7.1 Introduction

Note Title 10/14/2019

1.

Let $x_1 = u$, $x_2 = u'$, $\therefore$ $x_1' = x_2$, $x_2' + 0.5x_2 + 2x_1 = 0$

Or, $\boxed{\begin{aligned} x_1' &= x_2 \\ x_2' &= -2x_1 - 0.5x_2 \end{aligned}}$

2.

Let $x_1 = u$, $x_2 = u'$, $\therefore$ $x_1' = x_2$, $t^2 x_2' + t x_2 + (t^2 - 0.25)x_1 = 0$

Dividing by t^2, $t \neq 0$, $x_2' + t^{-1}x_2 + (1 - 0.25t^{-2})x_1 = 0$

Or, $\boxed{\begin{aligned} x_1' &= x_2 \\ x_2' &= -(1-0.25t^{-2})x_1 - t^{-1}x_2 \end{aligned}}$

3.

Let $x_1 = u$, $x_2 = u'$, $x_3 = u''$, $x_4 = u'''$, $x_4' = u^{(4)}$

$\therefore$ $x_1' = x_2$, $x_2' = x_3$, $x_3' = x_4$, $x_4' - x_1 = 0$

Or,

$$\boxed{\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= x_4 \\ x_4' &= x_1 \end{aligned}}$$

4.

Let $x_1(t) = u(t)$, $x_2(t) = u'(t) = x_1'(t)$, $x_2'(t) = u''(t)$

$\therefore$ $x_1(0) = u(0) = 1$, $x_2(0) = u'(0) = -2$

$$\therefore \boxed{\begin{aligned} x_1' &= x_2 \\ x_2' &= -4x_1 - 0.25x_2 + 2\cos(3t) \\ x_1(0) &= 1, \quad x_2(0) = -2 \end{aligned}}$$

5.

Let $x_1(t) = u(t)$, $x_2(t) = u'(t) = x_1'(t)$, $\therefore x_2' = u''$

$x_1(0) = u(0) = u_0$, $x_2(0) = u'(0) = u_0'$

$$\therefore \boxed{\begin{aligned} x_1' &= x_2 \\ x_2' &= -q(t)\,x_1 - p(t)\,x_2 + g(t) \\ x_1(0) &= u_0, \quad x_2(0) = u_0' \end{aligned}}$$

6.

(a)

$$x_2 = 2x_1 + x_1'$$

(b)

$$x_2' = 2x_1' + x_1''$$

$$\therefore 2x_1' + x_1'' = x_1 - 2(2x_1 + x_1') = x_1 - 4x_1 - 2x_1'$$

Or, $\boxed{x_1'' + 4x_1' + 3x_1 = 0}$

(c)

Characteristic equation: $r^2 + 4r + 3 = 0$, $(r+3)(r+1) = 0$

$\therefore r = -1, -3.$ $\quad \therefore \boxed{x_1(t) = c_1 e^{-t} + c_2 e^{-3t}}$

where c_1, c_2 are constants

(d)

$$x_2 = 2\left(c_1 e^{-t} + c_2 e^{-3t}\right) + \left(-c_1 e^{-t} - 3c_2 e^{-3t}\right)$$

$$\therefore \boxed{x_2 = c_1 e^{-t} - c_2 e^{-3t}}$$

7.

(a) $2x_2 = 3x_1 - x_1'$ $\quad \therefore 2\overbrace{x_2(0)}^{=\frac{1}{2}} = 3\overbrace{x_1(0)}^{=3} - x_1'(0)$, $\; x_1'(0) = 8$

$$x_2' = \frac{3}{2}x_1' - \frac{1}{2}x_1''$$

$$\therefore \frac{3}{2}x_1' - \frac{1}{2}x_1'' = 2x_1 - (3x_1 - x_1')$$

$$\therefore \frac{1}{2}x_1'' - \frac{1}{2}x_1' - x_1 = 0$$

$$\therefore \boxed{x_1'' - x_1' - 2x_1 = 0, \quad x_1(0) = 3, \; x_1'(0) = 8}$$

(b) Characteristic equation for x_1: $r^2 - r - 2 = 0$,

$(r-2)(r+1) = 0$, $\; r = 2, -1$.

$$\therefore \quad x_1(t) = c_1 e^{2t} + c_2 e^{-t}$$

$$x_1(0) = 3 = c_1 + c_2 \qquad \therefore \quad 3c_1 = 11, \quad c_1 = \frac{11}{3}$$

$$x_1'(0) = 8 = 2c_1 - c_2 \qquad\qquad c_2 = -\frac{2}{3}$$

$$\therefore \quad \boxed{x_1(t) = \frac{11}{3} e^{2t} - \frac{2}{3} e^{-t}}$$

From $2x_2 = 3x_1 - x_1'$,

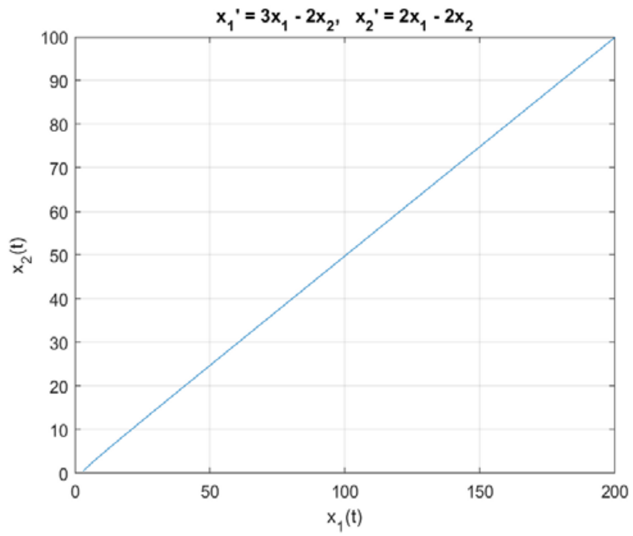
$$2x_2 = 3\left(\frac{11}{3} e^{2t} - \frac{2}{3} e^{-t}\right) - \left(\frac{22}{3} e^{2t} + \frac{2}{3} e^{-t}\right)$$

$$= 11 e^{2t} - 2 e^{-t} - \frac{22}{3} e^{2t} - \frac{2}{3} e^{-t}$$

$$\therefore \quad \boxed{x_2(t) = \frac{11}{6} e^{2t} - \frac{4}{3} e^{-t}}$$

(c) Using MATLAB,

```
clear, clc
t = 0:0.01:2;
x1 = (11/3)*exp(2*t) - (2/3)*exp(-t);
x2 = (11/6)*exp(2*t) - (4/3)*exp(-t);
plot(x1,x2)
grid on
xlabel 'x_1(t)', ylabel 'x_2(t)'
st = strcat('x_1'' = 3x_1 - 2x_2,', ...
            '   x_2'' = 2x_1 - 2x_2');
title(st)
```



8.

(a)

$x_1'' = 2x_2' = 2(-2x_1)$ $\quad x_1'(0) = 2x_2(0) = 2(4) = 8$

$\therefore \boxed{x_1'' + 4x_1 = 0 \quad x_1(0) = 3, \quad x_1'(0) = 8}$

(b)

Characteristic equation for x_1: $r^2 + 4 = 0$, $r = \pm 2i$

$\therefore x_1(t) = c_1 \cos(2t) + c_2 \sin(2t)$

$x_1(0) = c_1 = 3 \quad x_1'(0) = 2c_2 \cos(0) = 8, \; c_2 = 4$

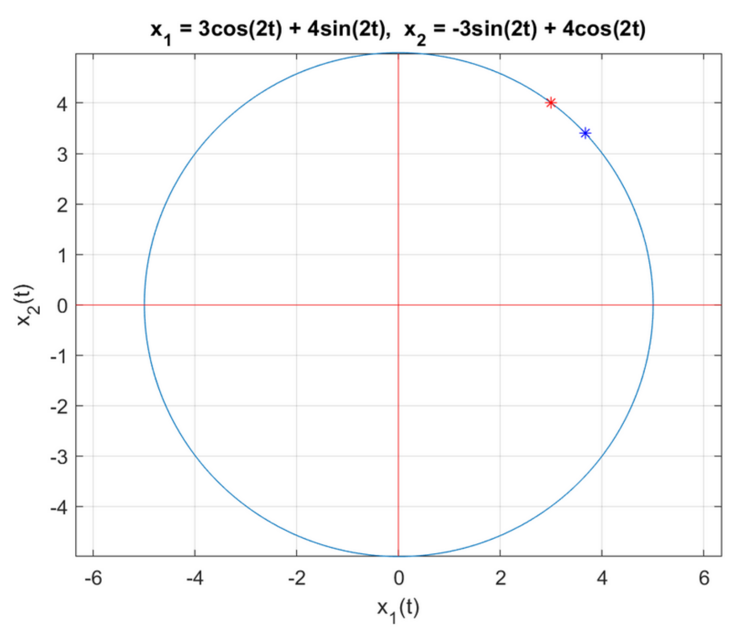
$\therefore \boxed{x_1(t) = 3\cos(2t) + 4\sin(2t)}$

$\frac{1}{2}x_1' = \boxed{x_2(t) = -3\sin(2t) + 4\cos(2t)}$

(c)

Using MATLAB, code and plot on next page.

```
clear, clc
t = 0:0.01:2*pi;
x1 = 3*cos(2*t) + 4*sin(2*t);
x2 = -3*sin(2*t) + 4*cos(2*t);
plot(x1,x2)
hold on      % plot sequential points to get direction
plot(x1(1),x2(1),'r*')   % plot 1st point in red
plot(x1(10),x2(10),'b*') % plot 10th point in blue
grid on
axis equal  % to see if circle or ellipse
xline(0, 'color', 'r');
yline(0, 'color', 'r');
xlabel 'x_1(t)', ylabel 'x_2(t)'
title('x_1 = 3cos(2t) + 4sin(2t),  x_2 = -3sin(2t) + 4cos(2t)')
```



red → blue points mean line is drawn clockwise

9.

(a)

$$x_2 = \frac{1}{2}x_1' + \frac{1}{4}x_1 \quad \therefore \quad x_2' = \frac{1}{2}x_1'' + \frac{1}{4}x_1'$$

$$\therefore \left(\frac{1}{2}x_1'' + \frac{1}{4}x_1'\right) = -2x_1 - \frac{1}{2}\left(\frac{1}{2}x_1' + \frac{1}{4}x_1\right) = -\frac{1}{4}x_1' - \frac{17}{8}x_1$$

$$x_1'(0) = -\frac{1}{2}\overset{=-2}{x_1(0)} + 2\overset{=2}{x_2(0)} = 5$$

$$\therefore \frac{1}{2}x_1'' + \frac{1}{2}x_1' + \frac{17}{8}x_1 = 0,$$

$$\text{Or,} \quad \boxed{x_1'' + x_1' + \frac{17}{4}x_1 = 0, \; x_1(0) = -2, \; x_1'(0) = 5}$$

(b)

x_1 characteristic equation: $r^2 + r + \frac{17}{4} = 0$, $r = \frac{-1 \pm \sqrt{1-17}}{2}$,

$r = -\frac{1}{2} \pm 2i$. $\therefore x_1 = c_1 e^{-t/2}\cos(2t) + c_2 e^{-t/2}\sin(2t)$

$$x(0) = c_1 = -2$$

$$x_1'(t) = e^{-t/2}\cos(2t) + 4e^{-t/2}\sin(2t)$$

$$- \frac{c_2}{2} e^{-t/2}\sin(2t) + 2c_2 e^{-t/2}\cos(2t)$$

$$\therefore x_1'(0) = 1 + 2c_2 = 5, \quad c_2 = 2$$

$$\therefore x_1'(t) = 5e^{-t/2}\cos(2t) + 3e^{-t/2}\sin(2t)$$

$$\therefore \boxed{x_1(t) = -2e^{-t/2}\cos(2t) + 2e^{-t/2}\sin(2t)}$$

$$x_2 = \frac{1}{2}x_1' + \frac{1}{4}x_1$$

$$= \frac{5}{2}e^{-t/2}\cos(2t) + \frac{3}{2}e^{-t/2}\sin(2t)$$

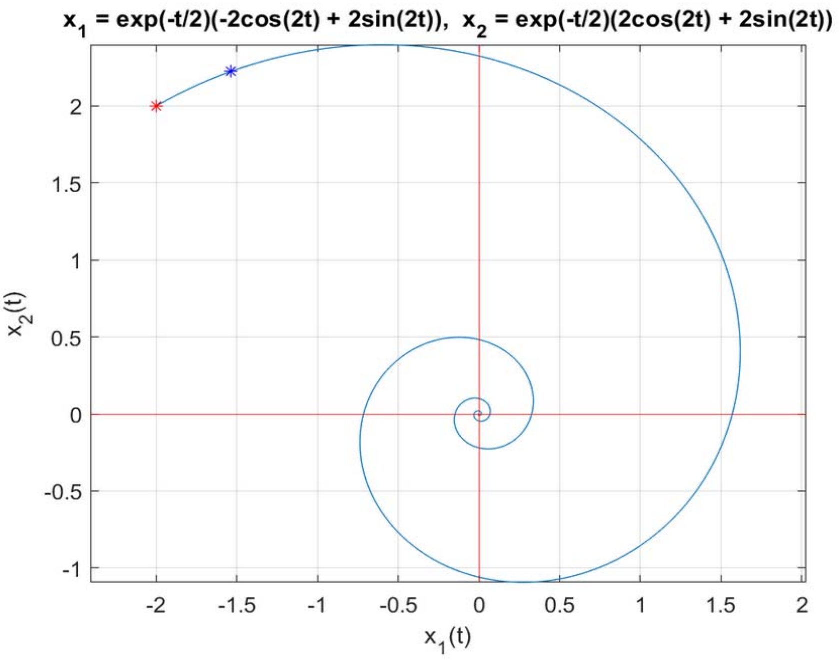
$$+ -\frac{1}{2}e^{-t/2}\cos(2t) + \frac{1}{2}e^{-t/2}\sin(2t)$$

$$\therefore \boxed{x_2(t) = 2e^{-t/2}\cos(2t) + 2e^{-t/2}\sin(2t)}$$

(c)

Using MATLAB,

```
clear, clc
t = 0:0.01:4*pi;
x1 = exp(-t/2).*(-2*cos(2*t) + 2*sin(2*t));
x2 = exp(-t/2).*(2*cos(2*t) + 2*sin(2*t));
plot(x1,x2)
hold on      % plot sequential points to get direction
plot(x1(1),x2(1),'r*')   % plot 1st point in red
plot(x1(10),x2(10),'b*') % plot 10th point in blue
grid on
axis equal
xline(0, 'color', 'r');
yline(0, 'color', 'r');
xlabel 'x_1(t)', ylabel 'x_2(t)'
title('x_1 = exp(-t/2)(-2cos(2t) + 2sin(2t)),  x_2 = exp(-t/2)(2cos(2t) + 2sin(2t))')
```



A spiral moving clockwise, toward the origin.

10.

$$\frac{d^2 I}{dt} = \frac{1}{L}\frac{dV}{dt}$$

$$= \frac{1}{L}\left[-\frac{I}{C} - \frac{V}{RC}\right] = -\frac{1}{LC} I - \frac{1}{LRC} V$$

$$= -\frac{1}{LC} I - \frac{1}{LRC}\left(L\frac{dI}{dt}\right) = -\frac{1}{LC} I - \frac{1}{RC}\frac{dI}{dt}$$

$$\therefore \boxed{\frac{d^2 I}{dt^2} + \left(\frac{1}{RC}\right)\frac{dI}{dt} + \left(\frac{1}{LC}\right) I = 0}$$

11.

(a) $a_{12} x_2 = x_1' - a_{11} x_1 - g_1(t)$ [1]

$\therefore a_{12} x_2(0) = x_1'(0) - a_{11} x_1(0) - g_1(0)$

$\therefore\ x_1'(0) = a_{12} x_2^0 + a_{11} x_1^0 + g_1(0)$

Differentiating the first given equation,

$$x_1'' = a_{11} x_1' + a_{12} x_2' + g_1'(t)$$

Now substituting for x_2',

$$\therefore\ x_1'' = a_{11} x_1' + a_{12}\left(a_{21} x_1 + a_{22} x_2 + g_2(t)\right) + g_1'(t)$$

Or, $x_1'' - a_{11} x_1' - a_{12} a_{21} x_1 = a_{12} a_{22} x_2 + a_{12} g_2(t) + g_1'(t)$

Substituting for $a_{12} x_2$ from [1],

$$\therefore\ x_1'' - a_{11} x_1' - a_{12} a_{21} x_1 = a_{22}\left(x_1' - a_{11} x_1 - g_1(t)\right) + a_{12} g_2(t) + g_1'(t)$$

$$\therefore\ \boxed{\begin{array}{l} x_1'' - (a_{11} + a_{22}) x_1' + (a_{11} a_{22} - a_{12} a_{21}) x_1 = \\ \qquad a_{12} g_2(t) - a_{22} g_1(t) + g_1'(t) \\ x_1(0) = x_1^0, \quad x_1'(0) = a_{12} x_2^0 + a_{11} x_1^0 + g_1(0) \end{array}}$$

This can be solved for $x_1(t)$, and using [1],

then $x_2(t)$ can be solved if $a_{12} \neq 0$

If $a_{12} = 0$, then do above procedure, solving for $x_2(t)$. In detail,

$$a_{21}x_1 = x_2' - a_{22}x_2 - g_2(t) \qquad [2]$$

$$\therefore\ a_{21}x_1(0) = x_2'(0) - a_{22}x_2(0) - g_2(0)$$

$$\therefore\ x_2'(0) = a_{21}x_1^0 + a_{22}x_2^0 + g_2(0)$$

Differentiating the second given equation,

$$x_2'' = a_{21}x_1' + a_{22}x_2' + g_2'(t)$$

Substituting for x_1',

$$x_2'' = a_{21}\left(a_{11}x_1 + a_{12}x_2 + g_1(t)\right) + a_{22}x_2' + g_2'(t)$$

Or, $x_2'' - a_{22}x_2' - a_{21}a_{12}x_2 = a_{21}a_{11}x_1 + a_{21}g_1(t) + g_2'(t)$

Substituting for $a_{21}x_1$ from [2],

$$x_2'' - a_{22}x_2' - a_{21}a_{12}x_2 = a_{11}\left(x_2' - a_{22}x_2 - g_2(t)\right) + a_{21}g_1(t) + g_2'(t)$$

$$\therefore\ \boxed{\begin{aligned} &x_2'' - (a_{11}+a_{22})x_2' + (a_{11}a_{22} - a_{12}a_{21})x_2 = \\ &\qquad\qquad a_{21}g_1(t) - a_{11}g_2(t) + g_2'(t) \\ &\qquad x_2(0) = x_2^0,\quad x_2'(0) = a_{21}x_1^0 + a_{22}x_2^0 + g_2(0) \end{aligned}}$$

After solving this for $x_2(t)$, $x_1(t)$ is solved using [2] if $a_{21} \neq 0$

$\therefore$ If a_{12} and a_{21} are not both 0, solutions for $x_1(t)$ and $x_2(t)$ can be found using one of the above two methods.

(b) If $a_{11}, \ldots, a_{22}$ are functions of t, a similar method can be performed, but to solve for $x_1(t)$ and $x_2(t)$, both $a_{12}(t)$ and $a_{21}(t)$ cannot be 0 anywhere in the interval of interest, in order to perform the needed substitutions, and the a_{ij} should be differentiable.

12.

Let $A = \begin{bmatrix} p_{11}(t) & p_{12}(t) \\ p_{21}(t) & p_{22}(t) \end{bmatrix}$. Since $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_1(t) \\ y_1(t) \end{bmatrix}$ is a solution, and $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$ is a solution,

$$A\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_1' \\ y_1' \end{bmatrix} \quad \text{and} \quad A\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_2' \\ y_2' \end{bmatrix}$$

$$\therefore A\begin{bmatrix} c_1 x_1 \\ c_1 y_1 \end{bmatrix} = \begin{bmatrix} c_1 x_1' \\ c_1 y_1' \end{bmatrix} \text{ and } A\begin{bmatrix} c_2 x_2 \\ c_2 y_2 \end{bmatrix} = \begin{bmatrix} c_2 x_2' \\ c_2 y_2' \end{bmatrix}$$

$$\therefore A\begin{bmatrix} c_1 x_1 + c_2 x_2 \\ c_1 y_1 + c_2 y_2 \end{bmatrix} = A\begin{bmatrix} c_1 x_1 \\ c_1 y_1 \end{bmatrix} + A\begin{bmatrix} c_2 x_2 \\ c_2 y_2 \end{bmatrix}$$

$$= \begin{bmatrix} c_1 x_1' \\ c_1 y_1' \end{bmatrix} + \begin{bmatrix} c_2 x_2' \\ c_2 y_2' \end{bmatrix}$$

$$= \begin{bmatrix} c_1 x_1' + c_2 x_2' \\ c_1 y_1' + c_2 y_2' \end{bmatrix}$$

$\therefore$ $x = c_1 x_1(t) + c_2 x_2(t)$, $y = c_1 y_1(t) + c_2 y_2(t)$ is a solution, where c_1, c_2 are any constants.

13.

Let $A = \begin{bmatrix} p_{11}(t) & p_{12}(t) \\ p_{21}(t) & p_{22}(t) \end{bmatrix}$ Since $\begin{bmatrix} x_1(t) \\ y_1(t) \end{bmatrix}$ and $\begin{bmatrix} x_2(t) \\ y_2(t) \end{bmatrix}$ are solutions, then $A\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} x_1' \\ y_1' \end{bmatrix}$

and $A\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} + \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} x_2' \\ y_2' \end{bmatrix}$

$$\therefore A\begin{bmatrix} x_1 - y_1 \\ x_2 - y_2 \end{bmatrix} = A\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} - \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right)$$

$$= A\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} - A\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

$$= \left(\begin{bmatrix} x_1' \\ y_1' \end{bmatrix} - \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}\right) - \left(\begin{bmatrix} x_2' \\ y_2' \end{bmatrix} - \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}\right)$$

$$= \begin{bmatrix} x_1' \\ y_1' \end{bmatrix} - \begin{bmatrix} x_2' \\ y_2' \end{bmatrix} \underbrace{- \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} + \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}}_{\text{cancel}}$$

$$= \begin{bmatrix} x_1' - x_2' \\ y_1' - y_2' \end{bmatrix}$$

$\therefore \begin{bmatrix} x_1 - y_1 \\ x_2 - y_2 \end{bmatrix}$ solves the homogeneous system:

$$A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

14.

For mass m_1, the acceleration is $\frac{d^2x_1}{dt^2}$ and the sum of forces acting on it is, using left as negative, $-k_1x_1 + k_2(x_2-x_1) + F_1(t)$

$$\therefore\ m_1\frac{d^2x_1}{dt^2} = -k_1x_1 + k_2(x_2-x_1) + F_1(t) \qquad [1]$$

Similarly for m_2, $m_2\frac{d^2x_2}{dt^2} = \Sigma F$, and

$$\Sigma F = -k_3x_2 - k_2(x_2-x_1) + F_2(t)$$

$$\therefore\ m_2\frac{d^2x_2}{dt^2} = -k_3x_2 - k_2(x_2-x_1) + F_2(t) \qquad [2]$$

Note for k_1, if $x_1<0$, the force is still $-k_1x_1$, and if $x_2<0$, the force is still $-k_3x_2$. Regardless of the sign of x_1 and x_2, if $x_2-x_1>0$, then k_2 is stretched, and so pulls m_1 to the right, m_2 to the left.

$\therefore$ its force on m_1 is $+k_2(x_2-x_1)$ and is $-k_2(x_2-x_1)$ on m_2.

If $x_2 - x_1 < 0$, then k_2 is compressed and pushes m_1 to the left and m_2 to the right.

$\therefore$ its force on m_1 $+k_2(x_2-x_1)$ and on m_2 its force is $-k_2(x_2-x_1)$

$\therefore$ [1] and [2] are correct regardless of the signs of x_1 and x_2.

15.

Restated for brevity, [1] and [2] in # 14 become

$$x_1'' = -\left(\frac{k_1}{m_1} + \frac{k_2}{m_1}\right)x_1 + \frac{k_2}{m_1}x_2 + F_1$$

$$x_2'' = \frac{k_2}{m_2}x_1 - \left(\frac{k_3}{m_2} + \frac{k_2}{m_2}\right)x_2 + F_2$$

Let $a = -\left(\frac{k_1}{m_1} + \frac{k_2}{m_1}\right)$, $b = \frac{k_2}{m_1}$, $c = \frac{k_2}{m_2}$, $d = -\left(\frac{k_3}{m_2} + \frac{k_2}{m_2}\right)$

$$\therefore \; x_1'' = a x_1 + b x_2 + F_1 \qquad [1']$$

$$x_2'' = c x_1 + d x_2 + F_2 \qquad [2']$$

With $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} y_3 \\ y_4 \end{bmatrix}$, then

$$\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = \begin{bmatrix} y_3' \\ y_4' \end{bmatrix} \text{ and } \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} y_3 \\ y_4 \end{bmatrix}$$

$$\therefore \; y_1' = y_3$$

$$y_2' = y_4$$

$$y_3' = a y_1 + b y_2 + F_1$$

$$y_4' = c y_1 + d y_2 + F_1$$

Or, converting back,

$$\boxed{\begin{aligned} y_1' &= y_3 \\ y_2' &= y_4 \\ y_3' &= -\left(\frac{k_1 + k_2}{m_1}\right) y_1 + \left(\frac{k_2}{m_1}\right) y_2 + F_1 \\ y_4' &= \left(\frac{k_2}{m_2}\right) y_1 - \left(\frac{k_2 + k_3}{m_2}\right) y_2 + F_2 \end{aligned}}$$

16.

(a)

In the upper loop, clockwise flow is in the direction of the arrow, so the voltage drop is positive, $+V_1$. Across the resistor, clockwise flow is opposite the arrow, so the voltage drop is negative, $-V_2$. By Kirchhoff's voltage law around the loop, $\Sigma V = 0$, so $V_1 + (-V_2) = 0$, or $\underline{V_1 - V_2 = 0}$.

Going clockwise around the lower loop, drop across R is $+V_2$ since the current arrow aligns with the clockwise direction, and the drop across L is $-V_3$, since the current arrow is opposite clockwise. $\therefore$ $\sum v = (+V_2) + (-V_3) = \underline{V_2 - V_3 = 0}$

(b)

Take the left node. Current going into the node, say from L, is $(-I_3)$, since it is opposite the arbitrarily assigned arrow. Similarly, the flow going into the node from R is $(-I_2)$ and from C is $(-I_1)$. By Kirchhoff's law, $\sum I = 0$,

$\therefore$ $(-I_1) + (-I_2) + (-I_3) = 0$, or $\underline{I_1 + I_2 + I_3 = 0}$

For the right node, all signs for currents are positive, so again, $I_1 + I_2 + I_3 = 0$.

(c)

For the capacitor, $C\dfrac{dV}{dt} = I_1$ $\therefore$ $C\dfrac{dV_1}{dt} = I_1$, or

$\underline{CV_1' = I_1}$

For the resistor, I_2 flows through it, $\therefore$ $\underline{V_2 = RI_2}$

For the inductor, $L\frac{dI}{dt} = V$, $\therefore$ $\underline{LI_3' = V_3}$

(d)

$I_1 + I_2 + I_3 \overset{\text{from (b)}}{=} 0 \Rightarrow I_1 = -I_2 - I_3$

$\therefore$ $CV_1' \overset{\text{from (c)}}{=} I_1 = -I_2 - I_3$

But $V_2 \overset{\text{from (c)}}{=} RI_2$, so $I_2 = \frac{V_2}{R}$

$\therefore$ $CV_1' = -I_3 - \frac{V_2}{R}$

But $V_1 - V_2 \overset{\text{from (a)}}{=} 0 \Rightarrow V_2 = V_1$

$\therefore$ $\boxed{CV_1' = -I_3 - \frac{V_1}{R}}$

Since $V_1 - V_2 \overset{\text{from (a)}}{=} 0$ and $V_2 - V_3 \overset{\text{from (a)}}{=} 0$, $V_1 = V_3$

$\therefore$ From $LI_3' \overset{\text{from (c)}}{=} V_3$, $\boxed{LI_3' = V_1}$

17.

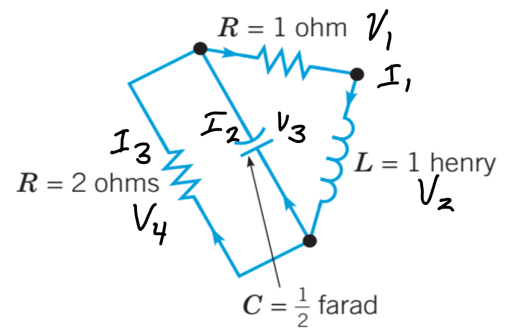


FIGURE 7.1.4 The circuit in Problem 17.

Using the directions of the arrows in the figure, Let V_1 = voltage drop across R_1 (= 1 ohm), V_2 = drop across L, V_3 drop across C, and V_4 drop across R_2 (= 2 ohms).

$\therefore \quad V_1 + V_2 + V_3 = 0 \qquad [1]$

$(-V_3) + V_4 = 0$ [2] (current opposite clockwise in loop)

Let I_1 = current across R and L, I_2 across C, and I_3 across $R = 2\,\Omega$.

$\therefore \quad I_1 = I_2 + I_3 \qquad [3]$

Now use $V = RI$, $C\dfrac{dV}{dt} = I$, $L\dfrac{dI}{dt} = V$

From [3], $I_1 = \underbrace{C\dfrac{dV_3}{dt}}_{=I_2} + \underbrace{\dfrac{V_4}{R_2}}_{=I_3}$

But $C = \frac{1}{2}$ and $R_2 = 2$

$\therefore \quad I_1 = \frac{1}{2}\frac{dV_3}{dt} + \frac{V_4}{2}$, or $I_1 - \frac{V_4}{2} = \frac{1}{2}\frac{dV_3}{dt}$

Or, $2I_1 - V_4 = \frac{dV_3}{dt}$

By [2], $V_3 = V_4$

$$\therefore \quad \underline{\frac{dV_3}{dt} = 2I_1 - V_3} \qquad [4]$$

From [1], $\overset{=V_1}{R_1 I_1} + \overset{=V_2}{L\frac{dI_1}{dt}} + V_3 = 0$

Using $R_1 = 1$, $L = 1$, $I_1 + \frac{dI_1}{dt} + V_3 = 0$

Or, $\underline{\frac{dI_1}{dt} = -I_1 - V_3}$ [5]

Dropping the subscripts from [4], [5],

$$\boxed{\frac{dV}{dt} = 2I - V, \quad \frac{dI}{dt} = -I - V}$$

where I is the curent through L, and V is the voltage drop across C.

18.

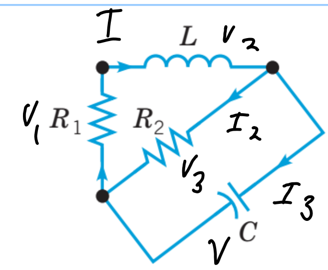


FIGURE 7.1.5 The circuit in Problem 18.

Use directions of arrows in the figure as the direction of current, and let

V_1 = voltage across R_1, V_2 voltage across L,

V_3 voltage across R_2, V voltage across C,

I = current through L, I_2 current through R_2,

I_3 current through C.

By Kirchhoff's laws,

$$V_1 + V_2 + V_3 = 0 \quad [1]$$

$$(-V_3) + V = 0 \quad [2] \quad \text{(current opposite clockwise in loop)}$$

$$I = I_2 + I_3 \quad [3]$$

And use $V = RI$, $C\frac{dV}{dt} = I$, $L\frac{dI}{dt} = V$

By [1], $\underbrace{R_1 I}_{=V_1} + \underbrace{L\frac{dI}{dt}}_{=V_2} + V_3 = 0$

$$\therefore \quad L\frac{dI}{dt} = -R_1 I - V_3$$

But by [3], $V = V_3$

$$\therefore \quad \boxed{L\frac{dI}{dt} = -R_1 I - V}$$

By [3], $I = \underbrace{\frac{V_3}{R_2}}_{=I_2} + \underbrace{C\frac{dV}{dt}}_{=I_3}$

$$\therefore \quad C\frac{dV}{dt} = I - \frac{V_3}{R_2}$$

But $V = V_3$ by [3]

$$\therefore \quad \boxed{C\frac{dV}{dt} = I - \frac{V}{R_2}}$$

where I = current through L, V = voltage across C.

19.

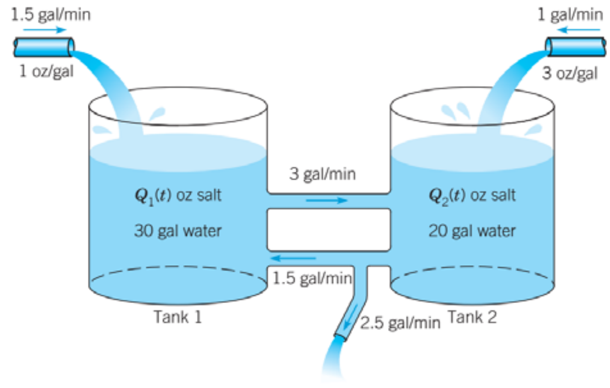


FIGURE 7.1.6 Two interconnected tanks (Problem 19).

(a) **a.** Let $Q_1(t)$ and $Q_2(t)$, respectively, be the amount of salt in each tank at time t. Write down differential equations and initial conditions that model the flow process. Observe that the system of differential equations is nonhomogeneous.

The concentration of salt in tank 1 at any time t is $\frac{Q_1(t)}{30 \text{ gallons}}$, and for tank 2 $\frac{Q_2(t)}{20 \text{ gallons}}$

Salt is dumped into tank 1 at a rate of $\left(\frac{1 \text{ oz}}{\text{gal}}\right)\left(\frac{1.5 \text{ gal}}{\text{min}}\right) = \frac{1.5 \text{ oz}}{\text{min}}$. Salt is dumped into tank 2 at a rate of $\left(\frac{3 \text{ oz}}{\text{gal}}\right)\left(\frac{1 \text{ gal}}{\text{min}}\right) = \frac{3 \text{ oz}}{\text{min}}$

Salt goes from tank 1 $\to$ tank 2 at a rate of $\frac{Q_1(t) \text{ oz}}{30 \text{ gal}} \cdot \frac{3 \text{ gal}}{\text{min}} = \frac{Q_1}{10}$, and from tank 2 $\to$ tank 1 at a rate of

$$\frac{Q_2(t)\text{ oz}}{20\text{ gal}} \cdot \frac{1.5\text{ gal}}{\text{min}} = \frac{3Q_2}{40}.$$ Salt leaves

tank 2 at a rate of $\frac{Q_2(t)\text{ oz}}{20\text{ gal}} \cdot \frac{4\text{ gal}}{\text{min}} = \frac{Q_2}{5}$

$\therefore$

$$\boxed{\begin{gathered}\frac{dQ_1}{dt} = 1.5 - \frac{Q_1}{10} + \frac{3Q_2}{40} \\ \frac{dQ_2}{dt} = 3 + \frac{Q_1}{10} - \frac{Q_2}{5} \\ \text{Initial conditions: } Q_1(0) = 25,\ Q_2(0) = 15\end{gathered}}$$

(b)

Take limit as $t \to \infty$, as $Q_1' \to 0$, $Q_2' \to 0$ as $t \to \infty$.

Prediction: Tank 2 reaches equilibrium faster since its flux is 4 gal/min (vs. 3 gal/min for Tank 1)

$$\therefore \quad 0 = 1.5 - \frac{Q_1^E}{10} + \frac{3}{40}Q_2^E$$

$$0 = 3 + \frac{Q_1^E}{10} - \frac{Q_2^E}{5}$$

Adding, $0 = 4.5 - \frac{5}{40} Q_2^E$, $\underline{Q_2^E = 36 \text{ oz}}$.

$\therefore\ 0 = 1.5 - \frac{Q_1^E}{10} + \frac{3}{40}(36)$, $\underline{Q_1^E = 42 \text{ oz}}$

(c)

$$X_1' = Q_1' = 1.5 - \frac{Q_1}{10} + \frac{3}{40} Q_2$$

$$= 1.5 - \frac{(X_1 + Q_1^E)}{10} + \frac{3}{40}(X_2 + Q_2^E)$$

$$= 1.5 - \frac{X_1}{10} - 4.2 + \frac{3}{40} X_2 + \frac{3}{40}(36) \quad = \frac{27}{10} = 2.7$$

$$\therefore\ X_1' = -\frac{X_1}{10} + \frac{3}{40} X_2$$

$$X_2' = Q_2' = 3 + \frac{Q_1}{10} - \frac{Q_2}{5}$$

$$= 3 + \frac{(X_1 + Q_1^E)}{10} - \frac{(X_2 + Q_2^E)}{5}$$

$$= 3 + \frac{X_1}{10} + 4.2 - \frac{X_2}{5} - 7.2 \quad = \frac{X_1}{10} - \frac{X_2}{5}$$

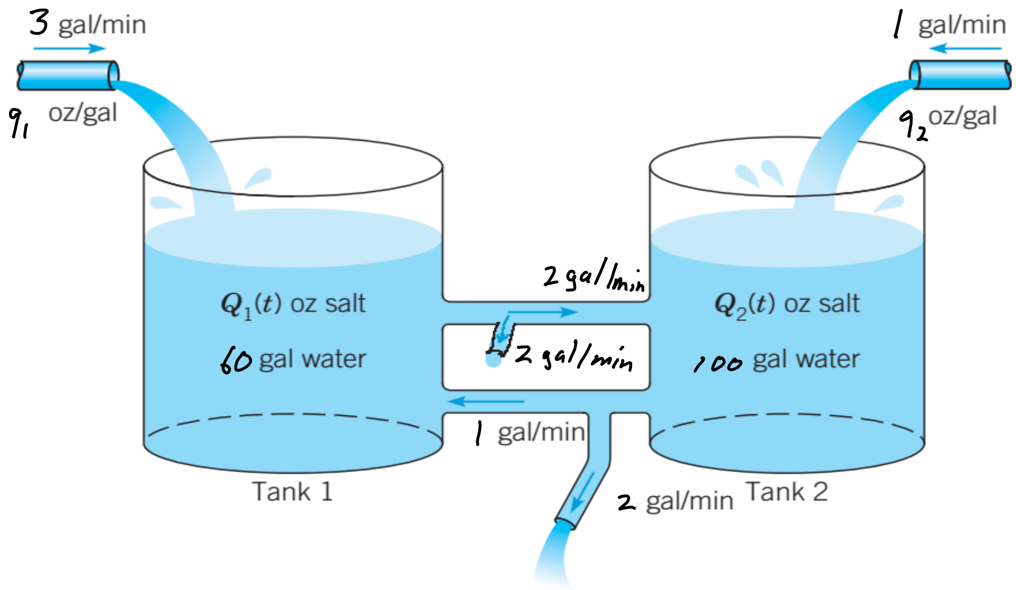
Note: $x_1(0) = Q_1(0) - Q_1^E = 25 - 42 = -17$

$x_2(0) = Q_2(0) - Q_2^E = 15 - 36 = -21$

$$\therefore \boxed{\begin{array}{ll} x_1' = -\dfrac{x_1}{10} + \dfrac{3}{40}x_2 & x_1(0) = -17 \\[2ex] x_2' = \dfrac{x_1}{10} - \dfrac{x_2}{5} & x_2(0) = -21 \end{array}}$$

20.

(a)



Salt concentration in Tank 1 at time t: $\frac{Q_1(t) \text{ oz}}{60 \text{ gal}}$

in Tank 2: $\frac{Q_2(t) \text{ oz}}{100 \text{ gal}}$

Salt enters Tank 1 at: $\left(q_1 \frac{\text{oz}}{\text{gal}}\right)\left(3 \frac{\text{gal}}{\text{min}}\right) = 3q_1$

leaves Tank 1 at: $\frac{Q_1}{60}(4 \text{gal/min}) = \frac{Q_1}{15}$

enters Tank 1 from 2 at: $\left(\frac{Q_2}{100}\right)\left(1 \frac{\text{gal}}{\text{min}}\right) = \frac{Q_2}{100}$

Salt enters Tank 2 at: $(q_2 \text{ oz/gal})(1 \text{ gal/min}) = q_2$

enters Tank 2 from 1 at: $\left(\frac{Q_1}{60}\right)(2 \text{ gal/min}) = \frac{Q_1}{30}$

leaves Tank 2 at: $\left(\frac{Q_2}{100}\right)(3 \text{ gal/min}) = \frac{3Q_2}{100}$

$$\therefore \boxed{\begin{aligned} \frac{dQ_1}{dt} &= 3q_1 - \frac{Q_1}{15} + \frac{Q_2}{100}, \quad Q_1(0) = Q_1^\circ \\ \frac{dQ_2}{dt} &= q_2 + \frac{Q_1}{30} - \frac{3Q_2}{100}, \quad Q_2(0) = Q_2^\circ \end{aligned}}$$

(b)

At equilibrium (limit as $t \to \infty$), $\frac{dQ_1}{dt} = \frac{dQ_2}{dt} = 0$

$$\therefore\ 0 = 3q_1 - \frac{Q_1^E}{15} + \frac{Q_2^E}{100} \quad [1]$$

$$0 = q_2 + \frac{Q_1^E}{30} - \frac{3Q_2^E}{100} \quad [2]$$

Multiply [1] by 3, add to [2]:

$$0 = 9q_1 + q_2 - \frac{5}{30}Q_1^E, \quad \underline{Q_1^E = 54q_1 + 6q_2}$$

Multiply [2] by 2, add to [1]:

$$0 = 3q_1 + 2q_2 - \frac{5}{100}Q_2^E, \quad \underline{Q_2^E = 60q_1 + 40q_2}$$

(c)

$$\begin{bmatrix} 54 & 6 \\ 60 & 40 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} Q_1^E \\ Q_2^E \end{bmatrix}.$$ Since $\det\begin{bmatrix} 54 & 6 \\ 60 & 40 \end{bmatrix} = 1800 \neq 0$,

Then $$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \frac{1}{1800}\begin{bmatrix} 40 & -6 \\ -60 & 54 \end{bmatrix}\begin{bmatrix} 60 \\ 50 \end{bmatrix} = \frac{1}{1800}\begin{bmatrix} 2100 \\ -900 \end{bmatrix}$$

$\therefore\ q_2 = -0.5$ oz/gal, which is impossible, since q_2 must be ≥ 0 (salt can't flow upward out of tank). $\therefore$ <u>No</u>, not possible.

