - \ldots$$

$$= x - \sum_{n=1}^{\infty} \frac{x^{2n+1}}{4^n (2n-1)(2n+1)}$$

Using MATLAB,

```
clear, clc
syms x y(x)
y(x) = x;
figure
hold on
grid on
xmin = -3.0; xmax = 3.0;
fplot(y(x), [xmin, xmax])
for n = 1:5
    y(x) = y(x) - (x^(2*n+1))/((4^n)*(2*n-1)*(2*n+1));
    fplot(y(x), [xmin, xmax])
end
xlabel 'x', ylabel 'y'
title '(4-x^2)y" + 2y = 0, y(0)=0, y''(0)=1'
legend('y=x', 'n=1', 'n=2', 'n=3', 'n=4', 'n=5', ...
       'Location', 'south')
```



22.

From #5 with $K=1$,

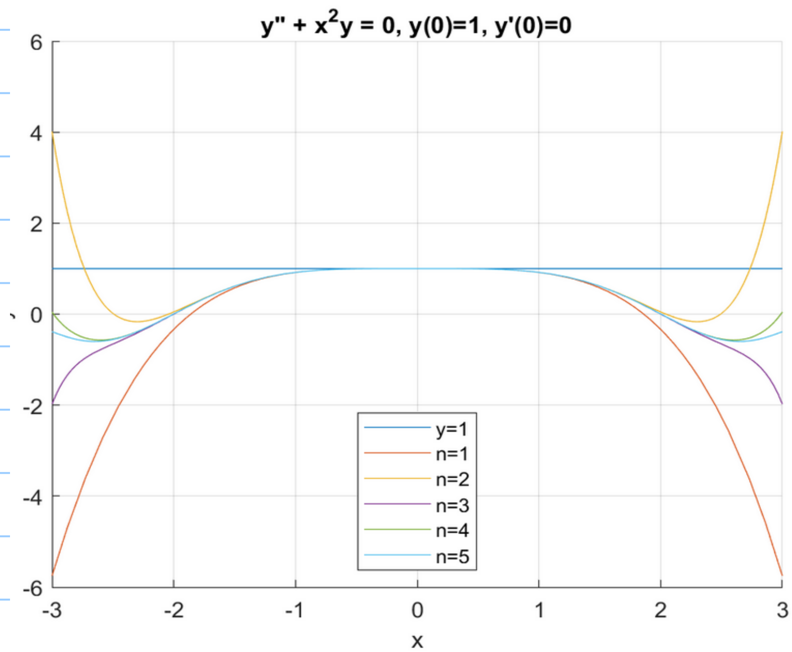
$$y(x) = a_0\left[1 - \frac{x^4}{4\cdot 3} + \frac{x^8}{8\cdot 7\cdot 4\cdot 3} + \cdots\right]$$

$$+ a_1\left[x - \frac{x^5}{5\cdot 4} + \frac{x^9}{9\cdot 8\cdot 5\cdot 4} + \cdots\right]$$

$y(0)=1 \Rightarrow a_0 = 1 \qquad y'(0)=0 \Rightarrow a_1 = 0$

$$\therefore \ y(x) = 1 - \frac{x^4}{4\cdot 3} + \frac{x^8}{8\cdot 7\cdot 4\cdot 3} + \cdots \frac{(-1)^n x^{4n}}{3\cdot 4\cdot 7\cdot 8\cdots(4n-1)(4n)}$$

```
clear, clc
syms x y(x)
y(x) = 1;
figure
hold on
grid on
xmin = -3.0; xmax = 3.0;
fplot(y(x), [xmin, xmax])
a = 1;
for n = 1:5
    a = -a/((4*n-1)*4*n);
    y(x) = y(x) + a*x^(4*n);
    fplot(y(x), [xmin, xmax])
end
xlabel 'x', ylabel 'y'
title 'y" + x^2y = 0, y(0)=1, y''(0)=0'
legend('y=1', 'n=1', 'n=2', 'n=3', 'n=4', 'n=5', ...
       'Location', 'south')
```



23.

Let $y = \sum_{n=0}^{\infty} a_n x^n \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

$$\therefore xy' = \sum_{n=1}^{\infty} n a_n x^n = \sum_{n=0}^{\infty} n a_n x^n$$

$$(1-x)y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1) a_n x^{n-1}$$

shift index ↓ shift index

$$= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} (n+1)(n) a_{n+1} x^n$$

and $n=0$, this entire term is zero

$$= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} (n+1)(n) a_{n+1} x^n$$

$$= \sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} - (n+1)(n) a_{n+1}\right] x^n$$

$\therefore (1-x)y'' + xy' - 2y =$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} - (n+1)(n) a_{n+1} + n a_n - 2 a_n\right] x^n$$

$$= \sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} - (n+1)(n) a_{n+1} + (n-2) a_n\right] x^n$$

$\therefore (1-x)y'' + xy' - 2y = 0 \Rightarrow$

$$(n+2)(n+1) a_{n+2} - (n+1)(n) a_{n+1} + (n-2) a_n = 0, \quad n \geq 0$$

$$\text{Or,}\quad a_{n+2} = \frac{n}{n+2}\,a_{n+1} - \frac{(n-2)}{(n+2)(n+1)}\,a_n\,,\quad n \geq 0$$

$$\text{Or,}\quad a_n = \left(\frac{n-2}{n}\right)a_{n-1} - \frac{(n-4)}{n(n-1)}\,a_{n-2}\,,\quad n \geq 2$$

$$\therefore\quad a_2 = a_0 \qquad\qquad a_3 = \frac{1}{3}a_2 + \frac{1}{6}a_1 = \frac{1}{3}a_0 + \frac{1}{6}a_1$$

$$a_4 = \frac{1}{2}a_3 = \frac{1}{6}a_0 + \frac{1}{12}a_1 \qquad a_5 = \frac{3}{5}a_4 - \frac{1}{5\cdot 4}a_3$$

Use MATLAB to compute the a_i coefficients.

Note for $y(x) = a_0 + a_1 x + a_2 x^2 + \dots$, $y(0) = 0 \Rightarrow a_0 = 0$,

and $y'(0) = 1 \Rightarrow a_1 = 1$

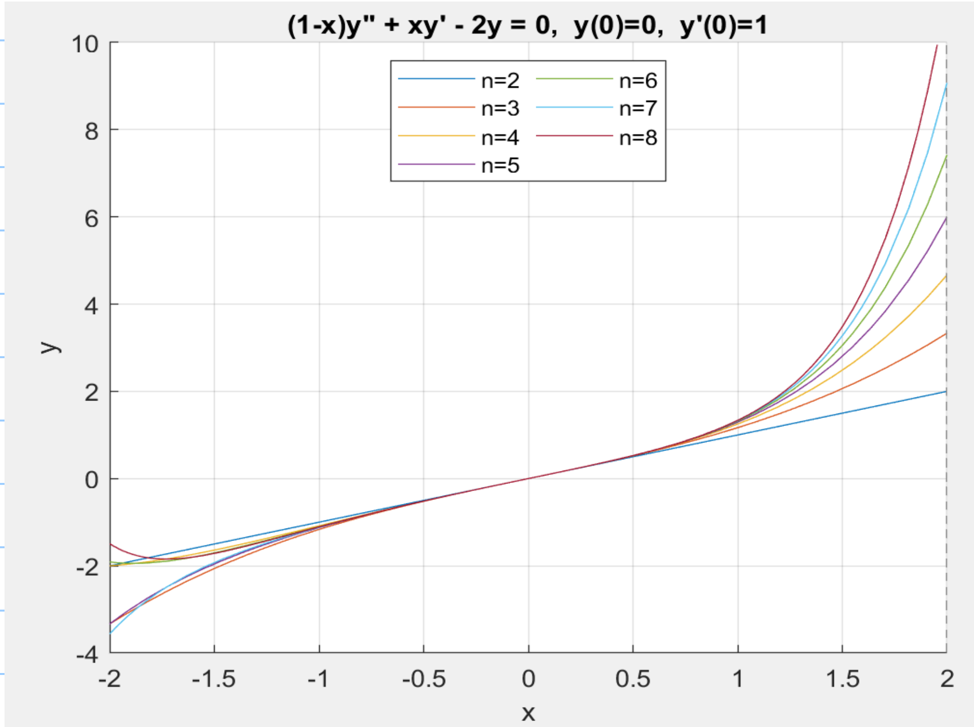
```
clear, clc
syms a0 a1 x y(x) f(x)
Nterms = 8;  %number of desired terms
x0 = 0;  % the ordinary point
%allocate space - array to hold coefficients
a = sym(zeros(Nterms,2));
y(x) = sym(zeros(Nterms+1,1));  % MATLAB likes to preallocate
f(x) = sym(zeros(Nterms+1,1));  % for these expanding polys
% initialize first entries for the for loop
% note a(0,:) = [a0 0], but MATLAB is 1-based
a(1,:) = [0 a1];
a(2,:) = [a0 0];
% n index gets coefficient for (x-x0)^n
for n = 3:Nterms
    a(n,:) = ((n-2)/n)*a(n-1,:) - ((n-4)/(n*(n-1)))*a(n-2,:);
end
a           % display coefficients
figure      % prepare to plot
hold on
grid on
xmin = -2.0; xmax = 2.0;
y(x) = a0 + a1*(x-x0);                 % initialize
for n = 2:Nterms
    y(x) = y(x) + (a(n,1)+a(n,2))*(x-x0)^n; % build y(x)
    f(x) = subs(y(x),[a0 a1],[0 1]);    % y(0)=0, y'(0)=1
    fplot(f(x), [xmin, xmax])
end
xlabel 'x', ylabel 'y'
title '(1-x)y" + xy'' - 2y = 0,  y(0)=0,  y''(0)=1'
legend('n=2', 'n=3', 'n=4', ...
       'n=5', 'n=6', 'n=7', 'n=8', ...
       'Location', 'north', 'NumColumns', 2)
y(x) = subs(y(x),[a0 a1],[0 1])    % show final result
```

a =

$$\begin{array}{cl}\left(\begin{array}{cc} 0 & a_1 \\ a_0 & 0 \\ \frac{a_0}{3} & \frac{a_1}{6} \\ \frac{a_0}{6} & \frac{a_1}{12} \\ \frac{a_0}{12} & \frac{a_1}{24} \\ \frac{2\,a_0}{45} & \frac{a_1}{45} \\ \frac{13\,a_0}{504} & \frac{13\,a_1}{1008} \\ \frac{163\,a_0}{10080} & \frac{163\,a_1}{20160} \end{array}\right) & \begin{array}{l} \to x^1 \\ \to x^2 \\ \to x^3 \\ \to x^4 \\ \to x^5 \\ \to x^6 \\ \to x^7 \\ \to x^8 \end{array}\end{array}$$

y(x) =

$$\frac{163\,x^8}{20160} + \frac{13\,x^7}{1008} + \frac{x^6}{45} + \frac{x^5}{24} + \frac{x^4}{12} + \frac{x^3}{6} + x$$



$n=2$ means $y(x) = a_0 + a_1 x + a_2 x^2 = x$

since $a_0 = 0$ and $a_2 = a_0 = 0$.

$n=3$: $y(x) = x + \frac{x^3}{6}$

5.3 Series Solutions Near an Ordinary Point, Part II

Note Title 3/8/2019

1.

$$\phi''(x) = -x\phi'(x) - \phi(x) \quad , \quad \phi(0) = 1, \ \phi'(0) = 0$$

$$\phi'''(x) = -\phi'(x) - x\phi''(x) - \phi'(x) = -2\phi'(x) - x\phi''(x)$$

$$\phi^{(4)}(x) = -\phi''(x) - \phi''(x) - x\phi'''(x) - \phi''(x)$$

$$= -3\phi''(x) - x\phi'''(x)$$

$$\therefore \ \phi''(0) = 0 - \phi(0) = \underline{-1}$$

$$\phi'''(0) = -2(0) - 0 = \underline{0}$$

$$\phi^{(4)}(0) = -3(-1) - 0 = \underline{3}$$

2.

$$\phi''(x) = -\frac{(1+x)}{x^2}\phi'(x) - \frac{3\ln(x)}{x^2}\phi(x)$$

Let MATLAB do all the computations:

```
clear, clc
syms x ph(x)
ph1 = diff(ph,x,1)
ph2 = (-(1+x)/x^2)*ph1 - (3*log(x)/x^2)*ph
ph3 = collect(diff(ph2,x,1), [ph, ph1, diff(ph,x,2)])
ph4 = collect(diff(ph3,x,1), [ph, ph1, diff(ph,x,2), diff(ph,x,3)])
ph_1 = 2; ph1_1 = 0;
% ph2_1(x) = ph2(1) should be a constant function
ph2_1 = subs(ph2,[x,ph(x),ph1(x)], [1,ph_1,ph1_1])
% ph3_1(x) = ph3(1) should be a constant function
ph3_1 = subs(ph3,[x,ph(x),ph1(x),diff(ph,x,2)], [1,ph_1,ph1_1,ph2_1])
% ph4_1(x) = ph4(1) should be a constant function
ph4_1 = subs(ph4,[x,ph(x),ph1(x),diff(ph,x,2),diff(ph,x,3)], [1,ph_1,ph1_1,ph2_1,ph3_1])
```

ph1(x) =

$$\frac{\partial}{\partial x}\,\mathrm{ph}(x)$$

ph2(x) =

$$-\frac{3\log(x)\,\mathrm{ph}(x)}{x^2}-\frac{(x+1)\,\frac{\partial}{\partial x}\,\mathrm{ph}(x)}{x^2}$$

ph3(x) =

$$\left(\frac{6\log(x)}{x^3}-\frac{3}{x^3}\right)\mathrm{ph}(x)+\left(\frac{2\,(x+1)}{x^3}-\frac{3\log(x)}{x^2}-\frac{1}{x^2}\right)\frac{\partial}{\partial x}\,\mathrm{ph}(x)+\left(-\frac{x+1}{x^2}\right)\frac{\partial^2}{\partial x^2}\,\mathrm{ph}(x)$$

ph4(x) =

$$\left(\frac{15}{x^4}-\frac{18\log(x)}{x^4}\right)\mathrm{ph}(x)+\left(\frac{12\log(x)}{x^3}-\frac{6\,(x+1)}{x^4}-\frac{2}{x^3}\right)\frac{\partial}{\partial x}\,\mathrm{ph}(x)+\left(\frac{4\,(x+1)}{x^3}-\frac{3\log(x)}{x^2}-\frac{2}{x^2}\right)\frac{\partial^2}{\partial x^2}\,\mathrm{ph}(x)+\left(-\frac{x+1}{x^2}\right)\frac{\partial^3}{\partial x^3}\,\mathrm{ph}(x)$$

ph2_1(x) = 0

ph3_1(x) = -6

ph4_1(x) = 42

Since $\phi''(x)$ was originally given in a formula, needed to use diff(ph,x,2) for $\phi''(x)$ and diff(ph,x,3) for $\phi'''(x)$ when constructing $\phi'''(x)$ and $\phi^{(4)}(x)$.

From MATLAB, $\phi''(1)=0$, $\phi'''(1)=-6$, $\phi^{(4)}(1)=42$

3.

$y'' = -x^2 y' - (\sin(x))y$, so $\phi''(x) = -x^2\phi'(x) - \sin(x)\phi(x)$

and $\phi(0) = a_0$, $\phi'(0) = a_1$.

Again use MATLAB,

```
clear, clc
syms x ph(x) a0 a1
ph1 = diff(ph,x,1)
ph2 = -(x^2)*ph1 - sin(x)*ph
ph3 = collect(diff(ph2,x,1), [ph, ph1, diff(ph,x,2)])
ph4 = collect(diff(ph3,x,1), [ph, ph1, diff(ph,x,2), diff(ph,x,3)])
ph_0 = a0; ph1_0 = a1;
% ph2_0(x) = ph2(0) should be a constant function
ph2_0 = subs(ph2,[x,ph(x),ph1(x)], [0,ph_0,ph1_0])
% ph3_0(x) = ph3(0) should be a constant function
ph3_0 = subs(ph3,[x,ph(x),ph1(x),diff(ph,x,2)], [0,ph_0,ph1_0,ph2_0])
% ph4_0(x) = ph4(0) should be a constant function
ph4_0 = subs(ph4,[x,ph(x),ph1(x),diff(ph,x,2),diff(ph,x,3)], [0,ph_0,ph1_0,ph2_0,ph3_0])
```

ph1(x) =

$$\frac{\partial}{\partial x}\,\mathrm{ph}(x)$$

ph2(x) =

$$-\mathrm{ph}(x)\sin(x) - x^2\frac{\partial}{\partial x}\,\mathrm{ph}(x)$$

ph3(x) =

$$(-\cos(x))\,\mathrm{ph}(x) + (-2\,x - \sin(x))\frac{\partial}{\partial x}\,\mathrm{ph}(x) + (-x^2)\frac{\partial^2}{\partial x^2}\,\mathrm{ph}(x)$$

ph4(x) =

$$\sin(x)\,\mathrm{ph}(x) + (-2\cos(x) - 2)\frac{\partial}{\partial x}\,\mathrm{ph}(x) + (-4\,x - \sin(x))\frac{\partial^2}{\partial x^2}\,\mathrm{ph}(x) + (-x^2)\frac{\partial^3}{\partial x^3}\,\mathrm{ph}(x)$$

ph2_0(x) = 0

ph3_0(x) = $-a_0$

ph4_0(x) = $-4\,a_1$

$\therefore\ \underline{\phi''(0) = 0}$, $\underline{\phi'''(0) = -a_0}$, $\phi^{(4)}(0) = \underline{-4a_1}$

4.

$Q(x) = 4$, $R(x) = 6x$ are analytic for all x.

$P(x) = 1$ has no zero. $\therefore$ radius of convergence

for $x_0 = 0$ and $x_0 = 4$ is $\underline{-\infty < x < \infty}$

5.

$x^2 - 2x - 3 = (x-3)(x+1)$, zeros are $x = 3, -1$

$\therefore p(x) = \frac{x}{x^2-2x-3}$ and $q(x) = \frac{4}{x^2-2x-3}$ are analytic

about $x_0 = 4, -4, 0$.

$x_0 = 4$: $|4-3| = 1$, $\underline{\rho = 1}$

$x_0 = -4$: $|-4-(-1)| = 3$, $\underline{\rho = 3}$

$x_0 = 0$: $|-1-0| = 1$, $\underline{\rho = 1}$

6.

Use MATLAB to find zeros of $x^3 + 1 = 0$

```
clear, clc
syms x
solve(x^3 + 1 == 0)
```

ans =

$$\begin{pmatrix} -1 \\ \frac{1}{2} - \frac{\sqrt{3}\,i}{2} \\ \frac{1}{2} + \frac{\sqrt{3}\,i}{2} \end{pmatrix}$$

$\therefore x = -1, \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

$\therefore p(x) = \frac{4x}{1+x^3}$ and $q(x) = \frac{1}{1+x^3}$ are analytic at

$x_0 = 0, 2$

$x_0 = 0: \quad \sqrt{(\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} = 1, \quad |0-(-1)| = 1 \quad \therefore \underline{\rho = 1}$

$x_0 = 2: \quad \sqrt{(2-\frac{1}{2}) + (0-\frac{\sqrt{3}}{2})^2} = \sqrt{3} \quad \therefore \underline{\rho = \sqrt{3}}$

7.

(a)

$P(x) = 1. \therefore p(x) = 0, q(x) = -1$, analytic for all x. $\therefore \underline{\rho = \infty}$

(b)

$P(x) = 1. \therefore p(x) = 3, q(x) = 0$, analytic for all x. $\therefore \underline{\rho = \infty}$

(c)

$P(x) = 1. \therefore p(x) = -x, q(x) = -1$, analytic for all x. $\therefore \underline{\rho = \infty}$

(d)

$P(x)=1$, $p(x)=-x$, $q(x)=-1$, analytic for all x. $\therefore \underline{\underline{\rho=\infty}}$

(e)

$P(x)=1$, $p(x)=0$, $q(x)=K^2x^2$, analytic for all x. $\therefore \rho=\underline{\infty}$

(f)

$P(x)=1-x$, zero of $x=1$. $\therefore q(x)=\frac{1}{1-x}$ is analytic at $x_0=0$. $|0-1|=1$. $\therefore \rho=\underline{1}$

(g)

$P(x)=1$, $p(x)=x$, $q(x)=2$, analytic for all x. $\therefore \underline{\underline{\rho=\infty}}$

(h)

$P(x)=x$ has zero at $x=0$. $p(x)=\frac{1}{x}$ is analytic at $x_0=1$, $q(x)=\frac{x}{x}=1$ is analytic at $x_0=1$.

$|1-0|=1$, $\therefore \rho=\underline{1}$

(i)

$P(x)=3-x^2$ has zeros at $x=\pm\sqrt{3}$

$\therefore p(x) = \frac{-3x}{3-x^2}$ and $q(x) = \frac{-1}{3-x^2}$ are analytic at $x_0 = 0$.

$|0 - \sqrt{3}| = \sqrt{3}$. $\therefore \underline{\rho = \sqrt{3}}$

(j)

$P(x) = 2$, $\therefore p(x) = \frac{x}{2}$, $q(x) = \frac{3}{2}$ are analytic for all x. $\therefore \underline{\rho = \infty}$

(k)

$P(x) = 2$. $\therefore p(x) = \frac{x+1}{2}$, $q(x) = \frac{3}{2}$, analytic for all x. $\therefore \underline{\rho = \infty}$

8.

(a)

Let $y = \sum_{n=0}^{\infty} a_n x^n$, $\alpha^2 y = \sum_{n=0}^{\infty} \alpha^2 a_n x^n$

$\therefore y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$, $xy' = \sum_{n=1}^{\infty} n a_n x^n = \sum_{n=0}^{\infty} n a_n x^n$

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}, \quad x^2 y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^n = \sum_{n=0}^{\infty} n(n-1)a_n x^n$$

$$\therefore (1-x^2)y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} n(n-1)a_n x^n$$

$\downarrow$ shift index

$$= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n - \sum_{n=0}^{\infty} n(n-1)a_n x^n$$

$$= \sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} - n(n-1)a_n\right] x^n$$

$$\therefore (1-x^2)y'' - xy' + \alpha^2 y =$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} \underbrace{- n(n-1)a_n - na_n}_{= -n^2 a_n} + \alpha^2 a_n\right] x^n$$

$$= \sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} - (n^2-\alpha^2)a_n\right] x^n$$

$$\therefore (n+2)(n+1)a_{n+2} - (n^2-\alpha^2)a_n = 0, \quad n \geq 0$$

Or, $$a_{n+2} = \frac{(n^2-\alpha^2)}{(n+2)(n+1)} a_n, \quad n \geq 0$$

Or, $$a_n = \frac{(n-2)^2-\alpha^2}{n(n-1)} a_{n-2}, \quad n \geq 2$$

$$\therefore \quad a_2 = -\frac{\alpha^2}{2} a_0 \qquad a_3 = \frac{(1-\alpha^2)}{3\cdot 2} a_1$$

$$a_4 = \frac{(2^2-\alpha^2)}{4\cdot 3}a_2 = -\frac{(2^2-\alpha^2)\alpha^2}{4\cdot 3\cdot 2\cdot 1}a_0 \qquad a_5 = \frac{(3^2-\alpha^2)}{5\cdot 4}\cdot a_3 = \frac{(3^2-\alpha^2)(1-\alpha^2)}{5\cdot 4\cdot 3\cdot 2\cdot 1}a_1$$

$$a_6 = \frac{(4^2-\alpha^2)}{6\cdot 5}a_4 = -\frac{(4^2-\alpha^2)(2^2-\alpha^2)\alpha^2}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}a_0 \qquad a_7 = \frac{(5^2-\alpha^2)}{7\cdot 6}a_5 = \frac{(5^2-\alpha^2)(3^2-\alpha^2)(1-\alpha^2)}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}a_1$$

$$\therefore a_{2n} = -\frac{[(2n-2)^2-\alpha^2]\cdots(4^2-\alpha^2)(2^2-\alpha^2)\alpha^2}{(2n)!}a_0, \quad n \geq 2$$

$$a_{2n+1} = \frac{[(2n-1)^2-\alpha^2]\cdots(3^2-\alpha^2)(1-\alpha^2)}{(2n+1)!}a_1, \quad n \geq 1$$

$$\therefore y(x) =$$

$$a_0\left[1 - \frac{\alpha^2}{2!}x^2 - \frac{(2^2-\alpha^2)\alpha^2}{4!}x^4 - \dots - \frac{[(2n-2)^2-\alpha^2]\cdots(2^2-\alpha^2)\alpha^2}{(2n)!}x^{2n} - \dots\right]$$

$$+ a_1\left[x + \frac{(1-\alpha^2)}{3!}x^3 + \frac{(3^2-\alpha^2)(1-\alpha^2)}{5!}x^5 + \dots + \frac{[(2n-1)^2-\alpha^2]\cdots(1-\alpha^2)}{(2n+1)!}x^{2n+1} + \dots\right]$$

$$\therefore \boxed{\begin{aligned} \underline{y_1(x)} &= 1 - \frac{\alpha^2}{2}x^2 - \frac{(2^2-\alpha^2)\alpha^2}{4!}x^4 - \dots - \frac{[(2n-2)^2-\alpha^2]\cdots(2^2-\alpha^2)\alpha^2}{(2n)!}x^{2n} - \dots \\ \underline{y_2(x)} &= x + \frac{(1-\alpha^2)}{3!}x^3 + \dots + \frac{[(2n-1)^2-\alpha^2]\cdots(1-\alpha^2)}{(2n+1)!}x^{2n+1} + \dots \end{aligned}}$$

$\rho = 1$ as zeros of $1-x^2$ are ± 1, so distance from

$x_0 = 0$ to nearest zero is 1.

$y_1(0) = 1$, $y_2(0) = 0$

$y_1'(x) = -\alpha^2 x + (\text{other powers in } x)$. $\therefore y_1'(0) = 0$

$y_2'(x) = 1 + (\text{other powers in } x)$. $\therefore y_2'(0) = 1$

$\therefore W[y_1, y_2](0) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$.

$\therefore y_1(x), y_2(x)$ form a fundamental set of solutions.

(b)

(1) If α is odd, then there is a positive integer K s.t. $2K-1 = \alpha$. $\therefore (2K-1)^2 = \alpha^2$

As shown in (a), all terms of $y_2(x)$ with power of x^{2K+1}, x^{2K+3}, x^{2K+5}, etc., have $[(2K-1)^2 - \alpha^2]$ as a term in their coefficients, and so will be zero. $\therefore y_2(x)$ is a finite sum with greatest power of $x^{2K-1} = x^\alpha$

(2) If α is even, then there is a positive integer K s.t. $2K = \alpha$. $\therefore$ Let $p = K+1$

$\therefore 2p = 2K+2$, so $2p-2 = 2K = \alpha$.

$\therefore$ For even positive integer α, there is a positive integer $2p-2 = \alpha$. As shown in (a), all terms of $y_1(x)$ with power of x^{2p}, x^{2p+2}, etc, have $[(2p-2)^2 - \alpha^2]$ as a term in their coefficients, and so will be zero.

$\therefore y_1(x)$ is a finite sum with greatest power $x^{2p-2} = x^{\alpha}$

(1) + (2) mean given positive integer α, there is a polynomial solution with greatest power x^{α}.

(c)

$\alpha = 0$: $x^{\alpha} = 1$, $\therefore y_1(x) = \underline{1}$

$\alpha = 1$: $x^{\alpha} = x$, $\therefore$ use $y_2(x) = \underline{x}$

$\alpha = 2$: $x^\alpha = x^2$, $\therefore$ use $y_1(x) = 1 - \frac{\alpha^2}{2} x^2 = \underline{1 - 2x^2}$

$\alpha = 3$: $x^\alpha = x^3$, $\therefore$ use $y_2(x) = x + \frac{(1-\alpha^2)}{3!} x^3$

$= \underline{x - \frac{4}{3} x^3}$

For problems 9-11, the following comments apply.

By Theorem 5.3.1, there exist two power series solutions $y_1(x)$, $y_2(x)$, analytic at $x_0 = 0$, s.t.

$y(x) = a_0 y_1(x) + a_1 y_2(x) = \sum_{n=0}^{\infty} a_n x^n$

$\therefore$ Let $y_1(x) = 1 + \sum_{n=2}^{\infty} b_n x^n$, $y_2(x) = x + \sum_{n=2}^{\infty} c_n x^n$

$\therefore$ $y_1(0) = 1$, $y_1'(0) = 0$ $\quad$ $y_2(0) = 0$, $y_2'(0) = 1$

By Theorem 3.2.1, $y_1(x)$, $y_2(x)$ exist and are unique.

Develop $y_1(x)$ and $y_2(x)$ separately.

9.

Let $y_1(x) = 1 + \sum_{n=2}^{\infty} b_n x^n$, so $\underline{b_0 = 1}$, $\underline{b_1 = 0}$

From $n!\, b_n = y_1^{(n)}(0)$, $b_2 = \frac{y''(0)}{2!}$

But $y_1''(x) = -\sin(x)\, y_1(x) \Rightarrow y_1''(0) = 0$. $\therefore \underline{b_2 = 0}$

$b_3 = \frac{y_1^{(3)}(0)}{3!}$, $y_1^{(3)}(x) = -\cos(x)\, y_1(x) - \sin(x)\, y_1'(x)$

$\therefore y_1^{(3)}(0) = -y_1(0) - 0 = -1$ as $y_1(0) = 1$

$\therefore b_3 = -\frac{1}{3!} = \underline{-\frac{1}{6}}$

At this point, let MATLAB do the computations. (code on next page). Answer shown →

b(x) =

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ -\frac{1}{6} \\ 0 \\ \frac{1}{120} \\ \frac{1}{180} \\ -\frac{1}{5040} \\ -\frac{13}{20160} \end{pmatrix}$$

$\therefore b_0 = 1 \quad b_1 = 0 \quad b_2 = 0 \quad b_3 = -\frac{1}{6}$

$b_4 = 0 \quad b_5 = \frac{1}{120} \quad b_6 = \frac{1}{180}$

$\therefore y_1(x) = \underline{1 - \frac{1}{6}x^3} + \frac{1}{120}x^5 + \frac{1}{180}x^6 + \ldots$

```
clear, clc
% call below function for coefficients
b = DispCoeffs(1,0)  % for y1(x)
c = DispCoeffs(0,1)  % for y2(x)

function a = DispCoeffs(a0, a1)
syms x y(x)
% compute all the derivatives
dy1 = diff(y,x,1);
dy2 = -sin(x)*y;  % from equation of problem
dy3 = diff(dy2,x,1);
dy4 = diff(dy3,x,1);
dy5 = diff(dy4,x,1);
dy6 = diff(dy5,x,1);
dy7 = diff(dy6,x,1);
dy8 = diff(dy7,x,1);
% use initial conditions a0, a1
% evaluate all derivatives at x = 0 to get a()*n!
dy20 = subs(dy2,[x,y(x),dy1(x)], ...
               [0,a0,a1]);
dy30 = subs(dy3,[x,y(x),dy1(x),diff(y,x,2)], ...
               [0,a0,a1,dy20]);
dy40 = subs(dy4,[x,y(x),dy1(x),diff(y,x,2),diff(y,x,3)], ...
               [0,a0,a1,dy20,dy30]);
dy50 = subs(dy5,[x,y(x),dy1(x),diff(y,x,2),diff(y,x,3),diff(y,x,4)], ...
               [0,a0,a1,dy20,dy30,dy40]);
dy60 = subs(dy6,[x,y(x),dy1(x),diff(y,x,2),diff(y,x,3), ...
                             diff(y,x,4),diff(y,x,5)], ...
               [0,a0,a1,dy20,dy30,dy40,dy50]);
dy70 = subs(dy7,[x,y(x),dy1(x),diff(y,x,2),diff(y,x,3), ...
                             diff(y,x,4),diff(y,x,5),diff(y,x,6)], ...
               [0,a0,a1,dy20,dy30,dy40,dy50,dy60]);
dy80 = subs(dy8,[x,y(x),dy1(x),diff(y,x,2),diff(y,x,3), ...
                             diff(y,x,4),diff(y,x,5),diff(y,x,6), ...
                             diff(y,x,7)], ...
               [0,a0,a1,dy20,dy30,dy40,dy50,dy60,dy70]);
% compute the coefficients a()
a2 = dy20/factorial(2);
a3 = dy30/factorial(3);
a4 = dy40/factorial(4);
a5 = dy50/factorial(5);
a6 = dy60/factorial(6);
a7 = dy70/factorial(7);
a8 = dy80/factorial(8);
a = [a0 a1 a2 a3 a4 a5 a6 a7 a8]';
end
```

c(x) =

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -\frac{1}{12} \\ 0 \\ \frac{1}{180} \\ \frac{1}{504} \\ -\frac{1}{6720} \end{pmatrix}$$

$\therefore\ c_0 = 0 \quad c_1 = 1 \quad c_2 = 0 \quad c_3 = 0$

$c_4 = -\frac{1}{12} \quad c_5 = 0 \quad c_6 = \frac{1}{180} \quad c_7 = \frac{1}{504}$

$$\therefore\ \underline{y_2(x)} = x - \frac{1}{12}x^4 + \frac{1}{180}x^6 + \frac{1}{504}x^7 + \ldots$$

Summary for $y'' + \sin(x)y = 0$:

$$\boxed{\begin{aligned} y_1(x) &= 1 - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \frac{1}{180}x^6 + \dots \\ y_2(x) &= x - \frac{1}{12}x^4 + \frac{1}{180}x^6 + \frac{1}{504}x^7 + \dots \end{aligned}}$$

These form a fundamental set of solutions

since $W[y_1, y_2] = \begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$

Since $\sin(x)$ is analytic for $-\infty < x < \infty$,

$\underline{\rho = \infty}$ for $y_1(x)$, $y_2(x)$

Alternate solution method - multiply series

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Let $y = \sum_{n=0}^{\infty} a_n x^n$

$$\therefore y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n$$

$\therefore y'' + \sin(x)y = 0$ becomes

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \left(\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}\right)\left(\sum_{n=0}^{\infty}a_n x^n\right) = 0$$

Use MATLAB to do the computations,

set $y(0)=1$, $y'(0)=0$ for $y_1(x)$ and

$y(0)=0$, $y'(0)=1$ for $y_2(x)$

Set n sufficiently large to get 4 nonzero terms,

and solve for the a_n.

```
clear, clc
syms x n a(n) a0
Nterms = 8;  %number of desired terms
f = ((-1)^n)*x^(2*n+1)/factorial(2*n+1);
s = symsum(f,n,0,Nterms);                  % sin(x)
y = a0 + symsum(a(n)*x^n,n,1,Nterms);    % y(x)
% d2y = y"(x)
d2y = symsum((n+2)*(n+1)*a(n+2)*x^n,n,0,Nterms);
% from diff eq, compute series up to Nterms
p = taylor(d2y + s*y,x,'Order',Nterms);
% initial conditions for y1(x)
y1 = subs(p,[a0 a(1)], [1 0]);
c1 = coeffs(y1,'All');
c1'  % display in column format
% initial conditions for y2(x)
y2 = subs(p,[a0 a(1)], [0 1]);
c2 = coeffs(y2,'All');
c2'  % display in column format
```

```
ans =
```

$$\begin{pmatrix} \frac{a(2)}{120} - \frac{a(4)}{6} + a(6) + 72\,a(9) - \frac{1}{5040} \\ a(5) - \frac{a(3)}{6} + 56\,a(8) \\ a(4) - \frac{a(2)}{6} + 42\,a(7) + \frac{1}{120} \\ a(3) + 30\,a(6) \\ a(2) + 20\,a(5) - \frac{1}{6} \\ 12\,a(4) \\ 6\,a(3) + 1 \\ 2\,a(2) \end{pmatrix} \quad C1$$

```
ans =
```

$$\begin{pmatrix} \frac{a(2)}{120} - \frac{a(4)}{6} + a(6) + 72\,a(9) \\ a(5) - \frac{a(3)}{6} + 56\,a(8) + \frac{1}{120} \\ a(4) - \frac{a(2)}{6} + 42\,a(7) \\ a(3) + 30\,a(6) - \frac{1}{6} \\ a(2) + 20\,a(5) \\ 12\,a(4) + 1 \\ 6\,a(3) \\ 2\,a(2) \end{pmatrix} \quad C2$$

$\therefore$ Set the coefficients to 0

using C1 for $y_1(x)$ and

C2 for $y_2(x)$

$y_1(x)$: using $a_0 = 1$, $a_1 = 0$

$2a_2 = 0 \Rightarrow a_2 = 0$ $\qquad$ $\overset{=0}{a_2} + 20a_5 - \frac{1}{6} = 0 \Rightarrow a_5 = \frac{1}{120}$

$6a_3 + 1 = 0 \Rightarrow a_3 = -\frac{1}{6}$ $\qquad$ $\overset{=-\frac{1}{6}}{a_3} + 30a_6 = 0 \Rightarrow a_6 = \frac{1}{180}$

$12a_4 = 0 \Rightarrow a_4 = 0$

$$\boxed{\therefore y_1(x) = \sum_{n=0}^{\infty} a_n x^n = 1 - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \frac{1}{180}x^6 + \dots}$$

$y_2(x)$: using $a_0 = 0$, $a_1 = 1$

$2a_2 = 0 \Rightarrow a_2 = 0$ $\qquad$ $\overset{=0}{a_2} + 20a_5 = 0 \Rightarrow a_5 = 0$

$6a_3 = 0 \Rightarrow a_3 = 0$ $\qquad$ $\overset{=0}{a_3} + 30a_6 - \frac{1}{6} = 0 \Rightarrow a_6 = 180$

$12a_4 + 1 = 0 \Rightarrow a_4 = -\frac{1}{12}$ $\qquad$ $\overset{=-\frac{1}{12}}{a_4} - \frac{\overset{=0}{a_2}}{6} + 42a_7 = 0 \Rightarrow a_7 = 504$

$$\boxed{\therefore y_2(x) = \sum_{n=0}^{\infty} a_n x^n = x - \frac{1}{12}x^4 + \frac{1}{180}x^6 + \frac{1}{504}x^7 + \dots}$$

Same answers as first method. Code of 1st method longer, but get direct results. Code of 2nd method shorter, but requires knowing Taylor series of $p(x)$, $q(x)$ and additional calculations to get coefficients.

10.

Use MATLAB and 1st method of #9 above.

Here, $y'' + xe^{-x}y = 0$, or $y''(x) = -xe^{-x}y(x)$

```
clear, clc
% call below function for coefficients
b = DispCoeffs(1,0)  % for y1(x)
c = DispCoeffs(0,1)  % for y2(x)

function a = DispCoeffs(a0, a1)
syms x y(x)
% compute all the derivatives
dy1 = diff(y,x,1);
dy2 = -x*exp(-x)*y;  % from equation of problem
dy3 = diff(dy2,x,1);
dy4 = diff(dy3,x,1);
dy5 = diff(dy4,x,1);
dy6 = diff(dy5,x,1);
dy7 = diff(dy6,x,1);
dy8 = diff(dy7,x,1);
% use initial conditions a0, a1
% evaluate all derivatives at x = 0 to get a()*n!
dy20 = subs(dy2,[x,y(x),dy1(x)], ...
               [0,a0,a1]);
dy30 = subs(dy3,[x,y(x),dy1(x),diff(y,x,2)], ...
               [0,a0,a1,dy20]);
dy40 = subs(dy4,[x,y(x),dy1(x),diff(y,x,2),diff(y,x,3)], ...
               [0,a0,a1,dy20,dy30]);
dy50 = subs(dy5,[x,y(x),dy1(x),diff(y,x,2),diff(y,x,3),diff(y,x,4)], ...
               [0,a0,a1,dy20,dy30,dy40]);
dy60 = subs(dy6,[x,y(x),dy1(x),diff(y,x,2),diff(y,x,3), ...
                             diff(y,x,4),diff(y,x,5)], ...
               [0,a0,a1,dy20,dy30,dy40,dy50]);
dy70 = subs(dy7,[x,y(x),dy1(x),diff(y,x,2),diff(y,x,3), ...
                             diff(y,x,4),diff(y,x,5),diff(y,x,6)], ...
               [0,a0,a1,dy20,dy30,dy40,dy50,dy60]);
dy80 = subs(dy8,[x,y(x),dy1(x),diff(y,x,2),diff(y,x,3), ...
                             diff(y,x,4),diff(y,x,5),diff(y,x,6), ...
                             diff(y,x,7)], ...
               [0,a0,a1,dy20,dy30,dy40,dy50,dy60,dy70]);
% compute the coefficients a()
a2 = dy20/factorial(2);
a3 = dy30/factorial(3);
a4 = dy40/factorial(4);
a5 = dy50/factorial(5);
a6 = dy60/factorial(6);
a7 = dy70/factorial(7);
a8 = dy80/factorial(8);
a = [a0 a1 a2 a3 a4 a5 a6 a7 a8]';
end
```

$y_1(x)$: $b_0 = 1 \quad b_1 = 0 \quad b_2 = 0 \quad b_3 = -\frac{1}{6}$

$b_4 = \frac{1}{12} \quad b_5 = -\frac{1}{40}$

$$\boxed{\therefore\ y_1(x) = 1 - \frac{1}{6}x^3 + \frac{1}{12}x^4 - \frac{1}{40}x^5 + \ldots}$$

b(x) =

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ -\frac{1}{6} \\ \frac{1}{12} \\ -\frac{1}{40} \\ \frac{1}{90} \\ -\frac{1}{144} \\ \frac{1}{280} \end{pmatrix}$$

$y_2(x)$: $c_0 = 0 \quad c_1 = 1 \quad c_2 = 0 \quad c_3 = 0$

$c_4 = -\frac{1}{12} \quad c_5 = \frac{1}{20} \quad c_6 = -\frac{1}{60}$

$$\boxed{\therefore\ y_2(x) = x - \frac{1}{12}x^4 + \frac{1}{20}x^5 - \frac{1}{60}x^6 + \ldots}$$

c(x) =

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -\frac{1}{12} \\ \frac{1}{20} \\ -\frac{1}{60} \\ \frac{1}{168} \\ -\frac{1}{320} \end{pmatrix}$$

$$W[y_1, y_2](0) = \begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$\therefore$ $y_1(x)$, $y_2(x)$ form a fundamental set of solutions.

For $y'' + xe^{-x}y = 0$, xe^{-x} is analytic over $-\infty < x < \infty$ since x and e^{-x} both are.

$\therefore$ $\underline{\rho = \infty}$ for $y_1(x)$ and $y_2(x)$

11.

Use MATLAB, 1st method of #9 above, but recoded to show indexing of functions. $y'' = -x\sec(x)y' + 2\sec(x)y$

```
clear, clc
% call below function for coefficients
n = 8;  % do for 8 terms of x^n
b = DispCoeffs(1,0,n)  % for y1(x)
c = DispCoeffs(0,1,n)  % for y2(x)

function a = DispCoeffs(y_0, dy_0, Nterms)
% Series solution of 2nd order linear diff eq
%   y_0  = y(0)
%   dy_0 = y'(0)
%   Nterms = highest order of x^n desired
% just change dy(2) below from problem
syms x y(x)
% create array of symbolic terms, 's' is a dummy
%   then assign derivatives to the terms
dy = sym('s',[1 Nterms]);
% compute all y derivatives based on y"
%   from equation of problem, store in dy()
dy(2) = -x*sec(x)*diff(y,x,1) + 2*sec(x)*y;
for i = 3:Nterms
    dy(i) = diff(dy(i-1),x,1);
end

% create symbolic array of terms, will assign
%   values of the derivatives at x=0 to terms
dy0 = sym('s',[1 Nterms]);
% use initial conditions y_0, dy_0 to evaluate
%   all y derivatives at x = 0 to get a()*n!
%   store in dy0() array
% preallocate memory for arrays in "subs" command
Aold = sym(zeros(1,Nterms+1));  % will contain "search for" values
Anew = sym(zeros(1,Nterms+1));  % will contain replacemnt values
Aold(1:2) = [x,y];        % intialize before "for" loop
Anew(1:2) = [0,y_0];
dy0(1) = dy_0;
for i = 2:Nterms
    Aold(i+1) = diff(y,x,i-1);
    Anew(i+1) = dy0(i-1);
    % Evaluate derivatives derived from y", dy(i), at x=0
    %   by replacing symbolic derivative functions with
    %   their values at x=0.  Store value in array dy0()
    dy0(i) = subs(dy(i), Aold, Anew);
end

% compute the Taylor coefficients, assign to symbolic array
a = sym('a', [1 Nterms]);
a(1) = dy_0;
for i = 2:Nterms
    a(i) = dy0(i)/factorial(i);
end
a = [y_0, a]';  % expand array to include y_0

end
```

b =

$$\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ \frac{1}{12} \\ 0 \\ \frac{1}{120} \\ 0 \\ \frac{19}{20160} \end{pmatrix}$$

c =

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{1}{6} \\ 0 \\ \frac{1}{60} \\ 0 \\ \frac{1}{560} \\ 0 \end{pmatrix}$$

$y_1(x)$: $b_0 = 1$ $b_1 = 0$ $b_2 = 1$ $b_3 = 0$

$b_4 = \frac{1}{12}$ $b_5 = 0$ $b_6 = \frac{1}{120}$

$$\boxed{\therefore\ y_1(x) = 1 + x^2 + \frac{1}{12}x^4 + \frac{1}{120}x^6 + \ldots}$$

b =

$$\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ \frac{1}{12} \\ 0 \\ \frac{1}{120} \\ 0 \\ \frac{19}{20160} \end{pmatrix}$$

$y_2(x)$: $c_0 = 0$ $c_1 = 1$ $c_2 = 0$ $c_3 = \frac{1}{6}$

$c_4 = 0$ $c_5 = \frac{1}{60}$ $c_6 = 0$ $c_7 = \frac{1}{560}$

$$\boxed{\therefore\ y_2(x) = x + \frac{1}{6}x^3 + \frac{1}{60}x^5 + \frac{1}{560}x^7 + \ldots}$$

c =

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{1}{6} \\ 0 \\ \frac{1}{60} \\ 0 \\ \frac{1}{560} \\ 0 \end{pmatrix}$$

$$W[y_1, y_2](0) = \begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$\therefore$ $y_1(x)$, $y_2(x)$ form a fundamental set of solutions

For $y'' + x\sec(x)y' - 2\sec(x)y = 0$,

$\sec(x)$ is defined over $-\frac{\pi}{2} < x < \frac{\pi}{2}$ and is

analytic there. $\therefore$ $\underline{\rho = \frac{\pi}{2}}$ for $y_1(x)$, $y_2(x)$

12.

Since $y = x^2$ is a solution, $y' = 2x$, $y'' = 2$,

and now substituting into the equation,

$2P(x) + 2Q(x)x + R(x)x^2 = 0$

$\therefore$ $2P(0) + 2Q(0)(0) + R(0)(0) = 0$, or $P(0) = 0$

$\therefore$ $Q(x)/P(x)$ and $R(x)/P(x)$ can't be analytic at $x = 0$

since they are undefined at $x = 0$ as $P(0) = 0$.

$\therefore$ $x = 0$ is a <u>singular</u> point.

13.

Let $y = \sum_{n=0}^{\infty} a_n x^n$ $\therefore$ $y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$

$$\therefore\ y' - y = \sum_{n=0}^{\infty} [(n+1)a_{n+1} - a_n]x^n = 0$$

$$\therefore\ (n+1)a_{n+1} - a_n = 0, \quad a_{n+1} = \frac{a_n}{n+1}, \quad n \geq 0$$

$\therefore$ All a_n, $n \geq 1$ are determined by a_0

$$a_1 = a_0 \qquad a_2 = \frac{a_1}{2} = \frac{a_0}{2}$$

$$a_3 = \frac{a_2}{3} = \frac{a_0}{3 \cdot 2 \cdot 1} \qquad a_4 = \frac{a_3}{4} = \frac{a_0}{4 \cdot 3 \cdot 2 \cdot 1}$$

$$\therefore\ a_n = \frac{a_0}{n!}, \quad n \geq 0$$

$$\therefore\ y(x) = \sum_{n=0}^{\infty} \frac{a_0}{n!} x^n = a_0 \underline{\left(\sum_{n=0}^{\infty} \frac{x^n}{n!} \right)}$$

But $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$\therefore\ \underline{y(x) = a_0 e^x}$, a_0 any constant

From Chapter 2, $y' = y$, $\frac{1}{y}\frac{dy}{dx} = 1$, $\int \frac{1}{y}\frac{dy}{dx}dx = \int dx$

$\therefore\ \ln(y(x)) = x + c$, c a constant

$\therefore\ y(x) = e^{x+c} = ae^x$, a a constant $= e^c$

14.