(d)

From (c), $\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \frac{1}{1800}\begin{bmatrix} 40 & -6 \\ -60 & 54 \end{bmatrix}\begin{bmatrix} Q_1^E \\ Q_2^E \end{bmatrix}$

$$\therefore \quad q_1 = \frac{1}{1800}\left(40Q_1^E - 6Q_2^E\right) \geq 0 \qquad [1']$$

$$q_2 = \frac{1}{1800}\left(-60Q_1^E + 54Q_2^E\right) \geq 0 \qquad [2']$$

$$\therefore \quad 20Q_1^E - 3Q_2^E \geq 0 \qquad [1'']$$

$$-10Q_1^E + 9Q_2^E \geq 0 \qquad [2'']$$

$$\therefore \quad 20Q_1^E \geq 3Q_2^E, \text{ or } \frac{Q_1^E}{Q_2^E} \geq \frac{3}{20}$$

$$9Q_2^E \geq 10Q_1^E, \text{ or } \frac{9}{10} \geq \frac{Q_1^E}{Q_2^E}$$

$$\therefore \quad \boxed{\frac{3}{20} \leq \frac{Q_1^E}{Q_2^E} < \frac{9}{10}}$$

7.2 Matrices

Note Title 10/24/2019

1.

(a)

$$\begin{bmatrix} 2+4 & -4+-2 & 0+3 \\ 6+(-1) & 4+5 & -2+0 \\ -4+6 & 2+1 & 6+2 \end{bmatrix} = \begin{bmatrix} 6 & -6 & 3 \\ 5 & 9 & -2 \\ 2 & 3 & 8 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 1-16 & -2-(-8) & 0-12 \\ 3-(-4) & 2-20 & -1-0 \\ -2-24 & 1-4 & 3-8 \end{bmatrix} = \begin{bmatrix} -15 & 6 & -12 \\ 7 & -18 & -1 \\ -26 & -3 & -5 \end{bmatrix}$$

(c)

$$\begin{bmatrix} 4+2+0 & -2-10+0 & 3+0+0 \\ 12-2-6 & -6+10-1 & 9+0-2 \\ -8-1+18 & 4+5+3 & -6+0+6 \end{bmatrix} = \begin{bmatrix} 6 & -12 & 3 \\ 4 & 3 & 7 \\ 9 & 12 & 0 \end{bmatrix}$$

(d)

$$\begin{bmatrix} 4-6-6 & -8-4+3 & 0+2+9 \\ -1+15+0 & 2+10+0 & 0-5+0 \\ 6+3-4 & -12+2+2 & 0-1+6 \end{bmatrix} = \begin{bmatrix} -8 & -9 & 11 \\ 14 & 12 & -5 \\ 5 & -8 & 5 \end{bmatrix}$$

2.

(a)

$$\begin{bmatrix} 1+i-(0+2i) & -1+2i-(6+0i) \\ 3+2i-(4+0i) & 2-i-(0-4i) \end{bmatrix} = \underline{\begin{bmatrix} 1-i & -7+2i \\ -1+2i & 2+3i \end{bmatrix}}$$

(b)

$$\begin{bmatrix} 3+3i+(0+i) & -3+6i+(3+0i) \\ 9+6i+(2+0i) & 6-3i+(0-2i) \end{bmatrix} = \underline{\begin{bmatrix} 3+4i & 6i \\ 11+6i & 6-5i \end{bmatrix}}$$

(c)

$$\begin{bmatrix} (1+i)(i)+(-1+2i)(2) & (1+i)(3)+(-1+2i)(-2i) \\ (3+2i)(i)+(2-i)(2) & (3+2i)(3)+(2-i)(-2i) \end{bmatrix}$$

$$= \begin{bmatrix} -1+i+(-2+4i) & 3+3i+(4+2i) \\ -2+3i+(4-2i) & 9+6i+(-2-4i) \end{bmatrix} = \underline{\begin{bmatrix} -3+5i & 7+5i \\ 2+i & 7+2i \end{bmatrix}}$$

(d)

This time use MATLAB

```
clear, clc
A = [1 + 1i, -1 + 2i;
     3 + 2i, 2 - 1i];
B = [1i, 3;
     2, -2i];
B*A
```

```
ans = 2×2 complex
    8.0000 + 7.0000i   4.0000 - 4.0000i
    6.0000 - 4.0000i  -4.0000 + 0.0000i
```

$$\therefore BA = \begin{bmatrix} 8+7i & 4-4i \\ 6-4i & -4 \end{bmatrix}$$

3.

(a) $A^T = \begin{bmatrix} -2 & 1 & 2 \\ 1 & 0 & -1 \\ 2 & -3 & 1 \end{bmatrix}$ (b) $B^T = \begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 1 \\ 3 & -1 & 0 \end{bmatrix}$

(c) $A^T + B^T = \begin{bmatrix} -2+1 & 1+3 & 2+(-2) \\ 1+2 & 0+(-1) & -1+1 \\ 2+3 & -3+(-1) & 1+0 \end{bmatrix} = \begin{bmatrix} -1 & 4 & 0 \\ 3 & -1 & 0 \\ 5 & -4 & 1 \end{bmatrix}$

(d) $(A+B)^T = \begin{bmatrix} -1 & 3 & 5 \\ 4 & -1 & -4 \\ 0 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} -1 & 4 & 0 \\ 3 & -1 & 0 \\ 5 & -4 & 1 \end{bmatrix}$

4.

(a) $A^T = \begin{bmatrix} 3-2i & 2-i \\ 1+i & -2+3i \end{bmatrix}$ (b) $\bar{A} = \begin{bmatrix} 3+2i & 1-i \\ 2+i & -2-3i \end{bmatrix}$

(c) $A^* = (\bar{A})^T = \begin{bmatrix} 3+2i & 2+i \\ 1-i & -2-3i \end{bmatrix}$

5.

(a) Using MATLAB,

```
clear, clc
syms a
A = [1, -2, 0;
     3, 2, -1;
     -2, 0, 3];
B = [2, 1, -1;
     -2, 3, 3;
     1, 0, 2];
C = [2, 1, 0;
     1, 2, 2;
     0, 1, -1];
L = (A*B)*C
R = A*(B*C);
isequal(L,R)
L = (A+B)+C
R = A+(B+C);
isequal(L,R)
L = A*(B+C)
R = A*B + A*C;
isequal(L,R)
L = a*(A+B)
R = a*A + a*B;
isequal(L,R)
```

```
L = 3×3

     7   -11    -3
    11    20    17
    -4     3   -12

ans = Logical
   1
L = 3×3

     5     0    -1
     2     7     4
    -1     1     4

ans = Logical
   1
L = 3×3

     6    -8   -11
     9    15     6
    -5    -1     5

ans = Logical
   1
L =
```

$$\begin{pmatrix} 3a & -a & -a \\ a & 5a & 2a \\ -a & 0 & 5a \end{pmatrix}$$

```
ans = Logical
   1
```

(a) $(AB)C = A(BC) = \begin{bmatrix} 7 & -11 & -3 \\ 11 & 20 & 17 \\ -4 & 3 & -12 \end{bmatrix}$

(b) $(A+B)+C = A+(B+C) = \begin{bmatrix} 5 & 0 & -1 \\ 2 & 7 & 4 \\ -1 & 1 & 4 \end{bmatrix}$

(c) $A(B+C) = AB+AC = \begin{bmatrix} 6 & -8 & -11 \\ 9 & 15 & 6 \\ -5 & -1 & 5 \end{bmatrix}$

(d) $\alpha(A+B) = \alpha A + \alpha B = \begin{bmatrix} 3\alpha & -\alpha & -\alpha \\ \alpha & 5\alpha & 2\alpha \\ -\alpha & 0 & 5\alpha \end{bmatrix}$

6.

Assume A, B, and C are $m \times n$ matrices over a field F, and $\alpha, \beta \in F$ for (a) $\to$ (d)

(a)

For all $1 \le i \le m$, $1 \le j \le n$,

$$(A+B)_{ij} = A_{ij} + B_{ij} = B_{ij} + A_{ij} = (B+A)_{ij}$$

↳ by commutative property of F

$\therefore A+B = B+A$

(b)

For all $1 \le i \le m$, $1 \le j \le n$,

$$[A + (B+C)]_{ij} = A_{ij} + (B+C)_{ij}$$

$$= A_{ij} + (B_{ij} + C_{ij})$$

$$= (A_{ij} + B_{ij}) + C_{ij}$$ by associative property of F

$$= (A+B)_{ij} + C_{ij}$$

$$= [(A+B) + C]_{ij}$$

$\therefore A + (B+C) = (A+B) + C$

(c)

For all $1 \le i \le m$, $1 \le j \le n$,

$$\begin{aligned}[\alpha(A+B)]_{ij} &= \alpha(A+B)_{ij} \\ &= \alpha(A_{ij} + B_{ij}) \\ &= \alpha A_{ij} + \alpha B_{ij} \quad \text{by distributive property of multiplication over addition for } F \\ &= [\alpha A + \alpha B]_{ij}\end{aligned}$$

$\therefore \alpha(A+B) = \alpha A + \alpha B$

(d)

For all $1 \le i \le m$, $1 \le j \le n$,

$$\begin{aligned}[(\alpha+\beta)A]_{ij} &= (\alpha+\beta)A_{ij} \\ &= \alpha A_{ij} + \beta A_{ij} \quad \text{by distributive property of multiplication over addition for } F \\ &= [\alpha A]_{ij} + [\beta A]_{ij} \\ &= [\alpha A + \beta A]_{ij}\end{aligned}$$

$\therefore (\alpha+\beta)A = \alpha A + \beta A$

(e)

Let A be $m \times n$, B be $n \times p$, C be $p \times r$ matrices over a field F.

For all $1 \le i \le m$, $1 \le j \le n$, $1 \le k \le p$, $1 \le l \le r$,

$$[A(BC)]_{il} = \sum_{j=1}^{n} A_{ij} (BC)_{jl}$$

$$= \sum_{j=1}^{n} A_{ij} \left(\sum_{k=1}^{p} B_{jk} C_{kl} \right)$$

$$= A_{i1}(B_{11}C_{1l} + B_{12}C_{2l} + \dots + B_{1p}C_{pl})$$
$$+ A_{i2}(B_{21}C_{1l} + B_{22}C_{2l} + \dots + B_{2p}C_{pl})$$
$$+ \vdots$$
$$+ A_{in}(B_{n1}C_{1l} + B_{n2}C_{2l} + \dots + B_{np}C_{pl})$$

$$= A_{i1}B_{11}C_{1l} + A_{i1}B_{12}C_{2l} + \dots + A_{i1}B_{1p}C_{pl}$$
$$+ A_{i2}B_{21}C_{1l} + A_{i2}B_{22}C_{2l} + \dots + A_{i2}B_{2p}C_{pl}$$
$$+ \vdots$$
$$+ A_{in}B_{n1}C_{1l} + A_{in}B_{n2}C_{2l} + \dots + A_{in}B_{np}C_{pl}$$

by distributive law of multiplication over addition

$$= A_{i1}B_{11}C_{1\ell} + A_{i2}B_{21}C_{1\ell} + \dots + A_{in}B_{n1}C_{1\ell}$$
$$+ A_{i1}B_{12}C_{2\ell} + A_{i2}B_{22}C_{2\ell} + \dots + A_{in}B_{n2}C_{2\ell}$$
$$+ \vdots$$
$$+ A_{i1}B_{1p}C_{p\ell} + A_{i2}B_{2p}C_{p\ell} + \dots + A_{in}B_{np}C_{p\ell}$$

by associative law of addition for F

$$= (A_{i1}B_{11} + A_{i2}B_{21} + \dots + A_{in}B_{n1})C_{1\ell}$$
$$+ (A_{i1}B_{12} + A_{i2}B_{22} + \dots + A_{in}B_{n2})C_{2\ell}$$
$$+ \vdots$$
$$+ (A_{i1}B_{1p} + A_{i2}B_{2p} + \dots A_{in}B_{np})C_{p\ell}$$

by distributive law of multiplication over addition

$$= \sum_{K=1}^{p} \left(\sum_{j=1}^{n} A_{ij}B_{jK} \right) C_{K\ell}$$

$$= \sum_{K=1}^{p} (AB)_{iK} C_{K\ell} = [(AB)C]_{i\ell}$$

$$\therefore [A(BC)]_{i\ell} = [(AB)C]_{i\ell}, \text{ all } 1 \le i \le m, 1 \le \ell \le r$$

$$\therefore A(BC) = (AB)C$$

(f)

Let A be $m \times n$, B, C be $n \times p$ matrices over a field F.

For all $1 \le i \le m$, $1 \le j \le n$, $1 \le k \le p$,

$$[A(B+C)]_{ik} = \sum_{j=1}^{n} A_{ij} (B+C)_{jk}$$

$$= \sum_{j=1}^{n} A_{ij} (B_{jk} + C_{jk})$$

$$= \sum_{j=1}^{n} A_{ij} B_{jk} + A_{ij} C_{jk}$$ distributive law

$$= \sum_{j=1}^{n} A_{ij} B_{jk} + \sum_{j=1}^{n} A_{ij} C_{jk}$$ associative law

$$= (AB)_{ik} + (AC)_{ik}$$

$$= [AB + AC]_{ik}$$

$\therefore A(B+C) = AB + AC$

Note: (a)-(f) rely on definition of equality of matrices: $A = B \iff A_{ij} = B_{ij}$, for all i, j.

7.

(a)

$$\begin{bmatrix} 2 & 3i & 1-i \end{bmatrix} \begin{bmatrix} -1+i \\ 2 \\ 3-i \end{bmatrix} = (-2+2i) + (6i) + (2-4i) = \underline{4i}$$

(b)

$$\begin{bmatrix} -1+i & 2 & 3-i \end{bmatrix} \begin{bmatrix} -1+i \\ 2 \\ 3-i \end{bmatrix} = (-2i) + (4) + (8-6i) = \underline{12-8i}$$

(c)

$$(2)(-1-i) + (3i)(2) + (1-i)(3+i) = (-2-2i) + (6i) + (4-2i)$$

$$= \underline{2+2i}$$

(d)

$$(-1+i)(-1-i) + (2)(2) + (3-i)(3+i) = (2) + (4) + (10) = \underline{16}$$

8.

det = 3 - (-8) = 11

$\therefore$ Inverse = $\frac{1}{11}\begin{bmatrix} 3 & -4 \\ 2 & 1 \end{bmatrix}$

9.

det = 6 - (-6) = 12

$\therefore$ Inverse = $\frac{1}{12}\begin{bmatrix} 2 & 1 \\ -6 & 3 \end{bmatrix} = \begin{bmatrix} 1/6 & 1/12 \\ -1/2 & 1/4 \end{bmatrix}$

10.

Using row reduction,

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 4 & 5 & 0 & 1 & 0 \\ 3 & 5 & 6 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & -1 & -2 & 1 & 0 \\ 0 & -1 & -3 & -3 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -3 & -3 & 0 & 1 \\ 0 & 0 & -1 & -2 & 1 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & -5 & 0 & 2 \\ 0 & -1 & -3 & -3 & 0 & 1 \\ 0 & 0 & -1 & -2 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & -5 & 0 & 2 \\ 0 & 1 & 3 & 3 & 0 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -3 & 2 \\ 0 & 1 & 3 & 3 & 0 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -3 & 2 \\ 0 & 1 & 0 & -3 & 3 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right]$$

$$\therefore A^{-1} = \underline{\begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}}$$

Using MATLAB,

```
clear, clc
A = [1, 2, 3;
     2, 4, 5;
     3, 5, 6];
det(A)
inv(A)
```

```
ans = -1
ans = 3×3
    1.0000   -3.0000    2.0000
   -3.0000    3.0000   -1.0000
    2.0000   -1.0000    0.0000
```

$$\therefore A^{-1} = \underline{\begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}}$$

11.

Using MATLAB,

```
clear, clc
A = [1, 2, 1;
     -2, 1, 8;
     1, -2, -7];
% MATLAB function to evaluate
% how close to zero det is
c = cond(A);
if c < 1000
    inv(A)
else
    c
    determinant =0
end
```

```
c = 1.8940e+16
determinant = 0
```

Determinant $= 0$, $\therefore$ <u>singular</u>

To manually show this,

$$\det = (1)\begin{vmatrix} 1 & 8 \\ -2 & -7 \end{vmatrix} - (2)\begin{vmatrix} -2 & 8 \\ 1 & -7 \end{vmatrix} + (1)\begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix}$$

$$= (1)(9) - (2)(6) + (1)(3) = 9 - 12 + 3 = \underline{0}$$

12.

Using MATLAB,

```
clear, clc
A = [2, 1, 0;
     0, 2, 1;
     0, 0, 2];
% MATLAB function to evaluate
% how close to zero det is
c = cond(A);
if c < 1000
    c
    InvA = rats(inv(A))
else
    c
    determinant =0
end
```

```
c = 2.0256

InvA = 3×42 char array
    '      1/2          -1/4           1/8     '
    '       0            1/2          -1/4     '
    '       0             0            1/2     '
```

$$\therefore A^{-1} = \begin{bmatrix} 1/2 & -1/4 & 1/8 \\ 0 & 1/2 & -1/4 \\ 0 & 0 & 1/2 \end{bmatrix}$$

13.

Using MATLAB,

```
clear, clc
A = [2, 3, 1;
     -1, 2, 1;
     4, -1, -1];
% MATLAB function to evaluate
% how close to zero det is
c = cond(A);
if c < 1000
    c
    InvA = rats(inv(A))
else
    c
    determinant =0
end
```

```
c = 1.4588e+16
determinant = 0
```

$\therefore$ Determinant $= 0 \Rightarrow$ singular

14.

```
clear, clc
A = [1, 0, 0, -1;
     0, -1, 1, 0;
     -1, 0, 1, 0;
     0, 1, -1, 1];
% MATLAB function to evaluate
% how close to zero det is
c = cond(A);
if c < 1000
    c
    InvA = rats(inv(A))
else
    c
    determinant =0
end
```

```
c = 7.1720
InvA = 4×56 char array
    '     1     1     0     1     '
    '     1     0     1     1     '
    '     1     1     1     1     '
    '     0     1     0     1     '
```

$$\therefore A^{-1} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

15.

Consider CAB.

$$C = CI = C(AB) = (CA)B = IB = B$$

16.

Using MATLAB,

```
clear, clc
syms t
A = [exp(t), 2*exp(-t), exp(2*t);
     2*exp(t), exp(-t), -exp(2*t);
     -exp(t), 3*exp(-t), 2*exp(2*t)];
B = [2*exp(t), exp(-t), 3*exp(2*t);
     -exp(t), 2*exp(-t), exp(2*t);
     3*exp(t), -exp(-t), -exp(2*t)];
A + 3*B
A*B
```

ans =

$$\begin{pmatrix} 7\,e^{t} & 5\,e^{-t} & 10\,e^{2t} \\ -e^{t} & 7\,e^{-t} & 2\,e^{2t} \\ 8\,e^{t} & 0 & -e^{2t} \end{pmatrix}$$

ans =

$$\begin{pmatrix} \sigma_3 + \sigma_1 - 2 & 4\,e^{-2t} - e^{t} + 1 & \sigma_1 - e^{4t} + 2\,e^{t} \\ 4\,e^{2t} - \sigma_1 - 1 & 2\,e^{-2t} + e^{t} + 2 & \sigma_2 + e^{4t} + e^{t} \\ \sigma_2 - \sigma_3 - 3 & 6\,e^{-2t} - 2\,e^{t} - 1 & 3\,e^{t} - 2\,e^{4t} - \sigma_1 \end{pmatrix}$$

where

$\sigma_1 = 3\,e^{3t}$

$\sigma_2 = 6\,e^{3t}$

$\sigma_3 = 2\,e^{2t}$

(a) $$A + 3B = \begin{bmatrix} 7e^{t} & 5e^{-t} & 10e^{2t} \\ -e^{t} & 7e^{-t} & 2e^{2t} \\ 8e^{t} & 0 & -e^{2t} \end{bmatrix}$$

(b) $$AB = \begin{bmatrix} -2 + 2e^{2t} + 3e^{3t} & 1 + 4e^{-2t} - e^{t} & 2e^{t} + 3e^{3t} - e^{4t} \\ -1 + 4e^{2t} - 3e^{3t} & 2 + 2e^{-2t} + e^{t} & e^{t} + 6e^{3t} + e^{4t} \\ -3 - 2e^{2t} + 6e^{3t} & -1 + 6e^{-2t} - 2e^{t} & 3e^{t} - 3e^{3t} - 2e^{4t} \end{bmatrix}$$

(c) Using MATLAB,

```
clear, clc
syms t
A = [exp(t), 2*exp(-t), exp(2*t);
     2*exp(t), exp(-t), -exp(2*t);
     -exp(t), 3*exp(-t), 2*exp(2*t)];
B = [2*exp(t), exp(-t), 3*exp(2*t);
     -exp(t), 2*exp(-t), exp(2*t);
     3*exp(t), -exp(-t), -exp(2*t)];
diff(A, t)
int(A, t, 0, 1)
```

ans =

$$\begin{pmatrix} e^t & -2\,e^{-t} & 2\,e^{2\,t} \\ 2\,e^t & -e^{-t} & -2\,e^{2\,t} \\ -e^t & -3\,e^{-t} & 4\,e^{2\,t} \end{pmatrix}$$

ans =

$$\begin{pmatrix} e-1 & 2-2\,e^{-1} & \frac{e^2}{2}-\frac{1}{2} \\ 2\,e-2 & 1-e^{-1} & \frac{1}{2}-\frac{e^2}{2} \\ 1-e & 3-3\,e^{-1} & e^2-1 \end{pmatrix}$$

$$\therefore \frac{d}{dt}A(t) = \begin{bmatrix} e^t & -2e^{-t} & 2e^{2t} \\ 2e^t & -e^{-t} & -2e^{2t} \\ -e^t & -3e^{-t} & 4e^{2t} \end{bmatrix}$$

(d) $$\int_0^1 A(t)\,dt = (e-1)\begin{bmatrix} 1 & 2/e & \frac{1}{2}(e+1) \\ 2 & 1/e & -\frac{1}{2}(e+1) \\ -1 & 3/e & (e+1) \end{bmatrix}$$

In each of Problems 17 and 18, verify that the given vector satisfies the given differential equation.

17.

$$x'(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t + 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} te^t$$

$$= \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^t + \begin{bmatrix} 2 \\ 2 \end{bmatrix} te^t \quad [1]$$

$$\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} X(t) = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} te^t$$

$$= \begin{bmatrix} 2 \\ 3 \end{bmatrix} e^t + \begin{bmatrix} 2 \\ 2 \end{bmatrix} te^t$$

$$\therefore \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} X(t) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t = \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^t + \begin{bmatrix} 2 \\ 2 \end{bmatrix} te^t \quad [2]$$

Since $[1] = [2]$, $X'(t) = A\,X(t) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t$

where $A = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$

18.

$$X' = (-1)\begin{pmatrix} 6 \\ -8 \\ -4 \end{pmatrix} e^{-t} + 2(2)\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{2t} = \begin{bmatrix} -6e^{-t} \\ 8e^{-t} + 4e^{2t} \\ 4e^{-t} - 4e^{2t} \end{bmatrix} \quad [1]$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 6e^{-t} \\ -8e^{-t} + 2e^{2t} \\ -4e^{-t} - 2e^{2t} \end{bmatrix} = \begin{bmatrix} 6e^{-t} + (-8e^{-t} + 2e^{2t}) + (-4e^{-t} - 2e^{2t}) \\ 12e^{-t} + (-8e^{-t} + 2e^{2t}) - (-4e^{-t} - 2e^{2t}) \\ -(-8e^{-t} + 2e^{2t}) + (-4e^{-t} - 2e^{2t}) \end{bmatrix}$$

$$= \begin{bmatrix} -6e^{-t} \\ 8e^{-t} + 4e^{2t} \\ 4e^{-t} - 4e^{2t} \end{bmatrix} \quad [2]$$

$[1] = [2]$, $\therefore \; x' = Ax$, $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$

19.

$$\psi'(t) = \begin{bmatrix} -3e^{-3t} & 2e^{2t} \\ 12e^{-3t} & 2e^{2t} \end{bmatrix} \quad [1]$$

$$\begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} e^{-3t} & e^{2t} \\ -4e^{-3t} & e^{2t} \end{bmatrix} = \begin{bmatrix} e^{-3t} + (-4e^{-3t}) & e^{2t} + e^{2t} \\ 4e^{-3t} + 8e^{-3t} & 4e^{2t} - 2e^{2t} \end{bmatrix}$$

$$= \begin{bmatrix} -3e^{-3t} & 2e^{2t} \\ 12e^{-3t} & 2e^{2t} \end{bmatrix} \quad [2]$$

$[1] = [2]$, $\therefore \; \psi'(t) = A\psi(t)$, $A = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}$

20.

Using MATLAB,

```
clear, clc
syms t
A = [1, -1, 4;
     3, 2, -1;
     2, 1, -1];
g(t) = [exp(t), exp(-2*t), exp(3*t);
        -4*exp(t), -exp(-2*t), 2*exp(3*t);
        -exp(t), -exp(-2*t), exp(3*t)];
R = A*g(t)
L = diff(g(t), t)
tf = isequal(L, R)  % 1 if true, 0 if false
```

R =

$$\begin{pmatrix} e^t & -2\,e^{-2t} & 3\,e^{3t} \\ -4\,e^t & 2\,e^{-2t} & 6\,e^{3t} \\ -e^t & 2\,e^{-2t} & 3\,e^{3t} \end{pmatrix}$$

L =

$$\begin{pmatrix} e^t & -2\,e^{-2t} & 3\,e^{3t} \\ -4\,e^t & 2\,e^{-2t} & 6\,e^{3t} \\ -e^t & 2\,e^{-2t} & 3\,e^{3t} \end{pmatrix}$$

tf = *logical*

1

Since $L = R$, $\Psi'(t) = A\Psi(t)$, $A = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix}$

Note Title 11/4/2019

7.3 Systems of Linear Algebraic Equations; Linear Independence, Eigenvalues, Eigenvectors

1.

$$\begin{bmatrix} 1 & 0 & -1 \\ 3 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

Using MATLAB,

```
clear, clc
A = [1, 0, -1;
     3, 1, 1;
     -1, 1, 2];
B = [0, 1, 2]';
C = [A, B]
rats(rref(C))
```

```
C = 3×4
     1     0    -1     0
     3     1     1     1
    -1     1     2     2

ans = 3×56 char array
   '       1             0             0           -1/3     '
   '       0             1             0            7/3     '
   '       0             0             1           -1/3     '
```

$$\therefore \boxed{x_1 = -\frac{1}{3}, \quad x_2 = \frac{7}{3}, \quad x_3 = -\frac{1}{3}}$$

2.

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Using MATLAB,

```
clear, clc
A = [1, 2, -1;
     2, 1, 1;
     1, -1, 2];
B = [1, 1, 1]';
C = [A, B]
rats(rref(C))
```

```
C = 3×4
     1     2    -1     1
     2     1     1     1
     1    -1     2     1

ans = 3×56 char array
  '      1            0            1            0      '
  '      0            1           -1            0      '
  '      0            0            0            1      '
```

Since last row of row-reduced echelon form of A is 0, and last element of B is 1, there is <u>no solution</u> to $(0,0,0)\cdot(x_1, x_2, x_3) = 1$

3.

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

Using MATLAB

```
clear, clc
A = [1, 2, -1;
     2, 1, 1;
     1, -1, 2];
B = [2, 1, -1]';
C = [A, B]
rats(rref(C))
```

```
C = 3×4
     1     2    -1     2
     2     1     1     1
     1    -1     2    -1

ans = 3×56 char array
  '      1            0            1            0      '
  '      0            1           -1            1      '
  '      0            0            0            0      '
```

There are two pivot columns, and one "free" column. Assigning $x_3 = c$, a constant, and solving from second row of A:

$(0, 1, -1) \cdot (x_1, x_2, c) = 1$ yields $x_2 = 1 + c$

Then solving from 1st row of A:

$(1, 0, 1) \cdot (x_1, 1+c, c) = 0$ yields $x_1 = -c$

$\therefore \; \underline{[x_1, x_2, x_3] = [-c, 1+c, c]}$, c any constant

4.

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Using MATLAB,

```
clear, clc
A = [1, 2, -1;
     2, 1, 1;
     1, -1, 2];
B = [0, 0, 0]';
C = [A, B]
rats(rref(C))
```

```
C = 3×4

     1     2    -1     0
     2     1     1     0
     1    -1     2     0

ans = 3×56 char array
    '       1              0              1              0     '
    '       0              1             -1              0     '
    '       0              0              0              0     '
```

Same as #3, but just the homogeneous solution.

Assigning $x_3 = c$, any constant, and solving:

$(0, 1, -1) \cdot (x_1, x_2, c) = 0, \quad x_2 = c$

Then solving $(1, 0, 1) \cdot (x_1, c, c) = 0, \quad x_1 = -c$

$\therefore \quad \underline{[x_1, x_2, x_3] = [-c, c, c]}$, c any constant

5.

$$\begin{bmatrix} 1 & 0 & -1 \\ 3 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Using MATLAB,

```
clear, clc
A = [1, 0, -1;
     3, 1, 1;
     -1, 1, 2];
B = [0, 0, 0]';
C = [A, B]
rats(rref(C))
```

```
C = 3×4

     1     0    -1     0
     3     1     1     0
    -1     1     2     0

ans = 3×56 char array
    '      1              0              0              0     '
    '      0              1              0              0     '
    '      0              0              1              0     '
```

This is the homogeneous version of #1. Since A is invertible, the only solution is $\underline{[x_1, x_2, x_3] = [0, 0, 0]}$

6.

Look at $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Using MATLAB,

```
clear, clc
A = [1, 0, 1;
     1, 1, 0;
     0, 1, 1];
B = [0, 0, 0]';
C = [A, B]
rats(rref(C))
```

```
C = 3×4
     1     0     1     0
     1     1     0     0
     0     1     1     0

ans = 3×56 char array
    '      1             0             0             0      '
    '      0             1             0             0      '
    '      0             0             1             0      '
```

$\therefore$ A is invertible, so $c_1 = c_2 = c_3 = 0$.

$\therefore$ Vectors are <u>linearly independent</u>

7.

$$\begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Last row is all zeros $\Rightarrow$ matrix singular.

$\therefore$ Vectors are linearly dependent

Using MATLAB,

```
clear, clc
A = [2, 0, -1;
     1, 1, 2;
     0, 0, 0];
B = [0, 0, 0]';
C = [A, B]
rats(rref(C))
```

```
C = 3×4
     2     0    -1     0
     1     1     2     0
     0     0     0     0

ans = 3×56 char array
   '       1              0             -1/2              0      '
   '       0              1              5/2              0      '
   '       0              0              0                0      '
```

$\therefore$ Setting $c_3 = 1$, $(0, 1, 5/2) \cdot (c_1, c_2, 1) = 0 \Rightarrow c_2 = -\frac{5}{2}$

$\therefore (1, 0, -\frac{1}{2}) \cdot (c_1, -\frac{5}{2}, 1) = 0 \Rightarrow c_1 = \frac{1}{2}$

$\therefore [c_1, c_2, c_3] = [\frac{1}{2}, -\frac{5}{2}, 1]$, or $[1, -5, 2]$

$$\therefore \quad \underline{x^{(1)} - 5x^{(2)} + 2x^{(3)} = 0}$$

8.

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 3 & 0 & -1 \\ -1 & 1 & 2 & 1 \\ 0 & -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Using MATLAB,

```
clear, clc
A = [1, 2, -1, 3;
     2, 3, 0, -1;
     -1, 1, 2, 1;
     0, -1, 2, 3];
B = [0, 0, 0, 0]';
C = [A, B]
rats(rref(C))
```

```
C = 4×5
     1     2    -1     3     0
     2     3     0    -1     0
    -1     1     2     1     0
     0    -1     2     3     0

ans = 4×70 char array
    '       1              0              0              0              0       '
    '       0              1              0              0              0       '
    '       0              0              1              0              0       '
    '       0              0              0              1              0       '
```

$\therefore$ A is invertible, $c_1 = c_2 = c_3 = c_4 = 0$

$\therefore$ Vectors are <u>linearly independent</u>

9.

Any 4 vectors in $\mathbb{R}^3$ must be <u>linearly dependent</u>

$$\begin{bmatrix} 1 & 3 & 2 & 4 \\ 2 & 1 & -1 & 3 \\ -2 & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Using MATLAB,

```
clear, clc
A = [1, 3, 2, 4;
     2, 1, -1, 3;
     -2, 0, 1, -2];
B = [0, 0, 0]';
C = [A, B]
rref(C)
```

```
C = 3×5
     1     3     2     4     0
     2     1    -1     3     0
    -2     0     1    -2     0

ans = 3×5
     1     0     0     1     0
     0     1     0     1     0
     0     0     1     0     0
```

$\therefore$ set $c_4 = 1$. $\therefore$ $c_3 = 0$, $c_2 = -1$, $c_1 = -1$

$\therefore$ $[c_1, c_2, c_3, c_4] = [-1, -1, 0, 1]$ or any multiple of that vector.

$\therefore -x^{(1)} - x^{(2)} + x^{(4)} = 0$

10.

Let $A = [x^{(1)}\ x^{(2)} \dots x^{(m)}]$, an $n \times m$ matrix

$\therefore A^T = \begin{bmatrix} x^{(1)T} \\ \vdots \\ x^{(m)T} \end{bmatrix}$, an $m \times n$ matrix, $n < m$

Let $C = [A^T\ B]$, where $B = m \times (m-n)$, a matrix of all zero entries.

$\therefore C$ is an $m \times m$ matrix with $m-n$ columns of zero entries.

$\therefore C^T = \begin{bmatrix} x^{(1)} & x^{(2)} & \dots & x^{(m)} \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & 0 \end{bmatrix}$, an $m \times m$ matrix with $m-n$ rows of zeros.

$\therefore \det(C^T) = 0$ because at least one row is all zeros. $\therefore$ The columns of C^T are linearly dependent, so that there is a non-zero vector D, of size $m \times 1$, s.t.

$$\begin{bmatrix} x^{(1)} & x^{(2)} & \cdots & x^{(m)} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & 0 \end{bmatrix} \begin{bmatrix} d_1 \\ \vdots \\ d_m \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}, \text{ an } m \times 1 \text{ vector}$$

$$\therefore \begin{bmatrix} x^{(1)} & x^{(2)} & \cdots & x^{(m)} \end{bmatrix} \begin{bmatrix} d_1 \\ \vdots \\ d_m \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}, \text{ an } n \times 1 \text{ vector}$$

$\therefore x^{(1)}, x^{(2)}, \ldots x^{(m)}$ are linearly dependent.

11.

Let $A = \begin{bmatrix} e^{-t} & e^{-t} & 3e^{-t} \\ 2e^{-t} & e^{-t} & 0 \end{bmatrix} = e^{-t} \begin{bmatrix} 1 & 1 & 3 \\ 2 & 1 & 0 \end{bmatrix}$

As in #10, these vectors are linearly dependent

Using MATLAB, solve $Ax = B$, $x^T = [c_1 \; c_2 \; c_3]$

```
clear, clc
A = [1, 1, 3;
     2, 1, 0];
B = [0, 0]';
C = [A, B]
rref(C)
```

```
C = 2×4
     1     1     3     0
     2     1     0     0

ans = 2×4
     1     0    -3     0
     0     1     6     0
```

$\therefore$ Let $c_3 = 1 \Rightarrow c_2 = -6$, $\therefore c_1 = 3$

$\therefore X^T = [3 \ -6 \ 1]$

$$\therefore \quad \underline{3x^{(1)}(t) - 6x^{(2)}(t) + x^{(3)}(t) = 0}$$

12.

Let $A = \begin{bmatrix} 2\sin(t) & \sin(t) \\ \sin(t) & 2\sin(t) \end{bmatrix} = \sin(t)\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

Since $\det(A) \neq 0$, then $x^{(1)}(t)$, $x^{(2)}(t)$ are <u>linearly independent</u>

13.

For any fixed t, $\det\begin{bmatrix} e^t & 1 \\ te^t & t \end{bmatrix} = te^t - te^t = 0$

$\therefore$ At each point, the vectors are linearly dependent.

However, suppose there are constants, x, y, such that

$$x\begin{bmatrix} e^t \\ te^t \end{bmatrix} + y\begin{bmatrix} 1 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Then $xe^t + y = 0$

$\therefore$ For $t=0$, $x+y=0 \Rightarrow x=-y$

For $t=1$, $xe+y=0 \Rightarrow xe=-y$

$\therefore$ $x = xe \Rightarrow x(e-1)=0 \Rightarrow x=0$. $\therefore$ $y=0$.

$\therefore$ The only values of x, y, for all values of t, are $x=0, y=0$. $\therefore$ $x^{(1)}(t)$ and $x^{(2)}(t)$ are linearly independent on $0 \le t \le 1$

14.

$$\begin{vmatrix} 5-\lambda & -1 \\ 3 & 1-\lambda \end{vmatrix} = (\lambda-5)(\lambda-1)+3 = \lambda^2-6\lambda+8 = (\lambda-4)(\lambda-2)=0$$

$\therefore$ $\lambda = 4, 2$

$\therefore$ $\lambda=4$: $\begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $x_1=1$, $x_2=1$

$\lambda=2$: $\begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $x_1=1$, $x_2=3$

$\therefore \lambda = 4, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\lambda = 2, \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

15.

$$\begin{vmatrix} 3-\lambda & -2 \\ 4 & -1-\lambda \end{vmatrix} = (\lambda-3)(\lambda+1)+8 = \lambda^2 - 2\lambda + 5$$

Using MATLAB,

```
clear, clc
syms x
eqn = x^2 - 2*x + 5 == 0;
eigv = solve(eqn, x)
A = [3, -2;
     4, -1];
B = [0, 0]';
for i = 1:2
    C = [A - eigv(i)*eye(2), B];
    rref(C)
end
```

eigv =

$$\begin{pmatrix} 1-2\,\mathrm{i} \\ 1+2\,\mathrm{i} \end{pmatrix}$$

ans =

$$\begin{pmatrix} 1 & -\frac{1}{2}+\frac{1}{2}\,\mathrm{i} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

ans =

$$\begin{pmatrix} 1 & -\frac{1}{2}-\frac{1}{2}\,\mathrm{i} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\therefore \lambda = 1-2i$: $\begin{bmatrix} 1, & -\frac{1}{2}+\frac{1}{2}i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $x_2 = 1$, $x_1 = \frac{1}{2} - \frac{1}{2}i$

$$\therefore \begin{bmatrix} \frac{1}{2}-\frac{1}{2}i \\ 1 \end{bmatrix} \overset{\times(2)}{=} \begin{bmatrix} 1-i \\ 2 \end{bmatrix} \overset{\times(1+i)}{=} \begin{bmatrix} 2 \\ 2+2i \end{bmatrix} \overset{\div(2)}{=} \begin{bmatrix} 1 \\ 1+i \end{bmatrix}$$

answer in back of book

$\lambda = 1+2i$: $\begin{bmatrix} 1, & -\frac{1}{2}-\frac{1}{2}i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $x_2 = 1$, $x_1 = \frac{1}{2} + \frac{1}{2}i$

$$\therefore \begin{bmatrix} \frac{1}{2}+\frac{1}{2}i \\ 1 \end{bmatrix} \overset{\times(2)}{=} \begin{bmatrix} 1+i \\ 2 \end{bmatrix} \overset{\times(1-i)}{=} \begin{bmatrix} 2 \\ 2-2i \end{bmatrix} \overset{\div(2)}{=} \begin{bmatrix} 1 \\ 1-i \end{bmatrix}$$

answer in book

$\therefore\ \lambda = 1-2i,\ \begin{bmatrix} \frac{1}{2}-\frac{1}{2}i \\ 1 \end{bmatrix}$ and $\lambda = 1+2i,\ \begin{bmatrix} \frac{1}{2}+\frac{1}{2}i \\ 1 \end{bmatrix}$

16.

$$\begin{vmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{vmatrix} = (\lambda+2)(\lambda+2)-1 = \lambda^2+4\lambda+3 = (\lambda+3)(\lambda+1) = 0$$

$$\therefore\ \lambda = -1, -3$$

$\lambda = -1:\ \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = -3:\ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ x = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \equiv \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\therefore\ \lambda = -1,\ \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\lambda = -3,\ \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

17.

$$\begin{vmatrix} 1-\lambda & \sqrt{3} \\ \sqrt{3} & -1-\lambda \end{vmatrix} = (\lambda-1)(\lambda+1)-3 = \lambda^2-4 = 0,\ \lambda = -2, 2$$

$\lambda = -2:\ \begin{bmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ x = \begin{bmatrix} -1 \\ \sqrt{3} \end{bmatrix} \overset{\times(-1)}{\equiv} \begin{bmatrix} 1 \\ -\sqrt{3} \end{bmatrix}$

$\lambda = 2:\ \begin{bmatrix} -1 & \sqrt{3} \\ \sqrt{3} & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ x = \begin{bmatrix} 3 \\ \sqrt{3} \end{bmatrix} \overset{\div(\sqrt{3})}{\equiv} \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}$

$\therefore \lambda = -2, \begin{bmatrix} -1 \\ \sqrt{3} \end{bmatrix}$ and $\lambda = 2, \begin{bmatrix} 3 \\ \sqrt{3} \end{bmatrix}$

18.