Let $y = \sum_{n=0}^{\infty} a_n x^n \quad \therefore \; xy = \sum_{n=0}^{\infty} a_n x^{n+1}$

$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$

$= a_1 + \sum_{n=1}^{\infty} (n+1) a_{n+1} x^n = a_1 + \sum_{n=0}^{\infty} (n+2) a_{n+2} x^{n+1}$

$\therefore \; y' - xy = a_1 + \sum_{n=0}^{\infty} \left[(n+2) a_{n+2} - a_n \right] x^{n+1} = 0$

$\therefore \; a_1 = 0 \,, \quad (n+2) a_{n+2} - a_n = 0 \,, \quad a_{n+2} = \frac{a_n}{n+2} \,, \quad n \geq 0$

$\therefore \; a_2 = \frac{a_0}{2} \qquad a_3 = \frac{a_1}{3} = 0$

$a_4 = \frac{a_2}{4} = \frac{a_0}{2 \cdot 4} \qquad a_5 = \frac{a_3}{5} = 0$

$\therefore \; a_{2n} = \frac{a_0}{2 \cdot 4 \cdots 2n} = \frac{a_0}{2^n (n!)} \qquad a_{2n+1} = 0 \,, \quad n = 0, 1, 2, \ldots$

$\therefore \; y(x) = a_0 \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n (n!)} \,, \quad a_0 \text{ any constant}$

From chapter 2, let $u = \int -x \, dx = -\frac{x^2}{2}$

$$y' - xy = e^{-x^2/2} y' - e^{-x^2/2} x y = 0$$

$$\therefore \frac{d}{dx}\left(e^{-x^2/2} y\right) = 0 \ , \quad e^{-x^2/2} y = c, \text{ a constant}$$

$$\therefore \ \underline{y = c e^{x^2/2}}$$

and note $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, so $e^{\frac{x^2}{2}} = \sum_{n=0}^{\infty} \frac{\left(\frac{x^2}{2}\right)^n}{n!}$

$$= \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n (n!)}$$

15.

Let $y = \sum_{n=0}^{\infty} a_n x^n \quad \therefore \ y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$

$$\therefore (1-x) y' = \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=1}^{\infty} n a_n x^n$$

$$= \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - \sum_{n=0}^{\infty} n a_n x^n$$

$$= \sum_{n=0}^{\infty} \left[(n+1) a_{n+1} - n a_n\right] x^n$$

$$\therefore (1-x) y' - y = \sum_{n=0}^{\infty} \left[(n+1) a_{n+1} - n a_n - a_n\right] x^n$$

$$= \sum_{n=0}^{\infty} [(n+1)a_{n+1} - (n+1)a_n] x^n$$

$$\therefore (1-x)y' - y = 0 \Rightarrow (n+1)a_{n+1} - (n+1)a_n = 0, \quad n \ge 0$$

$$\therefore a_{n+1} = a_n, \quad n \ge 0$$

$$\therefore a_n = a_0, \quad n \ge 0$$

$$\therefore \underline{y(x) = a_0 \sum_{n=0}^{\infty} x^n}, \quad a_0 \text{ any constant}$$

By chapter 2, $\frac{y'}{y} = \frac{1}{1-x}$, $\therefore \ln(y) = -\int \frac{dx}{x-1}$

$$\therefore \ln(y) = -\ln(x-1) + c, \quad c \text{ any constant}$$

$$\ln(y) = \ln \frac{1}{x-1} + c, \quad y = \frac{e^c}{x-1},$$

or $\underline{y = \frac{K}{1-x}}$, K any constant $= -e^{c}$

16.

Let $y = \sum_{n=0}^{\infty} a_n x^n$ $\quad \therefore y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1} x^n$

$$\therefore y' - y = x^2 \Rightarrow \sum_{n=0}^{\infty} [(n+1)a_{n+1} - a_n] x^n = x^2$$

$$\therefore\ a_{n+1} = \frac{a_n}{n+1}, \quad n \geq 0, \quad n \neq 2$$

$$\therefore\ a_1 - a_0 = 0, \quad a_1 = a_0$$

$$2a_2 - a_1 = 0, \quad a_2 = \frac{a_1}{2} = \frac{a_0}{2}$$

$$3a_3 - a_2 = 1, \quad a_3 = \frac{a_2+1}{3} = \frac{a_0/2 + 1}{3} = \frac{a_0}{3\cdot 2} + \frac{2}{3\cdot 2}$$

$$a_4 = \frac{a_3}{4} = \frac{a_0}{4\cdot 3\cdot 2} + \frac{2}{4\cdot 3\cdot 2}$$

$$\therefore\ a_n = \frac{a_0}{n!} + \frac{2}{n!}, \quad n \geq 3$$

$$\therefore\ y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_0 x + \frac{a_0}{2}x^2 + \sum_{n=3}^{\infty}\left(\frac{a_0}{n!} + \frac{2}{n!}\right)x^n$$

$$= a_0\left(1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots\right)$$

$$+ 2\left(\frac{x^3}{3!} + \frac{x^4}{4!} + \dots\right)$$

$$= a_0 e^x + 2\left(1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots\right) - 2 - 2x - x^2$$

$$= a_0 e^x + 2e^x - (x^2 + 2x + 2)$$

$$= (a_0 - 2)e^x - (x^2 + 2x + 2)$$

$$= \underline{c e^x - (x^2 + 2x + 2)}, \quad c \text{ a constant}$$

By Chapter 2, $y' - y = x^2$, $\therefore \int -1\, dx = -x$,

$$\therefore \frac{d}{dx}\left(e^{-x} y\right) = e^{-x} x^2$$

$$\therefore e^{-x} y = \int x^2 e^{-x}\, dx$$

Using $\int u^n e^{au}\, du = \frac{1}{a} u^n e^{au} - \frac{n}{a}\int u^{n-1} e^{au}\, du$ and $\int u e^{au}\, du = \frac{1}{a^2}(au - 1)e^{au} + C$

$$\int x^2 e^{-x}\, dx = -x^2 e^{-x} + 2\int x e^{-x}\, dx$$

$$= -x^2 e^{-x} + 2\left[(-x-1)e^{-x}\right] + c$$

$$= e^{-x}\left[-x^2 - 2x - 2\right] + c$$

$$\therefore y = \underline{\left[-x^2 - 2x - 2\right]} + c e^{x}$$, c a constant

same answer

17.

Let $y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + \sum_{n=2}^{\infty} a_n x^n$

$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \therefore 2xy' = \sum_{n=1}^{\infty} 2n a_n x^n = 2a_1 x + \sum_{n=2}^{\infty} 2n a_n x^n$

$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$

$\therefore (1-x^2) y'' = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=2}^{\infty} n(n-1) a_n x^n$

$$= 2a_2 + 6a_3 x + \sum_{n=2}^{\infty} \left[(n+2)(n+1)a_{n+2} - n(n-1)a_n\right] x^n$$

$$\therefore (1-x^2)y'' - 2xy + \alpha(\alpha+1)y =$$

$$2a_2 + 6a_3 x - 2a_1 x + \alpha(\alpha+1)a_0 + \alpha(\alpha+1)a_1 x \qquad (\alpha^2+\alpha-2)a_1$$

$$+ \sum_{n=2}^{\infty} \left[(n+2)(n+1)a_{n+2} - n(n-1)a_n - 2na_n + \alpha(\alpha+1)a_n\right] x^n$$

$$-n^2 + n - 2n + \alpha(\alpha+1) = -(n^2 + n - \alpha(\alpha+1))$$

$$= 2a_2 + \alpha(\alpha+1)a_0 + \left[6a_3 + (\alpha+2)(\alpha-1)a_1\right] x$$

$$+ \sum_{n=2}^{\infty} \left[(n+2)(n+1)a_{n+2} - (n^2+n-\alpha(\alpha+1))a_n\right] x^n$$

$$= [n-\alpha][n+(\alpha+1)]$$

$$\therefore a_2 = -\frac{\alpha(\alpha+1)}{2} a_0 \qquad a_3 = -\frac{(\alpha+2)(\alpha-1)}{6} a_1$$

$$a_{n+2} = -\frac{[\alpha-n][(\alpha+1)+n]}{(n+2)(n+1)} a_n, \quad n \geq 2$$

(and so also works for $n \geq 0$)

$n=2$: $m=2$ $$a_4 = -\frac{(\alpha-2)(\alpha+3)}{4\cdot 3} a_2 = \frac{\alpha(\alpha-2)(\alpha+1)(\alpha+3)}{4!} a_0$$

$n=4$: $m=3$ $$a_6 = -\frac{(\alpha-4)(\alpha+5)}{6\cdot 5} a_4 = -\frac{\alpha(\alpha-2)(\alpha-4)(\alpha+1)(\alpha+3)(\alpha+5)}{6!} a_0$$

$$\therefore a_{2m} = (-1)^m \frac{\alpha(\alpha-2)(\alpha-4)\cdots[\alpha-(2m-2)]\,(\alpha+1)(\alpha+3)\cdots[\alpha+(2m-1)]}{(2m)!} a_0$$

for $m \geq 1$

$n=3$, $m=2$: $a_5 = -\frac{(\alpha-3)(\alpha+4)}{5\cdot 4}\alpha_3 = \frac{(\alpha-1)(\alpha-3)(\alpha+2)(\alpha+4)}{5!}a_1$

$n=5$, $m=3$: $a_7 = -\frac{(\alpha-5)(\alpha+6)}{7\cdot 6}a_5 = -\frac{(\alpha-1)(\alpha-3)(\alpha-5)(\alpha+2)(\alpha+4)(\alpha+6)}{7!}a_1$

$$\therefore a_{2m+1} = (-1)^m \frac{(\alpha-1)(\alpha-3)\cdots[\alpha-(2m-1)](\alpha+2)(\alpha+4)\cdots(\alpha+2m)}{(2m+1)!}a_1$$

for $m \geq 1$

$$\therefore y(x) = a_0 + \sum_{m=1}^{\infty} a_{2m}x^{2m} + a_1x + \sum_{m=1}^{\infty} a_{2m+1}x^{2m+1}$$

$\therefore$ a_0 and a_1 are arbitrary, so, let $a_0=1$, $a_1=1$:

$$\boxed{\begin{aligned} y_1(x) &= 1 + \sum_{m=1}^{\infty}(-1)^m \frac{\alpha(\alpha-2)(\alpha-4)\cdots[\alpha-(2m-2)](\alpha+1)(\alpha+3)\cdots[\alpha+(2m-1)]}{(2m)!}x^{2m} \\ y_2(x) &= x + \sum_{m=1}^{\infty}(-1)^m \frac{(\alpha-1)(\alpha-3)\cdots[\alpha-(2m-1)](\alpha+2)(\alpha+4)\cdots(\alpha+2m)}{(2m+1)!}x^{2m+1} \end{aligned}}$$

18.

(a) If $\alpha = 0$, from #17, all terms beyond 1 contain

α in their coefficient, and so are 0. $\therefore$ $y_1(x) = 1$ which is a polynomial of degree $2(0) = 0$.

If $\alpha \neq 0$ but $\alpha = 2n$, there is an m s.t. $2m - 2 = 2n$. From #17, the coefficient for x^{2m} and every subsequent term contains the factor $[\alpha - (2m-2)]$, and so is 0. $\therefore$ This is a finite polynomial, and the previous non-zero term to x^{2m} is $x^{2m-2} = x^{2n}$.

(b) $y_1(x) = 1 - \frac{\alpha(\alpha+1)}{2}x^2 + \frac{\alpha(\alpha-2)(\alpha+1)(\alpha+3)}{4!}x^4 - \frac{\alpha(\alpha-2)(\alpha-4)(\alpha+1)(\alpha+3)(\alpha+5)}{6!}x^6$

$\therefore$ $\alpha = 0$: $\quad \underline{y_1(x) = 1}$

$\alpha = 2$: $\quad y_1(x) = 1 - \frac{\alpha(\alpha+1)}{2}x^2 = \underline{1 - 3x^2}$

$\alpha = 4$: $\quad y_1(x) = 1 - \frac{\alpha(\alpha+1)}{2}x^2 + \frac{\alpha(\alpha-2)(\alpha+1)(\alpha+3)}{4!}x^4$

$= \underline{1 - 10x^2 + \frac{35}{3}x^4}$

(c) If $\alpha = 2n+1$, n some positive integer, then since $2n < 2n+1$ and $2n-1 < 2n+1$, $y_2(x)$

will contain terms up to and including x^{2n+1}, by the expression for $y_2(x)$ in # 17. The next term (for n+1), contains the coefficient factor $\alpha - [2(n+1)-1]$ for $x^{2(n+1)+1}$, which is $\alpha-(2n+1)$, and so will be zero. This is true for all following terms. $\therefore$ $y_2(x)$ is finite of degree x^{2n+1}, and contains only odd powers of x by the formula in # 17.

(d) $y_2(x) = x - \frac{(\alpha-1)(\alpha+2)}{3!}x^3 + \frac{(\alpha-1)(\alpha-3)(\alpha+2)(\alpha+4)}{5!}x^5 + \dots$

$\alpha = 1$: $y_2(x) = \underline{x}$

$\alpha = 3$: $y_2(x) = x - \frac{(\alpha-1)(\alpha+2)}{3!}x^3 = \underline{x - \frac{5}{3}x^3}$

$\alpha = 5$: $y_2(x) = x - \frac{(\alpha-1)(\alpha+2)}{3!}x^3 + \frac{(\alpha-1)(\alpha-3)(\alpha+2)(\alpha+4)}{5!}x^5$

$= \underline{x - \frac{14}{3}x^3 + \frac{21}{5}x^7}$

19.

(a)

Using fundamental solutions $y_1(x), y_2(x)$ from #18,

$\alpha = 0: \ y_1(x) = 1 \quad \therefore y_1(1) = 1 \quad \therefore P_0(x) = y_1(x)$

$\alpha = 1: \ y_2(x) = x \quad \therefore y_2(1) = 1 \quad \therefore P_1(x) = y_2(x)$

$\alpha = 2: \ y_1(x) = 1 - 3x^2, \ \therefore y_1(1) = -2 \quad P_2(x) = \frac{y_1(x)}{-2}$

$\alpha = 3: \ y_2(x) = x - \frac{5}{3}x^3 \quad \therefore y_2(1) = -\frac{2}{3} \quad P_3(x) = \frac{y_2(x)}{-2/3}$

$\alpha = 4: \ y_1(x) = 1 - 10x^2 + \frac{35}{3}x^4 \quad \therefore y_1(1) = \frac{8}{3} \quad P_4(x) = \frac{y_1(x)}{8/3}$

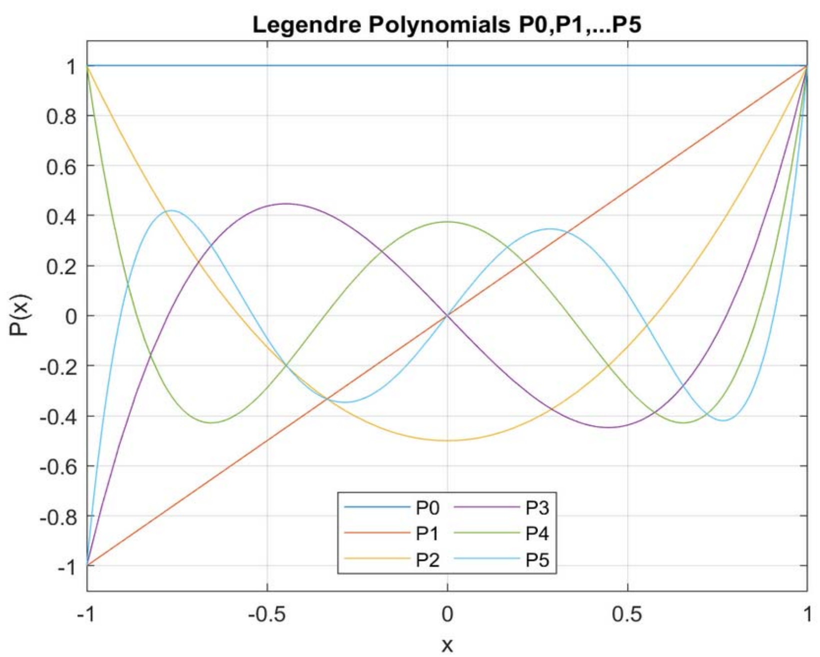
$\alpha = 5: \ y_2(x) = x - \frac{14}{3}x^3 + \frac{21}{5}x^5 \quad \therefore y_2(1) = \frac{8}{15} \quad P_5(x) = \frac{y_2(x)}{8/15}$

$$\therefore \boxed{\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= -\frac{1}{2} + \frac{3}{2}x^2 \\ P_3(x) &= -\frac{3}{2}x + \frac{5}{2}x^3 \\ P_4(x) &= \frac{3}{8} - \frac{15}{4}x^2 + \frac{35}{8}x^4 \\ P_5(x) &= \frac{15}{8}x - \frac{35}{4}x^3 + \frac{63}{8}x^5 \end{aligned}}$$

(b)

Using MATLAB

```
clear, clc
syms x
P0 = 1;
P = sym('P',[1 5]);
P(1) = x;
P(2) = -1/2 + (3/2)*x^2;
P(3) = (-3/2)*x + (5/2)*x^3;
P(4) = 3/8 - (15/4)*x^2 + (35/8)*x^4;
P(5) = (15/8)*x - (35/4)*x^3 + (63/8)*x^5;
fplot(P0)
hold on
grid on
xlim([-1 1])
ylim([-1.1 1.1])
for i = 1:5
    fplot(P(i))
    vpa(solve(P(i)==0,x))
end
xlabel 'x', ylabel 'P(x)'
title('Legendre Polynomials P0,P1,...P5')
legend('P0','P1','P2','P3','P4','P5','Location','south', ...
       'NumColumns',2)
```



ans = 0.0

ans =

$$\begin{pmatrix} -0.57735026918962576450914878050196 \\ 0.57735026918962576450914878050196 \end{pmatrix}$$

ans =

$$\begin{pmatrix} 0 \\ -0.77459666924148337703585307995648 \\ 0.77459666924148337703585307995648 \end{pmatrix}$$

ans =

$$\begin{pmatrix} -0.33998104358485626480266575910324 \\ -0.86113631159405257522394648889281 \\ 0.33998104358485626480266575910324 \\ 0.86113631159405257522394648889281 \end{pmatrix}$$

ans =

$$\begin{pmatrix} 0 \\ -0.53846931010568309103631442070021 \\ -0.90617984593866399279762687829939 \\ 0.53846931010568309103631442070021 \\ 0.90617984593866399279762687829939 \end{pmatrix}$$

(c)

$P_0(x) = 1$ has no zeros

$P_1(x) = x$ has 1 zero at $x = 0$

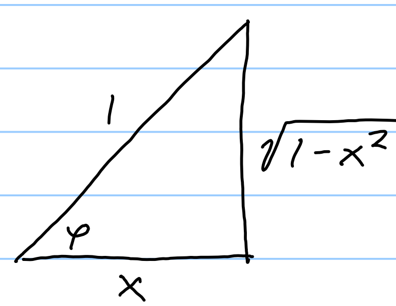
$P_2(x)$ zeros: ± 0.57735

$P_3(x)$ zeros: $0, \pm 0.774597$

$P_4(0)$ zeros: $\pm 0.339981, \pm 0.8611363$

$P_5(0)$ zeros: $0, \pm 0.5384693, \pm 0.9061798$

20.



Letting $x = \cos(\varphi)$, Then $\cot(\varphi) = \dfrac{x}{\sqrt{1-x^2}} = x\underline{(1-x^2)}^{-\frac{1}{2}}$

$\therefore \varphi = \arccos(x)$, $-1 < x < 1$ if $0 < \varphi < \pi$

$\therefore \dfrac{d\varphi}{dx} = -(1-x^2)^{-\frac{1}{2}}$ $\quad\therefore \dfrac{d^2\varphi}{dx^2} = -x(1-x^2)^{-\frac{3}{2}}$

$$\therefore \frac{df}{dx} = \frac{dF}{d\varphi}\cdot\frac{d\varphi}{dx}, \text{ so } \frac{dF}{d\varphi} = \frac{f'(x)}{d\varphi/dx} = \underline{-(1-x^2)^{\frac{1}{2}} f'(x)}$$

$$\therefore \frac{d^2 f}{dx^2} = \left[\frac{d}{dx}\frac{dF}{d\varphi}\right]\cdot\frac{d\varphi}{dx} + \frac{dF}{d\varphi}\cdot\frac{d^2\varphi}{dx^2}$$

$$= \left(\frac{d^2F}{d\varphi^2}\cdot\frac{d\varphi}{dx}\right)\cdot\frac{d\varphi}{dx} + \frac{dF}{d\varphi}\cdot\frac{d^2\varphi}{dx^2}$$

$$= \frac{d^2F}{d\varphi^2}\left(\frac{d\varphi}{dx}\right)^2 + \frac{dF}{d\varphi}\cdot\frac{d^2\varphi}{dx^2}$$

$$\therefore \frac{d^2F}{d\varphi^2} = \frac{\frac{d^2f}{dx^2} - \frac{dF}{d\varphi}\cdot\frac{d^2\varphi}{dx^2}}{\left(\frac{d\varphi}{dx}\right)^2}$$

$$= \frac{f''(x) - \left[-(1-x^2)^{\frac{1}{2}} f'(x)\right]\left[-x(1-x^2)^{-\frac{3}{2}}\right]}{\left[-(1-x^2)^{-\frac{1}{2}}\right]^2}$$

$$= \frac{f''(x) - x(1-x^2)^{-1} f'(x)}{(1-x^2)^{-1}} = (1-x^2) f''(x) - x f'(x)$$

$$\therefore \frac{d^2F(\varphi)}{d\varphi^2} + \cot(\varphi)\frac{dF(\varphi)}{d\varphi} + n(n+1)F(\varphi) =$$

$$(1-x^2) f''(x) - x f'(x) + \underbrace{x(1-x^2)^{-\frac{1}{2}}}_{=\cot(\varphi)}\underbrace{\left[-(1-x^2)^{\frac{1}{2}} f'(x)\right]}_{=\frac{dF(\varphi)}{d\varphi}} + n(n+1) f(x)$$

$$= (1-x^2) f''(x) - 2x f'(x) + n(n+1) f(x)$$

Or, using $y = f(x)$ and $\alpha = n$,

$$\underline{(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0}$$

21.

Assuming $\frac{d^0}{dx^0}$ means "no derivative at all",

$$P_0(x) = \frac{1}{2^0 0!}(x^2-1)^0 = \frac{1}{1\cdot 1}\cdot 1 = \underline{1}$$

$$P_1(x) = \frac{1}{2}\frac{d}{dx}(x^2-1) = \frac{1}{2}(2x) = \underline{x}$$

$$P_2(x) = \frac{1}{2^2 2!}\frac{d^2}{dx^2}(x^2-1)^2 = \frac{1}{8}\frac{d}{dx}\left[2(x^2-1)2x\right]$$

$$= \frac{1}{8}\left[4(x^2-1) + 4x(2x)\right] = \frac{1}{8}\left[4x^2 - 4 + 8x^2\right]$$

$$= \underline{\frac{3}{2}x^2 - \frac{1}{2}}$$

$$P_3(x) = \frac{1}{2^3 3!}\frac{d^3}{dx^3}(x^2-1)^3 = \frac{1}{48}\frac{d^2}{dx^2}\left[6x(x^2-1)^2\right]$$

$$= \frac{1}{48}\frac{d}{dx}\left[6(x^2-1)^2 + 24x^2(x^2-1)\right]$$

$$= \frac{1}{48}\left[24x(x^2-1) + 48x(x^2-1) + 24x^2(2x)\right]$$

$$= \frac{1}{48}\left[24x^3 - 24x + 48x^3 - 48x + 48x^3\right]$$

$$= \frac{1}{48}\left[120x^3 - 72x\right] = \underline{\frac{5}{2}x^3 - \frac{3}{2}x}$$

These answers agree with the results of # 19 above

22.

(a) $[(1-x^2)y']' = (-2x)y' + (1-x^2)y''$

$\therefore$ $[(1-x^2)y']' = -\alpha(\alpha+1)y$ becomes

$$-2xy' + (1-x^2)y'' = -\alpha(\alpha+1)y, \text{ or}$$

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0$$

(5) $[(1-x^2)P_n'(x)]'P_m(x) = -n(n+1)P_n(x)P_m(x)$ [1]

$[(1-x^2)P_m'(x)]'P_n(x) = -m(m+1)P_m(x)P_n(x)$ [2]

Integration by parts: $\int u'v = uv - \int uv'$

from $\int u'v + uv' = \int (uv)' = uv$

$\therefore$ From [1], $\int_{-1}^{1}[(1-x^2)P_n']'P_m\,dx =$

$$(1-x^2)P_n'P_m\Big|_{-1}^{1} - \int_{-1}^{1}(1-x^2)P_n'P_m'\,dx = \int_{-1}^{1} -n(n+1)P_nP_m\,dx \quad [3]$$

From [2], $\int_{-1}^{1}[(1-x^2)P_m']'P_n\,dx =$

$$(1-x^2)P_m'P_n\Big|_{-1}^{1} - \int_{-1}^{1}(1-x^2)P_m'P_n'\,dx = \int_{-1}^{1} -m(m+1)P_mP_n\,dx \quad [4]$$

Note $(1-x^2)P_n'P_m\Big|_{-1}^{1} = (1-x^2)P_m'P_n\Big|_{-1}^{1} = 0$ because of the $(1-x^2)$ factor.

$\therefore$ Subtracting [3] - [4]:

$$0 = -n(n+1)\int_{-1}^{1}P_nP_m\,dx + m(m+1)\int_{-1}^{1}P_mP_n\,dx$$

Or, $[m(m+1) - n(n+1)] \int_{-1}^{1} P_n(x) P_m(x)\, dx = 0$

$\therefore$ If $m \neq n$, $m(m+1) \neq n(n+1)$,

$$\therefore \underline{\int_{-1}^{1} P_n(x) P_m(x)\, dx = 0}$$

23.

$$\int_{-1}^{1} f(x) P_K(x)\, dx = \int_{-1}^{1} \left(\sum_{K=0}^{n} a_K P_K \right) P_K$$

$$= \int_{-1}^{1} (a_0 P_0 + a_1 P_1 + \dots + a_k P_K + \dots + a_n P_n) P_K$$

$$= \int_{-1}^{1} a_0 P_0 P_K + a_1 P_1 P_K + \dots + a_K P_K P_K + \dots + a_n P_n P_K$$

$$= a_0 \int_{-1}^{1} P_0 P_K + a_1 \int_{-1}^{1} P_1 P_K + \dots + a_K \int_{-1}^{1} P_K P_K + \dots a_n \int_{-1}^{1} P_n P_K$$

$$= 0 + 0 + \dots + a_K \int_{-1}^{1} P_K P_K + \dots + 0, \text{ by } \# 22$$

$$\therefore \int_{-1}^{1} f(x) P_K(x)\,dx = a_K \int_{-1}^{1} P_K P_K$$

Also by #22 above, $\int_{-1}^{1} P_K P_K = \frac{2}{2K+1}$

$$\therefore \int_{-1}^{1} f(x) P_K(x)\,dx = a_K \left(\frac{2}{2K+1}\right)$$

$$\therefore \underline{a_K = \frac{2K+1}{2} \int_{-1}^{1} f(x) P_K(x)\,dx}$$

5.4 Euler Equations; Regular Singular Points

Note Title 3/25/2019

In each of Problems 1 through 8, determine the general solution of the given differential equation that is valid in any interval not including the singular point.

1.

Let $y = x^r$. $\therefore\ x^2(r)(r-1)x^{r-2} + 4x\,r x^{r-1} + 2x^r = 0$, or

$x^r[r(r-1) + 4r + 2] = 0$, $\therefore\ r^2 + 3r + 2 = (r+2)(r+1) = 0$

$\therefore\ r = -1, -2$. $\therefore\ \underline{\underline{y = c_1 x^{-1} + c_2 x^{-2}}}$

2.

Let $y = (x+1)^r$ $\therefore\ y' = r(x+1)^{r-1}$ $y'' = r(r-1)(x+1)^{r-2}$

$\therefore\ (x+1)^r\left[r(r-1) + 3r + \frac{3}{4}\right] = 0$

$r^2 + 2r + \frac{3}{4} = 0$, $r = \dfrac{-2 \pm \sqrt{4 - 4(\frac{3}{4})}}{2} = -\frac{3}{2}, -\frac{1}{2}$

$\therefore\ y = \underline{\underline{c_1|x+1|^{-\frac{3}{2}} + c_2|x+1|^{-\frac{1}{2}}}}$

(since taking square roots, need absolute values).

3.

Let $y = x^r$ $\therefore$ $x^r[r(r-1) - 3r + 4] = 0$

$\therefore$ $r^2 - 4r + 4 = 0$, $(r-2)^2 = 0$, $r = 2, 2$

$\therefore$ $y = \underline{C_1 x^2 + C_2 x^2 \ln|x|}$

4.

Let $y = x^r$ $\therefore$ $x^r[r(r-1) - r + 1] = 0$,

$\therefore$ $r^2 - 2r + 1 = 0$, $(r-1)^2 = 0$, $r = 1, 1$

$\therefore$ $y = \underline{C_1 x + C_2 x \ln|x|}$

5.

Let $y = x^r$ $\therefore$ $x^r[r(r-1) + 6r - 1] = 0$

$\therefore$ $r^2 + 5r - 1 = 0$, $r = \dfrac{-5 \pm \sqrt{25+4}}{2} = -\dfrac{5}{2} \pm \dfrac{\sqrt{29}}{2}$

$\therefore$ $y = \underline{C_1 |x|^{\left(\frac{-5+\sqrt{29}}{2}\right)} + C_2 |x|^{\left(\frac{-5-\sqrt{29}}{2}\right)}}$

6.

Let $y = x^r$ $\therefore$ $x^r[2r(r-1) - 4r + 6] = 0$

$\therefore$ $2r^2 - 6r + 6 = 0$, $r = \dfrac{6 \pm \sqrt{36 - 4(2)(6)}}{4} = \dfrac{3}{2} \pm i\dfrac{\sqrt{3}}{2}$

$\therefore$ $\underline{y = |x|^{3/2}\left[c_1 \cos\left(\frac{\sqrt{3}}{2}\ln|x|\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}\ln|x|\right)\right]}$

7.

Let $y = x^r$ $\therefore$ $x^r[r(r-1) - 5r + 9] = 0$

$\therefore$ $r^2 - 6r + 9 = 0$, $(r-3)^2 = 0$, $r = 3, 3$

$\therefore$ $\underline{y = c_1 x^3 + c_2 x^3 \ln|x|}$

8.

Let $y = (x-2)^r$ $\therefore$ $(x-2)^r[r(r-1) + 5r + 8] = 0$

$\therefore$ $r^2 + 4r + 8 = 0$, $r = \dfrac{-4 \pm \sqrt{16 - 4(8)}}{2} = -2 \pm 2i$

$\therefore$ $\underline{y = (x-2)^{-2}\left[c_1 \cos(2\ln|x-2|) + c_2 \sin(2\ln|x-2|)\right]}$

9.

(a) Let $y = x^r$ $\therefore$ $x^r[2r(r-1) + r - 3] = 0$

$\therefore$ $2r^2 - r - 3 = 0$, $(2r-3)(r+1) = 0$, $r = \frac{3}{2}, -1$

$\therefore$ $y = c_1 x^{\frac{3}{2}} + c_2 x^{-1}$ $\quad y(1) = c_1 + c_2 = 1$

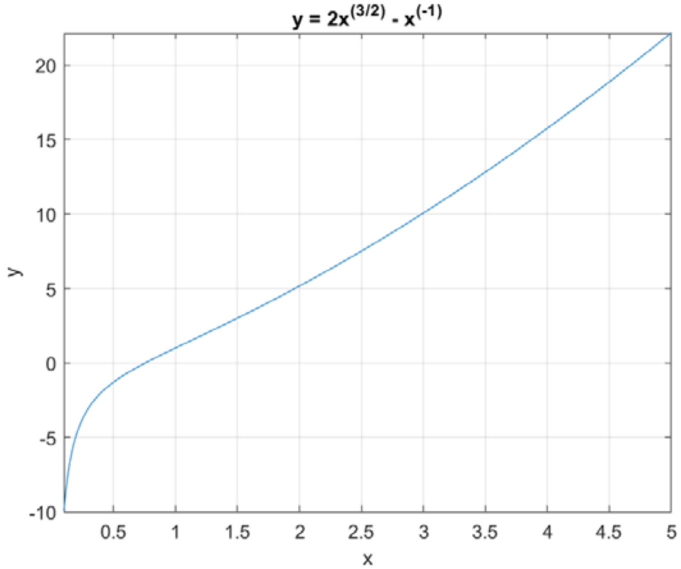
$y' = \frac{3}{2} c_1 x^{\frac{1}{2}} - c_2 x^{-2}$ $\quad y'(1) = \frac{3}{2} c_1 - c_2 = 4$

$\therefore$ $\frac{5}{2} c_1 = 5$, $c_1 = 2$ $\therefore c_2 = -1$

$\therefore$ $\underline{y(x) = 2x^{\frac{3}{2}} - x^{-1}}$

(b) Using MATLAB.

```
clear, clc
syms x
f = 2*x^(3/2) - x^(-1);
fplot(f)
grid on
xlabel 'x', ylabel 'y'
title('y = 2x^(^3^/^2^) - x^(^-^1^)')
```



$y \to 0$ as $x \to 0$

10.

(a) Let $y = x^r \therefore x^r[4r(r-1) + 8r + 17] = 0$

$\therefore 4r^2 + 4r + 17, \quad r = \frac{-4 \pm \sqrt{16 - 4(4)(17)}}{8} = -\frac{1}{2} \pm 2i$

$\therefore y = x^{-\frac{1}{2}}[c_1 \cos(2\ln(x)) + c_2 \sin(2\ln(x))]$

$y(1) = c_1 = 2$

$y' = -\frac{1}{2}x^{-\frac{3}{2}}[c_1 \cos(2\ln x) + c_2 \sin(2\ln x)]$

$\quad + x^{-\frac{1}{2}}[-c_1 \frac{2}{x} \sin(2\ln x) + c_2 \frac{2}{x} \cos(2\ln x)]$

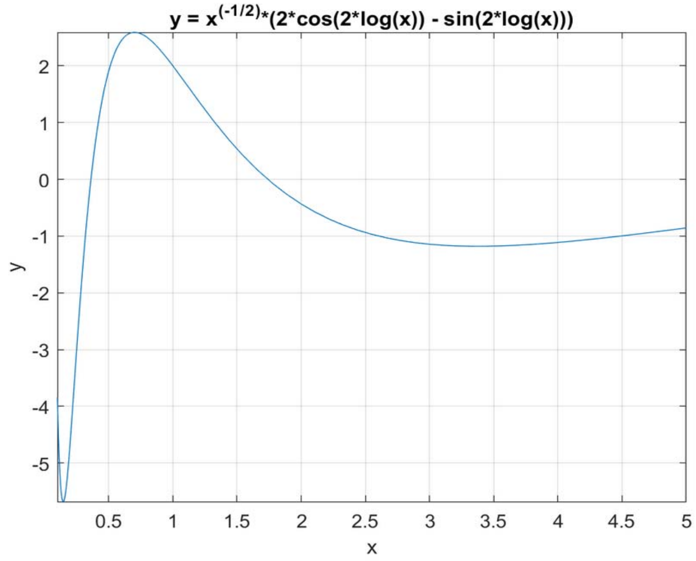
$\therefore y'(1) = -\frac{1}{2}[c_1 + 0] + [0 + 2c_2] = -\frac{c_1}{2} + 2c_2 = -3$

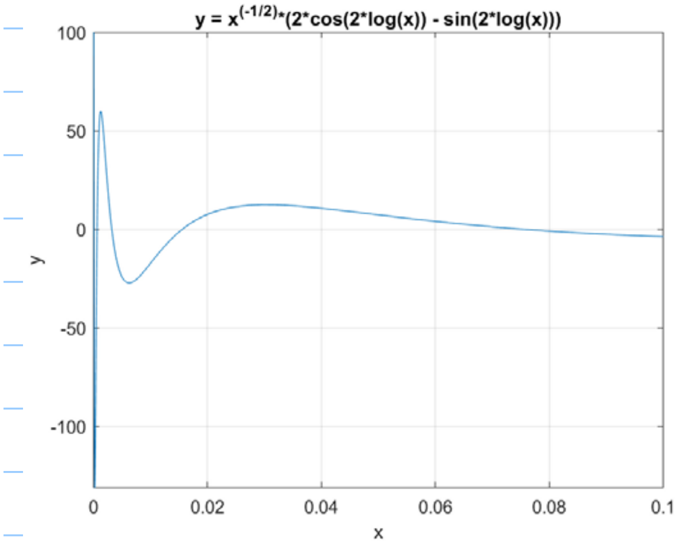
$\therefore -\frac{(2)}{2} + 2c_2 = -3, \quad c_2 = -1$

$\therefore \underline{y(x) = x^{-\frac{1}{2}}[2\cos(2\ln(x)) - \sin(2\ln(x))]}$

(b) Using MATLAB

```
clear, clc
syms x
f = x^(-1/2)*(2*cos(2*log(x)) - sin(2*log(x)));
fplot(f)
grid on
xlabel 'x', ylabel 'y'
title('y = x^(^-^1^/^2^)*(2*cos(2*log(x)) - sin(2*log(x)))')
```





↑ zooming in on $x=0$

As $x \to 0$, y oscillates unbounded

11.

Let $y = x^r \therefore x^r[r(r-1) - 3r + 4] = 0$

$\therefore r^2 - 4r + 4 = 0, \ (r-2)^2 = 0, \ r = 2, 2$

$\therefore y(x) = c_1 x^2 + c_2 x^2 \ln|x| \quad (\text{for } x < 0)$

$y(-1) = c_1 = 2$

Let $u = -x$ for $x < 0$, so $x(u) = -u$, $|x| = u$

$\therefore y(x(u)) = c_1 u^2 + c_2 u^2 \ln(u) \qquad (u > 0)$

$\therefore \dfrac{dy}{du} = \dfrac{dy}{dx} \cdot \dfrac{dx}{du} = 2c_1 u + 2c_2 u \ln(u) + c_2 u^2 \dfrac{1}{u}$

$$\therefore\ y'(x)(-1) = 2c_1 u + 2c_2 u \ln(u) + c_2 u$$

$$\text{or, } y'(x) = -2c_1 u - 2c_2 u \ln(u) - c_2 u$$

$$= -2c_1(-x) - 2c_2(-x)\ln(|x|) - c_2(-x)$$

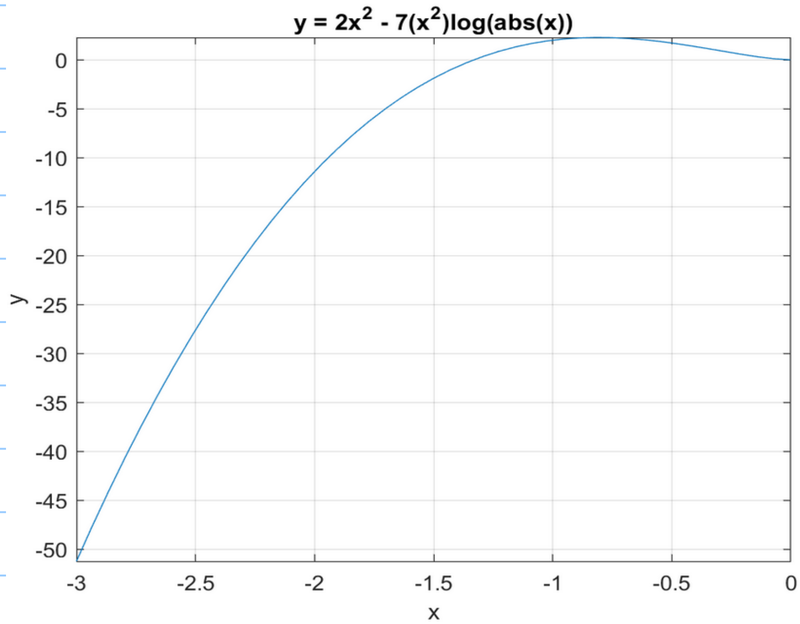
$$\therefore\ y'(x) = 2c_1 x + 2c_2 x \ln|x| + c_2 x$$

$$\therefore\ y'(-1) = -2c_1 - c_2 = 3,\quad -2(2) - c_2 = 3,\quad c_2 = -7$$

$$\therefore\ y(x) = \underline{2x^2 - 7x^2 \ln|x|}$$

(b) Using MATLAB,

```
clear, clc
syms x
f = 2*x^2 - 7*(x^2)*log(abs(x));
fplot(f,[-3 0])
grid on
xlabel 'x', ylabel 'y'
title('y = 2x^2 - 7(x^2)log(abs(x))')
```



$y \to 0$ as $x \to 0$

12.

$$y'' + \frac{1-x}{x} + y = 0 \quad \therefore \text{ Singular point: } x = 0$$

$$\lim_{x \to 0} x\left(\frac{1-x}{x}\right) = 1 \quad , \quad \lim_{x \to 0} x^2(1) = 0$$

$\therefore$ $\underline{x=0}$ is a $\underline{\text{regular}}$ singular point.

13.

$$y'' + \frac{2}{x(1-x)^2} y' + \frac{4}{x^2(1-x)^2} y = 0 \quad \therefore \text{ Singular points: } x = 0, 1$$

$$x=0: \lim_{x \to 0} x\left(\frac{2}{x(1-x)^2}\right) = 2 \qquad \lim_{x \to 0} x^2\left(\frac{4}{x^2(1-x)^2}\right) = 4$$

$\therefore$ $\underline{x=0}$ is $\underline{\text{regular}}$

$$x=1: \lim_{x \to 1} (x-1)\left[\frac{2}{x(1-x)^2}\right] = \lim_{x \to 1} \frac{2}{x(x-1)} = \infty$$

$\therefore$ $\underline{x=1 \text{ is}}$ irregular

14.

$$y'' + \frac{x-2}{x^2(1-x)} y' - \frac{3}{x(1-x)} y = 0 \quad \therefore \text{ Singular points: } x = 0, 1$$

$x=0$: $\lim_{x\to 0} x\left[\frac{x-2}{x^2(1-x)}\right] = \infty \quad \therefore \; \underline{x=0 \text{ is irregular}}$

$x=1$: $\lim_{x\to 1} (x-1)\left[\frac{x-2}{x^2(1-x)}\right] = 1$

$\lim_{x\to 1} (x-1)^2\left[\frac{x-2}{x^2(1-x)}\right] = \lim_{x\to 1} \frac{(x-1)(2-x)}{x^2} = 0$

$\therefore \; \underline{x=1 \text{ is regular}}$

15.

$y'' + \frac{2}{x^3(1-x^2)} y' + \frac{4}{x^2(1-x^2)} y = 0$. Singular points: $x = 0, \pm 1$

$x=0$: $\lim_{x\to 0} x\left[\frac{2}{x^3(1-x^2)}\right] = \infty \quad \therefore \; \underline{x=0 \text{ is irregular}}$

$x=1$: $\lim_{x\to 1} (x-1)\left[\frac{2}{x^3(1-x^2)}\right] = \lim_{x\to 1} \frac{-2}{x^3(x+1)} = -1$

$\lim_{x\to 1} (x-1)^2\left[\frac{4}{x^2(1-x^2)}\right] = \lim_{x\to 1} \frac{-4(x-1)}{x^2(x+1)} = 0$

$\therefore \; \underline{x=1 \text{ is regular}}$

$x=-1$: $\lim_{x\to -1} (x+1)\left[\frac{2}{x^3(1-x^2)}\right] = \lim_{x\to -1} \frac{2}{x^3(1-x)} = -1$

$$\lim_{x\to -1} (x+1)^2\left[\frac{4}{x^2(1-x^2)}\right] = \lim_{x\to -1} \frac{4(x+1)}{x^2(1-x)} = 0$$

$\therefore$ $x=-1$ is regular

16.

$$y'' + \frac{x}{(1+x)(1-x^2)}y' + \frac{1}{(1-x)(1-x^2)}y = 0 \quad \text{Singular points: } x=\pm 1$$

$$x=1: \lim_{x\to 1} (x-1)\left[\frac{x}{(1+x)(1-x^2)}\right] = \lim_{x\to 1} \frac{-x}{(1+x)(x+1)} = -\frac{1}{2}$$

$$\lim_{x\to 1} (x-1)^2\left[\frac{1}{(1-x)(1-x^2)}\right] = \lim_{x\to 1} \frac{1}{1+x} = \frac{1}{2}$$

$\therefore$ $x=1$ is regular

$$x=-1: \lim_{x\to -1} (x+1)\left[\frac{x}{(1+x)(1-x^2)}\right] = \infty$$

$\therefore$ $x=-1$ is irregular

17.

$$y'' + \frac{1}{x}y' + \frac{x^2-v^2}{x^2}y = 0 \quad \text{Singular points: } x=0$$

$$x=0: \lim_{x\to 0} x\left(\frac{1}{x}\right) = 1 \qquad \lim_{x\to 0} x^2\left(\frac{x^2-v^2}{x^2}\right) = -v^2$$

$\therefore$ $x = 0$ is regular

18.

$$y'' + \frac{3}{(x+2)^2} y' - \frac{2}{(x+2)(x-1)} y = 0$$ Singular points: $x = -2, 1$

$x = -2$: $$\lim_{x \to -2} (x+2)\left[\frac{3}{(x+2)^2}\right] = \infty$$

$\therefore$ $x = -2$ is irregular

$x = 1$: $$\lim_{x \to 1} (x-1)\left[\frac{3}{(x+2)^2}\right] = 0$$

$$\lim_{x \to 1} (x-1)^2\left[\frac{-2}{(x+2)(x-1)}\right] = 0$$

$\therefore$ $x = 1$ is regular

19.

$$y'' + \frac{x+1}{x(3-x)} y' - \frac{2}{x(3-x)} y = 0$$ Singular points: $x = 0, 3$

$x = 0$: $$\lim_{x \to 0} x\left[\frac{x+1}{x(3-x)}\right] = \frac{1}{3} \qquad \lim_{x \to 0} x^2\left[\frac{-2}{x(3-x)}\right] = 0$$

$\therefore$ $x = 0$ is regular

$$x = 3: \quad \lim_{x \to 3} (x-3)\left[\frac{x+1}{x(3-x)}\right] = -\frac{4}{3}$$

$$\lim_{x \to 3} (x-3)^2\left[\frac{-2}{x(3-x)}\right] = 0$$

$\therefore$ $x=3$ is regular

20.

$$y'' + \frac{e^x}{x} y' + \frac{3\cos(x)}{x} y = 0 \qquad \text{Singular points: } x=0$$

$x\left(\frac{e^x}{x}\right) = e^x$ is analytic at $x=0$

$x^2\left[\frac{3\cos(x)}{x}\right] = 3x\cos(x)$ is analytic at $x=0$

as both x and $\cos(x)$ are analytic at $x=0$.

$\therefore$ $x=0$ is regular

21.

$\frac{\ln|x|}{1}$ is not analytic at $x=0$. $\therefore$ Singular point: $x=0$

$\therefore$ Consider $x\ln|x|$. This function does not have a Taylor series about $x=0$. $\therefore$ $x=0$ is irregular.

22.

$y'' + \frac{x}{\sin(x)} y' + \frac{4}{\sin(x)} y = 0$ Singular points: $x = n\pi$,
$n = 0, \pm 1, \pm 2, \ldots$

Consider $n = 0 \therefore x = 0$: $x\left[\frac{x}{\sin(x)}\right] = \frac{x}{\frac{\sin(x)}{x}}$

Using the Taylor series for $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$

$\frac{\sin(x)}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!} - \ldots$

At $x = 0$, define $\frac{\sin(x)}{x} = 1$.

$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \therefore \frac{\sin(x)}{x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n+1)!}$

$\therefore \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{2n+2}}{(2n+3)!} \cdot \frac{(2n+1)!}{x^{2n}}\right| = \frac{x^2}{(2n+3)(2n+2)}$

$\therefore \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0$. $\therefore$ The series for $\frac{\sin(x)}{x}$

converges for all x. $\rho = \infty$.

$\therefore \frac{\sin(x)}{x}$ is analytic at $x = 0$.

$\therefore$ By statement #7 of text, p. 191,

$\frac{x}{\frac{\sin(x)}{x}} = \frac{x^2}{\sin(x)}$ is analytic at $x = 0$.

$\therefore \dfrac{4x^2}{\sin(x)}$ is analytic at $x=0$. $\therefore$ $\underline{x=0 \text{ is}}$ regular

Now consider $n \neq 0$ and $\therefore (x-n\pi)\dfrac{x}{\sin(x)}$ and $(x-n\pi)^2\dfrac{4}{\sin(x)}$

Using $\sin(a-b) = \sin(a)\cos(b) - \cos(a)\sin(b)$,

$\sin(x-n\pi) = \sin(x)(-1)^n$, $\therefore \sin(x) = (-1)^n \sin(x-n\pi)$

$$\therefore (x-n\pi)\frac{x}{\sin(x)} = (x-n\pi)\frac{x}{(-1)^n\sin(x-n\pi)}$$

$$= \frac{(x-n\pi)}{\sin(x-n\pi)} \cdot (-1)^n x$$

As shown above, $\dfrac{y}{\sin(y)}$ is analytic at $y=0$,

$\therefore \dfrac{(x-n\pi)}{\sin(x-n\pi)}$ is analytic at $x = n\pi$

For a given n, $(-1)^n x$ is also analytic at $x = n\pi$.

$\therefore (x-n\pi)\dfrac{x}{\sin(x)}$ is analytic at $x = n\pi$

Also, $(x-n\pi)^2\dfrac{4}{\sin(x)} = (x-n\pi)^2\dfrac{4}{(-1)^n\sin(x-n\pi)}$

$$= \frac{(x-n\pi)}{\sin(x-n\pi)} \cdot (-1)^n 4(x-n\pi)$$

For a given n, $(-1)^n 4(x-n\pi)$ is analytic at $x=n\pi$

$\therefore (x-n\pi)^2 \dfrac{4}{\sin(x)}$ is analytic at $x=n\pi$

$\therefore$ $\underline{x=n\pi}$ is regular, for $n=0,\pm1,\pm2,\ldots$

23.

$y'' + \dfrac{3}{x\sin(x)} y' + \dfrac{1}{\sin(x)} y = 0$ Singular points: $x = n\pi$, $n=0,\pm1,\pm2,\ldots$

$x=0$: $x\left[\dfrac{3}{x\sin(x)}\right] = \dfrac{3}{\sin(x)}$ This is not bounded as $x \to 0$, and so is not analytic

$\therefore$ $\underline{x=0 \text{ is irregular}}$

$x=n\pi$, $n \neq 0$: From #22 above, $\sin(x) = (-1)^n \sin(x-n\pi)$

$$\therefore (x-n\pi)\frac{3}{x\sin(x)} = \frac{(x-n\pi)}{\sin(x-n\pi)} \cdot \frac{(-1)^n 3}{x}$$

As shown in #22 above, $\dfrac{(x-n\pi)}{\sin(x-n\pi)}$ is analytic at $x=n\pi$. Since x is analytic at $x=n\pi$

and $x \neq 0$ at $x = n\pi$, $n \neq 0$, then $(x-n\pi)\dfrac{3}{x\sin(x)}$

is analytic at $x = n\pi$, $n \neq 0$, by statement #7

from text, p.191.

Also, $(x-n\pi)^2 \cdot \dfrac{1}{\sin(x)} = (x-n\pi)^2 \cdot \dfrac{1}{(-1)^n \sin(x-n\pi)}$

$= (-1)^n (x-n\pi) \cdot \dfrac{(x-n\pi)}{\sin(x-n\pi)}$. As shown in #22

above, $\dfrac{(x-n\pi)}{\sin(x-n\pi)}$ is analytic at $x = n\pi$, and so

is $(-1)^n(x-n\pi)$ for a given n.

$\therefore (x-n\pi)^2 \dfrac{1}{\sin(x)}$ is the product of analytic

functions at $x = n\pi$, $n \neq 0$, and so is

analytic at $x = n\pi$, $n \neq 0$.

$\therefore$ $\underline{x = n\pi,\ n \neq 0 \text{ is regular}}$, $n = \pm 1, \pm 2, \ldots$

24.

Let $y = x^r$ $\therefore$ $x^r\left[r(r-1) + ar + \frac{5}{2}\right] = 0$

$\therefore r^2 + (\alpha-1)r + \frac{5}{2} = 0$, $r = \dfrac{-(\alpha-1) \pm \sqrt{(\alpha-1)^2 - 10}}{2}$

(1) if $(\alpha-1)^2 < 10$, $y = |x|^{\frac{1-\alpha}{2}}\left[c_1 \cos(\mu \ln|x|) + c_2 \sin(\mu \ln|x|)\right]$

where $\mu = \dfrac{\sqrt{10-(\alpha-1)^2}}{2}$

$\therefore y \to 0$ as $x \to 0$ if $|x|^{\frac{1-\alpha}{2}} \to 0$

$\therefore$ if $1-\alpha > 0$, or $\alpha < 1$, then $y \to 0$

Since $(\alpha-1)^2 < 10 \Leftrightarrow -\sqrt{10} < \alpha - 1 < \sqrt{10}$

$\therefore$ if $\underline{1-\sqrt{10} < \alpha < 1}$ then $y \to 0$ as $x \to 0$

(2) if $(\alpha-1)^2 = 10$, then $y = |x|^{\frac{1-\alpha}{2}}\left[c_1 + c_2 \ln|x|\right]$

or $y = \dfrac{c_1 + c_2 \ln|x|}{|x|^{\frac{\alpha-1}{2}}}$ Using L'Hopital's rule,

$\lim\limits_{x\to 0} y = \lim\limits_{x \to 0} \dfrac{c_2 \frac{1}{x}}{\frac{\alpha-1}{2}|x|^{\frac{\alpha-1}{2}-1}} = k \lim\limits_{x\to 0} \dfrac{1}{|x|^{\frac{\alpha-1}{2}}}$, $= K \lim\limits_{x \to 0} |x|^{\frac{1-\alpha}{2}}$

K a constant

$\therefore K|x|^{\frac{1-\alpha}{2}} \to 0$ as $x \to 0$ if $\frac{1-\alpha}{2} > 0$,

or, $1 > \alpha$. $\therefore \alpha < 1$

$\therefore \alpha < 1$ and $(\alpha-1)^2 = 10 \Rightarrow \alpha - 1 = -\sqrt{10}$,

$\therefore$ if $\underline{\alpha = 1-\sqrt{10}}$ then $y \to 0$ as $x \to 0$.

(3) if $(\alpha-1)^2 > 10$, Then $y = c_1 |x|^{r_1} + c_2 |x|^{r_2}$,

$$r_1 = \frac{(1-\alpha)+\sqrt{(\alpha-1)^2-10}}{2}, \quad r_2 = \frac{(1-\alpha)-\sqrt{(\alpha-1)^2-10}}{2}$$

$y \to 0$ as $x \to 0$ if $r_1 > 0$ and $r_2 > 0$

$\therefore (1-\alpha)+\sqrt{(\alpha-1)^2-10} > 0$ and $1-\alpha > \sqrt{(\alpha-1)^2-10} > 0$

$\therefore \sqrt{(\alpha-1)^2-10} > \alpha - 1$ and $1-\alpha > 0 \Rightarrow \alpha < 1$

$\hookrightarrow$ only true if $\alpha - 1 < 0$

$\therefore \alpha < 1$ and $(\alpha-1)^2 > 10$, $\therefore \alpha < 1$ and $\alpha - 1 < -\sqrt{10}$

$\therefore$ if $\underline{\alpha} < 1-\sqrt{10}$ then $y \to 0$ as $x \to 0$

$\therefore$ (1), (2), (3) $\Rightarrow$

$$\{\alpha : 1-\sqrt{10} < \alpha < 1\} \cup \{\alpha : \alpha = 1-\sqrt{10}\} \cup \{\alpha : \alpha < 1-\sqrt{10}\}$$

i.e., if $\underline{\alpha < 1}$ Then $y \to 0$ if $x \to 0$

25.

Let $y = x^r$. $\therefore x^r[r(r-1)+\beta] = 0$

$\therefore\ r^2 - r + \beta = 0, \quad r = \dfrac{1 \pm \sqrt{1-4\beta}}{2}$

(1) $1 > 4\beta \quad \therefore\ y = c_1 |x|^{\frac{1+\sqrt{1-4\beta}}{2}} + c_2 |x|^{\frac{1-\sqrt{1-4\beta}}{2}}$

$y \to 0$ as $x \to 0$ if $1 - \sqrt{1-4\beta} > 0$, or $1 > \sqrt{1-4\beta}$

$\therefore\ 1 - 4\beta < 1, \quad \therefore\ \beta > 0$

$\therefore\ 1 > 4\beta$ and $\beta > 0 \ : \ 0 < \beta < \frac{1}{4}$

$\therefore\ y \to 0$ as $x \to 0$ if $\underline{0 < \beta < \frac{1}{4}}$

(2) $1 = 4\beta$ or $\beta = \frac{1}{4} \quad \therefore\ y = |x|^{\frac{1}{2}} [c_1 + c_2 \ln|x|]$

$y \to 0$ as $x \to 0$ if $\lim\limits_{x \to 0} |x|^{\frac{1}{2}} \ln|x|$

Using L'Hopital's rule, $\lim\limits_{x \to 0} |x|^{\frac{1}{2}} \ln|x| = \lim\limits_{x \to 0} \dfrac{\ln|x|}{|x|^{-\frac{1}{2}}}$

$= \lim\limits_{x \to 0} \dfrac{\frac{1}{x}}{-\frac{1}{2}|x|^{-\frac{3}{2}}} = -\frac{1}{2} \lim\limits_{x \to 0} |x|^{\frac{1}{2}} = 0$

$\therefore\ y \to 0$ as $x \to 0$ if $\underline{\beta = \frac{1}{4}}$

(3) $1 < 4\beta$, or $\beta > \frac{1}{4}$

$\therefore\ y = |x|^{\frac{1}{2}} [c_1 \cos(\mu \ln|x|) + c_2 \sin(\mu \ln|x|)]$

where $\mu = \dfrac{\sqrt{4\beta - 1}}{2}$

Since $c_1 \cos(\mu \ln|x|) + c_2 \sin(\mu \ln|x|) \leq |c_1| + |c_2|$

then $|y| \leq |x|^{\frac{1}{2}}(|c_1| + |c_2|)$

$\therefore y \to 0$ as $x \to 0$ for $\beta > \frac{1}{4}$

$\therefore (1), (2), (3) \Rightarrow y \to 0$ as $x \to 0$ if $\underline{\beta > 0}$

26.

Let $y = x^r \quad \therefore \quad x^r[r(r-1) - 2] = 0, \; r^2 - r - 2 = (r-2)(r+1) = 0$

$\therefore y = c_1 x^2 + c_2 x^{-1}, \quad y' = 2c_1 x - c_2 x^{-2}$

$$\therefore \left.\begin{aligned} y(1) &= c_1 + c_2 = 1 \\ y'(1) &= 2c_1 - c_2 = \gamma \end{aligned}\right\} \quad \begin{aligned} 3c_1 &= 1 + \gamma, \; c_1 = \frac{1+\gamma}{3} \\ 3c_2 &= 2 - \gamma, \; c_2 = \frac{2-\gamma}{3} \end{aligned}$$

$\therefore y = \left(\frac{1+\gamma}{3}\right) x^2 + \left(\frac{2-\gamma}{3}\right) \frac{1}{x}$

$\therefore$ as $x \to 0$, y will be bounded if $\frac{2-\gamma}{3} = 0$,

getting rid of the $\frac{1}{x}$ term.

$\therefore \underline{\gamma = 2}$

27.

Let $y = x^r \therefore x^r[r(r-1) + \alpha r + \beta] = 0$

$\therefore r^2 + (\alpha - 1)r + \beta = 0, \quad r = -\frac{(\alpha-1) \pm \sqrt{(\alpha-1)^2 - 4\beta}}{2}$

(a)

(1) $(\alpha-1)^2 - 4\beta > 0$

$\therefore y = c_1 |x|^{\frac{1-\alpha}{2} + \frac{\sqrt{(\alpha-1)^2 - 4\beta}}{2}} + c_2 |x|^{\frac{1-\alpha}{2} - \frac{\sqrt{(\alpha-1)^2 - 4\beta}}{2}}$

(1.1) $1 - \alpha > 0$

$\therefore y \to 0$ if $1 - \alpha > \sqrt{(\alpha-1)^2 - 4\beta} \Rightarrow (1-\alpha)^2 > (\alpha-1)^2 - 4\beta$

$\Rightarrow 4\beta > 0$, or $\beta > 0$

$\therefore$ if $\alpha < 1$ <u>and</u> $\beta > 0$ then $y \to 0$ as $x \to 0$

(1.2) $1 - \alpha = 0$

$\therefore \beta < 0$, and $y = c_1 |x|^{\sqrt{-\beta}} + c_2 |x|^{-\sqrt{-\beta}}$

$\therefore y$ will not $\to 0$ if $\alpha = 1$

(1.3) $1 - \alpha < 0$

$\therefore |x|^{\frac{1-\alpha}{2} - \frac{\sqrt{(\alpha-1)^2 - 4\beta}}{2}} \to \infty$ if $x \to 0$

$\therefore$ y will not $\to 0$ if $\alpha > 1$

$\therefore$ if $\alpha < 1$ and $\beta > 0$ and $(\alpha-1)^2 - 4\beta > 0$, $y \to 0$ as $x \to 0$

(2) $(\alpha-1)^2 - 4\beta = 0$

$$\therefore y = c_1 |x|^{\frac{1-\alpha}{2}} + c_2 |x|^{\frac{1-\alpha}{2}} \ln|x|$$

(2.1) $1-\alpha < 0$

Then $|x|^{\frac{1-\alpha}{2}}$ is unbounded as $x \to 0$

$\therefore$ $1-\alpha < 0$ can't be true

(2.2) $1-\alpha = 0$

Then $y = c_1 + c_2 \ln|x|$ $\therefore$ $y \not\to 0$ as $x \to 0$

$\therefore$ $1-\alpha = 0$ can't be true

(2.3) $1-\alpha > 0$

$$\therefore y = c_1 |x|^{\frac{1-\alpha}{2}} + c_2 |x|^{\frac{1-\alpha}{2}} \ln|x|$$

Using L'Hopital's rule, $\lim\limits_{x\to 0} |x|^{\frac{1-\alpha}{2}} \ln|x| =$

$$\lim_{x\to 0} \frac{\ln|x|}{|x|^{\frac{\alpha-1}{2}}} = \lim_{x\to 0} \frac{\frac{1}{x}}{\left(\frac{\alpha-1}{2}\right)|x|^{\frac{\alpha-3}{2}}} = K \lim_{x\to 0} |x|^{\frac{1-\alpha}{2}} = 0$$

K a constant

$\therefore$ if $\alpha < 1$ and $(\alpha-1)^2 - 4\beta = 0$, Then $y \to 0$ as $x \to 0$

$\alpha < 1 \Rightarrow (\alpha-1)^2 > 0 \Rightarrow 4\beta > 0$, or $\beta > 0$

$\therefore$ if $\underline{\alpha < 1 \text{ and}}\ \beta > 0$ and $(\alpha-1)^2 - 4\beta = 0$, $y \to 0$ as $x \to 0$

(3) $(\alpha-1)^2 - 4\beta < 0$

$$\therefore y = |x|^{\frac{1-\alpha}{2}}\left[c_1 \cos(\mu \ln|x|) + c_2 \sin(\mu \ln|x|)\right]$$

$$\text{where } \mu = \frac{\sqrt{4\beta - (\alpha-1)^2}}{2}$$

$\therefore y \to 0$ as $x \to 0$ if $1-\alpha > 0$, or $\alpha < 1$

But $\alpha < 1 \Rightarrow (\alpha-1)^2 > 0$, so $4\beta > (\alpha-1)^2 \Rightarrow \beta > 0$

$\therefore$ if $\underline{\alpha < 1 \text{ and}}\ \beta > 0$ and $(\alpha-1)^2 - 4\beta < 0$, $y \to 0$ as $x \to 0$

(1), (2), (3) $\Rightarrow$ $\boxed{\text{if } \underline{\alpha < 1 \text{ and}}\ \beta > 0, \text{ Then } y \to 0 \text{ as } x \to 0}$

(b)

Note solutions to (a) are included, so examine each step in (a) to include bounded solutions.

(1) $(\alpha-1)^2 - 4\beta > 0$

$$\therefore y = c_1 |x|^{\frac{1-\alpha}{2} + \frac{\sqrt{(\alpha-1)^2 - 4\beta}}{2}} + c_2 |x|^{\frac{1-\alpha}{2} - \frac{\sqrt{(\alpha-1)^2-4\beta}}{2}}$$

(1.1) $1-\alpha > 0$

The first term $\to 0$ as $x \to 0$

If $1-\alpha > \sqrt{(\alpha-1)^2 - 4\beta}$, the 2nd term $\to 0$ as $x \to 0$

$\therefore (1-\alpha)^2 > (\alpha-1)^2 - 4\beta \Rightarrow \beta > 0$

If $1-\alpha = \sqrt{(\alpha-1)^2 - 4\beta}$, then $y \to c_2$ as $x \to 0$

$\therefore (1-\alpha)^2 = (\alpha-1)^2 - 4\beta$, and so $\beta = 0$

$\therefore$ If $\underline{\alpha < 1 \text{ and }}$ $\beta \geq 0$, as $x \to 0$, y is bounded

(1.2) $1-\alpha = 0$

$\therefore \beta < 0$, and $y = c_1 |x|^{\sqrt{-\beta}} + c_2 |x|^{-\sqrt{-\beta}}$

$\therefore y \to \infty$ as $x \to 0$ when $c_2 \neq 0$

$\therefore$ Can't have $\alpha = 1$ and $\beta < 0$

(1.3) $1-\alpha < 0$

$\therefore |x|^{\frac{1-\alpha}{2} - \frac{\sqrt{(\alpha-1)^2-4\beta}}{2}} \to \infty$ if $x \to 0$

$\therefore$ Can't have $\alpha > 1$ for y to be bounded

$\therefore$ if $\underline{\alpha < 1 \text{ and}}$ $\beta \geq 0$ and $(\alpha-1)^2 - 4\beta > 0$, y is bounded as $x \to 0$

(2) $(\alpha-1)^2 - 4\beta = 0$

$\therefore y = c_1 |x|^{\frac{1-\alpha}{2}} + c_2 |x|^{\frac{1-\alpha}{2}} \ln|x|$

(2.1) $1-\alpha < 0$

Then $|x|^{\frac{1-\alpha}{2}}$ is unbounded as $x \to 0$

Also, $4\beta = (\alpha-1)^2 > 0$, so $\beta > 0$.

$\therefore$ $\alpha > 1$ and $\beta > 0$ can't be true

(2.2) $1-\alpha = 0$

Then $y = c_1 + c_2 \ln|x|$ $\therefore$ $y \to \infty$ as $x \to 0$

Also, $1-\alpha = 0 \Rightarrow \beta = 0$.

$\therefore$ $\alpha = 1$ and $\beta = 0$ can't be true

(2.3) $1-\alpha > 0$

$\therefore y = c_1 |x|^{\frac{1-\alpha}{2}} + c_2 |x|^{\frac{1-\alpha}{2}} \ln|x|$

The first term $\to 0$ as $x \to 0$

From (a) (2.3), the second term also $\to 0$

Also from (a)(2.3), $\beta > 0$ if $\alpha < 1$

$\therefore$ if $\underline{\alpha < 1 \text{ and }}$ $\beta > 0$ and $(\alpha-1)^2 - 4\beta = 0$, y is bounded as $x \to 0$

Note if $\alpha < 1$ and $\beta = 0$, then $(\alpha-1)^2 - 4\beta \neq 0$

(3) $(\alpha-1)^2 - 4\beta < 0$

$\therefore y = |x|^{\frac{1-\alpha}{2}} \left[c_1 \cos(\mu \ln|x|) + c_2 \sin(\mu \ln|x|)\right]$

where $\mu = \frac{\sqrt{4\beta - (\alpha-1)^2}}{2}$

(3.1) $1-\alpha > 0$

As in (a), $y \to 0$ as $x \to 0$, and $\alpha < 1$ also

means $\beta > 0$. $\therefore \alpha < 1$ and $\beta > 0 \Rightarrow y \to 0$

(3.2) $1-\alpha = 0$

$\therefore y = c_1 \cos(\mu \ln|x|) + c_2 \sin(\mu \ln|x|)$

$\therefore |y| \le |c_1| + |c_2|$ and so is bounded.

Note $\alpha = 1 \Rightarrow -4\beta < 0 \Rightarrow \beta > 0$

$\therefore$ if $\alpha = 1$ and $\beta > 0$, y is bounded

(3.3) $1-\alpha < 0$

$\therefore$ the term $\frac{1}{|x|^{\frac{\alpha-1}{2}}}$ causes $y \to \infty$ as $x \to 0$

Also, $1-\alpha < 0 \Rightarrow (\alpha-1)^2 > 0$, so $\beta > 0$.

$\therefore$ Can't have $\alpha > 1$ and $\beta > 0$

$\therefore$ $\underline{\alpha < 1 \text{ and}}$ $\beta > 0$ and $(\alpha-1)^2 - 4\beta < 0 \Rightarrow y$ is bounded

$\underline{\alpha = 1 \text{ and}}$ $\beta > 0$ and $(\alpha-1)^2 - 4\beta < 0 \Rightarrow y$ is bounded

But $\alpha = 1$ and $\beta > 0 \Rightarrow (\alpha-1)^2 - 4\beta > 0$

Summary for (6):

if $\alpha < 1$ and $\beta \ge 0$, then y is bounded as $x \to 0$

if $\alpha = 1$ and $\beta > 0$, then y is bounded as $x \to 0$

(c)

For $|x|^r \to 0$ as $x \to \infty$, want $r < 0$

(1) $(\alpha-1)^2 - 4\beta > 0$

$$y = c_1 |x|^{\frac{1-\alpha}{2} + \frac{\sqrt{(\alpha-1)^2-4\beta}}{2}} + c_2 |x|^{\frac{1-\alpha}{2} - \frac{\sqrt{(\alpha-1)^2-4\beta}}{2}}$$

(1.1) $1-\alpha > 0$

$\therefore \lim_{x\to\infty} c_1 |x|^r \to \infty \quad \therefore$ Can't have $\alpha < 1$

(1.2) $1-\alpha = 0$

$\therefore \lim_{x\to\infty} c_1 |x|^r \to \infty$ so $\alpha \neq 1$

(1.3) $1-\alpha < 0$

$\therefore \frac{1-\alpha}{2} - \frac{\sqrt{(\alpha-1)^2-4\beta}}{2} < 0$ so $\lim_{x\to\infty} c_2 |x|^r \to 0$

For $1-\alpha + \sqrt{(\alpha-1)^2 - 4\beta} < 0$, then

$\sqrt{(\alpha-1)^2-4\beta} < \alpha - 1 \Rightarrow (\alpha-1)^2 - 4\beta < (\alpha-1)^2$

$\Rightarrow \beta > 0$

$\therefore$ if $\alpha > 1$ and $\beta > 0$ and $(\alpha-1)^2 - 4\beta > 0$, $\lim_{x\to\infty} y = 0$

(2) $(\alpha-1)^2-4\beta=0$

$\therefore y = c_1|x|^{\frac{1-\alpha}{2}} + c_2|x|^{\frac{1-\alpha}{2}}\ln|x|$

(2.1) $1-\alpha>0$

$\therefore y \to \infty$ as $x\to\infty$. $\therefore$ Can't have $\alpha<1$

(2.2) $1-\alpha=0$

$\therefore y = c_1 + c_2\ln|x|$, so $y\to\infty$ as $x\to\infty$

$\therefore \alpha\neq 1$

(2.3) $1-\alpha<0$

$c_1|x|^{\frac{1-\alpha}{2}} \to 0$ as $x\to\infty$

Using L'Hopital's rule, $\lim\limits_{x\to\infty} c_2|x|^{\frac{1-\alpha}{2}}\ln|x| =$

$$c_2\lim_{x\to\infty}\frac{\ln|x|}{|x|^{\frac{\alpha-1}{2}}} = c_2\lim_{x\to\infty}\frac{1/x}{\left(\frac{\alpha-1}{2}\right)|x|^{\frac{\alpha-3}{2}}} =$$

$K\lim\limits_{x\to\infty}|x|^{\frac{1-\alpha}{2}} = 0$, K a constant

Note $1-\alpha<0 \Rightarrow 4\beta=(\alpha-1)^2>0 \Rightarrow \beta>0$

$\therefore$ if $\alpha>1$ and $\beta>0$ and $(\alpha-1)^2-4\beta=0$, $\lim\limits_{x\to\infty} y=0$

(3) $(\alpha-1)^2-4\beta<0$

$\therefore y=|x|^{\frac{1-\alpha}{2}}\left[c_1\cos(\mu\ln|x|)+c_2\sin(\mu\ln|x|)\right]$

where $\mu=\frac{\sqrt{4\beta-(\alpha-1)^2}}{2}$

$\lim\limits_{x\to\infty} c_1\cos(\mu\ln|x|)+c_2\sin(\mu\ln|x|)$ oscillates

between $\pm(|c_1|+|c_2|)$

(3.1) $1-\alpha<0$

$\therefore |x|^r\to 0$ so $y\to 0$ as $x\to\infty$

Note $1-\alpha<0 \Rightarrow (\alpha-1)^2>0$ so $4\beta>(\alpha-1)^2>0 \Rightarrow$

$\beta>0$. $\therefore$ if $\alpha>1$ and $\beta>0$, $y\to 0$

(3.2) $1-\alpha=0$

As above, y oscillates as $x\to\infty$

$\therefore \alpha\neq 1$

(3.3) $1-\alpha>0$

$\therefore |x|^{\frac{1-\alpha}{2}}\to\infty$ as $x\to\infty$, so $y\to\infty$.

$\therefore$ Can't have $\alpha<1$

$\therefore$ if $\alpha > 1$ and $\beta > 0$ and $(\alpha-1)^2 - 4\beta < 0$, then $\lim_{x\to\infty} y = 0$

$\therefore$ (1), (2), (3) $\Rightarrow$

$$\boxed{\text{if } \alpha > 1 \text{ and } \beta > 0 \text{ then } y \to 0 \text{ as } x \to \infty}$$

(d)

For $c|x|^r$ to be bounded as $x \to \infty$, want $r \leq 0$

(1) $(\alpha-1)^2 - 4\beta > 0$

$$y = c_1 |x|^{\frac{1-\alpha}{2} + \frac{\sqrt{(\alpha-1)^2 - 4\beta}}{2}} + c_2 |x|^{\frac{1-\alpha}{2} - \frac{\sqrt{(\alpha-1)^2 - 4\beta}}{2}}$$

As shown in (c) above, if $\alpha \leq 1$, then $y \to \infty$ as $x \to \infty$.

$1 - \alpha < 0$: then $c_2 |x|^r \to 0$ as $x \to \infty$

Consider term $c_1 |x|^{\frac{1-\alpha}{2} + \frac{\sqrt{(\alpha-1)^2 - 4\beta}}{2}}$

If $1 - \alpha + \sqrt{(\alpha-1)^2 - 4\beta} > 0$, then $\lim_{x\to\infty} c_1 |x|^r = \infty$

If $1 - \alpha + \sqrt{(\alpha-1)^2 - 4\beta} = 0$, then $\lim_{x\to\infty} c_1 |x|^r = c_1$

If $1 - \alpha + \sqrt{(\alpha-1)^2 - 4\beta} < 0$, then $\lim_{x\to\infty} c_1 |x|^r = 0$

Note $1-\alpha+\sqrt{(\alpha-1)^2-4\beta} \le 0 \Rightarrow \sqrt{(\alpha-1)^2-4\beta} \le \alpha-1$

and since $\sqrt{(\alpha-1)^2-4\beta} > 0$, $\therefore (\alpha-1)^2-4\beta \le (\alpha-1)^2$

$\Rightarrow 0 \le 4\beta$ or $\beta \ge 0$

$\therefore$ if $\underline{\alpha>1 \text{ and } \beta \ge 0}$ and $(\alpha-1)^2-4\beta>0$, then as $x\to\infty$,

y is bounded

(2) $(\alpha-1)^2-4\beta=0$

$$\therefore y = c_1|x|^{\frac{1-\alpha}{2}} + c_2|x|^{\frac{1-\alpha}{2}}\ln|x|$$

As shown in (c) above, can't have $\alpha \le 1$. So,

$\alpha>1$. Note this means $4\beta=(\alpha-1)^2>0 \Rightarrow \beta>0$

As shown in (c) (2.3), if $\underline{\alpha>1 \text{ and } \beta>0}$, $\lim_{x\to\infty} y = 0$

Note: if $\alpha>1$ and $\beta=0$, can't have $(\alpha-1)^2-4\beta=0$.

$\therefore \alpha>1$ and $\beta=0 \Rightarrow$ condition (1) or (3) below.

(3) $(\alpha-1)^2-4\beta<0$

$$\therefore y = |x|^{\frac{1-\alpha}{2}}\left[c_1\cos(\mu\ln|x|) + c_2\sin(\mu\ln|x|)\right]$$

where $\mu = \dfrac{\sqrt{4\beta-(\alpha-1)^2}}{2}$

$\lim_{x \to \infty} c_1 \cos(\mu \ln|x|) + c_2 \sin(\mu \ln|x|)$ oscillates

between $\pm(|c_1| + |c_2|)$

(3.1) $1-\alpha < 0$

$\therefore |x|^r \to 0$ so $y \to 0$ as $x \to \infty$

Note $1-\alpha < 0 \Rightarrow (\alpha-1)^2 > 0$ so $4\beta > (\alpha-1)^2 > 0 \Rightarrow$

$\beta > 0$. $\therefore$ if $\alpha > 1$ and $\beta > 0$, $y \to 0$

Note: $\alpha > 1$ and $\beta = 0 \Rightarrow (\alpha-1)^2 - 4\beta > 0$, so can't have condition (3), and so must have condition (1).

(3.2) $1-\alpha = 0$

$\therefore y = c_1 \cos(\mu \ln|x|) + c_2 \sin(\mu \ln|x|)$

$\therefore |y| \leq |c_1| + |c_2|$ for all x.

$\therefore$ As $x \to \infty$, y is bounded

Note $\alpha = 1 \Rightarrow -4\beta < 0 \Rightarrow \beta > 0$

$\therefore$ if $\alpha = 1, \beta > 0$, and $(\alpha-1)^2 - 4\beta < 0$, y is bounded

Note $\alpha = 1, \beta > 0 \Rightarrow (\alpha-1)^2 - 4\beta < 0$ which is (3.2)

(3.3) $1-\alpha > 0$

$|x|^r \to \infty$ as $x \to \infty$. $\therefore$ Can't have $\alpha < 1$

$\therefore$ if $\alpha > 1$ and $\beta > 0$ and $(\alpha-1)^2 - 4\beta < 0$, $\lim_{x \to \infty} y = 0$

$\therefore$ (1), (2), (3) $\Rightarrow$

if $\alpha > 1$ and $\beta \geq 0$, then y is bounded as $x \to \infty$
if $\alpha = 1$ and $\beta > 0$, Then y is bounded as $x \to \infty$

(e)

This is the intersection of (b) and (d)

(b) : $\{\alpha, \beta : (\alpha < 1 \text{ and } \beta \geq 0) \text{ or } (\alpha = 1 \text{ and } \beta > 0)\}$

(d) : $\{\alpha, \beta : (\alpha > 1 \text{ and } \beta \geq 0) \text{ or } (\alpha = 1 \text{ and } \beta > 0)\}$

$\therefore \{\alpha, \beta : \underline{\alpha = 1 \text{ and } \beta > 0}\}$

28.

Given $y = x^r$ is a solution. $\therefore y' = rx^{r-1}$, $y'' = r(r-1)x^{r-2}$

$$\therefore x^2 y'' + \alpha x y' + \beta y = x^r\left[r(r-1) + \alpha r + \beta\right] = 0$$

$$\therefore r(r-1) + \alpha r + \beta = r^2 + (\alpha - 1) r + \beta = 0 \quad [1]$$

But r a repeated root $\Rightarrow r = -\frac{(\alpha-1)}{2} \Rightarrow 2r + \alpha = 1 \quad [2]$

Now suppose $y = v(x) x^r$ is a solution

$$\therefore y' = v' x^r + r v x^{r-1}$$

$$\therefore y'' = v'' x^r + r v' x^{r-1} + r v' x^{r-1} + r(r-1) v x^{r-2}$$

$$= x^r v'' + 2r x^{r-1} v' + r(r-1) x^{r-2} v$$

$$\therefore x^2 y'' + \alpha x y' + \beta y =$$

$$x^{r+2} v'' + 2r x^{r+1} v' + r(r-1) x^r v$$

$$+ \alpha x^{r+1} v' + \alpha r x^r v$$

$$+ \beta x^r v$$

$$= x^{r+2} v'' + \underbrace{(\alpha + 2r)}^{=1 \text{ by } [2]} x^{r+1} v' + \overbrace{\left[r(r-1) + \alpha r + \beta\right]}^{=0 \text{ by } [1]} x^r v$$

$$= x^r(x^2v'' + xv') = 0 \qquad \text{Since } x>0,\ x^r \neq 0$$

$$\therefore\ x^2v'' + xv' = 0, \text{ or } \frac{v''}{v'} = -\frac{1}{x}$$

Integrating, $\ln(v') = -\ln(x) + c$, c a constant

or $v' = \dfrac{K_1}{x}$, K_1 a constant $= e^c$

Integrating again, $v(x) = K_1\ln(x) + K_2$

$$\therefore\ y = v(x)x^r = K_1x^r\ln(x) + K_2x^r$$

Since x^r was already a solution, an independent solution is $y = \underline{x^r\ln(x)}$

Note $W[x^r, x^r\ln(x)] = x^{2r-1} \neq 0$.

29.

$$\frac{d}{dx}x^\lambda\cos(\mu\ln x) = \lambda x^{\lambda-1}\cos(\mu\ln x) - x^\lambda\sin(\mu\ln x)\left[\frac{\mu}{x}\right]$$

$$\frac{d}{dx}x^\lambda\sin(\mu\ln x) = \lambda x^{\lambda-1}\sin(\mu\ln x) + x^\lambda\cos(\mu\ln x)\left[\frac{\mu}{x}\right]$$

$$\therefore \begin{vmatrix} x^\lambda\cos(\mu\ln x) & x^\lambda\sin(\mu\ln x) \\ \begin{array}{l}\lambda x^{\lambda-1}\cos(\mu\ln x) \\ \quad - x^\lambda\sin(\mu\ln x)\left[\frac{\mu}{x}\right]\end{array} & \begin{array}{l}\lambda x^{\lambda-1}\sin(\mu\ln x) \\ \quad + x^\lambda\cos(\mu\ln x)\left[\frac{\mu}{x}\right]\end{array}\end{vmatrix}$$

$$= \lambda x^{2\lambda-1}\cos(\mu\ln x)\sin(\mu\ln x) + \mu x^{2\lambda-1}\cos^2(\mu\ln x)$$

cancel $\qquad\qquad$ $\cos^2 + \sin^2 = 1$

$$- \lambda x^{2\lambda-1}\cos(\mu\ln x)\sin(\mu\ln x) + \mu x^{2\lambda-1}\sin^2(\mu\ln x)$$

$$= \underline{\underline{\mu x^{2\lambda-1}}}$$

30.

(a) Standard formation: $y'' + \frac{3}{2x}y' + \frac{1}{2}y = 0$

$$\lim_{x\to 0} x\left(\frac{3}{2x}\right) = \frac{3}{2} \qquad \lim_{x\to 0} x^2\left(\frac{1}{2}\right) = 0$$

Both limits are finite $\Rightarrow$ $x=0$ a regular singular point.

(b) Let $y = \sum_{n=0}^{\infty} a_n x^n \qquad \therefore\ xy = \sum_{n=0}^{\infty} a_n x^{n+1}$

$$\therefore\ y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n = a_1 + \sum_{n=1}^{\infty}(n+1)a_{n+1}x^n$$

$$= a_1 + \sum_{n=0}^{\infty}(n+2)a_{n+2}x^{n+1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} \quad , \quad 2xy'' = \sum_{n=2}^{\infty} 2n(n-1)a_n x^{n-1}$$

$$= \sum_{n=0}^{\infty} 2(n+2)(n+1)\, a_{n+2}\, x^{n+1}$$

$$\therefore\ 2xy'' + 3y' + xy = 0 \Rightarrow$$

$$3a_1 + \sum_{n=0}^{\infty} \left[2(n+2)(n+1)a_{n+2} + 3(n+2)a_{n+2} + a_n\right] x^{n+1}$$

$$= 3a_1 + \sum_{n=0}^{\infty} \left[(n+2)(2n+5)a_{n+2} + a_n\right] x^{n+1} = 0$$

$$\therefore\ a_1 = 0 \ \text{ and } \ (n+2)(2n+5)a_{n+2} + a_n = 0$$

$$a_{n+2} = -\frac{a_n}{(n+2)(2n+5)}$$

$$\therefore\ a_2 = -\frac{a_0}{2\cdot 5} \qquad a_3 = -\frac{a_1}{21} = 0$$

$$a_4 = -\frac{a_2}{4\cdot 9} = \frac{a_0}{2\cdot 4\cdot 5\cdot 9} \qquad a_5 = -\frac{a_3}{55} = 0$$

$$a_6 = -\frac{a_4}{6\cdot 13} = -\frac{a_0}{2\cdot 4\cdot 6\cdot 5\cdot 9\cdot 13}$$

$$\therefore\ y = a_0\left(1 - \frac{x^2}{2\cdot 5} + \frac{x^4}{2\cdot 4\cdot 5\cdot 9} - \frac{x^6}{2\cdot 4\cdot 6\cdot 5\cdot 9\cdot 13} + \cdots\right)$$

$\therefore$ The one solution only depends on a_0 and is a nonzero solution for $a_0 \neq 0$.

31.

(a) Standard form: $y'' + \frac{3}{2x} y' - \frac{1+x}{2x^2} y = 0$

$$\lim_{x \to 0} x \left(\frac{3}{2x} \right) = \frac{3}{2} \qquad \lim_{x \to 0} x^2 \left(- \frac{1+x}{2x^2} \right) = -\frac{1}{2}$$

Both limits are finite $\Rightarrow$ $x=0$ is a regular singular point.

(b) Let $y = \sum_{n=0}^{\infty} a_n x^n$

$$(1+x)y = \sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} a_n x^{n+1} = a_0 + a_1 x + \sum_{n=2}^{\infty} a_n x^n + \sum_{n=0}^{\infty} a_n x^{n+1}$$

$$= a_0 + (a_0 + a_1) x + \sum_{n=1}^{\infty} (a_{n+1} + a_n) x^{n+1}$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \therefore 3xy' = \sum_{n=1}^{\infty} 3n a_n x^n = 3a_1 + \sum_{n=2}^{\infty} 3n a_n x^n$$

$$\therefore 3xy' = 3a_1 + \sum_{n=1}^{\infty} 3(n+1) a_{n+1} x^{n+1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \quad \therefore 2x^2 y'' = \sum_{n=2}^{\infty} 2n(n-1) a_n x^n$$

$$\therefore\ 2x^2y'' = \sum_{n=1}^{\infty} 2(n+1)n\,a_{n+1}\,x^{n+1}$$

$$\therefore\ 2x^2y'' + 3xy' - (1+x)y = (3a_1 - a_0) - (a_0 + a_1)x + \sum_{n=1}^{\infty}\Big[\underbrace{2(n+1)n\,a_{n+1}}_{2n^2+2n} + \underbrace{3(n+1)a_{n+1}}_{3n+3} \underbrace{-(a_{n+1}}_{-1} + a_n)\Big]x^{n+1}$$

$$= (3a_1 - a_0) - (a_0 + a_1)x + \sum_{n=1}^{\infty}\left[(2n^2+5n+2)a_{n+1} - a_n\right]x^{n+1}$$

$\therefore\ 3a_1 - a_0 = 0$ and $a_0 + a_1 = 0 \Rightarrow a_0 = 0$ and $a_1 = 0$

$\therefore\ a_{n+1} = \dfrac{a_n}{2n^2+5n+2}$, $n \geq 1 \Rightarrow$ all $\underline{\underline{a_n = 0}}$

$\therefore\ y = 0$, so <u>no nonzero solution</u>

32.

For $\epsilon(x) = \frac{1}{x}$, let $u(\epsilon(x)) = y(x)$

So, for example, if $y(x) = 2x+1$, then $y(2) = 5$

so $u(\epsilon(2)) = u(\frac{1}{2}) = 5$, $\therefore u(n) = u\left(\frac{1}{(\frac{1}{n})}\right) = 2(\frac{1}{n})+1$

So $u(x) = \frac{2}{x} + 1$. Note $u(x) \neq y(x)$, but

$u(\frac{1}{x}) = y(x)$

$\therefore$ Using the chain rule, and using ϵ instead of ξ,

$$\frac{dy}{dx} = \frac{d\,u(\epsilon(x))}{dx} = \frac{du}{d\epsilon} \cdot \frac{d\epsilon}{dx} = \frac{du}{d\epsilon}\left(-\frac{1}{x^2}\right) = -\epsilon^2 \frac{du}{d\epsilon}$$

$$\frac{d^2y}{dx^2} = \frac{d}{d\epsilon}\left(-\epsilon^2 \cdot \frac{du}{d\epsilon}\right) \cdot \frac{d\epsilon}{dx} = \left(-2\epsilon \cdot \frac{du}{d\epsilon} - \epsilon^2 \cdot \frac{d^2u}{d\epsilon^2}\right)\left(-\frac{1}{x^2}\right)$$

$$= \left(-2\epsilon \frac{du}{d\epsilon} - \epsilon^2 \frac{d^2u}{d\epsilon^2}\right)(-\epsilon^2) = 2\epsilon^3 u' + \epsilon^4 u''$$

$\therefore\ P(x)\,y'' + Q(x)\,y' + R(x)\,y = 0$ becomes

$$P\left(\tfrac{1}{\epsilon}\right)\left[2\epsilon^3 u' + \epsilon^4 u''\right] + Q\left(\tfrac{1}{\epsilon}\right)\left[-\epsilon^2 u'\right] + R\left(\tfrac{1}{\epsilon}\right) u = 0$$

Or,

$$\epsilon^4 P\left(\tfrac{1}{\epsilon}\right) u'' + \left[2\epsilon^3 P\left(\tfrac{1}{\epsilon}\right) - \epsilon^2 Q\left(\tfrac{1}{\epsilon}\right)\right] u' + R\left(\tfrac{1}{\epsilon}\right) u = 0$$

Or, in standard form,

$$u'' + \frac{2\epsilon^3 P\left(\tfrac{1}{\epsilon}\right) - \epsilon^2 Q\left(\tfrac{1}{\epsilon}\right)}{\epsilon^4 P\left(\tfrac{1}{\epsilon}\right)} u' + \frac{R\left(\tfrac{1}{\epsilon}\right)}{\epsilon^4 P\left(\tfrac{1}{\epsilon}\right)} u = 0$$

<u>Ordinary point</u> at $\epsilon = 0$ (or $x = \infty$) if:

coefficient of u' and coefficient of u have Taylor expansions about $\epsilon = 0$.

i.e.,

$$\boxed{\frac{2\epsilon^3 P\left(\tfrac{1}{\epsilon}\right) - \epsilon^2 Q\left(\tfrac{1}{\epsilon}\right)}{\epsilon^4 P\left(\tfrac{1}{\epsilon}\right)} = \frac{1}{P\left(\tfrac{1}{\epsilon}\right)}\left[\frac{2P\left(\tfrac{1}{\epsilon}\right)}{\epsilon} - \frac{Q\left(\tfrac{1}{\epsilon}\right)}{\epsilon^2}\right] \text{ and } \frac{R\left(\tfrac{1}{\epsilon}\right)}{\epsilon^4 P\left(\tfrac{1}{\epsilon}\right)} \text{ have convergent Taylor expansions at } \epsilon = 0}$$

Regular singular point at $\epsilon = 0$ if:

$\epsilon = 0$ not an ordinary point; i.e., one of the above coefficients for u' and u does not have convergent Taylor expansions at $\epsilon = 0$, but both of the following do (ϵ times the first, ϵ^2 times the second):

$$\boxed{\frac{\epsilon}{P(\frac{1}{\epsilon})}\left[\frac{2P(\frac{1}{\epsilon})}{\epsilon} - \frac{Q(\frac{1}{\epsilon})}{\epsilon^2}\right] \text{ and } \frac{R(\frac{1}{\epsilon})}{\epsilon^2 P(\frac{1}{\epsilon})}}$$

33.

Let $\epsilon = \frac{1}{x}$, $u(\epsilon(x)) = u(\frac{1}{x}) = y(x)$

Here, $P(x) = 1$, $Q(x) = 0$, $R(x) = 1$

$\therefore P(\frac{1}{\epsilon}) = 1$, $Q(\frac{1}{\epsilon}) = 0$, $R(\frac{1}{\epsilon}) = 1$

$\therefore y'' + y = 0$ becomes

$$u'' + \frac{2\epsilon^3}{\epsilon^4}u' + \frac{1}{\epsilon^4}u = 0, \text{ or } u'' + \frac{2}{\epsilon}u' + \frac{1}{\epsilon^4}u = 0$$

Neither $\frac{2}{\epsilon}$ nor $\frac{1}{\epsilon^4}$ is analytic at $\epsilon = 0$.

$\therefore\ \epsilon = 0$ is a singular point.

$\epsilon\left(\frac{2}{\epsilon}\right) = 2$ is analytic at $\epsilon = 0$

But $\epsilon^2\left(\frac{1}{\epsilon^4}\right) = \frac{1}{\epsilon^2}$ is <u>not</u>.

$\therefore\ \epsilon = 0$ or $x = \infty$ is an <u>irregular singular point</u>.

34.

Let $\epsilon = \frac{1}{x}$, $u\left(\frac{1}{x}\right) = y(x)$

$\therefore\ P(x) = x^2,\ Q(x) = x,\ R(x) = -4$

$\therefore\ P\left(\frac{1}{\epsilon}\right) = \frac{1}{\epsilon^2},\ Q\left(\frac{1}{\epsilon}\right) = \frac{1}{\epsilon},\ R\left(\frac{1}{\epsilon}\right) = -4$

$\therefore\ x^2y'' + xy' - 4y = 0$ becomes:

$$u'' + \frac{2\epsilon^3 P\left(\frac{1}{\epsilon}\right) - \epsilon^2 Q\left(\frac{1}{\epsilon}\right)}{\epsilon^4 P\left(\frac{1}{\epsilon}\right)} u' + \frac{R\left(\frac{1}{\epsilon}\right)}{\epsilon^4 P\left(\frac{1}{\epsilon}\right)} u = 0$$

Or, $$u'' + \frac{2\epsilon^3\left(\frac{1}{\epsilon^2}\right) - \epsilon^2\left(\frac{1}{\epsilon}\right)}{\epsilon^4\left(\frac{1}{\epsilon^2}\right)} u' + \frac{-4}{\epsilon^4\left(\frac{1}{\epsilon^2}\right)} u = 0$$

Or, $$u'' + \frac{1}{\epsilon} u' - \frac{4}{\epsilon^2} u = 0$$

$\therefore\ p(\epsilon)$ and $q(\epsilon)$ not analytic at $\epsilon = 0$.