Using MATLAB,

```
clear, clc
syms x
A = [1, 0, 0;
     2, 1, -2;
     3, 2, 1];
p = det(A - x*eye(3))
eigv = solve(p==0,x);
B = [0, 0, 0]';
for i =1:3
    C = [A - eigv(i)*eye(3), B];
   disp(eigv(i)), disp(rref(C))
end
```

p = $-x^3 + 3x^2 - 7x + 5$

1

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$1 - 2\,\mathrm{i}$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \mathrm{i} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$1 + 2\,\mathrm{i}$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\mathrm{i} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\therefore \underline{\lambda = 1}: \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad X = \begin{bmatrix} 1 \\ -3/2 \\ 1 \end{bmatrix} \overset{\times(2)}{\equiv} \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}$$

$$\underline{\lambda = 1 - 2i}: \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & i \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad X = \begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix} \overset{\times(i)}{\equiv} \begin{bmatrix} 0 \\ 1 \\ i \end{bmatrix}$$

$$\underline{\lambda = 1 + 2i}: \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -i \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad X = \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix} \overset{\times(-i)}{\equiv} \begin{bmatrix} 0 \\ 1 \\ -i \end{bmatrix}$$

19.

Using MATLAB,

```
clear, clc
syms x
A = [3, 2, 2;
     1, 4, 1;
     -2, -4, -1];
p = det(A - x*eye(3))
eigv = solve(p==0,x);
B = [0, 0, 0]';
for i =1:3
    C = [A - eigv(i)*eye(3), B];
    disp(eigv(i)), disp(rref(C))
end
```

p = $-x^3+6\,x^2-11\,x+6$

1

$$\begin{pmatrix}1&0&1&0\\0&1&0&0\\0&0&0&0\end{pmatrix}$$

2

$$\begin{pmatrix}1&2&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix}$$

3

$$\begin{pmatrix}1&0&0&0\\0&1&1&0\\0&0&0&0\end{pmatrix}$$

$\underline{\lambda=1}$: $\begin{bmatrix}1&0&1\\0&1&0\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$, $x=\underline{\begin{bmatrix}-1\\0\\1\end{bmatrix}}\overset{\times(-1)}{\equiv}\begin{bmatrix}1\\0\\-1\end{bmatrix}$

$\underline{\lambda=2}$: $\begin{bmatrix}1&2&0\\0&0&1\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$, $x=\underline{\begin{bmatrix}-2\\1\\0\end{bmatrix}}$

$\underline{\lambda=3}$: $\begin{bmatrix}1&0&0\\0&1&1\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$, $x=\underline{\begin{bmatrix}0\\-1\\1\end{bmatrix}}\overset{\times(-1)}{\equiv}\begin{bmatrix}0\\1\\-1\end{bmatrix}$

20.

Using MATLAB,

```
clear, clc
syms x
A = [11/9, -2/9, 8/9;
     -2/9, 2/9, 10/9;
     8/9, 10/9, 5/9];
p = det(A - x*eye(3))
eigv = solve(p==0,x);
B = [0, 0, 0]';
for i =1:3
    C = [A - eigv(i)*eye(3), B];
    disp(eigv(i)), disp(rref(C))
end
```

p = $-x^3+2\,x^2+x-2$

-1

$$\begin{pmatrix} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

1

$$\begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

2

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\underline{\lambda=-1}$: $\begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $x = \underline{\begin{bmatrix} -\frac{1}{2} \\ -1 \\ 1 \end{bmatrix}} \overset{\times(-2)}{=} \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$

$\underline{\lambda=1}$: $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $x = \underline{\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}} \overset{\times(-1)}{=} \begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix}$

$\underline{\lambda=2}$: $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $x = \underline{\begin{bmatrix} 1 \\ \frac{1}{2} \\ 1 \end{bmatrix}} \overset{\times(2)}{=} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$

21.

(a) Let x, y be $n \times 1$ column vectors.

$\therefore$ Ax is a $n \times 1$ column vector.

$$(Ax, y) = \sum_{i=1}^{n} (Ax)_{i1} \, \bar{y}_{i1} \quad \text{Definition inner prod. [1]}$$

$$= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} A_{ij} \, x_{j1} \right) \bar{y}_{i1} \quad \text{Def. of matrix mult. [2]}$$

$$= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} A_{ij} \, x_{j1} \, \bar{y}_{i1} \right) \quad \text{Distributive law [3]}$$

$$= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} x_{j1} \, A_{ij} \, \bar{y}_{i1} \right) \quad \text{Commutative law [4]}$$

$$= \sum_{j=1}^{n} \left(\sum_{i=1}^{n} x_{j1} \, A_{ij} \, \bar{y}_{i1} \right) \quad \text{Associative law [5]}$$

$$= \sum_{j=1}^{n} x_{j1} \left(\sum_{i=1}^{n} A_{ij} \, \bar{y}_{i1} \right) \quad \text{Distributive law [6]}$$

$$= \sum_{j=1}^{n} x_{j1} \left(\sum_{i=1}^{n} (A^{T})_{ji} \, \bar{y}_{i1} \right) \quad \text{Def. of } A^{T} \quad [7]$$

$$= \sum_{j=1}^{n} x_{j1} \left(\sum_{i=1}^{n} \overline{(A^T)}_{ji} \overline{y}_{i1} \right)$$ A is real-valued [8]

$$= \sum_{j=1}^{n} x_{j1} \left(\sum_{i=1}^{n} \overline{(A^T)_{ji} \, y_{i1}} \right)$$ $\overline{ab} = \bar{a}\,\bar{b}$ [9]

$$= \sum_{j=1}^{n} x_{j1} \overline{\left(\sum_{i=1} (A^T)_{ji} \, y_{i1} \right)}$$ $\overline{a+b} = \bar{a} + \bar{b}$ [10]

$$= \sum_{j=1}^{n} x_{j1} \overline{(A^T y)}_{j1}$$ Def. matrix mult. [11]

$$= (x, A^T y)$$ Def. of inner product [12]

(b)

By definition, $A^*_{ij} = (\bar{A})^T_{ij} = \bar{A}_{ji}$

$\therefore \overline{A^*}_{ij} = A_{ji}$, or $A_{ij} = \overline{A^*}_{ji}$ [*]

$\therefore$ After step [6] in (a),

$$= \sum_{j=1}^{n} x_{j1} \left(\sum_{i=1}^{n} \overline{A^*}_{ji} \overline{y}_{i1} \right)$$ using [*] [7']

$$= \sum_{j=1}^{n} x_{j1} \left(\sum_{i=1}^{n} \overline{A^*_{ji} \, y_{i1}} \right)$$ $\overline{ab} = \bar{a}\,\bar{b}$ [9']

$$= \sum_{j=1}^{n} x_{j1} \overline{\left(\sum_{i=1} A^*_{ji} \, y_{i1} \right)} \qquad \overline{a+b} = \bar{a} + \bar{b} \; [10']$$

$$= \sum_{j=1}^{n} x_{j1} \overline{(A^* y)}_{j1} \qquad \text{Def. matrix mult. } [11']$$

$$= (x, A^* y) \qquad \text{Def. of inner product } [12']$$

(c)

Hermitian means $A^* = A$

$\therefore$ By (b), $(Ax, y) = (x, A^* y) = (x, Ay)$

22.

The problem assumes A is square, $n \times n$, for if A is, e.g., 2×3, with independent rows, then x could be nonzero and $Ax = 0$, but there is not y s.t. $A^T y = 0$ as the column vectors of A^T are independent. And the statement before #21 mentions dealing with $\det(A) = 0$.

Since there exists a nonzero x s.t. $Ax=0$, then let R = row reduction of A. $\therefore$ R must have at least one row of all zero entries, otherwise $Rx=0$ would imply $x=0$, which it isn't. $\therefore$ The rows of A are dependent $\Rightarrow$ there is a nonzero vector y s.t. $y^TA=0 \Rightarrow A^Ty=0$ since $(y^TA)^T = A^Ty$. Note A^Ty is $n\times 1$ and y is $n\times 1$

By definition $(A^Ty)_{i1} = \sum_{j=1}^{n} (A^T)_{ij}\, y_{j1}$, for each row i.

$$\therefore \sum_{j=1}^{n} (A^T)_{ij}\, y_{j1} = 0$$

$\vec{x}=0 \Rightarrow x_{j1}=0$

$$\overline{\sum_{j=1}^{n} (A^T)_{ij}\, y_{j1}} = \sum_{j=1}^{n} \overline{(A^T)}_{ij}\, \overline{y_{j1}}$$

$\overline{a+b} = \bar{a}+\bar{b}$

$$= \overline{0} = 0$$

$\therefore$ There exists a $\bar{y}$ s.t. $A^*\bar{y}=0$

$\overline{A^T} = \bar{A}^T$

23.

If y is s.t. $A^*y = 0$, then for any vector x,

$$(x, A^*y) = (x, 0) = 0$$

$\therefore$ Let x be s.t. $Ax = b$. Since $(Ax, y) = (x, A^*y)$

by #21(b), then $\underline{(b, y)} = (Ax, y) = (x, A^*y) \underline{= 0}$

In Example 2, p. 297 of the text,

$$A = \begin{bmatrix} 1 & -2 & 3 \\ -1 & 1 & -2 \\ 2 & -1 & 3 \end{bmatrix} \quad \therefore A^* = A^T = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 1 & -1 \\ 3 & -2 & 3 \end{bmatrix}$$

Using MATLAB,

```
clear, clc
A = [1, -2, 3;
     -1, 1, -2;
     2, -1, 3];
rref(A')
```

```
ans = 3×3
     1     0    -1
     0     1    -3
     0     0     0
```

$\therefore$ Solution to $A^*y = 0$ is $y = \begin{bmatrix} c \\ 3c \\ c \end{bmatrix}$, c any constant

But $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ is such that $b_1 + 3b_2 + b_3 = 0$ so that $Ax = b$ has a solution.

$\therefore (b_1, b_2, b_3) \cdot (1, 3, 1) = 0$, and so

$$(b_1, b_2, b_3) \cdot (c, 3c, c) = \sum_{i=1}^{3} b_i y_i = \sum_{i=1}^{3} b_i \bar{y}_i = 0$$

$\therefore \underline{(b, y) = 0}$

24.

$$A(x^{(0)} + \alpha \epsilon) = A x^{(0)} + A(\alpha \epsilon) = b + A(\alpha \epsilon)$$

$$= b + \alpha A(\epsilon) = b + \alpha \cdot 0 = b$$

25.

Assume A is $n \times n$ so that y is $n \times 1$.

$A^* y = 0 \Rightarrow \bar{A}^T y = 0 \Rightarrow \overline{\bar{A}^T y} = \bar{0} = 0 \Rightarrow \bar{\bar{A}}^T \bar{y} = 0$

$\Rightarrow A^T \bar{y} = 0 \Rightarrow \bar{y}^T A = 0$. Let $v = \bar{y}^T$ (v is $1 \times n$)

Note $(b, y) = \sum_{i=1}^{n} b_i \bar{y}_i = b^T \cdot \bar{y}$ (dot product)

$= \bar{y}^T \cdot b$ (dot prod. is commutative)

$= v \cdot b$

Note that b is $n \times 1$, and v is $1 \times n$

$\therefore (b, y) = 0 \iff v \cdot b = 0$, where $v = \bar{y}^T$

$\therefore$ Need to show if $v \cdot b = 0$ for every v such that $vA = 0$, then $Ax = b$ has a solution.

Note that vA is a linear combination of the rows of A producing a $1 \times n$ zero vector.

Let E = the product of the elimination matrices producing a row reduction of A. The restriction on whether $Ax = b$ has solutions is that for $EAx = Eb$, the zero rows of EA must correspond to zero rows of Eb; i.e., if $E_i A = 0$ and $E_i b \neq 0$ for row i, then there is no solution to $Ax=b$, because $E_i Ax = 0$ for all x, so $E_i Ax \neq E_i b$. But $E_i A = 0$ is equivalent to $vA=0$, where the elements of v are the elements of row E_i.

$\therefore$ $E_i A = 0$ and $E_i b = 0$ is equivalent to

$vA = 0$ and $v \cdot b = 0$

$\therefore$ For every y s.t. $A^* y = 0$, there is a $v = \bar{y}^T$, and if $(b, y) = 0$, then $v \cdot b = 0$, which implies that $EAx = Eb$ can be solved, which means $Ax=b$ can be solved.

26.

(1) If $\det(A) = 0$, then $Ax=0$ has a nonzero solution x. But $\lambda = 0 \Rightarrow \lambda x = 0$, so $Ax = \lambda x$, and $\therefore$ $\lambda = 0$ is an eigenvalue of A.

(2) If $\lambda = 0$ is an eigenvalue of A, then

$Ax = \lambda x = 0$ for a nonzero eigenvector x.

$Ax = 0$ for $x \neq 0 \Rightarrow \det(A) = 0$ as the columns of A are dependent.

27.

(a)

A Hermitian $\Rightarrow A^* = A$, or $\bar{A}^T = A$.

From #21(c), this means for any x and y,

$(Ax, y) = (x, Ay)$. $\therefore$ $(Ax, x) = (x, Ax)$

(b)

$$\lambda(x,x) = \lambda \sum_{i=1}^{n} x_i \bar{x}_i = \sum_{i=1}^{n} \lambda(x_i \bar{x}_i) = \sum_{i=1}^{n} (\lambda x_i)\bar{x}_i$$

$$= (\lambda x, x) = (Ax, x) = (x, Ax) = (x, \lambda x)$$

$$= \sum_{i=1}^{n} x_i \overline{(\lambda x_i)} = \sum_{i=1}^{n} x_i (\bar{\lambda} \bar{x}_i) = \sum_{i=1}^{n} \bar{\lambda} (x_i \bar{x}_i)$$

$$= \bar{\lambda} \sum_{i=1}^{n} x_i \bar{x}_i = \bar{\lambda}(x,x)$$

(c)

From (b), $\lambda(x,x) = \bar{\lambda}(x,x)$

Note that x is a nonzero vector.

$\therefore$ (x,x) is a nonzero real number, as

$(x,x) = \sum_{i=1}^{n} x_i \bar{x}_i = \sum_{i=1}^{n} |x_i|^2$, and at least

one of the $|x_i|$ is nonzero since $x \neq 0$.

$\therefore$ Divide both sides of $\lambda(x,x) = \bar{\lambda}(x,x)$ by

the nonzero real number (x,x) to get $\underline{\lambda = \bar{\lambda}}$

28.

Given $Ax^{(1)} = \lambda_1 x^{(1)}$ and $Ax^{(2)} = \lambda_2 x^{(2)}$, $\lambda_1 \neq \lambda_2$,

Then $\lambda_1 (x^{(1)}, x^{(2)}) = \lambda_1 \sum_{i=1}^{n} x_i^{(1)} \overline{x_i^{(2)}} = \sum_{i=1}^{n} \lambda_1 x_i^{(1)} \overline{x_i^{(2)}}$

$= (\lambda_1 x^{(1)}, x^{(2)}) = (Ax^{(1)}, x^{(2)})$

$= (x^{(1)}, Ax^{(2)})$ by #21(c)

$= (x^{(1)}, \lambda_2 x^{(2)})$

$$= \overline{\lambda_2} \left(x^{(1)}, x^{(2)}\right) \quad \text{by \#27(b)}$$

$$= \lambda_2 \left(x^{(1)}, x^{(2)}\right) \quad \text{by \#27(c)}$$

$$\therefore (\lambda_1 - \lambda_2)\left(x^{(1)}, x^{(2)}\right) = 0 \Rightarrow \underline{\left(x^{(1)}, x^{(2)}\right) = 0}$$

since $\lambda_1 - \lambda_2 \neq 0$

29.

Consider $c_1 x^{(1)} + c_2 x^{(2)} = 0$ [1]

$$\therefore A\left(c_1 x^{(1)} + c_2 x^{(2)}\right) = A\left(c_1 x^{(1)}\right) + A\left(c_2 x^{(2)}\right)$$

$$= c_1 A\left(x^{(1)}\right) + c_2 A\left(x^{(2)}\right)$$

$$= c_1 \lambda_1 x^{(1)} + c_2 \lambda_2 x^{(2)}$$

$$= A(0) = 0$$

$\therefore c_1 \lambda_1 x^{(1)} + c_2 \lambda_2 x^{(2)} = 0$ [2]

Multiply [1] by λ_1 to get,

$\lambda_1 c_1 x^{(1)} + \lambda_1 c_2 x^{2} = 0$ [3]

Subtract [3] from [2] to get,

$$c_2(\lambda_2-\lambda_1)\,x^{(2)}=0$$

$\Rightarrow c_2 x^{(2)}=0$ $\qquad \lambda_1 \neq \lambda_2$

$\Rightarrow \underline{\underline{c_2=0}}$ $\qquad x^{(2)} \neq 0$

Multiply [1] by λ_2 to get,

$$\lambda_2 c_1 x^{(1)} + \lambda_2 c_2 x^{(2)} = 0 \qquad [4]$$

Subtract [4] from [2] to get,

$$c_1(\lambda_1-\lambda_2)\,x^{(1)}=0$$

$\Rightarrow c_1 x^{(1)}=0$ $\qquad \lambda_1 \neq \lambda_2$

$\Rightarrow \underline{\underline{c_1=0}}$ $\qquad x^{(1)} \neq 0$

$\therefore c_1 x^{(1)} + c_2 x^{(2)} = 0 \Rightarrow c_1=0,\ c_2=0$

$\therefore x^{(1)}, x^{(2)}$ are <u>linearly independent</u>

7.4 Basic Theory of Systems of First-Order Linear Equations

Note Title 11/13/2019

1.

(a)

$$\frac{d}{dt}x^{(1)} = \begin{bmatrix} e^t \\ e^t \end{bmatrix} \qquad \frac{d}{dt}x^{(2)} = \begin{bmatrix} -e^{-t} \\ -3e^{-t} \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} x^{(1)} = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} e^t \\ e^t \end{bmatrix} = \begin{bmatrix} e^t \\ e^t \end{bmatrix} = \frac{d}{dt}x^{(1)}$$

$$\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} x^{(2)} = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} e^{-t} \\ 3e^{-t} \end{bmatrix} = \begin{bmatrix} -e^{-t} \\ -3e^{-t} \end{bmatrix} = \frac{d}{dt}x^{(2)}$$

$$\therefore \; x^{(1)\prime} = Ax^{(1)}, \quad x^{(2)\prime} = Ax^{(2)}$$

(b)

$$c_1x^{(1)} + c_2x^{(2)} = \begin{bmatrix} c_1e^t \\ c_1e^t \end{bmatrix} + \begin{bmatrix} c_2e^{-t} \\ 3c_2e^{-t} \end{bmatrix} = \begin{bmatrix} c_1e^t + c_2e^{-t} \\ c_1e^t + 3c_2e^{-t} \end{bmatrix}$$

$$\frac{d}{dt}\left(c_1x^{(1)} + c_2x^{(2)}\right) = \begin{bmatrix} c_1e^t - c_2e^{-t} \\ c_1e^t - 3c_2e^{-t} \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} x = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} c_1e^t + c_2e^{-t} \\ c_1e^t + 3c_2e^{-t} \end{bmatrix}$$

$$= \begin{bmatrix} 2c_1e^t + 2c_2e^{-t} - (c_1e^t + 3c_2e^{-t}) \\ 3c_1e^t + 3c_2e^{-t} - (2c_1e^t + 6c_2e^{-t}) \end{bmatrix}$$

$$= \begin{bmatrix} c_1e^t - c_2e^{-t} \\ c_1e^t - 3c_2e^{-t} \end{bmatrix} = \frac{d}{dt}\left(c_1x^{(1)} + c_2x^{(2)}\right)$$

$\therefore \; x' = Ax$

(c)

Show $x^{(1)}$ and $x^{(2)}$ are independent, so

$c_1x^{(1)} + c_2x^{(2)} = 0 \Rightarrow c_1 = c_2 = 0$

From (b), $c_1x^{(1)} + c_2x^{(2)} = \begin{bmatrix} c_1e^t + c_2e^{-t} \\ c_1e^t + 3c_2e^{-t} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For $A = \begin{bmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{bmatrix}$, $\det(A) = 2 \neq 0$

$$\therefore \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = A^{-1}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow c_1 = 0, \; c_2 = 0$$

$\therefore$ $x^{(1)}$ and $x^{(2)}$ are independent, and so $x^{(1)}$, $x^{(2)}$ form a fundamental set of solutions.

(d)

$$x(0) = c_1 x^{(1)}(0) + c_2 x^{(2)}(0)$$

$$= c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\therefore \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}, \quad c_1 = \frac{1}{2}, \; c_2 = \frac{1}{2}$$

$$\therefore \; x = \frac{1}{2} x^{(1)} + \frac{1}{2} x^{(2)} = \underline{\begin{bmatrix} \frac{1}{2} e^{t} + \frac{1}{2} e^{-t} \\ \frac{1}{2} e^{t} + \frac{3}{2} e^{-t} \end{bmatrix}}$$

(e)

$$W[x^{(1)}, x^{(2)}](t) = \det \begin{bmatrix} e^{t} & e^{-t} \\ e^{t} & 3e^{-t} \end{bmatrix} = 3 - 1 = \underline{2}$$

(f)

Here, $P(t) = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$, so that $p_{11}(t) = 2$, $p_{22}(t) = -2$

$$\therefore \; p_{11}(t) + p_{22}(t) = 2 + (-2) = 0$$

From (e), $W(t) = 2$, $\therefore$ $W'(t) = 0$

And $[p_{11}(t) + p_{22}(t)] W = (0)(2) = 0$.

$$\therefore \underline{W' = [p_{11}(t) + p_{22}(t)]W}$$

2.

Let $P = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}$

(a)

$$P x^{(1)} = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-3t} = \begin{bmatrix} -3 \\ 12 \end{bmatrix} e^{-3t}$$

$$P x^{(2)} = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} e^{2t}$$

$$\frac{d}{dt} x^{(1)} = \frac{d}{dt} \begin{bmatrix} e^{-3t} \\ -4e^{-3t} \end{bmatrix} = \begin{bmatrix} -3e^{-3t} \\ 12e^{-3t} \end{bmatrix} = \begin{bmatrix} -3 \\ 12 \end{bmatrix} e^{-3t}$$

$$\frac{d}{dt} x^{(2)} = \frac{d}{dt} \begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} e^{2t}$$

$$\therefore \underline{x^{(1)\prime} = P x^{(1)}}, \quad \underline{x^{(2)\prime} = P x^{(2)}}$$

(b)

$$c_1 x^{(1)} + c_2 x^{(2)} = \begin{bmatrix} c_1 e^{-3t} \\ -4c_1 e^{-3t} \end{bmatrix} + \begin{bmatrix} c_2 e^{2t} \\ c_2 e^{2t} \end{bmatrix} = \begin{bmatrix} c_1 e^{-3t} + c_2 e^{2t} \\ -4c_1 e^{-3t} + c_2 e^{2t} \end{bmatrix}$$

$$\therefore\ Px = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} c_1 e^{-3t} + c_2 e^{2t} \\ -4c_1 e^{-3t} + c_2 e^{2t} \end{bmatrix} = \begin{bmatrix} -3c_1 e^{-3t} + 2c_2 e^{2t} \\ 12 c_1 e^{-3t} + 2c_2 e^{2t} \end{bmatrix}$$

$$\frac{d}{dt}\left(c_1 x^{(1)} + c_2 x^{(2)}\right) = \frac{d}{dt} \begin{bmatrix} c_1 e^{-3t} + c_2 e^{2t} \\ -4c_1 e^{-3t} + c_2 e^{2t} \end{bmatrix} = \begin{bmatrix} -3c_1 e^{-3t} + 2c_2 e^{2t} \\ 12 c_1 e^{3t} + 2c_2 e^{2t} \end{bmatrix}$$

$$\therefore\ \underline{x' = Px}, \text{ for } x = c_1 x^{(1)} + c_2 x^{(2)}$$

(c)

$$\det\left(\left[x^{(1)}\ x^{(2)}\right]\right) = \det \begin{bmatrix} e^{-3t} & e^{2t} \\ -4e^{-3t} & e^{2t} \end{bmatrix} = e^{-t} - (-4e^{-t})$$

$$= 5e^{-t} \neq 0 \text{ for all } t.$$

$\therefore$ $x^{(1)}$ and $x^{(2)}$ are 2 linearly independent solutions to $x' = Px$, where P is 2x2.

$\therefore$ $x^{(1)}$ and $x^{(2)}$ form a fundamental set of solutions to $x' = Px$

(d)

$$x(0) = c_1 x^{(1)}(0) + c_2 x^{(2)}(0)$$

$$= \begin{bmatrix} c_1 & c_2 \\ -4c_1 & c_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\det\begin{bmatrix}1 & 1\\ -4 & 1\end{bmatrix} = 5 \quad \therefore \begin{bmatrix}c_1\\ c_2\end{bmatrix} = \frac{1}{5}\begin{bmatrix}1 & -1\\ 4 & 1\end{bmatrix}\begin{bmatrix}1\\ 2\end{bmatrix} = \begin{bmatrix}-1/5\\ 6/5\end{bmatrix}$$

$$\therefore \underline{x(t) = -\frac{1}{5}x^{(1)}(t) + \frac{6}{5}x^{(2)}(t)} = \begin{bmatrix}-\frac{1}{5}e^{-3t} + \frac{6}{5}e^{2t}\\ \frac{4}{5}e^{-3t} + \frac{6}{5}e^{2t}\end{bmatrix}$$

(e)

$$W[x^{(1)}, x^{(2)}](t) = \det\begin{bmatrix}e^{-3t} & e^{2t}\\ -4e^{-3t} & e^{2t}\end{bmatrix} = e^{-t} - (-4e^{-t})$$

$$= \underline{5e^{-t}}$$

(f)

From $P(t) = \begin{bmatrix}1 & 1\\ 4 & -2\end{bmatrix}$, $p_{11}(t) + p_{22}(t) = 1 + (-2) = -1$

$$\therefore (p_{11}(t) + p_{22}(t))W = (-1)(5e^{-t}) = -5e^{-t}$$

From (e), $W(t) = 5e^{-t}$. $\therefore W'(t) = -5e^{-t}$

$$\therefore \underline{W' = [p_{11}(t) + p_{22}(t)]W}$$

3.

Let $P(t) = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix}$

(a)

Using MATLAB.

```
clear, clc
syms t c1 c2
P = [2, -5;
     1, -2];
x1 = [5*cos(t);
      2*cos(t) + sin(t)];
x2 = [5*sin(t);
      2*sin(t) - cos(t)];
Px1 = P*x1
dx1 = diff(x1,t,1)
Px2 = P*x2
dx2 = diff(x2,t,1)
```

Px1 =

$$\begin{pmatrix} -5\sin(t) \\ \cos(t) - 2\sin(t) \end{pmatrix}$$

dx1 =

$$\begin{pmatrix} -5\sin(t) \\ \cos(t) - 2\sin(t) \end{pmatrix}$$

Px2 =

$$\begin{pmatrix} 5\cos(t) \\ 2\cos(t) + \sin(t) \end{pmatrix}$$

dx2 =

$$\begin{pmatrix} 5\cos(t) \\ 2\cos(t) + \sin(t) \end{pmatrix}$$

$$\therefore \underline{x^{(1)}(t)'} = Px^{(1)}(t), \quad \underline{x^{(2)}(t)' = Px^{(2)}(t)}$$

(b)

Using MATLAB, (code on next page, a continuation of above code),

$$\therefore \underline{x'(t) = Px(t)}, \quad \text{for } x(t) = c_1 x^{(1)}(t) + c_2 x^{(2)}(t)$$

```
clear, clc
syms t c1 c2
P = [2, -5;
     1, -2];
x1 = [5*cos(t);
      2*cos(t) + sin(t)];
x2 = [5*sin(t);
      2*sin(t) - cos(t)];
Px1 = P*x1
dx1 = diff(x1,t,1)
Px2 = P*x2
dx2 = diff(x2,t,1)
x = c1*x1 + c2*x2
Px = collect(P*x, [c1,c2])
dx = collect(diff(x,t,1), [c1,c2])
```

x =

$$\begin{pmatrix} 5\,c_1\cos(t)+5\,c_2\sin(t) \\ c_1\,(2\cos(t)+\sin(t))-c_2\,(\cos(t)-2\sin(t)) \end{pmatrix}$$

Px =

$$\begin{pmatrix} (-5\sin(t))\,c_1+(5\cos(t))\,c_2 \\ (\cos(t)-2\sin(t))\,c_1+(2\cos(t)+\sin(t))\,c_2 \end{pmatrix}$$

dx =

$$\begin{pmatrix} (-5\sin(t))\,c_1+(5\cos(t))\,c_2 \\ (\cos(t)-2\sin(t))\,c_1+(2\cos(t)+\sin(t))\,c_2 \end{pmatrix}$$

(c)

Using MATLAB, (continuation of above).

```
clear, clc
syms t c1 c2
P = [2, -5;
     1, -2];
x1 = [5*cos(t);
      2*cos(t) + sin(t)];
x2 = [5*sin(t);
      2*sin(t) - cos(t)];
% (a)
Px1 = P*x1
dx1 = diff(x1,t,1)
Px2 = P*x2
dx2 = diff(x2,t,1)
% (b)
x = c1*x1 + c2*x2
Px = collect(P*x, [c1,c2])
dx = collect(diff(x,t,1), [c1,c2])
% (c)
x1x2 = [x1, x2]
W = det(x1x2)
```

x1x2 =

$$\begin{pmatrix} 5\cos(t) & 5\sin(t) \\ 2\cos(t)+\sin(t) & 2\sin(t)-\cos(t) \end{pmatrix}$$

W = $-5\cos(t)^2-5\sin(t)^2$

$\therefore$ $W = -5$, so $x^{(1)}(t)$ and $x^{(2)}(t)$ are linearly independent, and so form a fundamental set of solutions to the 2x2 system.

(d)

Using MATLAB, (continuation of above),

```
clear, clc
syms t c1 c2
P = [2, -5;
     1, -2];
x1 = [5*cos(t);
      2*cos(t) + sin(t)];
x2 = [5*sin(t);
      2*sin(t) - cos(t)];
% (a)
Px1 = P*x1
dx1 = diff(x1,t,1)
Px2 = P*x2
dx2 = diff(x2,t,1)
% (b)
x = c1*x1 + c2*x2
Px = collect(P*x, [c1,c2])
dx = collect(diff(x,t,1), [c1,c2])
% (c)
x1x2 = [x1, x2]
W = det(x1x2)
% (d)
x0 = [subs(x1,t,0), subs(x2,t,0)]
b = [1,2]'
c = linsolve(x0,b)   % solve x0*c = b
x1x2*c
```

x0 =

$$\begin{pmatrix} 5 & 0 \\ 2 & -1 \end{pmatrix}$$

b = 2×1

1
2

c =

$$\begin{pmatrix} \frac{1}{5} \\ -\frac{8}{5} \end{pmatrix}$$

$$C = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

ans =

$$\begin{pmatrix} \cos(t) - 8\sin(t) \\ 2\cos(t) - 3\sin(t) \end{pmatrix}$$

$$\therefore \boxed{X(t) = \frac{1}{5} X^{(1)}(t) - \frac{8}{5} X^{(2)}(t) = \begin{bmatrix} \cos(t) - 8\sin(t) \\ 2\cos(t) - 3\sin(t) \end{bmatrix}}$$

(e)

From (c), $\underline{W = -5}$

(f)

From $P(t) = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix}$, $p_{11}(t) + p_{22}(t) = 2 + (-2) = 0$

From (e), $W(t) = -5$, so $W'(t) = 0$

$$\therefore \left[p_{11}(t) + p_{22}(t)\right] W = (0)(-5) = 0$$

$$\therefore \underline{W' = \left[p_{11}(t) + p_{22}(t)\right] W}$$

4.

Let $P = \begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix}$ Misprint: should be $x^{(2)} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} t - \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$x^{(2)}(t)$ as stated not a solution

Using MATLAB,

```
clear, clc
syms t c1 c2
P = [4, -2;
     8, -4];
x1 = [2,4]';
x2 = [2,4]'*t - [0,1]';
% (a)
Px1 = P*x1
dx1 = diff(x1,t,1)
Px2 = P*x2
dx2 = diff(x2,t,1)
% (b)
x = c1*x1 + c2*x2
Px = collect(P*x, [c1,c2])
dx = collect(diff(x,t,1), [c1,c2])
% (c)
x1x2 = [x1, x2]
W = det(x1x2)
% (d)
x0 = [subs(x1,t,0), subs(x2,t,0)]
b = [1,2]'
c = linsolve(x0,b)   % solve x0*c = b
x1x2*c
% (f)
dW = diff(W, t,1)
P11P22 = P(1,1) + P(2,2)
P11P22xW = P11P22*W
```

Px1 = 2×1

```
0
0
```

dx1 = $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Px2 = $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$

dx2 = $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$

x = $\begin{pmatrix} 2\,c_1 + 2\,c_2\,t \\ 4\,c_1 + c_2\,(4\,t - 1) \end{pmatrix}$

Px = $\begin{pmatrix} 2\,c_2 \\ 4\,c_2 \end{pmatrix}$

dx = $\begin{pmatrix} 2\,c_2 \\ 4\,c_2 \end{pmatrix}$

x1x2 = $\begin{pmatrix} 2 & 2\,t \\ 4 & 4\,t - 1 \end{pmatrix}$

W = -2

x0 = $\begin{pmatrix} 2 & 0 \\ 4 & -1 \end{pmatrix}$

b = 2×1

```
1
2
```

c = $\begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$

ans = $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

dW = 0

P11P22 = 0

P11P22xW = 0

(a)

$$\frac{d}{dt}x^{(1)}(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad P x^{(1)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \therefore \frac{d}{dt}x^{(1)} = P x^{(1)}$$

$$\frac{d}{dt}x^{(2)}(t) = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \qquad P x^{(2)} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \qquad \therefore \frac{d}{dt}x^{(2)} = P x^{(2)}$$

(b)

$$\frac{d}{dt}\left(c_1 x^{(1)} + c_2 x^{(2)}\right) = \begin{bmatrix} 2c_2 \\ 4c_2 \end{bmatrix}$$

$$P\left(c_1 x^{(1)} + c_2 x^{(2)}\right) = \begin{bmatrix} 2c_2 \\ 4c_2 \end{bmatrix}$$

$$\therefore x' = Px$$

(c)

$$\det\left[x^{(1)}, x^{(2)}\right] = \det\begin{bmatrix} 2 & 2t \\ 4 & 4t-1 \end{bmatrix} = -2$$

$\therefore$ $x^{(1)}$ and $x^{(2)}$ are linearly independent and so form a fundamental set of solutions to the 2x2 system.

(d)

$$x^{(1)}(0) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \quad x^{(2)}(0) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$\therefore \; c_1 x^{(1)}(0) + c_2 x^{(2)}(0) = \begin{bmatrix} 2 & 0 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

yields $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$

$$\therefore \; \boxed{x(t) = \frac{1}{2} x^{(1)}(t) + 0\, x^{(2)}(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}}$$

(e)

From (c), $W[x^{(1)}, x^{(2)}](t) = \det \begin{bmatrix} 2 & 2t \\ 4 & 4t-1 \end{bmatrix} = \underline{-2}$

(f)

From $W(t) = -2$, $W'(t) = 0$

From $P = \begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix}$, $\left(p_{11}(t) + p_{22}(t)\right) = 4 + (-4) = 0$

$$\therefore \; W' = 0 \text{ and } (p_{11} + p_{22})W = (0)(-2) = 0$$

5.

Let $P = \frac{1}{t}\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$ Rewrite as $x' = Px$

Using MATLAB,

```
clear, clc
syms t c1 c2
P = (1/t)*[2, -1;
           3, -2];
x1 = [1,1]'*t;
x2 = [1,3]'*(1/t);
% (a)
Px1 = P*x1
dx1 = diff(x1,t,1)
Px2 = P*x2
dx2 = diff(x2,t,1)
% (b)
x = c1*x1 + c2*x2
Px = collect(P*x, [c1,c2])
dx = collect(diff(x,t,1), [c1,c2])
% (c)
x1x2 = [x1, x2]
W = det(x1x2)
% (d)
%x0 = [subs(x1,t,0), subs(x2,t,0)]
%b = [1,2]'
%c = linsolve(x0,b)   % solve x0*c = b
%x1x2*c
% (f)
dW = diff(W, t,1)
P11P22 = P(1,1) + P(2,2)
P11P22xW = P11P22*W
```

Px1 =

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

dx1 =

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Px2 =

$$\begin{pmatrix} -\frac{1}{t^2} \\ -\frac{3}{t^2} \end{pmatrix}$$

dx2 =

$$\begin{pmatrix} -\frac{1}{t^2} \\ -\frac{3}{t^2} \end{pmatrix}$$

x =

$$\begin{pmatrix} c_1\,t + \frac{c_2}{t} \\ c_1\,t + \frac{3\,c_2}{t} \end{pmatrix}$$

Px =

$$\begin{pmatrix} c_1 + \left(-\frac{1}{t^2}\right) c_2 \\ c_1 + \left(-\frac{3}{t^2}\right) c_2 \end{pmatrix}$$

dx =

$$\begin{pmatrix} c_1 + \left(-\frac{1}{t^2}\right) c_2 \\ c_1 + \left(-\frac{3}{t^2}\right) c_2 \end{pmatrix}$$

x1x2 =

$$\begin{pmatrix} t & \frac{1}{t} \\ t & \frac{3}{t} \end{pmatrix}$$

W = 2

dW = 0

P11P22 = 0

P11P22xW = 0

(a)

$$Px^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \frac{d}{dt}x^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \therefore \frac{d}{dt}x^{(1)} = Px^{(1)}$$

$$Px^{(2)} = \begin{bmatrix} -1/t^2 \\ -3/t^2 \end{bmatrix} \quad \frac{d}{dt}x^{(2)} = \begin{bmatrix} -1/t^2 \\ -3/t^2 \end{bmatrix} \quad \therefore \frac{d}{dt}x^{(2)} = Px^{(2)}$$

(b)

$$\frac{d}{dt}\left(c_1 x^{(1)} + c_2 x^{(2)}\right) = \begin{bmatrix} c_1 - c_2/t^2 \\ c_1 - 3c_2/t^2 \end{bmatrix}$$

$$P\left(c_1 x^{(1)} + c_2 x^{(2)}\right) = \begin{bmatrix} c_1 - c_2/t^2 \\ c_1 - 3c_2/t^2 \end{bmatrix}$$

$$\therefore x' = Px$$

(c)

$$\det\left[x^{(1)}, x^{(2)}\right] = \det\begin{bmatrix} t & 1/t \\ t & 3/t \end{bmatrix} = 2$$

$\therefore$ $x^{(1)}$ and $x^{(2)}$ are linearly independent, are solutions to the 2×2 system by (a), and so form a fundamental set of solutions.

(d)

Since the system is undefined for $t = 0$, there is <u>no solution</u> for $x(0) = (1,2)^T$

(e)

From (c), $W[x^{(1)}, x^{(2)}](t) = \det[x^{(1)}(t), x^{(2)}(t)] = \underline{2}$

(f)

From $W(t) = 2$, $W'(t) = 0$

From $P = \frac{1}{t}\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$, $p_{11}(t) + p_{22}(t) = \frac{2}{t} - \frac{2}{t} = 0$

$\therefore 0 = (0)(2)$ so $W' = (p_{11} + p_{22})W$

6.