But $\epsilon\left(\frac{1}{\epsilon}\right) = 1$ and $\epsilon^2\left(-\frac{4}{\epsilon^2}\right) = -4$ are both analytic at $\epsilon = 0$. $\therefore$ $\epsilon = 0$ or $x = \infty$ is

a regular singular point.

35.

Let $\epsilon = \frac{1}{x}$, $u(\epsilon(x)) = u(\frac{1}{x}) = y(x)$

$P(x) = 1 - x^2$, $Q(x) = -2x$, $R(x) = \alpha(\alpha+1)$

$\therefore P(\frac{1}{\epsilon}) = 1 - \frac{1}{\epsilon^2} \quad Q(\frac{1}{\epsilon}) = -\frac{2}{\epsilon} \quad R(\frac{1}{\epsilon}) = \alpha(\alpha+1)$

$\therefore (1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0$ becomes:

$$u'' + \frac{2\epsilon^3 P(\frac{1}{\epsilon}) - \epsilon^2 Q(\frac{1}{\epsilon})}{\epsilon^4 P(\frac{1}{\epsilon})} u' + \frac{R(\frac{1}{\epsilon})}{\epsilon^4 P(\frac{1}{\epsilon})} u = 0$$

Or, $$u'' + \frac{2\epsilon^3(1 - \frac{1}{\epsilon^2}) - \epsilon^2(-\frac{2}{\epsilon})}{\epsilon^4(1 - \frac{1}{\epsilon^2})} u' + \frac{\alpha(\alpha+1)}{\epsilon^4(1-\frac{1}{\epsilon^2})} u = 0$$

Or, $$u'' + \frac{2\epsilon}{\epsilon^2 - 1} u' + \frac{\alpha(\alpha+1)}{\epsilon^4 - \epsilon^2} u = 0$$

Since $\lim_{\epsilon \to 0} \left| \frac{\alpha(\alpha+1)}{\epsilon^4 - \epsilon^2} \right| = \infty$ for $\alpha \neq 0$, not ordinary

$\therefore$ look at $\lim_{\epsilon \to 0} \epsilon \left(\frac{2\epsilon}{\epsilon^2 - 1} \right) = 2$

and $\lim_{\epsilon \to 0} \epsilon^2 \dfrac{\alpha(\alpha+1)}{\epsilon^4 - \epsilon^2} = \lim_{\epsilon \to 0} \dfrac{\alpha(\alpha+1)}{\epsilon^2 - 1} = -\alpha(\alpha+1)$

$\therefore$ $\epsilon = 0$ or $x = \infty$ is a regular singular point

36.

Let $\epsilon = \frac{1}{x}$, $u(\epsilon(x)) = u(\frac{1}{x}) = y(x)$

$P(x) = 1 \quad Q(x) = -2x \quad R(x) = \lambda$

$P(\frac{1}{\epsilon}) = 1 \quad Q(\frac{1}{\epsilon}) = -\frac{2}{\epsilon} \quad R(\frac{1}{\epsilon}) = \lambda$

$\therefore$ $y'' - 2xy' + \lambda y = 0$ becomes:

$$u'' + \frac{2\epsilon^3 P(\frac{1}{\epsilon}) - \epsilon^2 Q(\frac{1}{\epsilon})}{\epsilon^4 P(\frac{1}{\epsilon})} u' + \frac{R(\frac{1}{\epsilon})}{\epsilon^4 P(\frac{1}{\epsilon})} u = 0$$

Or, $u'' + \dfrac{2\epsilon^3(1) - \epsilon^2(-\frac{2}{\epsilon})}{\epsilon^4(1)} u' + \dfrac{\lambda}{\epsilon^4(1)} u = 0$

Or, $u'' + \dfrac{2\epsilon^2 + 2}{\epsilon^3} u' + \dfrac{\lambda}{\epsilon^4} u = 0$

$\lim_{\epsilon \to 0} \dfrac{\lambda}{\epsilon^4} = \infty$, $\therefore$ not ordinary

$\lim_{\epsilon \to 0} \epsilon \left(\dfrac{2\epsilon^2 + 2}{\epsilon^3}\right) = \infty \qquad \lim_{\epsilon \to 0} \epsilon^2 \left(\dfrac{\lambda}{\epsilon^4}\right) = \infty$

$\therefore$ $\epsilon = 0$ or $x = \infty$ is an <u>irregular</u> singular point

37.

Let $\epsilon = \frac{1}{x}$, $u(\epsilon(x)) = u(\frac{1}{x}) = y(x)$

$P(x) = 1 \quad Q(x) = 0 \quad R(x) = -x$

$P(\frac{1}{\epsilon}) = 1 \quad Q(\frac{1}{\epsilon}) = 0 \quad R(\frac{1}{\epsilon}) = -\frac{1}{\epsilon}$

$\therefore$ $y'' - xy = 0$ becomes:

$$u'' + \frac{2\epsilon^3 P(\frac{1}{\epsilon}) - \epsilon^2 Q(\frac{1}{\epsilon})}{\epsilon^4 P(\frac{1}{\epsilon})} u' + \frac{R(\frac{1}{\epsilon})}{\epsilon^4 P(\frac{1}{\epsilon})} u = 0$$

Or,
$$u'' + \frac{2\epsilon^3(1) - \epsilon^2(0)}{\epsilon^4(1)} u' + \frac{-\frac{1}{\epsilon}}{\epsilon^4(1)} u = 0$$

Or, $u'' + \frac{2}{\epsilon} u' - \frac{1}{\epsilon^5} u = 0$

$\therefore$ Not ordinary as $\lim\limits_{\epsilon \to 0} \frac{1}{\epsilon^5} = \infty$

$\lim\limits_{\epsilon \to 0} \epsilon\left(\frac{2}{\epsilon}\right) = 2$, but $\lim\limits_{\epsilon \to 0} \epsilon^2\left(-\frac{1}{\epsilon^5}\right) = \infty$

$\therefore$ $\epsilon = 0$ or $x = \infty$ is an <u>irregular</u> singular point

5.5 Series Solutions Near a Regular Singular Point, Part I

Note Title 4/15/2019

1.

(a) Standard form: $y'' + \frac{1}{2x}y' + \frac{1}{2}y = 0$

$x=0$ is a singular point.

$\lim_{x\to 0} x\left(\frac{1}{2x}\right) = \frac{1}{2}$ $\quad \lim_{x\to 0} x^2\left(\frac{1}{2}\right) = 0$, both finite

$\therefore$ $x=0$ is a regular singular point.

(b) Let $y = \sum_{n=0}^{\infty} a_n x^{r+n}$ $\quad xy = \sum_{n=0}^{\infty} a_n x^{r+n+1}$

$\therefore y' = \sum_{n=0}^{\infty} (r+n)a_n x^{r+n-1}$

$= r a_0 x^{r-1} + (r+1)a_1 x^r + \sum_{n=2}^{\infty} (r+n)a_n x^{r+n-1}$

$\downarrow$ shift index

$= r a_0 x^{r-1} + (r+1)a_1 x^r + \sum_{n=0}^{\infty} (r+n+2)a_{n+2} x^{r+n+1}$

$$y'' = \sum_{n=0}^{\infty} (r+n)(r+n-1) a_n x^{r+n-2}$$

$$2xy'' = \sum_{n=0}^{\infty} 2(r+n)(r+n-1) a_n x^{r+n-1}$$

$$= 2r(r-1)a_0 x^{r-1} + 2(r+1)r a_1 x^{r} + \sum_{n=2}^{\infty} 2(r+n)(r+n-1) a_n x^{r+n-1}$$

$\downarrow$ shift index

$$= 2r(r-1)a_0 x^{r-1} + 2(r+1)r a_1 x^{r} + \sum_{n=0}^{\infty} 2(r+n+2)(r+n+1) a_{n+2} x^{r+n+1}$$

$$\therefore 2xy'' + y' + xy = 0 \Rightarrow$$

$$a_0 x^{r-1}\left[2r(r-1) + r\right] + a_1 x^{r}\left[2(r+1) + (r+1)\right] +$$

$$\sum_{n=0}^{\infty} \left(\left[2(r+n+2)(r+n+1) + (r+n+2)\right] a_{n+2} + a_n \right) x^{r+n+1} = 0$$

$\therefore$ For all $x > 0$, $a_0 \neq 0$,

$$2r(r-1) + r = 2r^2 - r = r\underline{(2r-1) = 0}, \quad \underline{r = 0, \tfrac{1}{2}}$$

For a_1: $a_1\left[2(r+1) + (r+1)\right] = 0 \Rightarrow a_1 = 0$

$$\left[2(r+n+2)(r+n+1) + (r+n+2)\right] a_{n+2} + a_n = 0, \text{ or}$$

$$(r+n+2)\left[2r + 2n + 2 + 1\right] a_{n+2} + a_n = 0, \text{ or}$$

$$a_{n+2} = \frac{-a_n}{(r+n+2)\left[2r+2(n+2)-1\right]}, \quad n \geq 0$$

$$\text{or } \underline{a_n = \frac{-a_{n-2}}{(r+n)(2r+2n-1)}}, \quad n \geq 2$$

(every odd term is zero)

(c) With $r = \frac{1}{2}$

$$a_n = \frac{-a_{n-2}}{\left[\frac{2n+1}{2}\right]\left[2n\right]} = -\frac{a_{n-2}}{n(2n+1)}, \quad n \geq 2$$

$$\therefore \; a_2 = a_0\left[\frac{-1}{2(5)}\right] = a_0\left(-\frac{1}{10}\right)$$

$$a_4 = a_0\left(\frac{1}{2\cdot 5}\right)\frac{1}{4(9)} = a_0\frac{1}{(2\cdot 4)\cdot(5\cdot 9)}$$

$$a_6 = a_0\left(\frac{1}{2\cdot 4\cdot 5\cdot 9}\right)\left(-\frac{1}{6\cdot 13}\right) = a_0\frac{(-1)}{2\cdot 4\cdot 6\cdot 5\cdot 9\cdot 13}$$

$$\therefore \; a_{2n} = a_0\frac{(-1)^n}{(2\cdot 4\cdot 6\cdots 2n)(5\cdot 9\cdot 13\cdots(4n+1))}, \quad n \geq 1$$

$$= a_0\frac{(-1)^n}{2^n(n!)\left[5\cdot 9\cdot 13\cdots(4n+1)\right]}, \quad n \geq 1$$

$$\therefore y = x^r \sum_{n=0}^{\infty} a_n x^n = x^{\frac{1}{2}} \left[a_0 + \sum_{n=1}^{\infty} a_{2n} x^{2n} \right]$$

$$= x^{\frac{1}{2}} a_0 \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^n (n!) [5 \cdot 9 \cdot 13 \cdots (4n+1)]} \right]$$

Neglecting the a_0 (setting $a_0 = 1$)

$$\boxed{y = x^{\frac{1}{2}} \left[1 - \frac{x^2}{2 \cdot 5} + \frac{x^4}{2 \cdot 4 \cdot 5 \cdot 9} - \cdots + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^n (n!) [5 \cdot 9 \cdots (4n+1)]} \right]}$$

(d) $r = 0 \Rightarrow$ let $y = \sum_{n=0}^{\infty} a_n x^n \quad \therefore xy = \sum_{n=0}^{\infty} a_n x^{n+1}$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + \sum_{n=2}^{\infty} n a_n x^{n-1} = a_1 + \sum_{n=0}^{\infty} (n+2) a_{n+2} x^{n+1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \quad \therefore \quad 2xy'' = \sum_{n=2}^{\infty} 2n(n-1) a_n x^{n-1}$$

$$= \sum_{n=0}^{\infty} 2(n+2)(n+1) a_{n+2} x^{n+1}$$

$\therefore$ $2xy'' + y' + xy = 0 \Rightarrow$

$$a_1 + \sum_{n=0}^{\infty} \left[2(n+2)(n+1) a_{n+2} + (n+2) a_{n+2} + a_n \right] x^{n+1} = 0$$

$\therefore$ For all $x > 0$, $a_1 = 0$

$$[2(n+2)(n+1)+(n+2)]a_{n+2}+a_n=0, \quad n\geq 0$$

$$a_{n+2}=\frac{-a_n}{(n+2)(2n+3)}, \quad n\geq 0$$

$$\therefore a_2=\frac{-a_0}{2\cdot 3} \qquad a_4=\frac{1}{2\cdot 4\cdot 3\cdot 7}a_0$$

$$a_6=\frac{-1}{2\cdot 4\cdot 6\cdot 3\cdot 7\cdot 11} \qquad \therefore a_{2n}=\frac{(-1)^n}{(2\cdot 4\cdots 2n)(3\cdot 7\cdot 11\cdots(4n-1))}a_0$$

Or, $$a_{2n}=\frac{(-1)^n}{2^n(n!)[3\cdot 7\cdot 11\cdots(4n-1)]}a_0, \quad n\geq 1$$

Setting $a_0=1$,

$$\boxed{y=1-\frac{x^2}{2\cdot 3}+\frac{x^4}{2\cdot 4\cdot 3\cdot 7}-\cdots+\sum_{n=1}^{\infty}\frac{(-1)^n x^{2n}}{2^n(n!)[3\cdot 7\cdot 11\cdots(4n-1)]}}$$

2.

(a)

Standard form: $y''+\left(\frac{1}{x}\right)y'+\left(1-\frac{1}{9x^2}\right)y=0$

$\therefore x=0$ is a singular point

$\lim_{x\to 0} x\left(\frac{1}{x}\right)=1, \quad \lim_{x\to 0} x^2\left(1-\frac{1}{9x^2}\right)=-\frac{1}{9}$, both finite

$\therefore\ x=0$ is a regular singular point

(6)

Let $y=\sum_{n=0}^{\infty} a_n x^{r+n}$

$$\therefore\ \left(x^2-\frac{1}{9}\right)y=\sum_{n=0}^{\infty} a_n x^{r+n+2}-\sum_{n=0}^{\infty}\frac{a_n}{9}x^{r+n}$$

$$=-\frac{a_0}{9}x^r-\frac{a_1}{9}x^{r+1}+\sum_{n=0}^{\infty} a_n x^{r+n+2}-\sum_{n=2}^{\infty}\frac{a_n}{9}x^{r+n}$$

shift index

$$=-\frac{a_0}{9}x^r-\frac{a_1}{9}x^{r+1}+\sum_{n=0}^{\infty} a_n x^{r+n+2}-\sum_{n=0}^{\infty}\frac{a_{n+2}}{9}x^{r+n+2}$$

$$=-\frac{a_0}{9}x^r-\frac{a_1}{9}x^{r+1}+\sum_{n=0}^{\infty}\left(a_n-\frac{a_{n+2}}{9}\right)x^{r+n+2}$$

$$\therefore\ y'=\sum_{n=0}^{\infty}(r+n)a_n x^{r+n-1}$$

$$\therefore\ xy'=\sum_{n=0}^{\infty}(r+n)a_n x^{r+n}=ra_0x^r+(r+1)a_1x^{r+1}+\sum_{n=2}^{\infty}(r+n)a_n x^{r+n}$$

shift index

$$=ra_0x^r+(r+1)a_1x^{r+1}+\sum_{n=0}^{\infty}(r+n+2)a_{n+2}x^{r+n+2}$$

$$y''=\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n x^{r+n-2}$$

$$x^2 y'' = \sum_{n=0}^{\infty} (r+n)(r+n-1)\, a_n\, x^{r+n}$$

$$= r(r-1)a_0 x^r + (r+1)r\, a_1 x^{r+1} + \sum_{n=2}^{\infty} (r+n)(r+n-1) a_n\, x^{r+n}$$

$\downarrow$ shift index

$$= r(r-1)a_0 x^r + r(r+1)\, a_1\, x^{r+1} + \sum_{n=0}^{\infty} (r+n+2)(r+n+1) a_{n+2}\, x^{r+n+2}$$

$$\therefore\ x^2 y'' + x y' + \left(x^2 - \tfrac{1}{9}\right) y = 0 \Rightarrow$$

$$a_0 x^r \left[-\tfrac{1}{9} + r + r(r-1)\right] + a_1 x^{r+1}\left[-\tfrac{1}{9} + (r+1) + r(r+1)\right]$$

$$+ \sum_{n=0}^{\infty} \left[(r+n+2)(r+n+1) a_{n+2} + (r+n+2) a_{n+2} + \left(a_n - \frac{a_{n+2}}{9}\right)\right] x^{r+n+2} = 0$$

$\therefore$ For all $x > 0$, $a_0 \neq 0$,

$$-\tfrac{1}{9} + r + r(r-1) = \underline{r^2 - \tfrac{1}{9} = 0}, \quad \underline{r = \pm \tfrac{1}{3}}$$

Note for a_1: $a_1\left[-\tfrac{1}{9} + (r+1) + r(r+1)\right] = 0$, or

$$a_1\left[r^2 + 2r + \tfrac{8}{9}\right] = 0, \text{ for } r = \pm\tfrac{1}{3},\ \underline{a_1 = 0}$$

Also, $\left[(r+n+2)(r+n+1) a_{n+2} + (r+n+2) a_{n+2} + \left(a_n - \frac{a_{n+2}}{9}\right)\right] = 0$

$$\Rightarrow (r+n+2)(r+n+2)\, a_{n+2} + a_n - \frac{a_{n+2}}{9} = 0, \quad n \geq 0$$

$$\therefore\ a_{n+2}\left[(r+n+2)^2-\frac{1}{9}\right]+a_n=0\ ,\ n\geq 0$$

$$\text{Or,}\ a_{n+2}=\frac{-a_n}{(r+n+2)^2-\frac{1}{9}}\ ,\ n\geq 0$$

$$\text{Or,}\ a_n=\frac{-a_{n-2}}{(r+n)^2-\frac{1}{9}}\ ,\ n\geq 2$$

Since $a_1=0$, every odd term is 0.

(c)

$r=\frac{1}{3}$

$$\therefore\ a_2=\frac{-a_0}{\left(\frac{1}{3}+2\right)^2-\frac{1}{9}}=\frac{-a_0}{\left(2+\frac{1}{3}\right)^2-\left(\frac{1}{3}\right)^2}=\frac{-a_0}{\left(2+\frac{2}{3}\right)(2)}$$

$$a_4=\frac{-a_2}{\left(4+\frac{1}{3}\right)^2-\left(\frac{1}{3}\right)^2}=\frac{-a_2}{\left(4+\frac{2}{3}\right)(4)}=\frac{a_0}{\left(2+\frac{2}{3}\right)\left(4+\frac{2}{3}\right)2\cdot 4}$$

$$a_6=\frac{-a_4}{\left(6+\frac{1}{3}\right)^2-\left(\frac{1}{3}\right)^2}=\frac{-a_4}{\left(6+\frac{2}{3}\right)(6)}=\frac{-a_0}{\left(2+\frac{2}{3}\right)\left(4+\frac{2}{3}\right)\left(6+\frac{2}{3}\right)2\cdot 4\cdot 6}$$

$$\therefore\ a_{2n}=\frac{(-1)^n a_0}{\left(2+\frac{2}{3}\right)\left(4+\frac{2}{3}\right)\cdots\left(2n+\frac{2}{3}\right)\cdot 2\cdot 4\cdots(2n)}=\frac{(-1)^n a_0}{2^n\left(1+\frac{1}{3}\right)\left(2+\frac{1}{3}\right)\cdots\left(n+\frac{1}{3}\right)\cdot 2^n(n!)}$$

for $n\geq 1$

$$\therefore y = x^r \sum_{n=0}^{\infty} a_n x^n$$

$$= x^{\frac{1}{3}} a_0 \left[1 - \frac{x^2}{2(1+\frac{1}{3})(2)} + \frac{x^4}{2^2(1+\frac{1}{3})(2+\frac{1}{3}) \cdot 2^2 (1 \cdot 2)} - \cdots \right]$$

Neglecting the a_0 constant (set $a_0 = 1$), and noting $\frac{x^{2n}}{2^n \cdot 2^n} = \frac{x^{2n}}{2^{2n}} = \left(\frac{x}{2}\right)^{2n}$,

$$\boxed{y = x^{\frac{1}{3}} \left[1 - \frac{1}{(1+\frac{1}{3})} \left(\frac{x}{2}\right)^2 + \frac{1}{(1+\frac{1}{3})(2+\frac{1}{3})2!} \left(\frac{x}{2}\right)^4 - \cdots + \frac{(-1)^n}{(1+\frac{1}{3})(2+\frac{1}{3})\cdots(n+\frac{1}{3})n!} \left(\frac{x}{2}\right)^{2n} + \cdots \right]}$$

(d)

$r = -\frac{1}{3}$ From (b), $a_n = \frac{-a_{n-2}}{(r+n)^2 - \frac{1}{9}}$, $n \geq 2$

$$\therefore a_n = \frac{-a_{n-2}}{(n-\frac{1}{3})^2 - (\frac{1}{3})^2} = \frac{-a_{n-2}}{n(n-\frac{2}{3})}$$

$$\therefore a_2 = \frac{-a_0}{2(2-\frac{2}{3})} \qquad a_4 = \frac{-a_2}{4(4-\frac{2}{3})} = \frac{a_0}{2 \cdot 4 \cdot (2-\frac{2}{3})(4-\frac{2}{3})}$$

$$\therefore\ a_{2n} = \frac{(-1)^n a_0}{2\cdot 4\cdots(2n)(2-\frac{2}{3})(4-\frac{2}{3})\cdots(2n-\frac{2}{3})}$$

$$= \frac{(-1)^n a_0}{2^n(n!)\,(1-\frac{1}{3})(2-\frac{1}{3})\cdots(n-\frac{1}{3})2^n}$$

Again noting $\dfrac{x^{2n}}{2^n\cdot 2^n} = \dfrac{x^{2n}}{2^{2n}} = \left(\dfrac{x}{2}\right)^{2n}$

and setting $a_0 = 1$, $y(x) = x^r \sum_{n=0}^{\infty} a_n x^n$

$$\boxed{y = x^{-\frac{1}{3}}\left[1 - \frac{1}{(1-\frac{1}{3})}\left(\frac{x}{2}\right)^2 + \frac{1}{2!(1-\frac{1}{3})(2-\frac{1}{3})}\left(\frac{x}{2}\right)^4 - \dots + \frac{(-1)^n}{n!\,(1-\frac{1}{3})(2-\frac{1}{3})\cdots(n-\frac{1}{3})}\left(\frac{x}{2}\right)^{2n} + \dots\right.}$$

3.

(a)

Standard form: $y'' + \frac{1}{x}y = 0$. $\therefore\ x=0$ a singular point.

$\lim_{x\to 0} x^2\left(\frac{1}{x}\right) = 0$, finite.

$\therefore\ x=0$ is a regular singular point.

(b)

Let $y = \sum_{n=0}^{\infty} a_n x^{r+n}$

$$\therefore y'' = \sum_{n=0}^{\infty} (r+n)(r+n-1)\, a_n\, x^{r+n-2}$$

$$\therefore x y'' = \sum_{n=0}^{\infty} (r+n)(r+n-1)\, a_n\, x^{r+n-1}$$

$$= r(r-1) a_0 x^{r-1} + \sum_{n=1}^{\infty} (r+n)(r+n-1)\, a_n\, x^{r+n-1}$$

↓ shift index

$$= r(r-1) a_0 x^{r-1} + \sum_{n=0}^{\infty} (r+n+1)(r+n)\, a_{n+1}\, x^{r+n}$$

$\therefore xy'' + y = 0 \Rightarrow$

$$r(r-1) a_0 x^{r-1} + \sum_{n=0}^{\infty} \left[(r+n+1)(r+n) a_{n+1} + a_n \right] x^{r+n} = 0$$

$\therefore$ For all $x > 0$, $a_0 \neq 0$,

$$\underline{r(r-1) = 0}, \quad \underline{r = 0, 1}$$

$$\underline{a_{n+1} = \frac{-a_n}{(r+n+1)(r+n)}, \quad n \geq 0}$$

$$\text{Or, } a_n = \frac{-a_{n-1}}{(r+n)(r+n-1)}, \quad n \geq 1$$

(c)

$$r=1: \quad a_n = \frac{-a_{n-1}}{(n+1)n}, \quad n \geq 1$$

$$\therefore a_1 = \frac{-a_0}{2 \cdot 1} \qquad a_2 = \frac{-a_1}{3 \cdot 2} = \frac{a_0}{3 \cdot 2 \cdot 2 \cdot 1}$$

$$a_3 = \frac{-a_2}{4 \cdot 3} = -\frac{a_0}{4 \cdot 3 \cdot 3 \cdot 2 \cdot 2 \cdot 1}$$

$$\therefore a_n = \frac{(-1)^n a_0}{(n+1)!\, n!}$$

Setting $a_0 = 1$,

$$\boxed{y = x\left[1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^3}{144} + \dots + \frac{(-1)^n}{(n+1)!\, n!} x^n + \dots\right]}$$

(d)

The roots $r = 0, 1$ <u>do</u> differ by an integer.

4.

(a)

Standard form: $y'' + \frac{1}{x} y' - \frac{1}{x} y = 0$

$\therefore x = 0$ is a singular point

$\lim_{x \to 0} x\left(\frac{1}{x}\right) = 1$, $\lim_{x \to 0} x^2\left(\frac{1}{x}\right) = 0$, both finite

$\therefore x = 0$ is a regular singular point.

(b)

Let $y = \sum_{n=0}^{\infty} a_n x^{r+n}$

$$\therefore y' = \sum_{n=0}^{\infty} (r+n) a_n x^{r+n-1} = r a_0 x^{r-1} + \sum_{n=1}^{\infty} (r+n) a_n x^{r+n-1}$$

shift index

$$= r a_0 x^{r-1} + \sum_{n=0}^{\infty} (r+n+1) a_{n+1} x^{r+n}$$

$$y'' = \sum_{n=0}^{\infty} (r+n)(r+n-1) a_n x^{r+n-2}$$

$$\therefore xy'' = \sum_{n=0}^{\infty} (r+n)(r+n-1) a_n x^{r+n-1}$$

$$= r(r-1) a_0 x^{r-1} + \sum_{n=1}^{\infty} (r+n)(r+n-1) a_n x^{r+n-1}$$

shift index

$$= r(r-1) a_0 x^{r-1} + \sum_{n=0}^{\infty} (r+n+1)(r+n) a_{n+1} x^{r+n}$$

$\therefore\ xy'' + y' - y = 0 \Rightarrow$

$$[r(r-1)+r]a_0 x^{r-1} + \sum_{n=0}^{\infty}\left[(r+n+1)^2 a_{n+1} - a_n\right]x^{r+n} = 0$$

$\therefore$ For all $x>0$, $a_0 \neq 0$,

$$r(r-1)+r = \underline{r^2 = 0}, \quad \underline{r = 0, 0}$$

$$(r+n+1)^2 a_{n+1} - a_n = 0, \text{ or}$$

$$a_{n+1} = \frac{a_n}{(r+n+1)^2}, \text{ or } \underline{a_n = \frac{a_{n-1}}{(r+n)^2}}, \ n \geq 1$$

(c)

$$r=0: \quad a_n = \frac{a_{n-1}}{n^2}, \ n \geq 1$$

$$\therefore\ a_1 = \frac{a_0}{1} \quad a_2 = \frac{a_1}{2^2} = \frac{a_0}{2^2 \cdot 1^2} \quad a_3 = \frac{a_2}{3^2} = \frac{a_0}{3^2 \cdot 2^2 \cdot 1^2}$$

$$\therefore\ a_n = \frac{a_0}{(n!)^2}$$

Setting $a_0 = 1$, $x^r = 1$,

$$\boxed{y = 1 + x + \frac{x^2}{2^2} + \frac{x^3}{2^2 \cdot 3^2} + \dots + \frac{x^n}{(n!)^2} + \dots}$$

(d)

The roots, $r = 0, 0$, are equal.

5.

(a)

Standard form: $y'' + \frac{1}{x} y' + \frac{x-2}{x^2} y = 0$. $\therefore x = 0$ is a singular point.

$\lim_{x \to 0} x\left(\frac{1}{x}\right) = 1$, $\lim_{x \to 0} x^2\left(\frac{x-2}{x^2}\right) = -2$, both finite.

$\therefore x = 0$ is a regular singular point.

(b)

Let $y = \sum_{n=0}^{\infty} a_n x^{r+n}$

$$\therefore (x-2)y = \sum_{n=0}^{\infty} a_n x^{r+n+1} - \sum_{n=0}^{\infty} 2a_n x^{r+n}$$

$$= -2a_0 x^r + \sum_{n=0}^{\infty} a_n x^{r+n+1} - \sum_{n=1}^{\infty} 2a_n x^{r+n}$$

shift index

$$= -2a_0 x^r + \sum_{n=0}^{\infty} a_n x^{r+n+1} - \sum_{n=0}^{\infty} 2a_{n+1} x^{r+n+1}$$

$$= -2a_0 x^r + \sum_{n=0}^{\infty} (a_n - 2a_{n+1}) x^{r+n+1}$$

$$y' = \sum_{n=0}^{\infty} (r+n) a_n x^{r+n-1}$$

$$\therefore xy' = \sum_{n=0}^{\infty} (r+n) a_n x^{r+n} = r a_0 x^r + \sum_{n=1}^{\infty} (r+n) a_n x^{r+n}$$

shift index

$$= r a_0 x^r + \sum_{n=0}^{\infty} (r+n+1) a_{n+1} x^{r+n+1}$$

$$y'' = \sum_{n=0}^{\infty} (r+n)(r+n-1) a_n x^{r+n-2}$$

$$\therefore x^2 y'' = \sum_{n=0}^{\infty} (r+n)(r+n-1) a_n x^{r+n}$$

$$= r(r-1) a_0 x^r + \sum_{n=1}^{\infty} (r+n)(r+n-1) a_n x^{r+n}$$

shift index

$$= r(r-1) a_0 x^r + \sum_{n=0}^{\infty} (r+n+1)(r+n) a_{n+1} x^{r+n+1}$$

$$\therefore x^2 y'' + xy' + (x-2)y = 0 \Rightarrow$$

$$[r(r-1) + r - 2] a_0 x^r +$$

$$\sum_{n=0}^{\infty} \left[(r+n+1)(r+n) a_{n+1} + (r+n+1) a_{n+1} - 2a_{n+1} + a_n \right] x^{r+n+1} = 0$$

$$\Rightarrow (r^2-2)a_0 x^r + \sum_{n=0}^{\infty}\left(\left[(r+n+1)^2-2\right]a_{n+1} + a_n\right) x^{r+n+1} = 0$$

$\therefore$ For all $x > 0$, $a_0 \neq 0$,

$$\underline{r^2 - 2 = 0}, \quad \underline{r = \pm\sqrt{2}}, \quad \left[(r+n+1)^2 - 2\right]a_{n+1} + a_n = 0$$

$$\therefore \quad a_{n+1} = -\frac{a_n}{(r+n+1)^2 - 2}, \quad n \geq 0$$

Or, $\underline{a_n = -\dfrac{a_{n-1}}{(r+n)^2 - 2}}$, $n \geq 1$

(c)

$$r = \sqrt{2}: \quad a_n = -\frac{a_{n-1}}{(n+\sqrt{2})^2 - 2} = \frac{-a_{n-1}}{(n+\sqrt{2})^2 - (\sqrt{2})^2} = \underline{\frac{-a_{n-1}}{n(n+2\sqrt{2})}}, \quad n \geq 1$$

$$a_1 = \frac{-a_0}{1(1+2\sqrt{2})} \qquad a_2 = \frac{-a_1}{2(2+2\sqrt{2})} = \frac{a_0}{2\cdot 1\cdot(1+2\sqrt{2})(2+2\sqrt{2})}$$

$$a_3 = \frac{-a_2}{3(3+2\sqrt{2})} = \frac{-a_0}{3\cdot 2\cdot 1(1+2\sqrt{2})(2+2\sqrt{2})(3+2\sqrt{2})}$$

$$\therefore a_n = \frac{(-1)^n a_0}{n!(1+2\sqrt{2})(2+2\sqrt{2})\cdots(n+2\sqrt{2})}$$

$\therefore$ Setting $a_0 = 1$, $y = x^r \sum_{n=0}^{\infty} a_n x^n$,

$$\boxed{y = x^{\sqrt{2}}\left[1 - \frac{x}{(1+2\sqrt{2})} + \frac{x^2}{2!(1+2\sqrt{2})(2+2\sqrt{2})} - \cdots + \frac{(-1)^n x^n}{n!(1+2\sqrt{2})(2+2\sqrt{2})\cdots(n+2\sqrt{2})} + \cdots\right]}$$

(d)

$$r = -\sqrt{2}: \quad a_n = \frac{-a_{n-1}}{(n-\sqrt{2})^2 - (\sqrt{2})^2} = \frac{-a_{n-1}}{n(n-2\sqrt{2})}, \quad n \geq 1$$

$$\therefore a_1 = \frac{-a_0}{1\cdot(1-2\sqrt{2})} \qquad a_2 = \frac{-a_1}{2(2-2\sqrt{2})} = \frac{a_0}{2\cdot1(1-2\sqrt{2})(2-2\sqrt{2})}$$

$$a_3 = \frac{-a_2}{3(3-2\sqrt{2})} = \frac{-a_0}{3\cdot2\cdot1(1-2\sqrt{2})(2-2\sqrt{2})(3-2\sqrt{2})}$$

$$\therefore a_n = \frac{(-1)^n a_0}{n!(1-2\sqrt{2})(2-2\sqrt{2})\cdots(n-2\sqrt{2})}$$

Setting $a_0 = 1$, $y = x^r \sum_{n=0}^{\infty} a_n x^n$,

$$\boxed{y = x^{-\sqrt{2}}\left[1 - \frac{x}{(1-2\sqrt{2})} + \frac{x^2}{2\cdot1(1-2\sqrt{2})(2-2\sqrt{2})} - \frac{x^3}{3!(1-2\sqrt{2})(2-2\sqrt{2})(3-2\sqrt{2})} + \cdots \frac{(-1)^n x^n}{n!(1-2\sqrt{2})(2-2\sqrt{2})\cdots(n-2\sqrt{2})} + \cdots\right]}$$

6.

(a)

Standard form: $y'' + \left(\frac{1}{x} - 1\right)y' - \left(\frac{1}{x}\right)y = 0.$

$\therefore$ $x=0$ is a singular point.

$\lim_{x\to 0} x\left(\frac{1}{x}-1\right) = 1$, $\lim_{x\to 0} x^2\left(\frac{1}{x}\right) = 0$, both finite.

$\therefore$ $x=0$ is a regular singular point.

(b)

Let $y = \sum_{n=0}^{\infty} a_n x^{r+n}$

$$y' = \sum_{n=0}^{\infty} (r+n) a_n x^{r+n-1}$$

$$\therefore (1-x)y' = \sum_{n=0}^{\infty} (r+n) a_n x^{r+n-1} - \sum_{n=0}^{\infty} (r+n) a_n x^{r+n}$$

$$= r a_0 x^{r-1} + \sum_{n=1}^{\infty} (r+n) a_n x^{r+n-1} - \sum_{n=0}^{\infty} (r+n) a_n x^{r+n}$$

$\downarrow$ shift index

$$= r a_0 x^{r-1} + \sum_{n=0}^{\infty} (r+n+1) a_{n+1} x^{r+n} - \sum_{n=0}^{\infty} (r+n) a_n x^{r+n}$$

$$= r a_0 x^{r-1} + \sum_{n=0}^{\infty} \left[(r+n+1) a_{n+1} - (r+n) a_n \right] x^{r+n}$$

$$y'' = \sum_{n=0}^{\infty} (r+n)(r+n-1) a_n x^{r+n-2}$$

$$\therefore \; x y'' = \sum_{n=0}^{\infty} (r+n)(r+n-1) a_n x^{r+n-1}$$

$$= r(r-1) a_0 x^{r-1} + \sum_{n=1}^{\infty} (r+n)(r+n-1) a_n x^{r+n-1}$$

$\downarrow$ shift index

$$= r(r-1) a_0 x^{r-1} + \sum_{n=0}^{\infty} (r+n+1)(r+n) a_{n+1} x^{r+n}$$

$$\therefore \; x y'' + (1-x) y' - y = 0 \Rightarrow \left[r(r-1) + r \right] a_0 x^{r-1} +$$

$$\sum_{n=0}^{\infty} \left[(r+n+1)(r+n) a_{n+1} + (r+n+1) a_{n+1} - (r+n) a_n - a_n \right] x^{r+n} = 0$$

$\therefore$ For all $x > 0$, $a_0 \neq 0$,

$$r(r-1) + r = 0 \Rightarrow \underline{r^2 = 0}, \quad \underline{\underline{r = 0, 0}}$$

and $(r+n+1)(r+n) a_{n+1} + (r+n+1) a_{n+1} - (r+n) a_n - a_n = 0$, $n \geq 0$

or $\quad (r+n+1)^2 a_{n+1} - (r+n+1) a_n = 0, \quad n \geq 0$

or $\quad a_{n+1} = \dfrac{a_n}{r+n+1}, \quad n \geq 0$

or $a_n = \dfrac{a_{n-1}}{r+n}$, $n \geq 1$

(c)

$r=0$: $x^r = 1$, $a_n = \dfrac{a_{n-1}}{n}$

$\therefore a_1 = \dfrac{a_0}{1}$, $a_2 = \dfrac{a_1}{2} = \dfrac{a_0}{2\cdot 1}$, $a_3 = \dfrac{a_2}{3} = \dfrac{a_0}{3\cdot 2\cdot 1}$

$\therefore a_n = \dfrac{a_0}{n!}$

Setting $a_0 = 1$, $x^r = 1$,

$$\boxed{y = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots = e^x}$$

(d)

The roots, $r = 0, 0$, are equal.

7.

(a)

Let $y = \sum_{n=0}^{\infty} a_n (x-1)^{r+n}$ $\quad\therefore\ \alpha(\alpha+1)y = \sum_{n=0}^{\infty} \alpha(\alpha+1)a_n (x-1)^{r+n}$

$y' = \sum_{n=0}^{\infty} (r+n)a_n(x-1)^{r+n-1}$ $\qquad x = (x-1)+1$

$$\therefore\ 2xy' = 2(x-1)y' + 2y'$$

$$= \sum_{n=0}^{\infty} 2(r+n)a_n(x-1)^{r+n} + \sum_{n=0}^{\infty} 2(r+n)a_n(x-1)^{r+n-1}$$

$$= 2ra_0(x-1)^{r-1} + \sum_{n=0}^{\infty} 2(r+n)a_n(x-1)^{r+n} + \sum_{n=1}^{\infty} 2(r+n)a_n(x-1)^{r+n-1}$$

shift index

$$= 2ra_0(x-1)^{r-1} + \sum_{n=0}^{\infty} 2(r+n)a_n(x-1)^{r+n} + \sum_{n=0}^{\infty} 2(r+n+1)a_{n+1}(x-1)^{r+n}$$

$$= 2ra_0(x-1)^{r-1} + \sum_{n=0}^{\infty}\left[2(r+n+1)a_{n+1} + 2(r+n)a_n\right]x^{r+n}$$

$$y'' = \sum_{n=0}^{\infty}(r+n)(r+n-1)a_n(x-1)^{r+n-2}$$

Note $1-x^2 = 1-(x-1+1)^2 = 1-[(x-1)^2+2(x-1)+1]$

$$= -(x-1)^2 - 2(x-1)$$

$$\therefore (1-x^2)y'' = -(x-1)^2 y'' - 2(x-1)y''$$

$$= \sum_{n=0}^{\infty} -(r+n)(r+n-1)a_n(x-1)^{r+n} - \sum_{n=0}^{\infty} 2(r+n)(r+n-1)a_n(x-1)^{r+n-1}$$

$$= -2r(r-1)a_0(x-1)^{r-1} - \sum_{n=0}^{\infty}(r+n)(r+n-1)a_n(x-1)^{r+n} - \sum_{n=1}^{\infty} 2(r+n)(r+n-1)a_n(x-1)^{r+n-1}$$

$\downarrow$ shift index

$$= -2r(r-1)a_0(x-1)^{r-1} - \sum_{n=0}^{\infty}(r+n)(r+n-1)a_n(x-1)^{r+n} - \sum_{n=0}^{\infty} 2(r+n+1)(r+n)a_{n+1}(x-1)^{r+n}$$

$$= -2r(r-1)a_0(x-1)^{r-1} - \sum_{n=0}^{\infty}\left[2(r+n+1)(r+n)a_{n+1} + (r+n)(r+n-1)a_n\right]x^{r+n}$$

$\therefore (1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0 \Rightarrow$

$$[-2r(r-1)-2r]a_0(x-1)^{r-1} +$$

$$\sum_{n=0}^{\infty}\Big[-2(r+n+1)(r+n)a_{n+1} - (r+n)(r+n-1)a_n$$

$$-2(r+n+1)a_{n+1} - 2(r+n)a_n + \alpha(\alpha+1)a_n\Big]x^{r+n} = 0$$

$\therefore$ For all $x \neq 1$, $a_0 \neq 0$,

$$-2r(r-1)-2r = -2r^2+2r-2r = 0 \Rightarrow \underline{r^2=0},\ \underline{r=0,0}$$

(b)

From (a), for all $(x-1)>0$,

$$-2(r+n+1)(r+n)a_{n+1} - 2(r+n+1)a_{n+1} - (r+n)(r+n-1)a_n - 2(r+n)a_n + \alpha(\alpha+1)a_n$$

$$= -2(r+n+1)^2 a_{n+1} - \left[(r+n)(r+n+1) - \alpha(\alpha+1)\right]a_n = 0$$

For $r=0$,

$$2(n+1)^2 a_{n+1} + \left[n(n+1) - \alpha(\alpha+1)\right]a_n = 0$$

$$\therefore \quad a_{n+1} = \frac{\left[\alpha(\alpha+1) - n(n+1)\right]}{2(n+1)^2} a_n, \quad n \geq 0$$

$$\text{Or,} \quad a_n = \frac{\left[\alpha(\alpha+1) - (n-1)n\right]}{2n^2} a_{n-1}, \quad n \geq 1$$

$$\therefore \quad a_1 = \frac{\alpha(\alpha+1)}{2\cdot 1^2} a_0 \qquad a_2 = \frac{\alpha(\alpha+1) - 1\cdot 2}{2\cdot 2^2} a_1 = \frac{\alpha(\alpha+1)\left[\alpha(\alpha+1) - 1\cdot 2\right]}{2^2\cdot 1^2\cdot 2^2} a_0$$

$$a_3 = \frac{\alpha(\alpha+1) - 2\cdot 3}{2\cdot 3^2} a_2 = \frac{\alpha(\alpha+1)\left[\alpha(\alpha+1) - 1\cdot 2\right]\left[\alpha(\alpha+1) - 2\cdot 3\right]}{2^3\cdot 1^2\cdot 2^2\cdot 3^2} a_0$$

$$\therefore \quad a_n = \frac{\alpha(\alpha+1)\left[\alpha(\alpha+1) - 1\cdot 2\right]\left[\alpha(\alpha+1) - 2\cdot 3\right]\cdots\left[\alpha(\alpha+1) - (n-1)\cdot n\right]}{2^n (n!)^2} a_0$$

Setting $a_0 = 1$, $(x-1)^r = 1$,

$$\boxed{\begin{aligned} y = 1 &+ \frac{\alpha(\alpha+1)}{2\cdot 1^2}(x-1) + \frac{\alpha(\alpha+1)[\alpha(\alpha+1)-1\cdot 2]}{2^2\cdot 1^2\cdot 2^2}(x-1)^2 + \dots + \\ &\frac{\alpha(\alpha+1)[\alpha(\alpha+1)-1\cdot 2][\alpha(\alpha+1)-2\cdot 3]\cdots[\alpha(\alpha+1)-(n-1)n]}{2^n(n!)^2}(x-1)^n + \dots \end{aligned}}$$

8.

(a)

(1) Standard form: $y'' - \left(\frac{x}{1-x^2}\right)y' + \left(\frac{\alpha^2}{1-x^2}\right)y = 0$

$\therefore x = \pm 1$ is a singular point.

$$\lim_{x\to 1} x\left(-\frac{x}{1-x^2}\right) = \lim_{x\to 1} \frac{-x^2}{1-x^2} = \lim_{x\to 1} \frac{-1}{\frac{1}{x^2}-1} = 0$$

$$\lim_{x\to 1} x^2\left(\frac{\alpha^2}{1-x^2}\right) = \alpha^2 \lim_{x\to 1}\left(\frac{1}{\frac{1}{x^2}-1}\right) = 0$$

$$\lim_{x\to -1} x\left(\frac{-x}{1-x^2}\right) = \lim_{x\to -1}\left(\frac{-1}{\frac{1}{x^2}-1}\right) = 0$$

$$\lim_{x\to -1} x^2\left(\frac{\alpha^2}{1-x^2}\right) = \alpha^2 \lim_{x\to -1}\left(\frac{1}{\frac{1}{x^2}-1}\right) = 0$$

All limits finite. $\therefore x = \pm 1$ are <u>regular</u> singular points.

(2) Let $y = \sum_{n=0}^{\infty} a_n (x-1)^{r+n}$ for $x=1$

$y' = \sum_{n=0}^{\infty} (r+n) a_n (x-1)^{r+n-1}$ Note $x = (x-1)+1$

$\therefore -xy' = [-(x-1)-1] y'$

$= -\sum_{n=0}^{\infty} (r+n) a_n (x-1)^{r+n} - \sum_{n=0}^{\infty} (r+n) a_n (x-1)^{r+n-1}$ separate out $n=0$

$= -r a_0 (x-1)^{r-1} - \sum_{n=0}^{\infty} (r+n) a_n (x-1)^{r+n} - \sum_{n=1}^{\infty} (r+n) a_n (x-1)^{r+n-1}$

↓ shift index

$= -r a_0 (x-1)^{r-1} - \sum_{n=0}^{\infty} (r+n) a_n (x-1)^{r+n} - \sum_{n=0}^{\infty} (r+n+1) a_{n+1} (x-1)^{r+n}$

$= -r a_0 (x-1)^{r-1} - \sum_{n=0}^{\infty} \left[(r+n+1) a_{n+1} + (r+n) a_n \right] (x-1)^{r+n}$

$y'' = \sum_{n=0}^{\infty} (r+n)(r+n-1) a_n (x-1)^{r+n-2}$

Note $1-x^2 = -(x-1)(x+1) = -(x-1)(x-1+2) = -(x-1)^2 - 2(x-1)$

$\therefore (1-x^2) y'' = -(x-1)^2 y'' - 2(x-1) y''$

$= -\sum_{n=0}^{\infty} (r+n)(r+n-1) a_n (x-1)^{r+n} - \sum_{n=0}^{\infty} 2(r+n)(r+n-1) a_n (x-1)^{r+n-1}$ separate out $n=0$

$= -2r(r-1) a_0 (x-1)^{r-1} - \sum_{n=0}^{\infty} (r+n)(r+n-1) a_n (x-1)^{r+n} - \sum_{n=1}^{\infty} 2(r+n)(r+n-1) a_n (x-1)^{r+n-1}$

shift index

$$= -2r(r-1)a_0(x-1)^{r-1} - \sum_{n=0}^{\infty}(r+n)(r+n-1)a_n(x-1)^{r+n} - \sum_{n=0}^{\infty}2(r+n+1)(r+n)a_{n+1}(x-1)^{r+n}$$

$$= -2r(r-1)a_0(x-1)^{r-1} - \sum_{n=0}^{\infty}\left[2(r+n+1)(r+n)a_{n+1} + (r+n)(r+n-1)a_n\right](x-1)^{r+n}$$

$$\therefore (1-x^2)y'' - xy' + \alpha^2 y = 0 \Rightarrow \left[-2r(r-1) - r\right]a_0(x-1)^{r-1}$$

$$- \sum_{n=0}^{\infty}\Big[2(r+n+1)(r+n)a_{n+1} + (r+n)(r+n-1)a_n +$$

$$(r+n+1)a_{n+1} + (r+n)a_n - \alpha^2 a_n\Big](x-1)^{r+n} = 0$$

$\therefore$ For all $x \neq 1$, $a_0 \neq 0$,

$$-2r(r-1) - r = -(2r^2 - r) = -r(2r-1) = 0, \quad \underline{r = 0, \tfrac{1}{2}}$$

(3) Let $y = \sum_{n=0}^{\infty} a_n(x+1)^{r+n}$ for $x = -1$

$$y' = \sum_{n=0}^{\infty}(r+n)a_n(x-1)^{r+n-1} \qquad \text{Note } x = (x+1) - 1$$

$$\therefore -xy' = \left[-(x+1) + 1\right]y'$$

$$= -\sum_{n=0}^{\infty}(r+n)a_n(x+1)^{r+n} + \sum_{n=0}^{\infty}(r+n)a_n(x+1)^{r+n-1}$$

separate out $n=0$

$$= ra_0(x+1)^{r-1} - \sum_{n=0}^{\infty}(r+n)a_n(x+1)^{r+n} + \sum_{n=1}^{\infty}(r+n)a_n(x+1)^{r+n-1}$$

now shift index

$$= ra_0(x+1)^{r-1} - \sum_{n=0}^{\infty}(r+n)a_n(x+1)^{r+n} + \sum_{n=0}^{\infty}(r+n+1)a_{n+1}(x+1)^{r+n}$$

$$= ra_0(x+1)^{r-1} + \sum_{n=0}^{\infty}\left[(r+n+1)a_{n+1} - (r+n)a_n\right]x^{r+n}$$

$$y'' = \sum_{n=0}^{\infty}(r+n)(r+n-1)a_n(x+1)^{r+n-2}$$

Note $1-x^2 = -(x+1)(x-1) = -(x+1)(x+1-2) = -(x+1)^2 + 2(x+1)$

$$\therefore (1-x^2)y'' = -(x+1)^2 y'' + 2(x+1)y''$$

$$= -\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n(x+1)^{r+n} + 2\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n(x+1)^{r+n-1}$$

separate out $n=0$

$$= 2r(r-1)a_0(x+1)^{r-1} - \sum_{n=0}^{\infty}(r+n)(r+n-1)a_n(x+1)^{r+n}$$

$$+ 2\sum_{n=1}^{\infty}(r+n)(r+n-1)a_n(x+1)^{r+n-1}$$

now shift index

$$= 2r(r-1)a_0(x+1)^{r-1} - \sum_{n=0}^{\infty}(r+n)(r+n-1)a_n(x+1)^{r+n}$$

$$+ 2\sum_{n=0}^{\infty}(r+n+1)(r+n)a_{n+1}(x+1)^{r+n}$$

$$= 2r(r-1)a_0(x+1)^{r-1} + \sum_{n=0}^{\infty}\left[2(r+n+1)(r+n)a_{n+1} - (r+n)(r+n-1)a_n\right](x+1)^{r+n}$$

$$\therefore (1-x^2)y'' - xy' + \alpha^2 y = 0 \Rightarrow [2r(r-1)+r]a_0(x+1)^{r-1}$$

$$+\sum_{n=0}^{\infty}\left[2(r+n+1)(r+n)a_{n+1} - (r+n)(r+n-1)a_n + (r+n+1)a_{n+1} - (r+n)a_n + \alpha^2 a_n\right](x+1)^{r+n}$$

$$= [2r^2 - r]a_0(x+1)^{r-1} +$$

$$\sum_{n=0}^{\infty}\left[(r+n+1)(2r+2n+1)a_{n+1} + (\alpha^2 - (r+n)^2)a_n\right](x+1)^{r+n} = 0$$

$\therefore$ For all $x \neq -1$, $a_0 \neq 0$, $2r^2 - r = r(2r-1) = 0 \Rightarrow \underline{r = 0, \frac{1}{2}}$

$\therefore$ For both $x = 1, -1$, $\underline{r = 0, \frac{1}{2}}$

(b)

$$\underbrace{[2(r+n+1)(r+n) + (r+n+1)]}_{=(r+n+1)(2r+2n)}a_{n+1} + \underbrace{[(r+n)(r+n-1) + (r+n)]}_{=(r+n)^2}a_n - \alpha^2 a_n = 0$$

$$\therefore (r+n+1)(2r+2n+1)a_{n+1} + (r+n)^2 a_n - \alpha^2 a_n = 0, \quad n \geq 0$$

$$a_{n+1} = \frac{\alpha^2 - (r+n)^2}{(r+n+1)(2r+2n+1)}a_n, \quad n \geq 0$$

$$r = \frac{1}{2}: \ a_{n+1} = \frac{\alpha^2 - (\frac{1}{2}+n)^2}{(n+\frac{3}{2})(2n+2)}a_n = \frac{\alpha^2 - \frac{(1+2n)^2}{4}}{(2n+3)(n+1)}a_n$$

$$= \frac{4\alpha^2 - (2n+1)^2}{4(n+1)(2n+3)} a_n \quad , \quad n \geq 0$$

$$\text{Or, } a_n = \frac{(2\alpha)^2 - (2n-1)^2}{4n(2n+1)} a_{n-1} \quad , \quad n \geq 1$$

$$\text{Or, } a_n = \frac{(2\alpha + 2n-1)(2\alpha - (2n-1))}{4n(2n+1)} a_{n-1} \quad , \quad n \geq 1$$

$$\therefore a_1 = \frac{(2\alpha+1)(2\alpha-1)}{4 \cdot 1 \cdot 3} a_0$$

$$a_2 = \frac{(2\alpha+3)(2\alpha-3)}{4 \cdot 2 \cdot 5} a_1 = \frac{(2\alpha+1)(2\alpha+3)(2\alpha-1)(2\alpha-3)}{4 \cdot 4 \cdot 1 \cdot 2 \cdot 3 \cdot 5} a_0$$

$$= 4 \cdot 2 \cdot 4 \cdot 3 \cdot 5 = 4 \cdot 2 \cdot 3 \cdot 4 \cdot 5$$

$$\therefore a_n = \frac{(2\alpha+1)(2\alpha+3)\cdots[2\alpha+(2n-1)](2\alpha-1)(2\alpha-3)\cdots[2\alpha-(2n-1)]}{2^n(2n+1)!} a_0 \, , \quad n \geq 1$$

$\therefore$ Setting $a_0 = 1$, and since $r = \frac{1}{2}$, must use $|x-1|$,

$$\boxed{\begin{aligned} y_1(x) &= |x-1|^{\frac{1}{2}} \left[1 + \sum_{n=1}^{\infty} a_n (x-1)^n \right] \\ &= |x-1|^{\frac{1}{2}} \left[1 + \sum_{n=1}^{\infty} \frac{(2\alpha+1)\cdots[2\alpha+(2n-1)](2\alpha-1)\cdots[2\alpha-(2n-1)]}{2^n(2n+1)!} (x-1)^n \right] \end{aligned}}$$

$$r=0: \quad a_{n+1} = \frac{\alpha^2-(0+n)^2}{(0+n+1)(0+2n+1)}a_n, \quad n \geq 0$$

$$= \frac{\alpha^2-n^2}{(n+1)(2n+1)}a_n = \frac{(\alpha+n)(\alpha-n)}{(n+1)(2n+1)}a_n, \quad n \geq 0$$

Or, $a_n = \dfrac{(\alpha+n-1)(\alpha-(n-1))}{n(2n-1)}a_n, \quad n \geq 1$

$$\therefore a_1 = \frac{\alpha^2}{1\cdot 1}\alpha_0 \qquad a_2 = \frac{(\alpha+1)(\alpha-1)}{2\cdot 3}a_1 = \frac{\alpha(\alpha+1)\alpha(\alpha-1)}{1\cdot 2\cdot 1\cdot 3}a_0$$

$$a_3 = \frac{(\alpha+2)(\alpha-2)}{3\cdot 5}a_2 = \frac{\alpha(\alpha+1)(\alpha+2)\alpha(\alpha-1)(\alpha-2)}{1\cdot 2\cdot 3\cdot 1\cdot 3\cdot 5}a_0$$

$$\therefore a_n = \frac{\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)\alpha(\alpha-1)(\alpha-2)\cdots[\alpha-(n-1)]}{n!\,[1\cdot 3\cdot 5\cdots(2n-1)]}a_0, \quad n \geq 1$$

But $1\cdot 2\cdot 3\cdot 1\cdot 3\cdot 5 = \dfrac{2\cdot 4\cdot 6\cdot 1\cdot 3\cdot 5}{2\cdot 2\cdot 2} = \dfrac{6!}{2^3}$

$$\therefore n!\,[1\cdot 3\cdot 5\cdots(2n-1)] = \frac{(2n)!}{2^n}$$

$$\therefore a_n = \frac{\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)\alpha(\alpha-1)(\alpha-2)\cdots[\alpha-(n-1)]}{\frac{(2n)!}{2^n}}a_0, \quad n \geq 1$$

$\therefore$ Setting $a_0 = 1$, with $x^r = x^0 = 1$,

$$\boxed{\begin{aligned} y_2(x) &= x^0\left[1+\sum_{n=1}^{\infty} a_n (x-1)^n\right] \\ &= 1+\sum_{n=1}^{\infty} \frac{\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)\,\alpha(\alpha-1)\cdots[\alpha-(n-1)]\,2^n}{(2n)!}(x-1)^n \end{aligned}}$$

9.

(a)

Standard form: $y'' + \left(\frac{1-x}{x}\right)y' + \frac{\lambda}{x}y = 0$

$\therefore$ $x=0$ is a singular point.

$\lim_{x\to 0} x\left(\frac{1-x}{x}\right) = 1$, $\lim_{x\to 0} x^2\left(\frac{\lambda}{x}\right) = 0$, both finite.

$\therefore$ $x=0$ is a <u>regular</u> singular point.

(b)

Let $y = \sum_{n=0}^{\infty} a_n x^{r+n}$

$\therefore$ $y' = \sum_{n=0}^{\infty} (r+n) a_n x^{r+n-1}$

$$(1-x)y' = \sum_{n=0}^{\infty}(r+n)a_n x^{r+n-1} - \sum_{n=0}^{\infty}(r+n)a_n x^{r+n}$$

$$= ra_0 x^{r-1} + \sum_{n=1}^{\infty}(r+n)a_n x^{r+n-1} - \sum_{n=0}^{\infty}(r+n)a_n x^{r+n}$$

shift index

$$= ra_0 x^{r-1} + \sum_{n=0}^{\infty}(r+n+1)a_{n+1} x^{r+n} - \sum_{n=0}^{\infty}(r+n)a_n x^{r+n}$$

$$= ra_0 x^{r-1} + \sum_{n=0}^{\infty}\left[(r+n+1)a_{n+1} - (r+n)a_n\right] x^{r+n}$$

$$y'' = \sum_{n=0}^{\infty}(r+n)(r+n-1)a_n x^{r+n-2}$$

$$\therefore\ xy'' = \sum_{n=0}^{\infty}(r+n)(r+n-1)a_n x^{r+n-1}$$

$$= r(r-1)a_0 x^{r-1} + \sum_{n=1}^{\infty}(r+n)(r+n-1)a_n x^{r+n-1}$$

$$= r(r-1)a_0 x^{r-1} + \sum_{n=0}^{\infty}(r+n+1)(r+n)a_{n+1} x^{r+n}$$

$$\therefore\ xy'' + (1-x)y' + \lambda y = 0 \Rightarrow$$

$$\left[r(r-1) + r\right]a_0 x^{r-1} + \sum_{n=0}^{\infty}\Big(\left[(r+n+1)(r+n) + (r+n+1)\right]a_{n+1}$$

$$+ \left[\lambda - (r+n)\right]a_n\Big) x^{r+n} = 0$$

$\therefore$ For all $x \neq 0$, $a_0 \neq 0$,

$$r(r-1)+r = \underline{r^2 = 0}, \quad \underline{r = 0, 0}$$

$$[(r+n+1)(r+n) + (r+n+1)]\,a_{n+1} + [\lambda - (r+n)]\,a_n = 0, \quad n \geq 0$$

$$\therefore \quad (r+n+1)^2 a_{n+1} = [(r+n) - \lambda]\,a_n, \quad n \geq 0$$

$$a_{n+1} = \frac{(r+n) - \lambda}{(r+n+1)^2}\,a_n, \quad n \geq 0$$

Setting $r = 0$, $\quad a_{n+1} = \dfrac{n - \lambda}{(n+1)^2}\,a_n, \quad n \geq 0$

Or, $\quad \underline{a_n = \dfrac{n-1-\lambda}{n^2}}\,a_{n-1}, \quad n \geq 1$

(c)

(1) $\quad a_1 = \dfrac{-\lambda}{1^2}\,a_0 \qquad a_2 = \dfrac{1-\lambda}{2^2}\,a_1 = \dfrac{(-\lambda)(1-\lambda)}{1^2 \cdot 2^2}\,a_0$

$$a_3 = \frac{2-\lambda}{3^2}\,a_2 = \frac{(-\lambda)(1-\lambda)(2-\lambda)}{1^2 \cdot 2^2 \cdot 3^2}\,a_0$$

$$\therefore \quad a_n = \frac{(-\lambda)(1-\lambda)(2-\lambda)\cdots(n-1-\lambda)}{(n!)^2}\,a_0, \quad n \geq 1$$

Setting $a_0 = 1$, $x^r = 1$,

$$\boxed{y(x) = 1 + \sum_{n=1}^{\infty} \frac{(-\lambda)(1-\lambda)(2-\lambda)\cdots(n-1-\lambda)}{(n!)^2} x^n}$$

(2) If λ is a positive integer, eventually one of the terms of $(1-\lambda)(2-\lambda)(3-\lambda)\cdots$ will be zero, so that by the recurrence relation, all subsequent terms will be zero. Thus, $y(x)$ will have a finite number of terms, and so will be a polynomial.

Example: if $\lambda = 3$, then

$$a_4 = \frac{3-\lambda}{4^2} a_3 = \frac{3-3}{4^2} a_3 = 0 \quad \therefore a_5 = 0,\ a_6 = 0, \text{ etc.}$$

$$\therefore y(x) = 1 - \frac{\lambda}{1^2} x - \frac{\lambda(\lambda-1)}{1^2 \cdot 2^2} x^2 - \frac{\lambda(\lambda-1)(\lambda-2)}{1^2 \cdot 2^2 \cdot 3^2} x^3$$

$$= 1 - 3x - \frac{3}{2} x^2 - \frac{1}{6} x^3 = L_3(x)$$

10.

(a)

Standard form: $y'' + \left(\frac{1}{x}\right)y' + y = 0$

$\therefore$ x is a singular point.

$\lim_{x\to 0} x\left(\frac{1}{x}\right) = 1$, $\lim_{x\to 0} x^2(1) = 0$, both finite

$\therefore$ $x=0$ is a regular singular point.

(b)

Let $y = \sum_{n=0}^{\infty} a_n x^{r+n}$ $\therefore$ $x^2 y = \sum_{n=0}^{\infty} a_n x^{r+n+2}$

$y' = \sum_{n=0}^{\infty} (r+n) a_n x^{r+n-1}$ $\therefore$ $xy' = \sum_{n=0}^{\infty} (r+n) a_n x^{r+n}$

$= r a_0 x^r + (r+1) a_1 x^{r+1} + \sum_{n=2}^{\infty} (r+n) a_n x^{r+n}$

shift index

$= r a_0 x^r + (r+1) a_1 x^{r+1} + \sum_{n=0}^{\infty} (r+n+2) a_{n+2} x^{r+n+2}$

$y'' = \sum_{n=0}^{\infty} (r+n)(r+n-1) a_n x^{r+n-2}$

$$\therefore x^2 y'' = \sum_{n=0}^{\infty} (r+n)(r+n-1) a_n x^{r+n}$$

$$= r(r-1) a_0 x^r + (r+1)(r) a_1 x^{r+1} + \sum_{n=2}^{\infty} (r+n)(r+n-1) a_n x^{r+n}$$

$$= r(r-1) a_0 x^r + (r+1)(r) a_1 x^{r+1} + \sum_{n=0}^{\infty} (r+n+2)(r+n+1) a_{n+2} x^{r+n+2}$$

$\therefore x^2 y'' + x y' + x^2 y = 0 \Rightarrow$

$$[r(r-1) + r] a_0 x^r + [(r+1)(r) + (r+1)] a_1 x^{r+1} +$$

$$\sum_{n=0}^{\infty} [(r+n+2)(r+n+1) a_{n+2} + (r+n+2) a_{n+2} + a_n] x^{r+n+2} = 0$$

$\therefore$ For all $x \neq 0$, $a_0 \neq 0$,

$r(r-1) + r = \underline{r^2 = 0}$, $\underline{r = 0, 0}$

(c)

From (b), for a_1: $[(r+1)(r) + r + 1] a_1 = (r+1)^2 a_1 = a_1$

for $r = 0$. $\therefore$ For $a_1 x = 0$ for all $x > 0$, $\underline{\underline{a_1 = 0}}$

Also, $(r+n+2)(r+n+1) a_{n+2} + (r+n+2) a_{n+2} + a_n = 0$, $n \geq 0$

$$\therefore a_{n+2} = \frac{-a_n}{(r+n+2)^2}, \ n \geq 0 \qquad \text{Or, } a_n = \frac{-a_{n-2}}{(r+n)^2}, \ n \geq 2$$

For $r=0$, $a_n = \dfrac{-a_{n-2}}{n^2}$, $n \geq 2$

$$\therefore a_2 = \frac{-a_0}{2^2} \qquad a_3 = \frac{-a_1}{3^2} = 0 \text{ since } a_1 = 0$$

$$a_4 = \frac{-a_2}{4^2} = \frac{a_0}{2^2 \cdot 4^2} \qquad \therefore \text{all odd terms are } 0.$$

$$\therefore a_{2n} = \frac{(-1)^n a_0}{(2 \cdot 4 \cdots 2n)^2} = \frac{(-1)^n a_0}{(2^n \cdot 1 \cdot 2 \cdot 3 \cdots n)^2} = \frac{(-1)^n a_0}{2^{2n}(n!)^2}, \ n \geq 1$$

Setting $a_0 = 1$, $x^r = 1$,

$$\boxed{\begin{aligned} y(x) &= 1 + \left(\frac{-1}{2^2}\right)x^2 + \left(\frac{1}{2^2 \cdot 4^2}\right)x^4 + \cdots + \frac{(-1)^n}{2^{2n}(n!)^2} x^{2n} + \cdots \\ &= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{2^{2n}(n!)^2} x^{2n} = J_0(x) \end{aligned}}$$

(d)

Let $S_n = a_{2n} = \dfrac{(-1)^n}{2^{2n}(n!)^2} x^{2n}$

$$\therefore \left|\frac{S_{n+1}}{S_n}\right| = \left|\frac{(-1)^{n+1}\, x^{2(n+1)}}{2^{2(n+1)}\left[(n+1)!\right]^2} \cdot \frac{2^{2n}(n!)^2}{(-1)^n\, x^{2n}}\right|$$

$$= \left|2^2\left[\frac{n!}{(n+1)!}\right]^2 x^2\right| = \frac{x^2}{4(n+1)^2}$$

$$\therefore \lim_{n\to\infty}\left|\frac{S_{n+1}}{S_n}\right| = \lim_{n\to\infty}\frac{x^2}{4(n+1)^2} = 0, \text{ for all } x$$

$\therefore$ By ratio test, $J_0(x)$ <u>converges</u> for <u>all</u> x.

11.

Let $y_2(x) = v(x)\, J_0(x)$, where $J_0(x)$ is from #9 and is a solution to $x^2y'' + xy' + x^2y = 0$ for $x > 0$.

$$\therefore y_2' = v'J_0 + vJ_0', \quad y'' = v''J_0 + 2v'J_0' + vJ_0''$$

$$\therefore x^2y_2'' + xy_2' + x^2y_2 = x^2v''J_0 + 2x^2v'J_0' + \underline{x^2vJ_0''}$$
$$+ xv'J_0 + \underline{xvJ_0'} + \underline{x^2vJ_0}$$

$$= \underbrace{(x^2J_0'' + xJ_0' + x^2J_0)}_{=0}v + x^2J_0v'' + (2x^2J_0' + xJ_0)v'$$

$$= x^2J_0v'' + (2x^2J_0' + xJ_0)v' = 0$$

Dividing by x^2J_0,

$$v'' + \left(2\frac{J_0'}{J_0} + \frac{1}{x}\right)v' = 0$$

$$\therefore \frac{v''}{v'} = -\left(\frac{2J_0'}{J_0} + \frac{1}{x}\right)$$

$$\int \frac{v''}{v'}\,dx = -\int\left(2\frac{J_0'}{J_0} + \frac{1}{x}\right)dx$$

$$\therefore \ln(v') = -\left[\ln(J_0^2) + \ln(x)\right] = \ln\left(\frac{1}{xJ_0^2}\right)$$

$$\therefore v' = \frac{1}{xJ_0^2}$$

$$\therefore \text{Let } v(x) = \int\frac{dx}{xJ_0(x)^2}$$

$$\therefore y_2(x) = v(x)\,J_0(x) = J_0(x)\int\frac{dx}{x[J_0(x)]^2}$$

From #10 above, $J_0(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} + \ldots$

$$\therefore y_2(x) = \left[1 - \frac{x^2}{4} + \ldots\right]\int\frac{dx}{x\left(1 - \frac{x^2}{4} + \ldots\right)^2} = \left[1 - \frac{x^2}{4} + \ldots\right]\int\frac{1}{x}\left(1 - \frac{x^2}{4} + \ldots\right)^{-2}dx$$

now using $(1+z)^{-2} = 1 + (-2)z + \frac{(-2)(-3)}{2!}z^2 + \ldots$

$$= \left(1 - \frac{x^2}{4} + \ldots\right)\left[\int\frac{dx}{x} + \int\left(\frac{2x^2}{4x} + \ldots \text{higher terms in } x^n\right)\right]$$

$$= \int\frac{dx}{x} + (\text{terms in } x^n,\ n > 3)$$

$= \underline{\ln(x)} + (\text{terms in } x^n, n > 3)$

$\therefore$ $y_2(x)$ contains a logarithmic term

12.

(a)

Standard form: $y'' + \left(\frac{1}{x}\right) y' + \left(\frac{x^2-1}{x^2}\right) y = 0$

$\therefore$ $x=0$ is a singular point.

$\lim_{x \to 0} x\left(\frac{1}{x}\right) = 1$, $\lim_{x \to 0} x^2\left(\frac{x^2-1}{x^2}\right) = -1$, both finite.

$\therefore$ $x=0$ is a <u>regular</u> singular point.

(b)

Let $y = \sum_{n=0}^{\infty} a_n x^{r+n}$

$$\therefore (x^2-1)y = \sum_{n=0}^{\infty} a_n x^{r+n+2} - \sum_{n=0}^{\infty} a_n x^{r+n}$$

$$= -a_0 x^r - a_1 x^{r+1} + \sum_{n=0}^{\infty} a_n x^{r+n+2} - \sum_{n=2}^{\infty} a_n x^{r+n}$$

shift index

$$= -a_0 x^r - a_1 x^{r+1} + \sum_{n=0}^{\infty} a_n x^{r+n+2} - \sum_{n=0}^{\infty} a_{n+2} x^{r+n+2}$$

$$= -a_0 x^r - a_1 x^{r+1} + \sum_{n=0}^{\infty} (a_n - a_{n+2}) x^{r+n+2}$$

$$y' = \sum_{n=0}^{\infty} (r+n) a_n x^{r+n-1} \quad \therefore xy' = \sum_{n=0}^{\infty} (r+n) a_n x^{r+n}$$

$$= r a_0 x^r + (r+1) a_1 x^{r+1} + \sum_{n=2}^{\infty} (r+n) a_n x^{r+n}$$

$\downarrow$ shift index

$$= r a_0 x^r + (r+1) a_1 x^{r+1} + \sum_{n=0}^{\infty} (r+n+2) a_{n+2} x^{r+n+2}$$

$$y'' = \sum_{n=0}^{\infty} (r+n)(r+n-1) a_n x^{r+n-2}$$

$$\therefore x^2 y'' = \sum_{n=0}^{\infty} (r+n)(r+n-1) a_n x^{r+n}$$

$$= r(r-1) a_0 x^r + (r+1)(r) a_1 x^{r+1} + \sum_{n=2}^{\infty} (r+n)(r+n-1) a_n x^{r+n}$$

$$= r(r-1) a_0 x^r + (r+1)(r) a_1 x^{r+1} + \sum_{n=0}^{\infty} (r+n+2)(r+n+1) a_{n+2} x^{r+n+2}$$

$$\therefore x^2 y'' + xy' + (x^2-1)y = 0 \Rightarrow$$

$$[r(r-1) + r - 1] a_0 x^r + [(r+1)(r) + (r+1) - 1] a_1 x^{r+1} +$$

$$\sum_{n=0}^{\infty} \left[(r+n+2)(r+n+1)a_{n+2} + (r+n+2)a_{n+2} + (a_n - a_{n+2})\right] x^{r+n+2} = 0$$

$\therefore$ For all $x \neq 0$, $a_0 \neq 0$,

$$r(r-1) + r - 1 = r^2 - 1 = 0, \quad \underline{r = \pm 1}$$

For a_1: $(r+1)(r) + (r+1) - 1 = r^2 + 2r$

$\therefore$ For all $x \neq 0$, $(r^2+2r)a_1 x^{r+1} = 0$

For $r = 1$, $3a_1 x^{r+1} = 0 \Rightarrow a_1 = 0$

For $r = -1$, $-a_1 x^{r+1} = 0 \Rightarrow a_1 = 0$

$\therefore$ For $r = \pm 1$, $\underline{a_1 = 0}$

(c)

From (b), $a_1 = 0$, and

$$(r+n+2)(r+n+1)a_{n+2} + (r+n+2)a_{n+2} + (a_n - a_{n+2}) = 0, \quad n \geq 0$$

$$\therefore (r+n+2)(r+n+1)a_{n+2} + (r+n+1)a_{n+2} + a_n = 0, \quad n \geq 0$$

$$\therefore (r+n+1)(r+n+3)a_{n+2} + a_n = 0, \quad n \geq 0$$

$$\therefore \quad a_{n+2} = \frac{-a_n}{(r+n+1)(r+n+3)}, \quad n \geq 0$$

Let $r=1$: $\quad a_{n+2} = \dfrac{-a_n}{(n+2)(n+4)}, \quad n \geq 0$

Or, $\quad a_n = \dfrac{-a_{n-2}}{n(n+2)}, \quad n \geq 2$

$\therefore \quad a_2 = \dfrac{-a_0}{2 \cdot 4} \qquad a_3 = 0$ since $a_1 = 0$

$a_4 = \dfrac{-a_2}{4 \cdot 6} = \dfrac{a_0}{2 \cdot 4 \cdot 4 \cdot 6} \qquad a_5 = a_7 = a_9 = \ldots = 0$

$$a_6 = -\frac{a_4}{6 \cdot 8} = \frac{-a_0}{2 \cdot 4 \cdot 6 \cdot 4 \cdot 6 \cdot 8} = \frac{-a_0}{2^3 \underbrace{(1 \cdot 2 \cdot 3)}_{n!} \cdot 2^3 \underbrace{(2 \cdot 3 \cdot 4)}_{(n+1)!}}$$

$$\therefore \quad a_{2n} = \frac{(-1)^n a_0}{(2^n)^2 (n!)(n+1)!} = \frac{(-1)^n a_0}{(2^{2n})(n!)(n+1)!}, \quad n \geq 1$$

Note relation also works for $n=0$.

$\therefore$ Setting $a_0 = 1$, $x^r = x$, $\therefore y = x^r \sum_{n=0}^{\infty} a_n x^n = x \sum_{n=0}^{\infty} a_{2n} x^{2n}$

$$y(x) = x \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2^{2n})(n!)(n+1)!} \right]$$

To get $\frac{x}{2}$, set $a_0 = \frac{1}{2}$

$\therefore$

$$a_{2n} = \frac{(-1)^n\left(\frac{1}{2}\right)}{(2^{2n})(n!)(n+1)!} = \frac{1}{2}\,\frac{(-1)^n}{(2^{2n})(n!)(n+1)!}$$

$$\therefore\ y(x) = x^r \sum_{n=0}^{\infty} a_{2n} x^{2n} = x \sum_{n=0}^{\infty} \frac{1}{2}\cdot\frac{(-1)^n x^{2n}}{(2^{2n})(n!)(n+1)!}$$

$$= \frac{x}{2} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2^{2n})(n!)(n+1)!}$$

$$\therefore\ \boxed{J_1(x) = \frac{x}{2} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2^{2n})(n!)(n+1)!}}$$

(d)

From (c), $S_n = \dfrac{(-1)^n x^{2n+1}}{(2^{2n})(n!)(n+1)!}$, $n \geq 0$

$$\therefore\ \left|\frac{S_{n+1}}{S_n}\right| = \left|\frac{(-1)^{n+1} x^{2n+3}}{(2^{2n+2})(n+1)!\,(n+2)!}\cdot\frac{2^{2n}(n!)(n+1)!}{(-1)^n x^{2n+1}}\right|$$

$$= \left|\frac{x^2}{2^2(n+1)(n+2)}\right|$$

$$\therefore \lim_{n\to\infty}\left|\frac{S_{n+1}}{S_n}\right| = \lim_{n\to\infty}\left|\frac{x^2}{2^2(n+1)(n+2)}\right| = 0, \text{ for all } x.$$

$\therefore$ By ratio test,

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} J_1(x) \text{ exists for all } x$$

i.e., $J_1(x)$ converges for all x.

(e)

From (c), $(r+n+1)(r+n+3)a_{n+2} + a_n = 0$, $n \geq 0$

For $r=-1$, $n(n+2)a_{n+2} + a_n = 0$, $n \geq 0$

$\therefore$ For independent a_0, with $n=0$, $0(0+2)a_2 + a_0 = 0$,

or $0 \cdot a_2 + a_0 = 0$. $\therefore$ Can't determine a_2, so

can't determine $a_4, a_6, a_8, \ldots$.

$\therefore$ For $y = x^r \sum_{n=0}^{\infty} a_n x^n$, and $r=-1$, can't

determine $y(x)$.

5.6 Series Solutions Near a Regular Singular Point, Part II

Note Title 5/1/2019

1.

(a) Standard form: $y'' + 2y' + \frac{6e^x}{x} y = 0$ [1]

$\therefore$ $x = 0$ is a singular point.

$x(2)$ is analytic at $x=0$

$x^2\left(\frac{6e^x}{x}\right) = 6xe^x$ is analytic at $x=0$

$\therefore$ $x=0$ is a regular singular point.

(b) $\lim_{x\to 0} xp(x) = \lim_{x\to 0} 2x = 0 = p_0$

$\lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} x^2\left(\frac{6e^x}{x}\right) = 0 = q_0$

$\therefore$ $r(r-1) + p_0 r + q_0 = \underline{\underline{r(r-1) = 0}}$. $\therefore$ $\underline{\underline{r_1 = 1,\ r_2 = 0}}$

2.

(a) Standard form: $y'' - \frac{2+x}{x} y' + \frac{(2+x^2)}{x^2} y = 0$

$\therefore x=0$ is a singular point.

$x\left(-\frac{2+x}{x}\right) = (-2-x)$ is analytic at $x=0$

$x^2\left(\frac{2+x^2}{x^2}\right) = 2+x^2$ is analytic at $x=0$

$\therefore x=0$ is a regular singular point.

(b) $\lim_{x\to 0}(-2-x) = -2 = p_0$

$\lim_{x\to 0}(2+x^2) = 2 = q_0$

$\therefore r(r-1)-2r+2 = \underline{r^2-3r+2=0}$

$r^2-3r+2 = (r-2)(r-1) \quad \therefore \underline{r_1=2,\ r_2=1}$

3.

(a)

There are no singular points.

(b)

No singular points, $\therefore$ no indicial equation.

4.

(a)

Standard form: $y'' + \left(\frac{1}{2x(x+2)}\right)y' + \left(\frac{-1}{2(x+2)}\right)y = 0$

$\therefore$ $x = 0, -2$ are singular points

$x=0$: $\lim_{x\to 0} x\left(\frac{1}{2x(x+2)}\right) = \frac{1}{4} = p_0$

$\lim_{x\to 0} x^2\left(\frac{-1}{2(x+2)}\right) = 0 = q_0$

$\therefore$ $\underline{x=0}$ is a $\underline{\text{regular}}$ singular point

$x=-2$: $\lim_{x\to -2} (x+2)\left(\frac{1}{2x(x+2)}\right) = -\frac{1}{4} = p_0$

$\lim_{x\to -2} (x+2)^2\left(\frac{-1}{2(x+2)}\right) = 0 = q_0$

$\therefore$ $\underline{x=-2}$ is a $\underline{\text{regular}}$ singular point

(b)

$x=0$: $r(r-1) + \frac{1}{4}r = \underline{r\left(r-\frac{3}{4}\right) = 0}$ $\therefore$ $\underline{r_1 = \frac{3}{4}, r_2 = 0}$

$x=-2$: $r(r-1) - \frac{1}{4}r = \underline{r\left(r-\frac{5}{4}\right) = 0}$ $\therefore$ $\underline{r_1 = \frac{5}{4}, r_2 = 0}$

5.

(a)

Standard form: $y'' + \frac{x+\sin(x)}{2x^2} + \left(\frac{1}{x^2}\right) y = 0$

$\therefore$ $x=0$ is a singular point.

using L'Hopital's

$$\lim_{x\to 0} x\left(\frac{x+\sin(x)}{2x^2}\right) = \lim_{x\to 0} \frac{x+\sin(x)}{2x} = \lim_{x\to 0} \frac{1+\cos(x)}{2} = 1 = p_0$$

$$\lim_{x\to 0} x^2\left(\frac{1}{x^2}\right) = 1 = q_0$$

$\therefore$ $\underline{x=0}$ is a $\underline{\text{regular}}$ singular point

(b)

$r(r-1)+r+1 = \underline{\underline{r^2+1=0}}$ $\therefore$ $\underline{r_1 = i, \ r_2 = -i}$

6.

(a)

Standard form: $y'' + \left[\frac{1+x}{x^2(x-1)}\right] y' + \left[\frac{2}{x(1-x)}\right] y = 0$

$\therefore$ $x=0,1$ are singular points

$$x=0: \lim_{x\to 0} x\left[\frac{1+x}{x^2(x-1)}\right] = \infty \quad \therefore \text{ not regular}$$

$$x=1: \lim_{x\to 1} (x-1)\left[\frac{1+x}{x^2(x-1)}\right] = 2 = p_0$$

$$\lim_{x\to 1} (x-1)^2\left[\frac{2}{x(1-x)}\right] = 0 = q_0$$

$\therefore$ $\underline{x=1}$ is a $\underline{\text{regular}}$ singular point

(b)

$$r(r-1)+2r = \underline{r^2+r=0} \quad \therefore \underline{r_1=0,\ r_2=-1}$$

7.

(a)

Standard form: $y'' + \left[\dfrac{2x}{(x-2)^2(x+2)}\right]y' + \left[\dfrac{3}{(x-2)(x+2)}\right]y = 0$

$\therefore$ $x = 2, -2$ are singular points.

$$x=2: \lim_{x\to 2} (x-2)\left[\frac{2x}{(x-2)^2(x+2)}\right] = \infty \quad \therefore \text{ not regular}$$

$$x=-2: \lim_{x\to -2} (x+2)\left[\frac{2x}{(x-2)^2(x+2)}\right] = -\frac{1}{4} = p_0$$

$$\lim_{x \to -2} (x+2)^2 \left[\frac{3}{(x-2)(x+2)}\right] = 0 = q_0$$

$\therefore$ $\underline{x=-2}$ is a $\underline{\text{regular}}$ singular point

(b)

$r(r-1) - \frac{1}{4}r = \underline{r(r-\frac{5}{4})} = 0 \quad \therefore \underline{r_1 = \frac{5}{4},\ r_2 = 0}$

8.

(a)

Standard form: $y'' + \left(\frac{2x}{4-x^2}\right)y' + \left(\frac{3}{4-x^2}\right)y = 0$

$\therefore$ $x = 2, -2$ are singular points

$x=2$: $\displaystyle\lim_{x \to 2} (x-2)\left[\frac{2x}{(2-x)(2+x)}\right] = -1 = p_0$

$$\lim_{x \to 2} (x-2)^2 \left[\frac{3}{(2-x)(2+x)}\right] = 0 = q_0$$

$\therefore$ $\underline{x=2}$ is a $\underline{\text{regular}}$ singular point.

$x=-2$: $\displaystyle\lim_{x \to -2} (x+2)\left[\frac{2x}{(2-x)(2+x)}\right] = -1 = p_0$

$$\lim_{x\to -2} (x+2)^2 \left[\frac{3}{(2-x)(2+x)}\right] = 0 = q_0$$

$\therefore$ $\underline{x=-2}$ is a $\underline{\text{regular}}$ singular point.

(b)

$x=2$: $r(r-1)-r = \underline{r^2-2r=0}$ $\therefore$ $\underline{r_1=2,\ r_2=0}$

$x=-2$: $r(r-1)-r = \underline{r^2-2r=0}$ $\therefore$ $\underline{r_1=2,\ r_2=0}$

9.

(a) Standard form: $y'' + \left(\frac{1}{x}\right)y' - \left(\frac{1}{x}\right)y = 0.$

$\lim_{x\to 0} x\left(\frac{1}{x}\right) = 1$ $\quad \lim_{x\to 0} x^2\left(-\frac{1}{x}\right) = 0$, both finite.

$\therefore$ $x=0$ is a regular singular point.

(b) From (a), $p_0 = 1$, $q_0 = 0$

$\therefore$ $r(r-1)+r = r^2 = 0$, $\therefore$ $\underline{r_1=0}$, $\underline{r_2=0}$

(c) Let $y_1(x) = \sum_{n=0}^{\infty} a_n x^{r+n}$

$$\therefore y_1' = \sum_{n=0}^{\infty} (r+n) a_n x^{r+n-1} = r a_0 x^{r-1} + \sum_{n=1}^{\infty} (r+n) a_n x^{r+n-1}$$

$$= r a_0 x^{r-1} + \sum_{n=0}^{\infty} (r+n+1) a_{n+1} x^{r+n}$$

$$y_1'' = \sum_{n=0}^{\infty} (r+n)(r+n-1) a_n x^{r+n-2}$$

$$x y_1'' = \sum_{n=0}^{\infty} (r+n)(r+n-1) a_n x^{r+n-1} =$$

$$r(r-1) a_0 x^{r-1} + \sum_{n=1}^{\infty} (r+n)(r+n-1) a_n x^{r+n-1}$$

shift index

$$= r(r-1) a_0 x^{r-1} + \sum_{n=0}^{\infty} (r+n+1)(r+n) a_{n+1} x^{r+n}$$

$$\therefore x y_1'' + y_1' - y_1 = 0 \Rightarrow \left[r(r-1) a_0 + r a_0 \right] x^{r-1} +$$

$$\sum_{n=0}^{\infty} \left[(r+n+1)(r+n) a_{n+1} + (r+n+1) a_{n+1} - a_n \right] x^{r+n} = 0 \quad [y_1]$$

$$\therefore r^2 a_0 = 0, \quad (r+n+1)^2 a_{n+1} = a_n, \quad n \geq 0$$

$$\therefore a_{n+1} = \frac{a_n}{(r+n+1)^2}, \quad n \geq 0, \quad \text{or} \quad a_n = \frac{a_{n-1}}{(r+n)^2}, \quad n \geq 1$$

$$\therefore a_1 = \frac{a_0}{(r+1)^2}, \quad a_2 = \frac{a_1}{(r+2)^2} = \frac{a_0}{(r+2)^2(r+1)^2}$$

$$\therefore a_n(r) = \frac{a_0}{[(r+n)(r+n-1)\cdots(r+2)(r+1)]^2} \qquad [1]$$

$$\therefore a_n(0) = \frac{a_0}{(n!)^2}$$

$$\therefore y_1(x) = \sum_{n=0}^{\infty} a_n(0)\, x^n = \sum_{n=0}^{\infty} \frac{a_0}{(n!)^2} x^n, \text{ using } x^r = x^0 = 1$$

Setting $a_0 = 1$, $\underline{y_1(x) = \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2}}$

$$\therefore \boxed{y_1(x) = 1 + x + \frac{x^2}{4} + \frac{x^3}{36} + \cdots}$$

For $y_2(x)$, computing $\frac{d}{dr} a_n(r)$ from [1] looks messy.

$\therefore$ Assume $y_2(x) = y_1(x) \ln(x) + \sum_{n=1}^{\infty} b_n x^n$ ← note $n=1$ from Theorem 5.6.1

and let $s(x) = \sum_{n=1}^{\infty} b_n x^n$

$$\therefore y_2' = y_1' \ln(x) + y_1 \left(\frac{1}{x}\right) + s'$$

$$y_2'' = y_1''\ln(x) + y_1'\left(\frac{1}{x}\right) + y_1'\left(\frac{1}{x}\right) + y_1\left(-\frac{1}{x^2}\right) + s''$$

$$= y_1''\ln(x) + 2y_1'\left(\frac{1}{x}\right) - y_1\left(\frac{1}{x^2}\right) + s''$$

$$\therefore \; xy_2'' + y_2' - y_2 =$$

$$xy_1''\ln(x) + 2y_1' - y_1\left(\frac{1}{x}\right) + xs''$$

cancel

$$+ \; y_1'\ln(x) + y_1\left(\frac{1}{x}\right) + s' - y_1\ln(x) - s$$

$$= \ln(x)\underbrace{\left[xy_1'' + y_1' - y_1\right]}_{=0} + 2y_1' + xs'' + s' - s$$

$$= 2y_1' + xs'' + s' - s = 0 = 2y_1' + L[s] \qquad [2]$$

From $y_1(x) = \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2}$, $y_1' = \sum_{n=1}^{\infty} \frac{n x^{n-1}}{(n!)^2}$

$$\therefore \; 2y_1' = \sum_{n=1}^{\infty} \frac{2n}{(n!)^2} x^{n-1} = \sum_{n=0}^{\infty} \frac{2(n+1)}{[(n+1)!]^2} x^n$$

The development of $s(x)$ is the same as $y_1(x)$ using $L[y] = xy'' + y' - y$, where $y = \sum_{n=0}^{\infty} a_n x^{r+n}$. For $L[s]$, set $r=0$ and $b_0 = 0$ since $s = \sum_{n=1}^{\infty} b_n x^n$

$\therefore$ Using $[y_1]$, and setting $r=0$ and remembering $b_0 = 0$,

$$L[S] = \sum_{n=0}^{\infty} \left[(n+1)(n) b_{n+1} + (n+1) b_{n+1} - b_n \right] x^n$$

$$= \sum_{n=0}^{\infty} \left[(n+1)^2 b_{n+1} - b_n \right] x^n$$

$\therefore$ From [2], $2y_1' + L[S] = 0 \Rightarrow$

$$\sum_{n=0}^{\infty} \left[(n+1)^2 b_{n+1} - b_n \right] x^n = - \sum_{n=0}^{\infty} \frac{2(n+1)}{[(n+1)!]^2} x^n$$

$$\therefore \quad b_{n+1} = \frac{b_n - \frac{2(n+1)}{[(n+1)!]^2}}{(n+1)^2}, \quad n \geq 0, \quad b_0 = 0$$

$$\text{Or,} \quad b_n = \frac{b_{n-1} - \frac{2n}{(n!)^2}}{n^2}, \quad n \geq 1, \quad b_0 = 0$$

Using MATLAB,

```
clear, clc
b0 = 0;
NumTerms = 5;     %up to b5
b = zeros(NumTerms,1);
b_last = b0;
for n = 1:NumTerms
    b(n) = (b_last - (2*n/((factorial(n))^2))) / (n^2);
    b_last = b(n);
end
b0
rats(b)  % display as fractions
```

```
b0 = 0
ans = 5×14 char array
    '      -2      '
    '     -3/4     '
    '    -11/108   '
    '     -4/553   '
    '    -1/3153   '
```

$$\therefore \quad b_1 = -2 \quad b_2 = -\frac{3}{4} \quad b_3 = \frac{-11}{108} \quad b_4 = \frac{-4}{553} \quad b_5 = \frac{-1}{3153}$$

$$\therefore \quad S(x) = -2x - \frac{3}{4}x^2 - \frac{11}{108}x^3 - \frac{4}{553}x^4 - \cdots$$

$\therefore$

$$\boxed{\begin{aligned} y_1(x) &= 1 + x + \frac{x^2}{4} + \frac{x^3}{36} + \cdots \\ y_2(x) &= y_1(x)\ln(x) - 2x - \frac{3}{4}x^2 - \frac{11}{108}x^3 - \frac{4}{553}x^4 - \cdots \end{aligned}}$$

For the sake of completeness, Use MATLAB to compute

$\frac{d}{dr} a_n(r)$, where $a_n(r) = \dfrac{a_0}{[(r+n)(r+n-1)\cdots(r+2)(r+1)]^2}$, $n \geq 1$

so that $y_2(x)$ is from equation (17) in the text.

$$= y_1(x)\ln x + x^{r_1}\sum_{n=1}^{\infty} a_n'(r_1)x^n, \quad x > 0, \qquad (17)$$

Here, $x^{r_1} = x^0 = 1$

```
clear, clc
syms r
NumTerms = 4;  % degree x^4
a = sym('a', [NumTerms, 1])
s = 1;
for n = 1:NumTerms
    s = s*(r+n)^2;
    a(n) = diff(1/s,r,1);
end
subs(a,r,0)
```

a =

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}$$

ans =

$$\begin{pmatrix} -2 \\ -\frac{3}{4} \\ -\frac{11}{108} \\ -\frac{25}{3456} \end{pmatrix}$$

$$\therefore \quad x^{r_1}\sum_{n=1}^{\infty} a_n'(r)x^n = -2x - \frac{3}{4}x^2 - \frac{11}{108}x^3 - \frac{25}{3456}x^4 + \cdots$$

$\therefore$ coefficients are the same for each method.

Note: $\frac{25}{3456} = 0.007233796\ldots$

$\frac{4}{553} = 0.007233273\ldots$

The coefficients for the x^4 term are equivalent.

If "rats(6,20)" had been used in first method,

$-\frac{25}{3456}$ would have been obtained (i.e., higher precision).

10.

(a)

Standard form: $y'' + 2y' + \frac{6e^x}{x} y = 0$ [1]

$\lim_{x \to 0} x(2) = 0 \qquad \lim_{x \to 0} x^2 \left(\frac{6e^x}{x}\right) = 0$ both finite

$\therefore$ $x = 0$ is a regular singular point.

(b)

From (a), $p_0 = 0$, $q_0 = 0$.