Let $P = \frac{1}{t}\begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}$ Rewrite as $x' = Px$

Using MATLAB,

```
clear, clc
syms t c1 c2
P = (1/t)*[3, -2;
           2, -2];
x1 = [1,2]'*(1/t);
x2 = [2,1]'*t^2;
% (a)
Px1 = P*x1
dx1 = diff(x1,t,1)
Px2 = P*x2
dx2 = diff(x2,t,1)
% (b)
x = c1*x1 + c2*x2
Px = collect(P*x, [c1,c2])
dx = collect(diff(x,t,1), [c1,c2])
% (c)
x1x2 = [x1, x2]
W = det(x1x2)
% (d)
%x0 = [subs(x1,t,0), subs(x2,t,0)]
%b = [1,2]'
%c = linsolve(x0,b)   % solve x0*c = b
%x1x2*c
% (f)
dW = diff(W, t,1)
P11P22 = P(1,1) + P(2,2)
P11P22xW = P11P22*W
```

Px1 =

$$\begin{pmatrix} -\frac{1}{t^2} \\ -\frac{2}{t^2} \end{pmatrix}$$

dx1 =

$$\begin{pmatrix} -\frac{1}{t^2} \\ -\frac{2}{t^2} \end{pmatrix}$$

Px2 =

$$\begin{pmatrix} 4\,t \\ 2\,t \end{pmatrix}$$

dx2 =

$$\begin{pmatrix} 4\,t \\ 2\,t \end{pmatrix}$$

x =

$$\begin{pmatrix} \frac{c_1}{t} + 2\,c_2\,t^2 \\ \frac{2\,c_1}{t} + c_2\,t^2 \end{pmatrix}$$

Px =

$$\begin{pmatrix} \left(-\frac{1}{t^2}\right) c_1 + (4\,t)\,c_2 \\ \left(-\frac{2}{t^2}\right) c_1 + (2\,t)\,c_2 \end{pmatrix}$$

dx =

$$\begin{pmatrix} \left(-\frac{1}{t^2}\right) c_1 + (4\,t)\,c_2 \\ \left(-\frac{2}{t^2}\right) c_1 + (2\,t)\,c_2 \end{pmatrix}$$

x1x2 =

$$\begin{pmatrix} \frac{1}{t} & 2\,t^2 \\ \frac{2}{t} & t^2 \end{pmatrix}$$

W = $-3\,t$

dW = -3

P11P22 =

$$\frac{1}{t}$$

P11P22xW = -3

(a)

$$\frac{d}{dt}x^{(1)} = \begin{bmatrix} -1/t^2 \\ -2/t^2 \end{bmatrix} \quad Px^{(1)} = \begin{bmatrix} -1/t^2 \\ -2/t^2 \end{bmatrix} \quad \therefore \frac{d}{dt}x^{(1)} = Px^{(1)}$$

$$\frac{d}{dt}x^{(2)} = \begin{bmatrix} 4t \\ 2t \end{bmatrix} \quad Px^{(2)} = \begin{bmatrix} 4t \\ 2t \end{bmatrix} \quad \therefore \frac{d}{dt}x^{(2)} = Px^{(2)}$$

(b)

$$\frac{d}{dt}\left(c_1x^{(1)} + c_2x^{(2)}\right) = \begin{bmatrix} -c_1/t^2 + 4c_2t \\ -2c_1/t^2 + 2c_2t \end{bmatrix}$$

$$P\left(c_1x^{(1)} + c_2x^{(2)}\right) = \begin{bmatrix} -c_1/t^2 + 4c_2t \\ -2c_1/t^2 + 2c_2t \end{bmatrix}$$

$$\therefore x' = Px$$

(c)

$$\det\left[x^{(1)}, x^{(2)}\right] = \det\begin{bmatrix} 1/t & 2t^2 \\ 2/t & t^2 \end{bmatrix} = -3t \neq 0,\ t > 0$$

$\therefore$ The solutions $x^{(1)}(t)$ and $x^{(2)}(t)$ are linearly independent for $t > 0$, and so form a fundamental set of solutions of the 2×2 system.

(d)

Since the system is undefined for $t=0$, there is no solution for $x(0) = (1,2)^T$

(e)

From (c), $W[x^{(1)}, x^{(2)}](t) = \det[x^{(1)}(t), x^{(2)}(t)] = \underline{-3t}$

(f)

From $W(t) = -3t$, $W'(t) = -3$

From $P = \frac{1}{t}\begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}$, $p_{11}(t) + p_{22}(t) = \frac{1}{t}$

$\therefore W' = (p_{11} + p_{22})W$ as $-3 = \left(\frac{1}{t}\right)(-3t)$

7.

Essentially an induction argument.

(1) $n=1$: $x^{(1)}(t)$ is a solution to $x' = Px$.

Let c_1 be any constant. Then $\frac{d}{dt}(c_1 x^{(1)}(t))$

$= c_1 \frac{d}{dt} x^{(1)}(t)$ by linearity of differentiation.

Also, $c_1 P x^{(1)} = P(c_1 x^{(1)})$ by linearity

of matrix multiplication.

$\therefore \; x^{(1)\prime}(t) = P x^{(1)}(t) \Rightarrow c_1 x^{(1)\prime}(t) = c_1 P x^{(1)}(t)$

$\Rightarrow (c_1 x^{(1)}(t))' = P(c_1 x^{(1)}(t))$

$\therefore$ true for $n=1$

(2) Theorem 7.4.1 states true for $n=2$

(3) Suppose true for $n=k$, $k>2$.

Consider case for $k+1$, where all $x^{(i)}$ are

solutions to $x' = Px$, $1 \le i \le k+1$

and $(c_1 x^{(1)} + \ldots + c_k x^{(k)})' = P(c_1 x^{(1)} + \ldots + c_k x^{(k)})$

Let $x = c_1 x^{(1)} + \ldots + c_k x^{(k)} + c_{k+1} x^{(k+1)}$

Let $w = c_1 x^{(1)} + \ldots + c_k x^{(k)}$

$\therefore \; x = w + c_{k+1} x^{(k+1)}$

By assumption, $w' = Pw$ and $\therefore$ by Theorem 7.4.1, since w and $x^{(k+1)}$ are solutions to $x' = Px$, then so is $w + c_{k+1} x^{(k+1)}$

$$\therefore \; P\left(c_1 x^{(1)} + \dots + c_k x^{(k)} + c_{k+1} x^{(k+1)}\right)$$

$$= P\left(w + c_{k+1} x^{(k+1)}\right)$$

$$= \left(w + c_{k+1} x^{(k+1)}\right)' \quad \text{by Th. 7.4.1}$$

$$= \left(c_1 x^{(1)} + \dots + c_k x^{(k)} + c_{k+1} x^{(k+1)}\right)'$$

$\therefore$ When true for k, true for $k+1$

$\therefore$ Given any finite n, if $x^{(1)}, \dots, x^{(n)}$ are solutions to $x' = Px$, then $c_1 x^{(1)} + \dots + c_n x^{(n)}$ is also a solution.

8.

(a) **a.** Show that

$$\frac{dW}{dt} = \begin{vmatrix} \frac{dx_1^{(1)}}{dt} & \frac{dx_1^{(2)}}{dt} \\ x_2^{(1)} & x_2^{(2)} \end{vmatrix} + \begin{vmatrix} x_1^{(1)} & x_1^{(2)} \\ \frac{dx_2^{(1)}}{dt} & \frac{dx_2^{(2)}}{dt} \end{vmatrix}.$$

Let $x^{(1)} = \begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \end{bmatrix}$ and $x^{(2)} = \begin{bmatrix} x_1^{(2)} \\ x_2^{(2)} \end{bmatrix}$, in which

the $x_j^{(i)}$ are functions of t.

$$\therefore W[x^{(1)}, x^{(2)}](t) = \det \begin{bmatrix} x_1^{(1)} & x_1^{(2)} \\ x_2^{(1)} & x_2^{(2)} \end{bmatrix}$$

$$= \begin{vmatrix} x_1^{(1)} & x_1^{(2)} \\ x_2^{(1)} & x_2^{(2)} \end{vmatrix} = x_1^{(1)} x_2^{(2)} - x_2^{(1)} x_1^{(2)}$$

$$\therefore \frac{dW}{dt} = \frac{d}{dt}\left(x_1^{(1)} x_2^{(2)} - x_2^{(1)} x_1^{(2)} \right)$$

$$= \frac{d}{dt} x_1^{(1)} x_2^{(2)} + x_1^{(1)} \frac{d}{dt} x_2^{(2)}$$

$$- \frac{d}{dt} x_2^{(1)} x_1^{(2)} - x_2^{(1)} \frac{d}{dt} x_1^{(2)}$$

$$= \frac{d}{dt} x_1^{(1)} x_2^{(2)} - \frac{d}{dt} x_1^{(2)} x_2^{(1)} + x_1^{(1)} \frac{d}{dt} x_2^{(2)} - x_1^{(2)} \frac{d}{dt} x_2^{(1)}$$

$$= \begin{vmatrix} \frac{d}{dt}x_1^{(1)} & \frac{d}{dt}x_1^{(2)} \\ x_2^{(1)} & x_2^{(2)} \end{vmatrix} + \begin{vmatrix} x_1^{(1)} & x_1^{(2)} \\ \frac{d}{dt}x_2^{(1)} & \frac{d}{dt}x_2^{(2)} \end{vmatrix}$$

(b)

From (3), $\frac{d}{dt}x^{(1)} = P x^{(1)}$ and $\frac{d}{dt}x^{(2)} = P x^{(2)}$

Or, $$\begin{bmatrix} \frac{d}{dt}x_1^{(1)} \\ \frac{d}{dt}x_2^{(1)} \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \end{bmatrix} \quad [1]$$

and $$\begin{bmatrix} \frac{d}{dt}x_1^{(2)} \\ \frac{d}{dt}x_2^{(2)} \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} x_1^{(2)} \\ x_2^{(2)} \end{bmatrix} \quad [2]$$

$$\therefore \frac{d}{dt}x_1^{(1)} = p_{11}x_1^{(1)} + p_{12}x_2^{(1)} \quad [1.1]$$

$$\frac{d}{dt}x_2^{(1)} = p_{21}x_1^{(1)} + p_{22}x_2^{(1)} \quad [1.2]$$

from [1]

$$\frac{d}{dt}x_1^{(2)} = p_{11}x_1^{(2)} + p_{12}x_2^{(2)} \quad [2.1]$$

$$\frac{d}{dt}x_2^{(2)} = p_{21}x_1^{(2)} + p_{22}x_2^{(2)} \quad [2.2]$$

from [2]

Now substitute these from result in (a).

$$\frac{dW}{dt} = \begin{vmatrix} p_{11}x_1^{(1)} + p_{12}x_2^{(1)} & p_{11}x_1^{(2)} + p_{12}x_2^{(2)} \\ x_2^{(1)} & x_2^{(2)} \end{vmatrix} \quad \text{from } [1.1], [2.1]$$

$$+ \begin{vmatrix} x_1^{(1)} & x_1^{(2)} \\ p_{21}x_1^{(1)} + p_{22}x_2^{(1)} & p_{21}x_1^{(2)} + p_{22}x_2^{(2)} \end{vmatrix} \quad \text{from } [1.2], [2.2]$$

$$= \begin{vmatrix} p_{11}x_1^{(1)} & p_{11}x_1^{(2)} \\ x_2^{(1)} & x_2^{(2)} \end{vmatrix} + \begin{vmatrix} p_{12}x_2^{(1)} & p_{12}x_2^{(2)} \\ x_2^{(1)} & x_2^{(2)} \end{vmatrix}$$

$$+ \begin{vmatrix} x_1^{(1)} & x_1^{(2)} \\ p_{21}x_1^{(1)} & p_{21}x_2^{(2)} \end{vmatrix} + \begin{vmatrix} x_1^{(1)} & x_1^{(2)} \\ p_{22}x_2^{(1)} & p_{22}x_2^{(2)} \end{vmatrix}$$

$$\begin{vmatrix} a & b \\ c+c' & d+d' \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a & b \\ c' & d' \end{vmatrix}$$

$$= p_{11}\underbrace{\begin{vmatrix} x_1^{(1)} & x_1^{(2)} \\ x_2^{(1)} & x_2^{(2)} \end{vmatrix}}_{=W} + p_{12}\underbrace{\begin{vmatrix} x_2^{(1)} & x_2^{(2)} \\ x_2^{(1)} & x_2^{(2)} \end{vmatrix}}_{=0}$$

$$\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$+ p_{21}\underbrace{\begin{vmatrix} x_1^{(1)} & x_1^{(2)} \\ x_1^{(1)} & x_1^{(2)} \end{vmatrix}}_{=0} + p_{22}\underbrace{\begin{vmatrix} x_1^{(1)} & x_1^{(2)} \\ x_2^{(1)} & x_2^{(2)} \end{vmatrix}}_{=W}$$

$$= p_{11}W + p_{12}(0) + p_{21}(0) + p_{22}W$$

$$= (p_{11} + p_{22})W$$

$$\therefore \frac{dW}{dt} = (p_{11} + p_{22})W$$

(c)

$\frac{dW}{dt} = (p_{11}(t) + p_{22}(t))W$ is a first-order linear equation. By Theorem 2.4.1, the solution is

$W(t) = c\exp\left[\int (p_{11}(t) + p_{22}(t))\,dt\right]$, c a constant

Since $e^x \neq 0$ for all x, $W(t)$ depends on c, a constant, and so is nonzero for all t ($c \neq 0$), or zero for all t ($c = 0$).

(d)

[a] From (b), $\frac{d}{dt} x_i^{(j)} = \sum_{k=1}^{n} p_{ik} x_k^{(j)}$ [3]

that is, from $x' = Px$, the derivative of the i-th component of the $x^{(j)}$ solution is equal to the i-th row of P times the components of $x^{(j)}$

The above formula is substituted into the general form of $\frac{dW}{dt}$ to make the proof.

Prove by induction for an $n \times n$ determinant W, with $x^{(1)}, \dots, x^{(n)}$ column vector components:

$$\frac{dW}{dt} = \begin{vmatrix} \frac{d}{dt} x_1^{(1)} & \cdots & \frac{d}{dt} x_1^{(n)} \\ \vdots & & \vdots \\ x_n^{(1)} & \cdots & x_n^{(n)} \end{vmatrix} + \cdots + \begin{vmatrix} x_1^{(1)} & \cdots & x_1^{(n)} \\ \vdots & & \vdots \\ \frac{d}{dt} x_i^{(1)} & \cdots & \frac{d}{dt} x_i^{(n)} \\ \vdots & & \vdots \\ x_n^{(1)} & \cdots & x_n^{(n)} \end{vmatrix}$$

$$+ \begin{vmatrix} x_1^{(1)} & \cdots & x_1^{(n)} \\ \vdots & & \vdots \\ \frac{d}{dt} x_n^{(1)} & \cdots & \frac{d}{dt} x_n^{(n)} \end{vmatrix}$$

Thus, the derivative of an $n \times n$ determinant W is the sum of n $n \times n$ determinants, each of which is identical to W except at row i, which consists of the derivatives of the components of row i of W.

(1) (a) above proved this is true for a 2×2 determinant.

(2) Assume formula true for any $(n-1) \times (n-1)$

determinant, where $n \geq 3$, and consider the $n \times n$ case. By cofactor definition of W,

$$W = \begin{vmatrix} x_1^{(1)} & \cdots & x_1^{(j)} & \cdots & x_1^{(n)} \\ \vdots & & & & \\ x_i^{(1)} & \cdots & x_i^{(j)} & \cdots & x_i^{(n)} \\ \vdots & & \vdots & & \vdots \\ x_n^{1} & \cdots & x_n^{(j)} & \cdots & x_n^{(n)} \end{vmatrix} = x_1^{(1)} \begin{vmatrix} x_2^{(2)} & \cdots & x_2^{(n)} \\ \vdots & & \\ x_n^{(2)} & \cdots & x_n^{(n)} \end{vmatrix}$$

$$+ \cdots + (-1)^{1+j} x_1^{(j)} \begin{vmatrix} x_2^{(1)} & \cdots & x_2^{(n)} \\ \vdots & & \\ x_n^{(1)} & \cdots & x_n^{(n)} \end{vmatrix} + \cdots + (-1)^{1+n} x_1^{(n)} \begin{vmatrix} x_2^{(1)} & \cdots & x_2^{(n-1)} \\ \vdots & & \vdots \\ x_n^{(1)} & \cdots & x_n^{(n-1)} \end{vmatrix}$$

Letting $m = [x^{(1)}, \ldots x^{(j)}, \ldots x^{(n)}]$, an $n \times n$ matrix, define m_{1j} = submatrix formed from m by deleting row 1 and column j, an $(n-1) \times (n-1)$ matrix.

Define $W_j = \det(m_{1j})$, then

$$W = \sum_{j=1}^{n} (-1)^{1+j} x_1^{(j)} W_j \quad \text{(cofactor definition of determinant)}$$

$$\therefore \frac{dW}{dt} = \sum_{j=1}^{n} (-1)^{1+j} \left[\frac{d}{dt} x_1^{(j)} W_j + x_1^{(j)} \frac{dW_j}{dt} \right]$$

$$= \sum_{j=1}^{n} (-1)^{1+j} \frac{d}{dt} x_1^{(j)} W_j + \sum_{j=1}^{n} (-1)^{1+j} x_1^{(j)} \frac{dW_j}{dt}$$

$$= \begin{vmatrix} \frac{d}{dt} x_1^{(1)} & \cdots & \frac{d}{dt} x_1^{(n)} \\ \vdots & & \\ x_n^{(1)} & \cdots & x_n^{(n)} \end{vmatrix}$$ by cofactor definition of determinant

$$+ \sum_{j=1}^{n} (-1)^{1+j} x_1^{(j)} \frac{dW_j}{dt}$$

But by assumption, since W_j is an $(n-1) \times (n-1)$ determinant, $\frac{d}{dt} W_j$ is a sum of $(n-1) \times (n-1)$ determinants, whose row i is the derivatives of the components of row i of W_j.

$$\sum_{j=1}^{n} (-1)^{1+j} x_1^{(j)} \frac{dW_j}{dt} =$$ (n rows of $n-1$ terms)

$$x_1^{(1)} \begin{vmatrix} \frac{d}{dt} x_2^{(2)} & \cdots & \frac{d}{dt} x_2^{(n)} \\ \vdots & & \\ x_n^{(2)} & \cdots & x_n^{(n)} \end{vmatrix} + \cdots + x_1^{(1)} \begin{vmatrix} x_2^{(2)} & \cdots & x_2^{(n)} \\ \frac{d}{dt} x_n^{(2)} & \cdots & \frac{d}{dt} x_n^{(n)} \end{vmatrix}$$

$$+$$
$$\vdots$$
$$+$$

$\Updownarrow$ using cofactor definition of determinant $\Updownarrow$

$$(-1)^{1+n} x_1^{(n)} \begin{vmatrix} \frac{d}{dt}x_2^{(1)} & \cdots & \frac{d}{dt}x_2^{(n-1)} \\ \vdots & & \\ x_n^{(1)} & \cdots & x_n^{(n-1)} \end{vmatrix} + \cdots + (-1)^{1+n} x_1^{(n)} \begin{vmatrix} x_2^{(1)} & \cdots & x_2^{(n-1)} \\ \vdots & & \\ \frac{d}{dt}x_n^{(1)} & \cdots & \frac{d}{dt}x_n^{(n-1)} \end{vmatrix}$$

$\Downarrow$ collect vertical components $\qquad\qquad$ $\Downarrow$ collect vertical components

$$= \begin{vmatrix} x_1^{(1)} & \cdots & x_1^{(n)} \\ \frac{d}{dt}x_2^{(1)} & \cdots & \frac{d}{dt}x_2^{(n)} \\ \vdots & & \\ x_n^{(1)} & \cdots & x_n^{(n)} \end{vmatrix} + \cdots + \begin{vmatrix} x_1^{(1)} & \cdots & x_1^{(n)} \\ \vdots & & \vdots \\ \frac{d}{dt}x_n^{(1)} & \cdots & \frac{d}{dt}x_n^{(n)} \end{vmatrix}$$

$$\therefore \frac{dW}{dt} = \begin{vmatrix} \frac{dx_1^{(1)}}{dt} & \cdots & \frac{dx_1^{(n)}}{dt} \\ \vdots & & \vdots \\ x_n^{(1)} & \cdots & x_n^{(n)} \end{vmatrix} + \cdots + \begin{vmatrix} x_1^{(1)} & \cdots & x_1^{(n)} \\ \vdots & & \\ \frac{d}{dt}x_1^{(n)} & \cdots & \frac{d}{dt}x_n^{(n)} \end{vmatrix}$$

(1) & (2) $\Rightarrow$ formula true for any fixed $n \geq 2$

[6] Finally, substitute [3] into general formula for $\frac{dW}{dt}$, a formula consisting of n determinants.

$$\therefore \frac{dW}{dt} = \sum_{i=1}^{n} \frac{dW_i}{dt}, \text{ where } \frac{dW_i}{dt} \text{ is defined as:}$$

$$\frac{dW_i}{dt} = \begin{vmatrix} x_1^{(1)} & \cdots\cdots\cdots & x_1^{(n)} \\ \vdots & & \vdots \\ \frac{dx_i^{(1)}}{dt} & \cdots\cdots & \frac{dx_i^{(n)}}{dt} \\ \vdots & & \vdots \\ x_n^{(1)} & \cdots\cdots & x_n^{(n)} \end{vmatrix}$$

, i.e., $\frac{dW_i}{dt}$ is W except at row i, which has derivatives of the i-th row elements of W.

Substituting [3] which is $\frac{d}{dt} x_i^{(j)} = \sum_{k=1}^{n} p_{ik} x_k^{(j)}$

$$\frac{dW_i}{dt} = \begin{vmatrix} x_1^{(1)} & \cdots & x_1^{(j)} & \cdots & x_1^{(n)} \\ \vdots & & \vdots & & \vdots \\ \sum_{k=1}^{n} p_{ik} x_k^{(1)} & \cdots & \sum_{k=1}^{n} p_{ik} x_k^{(j)} & \cdots & \sum_{k=1}^{n} p_{ik} x_k^{(n)} \\ \vdots & & \vdots & & \vdots \\ x_n^{(1)} & \cdots & x_n^{(j)} & \cdots & x_n^{(n)} \end{vmatrix}$$

now use determinant property $\begin{vmatrix} a & b \\ c_1+c_2 & d_1+d_2 \end{vmatrix} = \begin{vmatrix} a & b \\ c_1 & d_1 \end{vmatrix} + \begin{vmatrix} a & b \\ c_2 & d_2 \end{vmatrix}$

break up into n determinant sums

$$= \begin{vmatrix} x_1^{(1)} & \cdots & x_1^{(j)} & \cdots & x_1^{(n)} \\ \vdots & & & & \\ p_{i1} x_1^{(1)} + \cdots p_{in} x_n^{(1)} & & p_{i1} x_1^{(j)} + \cdots + p_{in} x_n^{(j)} & & p_{i1} x_1^{(n)} + \cdots + p_{in} x_n^{(n)} \\ \vdots & & \vdots & & \vdots \\ x_n^{(1)} & \cdots & x_n^{(j)} & \cdots & x_n^{(n)} \end{vmatrix}$$

$$= \begin{vmatrix} x_1^{(1)} & \cdots & x_1^{(j)} & \cdots & x_1^{(n)} \\ \vdots & & \vdots & & \vdots \\ p_{i1} x_1^{(1)} & \cdots & p_{i1} x_1^{(j)} & \cdots & p_{i1} x_1^{(n)} \\ \vdots & & \vdots & & \vdots \\ x_n^{(1)} & & x_n^{(j)} & & x_n^{(n)} \end{vmatrix}$$

$$+ \cdots +$$

$$\begin{vmatrix} x_1^{(1)} & \cdots & x_1^{(j)} & \cdots & x_1^{(n)} \\ \vdots & & \vdots & & \vdots \\ p_{ij} x_j^{(1)} & & p_{ij} x_j^{(j)} & & p_{ij} x_j^{(n)} \\ \vdots & & \vdots & & \vdots \\ x_n^{(1)} & \cdots & x_n^{(j)} & \cdots & x_n^{(n)} \end{vmatrix}$$

$$+ \cdots +$$

$$\begin{vmatrix} x_1^{(1)} & x_1^{(j)} & x_1^{(n)} \\ \vdots & \vdots & \vdots \\ p_{in} x_n^{(1)} & p_{in} x_n^{(j)} & p_{in} x_n^{(n)} \\ x_n^{(1)} & x_n^{(j)} & x_n^{(n)} \end{vmatrix}$$

now use determinant property

$$\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$= p_{i1} \begin{vmatrix} X_1^{(1)} & \cdots & X_1^{(j)} & \cdots & X_1^{(n)} \\ \vdots & & \vdots & & \vdots \\ X_1^{(1)} & \cdots & X_1^{(j)} & \cdots & X_1^{(n)} \\ \vdots & & \vdots & & \vdots \\ X_n^{(1)} & \cdots & X_n^{(j)} & \cdots & X_n^{(n)} \end{vmatrix} = 0 \text{ since row } i = \text{row } 1$$

$$+ \cdots +$$

$$p_{ij} \begin{vmatrix} X_1^{(1)} & \cdots & X_1^{(j)} & \cdots & X_1^{(n)} \\ \vdots & & \vdots & & \vdots \\ X_j^{(1)} & \cdots & X_j^{(j)} & \cdots & X_j^{(n)} \\ \vdots & & \vdots & & \vdots \\ X_1^{(n)} & \cdots & X_n^{(j)} & \cdots & X_n^{(n)} \end{vmatrix} \begin{array}{l} = 0 \text{ if } i \neq j \text{ as} \\ \quad \text{row } i = \text{row}_j \\ = p_{ii} W \text{ if } i = j \end{array}$$

$$+ \cdots +$$

$$p_{in} \begin{vmatrix} X_1^{(1)} & \cdots & X_1^{(j)} & \cdots & X_1^{(n)} \\ \vdots & & \vdots & & \vdots \\ X_n^{(1)} & \cdots & X_n^{(j)} & \cdots & X_n^{(n)} \\ \vdots & & \vdots & & \vdots \\ X_n^{(1)} & \cdots & X_n^{(j)} & \cdots & X_n^{(n)} \end{vmatrix} = 0 \text{ since row } i = \text{row } n$$

$$\therefore \frac{dW_i}{dt} = p_{ii} W$$

$$\therefore \frac{dW}{dt} = \sum_{i=1}^{n} \frac{dW_i}{dt} = \sum_{i=1}^{n} p_{ii} W$$

$$= (p_{11} + \cdots + p_{nn}) W \qquad [4]$$

[C] $W(t)=0$ for all t is a solution to [4] in [6].

$\therefore$ Assume $W(t)\neq 0$ for some t. $\therefore$ $W(t)\neq 0$ in an interval around t since $W(t)$ is composed of continuous functions. From [4] above in [6],

$$\frac{1}{W(t)}\frac{dW}{dt} = p_{11}(t)+\dots+p_{nn}(t)$$ Integrating,

$$\therefore \ln(W(t)) = \int p_{11}(t)+\dots+p_{nn}(t)\,dt + K,$$ K a constant

$$\therefore W(t) = c\exp\left[\int(p_{11}+\dots+p_{nn})\,dt\right], \quad c=e^K,$$ a constant

$\therefore$ $c\neq 0$ since $W(t)$ is assumed nonzero on an interval. $\therefore$ $W(t)\neq 0$ for all t since $e^x\neq 0$.

9.

Let $W[x^{(1)},\dots,x^{(n)}](t) = c_1\exp\left(\int[p_{11}+\dots+p_{nn}]\,dt\right)$

and $W[y^{(1)},\dots,y^{(n)}](t) = c_2\exp\left(\int[p_{11}+\dots+p_{nn}]\,dt\right)$

$\therefore$ $W[x^{(1)},\dots,x^{(n)}](t) = \frac{c_1}{c_2}W[y^{(1)},\dots,y^{(n)}](t)$, if $c_2\neq 0$.

If $c_2 = 0$, then $W[y^{(1)}, \dots, y^{(n)}](t) = 0 \cdot W[x^{(1)}, \dots, x^{(n)}](t)$

10.

By definition, $W[y^{(1)}, y^{(2)}](t) = \begin{vmatrix} y^{(1)}(t) & y^{(2)}(t) \\ \frac{dy^{(1)}(t)}{dt} & \frac{dy^{(2)}(t)}{dt} \end{vmatrix}$

$= y^{(1)} \frac{dy^{(2)}}{dt} - y^{(2)} \frac{dy^{(1)}}{dt}$ [1]

where $y^{(1)}$ and $y^{(2)}$ are a fundamental solution to (18).

Let $r_1 = y^{(1)}$, $r_2 = \frac{d}{dt} y^{(1)}$ and $s_1 = y^{(2)}$, $s_2 = \frac{d}{dt} y^{(2)}$

Note $W[y^{(1)}, y^{(2)}] = \begin{vmatrix} r_1 & s_1 \\ r_2 & s_2 \end{vmatrix} = r_1 s_2 - r_2 s_1$ [2]

Since $y^{(1)}, y^{(2)}$ are a fundamental set, $W[y^{(1)}, y^{(2)}] \neq 0$

$\therefore \begin{vmatrix} r_1 & s_1 \\ r_2 & s_2 \end{vmatrix} \neq 0$, so $r^{(1)} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$ and $s^{(1)} = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}$ are

linearly independent vectors. $\therefore W[y^{(1)}, y^{(2)}] = W[r^{(1)}, s^{(1)}] \neq 0$

Note: $r_1' = r_2$ or, $\begin{bmatrix} r_1' \\ r_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -q & -p \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$

$r_2' = -q r_1 - p r_2$

Similarly, $s_1' = s_2$ or, $\begin{bmatrix} s_1' \\ s_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -q & -p \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}$

$s_2' = -q s_1 - p s_2$

$\therefore$ $r^{(1)}$ and $s^{(1)}$ are linearly independent solutions to $x' = Px$, $P = \begin{bmatrix} 0 & 1 \\ -q & -p \end{bmatrix}$, and so by

Theorem 7.4.3 and equation (15),

$W[r^{(1)}, s^{(1)}](t) = c \exp\left(\int -p(t)\,dt\right)$, c a nonzero constant since $W[r^{(1)}, s^{(1)}] \neq 0$.

Let $x^{(1)}, x^{(2)}$ be any other fundamental set of solutions to (19). Then $W[x^{(1)}, x^{(2)}] = K \exp\left(\int -p(t)\,dt\right)$, K a nonzero constant, since $x^{(1)}$ and $x^{(2)}$ are independent. $\therefore$ $W[r^{(1)}, s^{(1)}] = \frac{c}{K} W[x^{(1)}, x^{(2)}]$

$\therefore$ $W[y^{(1)}, y^{(2)}] = W[r^{(1)}, s^{(1)}] = \frac{c}{K}[x^{(1)}, x^{(2)}]$

11.

$x^{(p)}$ a solution means $\frac{d}{dt} x^{(p)} = P x^{(p)} + g$

$x^{(c)}$ a solution to the homogeneous equation means

$$\frac{d}{dt} x^{(c)} = P x^{(c)}$$

$$\therefore \frac{d}{dt}\left(x^{(p)} + x^{(c)}\right) = \frac{d}{dt} x^{(p)} + \frac{d}{dt} x^{(c)}$$

$$= P x^{(p)} + g + P x^{(c)}$$

$$= P\left(x^{(p)} + x^{(c)}\right) + g$$

$\therefore$ $x^{(p)} + x^{(c)}$ is a solution. If y is any other solution (both satisfying some initial conditions), then by the uniqueness theorem (7.1.2), $y = x^{(p)} + x^{(c)}$

12.

(a)

$$\begin{vmatrix} t & t^2 \\ 1 & 2t \end{vmatrix} = 2t^2 - t^2 = \underline{t^2}$$

(b)

where $t^2 \neq 0$. At every other value of t, they are independent. $\therefore$ For every interval, including an interval containing zero, they are independent.

To be dependent on an interval, there must be constants, c_1 and c_2, s.t. $c_1 x^{(1)}(t) + c_2 x^{(2)}(t) = 0$ for all t in that interval. This is not true for any interval containing $t=0$, the one value for which $W[x^{(1)}, x^{(2)}](t) = 0$. $\therefore$ $x^{(1)}$ and $x^{(2)}$ are independent on any interval.

(c)

The coefficients are the $p_{ij}(t)$ in array P.

From Theorem 7.4.3, $W(t) = c \exp\left(\int (p_{11} + \dots + p_{nn})\,dt\right)$,

so $W(t)$ either never vanishes (not true for

$x^{(1)}, x^{(2)}$ since at $t=0$, $W(t)=0$) or is zero on the enitire interval (also not true, since $W \neq 0$ at $t \neq 0$). $\therefore$ A premise the theorem is based on must not hold; i.e., one of the p_{ij} must be discontinuous at $t=0$.

Although not explicitly stated in 7.4.3, the proof depends upon the continuity of the $p_{ij}(t)$ in order to invoke Theorem 7.4.1, which gives an explicit formula for the solution. Actually, it is just the p_{ii} that need to be continuous, since $\frac{dW}{dt} = (p_{11} + \ldots + p_{nn})W$.

(d)

$$\frac{d}{dt} x^{(1)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} t \\ 1 \end{bmatrix} = \begin{bmatrix} p_{11} t + p_{12} \\ p_{21} t + p_{22} \end{bmatrix} \begin{matrix} [1] \\ [2] \end{matrix}$$

$$\frac{d}{dt} x^{(2)} = \begin{bmatrix} 2t \\ 2 \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} t^2 \\ 2t \end{bmatrix} = \begin{bmatrix} p_{11} t^2 + p_{12} 2t \\ p_{21} t^2 + p_{22} 2t \end{bmatrix} \begin{matrix} [3] \\ [4] \end{matrix}$$

$\therefore \quad 1 = t p_{11} + p_{12} \quad [1]$ multiply [1] by t,
$2t = t^2 p_{11} + 2t p_{12} \quad [3]$ then subtract

$\therefore \quad t = t p_{12}, \quad p_{12} = 1, \quad \therefore p_{11} = 0$

$0 = t p_{21} + p_{22} \quad [2]$ multiply [2] by t,

$2 = t^2 p_{21} + 2t p_{22} \quad [4]$ then subtract

$\therefore \quad 2 = t p_{22}, \quad p_{22} = \frac{2}{t}, \quad \therefore p_{21} = -\frac{2}{t^2}$

$$\therefore \quad P = \begin{bmatrix} 0 & 1 \\ -2/t^2 & 2/t \end{bmatrix}$$

Note $p_{22}(t)$ not continuous at $t = 0$.

$$\therefore \quad X' = \begin{bmatrix} 0 & 1 \\ -2/t^2 & 2/t \end{bmatrix} X$$

13.

(a)

$$\begin{vmatrix} t^2 & e^t \\ 2t & e^t \end{vmatrix} = t^2 e^t - 2t e^t = \underline{t(t-2)e^t}$$

(b)

At every point except $t=0$ and $t=2$, $x^{(1)}$ and $x^{(2)}$ are indeendent. $\therefore$ For any interval, including intervals containing $t=0$ and/or $t=2$, $x^{(1)}$ and $x^{(2)}$ are independent, since any interval containing those points will also contain values of t for which $W[x^{(1)}, x^{(2)}](t) \neq 0$, so $x^{(1)}, x^{(2)}$ can't be dependent on those intervals for all t.

(c)

The coefficients in the system are the components of array P, from $x' = Px$. $\therefore$ At least one of the diagonal elements, $p_{ii}(t)$, must be discontinuous at $t=0$ and $t=2$. If all $p_{ii}(t)$ were continuous at $t=0$ and $t=2$, then they would be continuous at intervals around

$t=0$ and $t=2$, and then from

$\frac{dW}{dt} = (p_{11}(t) + \dots + p_{nn}(t))W$ and Theorem 7.4.3,

$W[x^{(1)}, x^{(2)}]$ would be nonzero for all t, or zero for all t. And this is not true.

(d)

$$\frac{d}{dt}x^{(1)}(t) = \begin{bmatrix} 2t \\ 2 \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}\begin{bmatrix} t^2 \\ 2t \end{bmatrix} = \begin{bmatrix} t^2 p_{11} + 2t p_{12} \\ t^2 p_{21} + 2t p_{22} \end{bmatrix} \begin{matrix} [1] \\ [2] \end{matrix}$$

$$\frac{d}{dt}x^{(2)}(t) = \begin{bmatrix} e^t \\ e^t \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}\begin{bmatrix} e^t \\ e^t \end{bmatrix} = \begin{bmatrix} e^t p_{11} + e^t p_{12} \\ e^t p_{21} + e^t p_{22} \end{bmatrix} \begin{matrix} [3] \\ [4] \end{matrix}$$

$\therefore$ $2t = t^2 p_{11} + 2t p_{12}$ [1] Multiply [1] by e^t, [3] by

$e^t = e^t p_{11} + e^t p_{12}$ [3] $2t$, then subtract

$\therefore$ $0 = (t^2 e^t - 2t e^t) p_{11}$, for all t, $\Rightarrow$ $\underline{p_{11}(t) = 0}$

$\therefore$ From [1], $2t = 2t p_{12}$ $\Rightarrow$ $\underline{p_{12}(t) = 1}$

$2 = t^2 p_{21} + 2t p_{22}$ [2] Multiply [2] by e^t, [4] by

$e^t = e^t p_{21} + e^t p_{22}$ [4] $2t$, then subtract

$\therefore\ 2e^t - 2te^t = (t^2e^t - 2te^t)\, p_{21}$

$$2 - 2t = (t^2 - 2t)\, p_{21}\,,\quad \underline{p_{21} = \frac{2-2t}{t^2-2t}}$$

Multiplying [2] by e^t, [4] by t^2, then subtracting,

$$2e^t - t^2e^t = (2te^t - t^2e^t)\, p_{22}\,,\quad \underline{p_{22} = \frac{t^2-2}{t^2-2t}}$$

$$\therefore\ P = \begin{bmatrix} 0 & 1 \\ \frac{2-2t}{t^2-2t} & \frac{t^2-2}{t^2-2t} \end{bmatrix}$$

Note $p_{22}(t)$ is not continuous at $t = 0, 2$

$$\therefore\ \underline{X' = \begin{bmatrix} 0 & 1 \\ \frac{2-2t}{t^2-2t} & \frac{t^2-2}{t^2-2t} \end{bmatrix} X}$$

14.

(1) If $x^{(1)}, \ldots, x^{(m)}$ are linearly dependent on (α, β),

then by definition there are constants $c_1, \dots, c_m$, not all zero, s.t. $c_1 x^{(1)}(t) + \dots + c_m x^{(m)}(t) = 0$ for all $t \in (\alpha, \beta)$. $\therefore$ Since $t_0 \in (\alpha, \beta)$, then $c_1 x^{(1)}(t_0) + \dots + c_m x^{(m)}(t_0) = 0$, and $\therefore$ $x^{(1)}(t_0), \dots, x^{(m)}(t_0)$ are linearly dependent since not all of $c_1, \dots, c_m$ are zero.

(2) Suppose $x^{(1)}(t_0), \dots, x^{(m)}(t_0)$ are dependent for some $t_0 \in (\alpha, \beta)$, where $x^{(i)}(t)$ are solutions to $x' = Px$.

$\therefore$ There are constants, $c_1, \dots, c_m$, not all zero s.t. $c_1 x^{(1)}(t_0) + \dots + c_m x^{(m)}(t_0) = 0$ by definition of linearly dependent. Consider $c_1 x^{(1)}(t) + \dots + c_m x^{(m)}(t)$. Since $y(t) = 0$ is a solution to $x' = Px$, and $y(t_0) = 0$, then by the uniqueness theorem, $y(t) = c_1 x^{(1)}(t) + \dots + c_m x^{(m)}(t)$, since both are zero at $t = t_0$, and both are solutions to

$x' = Px$. $\therefore$ $y(t) = 0 \Rightarrow c_1 x^{(1)}(t) + \dots + c_m x^{(m)}(t) = 0$ for all $t \in (\alpha, \beta)$. $\therefore$ $x^{(1)}, \dots, x^{(m)}$ are linearly dependent on the interval (α, β).

15.

Assume P is $n \times n$.

(a)

Can't use uniqueness theorem using a method similar to the proof of Theorem 3.2.4, since for a $t_0 \in (\alpha, \beta)$, $x^{(1)}(t_0), \dots, x^{(n)}(t_0)$ may be dependent, and so may not be able to solve $z(t_0) = X(t_0)\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$. See Problem #13, Section 7.3.

(1) $\therefore$ Let $z(t)$ be any solution to $x' = Px$, and let t_0 be any point in (α, β). The $n+1$ column vectors $x^{(1)}(t_0), \dots, x^{(n)}(t_0), z(t_0)$ (all $n \times 1$) are linearly

dependent, using Problem #10, section 7.3. $\therefore$ There exist constants $c_1, \dots, c_n, c_{n+1}$, not all zero, s.t. $c_1 x^{(1)}(t_0) + \dots + c_n x^{(n)}(t_0) + c_{n+1} z(t_0) = 0$, by definition. Note that $c_{n+1} \neq 0$, because otherwise $c_1 x^{(1)}(t_0) + \dots + c_n x^{(n)}(t_0) = 0$. If one of the $c_1, \dots, c_n$ is nonzero, then the $x^{(1)}(t_0), \dots, x^{(n)}(t_0)$ are linearly dependent, and $\therefore$ by #14 above, the $x^{(1)}(t), \dots, x^{(n)}(t)$ are linearly dependent, a contradiction.