$\therefore r(r-1) = 0$. $\therefore$ $\underline{r_1 = 1,\ r_2 = 0}$

(c)

Multiplying [1] in (a) by x^2, we get

$$x^2 y'' + x(2x)y' + 6xe^x y = 0$$

i.e., $xp(x) = 2x \quad x^2 q(x) = 6xe^x$

Taylor expansion for $2x$ is $2x$

so $p_0 = 0$, $p_1 = 2$, $p_2 = p_3 = \dots p_n = 0$, $n \geq 2$

Taylor expansion for $6xe^x$ is

$$6x\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots + \frac{x^n}{n!} + \dots\right)$$

$$= 6x + 6x^2 + 3x^3 + x^4 + \dots + 6\frac{x^{n+1}}{n!} + \dots, \quad n \geq 0$$

so $q_0 = 0$, $q_1 = 6$, $q_2 = 6$, $q_3 = 3$, $q_4 = 1, \dots$

using $xp(x) = \sum_{n=0}^{\infty} p_n x^n$, $x^2 q(x) = \sum_{n=0}^{\infty} q_n x^n$

Let $y_1 = \sum_{n=0}^{\infty} a_n x^{r+n}$, $L[y] = x^2 y'' + x(xp(x))y' + x^2 q(x) y$

From $L[y_1] = 0$ and $F(r) = r(r-1) + p_0 r + q_0 = r(r-1)$

since $p_0 = q_0 = 0$, we get

$$F(r+n)a_n + \sum_{k=0}^{n-1} a_k((r+k)p_{n-k} + q_{n-k}) = 0, \quad n \geq 1. \tag{8}$$

$$\therefore a_n = \frac{-\sum_{k=0}^{n-1} a_k\left[(r+k)p_{n-k} + q_{n-k}\right]}{(r+n)(r+n-1)}, \quad n \geq 1$$

since $F(r) = r(r-1)$

$$\therefore n=1: \quad a_1 = \frac{-\sum_{k=0}^{0} a_k\left[(r+k)p_{1-k} + q_{1-k}\right]}{(r+1)r} = \frac{-a_0(rp_1+q_1)}{(r+1)r}$$

$$= -a_0\frac{(2r+6)}{(r+1)r} \qquad p_1 = 2 \quad q_1 = 6$$

$$= -\frac{(2+6)}{2(1)} = \underline{-4} \qquad a_0 = 1, \; r = 1$$

$$n=2: \quad a_2 = \frac{-\sum_{k=0}^{1} a_k\left[(r+k)p_{2-k} + q_{2-k}\right]}{(r+2)(r+1)}$$

$$= -\frac{\left[a_0(rp_2+q_2) + a_1\left[(r+1)p_1 + q_1\right]\right]}{(r+2)(r+1)}$$

$$= -\frac{\left[a_0(0+6) + (-4)\left[(r+1)2+6\right]\right]}{(r+2)(r+1)} \qquad \begin{matrix} p_2 = 0 & p_1 = 2 \\ q_2 = 6 & q_1 = 6 \end{matrix}$$

$$= -\frac{\left[6-40\right]}{(3)(2)} = \frac{34}{6} = \underline{\underline{\frac{17}{3}}} \qquad a_0 = 1, \; r = 1$$

As the computations are getting more involved, use MATLAB.

```
clear, clc
syms a0 r s sn frn
NumTerms = 4;
p0 = 0; q0 = 0;   % preload values of p(n), q(n)
p = [2 0 0 0];
q = [6 6 3 1];
a = sym('a', [NumTerms,1]);  % create symbolic array
for n = 1:NumTerms
    s = 0;              % initialize sum
    for k = 0:(n-1) % code for sigma from k=0 to n-1
        if k == 0  %MATLAB is 1-based, so a(0) not valid
            sn = a0*(r*p(n) + q(n));
        else
            sn = a(k)*((r+k)*p(n-k)+q(n-k));
        end
        s = s + sn;
    end
    frn = (r+n)*(r+n-1) + p0*(r+n) + q0;
    a(n) = -s/frn;
end
% display coefficients a(n), using a0 = 1, r = 1
subs(a,[a0 r],[1 1])
```

ans =

$$\begin{pmatrix} -4 \\ \frac{17}{3} \\ -\frac{47}{12} \\ \frac{191}{120} \end{pmatrix} \begin{matrix} = a_1 \\ = a_2 \\ = a_3 \\ = a_4 \end{matrix}$$

$\therefore\ a_1 = -4,\quad a_2 = \frac{17}{3},\quad a_3 = -\frac{47}{12},\quad a_4 = \frac{191}{120}$

$$\therefore\ y_1(x) = x^r \sum_{n=0}^{4} a_n x^n$$

$$= x\left(1 - 4x + \frac{17}{3}x^2 - \frac{47}{12}x^3 + \frac{191}{120}x^4 + \dots\right)$$

$$\boxed{y_1(x) = x - 4x^2 + \frac{17}{3}x^3 - \frac{47}{12}x^4 + \frac{191}{120}x^5 + \dots}$$

For $y_2(x)$, can't use above method substituting $r=0$, as $F(r+n) = (r+n)(r+n-1)$, so for $r=0$,

$F(r+n) = n(n-1)$, so for $n=1$, $F(r+n) = 0$.

This creates division by zero.

$\therefore$ Let $y_2(x) = a y_1(x)\ln(x) + x^0\left(1 + \sum_{n=1}^{\infty} c_n x^n\right)$

$= a y_1 \ln(x) + \left(1 + \sum_{n=1}^{\infty} c_n x^n\right)$

and substitute $y_2(x)$ into $xy'' + 2xy' + 6e^x y = 0$

$\therefore$ Using $L[y] = xy'' + 2xy' + 6e^x y$, $S = 1 + \sum_{n=1}^{\infty} c_n x^n$

so $y_2 = a y_1 \ln(x) + S$

$\therefore L[y_2] = L[a y_1 \ln(x)] + L[S]$

$= ax[y_1 \ln(x)]'' + 2ax[y_1 \ln(x)]' + 6ae^x[y_1 \ln(x)] + L[S]$

$= ax\left[y_1'' \ln(x) + 2y_1'\left(\frac{1}{x}\right) + y_1\left(-\frac{1}{x^2}\right)\right]$

$+ 2ax\left[y_1' \ln(x) + y_1\left(\frac{1}{x}\right)\right]$

$+ 6ae^x[y_1 \ln(x)] + L[S]$

$= a[xy_1'' + 2xy_1' + 6e^x y_1]\ln(x)$ $= a\underbrace{L[y_1]}_{=0}\ln(x) = 0$

$+ 2ay_1' + 2ay_1 - \frac{ay_1}{x} + L[S]$

$\therefore L[y_2] = 2ay_1' + 2ay_1 - \frac{ay_1}{x} + L[S] = 0$

$\therefore L\left[1 + \sum_{n=1}^{\infty} c_n x^n\right] = -a\left(2y_1' + 2y_1 - \frac{y_1}{x}\right)$

From above, $y_1 = x - 4x^2 + \frac{17}{3}x^3 - \frac{47}{12}x^4 + \frac{191}{120}x^5 + \dots$

Use MATLAB to do the accurate computations.

```
clear, clc
syms a n x y1 c
NumTerms = 5;

% compute -a*(2y1' + 2y1 - y1/x)
ay = [1, -4, 17/3, -47/12, 191/120];
y1 = 0;
for n = 1:NumTerms  % create y1(x)
    y1 = y1 + ay(n)*x^n;
end
% need the simplify(y1/x) to make a polynomial
pa = -a*(2*diff(y1,x,1) + 2*y1 - simplify(y1/x));
[coef_a, ta] = coeffs(pa,x,'All');
coef_a', ta' % display vertically

% now compute L[S]
% create symbolic array for S(x)
c = sym('c', [1, NumTerms]);
S = 1;
for n = 1:NumTerms
    S = S + c(n)*x^n;
end
S1 = diff(S,x,1);
S2 = diff(S,x,2);
e6 = taylor(6*exp(x), 'Order', NumTerms);
ps = x*S2 + 2*x*S1 + e6*S;  % L[y] = xy" + 2xy' + 6ey
[coefs, ts] = coeffs(ps,x,'All');
coefs', ts'  % display vertically
```

ans =

$$\begin{pmatrix} -\frac{191\,a}{60} \\ -\frac{779\,a}{120} \\ \frac{193\,a}{12} \\ -\frac{61\,a}{3} \\ 10\,a \\ -a \end{pmatrix}$$

ans =

$$\begin{pmatrix} x^5 \\ x^4 \\ x^3 \\ x^2 \\ x \\ 1 \end{pmatrix}$$

$-a\left(2y_1' + 2y_1 - \frac{y_1}{x}\right)$

ans =

$$\begin{pmatrix} \frac{c_5}{4} \\ \frac{c_4}{4} + c_5 \\ \frac{c_3}{4} + c_4 + 3\,c_5 \\ \frac{c_2}{4} + c_3 + 3\,c_4 + 6\,c_5 \\ \frac{c_1}{4} + c_2 + 3\,c_3 + 6\,c_4 + 16\,c_5 \\ c_1 + 3\,c_2 + 6\,c_3 + 14\,c_4 + 20\,c_5 + \frac{1}{4} \\ 3\,c_1 + 6\,c_2 + 12\,c_3 + 12\,c_4 + 1 \\ 6\,c_1 + 10\,c_2 + 6\,c_3 + 3 \\ 8\,c_1 + 2\,c_2 + 6 \\ 6 \end{pmatrix}$$

$L[S]$

ans =

$$\begin{pmatrix} x^9 \\ x^8 \\ x^7 \\ x^6 \\ x^5 \\ x^4 \\ x^3 \\ x^2 \\ x \\ 1 \end{pmatrix}$$

The first observation is that by equating the "1" terms, $-a = 6$, or $\underline{\underline{a = -6}}$

This could also be obtained by taking the limit $a = \lim_{r \to r_2} (r - r_2)\, a_N(r)$. Here, $r_2 = 0$

and $N = 1$, from $r_1 - r_2 = N = 1 - 0 = 1$.

From above, $a_n = \dfrac{-\sum_{k=0}^{n-1} a_k [(r+k)p_{n-k} + q_{n-k}]}{(r+n)(r+n-1)}$

$$\therefore a_N = a_1 = \frac{-\sum_{k=0}^{0} a_k [(r+k)p_{1-k} + q_{1-k}]}{(r+1)(r)} = \frac{-a_0(rp_1 + q_1)}{r(r+1)}$$

$$\therefore \lim_{r \to 0} r a_1 = \lim_{r \to 0} r \frac{[-a_0(rp_1 + q_1)]}{r(r+1)} = -a_0 q_1$$

From above, $a_0 = 1$ and $q_1 = 6$

$$\therefore a = \lim_{r \to 0} r a_1 = -(1)(6) = -6$$

The second observation is that there are more unknowns in c_i than equations in the $L[S]$ array.

For example, equating coefficients for the x^n terms, and using $a = -6$,

$$x: \quad 8c_1 + 2c_2 = -66$$

$$x^2: \quad 6c_1 + 10c_2 + 6c_3 = 119$$

$$x^3: \quad 3c_1 + 6c_2 + 12c_3 + 12c_4 = -\frac{195}{2}$$

With each additional equation, an additional variable is added. Note that to get more accurate equations, more terms of $y_1(x)$ need to be computed. Above, $y_1(x)$ was computed accurately to x^5. But $L[s]$ is equated to a term involving y_1', so is accurate to only x^4. The coefficient of x^4 for $L[s]$ has 5 unknowns in c_i. C_1 is in every equation up to x^4. If C_1 is set to 0, then the other coefficients can be calculated, and you get the answer in the back of the text. Let $C = \begin{bmatrix} c_2 \\ c_3 \\ c_4 \\ c_5 \end{bmatrix}$

and create array A from L[S] so

$AC = B - Xtra$, where $Xtra = \begin{bmatrix} 1/4 \\ 1 \\ 3 \\ 6 \end{bmatrix}$, the

constants w/o C_i coefficients in L[S] array.

Use MATLAB to solve the simultaneous equations.

```
clear, clc
syms a n x y1 c
NumTerms = 5;

% compute -a*(2y1' + 2y1 - y1/x)
ay = [1, -4, 17/3, -47/12, 191/120];  % y1(x) coeffs
y1 = 0;
for n = 1:NumTerms  % create y1(x)
    y1 = y1 + ay(n)*x^n;
end
% need the simplify(y1/x) to make a polynomial
pa = -a*(2*diff(y1,x,1) + 2*y1 - simplify(y1/x));
[coef_a, ta] = coeffs(pa,x,'All');
coef_a', ta' % display vertically

% now compute L[S]
% create symbolic array for S(x)
c = sym('c', [1, NumTerms]);
S = 1;
for n = 1:NumTerms
    S = S + c(n)*x^n;
end
S1 = diff(S,x,1);
S2 = diff(S,x,2);
e6 = taylor(6*exp(x), 'Order', NumTerms);
ps = x*S2 + 2*x*S1 + e6*S;  % L[y] = xy" + 2xy' + 6ey
[coefs, ts] = coeffs(ps,x,'All');
coefs', ts'  % display vertically

% solve for c(n) coefficients c2 -> c5, setting c1 = 0
v = subs(coef_a, a, -6)'; % evaluate using a = -6
B = v(2:(end-1)) % just the x terms: drop a assignment
C = c(1,2:end)'  % c2 -> c5
% create A from coefs array of L[S] to x^5, with c1 = 0
A = [3, 6, 14, 20;...
     6, 12, 12, 0;...
     10, 6, 0, 0;...
     2, 0, 0, 0];
% coefs array has some constants - subtract off
Xtra = [1/4, 1, 3, 6]';
% A*C = B - Xta
B - Xtra
A*C
C = A\(B - Xtra)
```

ans =

$$\begin{pmatrix} -\frac{191\,a}{60} \\ -\frac{779\,a}{120} \\ \frac{193\,a}{12} \\ -\frac{61\,a}{3} \\ 10\,a \\ -a \end{pmatrix}$$

ans =

$$\begin{pmatrix} x^5 \\ x^4 \\ x^3 \\ x^2 \\ x \\ 1 \end{pmatrix}$$

$-a\left(2y_1' + 2y_1 - \frac{y_1}{x}\right)$

ans =

$$\begin{pmatrix} \frac{c_5}{4} \\ \frac{c_4}{4} + c_5 \\ \frac{c_3}{4} + c_4 + 3\,c_5 \\ \frac{c_2}{4} + c_3 + 3\,c_4 + 6\,c_5 \\ \frac{c_1}{4} + c_2 + 3\,c_3 + 6\,c_4 + 16\,c_5 \\ c_1 + 3\,c_2 + 6\,c_3 + 14\,c_4 + 20\,c_5 + \frac{1}{4} \\ 3\,c_1 + 6\,c_2 + 12\,c_3 + 12\,c_4 + 1 \\ 6\,c_1 + 10\,c_2 + 6\,c_3 + 3 \\ 8\,c_1 + 2\,c_2 + 6 \\ 6 \end{pmatrix}$$

ans =

$$\begin{pmatrix} x^9 \\ x^8 \\ x^7 \\ x^6 \\ x^5 \\ x^4 \\ x^3 \\ x^2 \\ x \\ 1 \end{pmatrix}$$

L[S]

C =

$$\begin{pmatrix} c_2 \\ c_3 \\ c_4 \\ c_5 \end{pmatrix}$$

ans =

$$\begin{pmatrix} 3c_2 + 6c_3 + 14c_4 + 20c_5 \\ 6c_2 + 12c_3 + 12c_4 \\ 10c_2 + 6c_3 \\ 2c_2 \end{pmatrix}$$

$A * C$

B =

$$\begin{pmatrix} \frac{779}{20} \\ -\frac{193}{2} \\ 122 \\ -60 \end{pmatrix}$$

ans =

$$\begin{pmatrix} \frac{387}{10} \\ -\frac{195}{2} \\ 119 \\ -66 \end{pmatrix}$$

$B - Xtra$

$A\backslash(B - Xtra)$

C =

$$\begin{array}{cc} \begin{pmatrix} -33 \\ \frac{449}{6} \\ -\frac{1595}{24} \\ \frac{37147}{1200} \end{pmatrix} & \begin{array}{c} c_2 \\ c_3 \\ c_4 \\ c_5 \end{array} \end{array}$$

$$\therefore S(x) = 1 + \sum_{n=1}^{\infty} c_n x^n = 1 - 33x^2 + \frac{449}{6}x^3 - \frac{1594}{24}x^4 + \ldots$$

$\therefore$ Using $c_1 = 0$,

$$\boxed{y_2(x) = -6\,y_1(x)\ln(x) + 1 - 33x^2 + \frac{449}{6}x^3 - \frac{1594}{24}x^4 + \ldots}$$

Try another method: $c_n(r_2) = \frac{d}{dr}\left[(r - r_2)\,a_n(r)\right]\Big|_{r=r_2}$

$\therefore$ From above,

$$a_n = \frac{-\sum_{k=0}^{n-1} a_k\left[(r+k)p_{n-k} + q_{n-k}\right]}{(r+n)(r+n-1)}, \quad n \geq 1$$

$$\therefore n = 1: \; c_1 = \frac{d}{dr}\left[\frac{-r\sum_{k=0}^{0} a_k\left[(r+k)p_{1-k} + q_{1-k}\right]}{(r+1)\,r}\right]\Bigg|_{r=0}$$

$$= \frac{d}{dr}\left[-\frac{a_0\left[r p_1 + q_1\right]}{r+1}\right]\Bigg|_{r=0}$$

$p_1 = 2$

$q_1 = 6$

$q_0 = 1$

$$= \frac{d}{dr}\left[\frac{-2r - 6}{r+1}\right]\Bigg|_{r=0}$$

$$= \left. \frac{(r+1)(-2) - (-2r-6)(1)}{(r+1)^2} \right|_{r=0}$$

$$= \frac{-2+6}{1} = 4$$

Use MATLAB to compute all the c_n

```
clear, clc
syms a0 r s sn frn
r2 = 0;  % specify r2
NumTerms = 4;
p0 = 0; q0 = 0;   % preload values of p(n), q(n)
p = [2 0 0 0];
q = [6 6 3 1];
a = sym('a', [NumTerms,1]);  % create recurrence relation
for n = 1:NumTerms
    s = 0;              % initialize sum
    for k = 0:(n-1) % code for sigma from k=0 to n-1
        if k == 0  %MATLAB is 1-based, so a(0) not valid
            sn = a0*(r*p(n) + q(n));
        else
            sn = a(k)*((r+k)*p(n-k)+q(n-k));
        end
        s = s + sn;
    end
    frn = (r+n)*(r+n-1) + p0*(r+n) + q0;
    a(n) = -s/frn;
end
c = sym('c', [1, NumTerms])
for n = 1:NumTerms
    f = simplify((r-r2)*a(n));
    % take derivative of (r-r2)*a(n), evaluate at r = r2
    c(n) = subs(diff(f,r,1),[a0, r],[1, r2]);  % set a0 = 1, r = r2
end
c
```

c = $\begin{pmatrix} c_1 & c_2 & c_3 & c_4 \end{pmatrix}$

$$\therefore S(x) = 1 + \sum_{n=1}^{4} c_n x^n$$

c =

$\begin{pmatrix} 4 & -49 & \frac{195}{2} & -\frac{657}{8} \end{pmatrix}$

$$= 1 + 4x - 49x^2 + \frac{195}{2}x^3 - \frac{657}{8}x^4$$

$$\boxed{y_2(x) = -6\, y_1(x) \ln(x) + 1 + 4x - 49x^2 + \frac{195}{2}x^3 - \frac{657}{8}x^4 + \ldots}$$

11.

(a)

Standard form: $y'' + \left(\frac{1}{x}\right) y = 0$ [1]

$\lim_{x \to 0} x(0) = 0$, $\lim_{x \to 0} x^2\left(\frac{1}{x}\right) = 0$, both finite

$\therefore$ $x = 0$ is a regular singular point.

(b)

From (a), $p_0 = 0$, $q_0 = 0$

$\therefore$ $r(r-1) + p_0 r + q_0 = r(r-1) = 0 \Rightarrow r_1 = 1 \quad r_2 = 0$

(c)

$r_1 - r_2 = 1$

Multiplying [1] by x^2: $x^2 y'' + xy = 0$

$\therefore$ $xp(x) = 0$, $x^2 q(x) = x$

Using $xp(x) = \sum_{n=0}^{\infty} p_n x^n$, $x^2 q(x) = \sum_{n=0}^{\infty} q_n x^n$,

$p_0 = p_1 = p_2 = p_3 = p_4 = 0$

$q_0 = 0 \quad q_1 = 1 \quad q_2 = q_3 = q_4 = 0$

Let $y_1(x) = x^{r_1} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{r_1+n}$

$$L[y] = x^2 y'' + x(xp(x))y' + x^2 q(x) y$$

From $L[y_1] = 0$ and $F(r) = r(r-1) + p_0 r + q_0 = r(r-1)$

since $p_0 = q_0 = 0$. $\therefore F(r+n) = (r+n)(r+n-1)$

Also, using equation (8) from the text:

$$F(r+n)a_n + \sum_{k=0}^{n-1} a_k((r+k)p_{n-k} + q_{n-k}) = 0, \quad n \geq 1. \qquad (8)$$

We get $a_n = \dfrac{-\sum_{k=0}^{n-1} a_k[(r+k)p_{n-k} + q_{n-k}]}{(r+n)(r+n-1)}$, $n \geq 1$

$\therefore a_n = -\dfrac{\sum_{k=0}^{n-1} a_k q_{n-k}}{(r+n)(r+n-1)}$, $n \geq 1$ since all $p_i = 0$.

$\therefore a_1 = -\dfrac{a_0 q_1}{(r+1)(r)} = -\dfrac{(1)(1)}{2(1)} = -\dfrac{1}{2}$ $a_0 = 1$ $q_1 = 1$ $r = 1$

$a_2 = -\dfrac{[a_0 q_2 + a_1 q_1]}{(r+2)(r+1)} = -\dfrac{[0 + (-\frac{1}{2})]}{(3)(2)} = \dfrac{1}{12}$ $a_0 = 1$, $q_1 = 1$, $q_2 = 0$, $r = 1$

Use MATLAB for all the computations

```
clear, clc
syms a0 r s sn frn
r1 = 1;  % specify r1
NumTerms = 4;  % compute to x^4
p0 = 0; q0 = 0;   % preload values of p(n), q(n)
p = [0 0 0 0];
q = [1 0 0 0];
a = sym('a', [NumTerms,1]);  % create symbolic array
for n = 1:NumTerms
    s = 0;              % initialize sum
    for k = 0:(n-1) % code for sigma from k=0 to n-1
        if k == 0  %MATLAB is 1-based, so a(0) not valid
            sn = a0*(r*p(n) + q(n));
        else
            sn = a(k)*((r+k)*p(n-k)+q(n-k));
        end
        s = s + sn;
    end
    frn = (r+n)*(r+n-1) + p0*(r+n) + q0;
    a(n) = -s/frn;
end
subs(a,[a0 r],[1 r1])  % display coefficients a(n)
```

ans =

$$\begin{pmatrix} -\frac{1}{2} \\ \frac{1}{12} \\ -\frac{1}{144} \\ \frac{1}{2880} \end{pmatrix} \begin{matrix} = a_1 \\ = a_2 \\ = a_3 \\ = a_4 \end{matrix}$$

$$\therefore\ y_1(x) = x^{r_1} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+1} \quad \text{using } r_1 = 1,\ a_0 = 1$$

$$\therefore\ \boxed{y_1(x) = x - \frac{1}{2}x^2 + \frac{1}{12}x^3 - \frac{1}{144}x^4 + \frac{1}{2880}x^5 + \cdots}$$

Since $r_1 - r_2 = 1 - 0 = 1$, $x^0 = 1$,

$$y_2(x) = a\, y_1(x)\ln(x) + 1 + \sum_{n=1}^{\infty} c_n x^n$$

$$a = \lim_{r\to 0} r\, a_1(r) = \lim_{r\to 0} r\left[\frac{-a_0 q_1}{(r+1)r}\right] = \lim_{r\to 0} \frac{-1}{r+1} = \underline{-1} \qquad a_0 = 1,\ q_1 = 1$$

$$c_n = \frac{d}{dr}\left[r\, a_n(r)\right]\Big|_{r=0}, \quad a_n(r) = \frac{-\sum_{k=0}^{n-1} a_k\left[(r+k)p_{n-k} + q_{n-k}\right]}{(r+n)(r+n-1)}$$

Using MATLAB, compute for $n=1$ to $n=4$

```
clear, clc
syms a0 r s sn frn
r2 = 0;  % specify r2
NumTerms = 4;
p0 = 0; q0 = 0;   % preload values of p(n), q(n)
p = [0 0 0 0];
q = [1 0 0 0];
a = sym('a', [NumTerms,1]);  % create recurrence relation
for n = 1:NumTerms
    s = 0;            % initialize sum
    for k = 0:(n-1) % code for sigma from k=0 to n-1
        if k == 0  %MATLAB is 1-based, so a(0) not valid
            sn = a0*(r*p(n) + q(n));
        else
            sn = a(k)*((r+k)*p(n-k)+q(n-k));
        end
        s = s + sn;
    end
    frn = (r+n)*(r+n-1) + p0*(r+n) + q0;
    a(n) = -s/frn;
end
c = sym('c', [1, NumTerms])
for n = 1:NumTerms
    % simplify needed so common factors can be canceled, which
    %   avoids a division by zero error
    f = simplify((r-r2)*a(n));
    % take derivative of (r-r2)*a(n), evaluate at r = r2
    c(n) = subs(diff(f,r,1),[a0, r],[1, r2]);  % set a0 = 1, r = r2
end
c
```

c = $\begin{pmatrix} c_1 & c_2 & c_3 & c_4 \end{pmatrix}$

c =

$$\begin{pmatrix} 1 & -\frac{5}{4} & \frac{5}{18} & -\frac{47}{1728} \end{pmatrix}$$

$\therefore$

$$\boxed{y_2(x) = -y_1(x)\ln(x) + 1 + x - \frac{5}{4}x^2 + \frac{5}{18}x^3 - \frac{47}{1728}x^4 + \dots}$$

The answer in the back of the book uses the method of substituting $y_2(x)$ into $xy'' + y = 0$, and then computes the c_n coefficients of $1 + \sum_{n=1}^{\infty} c_n x^n$ by equating coefficients of similar x^n terms. Assigning $c_1 = 0$ then gives the book's coefficients.

MATLAB code using this method is shown below.

```
clear, clc
syms a n x y1 c
NumTerms = 5;  % y1 was evaluated to x^5

% compute -a*(2y1' - y1/x)
ay = [1, -1/2, 1/12, -1/144, 1/2880];  % y1(x) coeffs
y1 = 0;
for n = 1:NumTerms  % create y1(x)
    y1 = y1 + ay(n)*x^n;
end
% need the simplify(y1/x) to make a polynomial
pa = -a*(2*diff(y1,x,1) - simplify(y1/x));
[coef_a, ta] = coeffs(pa,x,'All');
coef_a', ta' % display vertically

% now compute L[S]
% create symbolic array for S(x)
c = sym('c', [1, NumTerms]);
S = 1;
% compute to x^4 since -a*(2y1' - y1/x) is degree 4
for n = 1:NumTerms-1
    S = S + c(n)*x^n;
end
S2 = diff(S,x,2);
ps = x*S2 + S;  % L[y] = xy" + y
[coefs, ts] = coeffs(ps,x,'All');
coefs', ts'  % display vertically

% solve for c(n) coefficients c2 -> c4, setting c1 = 0
v = subs(coef_a, a, -1)'; % evaluate using a = -1
B = v(2:(end-1)) % just the x terms: drop a assignment
C = c(1,2:end-1)'  % c2 -> c4
% create A from coefs array of L[S] to x^4, with c1 = 0
A = [0, 1, 12;...
     1, 6, 0;...
     2, 0, 0];
% A*C = B
A*C
C = A\B
```

ans =

$$\begin{pmatrix} -\frac{a}{320} \\ \frac{7\,a}{144} \\ -\frac{5\,a}{12} \\ \frac{3\,a}{2} \\ -a \end{pmatrix}$$

ans =

$$\begin{pmatrix} x^4 \\ x^3 \\ x^2 \\ x \\ 1 \end{pmatrix}$$

coeffs from $-a\left(2y_1' - \frac{y_1}{x}\right)$

ans =

$$\begin{pmatrix} c_4 \\ c_3 + 12\,c_4 \\ c_2 + 6\,c_3 \\ c_1 + 2\,c_2 \\ 1 \end{pmatrix}$$

ans =

$$\begin{pmatrix} x^4 \\ x^3 \\ x^2 \\ x \\ 1 \end{pmatrix}$$

coeffs from $L[S]$

$S = 1 + \sum_{n=1}^{\infty} c_n x^n$

B =

$$\begin{pmatrix} -\frac{7}{144} \\ \frac{5}{12} \\ -\frac{3}{2} \end{pmatrix}$$

coeffs from $-a\left(2y_1' - \frac{y_1}{x}\right)$ using $a = -1$

C =

$$\begin{pmatrix} c_2 \\ c_3 \\ c_4 \end{pmatrix}$$

ans =

$$\begin{pmatrix} c_3 + 12\,c_4 \\ c_2 + 6\,c_3 \\ 2\,c_2 \end{pmatrix}$$

A*C setting $c_1 = 0$

C =

$$\begin{pmatrix} -\frac{3}{4} \\ \frac{7}{36} \\ -\frac{35}{1728} \end{pmatrix} \begin{matrix} = c_2 \\ = c_3 \\ = c_4 \end{matrix}$$

from $C = A\backslash B$

$$\therefore \boxed{y_2(x) = -y_1(x)\ln(x) + 1 - \frac{3}{4}x^2 + \frac{7}{36}x^3 - \frac{35}{1728}x^4 + \ldots}$$

Thus, as shown in this problem and in #10 above, a different answer is obtained depending on the method used. However, the different results of $y_2(x)$ only differ by a constant multiple of $y_1(x)$. This can be shown as follows:

Let $y_2(x)$ and $y_2^*(x)$ be the different results

$\therefore L[y_2] = 0$ and $L[y_2^*] = 0$. $\therefore L[y_2 - y_2^*] = 0$

But $L[y_1] = 0$, so if s is some constant, $s \neq 0$, $L[sy_1] = 0$. $\therefore L[(y_2 - y_2^*) - sy_1] = 0$

But $(y_2 - y_2^*)$ subtracts the term containing $\ln(x)$.

$\therefore (y_2 - y_2^*)$ is just a power series, as is $s\,y_1(x)$.

$\therefore$ From equating coefficients, $(y_2 - y_2^*)$ is just a constant multiple of $y_1(x)$. $\therefore y_2^*$ does not introduce a 3rd independent solution.

12.

(a)

Standard form: $y'' + \left[\frac{\sin(x)}{x^2}\right]y' + \left[\frac{-\cos(x)}{x^2}\right]y = 0$

$\therefore \lim_{x \to 0} x\left[\frac{\sin(x)}{x^2}\right] = \lim_{x\to 0} \frac{\sin(x)}{x} = \lim_{x \to 0} \frac{\cos(x)}{1} = 1$

L'Hopital's rule

$\lim_{x\to 0} x^2\left[\frac{-\cos(x)}{x^2}\right] = -1$

$\therefore$ Both limits finite $\Rightarrow$ $x = 0$ a regular singular point

(b)

From (a), $p_0 = 1 \quad q_0 = -1$

$\therefore r(r-1) + p_0 r + q_0 = r^2 - 1 = 0, \quad r_1 = 1 \quad r_2 = -1$

(c)

From (a) $x p(x) = x\left[\frac{\sin(x)}{x^2}\right] = \frac{\sin(x)}{x}$

$x^2 q(x) = x^2\left[\frac{-\cos(x)}{x^2}\right] = -\cos(x)$

$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$

$$\therefore \frac{\sin(x)}{x} = 1 + 0\cdot x - \frac{x^2}{6} + 0\cdot x^3 + \frac{x^4}{120} + \cdots$$

$$\therefore p_0 = 1 \quad p_1 = 0 \quad p_2 = \frac{-1}{6} \quad p_3 = 0 \quad p_4 = \frac{1}{120}$$

$$-\cos(x) = -1 + 0\cdot x + \frac{x^2}{2} + 0\cdot x^3 - \frac{x^4}{24} + \cdots$$

$$\therefore q_0 = -1 \quad q_1 = 0 \quad q_2 = \frac{1}{2} \quad q_3 = 0 \quad q_4 = -\frac{1}{24}$$

Let $y_1(x) = x^{r_1} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{r_1+n}$

$$L[y] = x^2 y'' + x(x p(x)) y' + x^2 q(x) y$$

From $L[y_1] = 0$ and $F(r) = r(r-1) + p_0 r + q_0 = r^2 - 1$,

since $p_0 = 1$, $q_0 = -1$. $\therefore F(r+n) = (r+n)^2 - 1$

Also, using equation (8) from the text:

$$F(r+n)a_n + \sum_{k=0}^{n-1} a_k((r+k)p_{n-k} + q_{n-k}) = 0, \quad n \geq 1. \tag{8}$$

We get $a_n = -\dfrac{\sum_{k=0}^{n-1} a_k \left[(r+k)p_{n-k} + q_{n-k}\right]}{(r+n)^2 - 1}, \quad n \geq 1$

Using MATLAB with above values for p_i, q_i:

```
clear, clc
syms a0 r s sn frn
r1 = 1;  % specify r1
NumTerms = 4;  % compute to x^4
p0 = 1; q0 = -1;   % preload values of p(n), q(n)
p = [0 -1/6 0 1/120];
q = [0 1/2 0 -1/24];
a = sym('a', [NumTerms,1]);  % create symbolic array
for n = 1:NumTerms
    s = 0;           % initialize sum
    for k = 0:(n-1) % code for sigma from k=0 to n-1
        if k == 0  %MATLAB is 1-based, so a(0) not valid
            sn = a0*(r*p(n) + q(n));
        else
            sn = a(k)*((r+k)*p(n-k)+q(n-k));
        end
        s = s + sn;
    end
    frn = (r+n)*(r+n-1) + p0*(r+n) + q0;
    a(n) = -s/frn;
end
subs(a,[a0 r],[1 r1])  % display coefficients a(n)
```

ans =

$$\begin{pmatrix} 0 \\ -\frac{1}{24} \\ 0 \\ \frac{1}{720} \end{pmatrix} \begin{matrix} = a_1 \\ = a_2 \\ = a_3 \\ = a_4 \end{matrix}$$

$$\therefore \sum_{n=0}^{\infty} a_n x^n = 1 + 0\cdot x - \frac{1}{24}x^2 + 0\cdot x^3 + \frac{1}{720}x^4 + \ldots, \text{ using } a_0 = 1$$

$$\therefore \boxed{y_1(x) = x^1\left(\sum_{n=0}^{\infty} a_n x^n\right) = x - \frac{1}{24}x^3 + \frac{1}{720}x^5 + \ldots}$$

For $y_2(x)$, can't use $a_n(r)$ since for $r=-1$, $(r+n)^2-1$ becomes 0 for $n=2$, so get division by 0.

$r_1 - r_2 = 1-(-1) = 2$, so $N=2$, $x^{r_2} = x^{-1}$

$$\therefore y_2(x) = a\, y_1(x)\ln(x) + x^{-1}\left[1 + \sum_{n=1}^{\infty} c_n x^n\right]$$

$$a = \lim_{r\to r_2}(r-r_2)a_N(r) = \lim_{r\to -1}(r+1)a_2(r)$$

From above, with $N=2$, $p_{N-k} = p_{2-k}$, $q_{N-k} = q_{2-k}$

$$\text{and} \quad a_2(r) = \frac{-\sum_{k=0}^{1} a_k\left[(r+k)p_{2-k} + q_{2-k}\right]}{(r+2)^2 - 1}$$

$$= \frac{-\underset{=1}{a_0}\left[r\underset{=-\frac{1}{6}}{p_2} + \underset{=\frac{1}{2}}{q_2}\right] - a_1\left[(r+1)\underset{=0}{p_1} + \underset{=0}{q_1}\right]}{(r+2)^2 - 1}$$

$$= \frac{-\left[-\frac{r}{6} + \frac{1}{2}\right]}{r^2 + 4r + 3} = \frac{r-3}{6(r+1)(r+3)}$$

$$\therefore a = \lim_{r \to -1} (r+1)\frac{(r-3)}{6(r+1)(r+3)} = \lim_{r \to -1} \frac{r-3}{6(r+3)} = \underline{-\frac{1}{3}}$$

Now use MATLAB to compute $c_n = \frac{d}{dr}\left[(r+1)a_n(r)\right]\Big|_{r=-1}$

```
clear, clc
syms a0 r s sn frn
r2 = -1;  % specify r2
NumTerms = 4;
p0 = 1; q0 = -1;   % preload values of p(n), q(n)
p = [0 -1/6 0 1/120];
q = [0 1/2 0 -1/24];
a = sym('a', [NumTerms,1]);  % create recurrence relation
for n = 1:NumTerms
    s = 0;            % initialize sum
    for k = 0:(n-1) % code for sigma from k=0 to n-1
        if k == 0  %MATLAB is 1-based, so a(0) not valid
            sn = a0*(r*p(n) + q(n));
        else
            sn = a(k)*((r+k)*p(n-k)+q(n-k));
        end
        s = s + sn;
    end
    frn = (r+n)*(r+n-1) + p0*(r+n) + q0;
    a(n) = -s/frn;
end
c = sym('c', [1, NumTerms])
for n = 1:NumTerms
    % simplify needed so common factors can be canceled, which
    %   avoids a division by zero error
    f = simplify((r-r2)*a(n));
    % take derivative of (r-r2)*a(n), evaluate at r = r2
    c(n) = subs(diff(f,r,1),[a0, r],[1, r2]);  % set a0 = 1, r = r2
end
c
```

c = $\begin{pmatrix} c_1 & c_2 & c_3 & c_4 \end{pmatrix}$

c =

$$\begin{pmatrix} 0 & \frac{1}{4} & 0 & -\frac{31}{1440} \end{pmatrix}$$

$$\therefore \sum_{n=1}^{\infty} c_n x^n = 0\cdot x + \frac{1}{4}x^2 + 0\cdot x^3 - \frac{31}{1440}x^4 + \dots$$

$$\therefore x^{r_2}\left(1 + \sum_{n=1}^{\infty} c_n x^n\right) = x^{-1}\left(1 + \frac{1}{4}x^2 - \frac{31}{1440}x^4 + \dots\right)$$

$$\therefore \boxed{y_2(x) = -\frac{1}{3}y_1(x)\ln(x) + x^{-1} + \frac{1}{4}x - \frac{31}{1440}x^3 + \dots}$$

13.

(a) Standard form: $y'' + \frac{1}{2\ln(x)}y' + \frac{1}{\ln(x)}y = 0$

$\therefore$ $x = 1$ is a singular point as $\ln(1) = 0$

Consider $\lim_{x\to 1} \frac{(x-1)}{2\ln(x)} = \lim_{x\to 1} \frac{1}{2/x} = \frac{1}{2}$

L'Hopital's

$$\lim_{x\to 1} \frac{(x-1)^2}{\ln(x)} = \lim_{x\to 1} \frac{2(x-1)}{1/x} = 0$$

L'Hopital's

(b)

From (a), $p_0 = \frac{1}{2}$, $q_0 = 0$

$$\therefore\ r(r-1) + \tfrac{1}{2}r = r\left(r - \tfrac{1}{2}\right) = 0 \Rightarrow r = \underline{\tfrac{1}{2}, 0}$$

(c)

Write: $(x-1)^2 y'' + (x-1)\dfrac{(x-1)}{2\ln(x)} y' + (x-1)^2 y = 0$

Use MATLAB to get series of $p(x)$, $q(x)$:

```
clear, clc
syms x
p = (x-1)/(2*log(x));
q = (x-1)^2/log(x);
taylor(p,x,1)
taylor(q,x,1)
```

ans =

$$\frac{x}{4} - \frac{(x-1)^2}{24} + \frac{(x-1)^3}{48} - \frac{19\,(x-1)^4}{1440} + \frac{3\,(x-1)^5}{320} + \frac{1}{4}$$

ans =

$$x + \frac{(x-1)^2}{2} - \frac{(x-1)^3}{12} + \frac{(x-1)^4}{24} - \frac{19\,(x-1)^5}{720} - 1$$

Note $\dfrac{x}{4} + \dfrac{1}{4} = \dfrac{(x-1)}{4} + \dfrac{1}{2}$

$$\therefore\ xp(x) = \frac{1}{2} + \frac{(x-1)}{4} - \frac{(x-1)^2}{24} + \frac{(x-1)^3}{48} - \frac{19}{1440}(x-1)^4 + \dots$$

$$\therefore\ [p_0\ p_1\ p_2\ p_3\ p_4] = \left[\tfrac{1}{2}\ \ \tfrac{1}{4}\ \ -\tfrac{1}{24}\ \ \tfrac{1}{48}\ \ -\tfrac{19}{1440}\right]$$

$$x^2 q(x) = (x-1) + \frac{(x-1)^2}{2} - \frac{(x-1)^3}{12} + \frac{(x-1)^4}{24} + \dots$$

$$\therefore\ [q_0\ q_1\ q_2\ q_3\ q_4] = \left[0\ \ 1\ \ \tfrac{1}{2}\ \ -\tfrac{1}{12}\ \ \tfrac{1}{24}\right]$$

Using MATLAB using above and $r_1 = \frac{1}{2}$,

```
clear, clc
syms a0 r s sn frn
r1 = 1/2;  % specify r1
NumTerms = 4;  % compute to x^4
p0 = 1/2; q0 = 0;   % preload values of p(n), q(n)
p = [1/4 -1/24 1/48 -19/1440];
q = [1 1/2 -1/12 1/25];
a = sym('a', [NumTerms,1]);  % create symbolic array
for n = 1:NumTerms
    s = 0;              % initialize sum
    for k = 0:(n-1) % code for sigma from k=0 to n-1
        if k == 0  %MATLAB is 1-based, so a(0) not valid
            sn = a0*(r*p(n) + q(n));
        else
            sn = a(k)*((r+k)*p(n-k)+q(n-k));
        end
        s = s + sn;
    end
    frn = (r+n)*(r+n-1) + p0*(r+n) + q0;
    a(n) = -s/frn;
end
subs(a,[a0 r],[1 r1])  % display coefficients a(n)
```

ans =

$$\begin{pmatrix} -\frac{3}{4} \\ \frac{53}{480} \\ \frac{851}{40320} \\ -\frac{83729}{9676800} \end{pmatrix}$$

$$\therefore \sum_{n=0}^{\infty} a_n (x-1)^n = 1 - \frac{3}{4}(x-1) + \frac{53}{480}(x-1)^2 + \frac{851}{40320}(x-1)^3 + \dots$$

$$\therefore \boxed{y_1(x) = (x-1)^{\frac{1}{2}} \left[1 - \frac{3}{4}(x-1) + \frac{53}{480}(x-1)^2 + \frac{851}{40320}(x-1)^3 + \dots \right]}$$

(d)

Since $\ln(x)$ is only defined for $x > 0$, distance from $x = 0$ to $x = 1$ implies $\underline{\rho = 1}$

14.

(a)

Standard form: $y'' + \frac{[\gamma-(1+\alpha+\beta)x]}{x(1-x)}y' - \frac{\alpha\beta}{x(1-x)}y = 0$

$$\lim_{x\to 0} x\frac{[\gamma-(1+\alpha+\beta)x]}{x(1-x)} = \gamma = p_0$$

$$\lim_{x\to 0} x^2\frac{[-\alpha\beta]}{x(1-x)} = 0 = q_0$$

Both limits are finite. $\therefore$ $x=0$ is regular.

Indicial equation: $r(r-1)+\gamma r = r(r-1+\gamma)=0$

$\therefore$ $r=0,\ 1-\gamma$ are the roots.

(b)

$$\lim_{x\to 1} (x-1)\frac{[\gamma-(1+\alpha+\beta)x]}{x(1-x)} = -[\gamma-(1+\alpha+\beta)]$$

$$= 1+\alpha+\beta-\gamma = p_0$$

$$\lim_{x \to 1} (x-1)^2 \frac{[-\alpha\beta]}{x(1-x)} = 0 = q_0$$

Both limits are finite. $\therefore$ $x=1$ is regular

Indicial equation: $r(r-1) + (1+\alpha+\beta-\gamma)r =$

$$r(r+\alpha+\beta-\gamma) = 0 \Rightarrow r = 0,\ \gamma-\alpha-\beta \text{ are roots.}$$

(c)

At $x=0$, $r=0$ is one root, so $x^r = x^0 = 1$.

$$\therefore \text{Let } y_1(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + \sum_{n=2} a_n x^n$$

$$\therefore y_1' = \sum_{n=1}^{\infty} n a_n x^{n-1} \qquad y_1'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$\therefore [\gamma - (1+\alpha+\beta)x] y_1' =$$

$$\gamma \sum_{n=1}^{\infty} n a_n x^{n-1} - (1+\alpha+\beta) \sum_{n=1}^{\infty} n a_n x^n$$

$\downarrow$ shift index

$$= \gamma \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - (1+\alpha+\beta) \sum_{n=1}^{\infty} n a_n x^n$$

$$= \gamma a_1 + \sum_{n=1}^{\infty}\left[\gamma(n+1)a_{n+1} - (1+\alpha+\beta)na_n\right]x^n$$

$$= \gamma a_1 + \left[2\gamma a_2 - (1+\alpha+\beta)a_1\right]x + \sum_{n=2}^{\infty}\left[\gamma(n+1)a_{n+1} - (1+\alpha+\beta)na_n\right]x^n$$

$$x(1-x)y_1'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} - \sum_{n=2}^{\infty} n(n-1)a_n x^n$$

$$= 2a_2 x + \sum_{n=3}^{\infty} n(n-1)a_n x^{n-1} - \sum_{n=2}^{\infty} n(n-1)a_n x^n$$

$\downarrow$ shift index

$$= 2a_2 x + \sum_{n=2}^{\infty} (n+1)(n)a_{n+1} x^n - \sum_{n=2}^{\infty} n(n-1)a_n x^n$$

$$= 2a_2 x + \sum_{n=2}^{\infty}\left[n(n+1)a_{n+1} - n(n-1)a_n\right]x^n$$

$$\therefore\ x(1-x)y_1'' + \left[\gamma - (1+\alpha+\beta)x\right]y_1' - \alpha\beta y =$$

$$-\alpha\beta a_0 + \gamma a_1 + \left[2a_2(\gamma+1) - (1+\alpha+\beta)a_1 - \alpha\beta a_1\right]x$$

$$+ \sum_{n=2}^{\infty}\left[n(n+1)a_{n+1} - n(n-1)a_n + \gamma(n+1)a_{n+1} - (1+\alpha+\beta)na_n - \alpha\beta a_n\right]x^n$$

$$= -\alpha\beta a_0 + \gamma a_1 + \left[2(\gamma+1)a_2 - (1+\alpha+\beta+\alpha\beta)a_1\right]x$$

$$+ \sum_{n=2}^{\infty}\left[\underbrace{(n+1)(n+\gamma)}_{n(n+1)+\gamma(n+1)}a_{n+1} - \underbrace{\left(n(n+\alpha+\beta)+\alpha\beta\right)}_{n(n-1)+(1+\alpha+\beta)n+\alpha\beta}a_n\right]x^n = 0$$

Equating coefficients,

$$a_1 = \frac{\alpha\beta}{\gamma} a_0 \qquad a_2 = \frac{\overset{=(1+\alpha)(1+\beta)}{1+\alpha+\beta+\alpha\beta}}{2(\gamma+1)} a_1 = \frac{(1+\alpha)(1+\beta)\alpha\beta}{2\gamma(\gamma+1)} a_0$$

$$a_{n+1} = \frac{n(n+\alpha+\beta)+\alpha\beta}{(n+1)(n+\gamma)} a_n \quad , \quad n \geq 2$$

$$\text{Or,} \quad a_n = \frac{(n-1)(n-1+\alpha+\beta)+\alpha\beta}{n(n-1+\gamma)} a_{n-1} \quad , \quad n \geq 3$$

$$\therefore a_3 = \frac{2(2+\alpha+\beta)+\alpha\beta}{3(2+\gamma)} a_2 = \frac{[2(2+\alpha+\beta)+\alpha\beta](1+\alpha)(1+\beta)\alpha\beta}{3\cdot 2\cdot \gamma(\gamma+1)(\gamma+2)} a_0$$

$$2(2+\alpha+\beta)+\alpha\beta = 4+2\alpha+2\beta+\alpha\beta = (2+\alpha)(2+\beta)$$

$$= \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{3!\,\gamma(\gamma+1)(\gamma+2)} a_0$$

$$\therefore a_n = \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)\beta(\beta+1)\cdots(\beta+n-1)}{n!\,\gamma(\gamma+1)\cdots(\gamma+n-1)} a_0$$

$\therefore$ setting $a_0 = 1$,

$$\boxed{\begin{aligned} y_1(x) &= 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)\beta(\beta+1)\cdots(\beta+n-1)}{n!\,\gamma(\gamma+1)\cdots(\gamma+n-1)} x^n \\ &= 1 + \frac{\alpha\beta}{1!\,\gamma} x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{2!\,\gamma(\gamma+1)} x^2 + \cdots \end{aligned}}$$

$$x\,p(x) = \frac{\gamma - (1+\alpha+\beta)x}{1-x}, \text{ so } p = 1$$

$$x^2 q(x) = \frac{-\alpha\beta x}{1-x}, \text{ so } p = 1$$

$\therefore$ Expect ρ for $y_1(x)$ to be $\underline{\rho = 1}$

(d)

$1-\gamma$ is the second root for regular point $x=0$.