(2) Now consider $y(t) = -\frac{c_1}{c_{n+1}} x^{(1)}(t) - \dots - \frac{c_n}{c_{n+1}} x^{(n)}(t)$, division by c_{n+1} permitted because by (1), $c_{n+1} \neq 0$. Note that $y(t)$ is a solution to $x' = Px$, using superposition (theorem 7.4.1). Also, $z(t_0) = -\frac{c_1}{c_{n+1}} x^{(1)}(t_0) - \dots - \frac{c_n}{c_{n+1}} x^{(n)}(t_0)$. $\therefore$ By the uniqueness theorem (7.1.2), $z(t) = y(t)$, which proves the statement.

(b)

Let $z(t) = c_1 x^{(1)}(t) + \ldots + c_n x^{(n)}(t)$ [1] and

$z(t) = k_1 x^{(1)}(t) + \ldots + k_n x^{(n)}(t)$ [2] be two

expressions for $z(t)$.

Consider $(c_1 - k_1) x^{(1)}(t) + \ldots + (c_n - k_n) x^{(n)}(t) = 0$

for all $t \in (\alpha, \beta)$.

Since $x^{(1)}, \ldots, x^{(n)}$ are linearly independent, then

$c_i - k_i = 0$ for $i = 1, \ldots, n$; for if there was some

i s.t. $c_i - k_i \neq 0$ for all t, then by definition

$x^{(1)}(t), \ldots, x^{(n)}(t)$ would be linearly dependent.

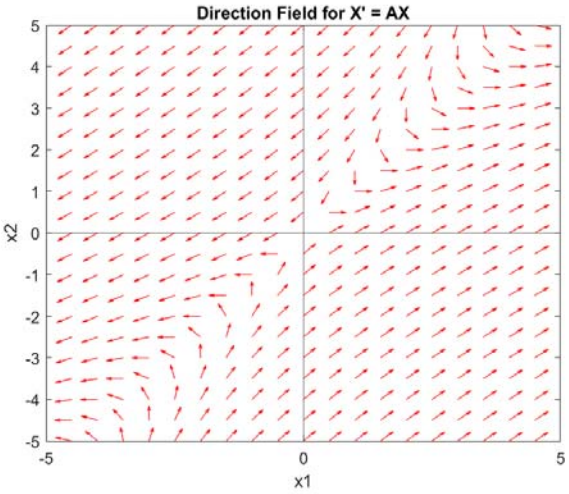
$\therefore$ The expression for $z(t)$ is unique.

7.5 Homogeneous Linear Systems with Constant Coefficients

Note Title 12/2/2019

1.

(a) Using MATLAB,




```
clear
% set plot boundaries
Xmin = -5; Xmax = 5;
Ymin = -5; Ymax = 5;
% set how fine to make the grid of values
Step = 0.5;
% total # points on x and y axes
Nx = round((Xmax-Xmin)/Step, 0) + 1;
Ny = round((Ymax-Ymin)/Step, 0) + 1;
% set the actual x and y coords
x = linspace(Xmin, Xmax, Nx);  % a row vector
y = linspace(Ymin, Ymax, Ny);  % a row vector
% preallocate memory for coord, slope arrays
%  to be used in quiver, must be same dimension
xcoord = zeros(Nx, Ny);
ycoord = zeros(Nx, Ny);
dx = zeros(Nx, Ny);
dy = zeros(Nx, Ny);
A = [3, -2;
     2, -2];
for i = 1:Nx
    for j = 1:Ny
        xcoord(i,j) = x(i);
        ycoord(i,j) = y(j);
        v = A*[x(i);y(j)];  % get the slope
        s = norm(v);  % length of v
        % make slope vector a unit vector
        dx(i,j) = v(1)/s;
        dy(i,j) = v(2)/s;
    end
end
% plot the slope vectors at (xcoord, ycoord)
quiver(xcoord, ycoord, dx, dy, 0.5, 'r')
xline(0);  % show x and y axes
yline(0);
axis([Xmin, Xmax, Ymin, Ymax])
xlabel 'x1', ylabel 'x2'
title 'Direction Field for X'' = AX'
```

(b) Using MATLAB,

```
clear
A = [3, -2;
     2, -2];
[V,D] = eig(A);
ev1 = D(1,1)   % eigenvalues
ev2 = D(2,2)
% normalized eigenvectors
if V(1,1) ~= 0
    evec1 = V(:,1)/V(1,1)
else
    evec1 = V(:,1)/V(2,1)
end
if V(1,2) ~= 0
    evec2 = V(:,2)/V(1,2)
else
    evec2 = V(:,2)/V(2,2)
end
```

```
ev1 = 2
ev2 = -1
evec1 = 2×1
        1.0000
        0.5000

evec2 = 2×1
        1.0000
        2.0000
```

$\therefore$ Eigenvalues : $\lambda = 2, -1$

Corresponding eigenvectors: $\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

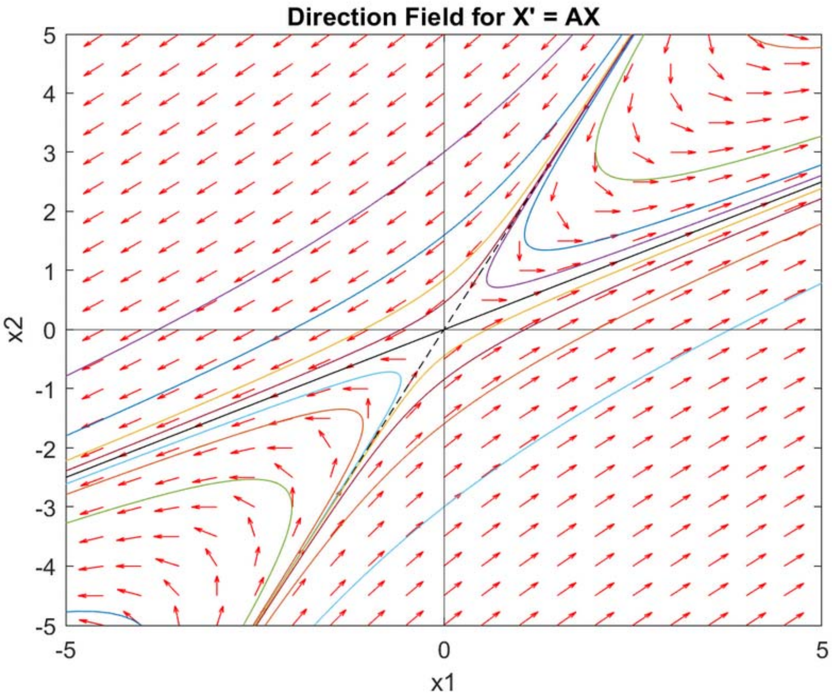
$\lambda_1 \neq \lambda_2 \Rightarrow$ eigenvectors independent

$$\therefore \boxed{\vec{x}(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t}}$$

As $t \to \infty$, $\vec{x}(t)$ approaches the extension to $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, or asymptotic line $x_2 = \frac{1}{2} x_1$.

(c) Using MATLAB, adding to code in (a), and shown on next page.

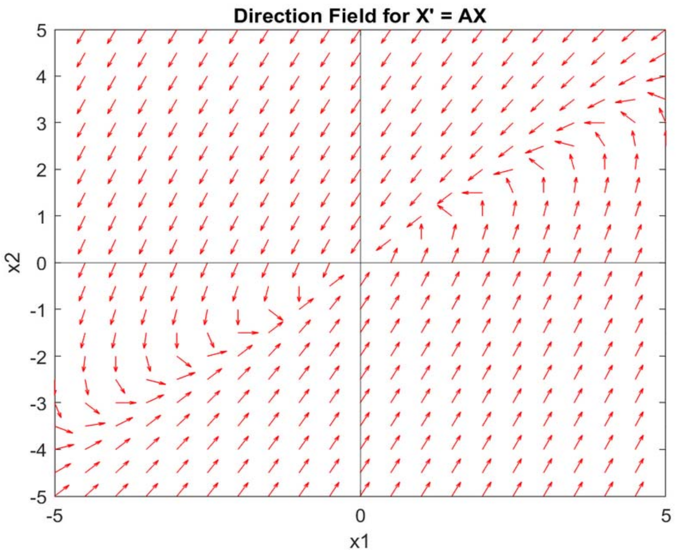
```
[V,D] = eig(A);
ev1 = D(1,1)   % eigenvalues
ev2 = D(2,2)
if V(1,1) ~= 0  % normalized eigenvectors
    evec1 = V(:,1)/V(1,1)
else
    evec1 = V(:,1)/V(2,1)
end
if V(1,2) ~= 0
    evec2 = V(:,2)/V(1,2)
else
    evec2 = V(:,2)/V(2,2)
end

hold on   %  plot phase portrait trajectories
m1 = evec1(2)/evec1(1)
m2 = evec2(2)/evec2(1)
plot(x,m1*x, 'k')      % asymptotic lines
plot(x,m2*x, '--k')
t = -2.5:0.05:2.5;
x1 = evec1*exp(ev1*t);
x2 = evec2*exp(ev2*t);
for c1 = [-2,-0.3,0.3,2]
    for c2 = [-2,-0.3,0.3,2]
        p = c1*x1 + c2*x2;
        u = p(1,:);
        v = p(2,:);
        plot(u,v)
    end
end
```



2.

(a)

Using MATLAB,




```
clear
% set plot boundaries
Xmin = -5; Xmax = 5;
Ymin = -5; Ymax = 5;
% set how fine to make the grid of values
Step = 0.5;
% total # points on x and y axes
Nx = round((Xmax-Xmin)/Step, 0) + 1;
Ny = round((Ymax-Ymin)/Step, 0) + 1;
% set the actual x and y coords
x = linspace(Xmin, Xmax, Nx);  % a row vector
y = linspace(Ymin, Ymax, Ny);  % a row vector
% preallocate memory for coord, slope arrays
%  to be used in quiver, must be same dimension
xcoord = zeros(Nx, Ny);
ycoord = zeros(Nx, Ny);
dx = zeros(Nx, Ny);
dy = zeros(Nx, Ny);
A = [1, -2;
     3, -4];
for i = 1:Nx
    for j = 1:Ny
        xcoord(i,j) = x(i);
        ycoord(i,j) = y(j);
        v = A*[x(i);y(j)];  % get the slope
        s = norm(v);  % length of v
        % make slope vector a unit vector
        dx(i,j) = v(1)/s;
        dy(i,j) = v(2)/s;
    end
end
% plot the slope vectors at (xcoord, ycoord)
quiver(xcoord, ycoord, dx, dy, 0.5, 'r')
xline(0);  % show x and y axes
yline(0);
axis([Xmin, Xmax, Ymin, Ymax])
xlabel 'x1', ylabel 'x2'
title 'Direction Field for X'' = AX'
```

(b)

Using MATLAB,

```
ev1 = -1.0000
ev2 = -2
evec1 = 2x1
        1.0000
        1.0000

evec2 = 2x1
        1.0000
        1.5000
```

```
clear
A = [1, -2;
     3, -4];
[V,D] = eig(A);
ev1 = D(1,1)   % eigenvalues
ev2 = D(2,2)
% normalized eigenvectors
if V(1,1) ~= 0
    evec1 = V(:,1)/V(1,1)
else
    evec1 = V(:,1)/V(2,1)
end
if V(1,2) ~= 0
    evec2 = V(:,2)/V(1,2)
else
    evec2 = V(:,2)/V(2,2)
end
```

$\therefore$ Eigenvalues: $\lambda_1 = -1$, $\lambda_2 = -2$

Corresponding eigenvectors: $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

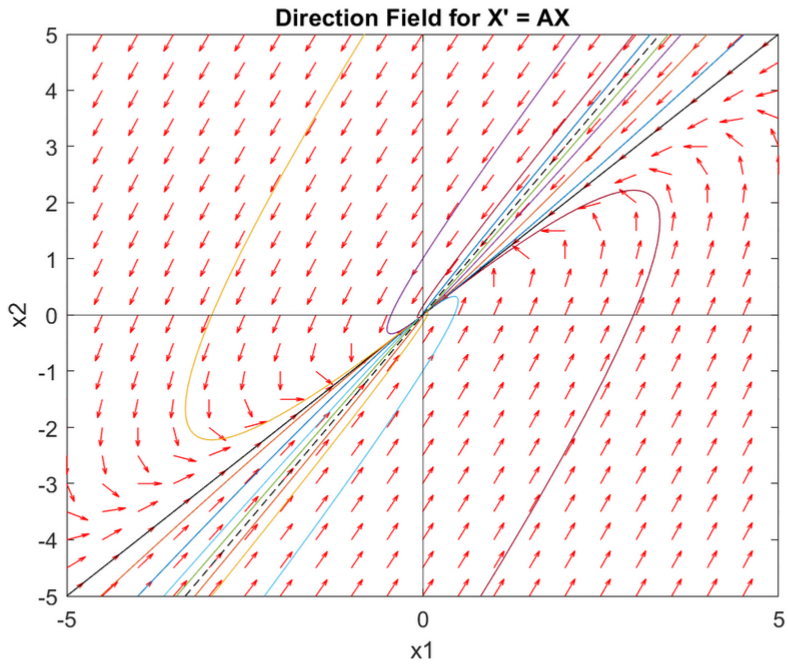
$\lambda_1 \neq \lambda_2 \Rightarrow$ eigenvectors independent.

$$\therefore \boxed{\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} e^{-2t}}$$

As $t \to \infty$, $\vec{x}(t)$ approaches the origin: $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

(c)

Using MATLAB, code on next page, an extension of code in (a)

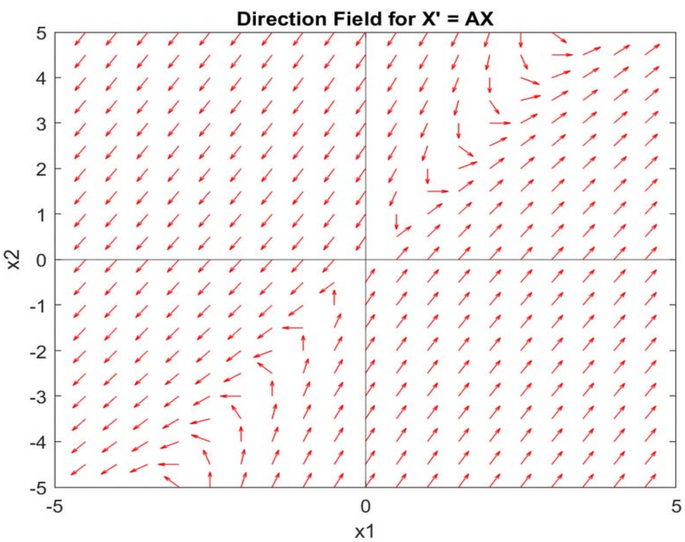


```
clear
% set plot boundaries
Xmin = -5; Xmax = 5;
Ymin = -5; Ymax = 5;
% set how fine to make the grid of values
Step = 0.5;
% total # points on x and y axes
Nx = round((Xmax-Xmin)/Step, 0) + 1;
Ny = round((Ymax-Ymin)/Step, 0) + 1;
% set the actual x and y coords
x = linspace(Xmin, Xmax, Nx);  % a row vector
y = linspace(Ymin, Ymax, Ny);  % a row vector
% preallocate memory for coord, slope arrays
%  to be used in quiver, must be same dimension
xcoord = zeros(Nx, Ny);
ycoord = zeros(Nx, Ny);
dx = zeros(Nx, Ny);
dy = zeros(Nx, Ny);
A = [1, -2;
     3, -4];
for i = 1:Nx
    for j = 1:Ny
        xcoord(i,j) = x(i);
        ycoord(i,j) = y(j);
        v = A*[x(i);y(j)];  % get the slope
        s = norm(v);  % length of v
        % make slope vector a unit vector
        dx(i,j) = v(1)/s;
        dy(i,j) = v(2)/s;
    end
end
% plot the slope vectors at (xcoord, ycoord)
quiver(xcoord, ycoord, dx, dy, 0.5, 'r')
xline(0);  % show x and y axes
yline(0);
axis([Xmin, Xmax, Ymin, Ymax])
xlabel 'x1', ylabel 'x2'
title 'Direction Field for X'' = AX'

[V,D] = eig(A);
ev1 = D(1,1)   % eigenvalues
ev2 = D(2,2)
if V(1,1) ~= 0  % normalized eigenvectors
    evec1 = V(:,1)/V(1,1)
else
    evec1 = V(:,1)/V(2,1)
end
if V(1,2) ~= 0
    evec2 = V(:,2)/V(1,2)
else
    evec2 = V(:,2)/V(2,2)
end

hold on   %  plot phase portrait trajectories
m1 = evec1(2)/evec1(1)
m2 = evec2(2)/evec2(1)
plot(x,m1*x, 'k')     % asymptotic lines
plot(x,m2*x, '--k')
t = -2.5:0.05:2.5;
x1 = evec1*exp(ev1*t);
x2 = evec2*exp(ev2*t);
for c1 = [-2,-0.3,0.3,2]
    for c2 = [-2,-0.3,0.3,2]
        p = c1*x1 + c2*x2;
        u = p(1,:);
        v = p(2,:);
        plot(u,v)
    end
end
```

3.

(a)

Using MATLAB,




```
clear
% set plot boundaries
Xmin = -5; Xmax = 5;
Ymin = -5; Ymax = 5;
% set how fine to make the grid of values
Step = 0.5;
% total # points on x and y axes
Nx = round((Xmax-Xmin)/Step, 0) + 1;
Ny = round((Ymax-Ymin)/Step, 0) + 1;
% set the actual x and y coords
x = linspace(Xmin, Xmax, Nx);  % a row vector
y = linspace(Ymin, Ymax, Ny);  % a row vector
% preallocate memory for coord, slope arrays
%  to be used in quiver, must be same dimension
xcoord = zeros(Nx, Ny);
ycoord = zeros(Nx, Ny);
dx = zeros(Nx, Ny);
dy = zeros(Nx, Ny);
A = [2, -1;
     3, -2];
for i = 1:Nx
    for j = 1:Ny
        xcoord(i,j) = x(i);
        ycoord(i,j) = y(j);
        v = A*[x(i);y(j)];  % get the slope
        s = norm(v);  % length of v
        % make slope vector a unit vector
        dx(i,j) = v(1)/s;
        dy(i,j) = v(2)/s;
    end
end
% plot the slope vectors at (xcoord, ycoord)
quiver(xcoord, ycoord, dx, dy, 0.5, 'r')
xline(0);  % show x and y axes
yline(0);
axis([Xmin, Xmax, Ymin, Ymax])
xlabel 'x1', ylabel 'x2'
title 'Direction Field for X'' = AX'
```

(b)

Using MATLAB,

```
ev1 = 1
ev2 = -1
evec1 = 2×1
     1
     1

evec2 = 2×1
    1.0000
    3.0000
```

```
clear
A = [2, -1;
     3, -2];
[V,D] = eig(A);
ev1 = D(1,1)   % eigenvalues
ev2 = D(2,2)
% normalized eigenvectors
if V(1,1) ~= 0
    evec1 = V(:,1)/V(1,1)
else
    evec1 = V(:,1)/V(2,1)
end
if V(1,2) ~= 0
    evec2 = V(:,2)/V(1,2)
else
    evec2 = V(:,2)/V(2,2)
end
```

Eigenvalues: $\lambda_1 = 1$, $\lambda_2 = -1$

Corresponding eigenvectors: $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

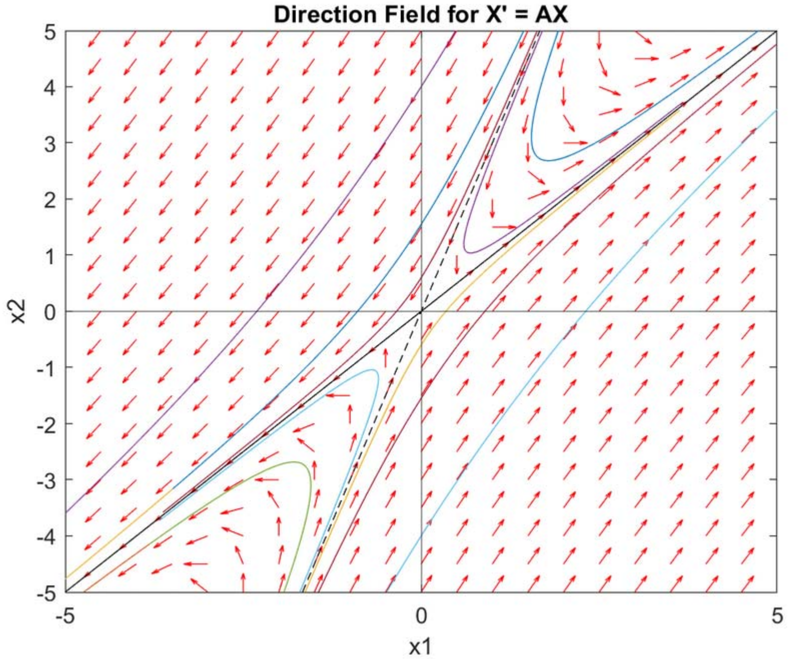
$\lambda_1 \neq \lambda_2 \Rightarrow$ eigenvectors independent

$$\therefore \boxed{\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-t}}$$

As $t \to \infty$, $\vec{x}(t)$ approaches asymptote $x_2 = x_1$

(c)

Using MATLAB, code on next page, an extension to code in (a).

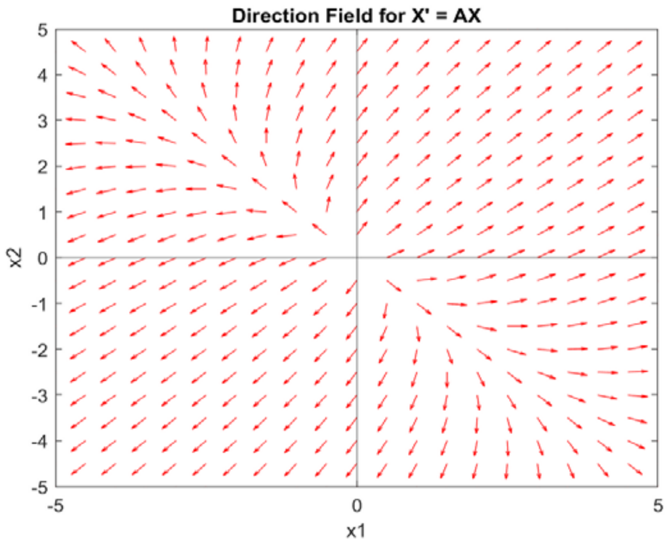


```
clear
% set plot boundaries
Xmin = -5; Xmax = 5;
Ymin = -5; Ymax = 5;
% set how fine to make the grid of values
Step = 0.5;
% total # points on x and y axes
Nx = round((Xmax-Xmin)/Step, 0) + 1;
Ny = round((Ymax-Ymin)/Step, 0) + 1;
% set the actual x and y coords
x = linspace(Xmin, Xmax, Nx);  % a row vector
y = linspace(Ymin, Ymax, Ny);  % a row vector
% preallocate memory for coord, slope arrays
%  to be used in quiver, must be same dimension
xcoord = zeros(Nx, Ny);
ycoord = zeros(Nx, Ny);
dx = zeros(Nx, Ny);
dy = zeros(Nx, Ny);
A = [2, -1;
     3, -2];
for i = 1:Nx
    for j = 1:Ny
        xcoord(i,j) = x(i);
        ycoord(i,j) = y(j);
        v = A*[x(i);y(j)];  % get the slope
        s = norm(v);  % length of v
        % make slope vector a unit vector
        dx(i,j) = v(1)/s;
        dy(i,j) = v(2)/s;
    end
end
% plot the slope vectors at (xcoord, ycoord)
quiver(xcoord, ycoord, dx, dy, 0.5, 'r')
xline(0);  % show x and y axes
yline(0);
axis([Xmin, Xmax, Ymin, Ymax])
xlabel 'x1', ylabel 'x2'
title 'Direction Field for X'' = AX'

[V,D] = eig(A);
ev1 = D(1,1)    % eigenvalues
ev2 = D(2,2)
if V(1,1) ~= 0  % normalized eigenvectors
    evec1 = V(:,1)/V(1,1)
else
    evec1 = V(:,1)/V(2,1)
end
if V(1,2) ~= 0
    evec2 = V(:,2)/V(1,2)
else
    evec2 = V(:,2)/V(2,2)
end

hold on   %  plot phase portrait trajectories
m1 = evec1(2)/evec1(1)
m2 = evec2(2)/evec2(1)
plot(x,m1*x, 'k')     % asymptotic lines
plot(x,m2*x, '--k')
t = -2.5:0.05:2.5;
x1 = evec1*exp(ev1*t);
x2 = evec2*exp(ev2*t);
for c1 = [-2,-0.3,0.3,2]
    for c2 = [-2,-0.3,0.3,2]
        p = c1*x1 + c2*x2;
        u = p(1,:);
        v = p(2,:);
        plot(u,v)
    end
end
```

4.

(a)

Using MATLAB,




```
clear
% set plot boundaries
Xmin = -5; Xmax = 5;
Ymin = -5; Ymax = 5;
% set how fine to make the grid of values
Step = 0.5;
% total # points on x and y axes
Nx = round((Xmax-Xmin)/Step, 0) + 1;
Ny = round((Ymax-Ymin)/Step, 0) + 1;
% set the actual x and y coords
x = linspace(Xmin, Xmax, Nx);  % a row vector
y = linspace(Ymin, Ymax, Ny);  % a row vector
% preallocate memory for coord, slope arrays
%  to be used in quiver, must be same dimension
xcoord = zeros(Nx, Ny);
ycoord = zeros(Nx, Ny);
dx = zeros(Nx, Ny);
dy = zeros(Nx, Ny);
A = [5/4, 3/4;
     3/4, 5/4];
for i = 1:Nx
    for j = 1:Ny
        xcoord(i,j) = x(i);
        ycoord(i,j) = y(j);
        v = A*[x(i);y(j)];  % get the slope
        s = norm(v);  % length of v
        % make slope vector a unit vector
        dx(i,j) = v(1)/s;
        dy(i,j) = v(2)/s;
    end
end
% plot the slope vectors at (xcoord, ycoord)
quiver(xcoord, ycoord, dx, dy, 0.5, 'r')
xline(0);  % show x and y axes
yline(0);
axis([Xmin, Xmax, Ymin, Ymax])
xlabel 'x1', ylabel 'x2'
title 'Direction Field for X'' = AX'
```

(b)

Using MATLAB,

```
clear
A = [5/4, 3/4;
     3/4, 5/4];
[V,D] = eig(A);
ev1 = D(1,1)   % eigenvalues
ev2 = D(2,2)
% normalized eigenvectors
if V(1,1) ~= 0
    evec1 = V(:,1)/V(1,1)
else
    evec1 = V(:,1)/V(2,1)
end
if V(1,2) ~= 0
    evec2 = V(:,2)/V(1,2)
else
    evec2 = V(:,2)/V(2,2)
end
```

```
ev1 = 0.5000
ev2 = 2
evec1 = 2×1
      1
     -1

evec2 = 2×1
      1
      1
```

$\therefore$ Eigenvalues: $\lambda_1 = \frac{1}{2}$, $\lambda_2 = 2$

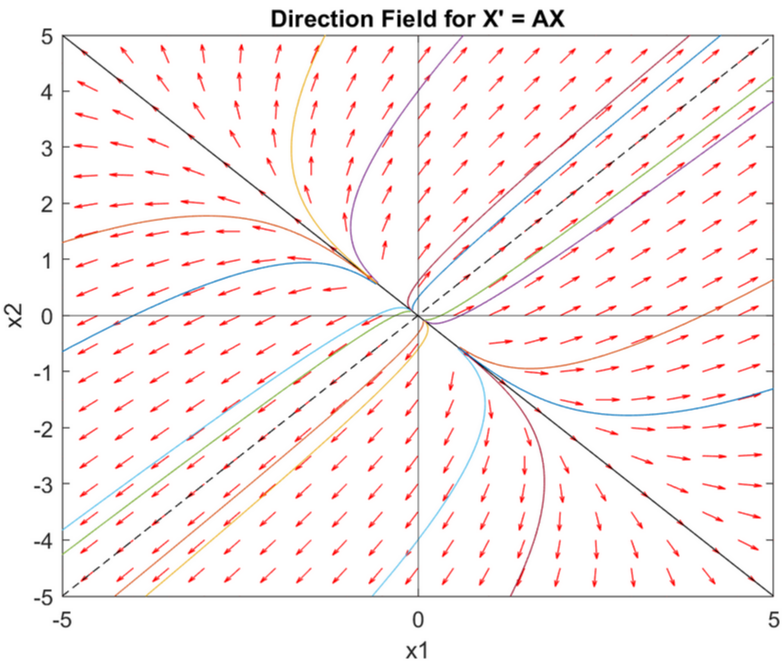
Corresponding eigenvectors: $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_1 \neq \lambda_2 \Rightarrow$ eigenvectors independent.

$$\therefore \boxed{\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{t/2} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}}$$

(c)

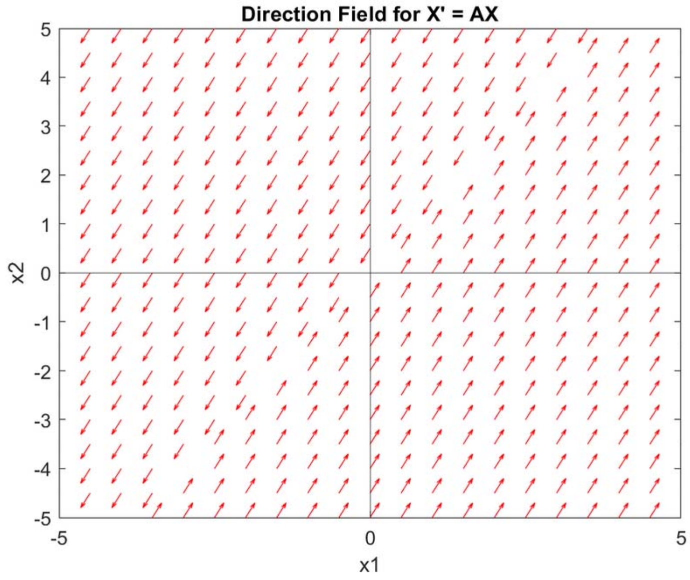
Using MATLAB, code on next page, an extension to code in (a).



```matlab
clear
% set plot boundaries
Xmin = -5; Xmax = 5;
Ymin = -5; Ymax = 5;
% set how fine to make the grid of values
Step = 0.5;
% total # points on x and y axes
Nx = round((Xmax-Xmin)/Step, 0) + 1;
Ny = round((Ymax-Ymin)/Step, 0) + 1;
% set the actual x and y coords
x = linspace(Xmin, Xmax, Nx);  % a row vector
y = linspace(Ymin, Ymax, Ny);  % a row vector
% preallocate memory for coord, slope arrays
%  to be used in quiver, must be same dimension
xcoord = zeros(Nx, Ny);
ycoord = zeros(Nx, Ny);
dx = zeros(Nx, Ny);
dy = zeros(Nx, Ny);
A = [5/4, 3/4;
     3/4, 5/4];
for i = 1:Nx
    for j = 1:Ny
        xcoord(i,j) = x(i);
        ycoord(i,j) = y(j);
        v = A*[x(i);y(j)];  % get the slope
        s = norm(v);  % length of v
        % make slope vector a unit vector
        dx(i,j) = v(1)/s;
        dy(i,j) = v(2)/s;
    end
end
% plot the slope vectors at (xcoord, ycoord)
quiver(xcoord, ycoord, dx, dy, 0.5, 'r')
xline(0);  % show x and y axes
yline(0);
axis([Xmin, Xmax, Ymin, Ymax])
xlabel 'x1', ylabel 'x2'
title 'Direction Field for X'' = AX'

[V,D] = eig(A);
ev1 = D(1,1)    % eigenvalues
ev2 = D(2,2)
if V(1,1) ~= 0  % normalized eigenvectors
    evec1 = V(:,1)/V(1,1)
else
    evec1 = V(:,1)/V(2,1)
end
if V(1,2) ~= 0
    evec2 = V(:,2)/V(1,2)
else
    evec2 = V(:,2)/V(2,2)
end

hold on   %  plot phase portrait trajectories
m1 = evec1(2)/evec1(1)
m2 = evec2(2)/evec2(1)
plot(x,m1*x, 'k')      % asymptotic lines
plot(x,m2*x, '--k')
t = -2.5:0.05:2.5;
x1 = evec1*exp(ev1*t);
x2 = evec2*exp(ev2*t);
for c1 = [-2,-0.3,0.3,2]
    for c2 = [-2,-0.3,0.3,2]
        p = c1*x1 + c2*x2;
        u = p(1,:);
        v = p(2,:);
        plot(u,v)
    end
end
```

5.

(a) Using MATLAB,




```
clear
% set plot boundaries
Xmin = -5; Xmax = 5;
Ymin = -5; Ymax = 5;
% set how fine to make the grid of values
Step = 0.5;
% total # points on x and y axes
Nx = round((Xmax-Xmin)/Step, 0) + 1;
Ny = round((Ymax-Ymin)/Step, 0) + 1;
% set the actual x and y coords
x = linspace(Xmin, Xmax, Nx);  % a row vector
y = linspace(Ymin, Ymax, Ny);  % a row vector
% preallocate memory for coord, slope arrays
%  to be used in quiver, must be same dimension
xcoord = zeros(Nx, Ny);
ycoord = zeros(Nx, Ny);
dx = zeros(Nx, Ny);
dy = zeros(Nx, Ny);
A = [4, -3;
     8, -6];
for i = 1:Nx
    for j = 1:Ny
        xcoord(i,j) = x(i);
        ycoord(i,j) = y(j);
        v = A*[x(i);y(j)];  % get the slope
        s = norm(v);  % length of v
        % make slope vector a unit vector
        dx(i,j) = v(1)/s;
        dy(i,j) = v(2)/s;
    end
end
% plot the slope vectors at (xcoord, ycoord)
quiver(xcoord, ycoord, dx, dy, 0.5, 'r')
xline(0);  % show x and y axes
yline(0);
axis([Xmin, Xmax, Ymin, Ymax])
xlabel 'x1', ylabel 'x2'
title 'Direction Field for X'' = AX'
```

(b) Using MATLAB,

```
clear
A = [4, -3;
     8, -6];
[V,D] = eig(A);
ev1 = D(1,1)   % eigenvalues
ev2 = D(2,2)
% normalized eigenvectors
if V(1,1) ~= 0
    evec1 = V(:,1)/V(1,1)
else
    evec1 = V(:,1)/V(2,1)
end
if V(1,2) ~= 0
    evec2 = V(:,2)/V(1,2)
else
    evec2 = V(:,2)/V(2,2)
end
```

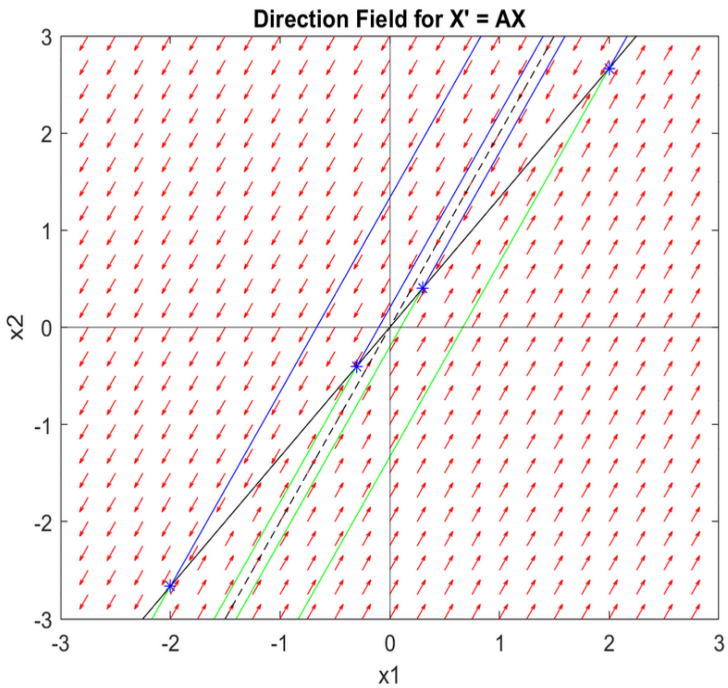
```
ev1 = 0
ev2 = -2
evec1 = 2×1
      1.0000
      1.3333

evec2 = 2×1
      1
      2
```

$\therefore$ Eigenvalues: $\lambda_1 = 0$, $\lambda_2 = -2$

Corresponding eigenvectors: $\begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$\therefore \boxed{\vec{x}(t) = c_1 \begin{bmatrix} 3 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-2t}}$$

(c) Using MATLAB, code on next page, an extension to code in (a), except changed boundaries to [-3, 3] for x, y axes, changed step to 0.25, and color-coded based on value of c_2.



For $c_2 = 0$, values stay on line $x_2 = \frac{4}{3}x_1$ (the '*' points)

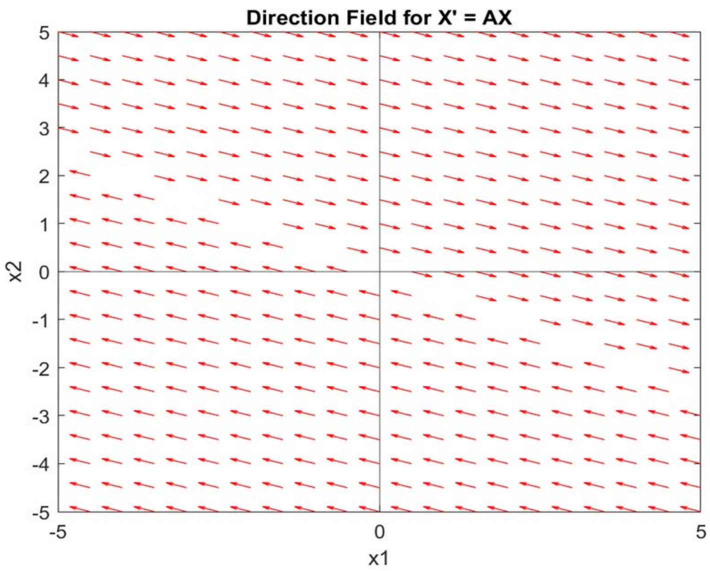
For $c_2 \neq 0$, solution values are on a line parallel to $x_2 = 2x_1$, and approach the line $x_2 = \frac{4}{3}x_1$

```
clear
% set plot boundaries
Xmin = -3; Xmax = 3;
Ymin = -3; Ymax = 3;
% set how fine to make the grid of values
Step = 0.25;
% total # points on x and y axes
Nx = round((Xmax-Xmin)/Step, 0) + 1;
Ny = round((Ymax-Ymin)/Step, 0) + 1;
% set the actual x and y coords
x = linspace(Xmin, Xmax, Nx);  % a row vector
y = linspace(Ymin, Ymax, Ny);  % a row vector
% preallocate memory for coord, slope arrays
%  to be used in quiver, must be same dimension
xcoord = zeros(Nx, Ny);
ycoord = zeros(Nx, Ny);
dx = zeros(Nx, Ny);
dy = zeros(Nx, Ny);
A = [4, -3;
     8, -6];
for i = 1:Nx
    for j = 1:Ny
        xcoord(i,j) = x(i);
        ycoord(i,j) = y(j);
        v = A*[x(i);y(j)];  % get the slope
        s = norm(v);  % length of v
        % make slope vector a unit vector
        dx(i,j) = v(1)/s;
        dy(i,j) = v(2)/s;
    end
end
% plot the slope vectors at (xcoord, ycoord)
quiver(xcoord, ycoord, dx, dy, 0.5, 'r')
xline(0);  % show x and y axes
yline(0);
axis([Xmin, Xmax, Ymin, Ymax])
xlabel 'x1', ylabel 'x2'
title 'Direction Field for X'' = AX'

[V,D] = eig(A);
ev1 = D(1,1)   % eigenvalues
ev2 = D(2,2)
if V(1,1) ~= 0  % normalized eigenvectors
    evec1 = V(:,1)/V(1,1)
else
    evec1 = V(:,1)/V(2,1)
end
if V(1,2) ~= 0
    evec2 = V(:,2)/V(1,2)
else
    evec2 = V(:,2)/V(2,2)
end

hold on   %  plot phase portrait trajectories
m1 = evec1(2)/evec1(1)
m2 = evec2(2)/evec2(1)
plot(x,m1*x, 'k')     % asymptotic lines
plot(x,m2*x, '--k')
t = -2.5:0.05:2.5;
x1 = evec1*exp(ev1*t);
x2 = evec2*exp(ev2*t);
for c1 = [-2, -0.3, 0.3, 2]
    for c2 = [-2,0,2]
        p = c1*x1 + c2*x2;
        u = p(1,:);
        v = p(2,:);
        if c2 > 0
            plot(u,v,'b')  % blue
        elseif c2 < 0
            plot(u,v,'g')  % green
        else
            plot(u,v,'*b')
        end
    end
end
```

6.