Let $y_2(x) = x^{1-\gamma}\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+1-\gamma}$

$$\therefore -\alpha\beta y_2 = \sum_{n=0}^{\infty} -\alpha\beta a_n x^{n+1-\gamma} \qquad [0]$$

(1) $\therefore y_2' = \sum_{n=0}^{\infty}(n+1-\gamma)a_n x^{n-\gamma}$ $\quad\therefore [\gamma - (1+\alpha+\beta)x]\,y_2' =$

$$\gamma\sum_{n=0}^{\infty}(n+1-\gamma)a_n x^{n-\gamma} - (1+\alpha+\beta)\sum_{n=0}^{\infty}(n+1-\gamma)a_n x^{n+1-\gamma}$$

separate out 1st term

$$= \gamma(1-\gamma)a_0 x^{-\gamma} + \sum_{n=1}^{\infty}\gamma(n+1-\gamma)a_n x^{n-\gamma} - \sum_{n=0}^{\infty}(1+\alpha+\beta)(n+1-\gamma)a_n x^{n+1-\gamma}$$

shift index

$$= \gamma(1-\gamma)a_0 x^{-\gamma} + \sum_{n=0}^{\infty}\left[\gamma(n+2-\gamma)a_{n+1} - (1+\alpha+\beta)(n+1-\gamma)a_n\right]x^{n+1-\gamma} \quad [1]$$

(2) $$y_2'' = \sum_{n=0}^{\infty}(n+1-\gamma)(n-\gamma)\,a_n\,x^{n-1-\gamma}$$

$$\therefore\ x(1-x)\,y_2'' =$$

$$\sum_{n=0}^{\infty}(n+1-\gamma)(n-\gamma)a_n x^{n-\gamma} - \sum_{n=0}^{\infty}(n+1-\gamma)(n-\gamma)a_n x^{n+1-\gamma}$$

separate out 1st term

$$= (1-\gamma)(-\gamma)a_0 x^{-\gamma} + \sum_{n=1}^{\infty}(n+1-\gamma)(n-\gamma)a_n x^{n-\gamma} - \sum_{n=0}^{\infty}(n+1-\gamma)(n-\gamma)a_n x^{n+1-\gamma}$$

shift index

$$= (1-\gamma)(-\gamma)a_0 x^{-\gamma} + \sum_{n=0}^{\infty}\left[(n+2-\gamma)(n+1-\gamma)a_{n+1} - (n+1-\gamma)(n-\gamma)a_n\right]x^{n+1-\gamma} \quad [2]$$

$\therefore$ [2] + [1] + [0] yields:

$$\left[\underbrace{(1-\gamma)(-\gamma) + \gamma(1-\gamma)}_{=0}\right]a_0 x^{-\gamma} +$$

$$\sum_{n=0}^{\infty}\Big[(n+2-\gamma)(n+1-\gamma)a_{n+1} - (n+1-\gamma)(n-\gamma)a_n$$

$$+\gamma(n+2-\gamma)a_{n+1} - (1+\alpha+\beta)(n+1-\gamma)a_n - \alpha\beta a_n\Big]x^{n+1-\gamma}$$

$$= \sum_{n=0}^{\infty}\Big[(n+2-\gamma)(n+1)a_{n+1} - \left[(n+1-\gamma)(n-\gamma) + (1+\alpha+\beta)(n+1-\gamma) + \alpha\beta\right]a_n\Big]x^{n+1-\gamma} = 0$$

Equating coefficients, $$a_{n+1} = \frac{(n+1-\gamma)(n-\gamma) + (1+\alpha+\beta)(n+1-\gamma) + \alpha\beta}{(n+1)(n+2-\gamma)}\,a_n$$

Or, $a_n = \dfrac{(n-\gamma)(n-1-\gamma) + (1+\alpha+\beta)(n-\gamma) + \alpha\beta}{n(n+1-\gamma)}\, a_{n-1},\ n \geq 1$

To simplify the numerator,

$(n-\gamma)(n-\gamma-1) + (n-\gamma)(1+\alpha+\beta) + \alpha\beta$

multiply out terms

$= (n-\gamma)(n-\gamma) - (n-\gamma) + (n-\gamma) + (n-\gamma)\alpha + (n-\gamma)\beta + \alpha\beta$

cancel

" $(n-\gamma+\alpha)\beta$

$= (n-\gamma)(n-\gamma) + (n-\gamma)\alpha + (n-\gamma+\alpha)\beta$

" $(n-\gamma)(n-\gamma+\alpha)$

$= (n-\gamma)(n-\gamma+\alpha) + (n-\gamma+\alpha)\beta$

$= (n-\gamma+\alpha)(n-\gamma+\beta)$, or $(\alpha-\gamma+n)(\beta-\gamma+n)$

$\therefore\ a_n = \dfrac{(\alpha-\gamma+n)(\beta-\gamma+n)}{n(n+1-\gamma)}\, a_{n-1},\ n \geq 1$

$\therefore\ a_1 = \dfrac{(\alpha-\gamma+1)(\beta-\gamma+1)}{1\cdot(2-\gamma)}\, a_0$

$a_2 = \dfrac{(\alpha-\gamma+2)(\beta-\gamma+2)}{2(3-\gamma)}\, a_1 = \dfrac{(\alpha-\gamma+1)(\alpha-\gamma+2)(\beta-\gamma+1)(\beta-\gamma+2)}{2\cdot 1\cdot(3-\gamma)(2-\gamma)}\, a_0$

$$\therefore\ \boxed{a_n = \dfrac{(\alpha-\gamma+1)(\alpha-\gamma+2)\cdots(\alpha-\gamma+n)(\beta-\gamma+1)(\beta-\gamma+2)\cdots(\beta-\gamma+n)}{n!\,(2-\gamma)(3-\gamma)\cdots(n+1-\gamma)}\, a_0}$$

Setting $a_0 = 1$,

$$\boxed{\begin{aligned} y_2(x) &= x^{1-\gamma}\sum_{n=0}^{\infty} a_n x^n \\ &= x^{1-\gamma}\left[1 + \frac{(\alpha-\gamma+1)(\beta-\gamma+1)}{(2-\gamma)}x \right. \\ &\qquad \left. + \frac{(\alpha-\gamma+1)(\alpha-\gamma+2)(\beta-\gamma+1)(\beta-\gamma+2)}{2!(2-\gamma)(3-\gamma)}x^2 + \cdots\right] \end{aligned}}$$

(e)

Using ϵ instead of ξ (easier to write),

let $\epsilon(x) = \frac{1}{x}$ and $y(x) = u(\epsilon(x))$ $\therefore$ $y(x) = u(\frac{1}{x})$

and $u(x) = y(\frac{1}{x})$.

Using computations of $u(\epsilon)$ from #32 of section 5.4,

$x(1-x)y'' + [\gamma - (1+\alpha+\beta)x]y' - \alpha\beta y = 0$ becomes

$\frac{1}{\epsilon}(1-\frac{1}{\epsilon})[2\epsilon^3 u' + \epsilon^4 u''] +$

$[\gamma - (1+\alpha+\beta)\frac{1}{\epsilon}][-\epsilon^2 u'] - \alpha\beta u = 0$

Or, $2(\epsilon^2 - \epsilon)u' + (\epsilon^3 - \epsilon^2)u'' + [-\gamma\epsilon^2 + (1+\alpha+\beta)\epsilon]u' - \alpha\beta u$

$= (\epsilon^3 - \epsilon^2)u'' + [(2-\gamma)\epsilon^2 + (-1+\alpha+\beta)\epsilon]u' - \alpha\beta u = 0$

In standard form,

$$u'' + \frac{\left[(2-\gamma)\epsilon^2 + (-1+\alpha+\beta)\epsilon\right]}{\epsilon^2(\epsilon-1)} u' - \frac{\alpha\beta}{\epsilon^2(\epsilon-1)} u = 0$$

$\therefore$ $\epsilon = 0$ is a singular point

$$\lim_{\epsilon \to 0} \epsilon\, p(\epsilon) = \lim_{\epsilon \to 0} \epsilon \frac{\left[(2-\gamma)\epsilon^2 + (-1+\alpha+\beta)\epsilon\right]}{\epsilon^2(\epsilon-1)}$$

$$= \lim_{\epsilon \to 0} \frac{\left[(2-\gamma)\epsilon + (-1+\alpha+\beta)\right]}{\epsilon-1} = 1-\alpha-\beta = p_0$$

$$\lim_{\epsilon \to 0} \epsilon^2 q(\epsilon) = \lim_{\epsilon \to 0} \epsilon^2 \left[\frac{-\alpha\beta}{\epsilon^2(\epsilon-1)}\right] = \alpha\beta = q_0$$

Both limits are finite, $\therefore$ $\epsilon = 0$ is a regular singular point (i.e., x is regular as $x \to \infty$)

Indicial equation: $r(r-1) + (1-\alpha-\beta)r + \alpha\beta = 0$

from $r(r-1) + p_0 r + q_0 = 0$

or $r^2 - (\alpha+\beta)r + \alpha\beta = (r-\alpha)(r-\beta) = 0$

$\therefore$ $r = \alpha, \beta$ are roots.

15.

(a)

Standard form: $y'' + \frac{\alpha}{x^2} y' + \frac{\beta}{x^3} y = 0$

$\therefore x = 0$ a singular point.

$\lim_{x \to 0} x p(x) = \lim_{x \to 0} x \left(\frac{\alpha}{x^2} \right) = \lim_{x \to 0} \frac{\alpha}{x} = \infty$, as $\alpha \neq 0$

$\therefore x = 0$ is not a regular singular point.

$\therefore x = 0$ is an irregular singular point.

(b)

Assume $y = \sum_{n=0}^{\infty} a_n x^{r+n}$

$\therefore y' = \sum_{n=0}^{\infty} (r+n) a_n x^{r+n-1}$

$y'' = \sum_{n=0}^{\infty} (r+n-1)(r+n) a_n x^{r+n-2}$

$\therefore\ x^3 y'' + \alpha x y' + \beta y =$

$$\sum_{n=0}^{\infty} (r+n-1)(r+n) a_n x^{r+n+1} + \sum_{n=0}^{\infty} \alpha (r+n) a_n x^{r+n} + \sum_{n=0}^{\infty} \beta a_n x^{r+n}$$

$\downarrow$ shift index

$$= \sum_{n=1}^{\infty} (r+n-2)(r+n-1) a_{n-1} x^{r+n} + \sum_{n=0}^{\infty} \alpha (r+n) a_n x^{r+n} + \sum_{n=0}^{\infty} \beta a_n x^{r+n}$$

separate out $n=0$ terms, combine

$$= (\alpha r + \beta) a_0 x^r + \sum_{n=1}^{\infty} \left[(r+n-2)(r+n-1) a_{n-1} + (\alpha(r+n) + \beta) a_n \right] x^{r+n} = 0$$

Setting coefficients to zero: $\alpha r + \beta = 0$, or $\underline{r = -\frac{\beta}{\alpha}}$

as $\alpha \neq 0$.

and $a_n = \dfrac{-(r+n-2)(r+n-1)}{\alpha(r+n) + \beta} a_{n-1}, \quad n \geq 1$

assuming $\alpha(r+n) + \beta \neq 0$.

$\therefore$ Indicial equation: $\underline{\alpha r + \beta = 0}$

(c)

(1) Substituting $r = \frac{-\beta}{\alpha}$ into the recurrence relation from (6),

$$a_n = -\frac{\left(-\frac{\beta}{\alpha}+n-2\right)\left(-\frac{\beta}{\alpha}+n-1\right)}{\alpha\left(-\frac{\beta}{\alpha}+n\right)+\beta} a_{n-1}$$

$$= -\frac{\left(n-2-\frac{\beta}{\alpha}\right)\left(n-1-\frac{\beta}{\alpha}\right)}{\alpha n} a_{n-1} \quad , \quad n \geq 1$$

note: $\alpha \neq 0$

$\therefore$ $a_n = 0$ when $n-2-\frac{\beta}{\alpha} = 0$ or $n-1-\frac{\beta}{\alpha} = 0$,

that is when $n-2 = \frac{\beta}{\alpha}$ or $n-1 = \frac{\beta}{\alpha}$, $n \geq 1$.

$\therefore \left\{\frac{\beta}{\alpha} : \frac{\beta}{\alpha} = 1-2, 2-2, 3-2, \ldots\right\} \cup \left\{\frac{\beta}{\alpha} : \frac{\beta}{\alpha} = 1-1, 2-1, 3-1, \ldots\right\}$

i.e., $\underline{\left\{\frac{\beta}{\alpha} : \frac{\beta}{\alpha} = -1, 0, 1, 2, 3, \ldots\right\}}$

When that happens, $a_n = 0$, $\therefore a_{n+1} = a_{n+2} = a_{n+3} = \ldots = 0$.

$\therefore y(x) = \sum_{n=0}^{\infty} a_n x^{r+n}$ is finite and so converges for all x.

(2) For other values of $\frac{\beta}{\alpha}$, from (6),

$$a_n = \frac{-(r+n-2)(r+n-1)}{\alpha(r+n)+\beta} a_{n-1}, \quad n \geq 1$$

and the numerator is never 0.

$$\therefore a_1 = \frac{-(r-1)(r)}{\alpha(r+1)+\beta} a_0$$

$$a_2 = \frac{-(r)(r+1)}{\alpha(r+2)+\beta} a_1 = \frac{(r)(r+1)(r-1)(r)}{[\alpha(r+2)+\beta][\alpha(r+1)+\beta]} a_0$$

$$a_3 = \frac{-(r+1)(r+2)}{\alpha(r+3)+\beta} a_2 = \frac{-(r+1)(r+2)(r)(r+1)(r-1)(r)}{[\alpha(r+3)+\beta][\alpha(r+2)+\beta][\alpha(r+1)+\beta]} a_0$$

$$= \frac{-(r+2)(r+1)(r)\,(r+1)(r)(r-1)}{[\alpha(r+2)+\beta][\alpha(r+2)+\beta]\{\alpha(r+1)+\beta]} a_0$$

$$\therefore a_n = \frac{(-1)^n (r+n-1)\cdots(r+1)(r)\,(r+n-2)\cdots(r)(r-1)}{[\alpha(r+n)+\beta]\cdots[\alpha(r+1)+\beta]} a_0$$

Using the ratio test, $\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| = |x| \left|\frac{a_{n+1}}{a_n}\right|$

$$= |x| \left| \frac{(r+n)\cdots(r+1)(r)\,(r+n-1)\cdots(r)(r-1)}{[\alpha(r+n+1)+\beta]\cdots\{\alpha(r+1)+\beta\}} \cdot \frac{[\alpha(r+n)+\beta]\cdots[\alpha(r+1)+\beta]}{(r+n-1)\cdots(r)(r+n-2)\cdots(r-1)} \right|$$

$$= |x| \left| \frac{(r+n)(r+n-1)}{\alpha(r+n+1)+\beta} \right|$$

$$\therefore \lim_{n\to\infty}\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| = \lim_{n\to\infty}|x|\left|\frac{(r+n)(r+n-1)}{\alpha(r+n+1)+\beta}\right|$$

$$= |x| \lim_{n\to\infty}\left|\frac{(r+n)(\frac{r}{n}+1-\frac{1}{n})}{\alpha(\frac{r}{n}+1+\frac{1}{n})+\frac{\beta}{n}}\right|$$

$$= |x| \lim_{n\to\infty}\left|\frac{(r+n)}{\alpha}\right| = \underline{\infty} \text{ for any } x \neq 0.$$

$\therefore$ The series diverges for all other values of $\frac{\beta}{\alpha} \Rightarrow \sum_{n=0}^{\infty} a_n x^{r+n}$ is not an actual solution.

16.

(a)

$$\lim_{x\to 0} x\left(\frac{\alpha}{x^s}\right) = \lim_{x\to 0}\frac{\alpha}{x^{s-1}} = \infty \text{ if } s>1, \text{ so } s-1>0$$

Similarly, $\lim_{x\to 0} x^2\left(\frac{\beta}{x^t}\right) = \lim_{x\to 0}\frac{\beta}{x^{t-2}} = \infty$, if $t>2$

$\therefore$ if either $s>1$ or $t>2$, $x=0$ is <u>irregular</u>

(b)

$$y' = \sum_{n=0}^{\infty} (r+n) a_n x^{r+n-1} \qquad y'' = \sum_{n=0}^{\infty} (r+n)(r+n-1) a_n x^{r+n-2}$$

$$\therefore y'' + \alpha x^{-s} y' + \beta x^{-t} y =$$

$$\sum_{n=0}^{\infty} (r+n)(r+n-1) a_n x^{r+n-2}$$

$$+ \sum_{n=0}^{\infty} \alpha (r+n) a_n x^{r+n-1-s} + \sum_{n=0}^{\infty} \beta a_n x^{r+n-t} = 0$$

If $s=2$ and $t=2$, then the first term from each series is: $r(r-1)a_0 x^{r-2} + \alpha r a_0 x^{r-3} + \beta a_0 x^{r-2}$

$$= (r^2 - r + \beta) a_0 x^{r-2} + \alpha r a_0 x^{r-3} = 0$$

Setting the coefficients to 0,

$r^2 - r + \beta = 0$ and $\alpha r a_0 = 0$.

Assuming $a_0 \neq 0$, and since $\alpha \neq 0$, $\therefore r = 0$ from $\alpha r a_0 = 0$.

$\therefore$ There is only one value of r $(r=0)$

for a possible solution of form $y=\sum_{n=0}^{\infty} a_n x^{r+n}$

Note: for $y=\sum_{n=0}^{\infty} a_n x^n$, 1st term is $\alpha a_1 + \beta a_0 = 0$

(c)

For $s=1, t=3$, $y'' + \alpha x^{-s} y' + \beta x^{-t} y =$

$$y'' + \alpha x^{-1} + \beta x^{-3} y$$

$$\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n x^{r+n-2} + \sum_{n=0}^{\infty}\alpha(r+n)a_n x^{r+n-2} + \sum_{n=0}^{\infty}\beta a_n x^{r+n-3} = 0$$

The first term from each series yields:

$$r(r-1)a_0 x^{r-2} + \alpha r a_0 x^{r-2} + \beta a_0 x^{r-3}$$

$$= [r^2 + (\alpha-1)r]a_0 x^{r-2} + \beta a_0 x^{r-3} = 0$$

Setting coefficients to 0 means:

$\beta a_0 = 0$, which is impossible since $\beta \neq 0$, $a_0 \neq 0$.

$\therefore$ There is no value of r for which there is a solution of the form $y=\sum_{n=0}^{\infty} a_n x^{r+n}$

(d)

From (6), $y'' + \alpha x^{-s} y' + \beta x^{-t} y =$

$$\sum_{n=0}^{\infty} (r+n)(r+n-1) a_n x^{r+n-2} + \sum_{n=0}^{\infty} \alpha (r+n) a_n x^{r+n-1-s} + \sum_{n=0}^{\infty} \beta a_n x^{r+n-t} = 0$$

$\therefore$ The first term from each series is:

$$r(r-1) a_0 x^{r-2} + \alpha r a_0 x^{r-1-s} + \beta a_0 x^{r-t} = 0$$

Note s, t are positive integers, and the indicial equation comes from the lowest power of x.

For quadratic in r, this means x^{r-2} must be lowest power since it's coefficient is the quadratic term $r(r-1)a_0$ (and $a_0 \neq 0$).

$\therefore$ $r-1-s \geq r-2$, so $1 \geq s$

and $r-t \geq r-2$, so $2 \geq t$

$\therefore$ Maximum values are $\underline{s=1}$ and $\underline{t=2}$.

1.

Standard Form: $y'' + \frac{2}{x}y' + \frac{1}{x}y = 0$

(a) $\lim_{x\to 0} x\left(\frac{2}{x}\right) = 2$ $\quad \lim_{x\to 0} x^2\left(\frac{1}{x}\right) = 0$

Both limits finite, $\therefore$ regular singular point.

(b) $xp(x) = 2 \therefore p_0 = 2$ $\quad x^2 q(x) = x, \therefore q_0 = 0$

$\therefore$ Indicial equation: $r(r-1) + 2r = 0$,

$r(r+1) = 0 \Rightarrow r = 0, -1$

(1) Let $y_1(x) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}$, $a_0 \neq 0$

$\therefore x^2 y_1'' + 2x y_1' + x y_1 = 0 \Rightarrow$

$$\sum_{n=2}^{\infty} (n+r)(n+r-1)a_n x^{n+r} + \sum_{n=1}^{\infty} 2(n+r)a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+1}$$

shift index $\quad$ shift index

$$= \sum_{n=1}^{\infty}(n+r+1)(n+r)a_{n+1}x^{n+r+1} + 2(1+r)a_1 x^{r+1} + \sum_{n=1}^{\infty} 2(n+r+1)a_{n+1} x^{n+r+1}$$

$$+ a_0 x^{r+1} + \sum_{n=1}^{\infty} a_n x^{n+r+1}$$

$$= \left[a_0 + 2(r+1)a_1\right] x^{r+1} + \sum_{n=1}^{\infty} \left[(n+r+1)(n+r+2)a_{n+1} + a_n\right] x^{r+n+1} = 0$$

$$\therefore 2(r+1)a_1 = -a_0, \quad a_{n+1} = \frac{-a_n}{(n+r+2)(n+r+1)}, \quad n \geq 1$$

For $r=0$, $a_1 = -\dfrac{a_0}{2}$, $\quad a_n = -\dfrac{a_{n-1}}{(n+1)\,n}$, $n \geq 2$

$$\therefore a_2 = \frac{-a_1}{3\cdot 2} = \frac{a_0}{3\cdot 2\cdot 2} \qquad a_3 = \frac{-a_2}{4\cdot 3} = -\frac{a_0}{4\cdot 3\cdot 3\cdot 2\cdot 2}$$

$$a_4 = -\frac{a_3}{5\cdot 4} = \frac{a_0}{5\cdot 4\cdot 4\cdot 3\cdot 3\cdot 2\cdot 2} = \frac{a_0}{(5\cdot 4\cdot 3\cdot 2)(4\cdot 3\cdot 2\cdot 1)}$$

$\therefore a_n = \dfrac{(-1)^n a_0}{(n+1)!\,n!}$, $n \geq 2$, and works for $n=1$

$$\therefore \boxed{y_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(n+1)!\,n!}}$$

(2) Let $y_2(x) = a y_1(x) \ln(x) + \sum_{n=0}^{\infty} c_n x^{n-1}$, $c_0 = 1$

$$\therefore 2xy_2' = 2ax\left[y_1' \ln(x) + y_1\left(\tfrac{1}{x}\right)\right] + 2x\sum_{n=0}^{\infty} (n-1)c_n x^{n-2}$$

$$x^2 y_2'' = ax^2\left[y_1'' \ln(x) + 2y_1'\left(\tfrac{1}{x}\right) + y_1\left(-\tfrac{1}{x^2}\right)\right] + x^2\sum_{n=0}^{\infty} (n-2)(n-1)c_n x^{n-3}$$

$$x y_2 = a x y_1 \ln(x) + x \sum_{n=0}^{\infty} c_n x^{n-1}$$

$$\therefore L[y_2](x) = a \ln(x) [x^2 y_1'' + 2x y_1' + \underbrace{x y_1}_{=0}]$$

$$+ 2a y_1 + 2ax y_1' - a y_1$$

$$+ \sum_{n=0}^{\infty} (n-2)(n-1) c_n x^{n-1} + \sum_{n=0}^{\infty} 2(n-1) c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n$$

{extract 1st term, shift index}

$$= a y_1 + 2axy_1' + 2c_0 x^{-1} - 2c_0 x^{-1}$$

$$+ \sum_{n=0}^{\infty} [(n-1)n c_{n+1} + 2n c_{n+1} + c_n] x^n = 0$$

$$\therefore \sum_{n=0}^{\infty} [n(n+1) c_{n+1} + c_n] x^n = -a (y_1 + 2x y_1'), \quad c_0 = 1$$

$$y_1 = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(n+1)! n!}, \quad 2x y_1' = \sum_{n=1}^{\infty} \frac{(-1) 2n x^n}{(n+1)! n!}$$

$$\therefore y_1 + 2x y_1' = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n (2n+1) x^n}{(n+1)! n!}$$

$$\therefore 0 \cdot c_1 + c_0 + \sum_{n=1}^{\infty} [n(n+1) c_{n+1} + c_n] x^n = -a - a \sum_{n=1}^{\infty} \frac{(-1)^n (2n+1)}{(n+1)! n!} x^n$$

$\therefore c_0 = 1 = -a$, $\underline{a = -1}$ $\quad$ c_1 is undetermined

$$n(n+1) c_{n+1} + c_n = \frac{(-1)^n (2n+1)}{(n+1)! n!}, \quad n \geq 1$$

or $(n-1)n\, c_n + c_{n-1} = \dfrac{(-1)^{n-1}(2n-1)}{n!\,(n-1)!}$, $n \geq 2$

$$\therefore\ c_n = \frac{\dfrac{(-1)^{n-1}(2n-1)}{n!\,(n-1)!} - c_{n-1}}{n(n-1)}, \quad n \geq 2$$

$$c_2 = \frac{-\frac{3}{2} - c_1}{2\cdot 1} \quad \therefore \text{Let } c_1 = 1$$

$$\therefore\ c_2 = -\frac{(1+\frac{1}{2})+1}{2\cdot 1} = -\frac{(H_2+H_1)}{2\cdot 1} = -\frac{(H_2+H_1)}{2!\,1!}$$

$$c_3 = \frac{\frac{5}{3!\,2!} - c_2}{3\cdot 2} = \frac{\frac{5}{6\cdot 2} + \frac{H_2+H_1}{2}}{3\cdot 2}$$

$$= \frac{\frac{1}{2}\left(\frac{1}{3}+\frac{1}{2} + H_2 + H_1\right)}{3\cdot 2} = \frac{(H_3+H_2)}{3\cdot 2\cdot 2}$$

$$= \frac{(H_3+H_2)}{3!\,2!}$$

Note: $\dfrac{2n-1}{n(n-1)} = \dfrac{n+n-1}{n(n-1)} = \dfrac{1}{n} + \dfrac{1}{n-1}$

$\therefore$ Letting $a_n = \dfrac{(-1)^{n-1}}{n!\,(n-1)!}$ $\quad \therefore \dfrac{a_n}{(n+1)n} = -a_{n+1}$

$$\therefore\ c_3 = a_3(H_3+H_2)$$

Also, $\frac{1}{n} + \frac{1}{n-1} + H_{n-1} + H_{n-2} = H_n + H_{n-1}$

$$\therefore\ C_4 = a_4\left(\frac{1}{4}+\frac{1}{3}\right) - \frac{C_3}{4\cdot 3}$$

$$= a_4\left(\frac{1}{4}+\frac{1}{3}\right) - \frac{a_3}{4\cdot 3}\left(H_3+H_2\right)$$

$$= a_4\left(\frac{1}{4}+\frac{1}{3}\right) + a_4\left(H_3+H_2\right)$$

$$= a_4\left(H_3+\frac{1}{4}+H_2+\frac{1}{3}\right) = a_4\left(H_4+H_3\right)$$

$$\therefore\ C_n = \frac{\dfrac{(-1)^{n-1}(2n-1)}{n!\,(n-1)!} - C_{n-1}}{n(n-1)} \quad ,\ n \geq 4$$

$$= a_n\left(\frac{1}{n}+\frac{1}{n-1}\right) - \frac{C_{n-1}}{n(n-1)}$$

$$= a_n\left(\frac{1}{n}+\frac{1}{n-1}\right) - \frac{a_{n-1}\left(H_{n-1}+H_{n-2}\right)}{n(n-1)}$$

$$= a_n\left(\frac{1}{n}+\frac{1}{n-1}\right) + a_n\left(H_{n-1}+H_{n-2}\right)$$

$$= a_n\left(H_{n-1}+\frac{1}{n}+H_{n-2}+\frac{1}{n-1}\right)$$

$$= a_n\left(H_n+H_{n-1}\right) \quad ,\ n \geq 4$$

and this works for $n=3$, $n=2$ from above.

Setting $n=1$, $a_1\left(H_1+H_0\right) = \dfrac{(-1)^{1-1}}{1!\,0!}(1+0) = 1$

using $0!=1$, $H_0=0$

$\therefore$ Since set $c_1 = 1$, this also works for $n=1$

$$\therefore x^{-1}\sum_{n=0}^{\infty} c_n x^n = x^{-1}\left[1 + \sum_{n=1}^{\infty} c_n x^n\right] \quad \text{as } c_0 = 1$$

$$= x^{-1}\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(H_n + H_{n-1})}{n!\,(n-1)!} x^n\right]$$

$$\therefore \boxed{y_2(x) = -y_1(x)\ln(x) + \frac{1}{x}\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(H_n + H_{n-1})}{n!\,(n-1)!} x^n\right]}$$

2.

(a) standard form: $y'' + \frac{3}{x}y' + \left(\frac{1+x}{x^2}\right)y = 0$

$$\therefore \lim_{x\to 0} x\left(\frac{3}{x}\right) = 3 \qquad \lim_{x\to 0} x^2\left(\frac{1+x}{x^2}\right) = 1$$

Both limits are finite. $\therefore$ $x=0$ is <u>regular</u> singular

(b) $xp(x) = 3 \Rightarrow p_0 = 3 \qquad x^2 q(x) = 1 + x \Rightarrow q_0 = 1$

$$\therefore r(r-1) + 3r + 1 = (r+1)^2 = 0 \Rightarrow r = -1, -1$$

(1) $\therefore$ Let $y_1(x) = x^{-1}\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n-1}$

$$\therefore (1+x)y_1 = \sum_{n=0}^{\infty} a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n$$

$$= a_0 x^{-1} + \sum_{n=0}^{\infty} (a_{n+1} + a_n) x^n$$

$$3xy_1' = \sum_{n=0}^{\infty} 3(n-1) a_n x^{n-1} = -3a_0 x^{-1} + \sum_{n=1}^{\infty} 3(n-1) a_n x^{n-1}$$

$$= -3a_0 x^{-1} + \sum_{n=0}^{\infty} 3n a_{n+1} x^n$$

$$x^2 y_1'' = \sum_{n=0}^{\infty} (n-1)(n-2) a_n x^{n-1} = 2a_0 x^{-1} + \sum_{n=1}^{\infty} (n-1)(n-2) a_n x^{n-1}$$

$$= 2a_0 x^{-1} + \sum_{n=0}^{\infty} n(n-1) a_{n+1} x^n$$

$$a_0 x^{-1} - 3a_0 x^{-1} + 2a_0 x^{-1} = 0$$

$$a_{n+1} + a_n + 3n a_{n+1} + n(n-1) a_{n+1} = (n+1)^2 a_{n+1} + a_n$$

$$\therefore x^2 y_1'' + 3xy_1' + (1+x) y_1 = 0 \Rightarrow$$

$$\sum_{n=0}^{\infty} \left[(n+1)^2 a_{n+1} + a_n \right] x^n = 0$$

$$\therefore a_{n+1} = \frac{-a_n}{(n+1)^2}, \quad n \geq 0$$

$$\therefore a_1 = -\frac{a_0}{1^2} \qquad a_2 = -\frac{a_1}{2^2} = \frac{a_0}{2^2 \cdot 1^2}$$

$$\therefore a_n = \frac{(-1)^n a_0}{(n!)^2}, \quad n \geq 1$$

$$\therefore \quad \boxed{y_1(x) = x^{-1} \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} x^n} \quad , \text{assigning } a_0 = 1$$

(2) Assume $y_2(x) = y_1(x) \ln(x) + x^{-1} \sum_{n=1}^{\infty} b_n x^n$

$$(1+x) y_2 = (1+x) y_1 \ln(x) + \sum_{n=1}^{\infty} b_n x^{n-1} + \sum_{n=1}^{\infty} b_n x^n$$

$$= (1+x) y_1 \ln(x) + b_1 + \sum_{n=1}^{\infty} (b_{n+1} + b_n) x^n$$

$$= (1+x) y_1 \ln(x) + b_1 + (b_2 + b_1) x + \sum_{n=2}^{\infty} (b_{n+1} + b_n) x^n$$

$$3x y_2' = 3x y_1' \ln(x) + 3y_1 + \sum_{n=2}^{\infty} 3(n-1) b_n x^{n-1}$$

$$= 3x y_1' \ln(x) + 3y_1 + 3b_2 x + \sum_{n=2}^{\infty} 3n b_{n+1} x^n$$

$$x^2 y_2'' = x^2 y_1'' \ln(x) + 2x y_1' - y_1 + \sum_{n=3}^{\infty} (n-1)(n-2) b_n x^{n-1}$$

$$= x^2 y_1'' \ln(x) + 2x y_1' - y_1 + \sum_{n=2}^{\infty} n(n-1) b_{n+1} x^n$$

$$\therefore x^2 y_2'' + 3x y_2' + (1+x) y_2 = 0 \Rightarrow$$

$$\underbrace{[x^2 y_1'' + 3x y_1' + (1+x) y_1]}_{=0} \ln(x) + 2x y_1' + 2y_1$$

$$+ b_1 + (4b_2 + b_1) x + \sum_{n=2}^{\infty} [n(n-1) b_{n+1} + 3n b_{n+1} + (b_{n+1} + b_n) x^n$$

$$= 2xy_1' + 2y_1 + b_1 + (4b_2 + b_1)x + \sum_{n=2}^{\infty} [(n+1)^2 b_{n+1} + b_n] x^n = 0$$

From above,

$$y_1 = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n-1}}{(n!)^2} \qquad \therefore\ y_1' = \sum_{n=0}^{\infty} \frac{(-1)^n (n-1) x^{n-2}}{(n!)^2}$$

$$\therefore\ 2xy_1' + 2y_1 = \sum_{n=0}^{\infty} \frac{(-1)^n 2n}{(n!)^2} x^{n-1} = \sum_{n=1}^{\infty} \frac{(-1)^n 2n}{(n!)^2} x^{n-1}$$

$$= -2 + x + \sum_{n=3}^{\infty} \frac{(-1)^n 2n}{(n!)^2} x^{n-1} = -2 + x + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} 2(n+1)}{[(n+1)!]^2} x^n$$

$$\therefore\ b_1 + (4b_2 + b_1)x + \sum_{n=2}^{\infty} [(n+1)^2 b_{n+1} + b_n] x^n = 2 - x + \sum_{n=2}^{\infty} \frac{(-1)^n 2(n+1)}{[(n+1)!]^2} x^n$$

$$\therefore\ b_1 = 2 \qquad 4b_2 + b_1 = -1,\ b_2 = -\frac{3}{4}$$

$$(n+1)^2 b_{n+1} + b_n = \frac{(-1)^n 2(n+1)}{[(n+1)!]^2} = \frac{(-1)^n 2}{n!\,(n+1)!}, \quad n \geq 2$$

$$b_{n+1} = \frac{-b_n}{(n+1)^2} + \frac{(-1)^n 2}{[(n+1)!]^2 (n+1)}, \quad n \geq 2$$

$$\text{Or, } b_n = \frac{-b_{n-1}}{n^2} - \frac{(-1)^n 2}{(n!)^2 n}, \quad n \geq 3$$

Note the above also works for $n = 2$ since

$$b_2 = -\frac{(2)}{4} - \frac{2}{(2)^2 2} = -\frac{3}{4}.$$

As the goal of these exercises seems to be to come up with a "pretty" formula involving H_n, to facilitate this, let $a_n = \frac{(-1)^n 2}{(n!)^2}$, $n \geq 1$

$$\therefore\ a_n = \frac{-(-1)^{n-1}2}{n^2[(n-1)!]^2} = \frac{-1}{n^2} a_{n-1}, \quad n \geq 2$$

$$\therefore\ b_n = \frac{-b_{n-1}}{n^2} - \frac{(-1)^n 2}{(n!)^2 n} = \frac{-b_{n-1}}{n^2} - \frac{a_n}{n}$$

$$\therefore\ b_n = \frac{-b_{n-1}}{n^2} + \left(\frac{1}{n}\right)\frac{a_{n-1}}{n^2}, \quad n \geq 2$$

Note: $b_1 = 2 = -\frac{(-1)^1 2}{(1!)^2} = -H_1 a_1$ [1]

$$\therefore\ b_2 = -\frac{b_1}{2^2} + \frac{1}{2}\cdot\frac{a_1}{2^2} = \frac{H_1 a_1}{2^2} + \frac{1}{2}\frac{a_1}{2^2}$$

$$= \left(H_1 + \frac{1}{2}\right)\frac{a_1}{2^2} = -H_2 a_2$$

$\therefore$ Assuming $b_{n-1} = -H_{n-1}a_{n-1}$,

$$b_n = -\frac{(-H_{n-1}a_{n-1})}{n^2} + \left(\frac{1}{n}\right)\frac{a_{n-1}}{n^2}$$

$$= \left(H_{n-1} + \frac{1}{n}\right)\frac{a_{n-1}}{n^2} = -H_n a_n \quad [2]$$

$\therefore$ By induction, $b_n = -H_n a_n$, $n \geq 1$

Or, $b_n = -H_n \dfrac{(-1)^n 2}{(n!)^2} = \dfrac{-2(-1)^n H_n}{(n!)^2}$, $n \geq 1$

$$\therefore \boxed{y_2(x) = y_1(x)\ln(x) - \frac{2}{x}\sum_{n=1}^{\infty} \frac{(-1)^n H_n}{(n!)^2} x^n}$$

3.

(a) Standard form: $y'' + \left(\frac{1}{x}\right)y' + \left(\frac{2}{x}\right)y = 0$

$\therefore \lim_{x\to 0} x\left(\frac{1}{x}\right) = 1 \quad \lim_{x\to 0} x^2\left(\frac{2}{x}\right) = 0$

Both limits are finite. $\therefore$ $x=0$ is a regular singular point

(b) $xp(x) = 1 = p_0 \quad x^2 q(x) = 2x \Rightarrow q_0 = 0$

Indicial equation: $r(r-1) + r = 0$, $r^2 = 0$, $r = 0, 0$

(1) Let $y_1 = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}$

$$\therefore 2xy_1 = \sum_{n=0}^{\infty} 2a_n x^{n+r+1}$$

$$xy_1' = \sum_{n=0}^{\infty} (n+r)a_n x^{n+r} = ra_0 x^r + \sum_{n=0}^{\infty} (n+r+1)a_{n+1} x^{n+r+1}$$

$$x^2 y_1'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r} = r(r-1) a_0 x^r + \sum_{n=0}^{\infty} (n+r+1)(n+r) a_{n+1} x^{n+r+1}$$

For $r=0$, $x^2 y'' + xy' + 2xy = 0 \Rightarrow$

$$\sum_{n=0}^{\infty} \left[(n+1) n a_{n+1} + (n+1) a_{n+1} + 2a_n \right] x^{n+1} =$$

$$\sum_{n=0}^{\infty} \left[(n+1)^2 a_{n+1} + 2a_n \right] x^{n+1} = 0$$

$$\therefore \quad a_{n+1} = \frac{-2a_n}{(n+1)^2}, \quad n \geq 0$$

Or $$a_n = \frac{-2a_{n-1}}{n^2}, \quad n \geq 1$$

$$\therefore \quad a_1 = \frac{-2a_0}{1^2} \qquad a_2 = \frac{-2a_1}{2^2} = \frac{2^2 a_0}{1^2 \cdot 2^2}$$

$$a_3 = \frac{-2a_2}{3^2} = \frac{-2^3 a_0}{1^2 \cdot 2^2 \cdot 3^2}$$

$$\therefore \quad a_n = \frac{(-1)^n 2^n a_0}{(n!)^2}, \quad n \geq 1$$

Let $a_0 = 1$. $\therefore$ $a_n = \dfrac{(-1)^n 2^n}{(n!)^2}, \quad n \geq 0$

$$\therefore \quad \boxed{y_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{(n!)^2} x^n}$$

(2) Given double root of $r = 0, 0$

let $$y_2(x) = y_1(x)\ln(x) + \sum_{n=1}^{\infty} b_n x^n$$

$$\therefore\ 2xy_2 = 2xy_1\ln(x) + \sum_{n=1}^{\infty} 2b_n x^{n+1}$$

$$xy_2' = xy_1'\ln(x) + y_1 + \sum_{n=1}^{\infty} n b_n x^n$$

$$x^2 y_2'' = x^2 y_1''\ln(x) + 2xy_1' - y_1 + \sum_{n=2}^{\infty} n(n-1) b_n x^n$$

$$\therefore\ x^2 y_2'' + xy_2' + 2xy_2 = 0 \Rightarrow$$

$$\underbrace{[x^2 y_1'' + xy_1' + 2xy_1]}_{=0}\ln(x) + 2xy_1' +$$
$$\sum_{n=2}^{\infty} n(n-1) b_n x^n + \sum_{n=1}^{\infty} n b_n x^n + \sum_{n=1}^{\infty} 2b_n x^{n+1}$$

$$= 2xy_1' + b_1 x + \sum_{n=1}^{\infty} \left[(n+1)n\, b_{n+1} + (n+1) b_{n+1} + 2b_n\right] x^{n+1}$$

$$= 2xy_1' + b_1 x + \sum_{n=1}^{\infty} \left[(n+1)^2 b_{n+1} + 2b_n\right] x^{n+1} = 0$$

From (1) above, $$y_1' = \sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{(n!)^2} n x^{n-1}$$

$$\therefore 2xy_1' = \sum_{n=1}^{\infty} \frac{(-1)^n 2^{n+1}}{(n-1)!\,n!} x^n$$

$$= -4x + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2^{n+2}}{n!\,(n+1)!} x^{n+1}$$

$$\therefore b_1 x + \sum_{n=1}^{\infty} \left[(n+1)^2 b_{n+1} + 2b_n\right] x^{n+1} = 4x + \sum_{n=1}^{\infty} \frac{(-1)^n 2^{n+2}}{n!(n+1)!} x^{n+1}$$

$$\therefore b_1 = 4 \qquad (n+1)^2 b_{n+1} + 2b_n = \frac{(-1)^n 2^{n+2}}{n!\,(n+1)!}, \quad n \geq 1$$

$$b_{n+1} = \frac{-2b_n}{(n+1)^2} + \frac{(-1)^n 2^{n+2}}{[(n+1)!]^2 (n+1)}, \quad n \geq 1$$

$$\text{Or, } b_n = \frac{-2b_{n-1}}{n^2} - \frac{(-1)^n 2^{n+1}}{(n!)^2 n}, \quad n \geq 2$$

$$\therefore b_n = -\frac{2}{n}\left[\frac{b_{n-1}}{n} + \frac{(-1)^n 2^n}{(n!)^2}\right], \quad n \geq 2$$

For simplicity, let $a_n = \dfrac{(-1)^n 2^n}{(n!)^2}$

$$\therefore b_n = -\frac{2}{n}\left[\frac{b_{n-1}}{n} + a_n\right]$$

Note $a_n = \dfrac{(-1)^{n-1} 2^{n-1}}{[(n-1)!]^2} \cdot \dfrac{(-1)2}{n^2} = \dfrac{-2a_{n-1}}{n^2}$

$$\therefore b_n = -2\left[\frac{b_{n-1}}{n^2} + \left(\frac{1}{n}\right)\frac{-2a_{n-1}}{n^2}\right]$$

$$\therefore b_2 = -2\left[\frac{b_1}{2^2} + \left(\frac{1}{2}\right)\frac{-2\overset{=a_2}{a_1}}{2^2}\right]$$

$$= -2\left[\frac{4}{4} + \frac{1}{2}a_2\right] = -2\left[a_2 + \frac{1}{2}a_2\right] \qquad a_2 = \frac{4}{4}$$

$$= -2a_2 H_2$$

$$\therefore b_3 = -2\left[\frac{-2\overset{=a_3}{a_2} H_2}{3^2} + \left(\frac{1}{3}\right)\frac{-2\overset{=a_3}{a_2}}{3^2}\right]$$

$$= -2\left[H_2 a_3 + \frac{1}{3}a_3\right] = -2a_3\left(H_2 + \frac{1}{3}\right)$$

$$= -2a_3 H_3$$

And by induction, assuming $b_{n-1} = -2a_{n-1}H_{n-1}$,

$$b_n = -2\left[\frac{b_{n-1}}{n^2} + \left(\frac{1}{n}\right)\left(\frac{-2a_{n-1}}{n^2}\right)\right]$$

$$= -2\left[\left(\frac{-2a_{n-1}}{n^2}\right)H_{n-1} + \left(\frac{1}{n}\right)\left(\frac{-2a_{n-1}}{n^2}\right)\right]$$

$$= -2\left[a_n H_{n-1} + \left(\frac{1}{n}\right)a_n\right] = -2a_n\left[H_{n-1} + \frac{1}{n}\right]$$

$$= -2a_n H_n$$

$$\therefore y_2(x) = y_1(x)\ln(x) - 2\sum_{n=1}^{\infty} a_n H_n$$

$$\therefore \quad \boxed{y_2(x) = y_1(x)\ln(x) - 2\sum_{n=1}^{\infty} \frac{(-1)^n 2^n H_n}{(n!)^2} x^n}$$

4.

$$\text{Let } y = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$\left(x^2 - \frac{9}{4}\right)y = \sum_{n=0}^{\infty} a_n x^{n+r+2} - \frac{9}{4}a_0 x^r - \frac{9}{4}a_1 x^{r+1} - \sum_{n=2}^{\infty} \frac{9}{4} a_n x^{n+r}$$

$$= -\frac{9}{4}a_0 x^r - \frac{9}{4}a_1 x^{r+1} + \sum_{n=0}^{\infty}\left(a_n - \frac{9}{4}a_{n+2}\right) x^{n+r+2}$$

$$xy' = \sum_{n=0}^{\infty}(n+r)a_n x^{n+r} = r a_0 x^r + (r+1)a_1 x^{r+1} + \sum_{n=2}^{\infty}(n+r)a_n x^{n+r}$$

$$= r a_0 x^r + (r+1)a_1 x^{r+1} + \sum_{n=0}^{\infty}(n+r+2)a_{n+2} x^{n+r+2}$$

$$x^2 y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} = r(r-1)a_0 x^r + (r+1) r a_1 x^{r+1} +$$
$$\sum_{n=2}^{\infty}(n+r)(n+r-1)a_n x^{n+r}$$

$$= r(r-1)a_0 x^r + (r+1) r a_1 x^{r+1} + \sum_{n=0}^{\infty}(n+r+2)(n+r+1)a_{n+2} x^{n+r+2}$$

$\therefore\ x^2y'' + xy' + \left(x^2 - \frac{9}{4}\right)y = 0 \Rightarrow$

$$\left[r(r-1) + r - \frac{9}{4}\right]a_0x^r + \left[(r+1)r + (r+1) - \frac{9}{4}\right]a_1x^{r+1} +$$

$$\sum_{n=0}^{\infty}\left[(n+r+2)(n+r+1)a_{n+2} + (n+r+2)a_{n+2} + a_n - \frac{9}{4}a_{n+2}\right]x^{n+r+2} = 0 \quad [0]$$

Indicial equation: $r^2 - \frac{9}{4} = 0$, $r = \pm\frac{3}{2}$

(1) $r = \frac{3}{2}$: $(r+1)r + (r+1) - \frac{9}{4} = (r+1)^2 - \frac{9}{4} = \left(\frac{3}{2}+1\right)^2 - \frac{9}{4} = 4$

$$\therefore\ 4a_1x^{r+1} = 0 \Rightarrow \underline{a_1 = 0}$$

Recurrence: $\left[(n+r+2)^2 - \frac{9}{4}\right]a_{n+2} + a_n = 0$, $n \geq 0$

Or $\left[\left(n+\frac{7}{2}\right)^2 - \frac{9}{4}\right]a_{n+2} + a_n = 0$, or

$$(n+2)(n+5)a_{n+2} + a_n = 0,\ n \geq 0$$

$$\therefore\ a_{n+2} = \frac{-a_n}{(n+2)(n+5)},\ n \geq 0$$

Or, $a_n = -\frac{a_{n-2}}{n(n+3)}$, $n \geq 2$

$\therefore$ Since $a_1 = 0$, $a_3 = a_5 = \dots 0$

Let $n = 2m$, $m = 1, 2, 3, \dots$

$$\therefore\ a_{2m} = -\frac{a_{2m-2}}{2m(2m+3)} = -\frac{a_{2m-2}}{2^2m\left(m+\frac{3}{2}\right)},\quad m \geq 1$$

$\therefore\ a_2 = -\dfrac{a_0}{2^2 \cdot 1(1+\frac{3}{2})} \qquad a_4 = -\dfrac{a_2}{2^2 \cdot 2(2+\frac{3}{2})} = \dfrac{a_0}{2^4\, 2!(1+\frac{3}{2})(2+\frac{3}{2})}$

$$\therefore\ a_{2m} = \frac{(-1)^m a_0}{2^{2m}\, m!(1+\frac{3}{2})(2+\frac{3}{2})\cdots(m+\frac{3}{2})}, \quad m \geq 1$$

$\therefore$ Letting $a_0 = 1$, and $\dfrac{x^{2m}}{2^{2m}} = \left(\dfrac{x}{2}\right)^{2m}$

$$\boxed{y_1(x) = x^{\frac{3}{2}}\left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{m!(1+\frac{3}{2})(2+\frac{3}{2})\cdots(m+\frac{3}{2})}\left(\frac{x}{2}\right)^{2m}\right]}$$

(2) $r = -\frac{3}{2}$: From [0] and the term for x^{r+1},

$$\left[(r+1)r + (r+1) - \tfrac{9}{4}\right]a_1 x^{r+1} \Rightarrow \left[\left(-\tfrac{3}{2}+1\right)^2 - \tfrac{9}{4}\right]a_1 = -2a_1$$

$\Rightarrow \underline{a_1 = 0}$

Recurrence: $\left[(n+r+2)^2 - \frac{9}{4}\right]a_{n+2} + a_n = 0, \quad n \geq 0$

Or, $\left[(n+\frac{1}{2})^2 - \frac{9}{4}\right]a_{n+2} + a_n = 0, \quad n \geq 0$

Or, $(n+2)(n-1)a_{n+2} + a_n = 0, \quad n \geq 0$

For $n = 1$, $(3\cdot 0)a_3 + a_1 = 0$, and $a_1 = 0$

$\Rightarrow 0 \cdot a_3 = 0 \ \therefore\ a_3$ can be any number

Choose $\underline{a_3 = 0} \Rightarrow a_1 = a_3 = a_5 = a_7 = \dots = 0$

Note for any other choice of a_3, giving y_2^*,

$L[y_2 - y_2^*] = 0$, since $L[y_2] = 0$, $L[y_2^*] = 0$.