(a)

Using MATLAB,




```
clear
% set plot boundaries
Xmin = -5; Xmax = 5;
Ymin = -5; Ymax = 5;
% set how fine to make the grid of values
Step = 0.5;
% total # points on x and y axes
Nx = round((Xmax-Xmin)/Step, 0) + 1;
Ny = round((Ymax-Ymin)/Step, 0) + 1;
% set the actual x and y coords
x = linspace(Xmin, Xmax, Nx);  % a row vector
y = linspace(Ymin, Ymax, Ny);  % a row vector
% preallocate memory for coord, slope arrays
%  to be used in quiver, must be same dimension
xcoord = zeros(Nx, Ny);
ycoord = zeros(Nx, Ny);
dx = zeros(Nx, Ny);
dy = zeros(Nx, Ny);
A = [3, 6;
     -1, -2];
for i = 1:Nx
    for j = 1:Ny
        xcoord(i,j) = x(i);
        ycoord(i,j) = y(j);
        v = A*[x(i);y(j)];  % get the slope
        s = norm(v);  % length of v
        % make slope vector a unit vector
        dx(i,j) = v(1)/s;
        dy(i,j) = v(2)/s;
    end
end
% plot the slope vectors at (xcoord, ycoord)
quiver(xcoord, ycoord, dx, dy, 0.5, 'r')
xline(0);  % show x and y axes
yline(0);
axis([Xmin, Xmax, Ymin, Ymax])
xlabel 'x1', ylabel 'x2'
title 'Direction Field for X'' = AX'
```

(b)

Using MATLAB,

```
clear
A = [3, 6;
     -1, -2];
[V,D] = eig(A);
ev1 = D(1,1)   % eigenvalues
ev2 = D(2,2)
% normalized eigenvectors
if V(1,1) ~= 0
    evec1 = V(:,1)/V(1,1)
else
    evec1 = V(:,1)/V(2,1)
end
if V(1,2) ~= 0
    evec2 = V(:,2)/V(1,2)
else
    evec2 = V(:,2)/V(2,2)
end
```

```
ev1 = 1.0000
ev2 = -2.2204e-16
evec1 = 2×1
     1.0000
    -0.3333

evec2 = 2×1
     1.0000
    -0.5000
```

$\therefore$ Eigenvalues: $\lambda_1 = 1$, $\lambda_2 = 0$

Manually: $\det \begin{bmatrix} 3-\lambda & 6 \\ -1 & -2-\lambda \end{bmatrix} = 0$

$$\iff \underbrace{(3-\lambda)}^{\lambda-3}\underbrace{(-2-\lambda)}^{\lambda+2} - (-6) = 0$$

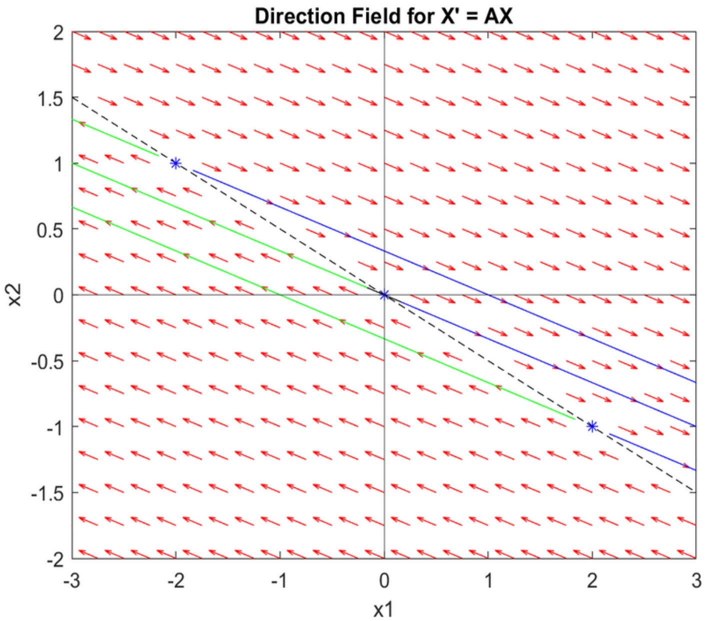
$$\iff \lambda^2 - \lambda = 0 \iff \lambda(\lambda - 1) = 0 \iff \lambda = 0, 1$$

Corresponding Eigenvectors: $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$$\therefore \boxed{\vec{x}(t) = c_1 \begin{bmatrix} 3 \\ -1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 2 \\ -1 \end{bmatrix}}$$

(c)

Using MATLAB, as in #5(c), altering code to account for c_1, c_2 differences, and y-axis [-2, 2]



For $c_1 = 0$, values stay on $x_2 = -\frac{1}{2}x_1$ line ('*').

For $c_1 \neq 0$, solution values move on line parallel to evec1: $x_2 = -\frac{1}{3}x_1$, and away from the $x_2 = -\frac{1}{2}x_1$ line.

Code for

previous page

plot of some

trajectories.

```
clear
% set plot boundaries
Xmin = -3; Xmax = 3;
Ymin = -2; Ymax = 2;
% set how fine to make the grid of values
Step = 0.25;
% total # points on x and y axes
Nx = round((Xmax-Xmin)/Step, 0) + 1;
Ny = round((Ymax-Ymin)/Step, 0) + 1;
% set the actual x and y coords
x = linspace(Xmin, Xmax, Nx);  % a row vector
y = linspace(Ymin, Ymax, Ny);  % a row vector
% preallocate memory for coord, slope arrays
%  to be used in quiver, must be same dimension
xcoord = zeros(Nx, Ny);
ycoord = zeros(Nx, Ny);
dx = zeros(Nx, Ny);
dy = zeros(Nx, Ny);
A = [3, 6;
     -1, -2];
for i = 1:Nx
    for j = 1:Ny
        xcoord(i,j) = x(i);
        ycoord(i,j) = y(j);
        v = A*[x(i);y(j)];  % get the slope
        s = norm(v);  % length of v
        % make slope vector a unit vector
        dx(i,j) = v(1)/s;
        dy(i,j) = v(2)/s;
    end
end
% plot the slope vectors at (xcoord, ycoord)
quiver(xcoord, ycoord, dx, dy, 0.5, 'r')
xline(0);  % show x and y axes
yline(0);
axis([Xmin, Xmax, Ymin, Ymax])
xlabel 'x1', ylabel 'x2'
title 'Direction Field for X'' = AX'

[V,D] = eig(A);
ev1 = D(1,1)   % eigenvalues
ev2 = D(2,2)
if V(1,1) ~= 0  % normalized eigenvectors
    evec1 = V(:,1)/V(1,1)
else
    evec1 = V(:,1)/V(2,1)
end
if V(1,2) ~= 0
    evec2 = V(:,2)/V(1,2)
else
    evec2 = V(:,2)/V(2,2)
end

hold on   %  plot phase portrait trajectories
m1 = evec1(2)/evec1(1)
m2 = evec2(2)/evec2(1)
plot(x,m1*x, 'k')     % asymptotic lines
plot(x,m2*x, '--k')
t = -2.5:0.05:2.5;
x1 = evec1*exp(ev1*t);
x2 = evec2*exp(ev2*t);
for c1 = [-2, 0, 2]
    for c2 = [-2,0,2]
        p = c1*x1 + c2*x2;
        u = p(1,:);
        v = p(2,:);
        if c1 > 0
            plot(u,v,'b')  % blue
        elseif c1 < 0
            plot(u,v,'g')  % green
        else
            plot(u,v,'*b')
        end
    end
end
```

7.

Using MATLAB,

```
clear, clc
n = 3;  % size of system
A = [1, 1, 2;
     1, 2, 1;
     2, 1, 1];
[V,D] = eig(A);  % eigenvectors, values
for i =1:n
    D(i,i)   % display eigenvalues
    rref(A - D(i,i)*eye(n))
end
```

```
ans = -1.0000
 ans = 3×3
      1.0000         0    1.0000
           0    1.0000    0.0000
           0         0         0
```

$A-\lambda I$

```
ans = 1.0000
 ans = 3×3
      1.0000         0   -1.0000
           0    1.0000    2.0000
           0         0         0
```

$A-\lambda I$

```
ans = 4.0000
 ans = 3×3
      1.0000         0   -1.0000
           0    1.0000   -1.0000
           0         0         0
```

$A-\lambda I$

$\therefore$ Eigenvalues: $\lambda_1 = -1$, $\lambda_2 = 1$, $\lambda_3 = 4$

Corresponding eigenvectors from the row reduced echelon form of $A-\lambda I$: $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$\therefore \quad \boxed{\vec{x}(t) = c_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} e^{t} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{4t}}$$

8.

Using MATLAB,

```
clear, clc
n = 3;  % size of system
A = [3, 2, 4;
     2, 0, 2;
     4, 2, 3];
[V,D] = eig(A);  % eigenvectors, values
for i =1:n
    D(i,i)   % display eigenvalues
    rref(A - D(i,i)*eye(n))
end
```

```
ans = -1.0000
 ans = 3×3
      1.0000    0.5000    1.0000
           0         0         0
           0         0         0
```

$A-\lambda I$

```
ans = -1.0000
 ans = 3×3
      1.0000    0.5000    1.0000
           0         0         0
           0         0         0
```

$A-\lambda I$

```
ans = 8
 ans = 3×3
      1.0000         0   -1.0000
           0    1.0000   -0.5000
           0         0         0
```

$A-\lambda I$

$\therefore$ Eigenvalues : $\lambda_1 = -1$, $\lambda_2 = -1$, $\lambda_3 = 8$

Corresponding eigenvectors from row reduced echelon form of $A-\lambda I$: $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$

$$\therefore \boxed{\vec{x}(t) = c_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} e^{-t} + c_3 \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} e^{8t}}$$

9.

Using MATLAB,

```
clear, clc
n = 3;  % size of system
A = [1, -1, 4;
     3, 2, -1;
     2, 1, -1];
[V,D] = eig(A);  % eigenvectors, values
for i =1:n
    D(i,i)   % display eigenvalues
    rref(A - D(i,i)*eye(n))
end
```

```
ans = 3.0000
ans = 3×3
    1.0000         0   -1.0000
         0    1.0000   -2.0000
         0         0         0
```

$A-\lambda I$

```
ans = -2.0000
ans = 3×3
    1.0000         0    1.0000
         0    1.0000   -1.0000
         0         0         0
```

$A-\lambda I$

```
ans = 1
ans = 3×3
     1     0     1
     0     1    -4
     0     0     0
```

$A-\lambda I$

$\therefore$ Eigenvalues: $\lambda_1 = 3$, $\lambda_2 = -2$, $\lambda_3 = 1$

Corresponding eigenvectors from the row reduced echelon form of $A-\lambda I$: $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}$

$\therefore$

$$\boxed{\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} e^{-2t} + c_3 \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} e^{t}}$$

10.

Using MATLAB,

```
clear, clc
n = 2;  % size of system
A = [5, -1;
     3, 1];
[V,D] = eig(A);  % eigenvectors, values
for i =1:n
    D(i,i)   % display eigenvalues
    rref(A - D(i,i)*eye(n))
end
```

```
ans = 4
ans = 2×2
     1    -1
     0     0

ans = 2
ans = 2×2
    1.0000   -0.3333
         0         0
```

$\therefore$ Eigenvalues: $\lambda_1 = 4$, $\lambda_2 = 2$

Corresponding eigenvectors: $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$$\therefore \vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{2t}$$

$$\therefore \vec{x}(0) = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

From MATLAB,

```
R = [1,1;
     1,3];
X = [2;-1];
linsolve(R,X)
```

```
ans = 2×1
    3.5000
   -1.5000
```

$$\therefore \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 7/2 \\ -3/2 \end{bmatrix}$$

$$\therefore \boxed{\vec{x}(t) = \frac{7}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} - \frac{3}{2} \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{2t}}$$

11.

Using MATLAB,

```
clear, clc
n = 2;  % size of system
A = [-2, 1;
     -5, 4];
[V,D] = eig(A);  % eigenvectors, values
for i =1:n
    D(i,i)   % display eigenvalues
    rref(A - D(i,i)*eye(n))
end
```

```
ans = -1
ans = 2×2
     1    -1
     0     0

ans = 3
ans = 2×2
    1.0000   -0.2000
         0         0
```

$\therefore$ Eigenvalues : $\lambda_1 = -1, \ \lambda_2 = 3$

Corresponding eigenvectors: $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

$$\therefore \vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ 5 \end{bmatrix} e^{3t}$$

$$\therefore \vec{x}(0) = \begin{bmatrix} 1 & 1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

From MATLAB,

```
R = [1,1;
     1,5];
X = [1;3];
linsolve(R,X)
```

```
ans = 2×1
    0.5000
    0.5000
```

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

$$\therefore \vec{x}(t) = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + \frac{1}{2} \begin{bmatrix} 1 \\ 5 \end{bmatrix} e^{3t}$$

12.

Using MATLAB,

```
clear, clc
n = 3;  % size of system
A = [0, 0, -1;
     2, 0, 0;
     -1, 2, 4];
[V,D] = eig(A);  % eigenvectors, values
for i =1:n
    D(i,i)   % display eigenvalues
    rref(A - D(i,i)*eye(n))
end
```

```
ans = -1
ans = 3x3
     1     0    -1
     0     1     2
     0     0     0

ans = 1
ans = 3x3
     1     0     1
     0     1     2
     0     0     0

ans = 4
ans = 3x3
    1.0000         0    0.2500
         0    1.0000    0.1250
         0         0         0
```

$\therefore \lambda_1 = -1, \lambda_2 = 1, \lambda_3 = 4$

Corresponding eigenvectors: $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -8 \end{bmatrix}$

$$\therefore \vec{x}(t) = c_1 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} e^{t} + c_3 \begin{bmatrix} 2 \\ 1 \\ -8 \end{bmatrix} e^{4t}$$

$$\vec{x}(0) = \begin{bmatrix} 1 & 1 & 2 \\ -2 & 2 & 1 \\ 1 & -1 & -8 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \\ 5 \end{bmatrix}$$

From MATLAB,

```
R = [1, 1, 2;
     -2, 2, 1;
     1, -1, -8];
X = [7;5;5];
linsolve(R,X)
```

```
ans = 3x1
     3
     6
    -1
```

$$\therefore \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ -1 \end{bmatrix}$$

$$\therefore \boxed{\vec{x}(t) = 3\begin{bmatrix}1\\-2\\1\end{bmatrix}e^{-t} + 6\begin{bmatrix}1\\2\\-1\end{bmatrix}e^{t} - \begin{bmatrix}2\\1\\-8\end{bmatrix}e^{4t}}$$

13.

Nontrivial $\vec{x}$ means $\vec{x} \neq \vec{0}$, so $t \neq 0$ for $\vec{x} = \vec{\epsilon}t^r$

$\vec{x}' = r\vec{\epsilon}t^{r-1}$, $\therefore t\vec{x}' = r\vec{\epsilon}t^r$

$$\therefore A\vec{x} = t\vec{x}' \iff A\vec{x} = tI\vec{x}'$$

$$\iff A(\vec{\epsilon}t^r) - tI\vec{x}' = \vec{0}$$

$$\iff A(\vec{\epsilon}t^r) - I(t\vec{x}') = \vec{0}$$

$$\iff A(\vec{\epsilon}t^r) - I(r\vec{\epsilon}t^r) = \vec{0}$$

$$\iff A(\vec{\epsilon}t^r) - (rt^r)I\vec{\epsilon} = \vec{0}$$

$$\iff t^r\left[A\vec{\epsilon} - rI\vec{\epsilon}\right] = \vec{0}$$

$$\iff A\vec{\epsilon} - rI\vec{\epsilon} = \vec{0}, \quad \text{since } t \neq 0$$

$$\iff \underline{(A - rI)\vec{\epsilon} = \vec{0}}$$

14.

Assume $\vec{x} = \vec{\epsilon} t^r$. By #13 above, if $\vec{x}$ is a solution, then $(A - rI)\vec{\epsilon} = \vec{0}$. For $\vec{\epsilon}$ to be a nontrivial solution, $\det(A - rI) = 0$

$$\therefore \begin{vmatrix} 2-r & -1 \\ 3 & -2-r \end{vmatrix} = (r+2)(r-2)+3 = r^2 - 1 = 0, \quad r = 1, -1$$

Using MATLAB to solve $(A - rI)\vec{\epsilon} = \vec{0}$,

```
clear, clc
n = 2; % size of system
A = [2, -1;
     3, -2];
for r = [-1, 1]
    r
    rref(A - r*eye(n))
end
```

```
r = -1
ans = 2×2
    1.0000   -0.3333
         0         0

r = 1
ans = 2×2
     1    -1
     0     0
```

$\therefore$ For $r = -1$, $\vec{\epsilon} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} t^{-1}$

For $r = 1$, $\vec{\epsilon} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} t$

$$\therefore \boxed{\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} t^{-1} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} t}$$

c_1, c_2 constants

15.

Let $\vec{x} = \vec{\epsilon} t^r$. As in #14, using MATLAB,

```
clear, clc
syms r
n = 2; % size of system
A = [5, -1;
     3, 1];
s = solve(det(A - r*eye(n))==0);
for i = 1:size(s)
    s(i)
    rref(A - s(i)*eye(n))
end
```

ans = 2

ans =

$$\begin{pmatrix} 1 & -\frac{1}{3} \\ 0 & 0 \end{pmatrix}$$

ans = 4

ans =

$$\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

$\therefore$ For $r=2$, $\vec{\epsilon} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} t^2$

For $r=4$, $\vec{\epsilon} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} t^4$

$\therefore$ $\boxed{\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} t^2 + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} t^4}$ c_1, c_2 constants

16.

Let $\vec{x} = \vec{\epsilon} t^r$. Use MATLAB to solve $(A - rI)\vec{\epsilon} = \vec{0}$,

```
clear, clc
syms r
n = 2; % size of system
A = [4, -3;
     8, -6];
s = solve(det(A - r*eye(n))==0);
for i = 1:size(s)
    s(i)
    rref(A - s(i)*eye(n))
end
```

ans = -2

ans =

$$\begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{pmatrix}$$

ans = 0

ans =

$$\begin{pmatrix} 1 & -\frac{3}{4} \\ 0 & 0 \end{pmatrix}$$

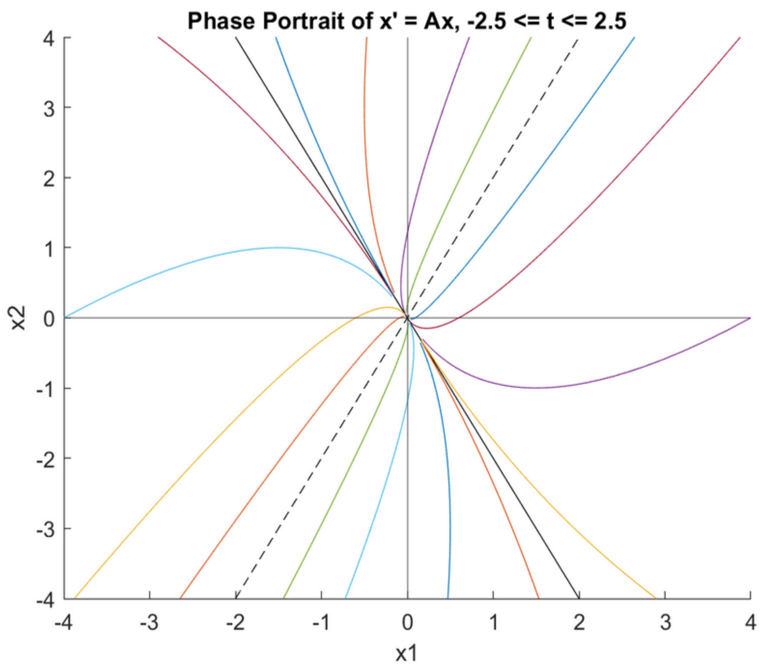
$\therefore$ For $r = -2$, $\vec{\epsilon} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} t^{-2}$

For $r = 0$, $\vec{\epsilon} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} t^{0}$

$$\therefore \boxed{\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} t^{-2} + c_2 \begin{bmatrix} 3 \\ 4 \end{bmatrix}}$$

17.

(a) Using MATLAB, plot $\vec{x}(t) = c_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-2t}$



Code on next page.

Note: $-2.5 \le t \le 2.5$

```
clear, clc
% set plot boundaries
Xmin = -4; Xmax = 4;
Ymin = -4; Ymax = 4;
ev1 = -1;    % eigenvalues
ev2 = -2;
evec1 = [-1; 2];  % eigenvectors
evec2 = [1; 2];
% compute slopes of asymptotic lines
m1 = evec1(2)/evec1(1);
m2 = evec2(2)/evec2(1);
t = -2.5:0.05:2.5;
figure
xlim([Xmin,Xmax])
ylim([Ymin,Ymax])
hold on
xline(0); yline(0);  % axes
plot(t,m1*t, 'k')  % asymptotic lines
plot(t,m2*t, '--k')
s1 = evec1*exp(ev1*t);
s2 = evec2*exp(ev2*t);
% plot trajectories for various c1,c2
for c1 = [-2,-0.3,0.3,2]
    for c2 = [-2,-0.3,0.3,2]
        x = c1*s1 + c2*s2;
        x1 = x(1,:);
        x2 = x(2,:);
        plot(x1,x2)
    end
end
xlabel 'x1', ylabel 'x2'
title 'Phase Portrait of x'' = Ax, -2.5 <= t <= 2.5'
```

Assume "initial" means $t=0$.

$\therefore$ Solve $\begin{bmatrix} 2 \\ 3 \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

Using MATLAB, code on next page,

$$\vec{x}(t) = -\frac{1}{4} \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{-t} + \frac{7}{4} \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-2t}$$

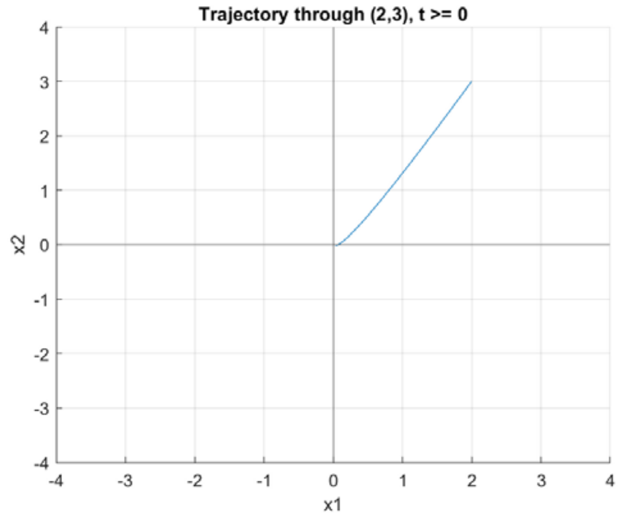
Note, as $t \to \infty$, $\vec{x}(t) \to (0,0)$

```
M = [evec1, evec2];
B = [2; 3];
C = linsolve(M,B)
t = 0:0.05:2.5;  % now just look at t>=0
s1 = evec1*exp(ev1*t);
s2 = evec2*exp(ev2*t);
x = C(1)*s1 + C(2)*s2;
x1 = x(1,:);
x2 = x(2,:);
figure
hold on
grid on
xlim([Xmin,Xmax])
ylim([Ymin,Ymax])
xline(0); yline(0);
plot(x1,x2)
xlabel 'x1', ylabel 'x2'
title 'Trajectory through (2,3), t >= 0'
```

code is a continuation of code in (a)

Note: $t \geq 0$

```
C = 2×1
   -0.2500
    1.7500
```



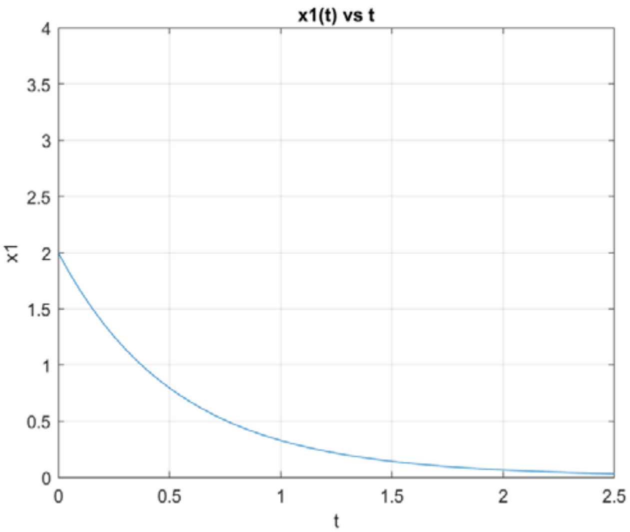
(c)

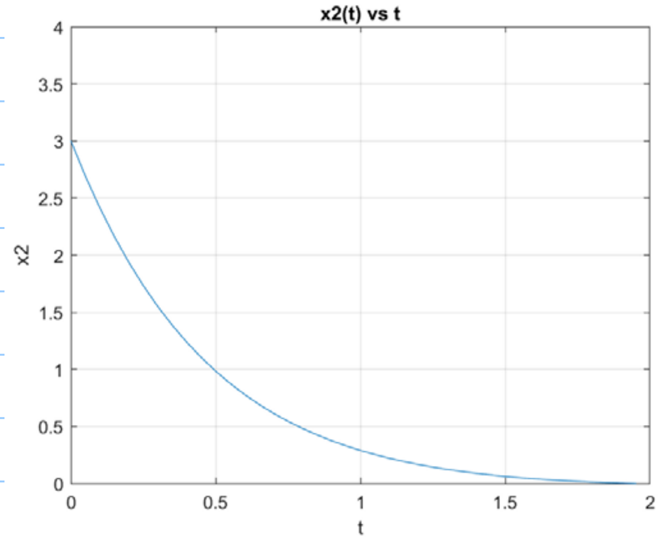
Using MATLAB,

```
figure
plot(t,x1)
ylim([0, 4])
grid on
xlabel 't', ylabel 'x1'
title 'x1(t) vs t'

figure
plot(t,x2)
ylim([0, 4])
grid on
xlabel 't', ylabel 'x2'
title 'x2(t) vs t'
```

Code a continuation of code from (a), (b), plots on next page.

$$x_1(t) = \frac{1}{4}e^{-t} + \frac{7}{4}e^{-2t}$$
$$x_2(t) = -\frac{1}{2}e^{-t} + \frac{7}{2}e^{-2t}$$

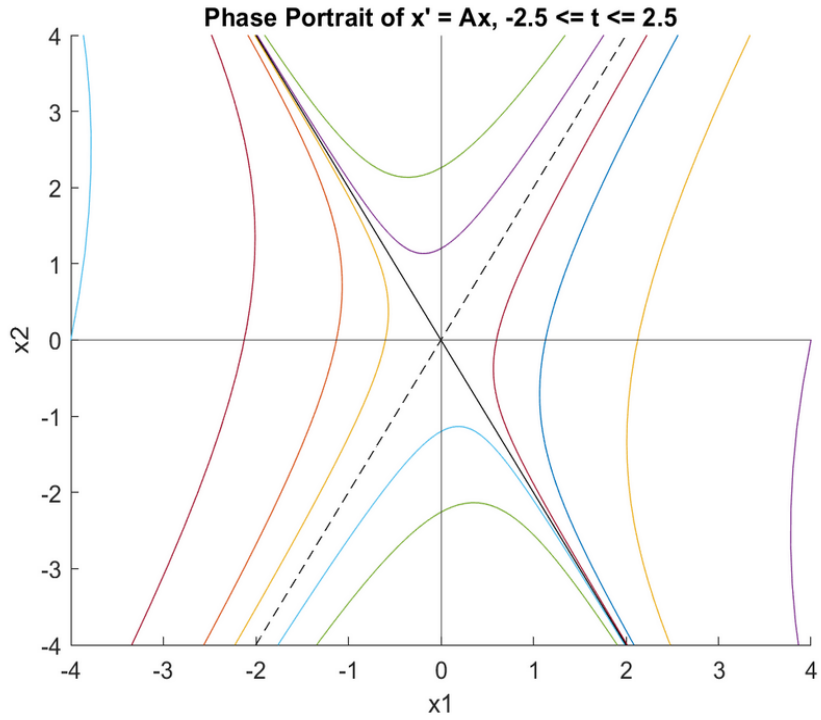




18.

(a)

Using MATLAB, plot $\vec{x}(t) = c_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-2t}$



Code on next page

Note: $-2.5 \le t \le 2.5$

```
clear, clc
% set plot boundaries
Xmin = -4; Xmax = 4;
Ymin = -4; Ymax = 4;
ev1 = 1;     % eigenvalues
ev2 = -2;
evec1 = [-1; 2];  % eigenvectors
evec2 = [1; 2];
% compute slopes of asymptotic lines
m1 = evec1(2)/evec1(1);
m2 = evec2(2)/evec2(1);
t = -2.5:0.05:2.5;
figure
xlim([Xmin,Xmax])
ylim([Ymin,Ymax])
hold on
xline(0); yline(0);  % axes
plot(t,m1*t, 'k')  % asymptotic lines
plot(t,m2*t, '--k')
s1 = evec1*exp(ev1*t);
s2 = evec2*exp(ev2*t);
% plot trajectories for various c1,c2
for c1 = [-2,-0.3,0.3,2]
    for c2 = [-2,-0.3,0.3,2]
        x = c1*s1 + c2*s2;
        x1 = x(1,:);
        x2 = x(2,:);
        plot(x1,x2)
    end
end
xlabel 'x1', ylabel 'x2'
title 'Phase Portrait of x'' = Ax, -2.5 <= t <= 2.5'
```

(b)

Assume "initial" means $t=0$.

$\therefore$ Solve $\begin{bmatrix} 2 \\ 3 \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

Using MATLAB, code on next page,

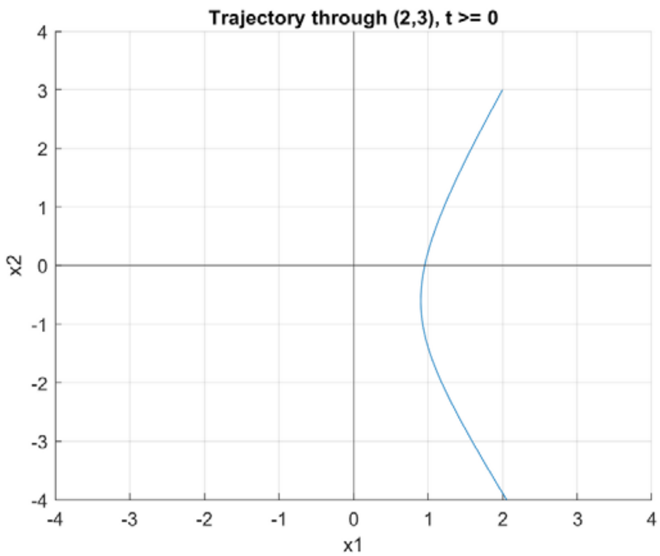
$$\vec{x}(t) = -\frac{1}{4} \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{t} + \frac{7}{4} \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-2t}$$

Note: as $t \to \infty$, $\vec{x}(t) \to -\infty$. As $t \to -\infty$, $\vec{x}(t) \to +\infty$

```
M = [evec1, evec2];
B = [2; 3];
C = linsolve(M,B)
t = 0:0.05:2.5;  % now just look at t>=0
s1 = evec1*exp(ev1*t);
s2 = evec2*exp(ev2*t);
x = C(1)*s1 + C(2)*s2;
x1 = x(1,:);
x2 = x(2,:);
figure
hold on
grid on
xlim([Xmin,Xmax])
ylim([Ymin,Ymax])
xline(0); yline(0);
plot(x1,x2)
xlabel 'x1', ylabel 'x2'
title 'Trajectory through (2,3), t >= 0'
```

Code is a continuation of code in (a)

```
C = 2×1
   -0.2500
    1.7500
```



Note: $t \geq 0$

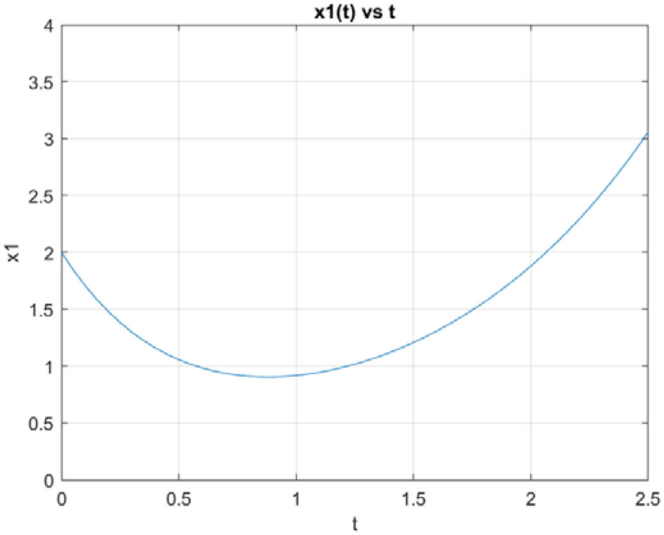
(c)

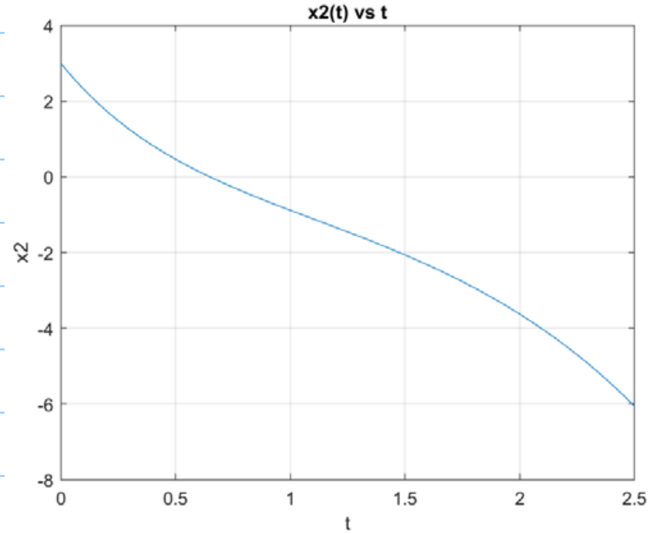
Using MATLAB, Code a continuation of code in (a), (b), plots on next page.

```
figure
plot(t,x1)
ylim([0, 4])
grid on
xlabel 't', ylabel 'x1'
title 'x1(t) vs t'

figure
plot(t,x2)
ylim([-8, 4])
grid on
xlabel 't', ylabel 'x2'
title 'x2(t) vs t'
```

$$x_1(t) = \frac{1}{4}e^{t} + \frac{7}{4}e^{-2t}$$

$$x_2(t) = -\frac{1}{2}e^{t} + \frac{7}{2}e^{-2t}$$

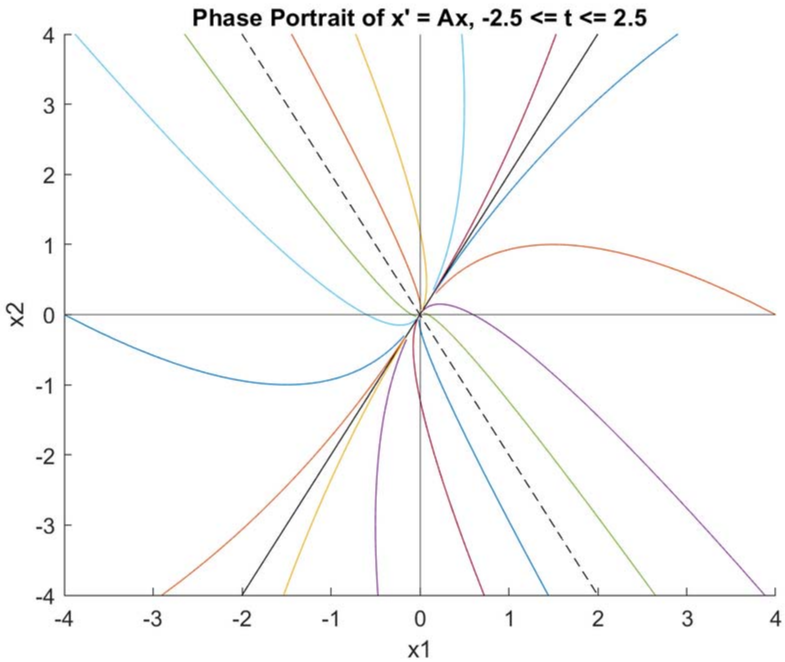




19.

(a)

Using MATLAB, plot $\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{2t}$



Code on next page.

Note: $-2.5 \le t \le 2.5$

```
clear, clc
% set plot boundaries
Xmin = -4; Xmax = 4;
Ymin = -4; Ymax = 4;
ev1 = 1;    % eigenvalues
ev2 = 2;
evec1 = [1; 2];  % eigenvectors
evec2 = [1; -2];
% compute slopes of asymptotic lines
m1 = evec1(2)/evec1(1);
m2 = evec2(2)/evec2(1);
t = -2.5:0.05:2.5;
figure
xlim([Xmin,Xmax])
ylim([Ymin,Ymax])
hold on
xline(0); yline(0);  % axes
plot(t,m1*t, 'k')  % asymptotic lines
plot(t,m2*t, '--k')
s1 = evec1*exp(ev1*t);
s2 = evec2*exp(ev2*t);
% plot trajectories for various c1,c2
for c1 = [-2,-0.3,0.3,2]
    for c2 = [-2,-0.3,0.3,2]
        x = c1*s1 + c2*s2;
        x1 = x(1,:);
        x2 = x(2,:);
        plot(x1,x2)
    end
end
xlabel 'x1', ylabel 'x2'
title 'Phase Portrait of x'' = Ax, -2.5 <= t <= 2.5'
```

(b)

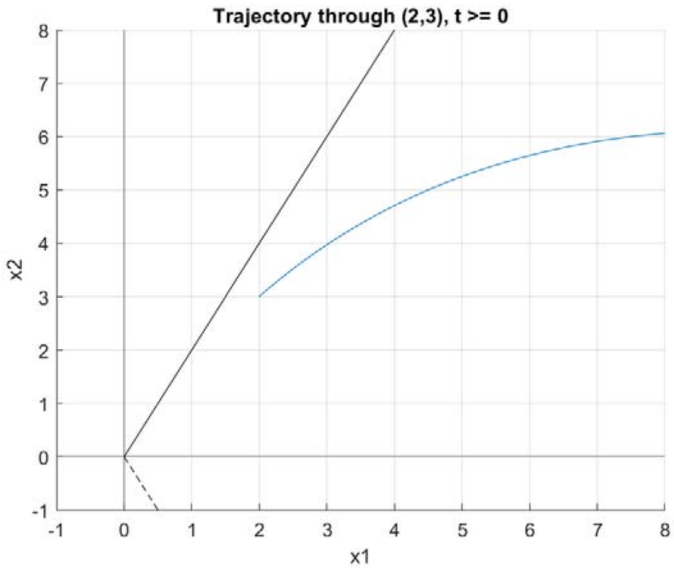
Using MATLAB, assuming "initial" means $t=0$.

Code is a continuation of code in (a)

```
M = [evec1, evec2];
B = [2; 3];
C = linsolve(M,B)
t = 0:0.05:4;  % now just look at t>=0
s1 = evec1*exp(ev1*t);
s2 = evec2*exp(ev2*t);
x = C(1)*s1 + C(2)*s2;
x1 = x(1,:);
x2 = x(2,:);
figure
hold on
grid on
xlim([-1,8])
ylim([-1,8])
xline(0); yline(0);
plot(t,m1*t, 'k')  % asymptotic lines
plot(t,m2*t, '--k')
plot(x1,x2)
xlabel 'x1', ylabel 'x2'
title 'Trajectory through (2,3), t >= 0'
```

```
C = 2×1

    1.7500
    0.2500
```

Note $t \geq 0$



Note: $t \geq 0$

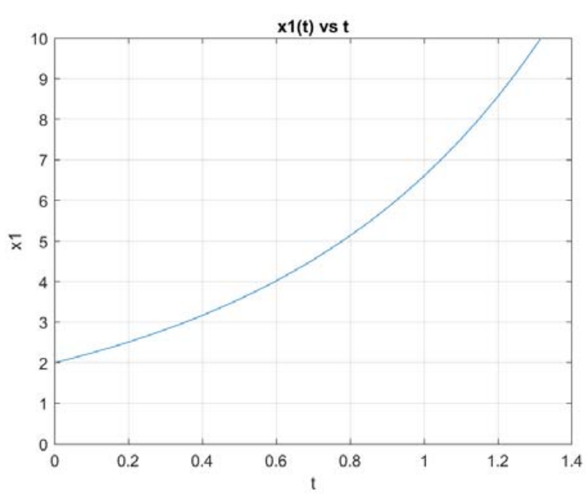
(c)

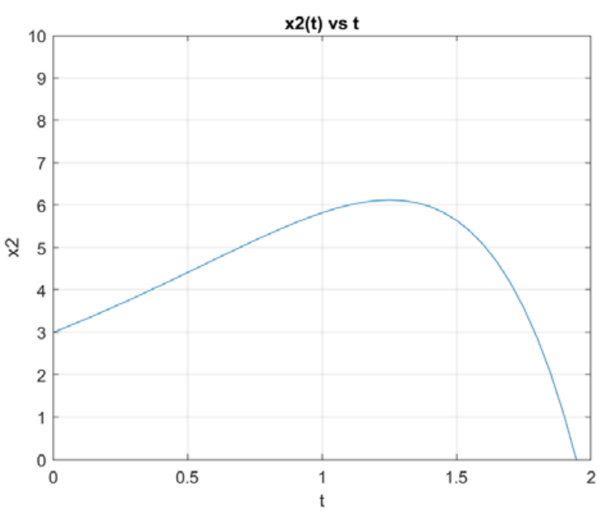
Using MATLAB,

```
figure
plot(t,x1)
ylim([0, 10])
grid on
xlabel 't', ylabel 'x1'
title 'x1(t) vs t'

figure
plot(t,x2)
ylim([0, 10])
grid on
xlabel 't', ylabel 'x2'
title 'x2(t) vs t'
```

Code is a continuation of code in (a), (b)

$$x_1(t) = \frac{7}{4}e^{t} + \frac{1}{4}e^{2t}$$
$$x_2(t) = \frac{7}{2}e^{t} - \frac{1}{2}e^{2t}$$





20.

(a)

If we seek solutions to $\vec{x}' = A\vec{x}$ of the form $\vec{x}(t) = \vec{\epsilon} e^{rt}$, where $\vec{\epsilon}$ is a 2×1 constant vector, then $\vec{x}'(t) = r\vec{\epsilon} e^{rt}$, so $\vec{x}' = A\vec{x}$ becomes $r\vec{\epsilon} e^{rt} = A(\vec{\epsilon} e^{rt})$, or since $e^{rt} \neq 0$, $r\vec{\epsilon} = A\vec{\epsilon}$. This is equivalent to $rI\vec{\epsilon} = A\vec{\epsilon}$, or $A\vec{\epsilon} - rI\vec{\epsilon} = 0$, or $(A - rI)\vec{\epsilon} = \vec{0}$. Nonzero vectors $\vec{\epsilon}$ satisfy $(A-rI)\vec{\epsilon} = \vec{0} \Leftrightarrow \det(A - rI) = 0$. For a 2×2 system, this means solving a polynomial equation of degree 2 in r.

$\therefore$ There are two roots, r_1 and r_2. We are assuming $r_1 \neq r_2$. $\therefore$ We seek nonzero vectors $\vec{\epsilon}^{(1)}$ and $\vec{\epsilon}^{(2)}$ that satisfy $(A - r_1 I)\vec{\epsilon}^{(1)} = \vec{0}$ and

$(A - r_2 I)\vec{\epsilon}^{(2)} = \vec{0}.$

Since $\det(A - r_1 I) = 0$, there is a nonzero vector $\vec{\epsilon}^{(1)}$ s.t. $(A - r_1 I)\vec{\epsilon}^{(1)} = \vec{0}$.

Since $\det(A - r_2 I) = 0$, there is a nonzero vector $\vec{\epsilon}^{(2)}$ s.t. $(A - r_2 I)\vec{\epsilon}^{(2)} = \vec{0}$.

(b)

$$(A - r_2 I)\vec{\epsilon}^{(1)} = A\vec{\epsilon}^{(1)} - r_2 I\vec{\epsilon}^{(1)}$$
$$= r_1\vec{\epsilon}^{(1)} - r_2\vec{\epsilon}^{(1)}$$
$$= (r_1 - r_2)\vec{\epsilon}^{(1)}$$

(c)

Since $c_1\vec{\epsilon}^{(1)} + c_2\vec{\epsilon}^{(2)} = \vec{0}$, then

$$(A - r_2 I)(c_1\vec{\epsilon}^{(1)} + c_2\vec{\epsilon}^{(2)}) = (A - r_2 I)\vec{0} = \vec{0} \quad [1]$$

But $(A - r_2 I)(c_1\vec{\epsilon}^{(1)} + c_2\vec{\epsilon}^{(2)}) =$

$$(A - r_2 I)c_1\vec{\epsilon}^{(1)} + (A - r_2 I)c_2\vec{\epsilon}^{(2)}$$

$$= c_1(r_1 - r_2)\vec{\epsilon}^{(1)} \quad \text{by (6)}$$

$$+ \underbrace{A c_2 \vec{\epsilon}^{(2)}}_{= c_2 r_2 \vec{\epsilon}^{(2)}} - r_2 I c_2 \vec{\epsilon}^{(2)}$$

$$= c_1(r_1 - r_2)\vec{\epsilon}^{(1)} + c_2 r_2 \vec{\epsilon}^{(2)} - r_2 c_2 \vec{\epsilon}^{(2)}$$

$$= c_1(r_1 - r_2)\vec{\epsilon}^{(1)} \qquad [2]$$

By [1], [2]: $(A - r_2 I)(c_1\vec{\epsilon}^{(1)} + c_2\vec{\epsilon}^{(2)}) = \vec{0}$

and $(A - r_2 I)(c_1\vec{\epsilon}^{(1)} + c_2\vec{\epsilon}^{(2)}) = c_1(r_1 - r_2)\vec{\epsilon}^{(1)}$

$\therefore c_1(r_1 - r_2)\vec{\epsilon}^{(1)} = \vec{0}$

Since $r_1 - r_2 \neq 0$ and $\vec{\epsilon}^{(1)}$ is nonzero, $\therefore c_1 = 0$,

a contradiction to the assumption that $c_1 \neq 0$.

$\therefore \vec{\epsilon}^{(1)}$ and $\vec{\epsilon}^{(2)}$ are linearly independent.

(d)

Use $(A - r_1 I)$ with $c_1\vec{\epsilon}^{(1)} + c_2\vec{\epsilon}^{(2)} = \vec{0}$, assuming $c_2 \neq 0$.

$\therefore (A - r_1 I)(c_1\vec{\epsilon}^{(1)} + c_2\vec{\epsilon}^{(2)}) = \vec{0} \qquad [1]$

But $(A - r_1 I)(c_1\vec{\epsilon}^{(1)} + c_2\vec{\epsilon}^{(2)}) =$

$$(A - r_1 I)c_1\vec{\epsilon}^{(1)} + (A - r_1 I)c_2\vec{\epsilon}^{(2)}$$

$$= A c_1 \vec{e}^{(1)} - r_1 c_1 \vec{e}^{(1)} + A c_2 \vec{e}^{(2)} - r_1 c_2 \vec{e}^{(2)}$$

$$= c_1 r_1 \vec{e}^{(1)} - r_1 c_1 \vec{e}^{(1)} + c_2 r_2 \vec{e}^{(2)} - r_1 c_2 \vec{e}^{(2)}$$

$$= \vec{0} + c_2 (r_2 - r_1) \vec{e}^{(2)} = c_2 (r_2 - r_1) \vec{e}^{(2)} \quad [2]$$

By [1], [2]: $(A - r_1 I)(c_1 \vec{e}^{(1)} + c_2 \vec{e}^{(2)}) = \vec{0}$

and $(A - r_1 I)(c_1 \vec{e}^{(1)} + c_2 \vec{e}^{(2)}) = c_2 (r_2 - r_1) \vec{e}^{(2)}$

$\therefore c_2 (r_2 - r_1) \vec{e}^{(2)} = \vec{0}$.

Since $r_2 \neq r_1$ and $\vec{e}^{(2)}$ is nonzero, $\therefore c_2 = 0$,

a contradiction to the assumption that $c_2 \neq 0$.

$\therefore \vec{e}^{(1)}$ and $\vec{e}^{(2)}$ are linearly independent.

(e)

Seeking solutions to $x' = Ax$ of the form $\vec{e} e^{rt}$, we need solutions to $\det(A - rI) = 0$, a polynomial equation of degree 3 in r. Assume we have r_1, r_2, r_3 as solutions s.t. $r_1 \neq r_2$, $r_2 \neq r_3$, $r_1 \neq r_3$. Because $\det(A - r_i I) = 0$, there exist nonzero vectors

$\vec{\epsilon}^{(1)}, \vec{\epsilon}^{(2)}, \vec{\epsilon}^{(3)}$ s.t. $(A - r_i I)\epsilon^{(i)} = \vec{0}$.

Suppose $c_1 \vec{\epsilon}^{(1)} + c_2 \vec{\epsilon}^{(2)} + c_3 \vec{\epsilon}^{(3)} = \vec{0}$ [1]

$$\therefore A\left(c_1 \vec{\epsilon}^{(1)} + c_2 \vec{\epsilon}^{(2)} + c_3 \vec{\epsilon}^{(3)}\right) = A\vec{0} = \vec{0}$$

$$= c_1 r_1 \vec{\epsilon}^{(1)} + c_2 r_2 \vec{\epsilon}^{(2)} + c_3 r_3 \vec{\epsilon}^{(3)} = \vec{0} \quad [2]$$

Multiplying [1] by r_3:

$$c_1 r_3 \vec{\epsilon}^{(1)} + c_2 r_3 \vec{\epsilon}^{(2)} + c_3 r_3 \vec{\epsilon}^{(3)} = \vec{0} \quad [3]$$

Subtracting [2] − [3] yields: (same as $A - r_3 I$)

$$c_1 (r_1 - r_3) \vec{\epsilon}^{(1)} + c_2 (r_2 - r_3) \vec{\epsilon}^{(2)} = \vec{0} \quad [4]$$

Now operate on [4] by A:

$$c_1 (r_1 - r_3) r_1 \vec{\epsilon}^{(1)} + c_2 (r_2 - r_3) r_2 \vec{\epsilon}^{(2)} = \vec{0} \quad [5]$$

And multiply [4] by r_2:

$$c_1 (r_1 - r_3) r_2 \vec{\epsilon}^{(1)} + c_2 (r_2 - r_3) r_2 \vec{\epsilon}^{(2)} = \vec{0} \quad [6]$$

Subtract [5] − [6]: (same as $A - r_2 I$)

$$c_1 (r_1 - r_3)(r_1 - r_2) \vec{\epsilon}^{(1)} = \vec{0}$$

Since $r_i \neq r_j$ for $i \neq j$ and $\vec{\epsilon}^{(i)} \neq \vec{0}$, this means $c_1 = 0$. $\therefore$ [4] becomes

$$c_2(r_2 - r_3)\vec{e}^{(2)} = \vec{0}, \text{ and so } c_2 = 0$$

$\therefore$ [1] becomes $c_3 \vec{e}^{(3)} = \vec{0}$, which means

$c_3 = 0$

$\therefore$ [1] implies $c_1 = c_2 = c_3 = 0$, and so

$\vec{e}^{(1)}$, $\vec{e}^{(2)}$, and $\vec{e}^{(3)}$ are linearly independent.

21.

(a)

Let $x_1 = y$, $x_2 = y'$

$\therefore x_1' = y' = x_2$, $x_2' = y''$

$\therefore a x_2' + b x_2 + c x_1 = 0$,

or $a x_2' = -c x_1 - b x_2$

Since $a \neq 0$,

$$\boxed{\begin{aligned} x_1' &= x_2 \\ x_2' &= -\frac{c}{a}x_1 - \frac{b}{a}x_2 \end{aligned}}$$

Or, $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -c/a & -b/a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

(b)

Using $\det(A - \lambda I) = 0$,

$A - \lambda I = \begin{bmatrix} -\lambda & 1 \\ -c/a & -b/a - \lambda \end{bmatrix}$

$\therefore \det(A - \lambda I) = 0 \Rightarrow \lambda^2 + \frac{b}{a}\lambda + \frac{c}{a} = 0$,

or, $a\lambda^2 + b\lambda + c = 0$, same as $ar^2 + br + c = 0$

22.

(a)

Assuming a solution in exponential form, find eigenvalues : $\det\begin{bmatrix} -\frac{1}{10}-\lambda & 3/40 \\ 1/10 & -\frac{1}{5}-\lambda \end{bmatrix} = 0$

Using MATLAB,

```
clear
syms r
A = [-1/10, 3/40;
     1/10, -1/5];
s = solve(det(A-r*eye(2))==0,r);
for i=1:2
   s(i)   % show eigenvalue
   % rref to compute eigenvector
   rref(A-s(i)*eye(2))
end
```

ans =

$-\frac{1}{4}$

ans =

$\begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{pmatrix}$

ans =

$-\frac{1}{20}$

ans =

$\begin{pmatrix} 1 & -\frac{3}{2} \\ 0 & 0 \end{pmatrix}$

$\therefore \vec{x}(t) = c_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t/4} + c_2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{-t/20}$

$\therefore$ solve $\begin{bmatrix} 1 & 3 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -17 \\ -21 \end{bmatrix} = \vec{x}(0)$

Using MATLAB,

```
B = [-17; -21];
V = [1, 3;
     -2, 2];
format rat
C = linsolve(V, B)
```

```
C = 2×1

      29/8
     -55/8
```

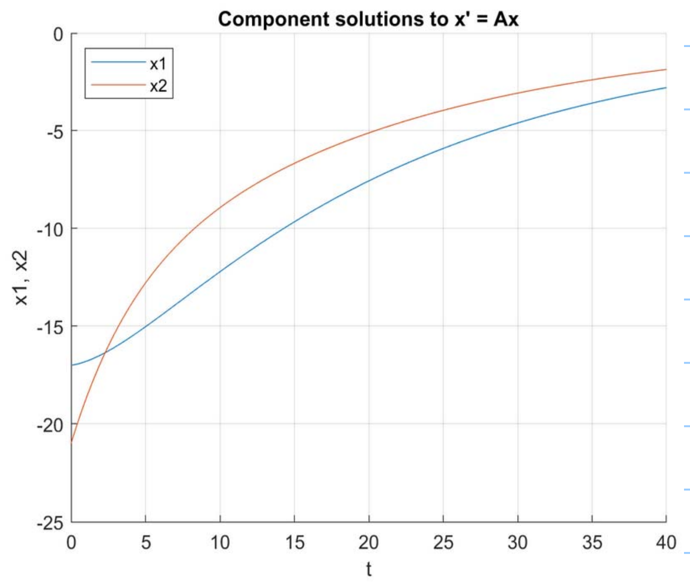
$\therefore\ C_1 = \frac{29}{8}, \quad C_2 = -\frac{55}{8}$

$$\therefore\ \boxed{\vec{x}(t) = \frac{29}{8}\begin{bmatrix}1\\-2\end{bmatrix}e^{-t/4} - \frac{55}{8}\begin{bmatrix}3\\2\end{bmatrix}e^{-t/20}}$$

(b)

Using MATLAB, continuation of above code.

```
t = 0:0.05:40;
evec1 = V(:,1);
evec2 = V(:,2);
x = C(1)*evec1*exp(s(1)*t) + ...
    C(2)*evec2*exp(s(2)*t);
x1 = x(1,:);
x2 = x(2,:);
figure
hold on
plot(t, x1)
plot(t, x2)
grid on
xlabel 't', ylabel 'x1, x2'
title 'Component solutions to x'' = Ax'
legend('x1','x2', 'Location', 'northwest')
```



(c)

From the plot in (b), both $x_1(t)$ and $x_2(t)$

are monotonically increasing functions.

$\therefore$ If $T_1 < T_2$, then $X_i(T_1) < X_i(T_2)$, for $i = 1, 2$.

Note that the $e^{-t/20}$ term dominates over $e^{-t/4}$

Since $t/20 < t/4$ for $t > 0$, so that

$$e^{t/20} < e^{t/4} \Rightarrow e^{-t/4} < e^{-t/20}$$

$\therefore$ Since both $e^{-t/20}$ and $e^{t/4} \to 0$

as $t \to \infty$, for sufficiently large $t > 0$,

the $e^{-t/4}$ term contributes a negligible

amount to the overall sum of $X_i(t)$.

$\therefore$ Can intially ignore the $e^{-t/4}$ term.

$\therefore$ Find t s.t. $X_1(t) \approx -\frac{165}{8} e^{-t/20} > -0.50$

and $X_2(t) \approx -\frac{110}{8} e^{-t/20} > -0.50$

Or, $\frac{165}{8} e^{-t/20} < 0.50$ and $\frac{110}{8} e^{-t/20} < 0.50$

Or, $e^{-t/20} > \frac{8}{165}(0.50)$ and $e^{-t/20} > \frac{8}{110}(0.50)$

Or, $-t < 20 \ln\left[\frac{8}{165}(0.50)\right]$ and $-t < 20 \ln\left[\frac{8}{110}(0.50)\right]$

Or, $t > 74.39$ and $t > 66.28$

$\therefore$ Choose $\underline{T \approx 74.39}$

As a check, using MATLAB, using code that continues from above:

```
for i = 74.39:0.01:74.40
    y = vpa(C(1)*evec1*exp(s(1)*i) + ...
        C(2)*evec2*exp(s(2)*i))
end
```

y =

$$\begin{pmatrix} -0.50007553172439389276451504668 55 \\ -0.33338376881628070615764293858262 \end{pmatrix}$$

y =

$$\begin{pmatrix} -0.49982555651821519025521854014191 \\ -0.33321711847658110679147841428232 \end{pmatrix}$$

$\therefore T = 74.39 \Rightarrow x_1 = -0.5001,\ x_2 = -0.3334$

$T = 74.40 \Rightarrow x_1 = -0.4998,\ x_2 = -0.3332$

$\therefore 74.39 < T < 74.40$

23.

(a)

Using MATLAB,

```
clear
syms a
A = [-1, -1;
     -a, -1];
% assign a = 1/2
[V,D] = eig(subs(A,a,1/2))
```

V =

$$\begin{pmatrix} \sqrt{2} & -\sqrt{2} \\ 1 & 1 \end{pmatrix}$$

D =

$$\begin{pmatrix} -\frac{\sqrt{2}}{2}-1 & 0 \\ 0 & \frac{\sqrt{2}}{2}-1 \end{pmatrix}$$

$\therefore$ Eigenvalues: $-\frac{\sqrt{2}}{2}-1$, $\frac{\sqrt{2}}{2}-1$, both are negative

$$\therefore \vec{x}(t) = c_1 \begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix} e^{\frac{-\sqrt{2}-2}{2}t} + c_2 \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix} e^{\frac{\sqrt{2}-2}{2}t}$$

$\lim_{t \to \infty} \vec{x}(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. $\therefore$ the equilibrium point is the origin, $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, and is a stable node since, over time, as t increases, $\vec{x}(t) \to (0,0)$

(b)

Using MATLAB,

```
clear
syms a
A = [-1, -1;
     -a, -1];
% assign a = 2
[V,D] = eig(subs(A,a,2))
```

V =

$$\begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 1 & 1 \end{pmatrix}$$

or $\begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}$, $\begin{bmatrix} -1 \\ \sqrt{2} \end{bmatrix}$

D =

$$\begin{pmatrix} -\sqrt{2}-1 & 0 \\ 0 & \sqrt{2}-1 \end{pmatrix}$$

$\therefore$ Eigenvalues: $-\sqrt{2}-1$, $\sqrt{2}-1$, opposite signs

$$\therefore \vec{x}(t) = c_1 \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} e^{-(\sqrt{2}+1)t} + c_2 \begin{bmatrix} -1 \\ \sqrt{2} \end{bmatrix} e^{(\sqrt{2}-1)t}$$

$$\lim_{t \to \infty} x_1(t) = -\infty \text{ if } c_2 > 0, \ +\infty \text{ if } c_2 < 0$$

$$\lim_{t \to \infty} x_2(t) = +\infty \text{ if } c_2 > 0, \ -\infty \text{ if } c_2 < 0$$

$\therefore$ Origin represents an unstable saddle point. As t increases, $\vec{x}(t)$ moves away from the origin, either foward quadrant II ($c_2 > 0$) or quadrant IV ($c_2 < 0$).

If $c_2 = 0$, then origin is stable as $e^{-(\sqrt{2}+1)t} \to 0$ as $t \to \infty$.

(c)

$$\det \begin{bmatrix} -1-\lambda & -1 \\ -\alpha & -1-\lambda \end{bmatrix} = \lambda^2 + 2\lambda + 1 - \alpha = 0$$

$$\therefore \lambda = \frac{-2 \pm \sqrt{4 - 4(1-\alpha)}}{2} = -1 \pm \sqrt{1-(1-\alpha)}$$

$$= -1 \pm \sqrt{\alpha}$$

As long as $\sqrt{\alpha} < 1$, $-1 \pm \sqrt{\alpha} < 0$, so $\lambda < 0$ for both eigenvalues.

If $\sqrt{\alpha} > 1$, then λ has values of opposite sign

$\therefore$ When $\sqrt{\alpha} = 1$, $\lambda = -2, 0$

$$\vec{x}(t) = c_1 \vec{v}_1 e^{-2t} + c_2 \vec{v}_1$$

As $t \to \infty$, $\vec{x}(t) \to c_2 \vec{v}_1$

$\therefore$ $\underline{\alpha = 1}$ is the bifurcation value.

24.

(a) Given the values, $A = \begin{bmatrix} -1/2 & -1/2 \\ 3/2 & -5/2 \end{bmatrix}$

Using MATLAB,

```
clear
syms a
% Trick to get rational output
% Use 'syms', then 'subs'
A = [-1/2, -1/2;
     3/2, a];
% assign a = -5/2
[V, D] = eig(subs(A,a,-5/2))
```

V =

$$\begin{pmatrix} \frac{1}{3} & 1 \\ 1 & 1 \end{pmatrix}$$

D =

$$\begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}$$

or $\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\therefore \boxed{x(t) = \begin{bmatrix} I(t) \\ V(t) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}}$$

(b)

c_1 and c_2 are determined by I_0, V_0.

$$\lim_{t \to \infty} I(t) = \lim_{t \to \infty} \left[c_1 e^{-2t} + c_2 e^{-t} \right] = 0$$

$$\text{as} \lim_{t\to\infty} e^{-2t} = \lim_{t\to\infty} e^{-t} = 0.$$

$$\lim_{t\to\infty} V(t) = \lim_{t\to\infty} \left[3c_1 e^{-2t} + c_2 e^{-t}\right] = 0$$

$$\text{as} \lim_{t\to\infty} e^{-2t} = \lim_{t\to\infty} e^{-t} = 0.$$

25.

(a)

$$\det\begin{bmatrix} -\frac{R_1}{L} - \lambda & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{CR_2} - \lambda \end{bmatrix} = \left(\lambda + \frac{R_1}{L}\right)\left(\lambda + \frac{1}{CR_2}\right) + \frac{1}{LC}$$

$$= \lambda^2 + \left(\frac{R_1}{L} + \frac{1}{CR_2}\right)\lambda + \frac{R_1}{LCR_2} + \frac{1}{LC} = 0 \quad [1]$$

For the eigenvalues to be real and different, discriminant must be > 0.

$$\therefore \left(\frac{R_1}{L} + \frac{1}{CR_2}\right)^2 - 4\left(\frac{R_1}{LCR_2} + \frac{1}{LC}\right) > 0$$

$$\text{Or, } \left(\frac{R_1}{L}\right)^2 + \left(\frac{1}{CR_2}\right)^2 + 2\frac{R_1}{LCR_2} - \frac{4R_1}{LCR_2} - \frac{4}{LC} > 0$$

$$\text{Or, } \left(\frac{R_1}{L}\right)^2 + \left(\frac{1}{CR_2}\right)^2 - \frac{2R_1}{LCR_2} - \frac{4}{LC} > 0$$

$$\text{Or, } \boxed{\left(\frac{R_1}{L} - \frac{1}{CR_2}\right)^2 - \frac{4}{LC} > 0}$$

(b)

(1) Solving the quadratic equation [1] in (a),

$$\lambda = \frac{-\left(\frac{R_1}{L} + \frac{1}{CR_2}\right) \pm \sqrt{\left(\frac{R_1}{L} + \frac{1}{CR_2}\right)^2 - 4\left(\frac{R_1}{LCR_2} + \frac{1}{LC}\right)}}{2}$$

Since $\frac{R_1}{LCR_2} + \frac{1}{LC} > 0$, then

$$\left(\frac{R_1}{L} + \frac{1}{CR_2}\right) > \sqrt{\left(\frac{R_1}{L} + \frac{1}{CR_2}\right)^2 - 4\left(\frac{R_1}{LCR_2} + \frac{1}{LC}\right)}$$

$$\therefore -\left(\frac{R_1}{L} + \frac{1}{CR_2}\right) \pm \sqrt{\left(\frac{R_1}{L} + \frac{1}{CR_2}\right)^2 - 4\left(\frac{R_1}{LCR_2} + \frac{1}{LC}\right)} < 0$$

$\therefore$ λ_1 and λ_2 are negative.

(2) Let $F = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$ and $G = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$ be the corresponding eigenvectors for λ_1 and λ_2.

$$\therefore \begin{bmatrix} I(t) \\ V(t) \end{bmatrix} = c_1 \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} e^{\lambda_1 t} + c_2 \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} e^{\lambda_2 t}$$

where c_1, c_2 are constants, and

F, G are constant vectors.

$$\therefore \lim_{t\to\infty} I(t) = \lim_{t\to\infty} \left(c_1 f_1 e^{\lambda_1 t} + c_2 g_1 e^{\lambda_2 t}\right) = 0$$

$$\lim_{t\to\infty} V(t) = \lim_{t\to\infty} \left(c_1 f_2 e^{\lambda_1 t} + c_2 g_2 e^{\lambda_2 t}\right) = 0$$

since $\lim_{t\to\infty} e^{\lambda_1 t} = \lim_{t\to\infty} e^{\lambda_2 t} = 0$ since $\lambda_1 < 0, \lambda_2 < 0$

Initial conditions only affect the values of c_1, c_2.

(c)

Analagous to second-order linear equations, for repeated roots, the general solution had form $c_1 e^{\lambda t} + c_2 t e^{\lambda t}$, and note $\lambda < 0$.

Since $\lim_{t\to\infty} e^{\lambda t} = 0$ and $\lim_{t\to\infty} t e^{\lambda t} \to 0$ for $\lambda < 0$, then for repeated roots, $I(t)$, $V(t)$

probably $\to 0$ as $t \to \infty$.

For complex eigenvalues, like second-order equations, the solution is likely of form $c_1 \begin{bmatrix} \\ \end{bmatrix} e^{\lambda t} \cos(\theta t) + c_2 \begin{bmatrix} \\ \end{bmatrix} e^{\lambda t} \sin(\theta t)$, and again $\lambda < 0$. So $I(t), V(t)$ likely oscillate to zero as $t \to \infty$.

7.6 Complex-Valued Eigenvalues

Note Title 12/17/2019

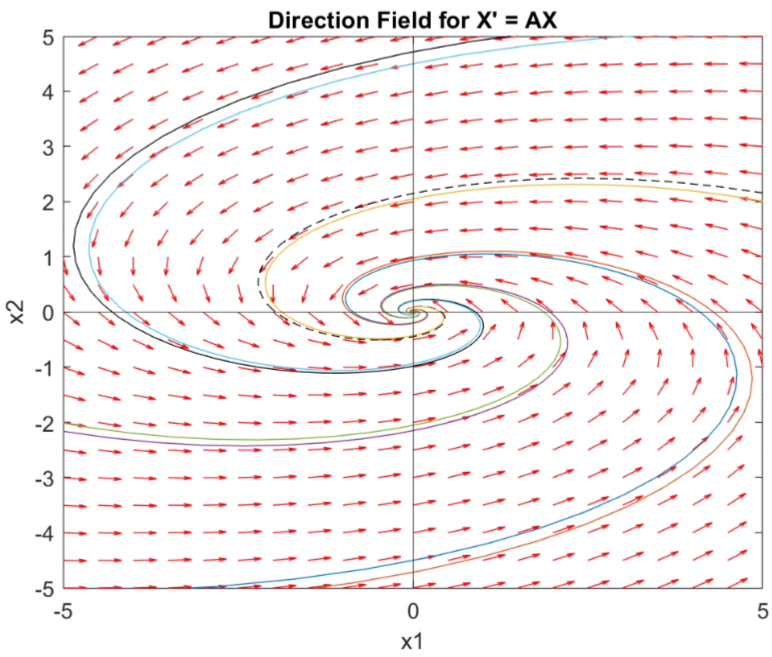
1.

(a) Using MATLAB,

```
clear
% Direction field - set plot boundaries
Xmin = -5; Xmax = 5;
Ymin = -5; Ymax = 5;
% set how fine to make the grid of values
Step = 0.5;
% total # points on x and y axes
Nx = round((Xmax-Xmin)/Step, 0) + 1;
Ny = round((Ymax-Ymin)/Step, 0) + 1;
% set the actual x and y coords
x = linspace(Xmin, Xmax, Nx);  % a row vector
y = linspace(Ymin, Ymax, Ny);  % a row vector
% preallocate memory for coord, slope arrays
%  to be used in quiver, must be same dimension
xcoord = zeros(Nx, Ny);
ycoord = zeros(Nx, Ny);
dx = zeros(Nx, Ny);
dy = zeros(Nx, Ny);
A = [-1, -4;          % Enter A
     1, -1];
for i = 1:Nx
    for j = 1:Ny
        xcoord(i,j) = x(i);
        ycoord(i,j) = y(j);
        v = A*[x(i);y(j)];  % get the slope
        s = norm(v);  % length of v
        % make slope vector a unit vector
        dx(i,j) = v(1)/s;
        dy(i,j) = v(2)/s;
    end
end
% plot the slope vectors at (xcoord, ycoord)
quiver(xcoord, ycoord, dx, dy, 0.5, 'r')
xline(0);  % show x and y axes
yline(0);
axis([Xmin, Xmax, Ymin, Ymax])  % plot bounds
xlabel 'x1', ylabel 'x2'
title 'Direction Field for X'' = AX'
```

Plot on next page

```
% get evalues, evectors for trajectories
[V,D] = eig(A);
ev1 = D(1,1)   % eigenvalues
ev2 = D(2,2)
syms z
for n = 1:2    % eigenvectors
    R = rref(A - D(n,n)*eye(2));
    % free variable = 1, S for  pivot
    S = solve(R(1,1)*z + R(1,2) == 0,z);
    if n == 1
        evec1 = [S;1]
    else
        evec2 = [S;1]
    end
end

%  plot phase portrait trajectories
%    over direction field
hold on
% size interval to exceed plot bounds
t = -5:0.05:5;
if ~isreal(ev1)
    x = evec1*exp(ev1*t);
    x1 = real(x);
    x2 = imag(x);
else
    x1 = evec1*exp(ev1*t);
    x2 = evec2*exp(ev2*t);
end
% create array of various coeff values
c = [1, 0, -1, -1, -1,  0,  1, 1;
     0, 1, -1,  0,  1, -1, -1, 1];
for n = 1:size(c,2)  % up to # columns of c[]
    p = c(1,n)*x1 + c(2,n)*x2;
    u = p(1,:);  % top row
    v = p(2,:);  % bottom row
    if (c(1,n)==1) && (c(2,n)==0)
        plot(u,v, '-k')
    elseif (c(1,n)==0) && (c(2,n)==1)
        plot(u,v, '--k')
    else
        plot(u,v)
    end
end
```




```
ev1 = -1.0000 + 2.0000i

ev2 = -1.0000 - 2.0000i

evec1 =
```

$$\begin{pmatrix} 2i \\ 1 \end{pmatrix}$$

```
evec2 =
```

$$\begin{pmatrix} -2i \\ 1 \end{pmatrix}$$

(b) From MATLAB code, eigenvalues: $-1+2i, -1-2i$

Corresponding eigenvectors: $\begin{bmatrix} 2i \\ 1 \end{bmatrix}, \begin{bmatrix} -2i \\ 1 \end{bmatrix}$

$$\vec{x}^{(1)}(t) = \begin{bmatrix} 2i \\ 1 \end{bmatrix} e^{-t}\left[\cos(2t) + i\sin(2t)\right]$$

$$= e^{-t}\underbrace{\begin{bmatrix} -2\sin(2t) \\ \cos(2t) \end{bmatrix}}_{\text{real}} + i\,e^{-t}\underbrace{\begin{bmatrix} 2\cos(2t) \\ \sin(2t) \end{bmatrix}}_{\text{imaginary}}$$

Since $\det\begin{bmatrix} -e^{-t}2\sin(2t) & e^{-t}2\cos(2t) \\ e^{-t}\cos(2t) & e^{-t}\sin(2t) \end{bmatrix} = -2e^{-2t} \neq 0$,

the real and imaginary parts of $\vec{x}^{(1)}(t)$

are independent and form a fundamental

solution set.

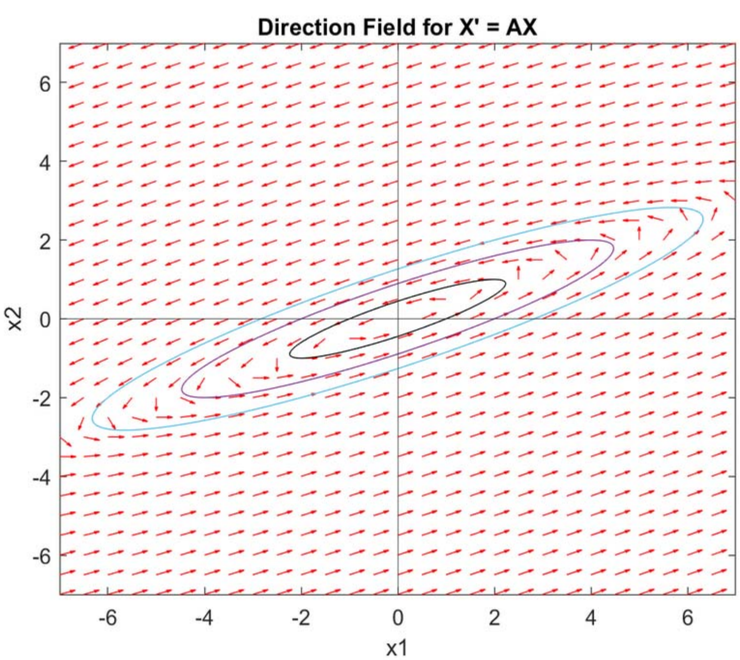
$$\therefore \quad \boxed{\vec{X}(t) = c_1 e^{-t}\begin{bmatrix} -2\sin(2t) \\ \cos(2t) \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 2\cos(2t) \\ \sin(2t) \end{bmatrix}}$$

(c) As $t \to \infty$, because of the e^{-t} factor, each component $\to \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. $\therefore$ $\vec{X}(t)$ spirals to origin.

2.

(a)

Using MATLAB, code on next page.



ev1 = -0.0000 + 1.0000i

ev2 = -0.0000 - 1.0000i

evec1 =

$$\begin{pmatrix} 2+i \\ 1 \end{pmatrix}$$

evec2 =

$$\begin{pmatrix} 2-i \\ 1 \end{pmatrix}$$

```
clear
% Direction field - set plot boundaries
Xmin = -7; Xmax = 7;
Ymin = -7; Ymax = 7;
% set how fine to make the grid of values
Step = 0.5;
% total # points on x and y axes
Nx = round((Xmax-Xmin)/Step, 0) + 1;
Ny = round((Ymax-Ymin)/Step, 0) + 1;
% set the actual x and y coords
x = linspace(Xmin, Xmax, Nx);  % a row vector
y = linspace(Ymin, Ymax, Ny);  % a row vector
% preallocate memory for coord, slope arrays
%  to be used in quiver, must be same dimension
xcoord = zeros(Nx, Ny);
ycoord = zeros(Nx, Ny);
dx = zeros(Nx, Ny);
dy = zeros(Nx, Ny);
A = [2, -5;          % Enter A
     1, -2];
for i = 1:Nx
    for j = 1:Ny
        xcoord(i,j) = x(i);
        ycoord(i,j) = y(j);
        v = A*[x(i);y(j)];  % get the slope
        s = norm(v);  % length of v
        % make slope vector a unit vector
        dx(i,j) = v(1)/s;
        dy(i,j) = v(2)/s;
    end
end
% plot the slope vectors at (xcoord, ycoord)
quiver(xcoord, ycoord, dx, dy, 0.5, 'r')
xline(0);  % show x and y axes
yline(0);
axis([Xmin, Xmax, Ymin, Ymax])  % plot bounds
xlabel 'x1', ylabel 'x2'
title 'Direction Field for X'' = AX'
```

```
% get evalues, evectors for trajectories
[V,D] = eig(A);
ev1 = D(1,1)    % eigenvalues
ev2 = D(2,2)
syms z
for n = 1:2     % eigenvectors
    R = rref(A - D(n,n)*eye(2));
    % free variable = 1, S for  pivot
    S = solve(R(1,1)*z + R(1,2) == 0,z);
    if n == 1
        evec1 = [S;1]
    else
        evec2 = [S;1]
    end
end

%  plot phase portrait trajectories
%    over direction field
hold on
% size interval to exceed plot bounds
t = -5:0.05:5;
if ~isreal(ev1)
    x = evec1*exp(ev1*t);
    x1 = real(x);
    x2 = imag(x);
else
    x1 = evec1*exp(ev1*t);
    x2 = evec2*exp(ev2*t);
end
% create array of various coeff values
c = [1, 0, -2, -2, -2,  0,  2, 2;
     0, 1, -2,  0,  2, -2, -2, 2];
for n = 1:size(c,2)  % up to # columns of c[]
    p = c(1,n)*x1 + c(2,n)*x2;
    u = p(1,:);  % top row
    v = p(2,:);  % bottom row
    if (c(1,n)==1) && (c(2,n)==0)
        plot(u,v, '-k')
    elseif (c(1,n)==0) && (c(2,n)==1)
        plot(u,v, '--k')
    else
        plot(u,v)
    end
end
```

(b)

From MATLAB code, eigenvalues: $i, -i$

Corresponding eigenvectors: $\begin{bmatrix} 2+i \\ 1 \end{bmatrix}, \begin{bmatrix} 2-i \\ 1 \end{bmatrix}$

$$\therefore \vec{X}^{(1)}(t) = \begin{bmatrix} 2+i \\ 1 \end{bmatrix} e^{0t}\left[\cos(t) + i\sin(t)\right]$$

$$= \begin{bmatrix} 2\cos(t) - \sin(t) \\ \cos(t) \end{bmatrix} + i \begin{bmatrix} \cos(t) + 2\sin(t) \\ \sin(t) \end{bmatrix}$$

Since $\det \begin{bmatrix} 2\cos(t) - \sin(t) & \cos(t) + 2\sin(t) \\ \cos(t) & \sin(t) \end{bmatrix}$

$= 2\cos(t)\sin(t) - \sin^2(t) - \cos^2(t) - 2\cos(t)\sin(t)$

$= -1 \neq 0$, real and imaginary components are independent.

$$\therefore \boxed{\vec{x}(t) = c_1 \begin{bmatrix} 2\cos(t) - \sin(t) \\ \cos(t) \end{bmatrix} + c_2 \begin{bmatrix} \cos(t) + 2\sin(t) \\ \sin(t) \end{bmatrix}}$$

Note: multiplying evector $\begin{bmatrix} 2+i \\ 1 \end{bmatrix}$ by $2-i$ yields $\begin{bmatrix} 5 \\ 2-i \end{bmatrix}$ which will yield the solution set in back of book.

(c)

Since all the components of $\vec{x}(t)$ have a period of 2π, then $\vec{x}(t)$ moves in a closed loop, an ellipse as shown on the phase portrait in (a).

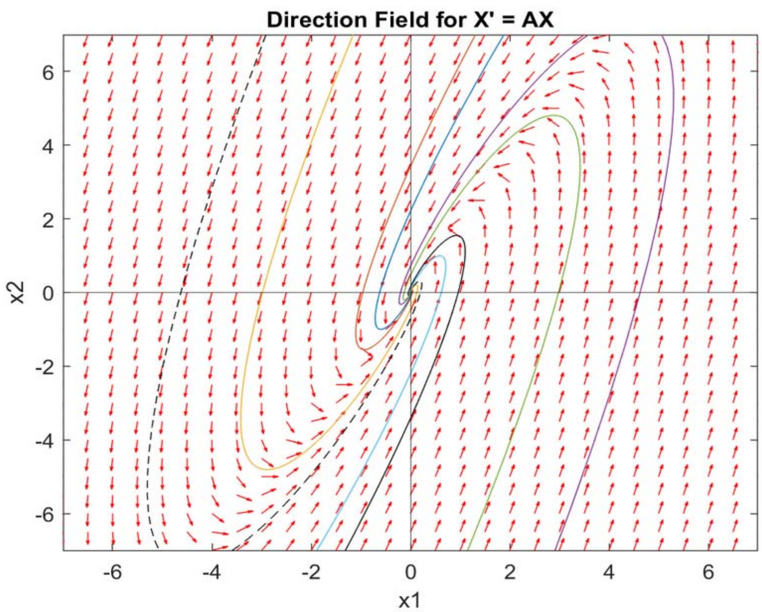
3.