If $s \neq 0$, Then since $L[sy_1] = 0$,

$L[(y_2 - y_2^*) - sy_1] = 0$, and y_1 is independent of y_2 and y_2^* since y_1 has $x^{3/2}$ as initial term, whereas y_2 and y_2^* have $x^{-\frac{3}{2}}$.

$\therefore$ Any other choice for a_3 besides $a_3 = 0$ gives y_2 (with $a_3 = 0$) plus a multiple of y_1.

$\therefore$ $a_3 = 0$ is the simplest choice and is justified.

$\therefore$ $(n+2)(n-1)a_{n+2} + a_n = 0$, $n = 0, 2, 4, 6, \ldots$

Or,
$$a_{n+2} = \frac{-a_n}{(n+2)(n-1)}, \quad n = 0, 2, 4, 6, \ldots$$

Or,
$$a_n = \frac{-a_{n-2}}{n(n-3)}, \quad n = 0, 2, 4, \ldots$$

Let $m = 2n$, $m = 1, 2, 3, \ldots$

$$\therefore \; a_{2m} = \frac{-a_{2m-2}}{2m(2m-3)} = \frac{-a_{2m-2}}{2^2 m(m-\frac{3}{2})}, \quad m = 1, 2, 3, \ldots$$

$$\therefore\ a_2 = \frac{-a_0}{2^2\cdot 1(1-\frac{3}{2})}, \quad a_4 = \frac{-a_2}{2^2\cdot 2(2-\frac{3}{2})} = \frac{a_0}{2^4(2!)(1-\frac{3}{2})(2-\frac{3}{2})}$$

$$\therefore\ a_{2m} = \frac{(-1)^n a_0}{2^{2m}(m!)(1-\frac{3}{2})(2-\frac{3}{2})\cdots(m-\frac{3}{2})}, \quad m = 1, 2, 3, \ldots$$

$\therefore$ Letting $a_0 = 1$, and noting $\frac{x^{2m}}{2^{2m}} = \left(\frac{x}{2}\right)^{2m}$,

$$\boxed{y_2(x) = x^{-\frac{3}{2}}\left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{m!(1-\frac{3}{2})(2-\frac{3}{2})\cdots(m-\frac{3}{2})}\left(\frac{x}{2}\right)^{2m}\right]}$$

5.

Let $y = x^{-\frac{1}{2}} v(x)$

$$\therefore\ (x^2 - \tfrac{1}{4})y = x^2(x^{-\frac{1}{2}}v) - \tfrac{1}{4}x^{-\frac{1}{2}}v = x^{\frac{3}{2}}v - \tfrac{1}{4}x^{-\frac{1}{2}}v$$

$$xy' = x\left(-\tfrac{1}{2}x^{-\frac{3}{2}}v + x^{-\frac{1}{2}}v'\right) = -\tfrac{1}{2}x^{-\frac{1}{2}}v + x^{\frac{1}{2}}v'$$

$$x^2y'' = x^2\left(\tfrac{3}{4}x^{-\frac{5}{2}}v - \tfrac{1}{2}x^{-\frac{3}{2}}v' - \tfrac{1}{2}x^{-\frac{3}{2}}v' + x^{-\frac{1}{2}}v''\right)$$

$$= \tfrac{3}{4}x^{-\frac{1}{2}}v - x^{\frac{1}{2}}v' + x^{\frac{3}{2}}v''$$

$\therefore x^2y'' + xy' + (x^2 - \frac{1}{4})y = 0 \Rightarrow$

$$x^{\frac{3}{2}}v'' + \underbrace{\left(-x^{\frac{1}{2}} + x^{\frac{1}{2}}\right)}_{\text{cancel}}v' + \left(\underbrace{\frac{3}{4}x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{4}x^{-\frac{1}{2}}}_{\text{cancel}} + x^{\frac{3}{2}}\right)v$$

$$= x^{\frac{3}{2}}v'' + x^{3/2}v = x^{3/2}(v'' + v) = 0$$

Since $x > 0$, divide by $x^{3/2}$ to get:

$\underline{v'' + v = 0}$

Since $v(x) = \cos(x)$ and $v(x) = \sin(x)$ are solutions,

$\therefore y = x^{-\frac{1}{2}}v = \underline{x^{-\frac{1}{2}}\cos(x)}$ and $y = \underline{x^{-\frac{1}{2}}\sin(x)}$

are solutions.

6.

Equation (7) is: $$y_1(x) = a_0\left(1 + \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}(m!)^2}\right), \quad x > 0. \tag{7}$$

$$\therefore \left|\frac{a_{m+1}}{a_m}\right| = \left|\frac{x^{2m+2}}{2^{2m+2}[(m+1)!]^2} \cdot \frac{2^{2m}(m!)^2}{x^{2m}}\right|$$

$$= \left|\frac{x^2}{2^2(m+1)^2}\right|$$

$$\therefore \lim_{m\to\infty}\left|\frac{a_{m+1}}{a_m}\right| = \lim_{m\to\infty}\left|\frac{x^2}{2^2(m+1)^2}\right| = 0, \text{ for all } x$$

$\therefore$ By ratio test, $y_1(x)$ converges absolutely for all x.

7.

From the text,
$$J_1(x) = \frac{x}{2}\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m}}{2^{2m}(m+1)!m!}. \tag{27}$$

(1)
$$\left|\frac{a_{m+1}}{a_m}\right| = \left|\frac{x^{2m+2}}{2^{2m+2}(m+2)!(m+1)!}\cdot\frac{x}{2}\cdot\frac{2^{2m}(m+1)!m!}{x^{2m}}\frac{2}{x}\right|$$

$$= \left|\frac{x^2}{2^2(m+2)(m+1)}\right|$$

$$\therefore \lim_{m\to\infty}\left|\frac{a_{m+1}}{a_m}\right| = \lim_{m\to\infty}\left|\frac{x^2}{2^2(m+2)(m+1)}\right| = 0 \text{ for all } x.$$

$\therefore$ $J_1(x)$ converges absolutely for all x, by the ratio test.

(2) From text equation (7),

$$J_0(x) = 1 + \sum_{m=1}^{\infty}\frac{(-1)^m x^{2m}}{2^{2m}(m!)^2}$$

$$\therefore J_0'(x) = \sum_{m=1}^{\infty}\frac{(-1)^m(2m)x^{2m-1}}{2^{2m}(m!)^2}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^{m+1}(2m+2)\,x^{2m+1}}{2^{2m+2}(m+1)!\,(m+1)!}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)(-1)^{m}\,2(m+1)\,x^{2m}\cdot x}{2^{2m}\cdot 2^{2}(m+1)!\,(m+1)!}$$

$$= -\frac{x}{2}\sum_{m=0}^{\infty} \frac{(-1)^{m}\,x^{2m}}{2^{2m}(m!)(m+1)!}$$

$$= -J_1(x)$$

8.

(a)

Standard form: $y'' + \left(\frac{1}{x}\right)y' + \left(\frac{x^2-v^2}{x^2}\right)y = 0$

$\lim_{x\to 0} x\left(\frac{1}{x}\right) = 1 = p_0 \qquad \lim_{x\to 0} x^2\left(\frac{x^2-v^2}{x^2}\right) = -v^2 = q_0$

$\therefore$ Both limits are finite, $\therefore$ $x=0$ is a regular singular point.

$\therefore r(r-1) + p_0 r + q_0 = r^2 - r + r - v^2 = r^2 - v^2 = 0$

$\therefore$ $r = \pm v$, as $v > 0$.

(6)

Let $y(x) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}$, $a_0 \neq 0$.

$$\therefore (x^2 - v^2) y = \sum_{n=0}^{\infty} a_n x^{n+r+2} - v^2 a_0 x^r - v^2 a_1 x^{r+1} - \sum_{n=2}^{\infty} v^2 a_n x^{n+r}$$

$$= -v^2 a_0 x^r - v^2 a_1 x^{r+1} + \sum_{n=0}^{\infty} (-v^2 a_{n+2} + a_n) x^{n+r+2}$$

$$xy' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r} = r a_0 x^r + (r+1) a_1 x^{r+1} + \sum_{n=2}^{\infty} (n+r) a_n x^{n+r}$$

$$= r a_0 x^r + (r+1) a_1 x^{r+1} + \sum_{n=0}^{\infty} (n+r+2) a_{n+2} x^{n+r+2}$$

$$x^2 y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r} = r(r-1) a_0 x^r + (r+1) r a_1 x^{r+1} + \sum_{n=2}^{\infty} (n+r)(n+r-1) a_n x^{n+r}$$

$$= r(r-1) a_0 x^r + (r+1) r a_1 x^{r+1} + \sum_{n=0}^{\infty} (n+r+2)(n+r+1) a_{n+2} x^{n+r+2}$$

$$\therefore x^2 y'' + xy' + (x^2 - v^2) y = 0 \Rightarrow$$

$$[r(r-1)+r-v^2]a_0x^r+[r(r+1)+(r+1)-v^2]a_1x^{r+1}+$$

$$\sum_{n=0}^{\infty}\left[(n+r+2)(n+r+1)a_{n+2}+(n+r+2)a_{n+2}-v^2a_{n+2}+a_n\right]x^{n+r+2}=0$$

Since $a_0\neq 0$, $r(r-1)+r-v^2=r^2-v^2=0$, $r=\pm v$, the indicial equation

For a_1x^{r+1}, $r(r+1)+(r+1)-v^2=(r+1)^2-v^2=(r+1+v)(r+1-v)$

$\therefore\ [(r+1)^2-v^2]a_1x^{r+1}=(r+1+v)(r+1-v)a_1x^{r+1}$

For the larger root, $r=v$, $\therefore\ (1+2v)a_1x^{r+1}=0 \Rightarrow$

$\underline{a_1=0}$, since $v>0$.

Recurrence: $[(n+r+2)^2-v^2]a_{n+2}+a_n=0$, $n\geq 0$

For $r=v$, $[(n+2+v)^2-v^2]a_{n+2}+a_n=0$, $n\geq 0$

Or, $(n+2+2v)(n+2)a_{n+2}+a_n=0$, $n\geq 0$

Or, $(n+2v)n\,a_n+a_{n-2}=0$, $n\geq 2$

$$\therefore\ a_n=\frac{-a_{n-2}}{n(n+2v)},\quad n\geq 2$$

$\therefore\ a_1=a_3=a_5=\dots=0$

$$a_n = \frac{-a_{n-2}}{n(n+2\nu)}, \quad n = 2, 4, 6, \ldots$$

Let $n = 2m$, $m = 1, 2, 3, \ldots$

$$\therefore \quad a_{2m} = \frac{-a_{2m-2}}{2m(2m+2\nu)} = \frac{-a_{m-2}}{2^2\, m(m+\nu)}, \quad m = 1, 2, 3, \ldots$$

$$\therefore \quad a_2 = \frac{-a_0}{2^2 \cdot 1 \cdot (1+\nu)}, \quad a_4 = \frac{-a_2}{2^2 \cdot 2(2+\nu)} = \frac{a_0}{2^4\, 2!\,(1+\nu)(2+\nu)}$$

$$\therefore \quad a_{2m} = \frac{(-1)^m\, a_0}{2^{2m}\, m!\,(1+\nu)(2+\nu)\cdots(m+\nu)}, \quad m = 1, 2, 3, \ldots$$

$\therefore$ For $a_0 = 1$, $r = \nu$, and noting $\frac{x^{2m}}{2^{2m}} = \left(\frac{x}{2}\right)^{2m}$

$$\boxed{y_1(x) = x^\nu \left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{m!\,(1+\nu)(2+\nu)\cdots(m+\nu)} \left(\frac{x}{2}\right)^{2m}\right]}$$

(c)

From (6), the x^{r+1} term is $(r+1+\nu)(r+1-\nu)a_1 x^{r+1}$

$\therefore$ For $r = -\nu$, this becomes $(1-2\nu)a_1 x^{r+1}$

But if $v = \frac{1}{2}$, then $r_1 - r_2 = 2v = 1$, an integer

Since this is assumed not the case, then

$1 - 2v \neq 0$, so $\underline{a_1 = 0}$

Also from (6), the recurrence relation is:

$$[(n+r+2)^2 - v^2]\, a_{n+2} + a_n = 0, \quad n \geq 0$$

Or, $(n+r+2+v)(n+r+2-v)\, a_{n+2} + a_n = 0$

and for $r = -v$, $(n+2)(n+2-2v)\, a_{n+2} + a_n = 0$, $n \geq 0$

Or, $a_n = \dfrac{-a_{n-2}}{n(n-2v)}$, $n \geq 2$ (since $2v$ is not an integer, $n-2v$ is never 0).

So, $a_1 = a_3 = a_5 = \cdots = 0$ Let $n = 2m$, $m = 1, 2, 3, \ldots$

$$\therefore a_{2m} = \frac{-a_{2m-2}}{2m(2m-2v)} = \frac{-a_{2m-2}}{2^2 m(m-v)}, \quad m = 1, 2, 3, \ldots$$

$$\therefore a_2 = \frac{-a_0}{2^2 \cdot 1 \cdot (1-v)} \qquad a_4 = \frac{-a_2}{2^2 \cdot 2 \cdot (2-v)} = \frac{a_0}{2^4 \cdot 2!(1-v)(2-v)}$$

$$\therefore a_{2m} = \frac{(-1)^m a_0}{2^{2m} m!(1-v)(2-v)\cdots(m-v)}, \quad m = 1, 2, 3, \ldots$$

Letting $a_0 = 1$, for $r = -v$, and noting $\dfrac{x^{2m}}{2^{2m}} = \left(\dfrac{x}{2}\right)^{2m}$,

$$\boxed{y_2(x) = x^{-v}\left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{m!(1-v)(2-v)\cdots(m-v)}\left(\frac{x}{2}\right)^{2m}\right]}$$

Since $\lim_{x\to 0}\left(\frac{x}{2}\right)^{2m} = 0$, $\lim_{x\to 0} x^v = 0$ since $v > 0$

and $\lim_{x\to 0} x^{-v} = \infty$, $\therefore$ $\lim_{x\to 0} y_1(x) = 0$, $\lim_{x\to 0} y_2(x) = \infty$

(d)

(1) $y_1(x)$: $\left|\frac{a_{m+1}}{a_m}\right| = \left|\frac{m!(1+v)\cdots(m+v)}{(m+1)!(1+v)\cdots(m+1+v)}\left(\frac{x}{2}\right)^{2m+2}\left(\frac{2}{x}\right)^{2m}\right|$

$$= \left|\frac{1}{(m+1)(m+1+v)}\left(\frac{x}{2}\right)^2\right|$$

$$\therefore \lim_{m\to\infty}\left|\frac{a_{m+1}}{a_m}\right| = \left(\frac{x}{2}\right)^2 \lim_{m\to\infty}\left|\frac{1}{(m+1)(m+1+v)}\right| = 0, \text{ all } x$$

$\therefore$ $y_1(x)$ converges absolutely for all x.

(2) $y_2(x)$: $\left|\frac{a_{m+1}}{a_m}\right| = \left|\frac{m!(1-v)\cdots(m-v)}{(m+1)!(1-v)\cdots(m+1-v)}\left(\frac{x}{2}\right)^{2m+2}\left(\frac{2}{x}\right)^{2m}\right|$

$$= \left|\frac{1}{(m+1)(m+1-v)}\left(\frac{x}{2}\right)^2\right|$$

$\therefore \lim_{x\to\infty} \left|\frac{a_{m+1}}{a_m}\right| = \left(\frac{x}{2}\right)^2 \lim \left|\frac{1}{(m+1)(m+1-\nu)}\right| = 0$, all x.

$\therefore$ $y_2(x)$ converges absolutely for all x.

(3) If ν is an integer, then $r_1 - r_2 = \nu - (-\nu) = 2\nu$ is an integer and so $y_2(x)$ can't be of the form $x^{r_2} \sum_{n=0}^{\infty} a_n x^n$. Note the above solution for $y_2(x)$ contains $(1-\nu)(2-\nu)\cdots(m-\nu)$ in the denominator.

$\therefore$ One factor would eventually be zero if ν is an integer, creating division by zero.

9.

(a)

$$x^2 y_2(x) = x^2 J_0 \ln(x) + \sum_{n=1}^{\infty} b_n x^{n+2}$$

$$x y_2' = x\left[J_0' \ln(x) + J_0\left(\frac{1}{x}\right) + \sum_{n=1}^{\infty} n b_n x^{n-1} \right]$$

$$= x J_0' \ln(x) + J_0 + \sum_{n=1}^{\infty} n b_n x^{n}$$

$$x^2 y_2'' = x^2\left[J_0'' \ln(x) + J_0'\left(\frac{1}{x}\right) + J_0'\left(\frac{1}{x}\right) + J_0\left(-\frac{1}{x^2}\right) + \sum_{n=2}^{\infty} n(n-1) b_n x^{n-2} \right]$$

$$= x^2 J_0'' \ln(x) + 2x J_0' - J_0 + \sum_{n=2}^{\infty} n(n-1) b_n x^{n}$$

$$\therefore L[y_2](x) = x^2 y_2'' + x y_2' + x^2 y_2$$

$$= \underbrace{\left[x^2 J_0'' + x J_0' + x^2 J_0 \right]}_{=0} \ln(x) \qquad \text{since } L[J_0] = 0$$

$$+ 2x J_0' \underbrace{- J_0 + J_0}_{\text{cancel}}$$

$$+ \sum_{n=2}^{\infty} n(n-1) b_n x^n + \sum_{n=1}^{\infty} n b_n x^n + \sum_{n=1}^{\infty} b_n x^{n+2}$$

$$\therefore \boxed{L[y_2](x) = 2x J_0'(x) + \sum_{n=2}^{\infty} n(n-1) b_n x^n + \sum_{n=1}^{\infty} n b_n x^n + \sum_{n=1}^{\infty} b_n x^{n+2}}$$

(b)

From equation (7) of the text on p. 231,

$$J_0(x) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n \, x^{2n}}{2^{2n}\,(n!)^2} \quad , \quad x > 0$$

$$\therefore \; 2x\,J_0'(x) = 2x\left[\sum_{n=1}^{\infty} \frac{(-1)^n\, 2n\, x^{2n-1}}{2^{2n}(n!)^2}\right]$$

$$= 2\sum_{n=1}^{\infty} \frac{(-1)^n\, 2n\, x^{2n}}{2^{2n}\,(n!)^2}$$

From $L[y_2](x) = 0$ and the result in (a),

$$\sum_{n=2}^{\infty} n(n-1)\,b_n x^n + \sum_{n=1}^{\infty} n\,b_n x^n + \sum_{n=1}^{\infty} b_n x^{n+2} = -2x\,J_0'(x)$$

extract 1st term

extract 1st 2 terms

$$\therefore \; 2b_2x^2 + \sum_{n=3}^{\infty} n(n-1)b_n x^n + b_1 x + 2b_2 x^2 + \sum_{n=3}^{\infty} n\,b_n\,x^n$$

$$\sum_{n=3}^{\infty} b_{n-2}\, x^n = -2x\,J_0'(x)$$

shifted index

$$\therefore \; b_1 x + 4b_2x^2 + \sum_{n=3}^{\infty}\Big[\underbrace{n(n-1)\,b_n + n\,b_n}_{=n^2 b_n} + b_{n-2}\Big]x^n = -2x\,J_0'(x)$$

$$\therefore \; \boxed{b_1 x + 2^2 b_2 x^2 + \sum_{n=3}^{\infty}\left[n^2 b_n + b_{n-2}\right]x^n = -2\sum_{n=1}^{\infty}\frac{(-1)^n 2n\, x^{2n}}{2^{2n}(n!)^2}}$$

(c)

The first term of $-2\sum_{n=1}^{\infty} \frac{(-1)^n 2n\, x^{2n}}{2^{2n}(n!)^2}$ is $-2\left[\frac{(-1)\,2\,x^2}{2^2(1!)^2}\right]$,

or $\frac{2^2}{2^2(1!)^2}x^2 = x^2$. There is no x term.

$\therefore \underline{b_1 = 0}$ and $2^2 b_2 x^2 = \frac{2^2}{2^2(1!)^2}x^2 \Rightarrow \underline{b_2 = \frac{1}{2^2(1!)^2}} = \frac{1}{4}$

Since the right side of the equation only has even powers, $n^2 b_n + b_{n-2} = 0$, $n = 3, 5, 7, \ldots$

$\therefore b_n = -\frac{b_{n-2}}{n^2}$, $n = 3, 5, 7, \ldots$

$\therefore b_3 = \frac{-b_1}{3^2} = 0$ since $b_1 = 0$ $\quad \therefore b_5 = -\frac{b_3}{5^2} = 0$

$\therefore \underline{b_1 = b_3 = b_5 = \ldots = 0}$

$$\therefore \sum_{n=4}^{\infty} \left[n^2 b_n + b_{n-2}\right] x^n = -2\sum_{n=2}^{\infty} \frac{(-1)^n 2n\, x^{2n}}{2^{2n}(n!)^2} \qquad [1]$$

For the left side, let $n = 2m$, $m = 2, 3, 4, \ldots$

$$\therefore \sum_{m=2}^{\infty} \left[(2m)^2 b_{2m} + b_{2m-2} \right] x^{2m}$$ and so rewrite [1] as

$$\sum_{n=2}^{\infty} \left[(2n)^2 b_{2n} + b_{2n-2} \right] x^{2n} = -2 \sum_{n=2}^{\infty} \left[\frac{(-1)^n 2n}{2^{2n}(n!)^2} \right] x^{2n}$$

$$\therefore \boxed{(2n)^2 b_{2n} + b_{2n-2} = -2 \left[\frac{(-1)^n 2n}{2^{2n}(n!)^2} \right], \quad n = 2, 3, 4, \ldots}$$

From above, $b_2 = \frac{1}{2^2(1!)^2} = \frac{1}{2^{2n}(n!)^2} H_n$ for $n = 1$

$$b_{2n} = -\frac{1}{(2n)^2} \left[b_{2n-2} + 2 \frac{(-1)^n 2n}{2^{2n}(n!)^2} \right], \quad n = 2, 3, 4, \ldots \quad [2]$$

$$\therefore b_4 = -\frac{1}{4^2} \left[\frac{1}{2^2(1!)^2} + \frac{2(4)}{2^4(2!)^2} \right]$$

$$= -\frac{1}{2^2 4^2} \left[1 + \frac{2(4)}{2^2(2!)^2} \right] = -\frac{1}{2^2 4^2} \left[1 + \frac{1}{2} \right]$$

$$= -\frac{1}{[2(1)2(2)]^2} \left[1 + \frac{1}{2} \right] = -\frac{1}{2^2(2!)^2} H_2$$

$\therefore$ Assume $b_{2n} = \frac{(-1)^{n+1} H_n}{2^{2n}(n!)^2}$ for any $n > 2$ [3]

$\therefore$ From [2],

$$b_{2(n+1)} = -\frac{1}{[2(n+1)]^2} \left[b_{2n} + 2 \frac{(-1)^{n+1} 2(n+1)}{2^{2n+2}[(n+1)!]^2} \right]$$

$$= -\frac{1}{2^2(n+1)^2}\left[\frac{(-1)^{n+1}H_n}{2^{2n}(n!)^2} + \frac{(-1)^{n+1}}{2^{2n}(n!)(n!)(n+1)}\right]$$

$$= \frac{-(-1)^{n+1}}{2^2(n+1)^2\,2^{2n}(n!)^2}\Big[\underbrace{H_n + \frac{1}{n+1}}_{=\,H_{n+1}}\Big]$$

$$= \frac{(-1)^{n+2}}{2^{2n+2}[(n+1)!]^2}H_{n+1}$$

$\therefore$ $\boxed{b_{2n} = \frac{(-1)^{n+1}H_n}{2^{2n}(n!)^2}}$ is true for $n=1$, $n=2$,

and when true for any $n>2$, it is true for $n+1$. $\therefore$ True for all $n \geq 1$

$\therefore$

$$\boxed{y_2(x) = J_0(x)\ln(x) + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}H_n}{2^{2n}(n!)^2}x^{2n}}$$

10.

(1) From equation (24):

$$L[\phi](r, x) = a_0(r^2 - 1)x^r + a_1\Big((r+1)^2 - 1\Big)x^{r+1} + \sum_{n=2}^{\infty}\Big(\big((r+n)^2 - 1\big)a_n + a_{n-2}\Big)x^{r+n} = 0. \qquad (24)$$

$$a_1\left[(r+1)^2 - 1\right] = 0, \quad \therefore\ a_1 = \frac{0}{(r+1)^2 - 1}$$

$$\therefore\ a_1(r) = 0, \quad a_1'(r) = 0 \Rightarrow \underline{a_1(-1) = 0} \text{ and } \underline{a_1'(-1) = 0}.$$

(2) From p. 228 of the text,

$$c_n(r_2) = \frac{d}{dr}[(r - r_2)a_n(r)]\Big|_{r=r_2}, \quad n = 1, 2, \ldots, \qquad (19)$$

$$\therefore\ c_n(-1) = \frac{d}{dr}\left[(r - (-1))\, a_n(r)\right]\Big|_{r=-1}$$

But $\frac{d}{dr}(r+1)a_n(r) = a_n(r) + (r+1)a_n'(r) = 0$

$\therefore\ c_n(-1) = a_n(-1) + (r+1)a_n'(-1)$ [2]

For $n=1$, this means

$\underline{c_1(-1)} = 0 + (r+1)\cdot 0 = \underline{0}$, using (1)

(3) From equation (24),

$$a_n(r) = \frac{-a_{n-2}(r)}{(r+n)^2-1}, \quad n \geq 2 \qquad [3]$$

$$\therefore\ a_n'(r) = \frac{[(r+n)^2-1][-a_{n-2}'(r)] - [-a_{n-2}(r)][2(r+n)]}{[(r+n)^2-1]^2}$$

$\therefore$ Whenever $a_{n-2}'(r)=0$ and $a_{n-2}(r)=0$, then

$a_n(r)=0$ and $a_n'(r)=0$

$\therefore$ Since $a_1(-1)=0$ and $a_1'(-1)=0$, then

$a_3(-1) = a_5(-1) = a_7(-1) = \dots = 0$

and $a_3'(-1) = a_5(-1) = a_7(-1) = \dots = 0$

$\therefore$ Using [2], $c_1(-1) = c_3(-1) = c_5(-1) = \dots = 0$

(4) From [3], since $N = 1-(-1) = 2$, $a_N = a_2$,

so $a_2(r) = \dfrac{-a_0(r)}{(r+2)^2-1}$, and $a_0(r) = 1$ by definition of a_0.

$\therefore a = \lim\limits_{r\to -1} (r+1) \cdot \dfrac{-1}{(r+2)^2-1} = \lim\limits_{r\to -1} \dfrac{-(r+1)}{(r+1)(r+3)} = -\dfrac{1}{2}$

$\therefore \underline{a = -\frac{1}{2}}$

(5) Using [3], $a_2 = \dfrac{-a_0}{(r+2)^2-1} = \dfrac{-a_0}{r^2+4r+3} = \dfrac{-a_0}{(r+1)(r+3)}$

$a_4 = \dfrac{-a_2}{(r+4)^2-1} = \dfrac{-a_2}{r^2+8r+15} = \dfrac{-a_2}{(r+3)(r+5)} = \dfrac{a_0}{(r+1)(r+3)(r+3)(r+5)}$

From $a_n = \dfrac{-a_{n-2}}{(r+n)^2-1}$, $n = 2, 4, 6, 8, \ldots$

let $n = 2m$, $m = 1, 2, 3, \ldots$

$\therefore a_{2m} = \dfrac{-a_{2m-2}}{(r+2m)^2-1} = \dfrac{-a_{2m-2}}{(r+2m-1)(r+2m+1)}$, $m = 1, 2, 3, \ldots$

$\therefore$ By induction,

$$a_{2m} = \frac{(-1)^m a_0}{(r+1)(r+3)\cdots(r+2m-1)(r+3)(r+5)\cdots(r+2m+1)}, \quad m = 1, 2, 3$$

(6) From $$c_n(r_2) = \frac{d}{dr}[(r-r_2)a_n(r)]\bigg|_{r=r_2}, \quad n = 1, 2, \ldots, \qquad (19)$$

$$c_{2m}(-1) = \frac{d}{dr}\left[(r+1)a_{2m}(r)\right]\Big|_{r=-1}, \quad m = 1, 2, 3, \ldots$$

$$(r+1)a_{2m}(r) = \frac{(-1)^m a_0}{(r+3)(r+5)\cdots(r+2m-1)(r+3)(r+5)\cdots(r+2m+1)}$$

$$= \frac{(-1)^m a_0}{(r+3)^2(r+5)^2\cdots(r+2m-1)^2(r+2m+1)}, \quad m = 2, 3, 4, \ldots$$

$$= (-1)^m a_0 (r+3)^{-2}(r+5)^{-2}\cdots(r+2m-1)^{-2}(r+2m+1)^{-1}$$

Need $m \geq 2$ since factored out $(r+1)$: $(r+2m-1) = 0$, $m=1, r=-1$

Let $x = (r+3)^{-2}(r+5)^{-2}\cdots(r+2m-1)^{-2}(r+2m+1)^{-1}$

Note if $x = a \cdot b \cdot c$, $\log x = \log a + \log b + \log c$

$$\therefore \frac{1}{x} \cdot x' = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}, \quad x' = \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) abc$$

$$\therefore \frac{d}{dr}(r+1)a_{2m}(r) = (-1)^m a_0 \frac{d}{dr} x$$

$$= (-1)^m a_0 \left[\frac{-2}{r+3} + \frac{-2}{r+5} + \cdots + \frac{-2}{r+2m-1} + \frac{-1}{r+2m+1}\right](r+3)^{-2}\cdots(r+2m-1)^{-2}(r+2m+1)^{-1}$$

$m = 2, 3, 4, \ldots$

$$\therefore \frac{d}{dr}(r+1)a_{2m}(r)\Big|_{r=-1} = \frac{-(-1)^m a_0 \left[\frac{2}{2} + \frac{2}{4} + \cdots \frac{2}{2m-2} + \frac{1}{2m}\right]}{2^2 \cdot 4^2 \cdot 6^2 \cdots (2m-2)^2 \cdot 2m}$$

$$= (-1)^{m+1} \frac{a_0 \frac{1}{2}\left[\frac{2}{1} + \frac{2}{2} + \frac{2}{3} + \cdots + \frac{2}{m-1} + \frac{1}{m}\right]}{2^2 \cdot 1^2 \cdot 2^2 \cdot 2^2 \cdot 2^2 \cdot 3^2 \cdots 2^2 (m-1)^2 \cdot 2m}, \quad m \geq 2$$

$$= (-1)^{m+1} \frac{a_0 \left[1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m-1} + 1 + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{m-1} + \frac{1}{m}\right]}{\underbrace{2^2 \cdot 2^2 \cdots 2^2}_{m \text{ times}} \cdot 1^2 \cdot 2^2 \cdots (m-1)^2 \cdot m}, \quad m \geq 2$$

$$= (-1)^{m+1} \frac{a_0 \left[H_{m-1} + H_m\right]}{2^{2m} (m-1)! \, m!}, \quad m \geq 2$$

Look at $m = 1$: Assume $H_0 = 0$. By this formula,

$$C_2(-1) = \frac{a_0 (0 + 1)}{2^2 \, 0! \, 1!} = \frac{a_0}{4}$$

$$(r+1) a_2(r) = (r+1) \frac{-a_0}{(r+1)(r+3)} \quad \text{from above}$$

$$\therefore C_2(-1) = \frac{d}{dr}\left[\frac{-a_0 (r+1)}{(r+1)(r+3)}\right]_{r=-1} = \frac{d}{dr}\left[\frac{-a_0}{(r+3)}\right]\Bigg|_{r=-1}$$

$$= \frac{a_0}{(r+3)^2}\Bigg|_{r=-1} = \frac{a_0}{4}$$

$\therefore$ Formula also works for $m = 1$, with $H_0 = 0$

$$\therefore \quad \boxed{C_{2m}(-1) = \frac{(-1)^{m+1}\left[H_{m-1} + H_m\right]}{2^{2m} (m-1)! \, m!}, \quad m \geq 1, a_0 = 1}$$

(7) $\therefore\ y_2(x) = a\, y_1(x) \ln(x) + x^{-1}\left(1 + \sum_{n=1}^{\infty} c_n x^n\right)$

$$= -\frac{1}{2} y_1(x) \ln(x) + x^{-1}\left[1 + \sum_{m=1}^{\infty} \frac{(-1)^{m+1}(H_{m-1} + H_m)}{2^{2m}(m-1)!\, m!} x^{2m}\right]$$

Note: in the text, $a = -1$. But $J_1(x)$ was developed using $a_0 = \frac{1}{2}$. $\therefore$ Here, $\frac{1}{2} y_1 = J_1$ so the two methods give the same result.

11.

$$y' = \frac{1}{2} x^{-\frac{1}{2}} f(\alpha x^\beta) + x^{\frac{1}{2}} f'(\alpha x^\beta)\, \alpha \beta x^{\beta-1}$$

$$y'' = -\frac{1}{4} x^{-\frac{3}{2}} f(\alpha x^\beta) + \underbrace{\frac{1}{2} x^{-\frac{1}{2}} f'(\alpha x^\beta)\, \alpha\beta x^{\beta-1}}$$

$$+ \underbrace{\frac{1}{2} x^{-\frac{1}{2}} f'(\alpha x^\beta)\, \alpha\beta x^{\beta-1}} + x^{\frac{1}{2}} f''(\alpha x^\beta)\, \alpha^2 \beta^2 x^{2\beta-2}$$

$$+ \underbrace{x^{\frac{1}{2}} f'(\alpha x^\beta)\, \alpha\beta(\beta-1) x^{\beta-2}}$$

Note: $\frac{1}{2} x^{-\frac{1}{2}} f'(\alpha x^\beta)\alpha\beta x^{\beta-1} + \frac{1}{2} x^{-\frac{1}{2}} f'(\alpha x^\beta)\alpha\beta x^{\beta-1} =$

$$x^{-\frac{1}{2}} f'(\alpha x^\beta)\alpha\beta x^{\beta-1}$$

And $x^{\frac{1}{2}} f'(\alpha x^\beta)\alpha\beta(\beta-1)x^{\beta-2} = x^{-\frac{1}{2}} f'(\alpha x^\beta)\alpha\beta(\beta-1)x^{\beta-1}$

$$\therefore x^{-\frac{1}{2}} f'(\alpha x^\beta)\alpha\beta x^{\beta-1} + x^{-\frac{1}{2}} f'(\alpha x^\beta)\alpha\beta(\beta-1)x^{\beta-1} =$$

$$x^{-\frac{1}{2}} f'(\alpha x^\beta)\alpha\beta^2 x^{\beta-1}$$

$$\therefore y'' = -\frac{1}{4}x^{-\frac{3}{2}} f(\alpha x^\beta) + x^{-\frac{1}{2}} f'(\alpha x^\beta)\alpha\beta^2 x^{\beta-1} + x^{\frac{1}{2}} f''(\alpha x^\beta)\alpha^2\beta^2 x^{2\beta-2}$$

$$= f''(\alpha x^\beta)\alpha^2\beta^2 x^{2\beta-\frac{3}{2}} + f'(\alpha x^\beta)\alpha\beta^2 x^{\beta-\frac{3}{2}} - \frac{1}{4} f(\alpha x^\beta)x^{-\frac{3}{2}}$$

$$\therefore x^2 y'' = f''(\,)\alpha^2\beta^2 x^{2\beta+\frac{1}{2}} + f'(\,)\alpha\beta^2 x^{\beta+\frac{1}{2}} - \frac{1}{4} f(\,)x^{\frac{1}{2}}$$

$$= (\alpha x^\beta)^2 f''(\alpha x^\beta)\beta^2 x^{\frac{1}{2}} + (\alpha x^\beta) f'(\alpha x^\beta)\beta^2 x^{\frac{1}{2}} - \frac{1}{4} f(\alpha x^\beta)x^{\frac{1}{2}}$$

$$= \epsilon^2 f''(\epsilon)\beta^2 x^{\frac{1}{2}} + (\epsilon) f'(\epsilon)\beta^2 x^{\frac{1}{2}} - \frac{1}{4} f(\epsilon)x^{\frac{1}{2}} \qquad \epsilon = \alpha x^\beta$$

$$(\alpha^2\beta^2 x^{2\beta} + \tfrac{1}{4} - v^2\beta^2)y = (\epsilon^2\beta^2 + \tfrac{1}{4} - v^2\beta^2)x^{\frac{1}{2}} f(\alpha x^\beta)$$

$$= (\epsilon^2 - v^2)\beta^2 x^{\frac{1}{2}} f(\epsilon) + \frac{1}{4}x^{\frac{1}{2}} f(\epsilon)$$

$$\therefore x^2 y'' + (\alpha^2\beta^2 x^{2\beta} + \tfrac{1}{4} - v^2\beta^2)y =$$

$$\epsilon^2 f''(\epsilon)\beta^2 x^{\frac{1}{2}} + \epsilon f'(\epsilon)\beta^2 x^{\frac{1}{2}} - \frac{1}{4} f(\epsilon)x^{\frac{1}{2}}$$

cancel

$$+ (\epsilon^2 - v^2) f(\epsilon)\beta^2 x^{\frac{1}{2}} + \frac{1}{4} f(\epsilon)x^{\frac{1}{2}}$$

$$= \underbrace{\left[\epsilon^2 f''(\epsilon) + \epsilon f'(\epsilon) + (\epsilon^2 - v^2) f(\epsilon)\right]}_{=\,0}\beta^2 x^{\frac{1}{2}}$$

0 since $f(\epsilon)$ a solution to Bessel equation of order v

$$= 0 \cdot \beta^2 x^{\frac{1}{2}} = 0$$

$\therefore$ $y = x^{\frac{1}{2}} f(\alpha x^{\beta})$ is a solution to

$$x^2 y'' + \left(\alpha^2\beta^2 x^{2\beta} + \tfrac{1}{4} - v^2\beta^2\right) y = 0 \quad , \quad x > 0$$

12.

Using $\alpha x^{\beta} = \frac{2}{3} i \, x^{\frac{3}{2}}$, let $\alpha = \frac{2}{3} i$, $\beta = \frac{3}{2}$

From #11, $x^{\frac{1}{2}} f(\alpha x^{\beta})$ is a solution to

$$x^2 y'' + \left(\alpha^2\beta^2 x^{2\beta} + \tfrac{1}{4} - v^2\beta^2\right) y = 0 \qquad [1]$$

$\therefore$ with $v = \frac{1}{3}$,

$$\alpha^2\beta^2 x^{2\beta} + \tfrac{1}{4} - \tfrac{1}{9}\beta^2 = \left(\tfrac{2}{3} i\right)^2 \left(\tfrac{3}{2}\right)^2 x^{2\left(\frac{3}{2}\right)} + \tfrac{1}{4} - \tfrac{1}{9}\left(\tfrac{3}{2}\right)^2$$

$$= \left(-\tfrac{4}{9}\right)\left(\tfrac{9}{4}\right) x^3 + \tfrac{1}{4} - \tfrac{1}{9}\left(\tfrac{9}{4}\right) = -x^3$$

$\therefore$ [1] becomes $x^2 y'' - x^3 y = 0$, or $y'' - xy = 0$

$\therefore$ $y = x^{\frac{1}{2}} f\left(\frac{2}{3} i \, x^{\frac{3}{2}}\right)$ is a solution to $y'' - xy = 0$

$\therefore$ if $f_1()$ and $f_2()$ are fundamental solutions to $x^2y'' + xy + (x^2 - (\frac{1}{3})^2)y = 0$, then by above and #11, $y = c_1 x^{\frac{1}{2}} f_1(\frac{2}{3} i x^{\frac{3}{2}}) + c_2 x^{\frac{1}{2}} f_2(\frac{2}{3} i x^{3/2})$ is the general solution to $y'' - xy = 0$, $x > 0$.

13.

(1) $y' = \lambda_j J_0'(\lambda_j x)$ $\quad y'' = \lambda_j^2 J_0''(\lambda_j x)$

$\therefore y'' + \frac{1}{x} y' + \lambda_j^2 y =$

$\lambda_j^2 J_0''(\lambda_j x) + \frac{\lambda_j}{x} J_0'(\lambda_j x) + \lambda_j^2 J_0(\lambda_j x)$

Multiplying both sides by x^2,

$x^2 y'' + xy' + \lambda_j^2 x^2 y =$

$(\lambda_j x)^2 J_0''(\lambda_j x) + (\lambda_j x) J_0'(\lambda_j x) + (\lambda_j x)^2 J_0(\lambda_j x)$

The latter is 0 since $J_0(\epsilon)$ is a solution to $\epsilon^2 J_0''(\epsilon) + \epsilon J_0'(\epsilon) + \epsilon^2 J_0(\epsilon) = 0$, and now just substitute $\lambda_j x$ for ϵ.

$\therefore$ For $y(x) = J_0(\lambda_j x)$, $x^2 y'' + xy' + \lambda_j^2 x^2 y = 0$, and since $x > 0$, and dividing by x^2, $y(x) = J_0(\lambda_j x)$ solves $y'' + \frac{1}{x} y' + \lambda_j^2 y = 0$.

(2) Since $\lambda_j^2 J_0''(\lambda_j x) + \frac{\lambda_j}{x} J_0'(\lambda_j x) + \lambda_j^2 J_0(\lambda_j x) = 0$

then multiplying by $x J_0(\lambda_i x)$ gives:

[1] $x\lambda_j^2 J_0''(\lambda_j x) J_0(\lambda_i x) + \lambda_j J_0'(\lambda_j x) J_0(\lambda_i x) + x\lambda_j^2 J_0(\lambda_j x) J_0(\lambda_i x) = 0$

Similarly,

[2] $x\lambda_i^2 J_0''(\lambda_i x) J_0(\lambda_j x) + \lambda_i J_0'(\lambda_i x) J_0(\lambda_j x) + x\lambda_i^2 J_0(\lambda_i x) J_0(\lambda_j x) = 0$

Subtracting [2] − [1]:

$$(\lambda_i^2 - \lambda_j^2) x J_0(\lambda_i x) J_0(\lambda_j x) =$$

$$- x\lambda_i^2 J_0''(\lambda_i x) J_0(\lambda_j x) - \lambda_i J_0'(\lambda_i x) J_0(\lambda_j x) - \lambda_i \lambda_j J_0'(\lambda_i x) J_0'(\lambda_j x)$$

$$+ x\lambda_j^2 J_0''(\lambda_j x) J_0(\lambda_i x) + \lambda_j J_0'(\lambda_j x) J_0(\lambda_i x) + \lambda_i \lambda_j J_0'(\lambda_j x) J_0'(\lambda_i x)$$

$$= -\frac{d}{dx}\left[\lambda_i x J_0'(\lambda_i x) J_0(\lambda_j x)\right] + \frac{d}{dx}\left[\lambda_j x J'(\lambda_j x) J_0(\lambda_i x)\right]$$

$$\therefore\ (\lambda_i^2 - \lambda_j^2)\int_0^1 x\, J_0(\lambda_i x)\, J_0(\lambda_j x)\, dx =$$

$$-\int_0^1 \frac{d}{dx}\left[\lambda_i x J_0'(\lambda_i x) J_0(\lambda_j x)\right] dx + \int_0^1 \frac{d}{dx}\left[\lambda_j x J_0'(\lambda_j x) J_0(\lambda_i x)\right] dx$$

$$= -x\lambda_i J_0'(\lambda_i x) J_0(\lambda_j x)\Big|_0^1 + x\lambda_j J_0'(\lambda_j x) J_0(\lambda_i x)\Big|_0^1$$

$$= -\left[\lambda_i J'(\lambda_i) \overbrace{J_0(\lambda_j)}^{=0} - 0\cdot\lambda_i J_0'(0) J_0(0)\right]$$

$$+ \left[\lambda_j J_0'(\lambda_j) \overbrace{J_0(\lambda_i)}^{=0} - 0\cdot\lambda_j J_0'(0) J_0(0)\right]$$

$$= -[0-0] + [0-0] = 0$$

$$\therefore\ (\lambda_i^2 - \lambda_j^2)\int_0^1 x\, J_0(\lambda_i x)\, J_0(\lambda_j x)\, dx = 0$$

Since $\lambda_i \neq \lambda_j$, $\lambda_i^2 - \lambda_j^2 \neq 0$,

$$\therefore\ \boxed{\int_0^1 x\, J_0(\lambda_i x)\, J_0(\lambda_j x)\, dx = 0}$$