(a)

Using MATLAB,

```
clear
% Direction field - set plot boundaries
Xmin = -7; Xmax = 7;
Ymin = -7; Ymax = 7;
% set how fine to make the grid of values
Step = 0.5;
% total # points on x and y axes
Nx = round((Xmax-Xmin)/Step, 0) + 1;
Ny = round((Ymax-Ymin)/Step, 0) + 1;
% set the actual x and y coords
x = linspace(Xmin, Xmax, Nx);  % a row vector
y = linspace(Ymin, Ymax, Ny);  % a row vector
% preallocate memory for coord, slope arrays
%  to be used in quiver, must be same dimension
xcoord = zeros(Nx, Ny);
ycoord = zeros(Nx, Ny);
dx = zeros(Nx, Ny);
dy = zeros(Nx, Ny);
A = [1, -1;         % Enter A
     5, -3];
for i = 1:Nx
    for j = 1:Ny
        xcoord(i,j) = x(i);
        ycoord(i,j) = y(j);
        v = A*[x(i);y(j)];  % get the slope
        s = norm(v);  % length of v
        % make slope vector a unit vector
        dx(i,j) = v(1)/s;
        dy(i,j) = v(2)/s;
    end
end
% plot the slope vectors at (xcoord, ycoord)
quiver(xcoord, ycoord, dx, dy, 0.5, 'r')
xline(0);  % show x and y axes
yline(0);
axis([Xmin, Xmax, Ymin, Ymax])  % plot bounds
xlabel 'x1', ylabel 'x2'
title 'Direction Field for X'' = AX'
% get evalues, evectors for trajectories
[V,D] = eig(A);
ev1 = D(1,1)   % eigenvalues
ev2 = D(2,2)
syms z
for n = 1:2    % eigenvectors
    R = rref(A - D(n,n)*eye(2));
    % free variable = 1, S for  pivot
    S = solve(R(1,1)*z + R(1,2) == 0,z);
    if n == 1
        evec1 = [S;1]
    else
        evec2 = [S;1]
    end
end
%  plot phase portrait trajectories
%    over direction field
hold on
% size interval to exceed plot bounds
t = -5:0.05:5;
if ~isreal(ev1)
    x = evec1*exp(ev1*t);
    x1 = real(x);
    x2 = imag(x);
else
    x1 = evec1*exp(ev1*t);
    x2 = evec2*exp(ev2*t);
end
% create array of various coeff values
c = [1, 0, -1, -1, -1,  0,  1, 1;
     0, 1, -1,  0,  1, -1, -1, 1];
for n = 1:size(c,2)  % up to # columns of c[]
    p = c(1,n)*x1 + c(2,n)*x2;
    u = p(1,:);  % top row
    v = p(2,:);  % bottom row
    if (c(1,n)==1) && (c(2,n)==0)
        plot(u,v, '-k')
    elseif (c(1,n)==0) && (c(2,n)==1)
        plot(u,v, '--k')
    else
        plot(u,v)
    end
end
```

Plot on next page,



ev1 = -1.0000 + 1.0000i

ev2 = -1.0000 - 1.0000i

evec1 =

$$\begin{pmatrix} \frac{2}{5} + \frac{1}{5}\,\mathrm{i} \\ 1 \end{pmatrix}$$

evec2 =

$$\begin{pmatrix} \frac{2}{5} - \frac{1}{5}\,\mathrm{i} \\ 1 \end{pmatrix}$$

(b) **b.** Express the general solution of the given system of equations in terms of real-valued functions.

From MATLAB, eigenvalues: $-1+i, -1-i$

Corresponding eigenvectors: $\begin{bmatrix} 2+i \\ 5 \end{bmatrix}, \begin{bmatrix} 2-i \\ 5 \end{bmatrix}$

after scaling by factor of 5.

$$\therefore \vec{x}^{(1)}(t) = \begin{bmatrix} 2+i \\ 5 \end{bmatrix} e^{-t}\left[\cos(t) + i\sin(t)\right]$$

$$= \underbrace{e^{-t}\begin{bmatrix} 2\cos(t) - \sin(t) \\ 5\cos(t) \end{bmatrix}}_{\text{real}} + i\,\underbrace{e^{-t}\begin{bmatrix} \cos(t) + 2\sin(t) \\ 5\sin(t) \end{bmatrix}}_{\text{imaginary}}$$

Since $\det\begin{bmatrix} e^{-t}(2\cos(t) - \sin(t)) & e^{-t}(\cos(t) + 2\sin(t)) \\ e^{-t}\,5\cos(t) & e^{-t}\,5\sin(t) \end{bmatrix}$

$= -5e^{-2t} \neq 0$, real and imaginary components are independent.

$$\therefore \boxed{\vec{x}(t) = c_1 e^{-t}\begin{bmatrix} 2\cos(t) - \sin(t) \\ 5\cos(t) \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} \cos(t) + 2\sin(t) \\ 5\sin(t) \end{bmatrix}}$$

Note: multiplying evector $\begin{bmatrix} 2+i \\ 5 \end{bmatrix}$ by $2-i$ yields $\begin{bmatrix} 5 \\ 10-5i \end{bmatrix}$, or $\begin{bmatrix} 1 \\ 2-i \end{bmatrix}$. Using this evector with $\lambda = -1+i$ yields form in back of book.

(c) **c.** Describe the behavior of the solutions as $t \to \infty$.

As $t \to \infty$, $\vec{x}(t) \to \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ because of the e^{-t} factor. $\therefore$ solution spirals towards the origin.

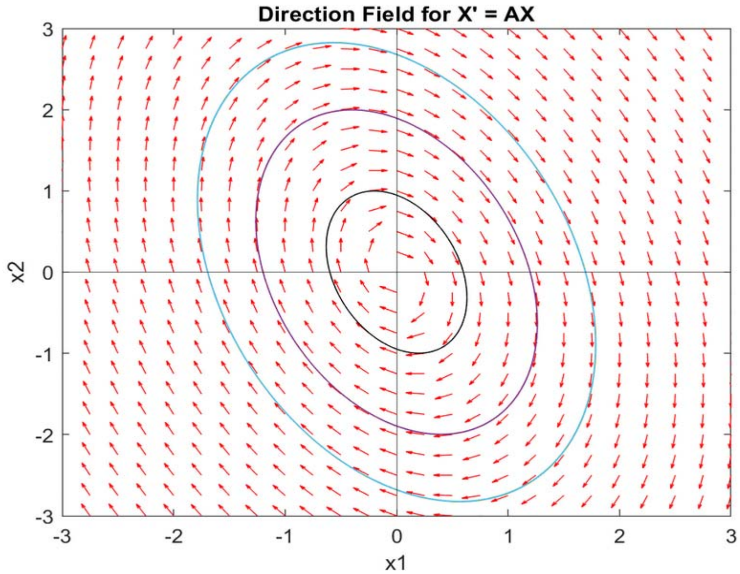
4.

(a)

Using MATLAB,

```
clear
% Direction field - set plot boundaries
Xmin = -3; Xmax = 3;
Ymin = -3; Ymax = 3;
% set how fine to make the grid of values
Step = 0.25;
% total # points on x and y axes
Nx = round((Xmax-Xmin)/Step, 0) + 1;
Ny = round((Ymax-Ymin)/Step, 0) + 1;
% set the actual x and y coords
x = linspace(Xmin, Xmax, Nx);  % a row vector
y = linspace(Ymin, Ymax, Ny);  % a row vector
% preallocate memory for coord, slope arrays
%  to be used in quiver, must be same dimension
xcoord = zeros(Nx, Ny);
ycoord = zeros(Nx, Ny);
dx = zeros(Nx, Ny);
dy = zeros(Nx, Ny);
A = [1, 2;          % Enter A
     -5, -1];
for i = 1:Nx
    for j = 1:Ny
        xcoord(i,j) = x(i);
        ycoord(i,j) = y(j);
        v = A*[x(i);y(j)];  % get the slope
        s = norm(v);  % length of v
        % make slope vector a unit vector
        dx(i,j) = v(1)/s;
        dy(i,j) = v(2)/s;
    end
end
% plot the slope vectors at (xcoord, ycoord)
quiver(xcoord, ycoord, dx, dy, 0.5, 'r')
xline(0);  % show x and y axes
yline(0);
axis([Xmin, Xmax, Ymin, Ymax])  % plot bounds
xlabel 'x1', ylabel 'x2'
title 'Direction Field for X'' = AX'
% get evalues, evectors for trajectories
[V,D] = eig(A);
ev1 = D(1,1)   % eigenvalues
ev2 = D(2,2)
syms z
for n = 1:2    % eigenvectors
    R = rref(A - D(n,n)*eye(2));
    % free variable = 1, S for  pivot
    S = solve(R(1,1)*z + R(1,2) == 0,z);
    if n == 1
        evec1 = [S;1]
    else
        evec2 = [S;1]
    end
end
%  plot phase portrait trajectories
%    over direction field
hold on
% size interval to exceed plot bounds
t = -5:0.05:5;
if ~isreal(ev1)
    x = evec1*exp(ev1*t);
    x1 = real(x);
    x2 = imag(x);
else
    x1 = evec1*exp(ev1*t);
    x2 = evec2*exp(ev2*t);
end
% create array of various coeff values
c = [1, 0, -2, -2, -2,  0,  2, 2;
     0, 1, -2,  0,  2, -2, -2, 2];
for n = 1:size(c,2)  % up to # columns of c[]
    p = c(1,n)*x1 + c(2,n)*x2;
    u = p(1,:);  % top row
    v = p(2,:);  % bottom row
    if (c(1,n)==1) && (c(2,n)==0)
        plot(u,v, '-k')
    elseif (c(1,n)==0) && (c(2,n)==1)
        plot(u,v, '--k')
    else
        plot(u,v)
    end
end
```

Plot on next page.



ev1 = -0.0000 + 3.0000i

ev2 = -0.0000 - 3.0000i

evec1 =

$$\begin{pmatrix} -\frac{1}{5} - \frac{3}{5}\mathrm{i} \\ 1 \end{pmatrix}$$

evec2 =

$$\begin{pmatrix} -\frac{1}{5} + \frac{3}{5}\mathrm{i} \\ 1 \end{pmatrix}$$

(b)

From MATLAB, eigenvalues: $3i, -3i$

Corresponding eigenvectors: $\begin{bmatrix} -1-3i \\ 5 \end{bmatrix}, \begin{bmatrix} -1+3i \\ 5 \end{bmatrix}$

scaling by 5.

$$\therefore \vec{X}^{(1)}(t) = \begin{bmatrix} -1-3i \\ 5 \end{bmatrix} e^{0t} \left[\cos(3t) + i\sin(3t)\right]$$

$$= \begin{bmatrix} -\cos(3t) + 3\sin(3t) \\ 5\cos(3t) \end{bmatrix} + i \begin{bmatrix} -3\cos(3t) - \sin(3t) \\ 5\sin(3t) \end{bmatrix}$$

$$\det\left(\vec{X}^{(1)}(t)\right) = -5\cos(3t)\sin(3t) + 15\sin^2(3t) + 15\cos^2(3t) + 5\cos(3t)\sin(3t)$$

$= 15 \neq 0$. $\therefore$ real and imaginary components are independent.

$$\therefore \quad \boxed{\vec{x}(t) = c_1 \begin{bmatrix} -\cos(3t) + 3\sin(3t) \\ 5\cos(3t) \end{bmatrix} + c_2 \begin{bmatrix} -3\cos(3t) - \sin(3t) \\ 5\sin(3t) \end{bmatrix}}$$

Note: Multiplying evector $\begin{bmatrix} -1-3i \\ 5 \end{bmatrix}$ by $-1+3i$ yields $\begin{bmatrix} 10 \\ -5+15i \end{bmatrix}$ or scaling by $1/5$, $\begin{bmatrix} 2 \\ -1+3i \end{bmatrix}$. Using this eigenvector with $\lambda = 3i$ yields answer in back of book.

(c)

As $t \to \infty$, $\vec{x}(t)$ moves in a closed loop (an ellipse) since each component of $\vec{x}(t)$ has a period of $\frac{2\pi}{3}$.

5.

Using MATLAB,

```
clear
A = sym([1, 0, 0;
         2, 1, -2;
         3, 2, 1]);
% get evectors, evalues
[V,D] = eig(A)
```

V =

$$\begin{pmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & -i & i \\ 1 & 1 & 1 \end{pmatrix}$$

D =

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-2i & 0 \\ 0 & 0 & 1+2i \end{pmatrix}$$

Using $\lambda_1 = 1$, and scaling $\begin{bmatrix} 1 \\ -3/2 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}$,

$$\vec{x}^{(1)}(t) = e^{-t} \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}$$

Using $\lambda_2 = 1-2i$, $\begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix}$,

$$\vec{x}^{(2)}(t) = \begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix} e^{t} \left[\cos(2t) - i\sin(2t)\right]$$

$$= e^{t} \left(\underbrace{\begin{bmatrix} 0 \\ -\sin(2t) \\ \cos(2t) \end{bmatrix}}_{\text{real}} + i \underbrace{\begin{bmatrix} 0 \\ -\cos(2t) \\ -\sin(2t) \end{bmatrix}}_{\text{imaginary}} \right)$$

Note $\det\begin{bmatrix} 2 & 0 & 0 \\ -3 & -\sin(2t) & -\cos(2t) \\ 2 & \cos(2t) & -\sin(2t) \end{bmatrix} = 2(\sin^2(2t)+\cos^2(2t)) = 2 \neq 0$

$\therefore$ The real component eigenvectors are independent.

$\therefore$ $$\boxed{\vec{x}(t) = c_1 e^{-t}\begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix} + c_2 e^{t}\begin{bmatrix} 0 \\ -\sin(2t) \\ \cos(2t) \end{bmatrix} + c_3 e^{t}\begin{bmatrix} 0 \\ \cos(2t) \\ \sin(2t) \end{bmatrix}}$$

where c_1, c_2, c_3 are constants, which may incorporate some of the minus signs above.

6.

Using MATLAB,

```
clear
A = sym([-3, 0, 2;
          1, -1, 0;
          -2, -1, 0]);
% get evectors, evalues
[V,D] = eig(A)
```

V =

$$\begin{pmatrix} 2 & \frac{2}{3}+\frac{\sqrt{2}\,i}{3} & \frac{2}{3}-\frac{\sqrt{2}\,i}{3} \\ -2 & -\frac{1}{3}+\frac{\sqrt{2}\,i}{3} & -\frac{1}{3}-\frac{\sqrt{2}\,i}{3} \\ 1 & 1 & 1 \end{pmatrix}$$

D =

$$\begin{pmatrix} -2 & 0 & 0 \\ 0 & -1-\sqrt{2}\,i & 0 \\ 0 & 0 & -1+\sqrt{2}\,i \end{pmatrix}$$

Using $\lambda_1 = -2$, $\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$, $\vec{x}^{(1)}(t) = e^{-2t}\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

Using $\lambda_3 = -1+\sqrt{2}\,i$, scaling $\begin{bmatrix} 2/3 - \sqrt{2}/3\, i \\ -1/3 - \sqrt{2}/3\, i \\ 1 \end{bmatrix}$ to

$$\begin{bmatrix} 2-\sqrt{2}\,i \\ -1-\sqrt{2}\,i \\ 3 \end{bmatrix} \xrightarrow{\times(2+\sqrt{2}\,i)} \begin{bmatrix} 6 \\ -3\sqrt{2}\,i \\ 6+3\sqrt{2}\,i \end{bmatrix} \xrightarrow{\div 3} \begin{bmatrix} 2 \\ -\sqrt{2}\,i \\ 2+\sqrt{2}\,i \end{bmatrix} \xrightarrow{\times \frac{\sqrt{2}}{2}} \begin{bmatrix} \sqrt{2} \\ -i \\ \sqrt{2}+i \end{bmatrix}$$

$$\therefore \vec{x}^{(3)}(t) = e^{-t} \begin{bmatrix} \sqrt{2} \\ -i \\ \sqrt{2}+i \end{bmatrix} \left(\cos(\sqrt{2}t) + i\sin(\sqrt{2}t)\right)$$

$$= e^{-t} \begin{bmatrix} \sqrt{2}\cos(\sqrt{2}t) \\ \sin(\sqrt{2}t) \\ \sqrt{2}\cos(\sqrt{2}t) - \sin(\sqrt{2}t) \end{bmatrix}$$

$$+\, i e^{-t} \begin{bmatrix} \sqrt{2}\sin(\sqrt{2}t) \\ -\cos(\sqrt{2}t) \\ \cos(\sqrt{2}t) + \sqrt{2}\sin(\sqrt{2}t) \end{bmatrix}$$

Using MATLAB,

$$\det \begin{bmatrix} 2 & \sqrt{2}\cos(\sqrt{2}t) & \sqrt{2}\sin(\sqrt{2}t) \\ -2 & \sin(\sqrt{2}t) & -\cos(\sqrt{2}t) \\ 1 & \sqrt{2}\cos(\sqrt{2}t) - \sin(\sqrt{2}t) & \cos(\sqrt{2}t) + \sqrt{2}\sin(\sqrt{2}t) \end{bmatrix} = 3\sqrt{2}$$

```
syms t
s = sqrt(2);
A = [ 2,      s*cos(s*t),          s*sin(s*t);
     -2,        sin(s*t),            -cos(s*t)
      1, s*cos(s*t)-sin(s*t), cos(s*t)+s*sin(s*t)];
det(A)
```

ans = $3\sqrt{2}\cos(\sqrt{2}\,t)^2 + 3\sqrt{2}\sin(\sqrt{2}\,t)^2$

$\therefore$ Real component vectors are independent.

$\therefore$

$$\boxed{\begin{aligned}\vec{x}(t) = c_1 e^{-2t}\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} &+ c_2 e^{-t}\begin{bmatrix} \sqrt{2}\cos(\sqrt{2}t) \\ \sin(\sqrt{2}t) \\ \sqrt{2}\cos(\sqrt{2}t) - \sin(\sqrt{2}t) \end{bmatrix} \\ &+ c_3 e^{-t}\begin{bmatrix} \sqrt{2}\sin(\sqrt{2}t) \\ -\cos(\sqrt{2}t) \\ \cos(\sqrt{2}t) + \sqrt{2}\sin(\sqrt{2}t) \end{bmatrix}\end{aligned}}$$

7.

(1) Use MATLAB to find eigenvalues, eigenvectors.

```
clear
% sym gives rational output
A = sym([1, -5;
         1, -3]);
[V,D] = eig(A)
```

V =

$$\begin{pmatrix} 2-i & 2+i \\ 1 & 1 \end{pmatrix}$$

D =

$$\begin{pmatrix} -1-i & 0 \\ 0 & -1+i \end{pmatrix}$$

(2) Find $\vec{x}(t)$

$$\vec{x}^{(1)}(t) = \begin{bmatrix} 2-i \\ 1 \end{bmatrix} e^{-t}\left[\cos(t) - i\sin(t)\right]$$

$$= \underbrace{e^{-t}\begin{bmatrix} 2\cos(t) - \sin(t) \\ \cos(t) \end{bmatrix}}_{\text{real}} + i\underbrace{e^{-t}\begin{bmatrix} -\cos(t) - 2\sin(t) \\ -\sin(t) \end{bmatrix}}_{\text{imaginary}}$$

$$\det\begin{bmatrix} e^{-t}(2\cos(t)-\sin(t)) & e^{-t}(-\cos(t)-2\sin(t)) \\ e^{-t}\cos(t) & -e^{-t}\sin(t) \end{bmatrix}$$

$$= e^{-2t}\left[-2\cos(t)\sin(t)+\sin^2(t)+\cos^2(t)+2\cos(t)\sin(t)\right]$$

$= e^{-2t} \neq 0$. $\therefore$ Real and imaginary components are independent.

$$\therefore\ \vec{x}(t) = c_1 e^{-t}\begin{bmatrix} 2\cos(t)-\sin(t) \\ \cos(t) \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} \cos(t)+2\sin(t) \\ \sin(t) \end{bmatrix}$$

(3) Find c_1, c_2 constants from initial condition

$$\vec{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = c_1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Using MATLAB,

```
A = sym([2, 1;
         1, 0]);
B = [1; 1];
linsolve(A,B)  % solve Ax=B
```

ans =

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$\therefore\ c_1 = 1,\ c_2 = -1$

$$\therefore\ \vec{x}(t) = e^{-t}\begin{bmatrix} 2\cos(t)-\sin(t) \\ \cos(t) \end{bmatrix} - e^{-t}\begin{bmatrix} \cos(t)+2\sin(t) \\ \sin(t) \end{bmatrix}$$

Or, $\boxed{\vec{x}(t) = e^{-t}\begin{bmatrix} \cos(t)-3\sin(t) \\ \cos(t)-\sin(t) \end{bmatrix}}$

As $t \to \infty$, $\vec{x}(t)$ spirals toward $(0,0)$, the origin.

8.

(1) Using MATLAB to get eigenvalues, eigenvectors:

```
clear
% sym gives rational output
A = sym([-3, 2;
         -1, -1]);
[V,D] = eig(A)
```

V =

$$\begin{pmatrix} 1+i & 1-i \\ 1 & 1 \end{pmatrix}$$

D =

$$\begin{pmatrix} -2-i & 0 \\ 0 & -2+i \end{pmatrix}$$

(2) Find $\vec{x}(t)$

$$\vec{x}^{(1)}(t) = \begin{bmatrix} 1+i \\ 1 \end{bmatrix} e^{-2t}\left[\cos(t) - i\sin(t)\right]$$

$$= \underbrace{e^{-2t}\begin{bmatrix} \cos(t)+\sin(t) \\ \cos(t) \end{bmatrix}}_{\text{real}} + i\,\underbrace{e^{-2t}\begin{bmatrix} \cos(t)-\sin(t) \\ -\sin(t) \end{bmatrix}}_{\text{imaginary}}$$

$$\det\begin{bmatrix} e^{-2t}(\cos(t)+\sin(t)) & e^{-2t}(\cos(t)-\sin(t)) \\ e^{-2t}\cos(t) & -e^{-2t}\sin(t) \end{bmatrix} = -e^{-4t} \neq 0$$

$\therefore$ real, imaginary component vectors independent.

$$\therefore \quad \vec{x}(t) = c_1 e^{-2t}\begin{bmatrix} \cos(t)+\sin(t) \\ \cos(t) \end{bmatrix} + c_2 e^{-2t}\begin{bmatrix} \cos(t)-\sin(t) \\ -\sin(t) \end{bmatrix}$$

(3) Find c_1, c_2 constants from initial condition

$$\vec{x}(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$\therefore$ $c_1 = -2$, $c_2 = 3$

$$\therefore \quad \vec{x}(t) = -2e^{-2t}\begin{bmatrix} \cos(t) + \sin(t) \\ \cos(t) \end{bmatrix} + 3e^{-2t}\begin{bmatrix} \cos(t) - \sin(t) \\ -\sin(t) \end{bmatrix}$$

Or,

$$\boxed{\vec{x}(t) = e^{-2t}\begin{bmatrix} \cos(t) - 5\sin(t) \\ -2\cos(t) - 3\sin(t) \end{bmatrix}}$$

As $t \to \infty$, $\vec{x}(t)$ spirals toward $(0,0)$, the origin.

9.

(a)

Using MATLAB,

```
clear
% sym gives rational output
A = sym([3/4, -2;
         1, -5/4]);
[V,D] = eig(A)
```

V =

$$\begin{pmatrix} 1-\mathrm{i} & 1+\mathrm{i} \\ 1 & 1 \end{pmatrix}$$

D =

$$\begin{pmatrix} -\frac{1}{4}-\mathrm{i} & 0 \\ 0 & -\frac{1}{4}+\mathrm{i} \end{pmatrix}$$

$\therefore$ Eigenvalues: $-\frac{1}{4} - i$, $\underline{-\frac{1}{4} + i}$

(b)

From (a), $\vec{x}^{(2)}(t) = \begin{bmatrix} 1+i \\ 1 \end{bmatrix} e^{-t/4}(\cos(t) + i\sin(t))$

$$= e^{-t/4}\begin{bmatrix} \cos(t) - \sin(t) \\ \cos(t) \end{bmatrix} + ie^{-t/4}\begin{bmatrix} \cos(t) + \sin(t) \\ \sin(t) \end{bmatrix}$$

real imaginary

$$\det\begin{bmatrix} e^{-t/4}(\cos(t) - \sin(t)) & e^{-t/4}(\cos(t) + \sin(t)) \\ e^{-t/4}\cos(t) & e^{-t/4}\sin(t) \end{bmatrix}$$

$= -e^{-t/2} \neq 0$. $\therefore$ Real and imaginary component vectors are independent.

$$\therefore \vec{x}(t) = c_1 e^{-t/4}\begin{bmatrix} \cos(t) - \sin(t) \\ \cos(t) \end{bmatrix} + c_2 e^{-t/4}\begin{bmatrix} \cos(t) + \sin(t) \\ \sin(t) \end{bmatrix}$$

$$\vec{x}(0) = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

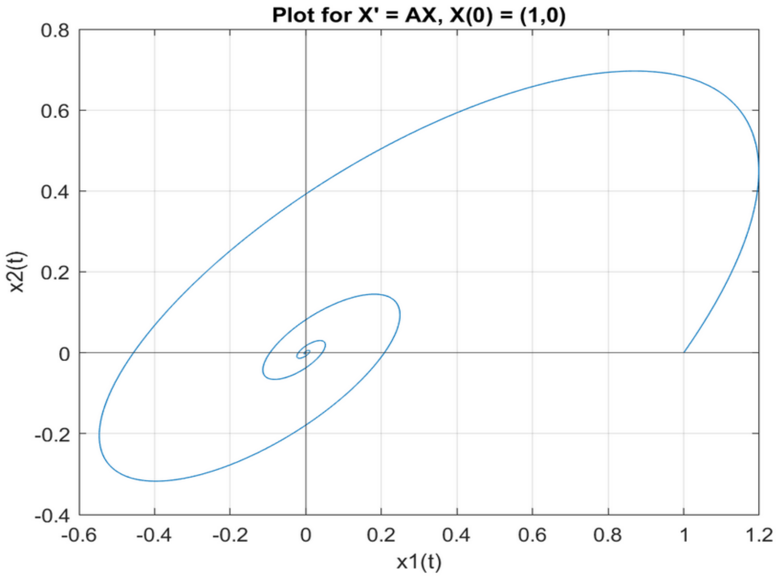
Choose $c_1 = 0$, $c_2 = 1$ so $\underline{\underline{\vec{x}(0) = (1, 0)}}$

$$\therefore x_1(t) = e^{-t/4}(\cos(t) + \sin(t))$$

$$x_2(t) = e^{-t/4}\sin(t)$$

Using MATLAB,

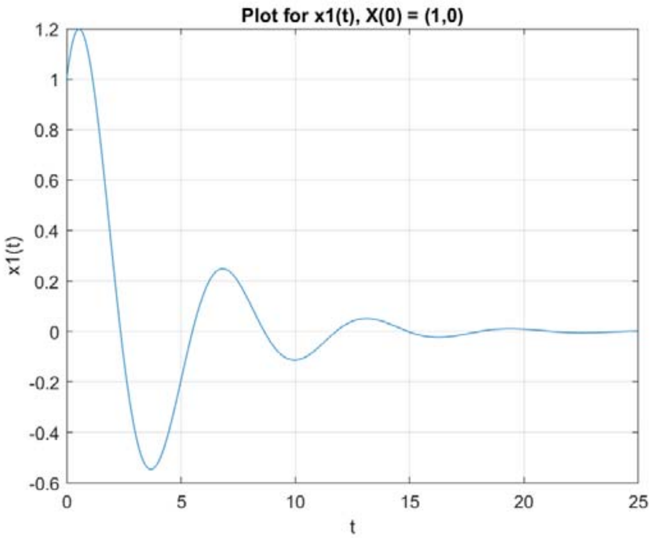
```
t = 0:0.01:25;
x1 = exp(-t/4).*(cos(t) + sin(t));
x2 = exp(-t/4).*sin(t);
plot(x1,x2)
yline(0);  % plot axes
xline(0);
grid on
xlabel 'x1(t)', ylabel 'x2(t)'
title 'Plot for X'' = AX, X(0) =
(1,0)'
```

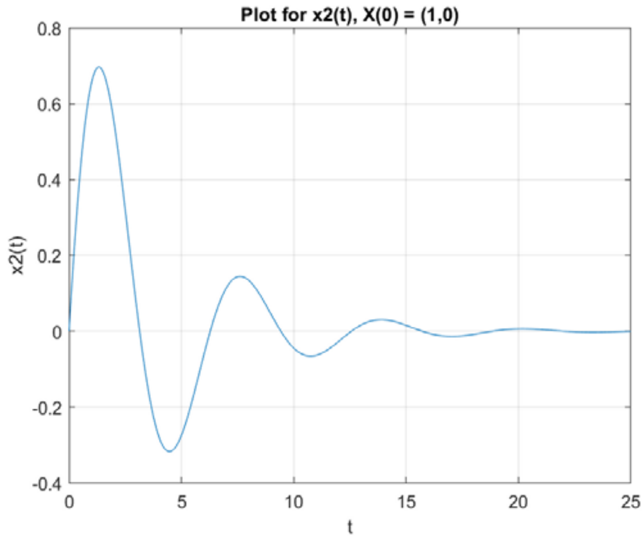


(c)

Using MATLAB and continuing code from above,

```
plot(t,x1)
grid on
xlabel 't', ylabel 'x1(t)'
title 'Plot for x1(t), X(0) = (1,0)'

plot(t,x2)
grid on
xlabel 't', ylabel 'x2(t)'
title 'Plot for x2(t), X(0) = (1,0)'
```

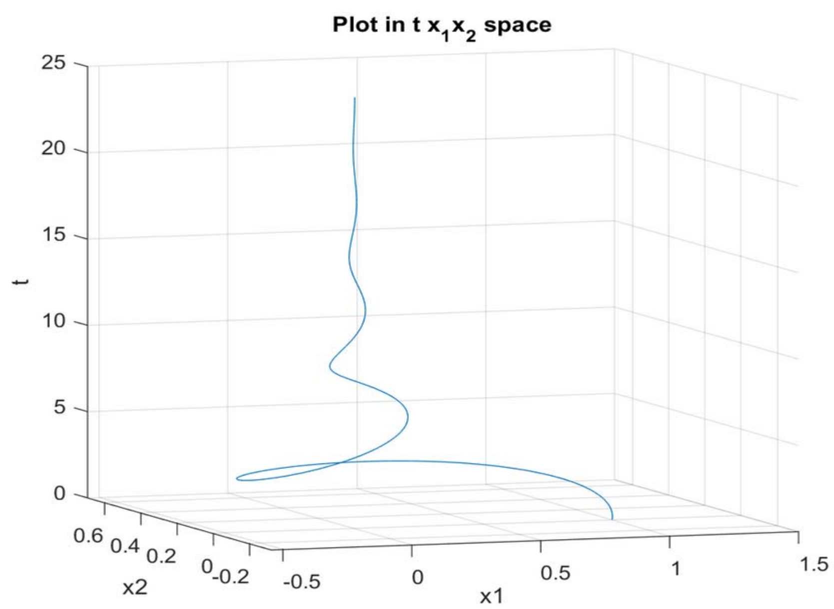




(d)

Using MATLAB, and manually rotating 3D plot, and continuing code from above.

```
plot3(x1,x2,t)
grid on
xlabel 'x1', ylabel 'x2', zlabel 't'
title 'Plot in t x_1x_2 space'
view([-19.2 7.3])  % after manual rotation, MATLAB supplies code
```



10.

(a)

Using MATLAB,

```
clear
% sym gives rational output
A = sym([-4/5, 2;
         -1, 6/5]);
[V,D] = eig(A)
```

V =

$$\begin{pmatrix} 1+i & 1-i \\ 1 & 1 \end{pmatrix}$$

D =

$$\begin{pmatrix} \frac{1}{5}-i & 0 \\ 0 & \frac{1}{5}+i \end{pmatrix}$$

Eigenvalues: $\underline{\frac{1}{5}-i, \ \frac{1}{5}+i}$

(b)

From (a), $\vec{x}^{(2)}(t) = \begin{bmatrix} 1-i \\ 1 \end{bmatrix} e^{t/5}(\cos(t) + i\sin(t))$

$$= \underbrace{e^{t/5}\begin{bmatrix} \cos(t)+\sin(t) \\ \cos(t) \end{bmatrix}}_{\text{real}} + i\underbrace{e^{t/5}\begin{bmatrix} -\cos(t)+\sin(t) \\ \sin(t) \end{bmatrix}}_{\text{imaginary}}$$

$$\det\begin{bmatrix} e^{t/5}(\cos(t)+\sin(t)) & e^{t/5}(-\cos(t)+\sin(t)) \\ e^{t/5}\cos(t) & e^{t/5}\sin(t) \end{bmatrix}$$

$= e^{2t/5} \neq 0$. $\therefore$ Real and imaginary component vectors are independent.

$$\therefore\ \vec{X}(t) = c_1 e^{t/5}\begin{bmatrix}\cos(t)+\sin(t)\\ \cos(t)\end{bmatrix} + c_2 e^{t/5}\begin{bmatrix}-\cos(t)+\sin(t)\\ \sin(t)\end{bmatrix}$$

$$\vec{X}(0) = c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}-1\\0\end{bmatrix}$$

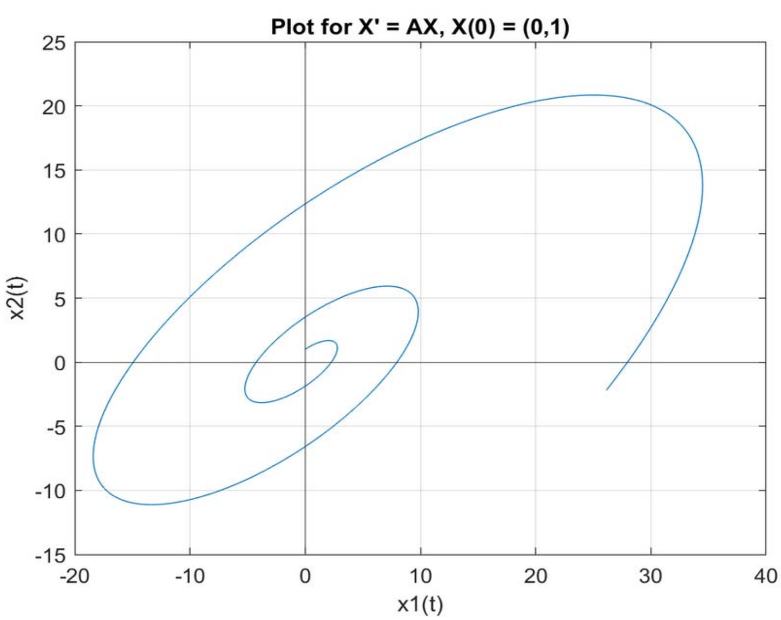
Choose $c_1 = c_2 = 1$ so $\vec{X}(0) = \underline{(0,1)}$

$$\therefore\ x_1(t) = 2e^{t/5}\sin(t)$$

$$x_2(t) = e^{t/5}\left[\cos(t)+\sin(t)\right]$$

Using MATLAB,

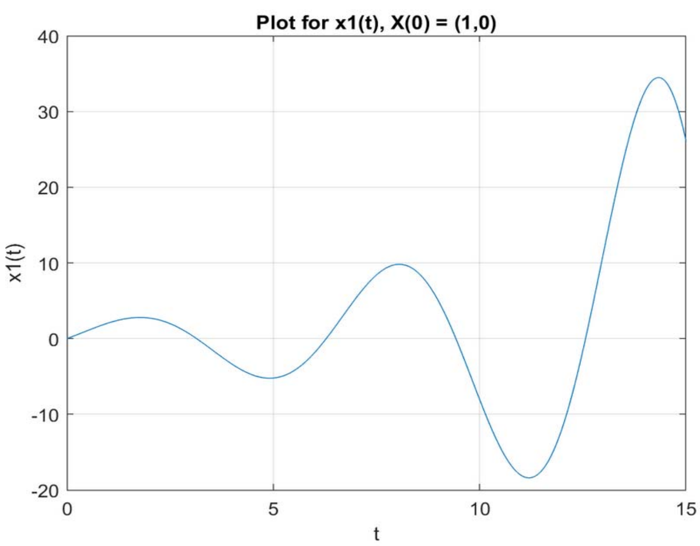
```
t = 0:0.01:15;
x1 = 2*exp(t/5).*sin(t);
x2 = exp(t/5).*(cos(t) + sin(t));
plot(x1,x2)
yline(0);  % plot axes
xline(0);
grid on
xlabel 'x1(t)', ylabel 'x2(t)'
title 'Plot for X'' = AX, X(0) = (0,1)'
```

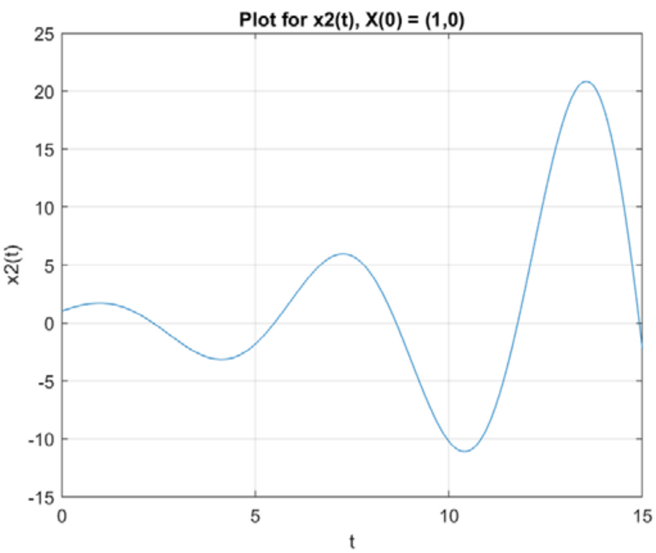


(c)

Using MATLAB, and continuing code from above,

```
plot(t,x1)
grid on
xlabel 't', ylabel 'x1(t)'
title 'Plot for x1(t), X(0) = (1,0)'

plot(t,x2)
grid on
xlabel 't', ylabel 'x2(t)'
title 'Plot for x2(t), X(0) = (1,0)'
```

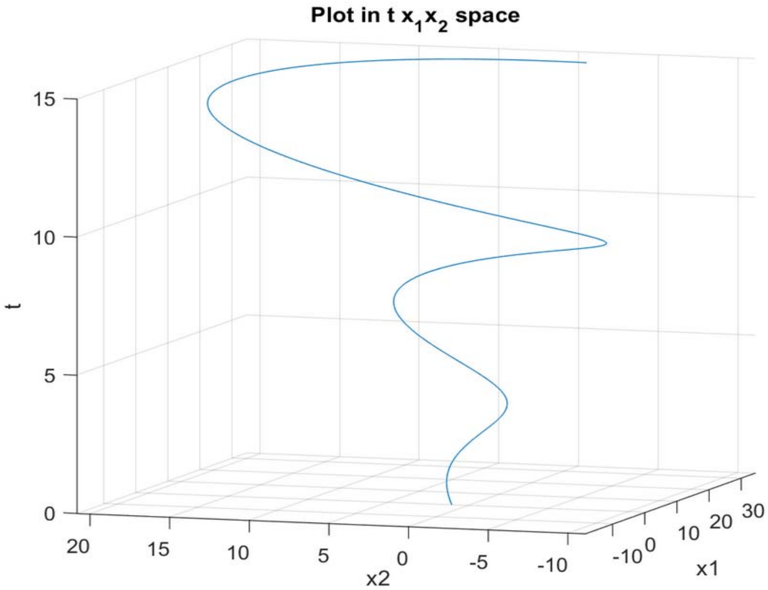




(d)

Using MATLAB, and manually rotating 3D plot, and continuing code from above,

```
plot3(x1,x2,t)
grid on
xlabel 'x1', ylabel 'x2', zlabel 't'
title 'Plot in t x_1x_2 space'
view([-71.5 8.7])  % after manual rotation, MATLAB supplies code
```



11.

(a) $\det\begin{bmatrix} \alpha-\lambda & 1 \\ -1 & \alpha-\lambda \end{bmatrix} = 0 \Rightarrow (\alpha-\lambda)^2+1=0, \quad \lambda^2 - 2\alpha\lambda + \alpha^2+1=0$

Solving for λ, $\lambda = \dfrac{2\alpha \pm \sqrt{4\alpha^2 - 4(\alpha^2+1)}}{2} = \underline{\alpha \pm i}$

Or, using MATLAB,

```
clear
syms a
A = [a, 1;
     -1, a];
[V,D] = eig(A)
```

V =

$$\begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$$

D =

$$\begin{pmatrix} a-i & 0 \\ 0 & a+i \end{pmatrix}$$

$\therefore$ Eigenvalues : $\alpha - i, \alpha + i$

(b)

For $\lambda = \alpha \pm i$, $\vec{x}^{(1)}(t) = [\ ] e^{\alpha t}(\cos(t) \pm i\sin(t))$

$\therefore$ For $\alpha > 0$, solutions spiral to ∞ as $t \to \infty$.

For $\alpha < 0$, $e^{\alpha t}$ causes solutions to spiral toward $(0,0)$ as $t \to \infty$.

$\therefore$ Bifurcation values for $\underline{\alpha = 0}$.

(c)

Choose $\alpha = \pm 0.1$, use $\lambda = \alpha + i$, eigenvector $\begin{bmatrix} -i \\ 1 \end{bmatrix}$

$$\therefore \vec{x}(t) = \begin{bmatrix} -i \\ 1 \end{bmatrix} e^{\alpha t}(\cos(t) + i\sin(t))$$

$$= \underbrace{e^{\alpha t}\begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}}_{\text{real}} + i\underbrace{e^{\alpha t}\begin{bmatrix} -\cos(t) \\ \sin(t) \end{bmatrix}}_{\text{imaginary}}$$

$$\det \begin{bmatrix} e^{\alpha t}\sin(t) & -e^{\alpha t}\cos(t) \\ e^{\alpha t}\cos(t) & e^{\alpha t}\sin(t) \end{bmatrix} = e^{2\alpha t} \neq 0.$$

$\therefore$ Real and imaginary components are independent.

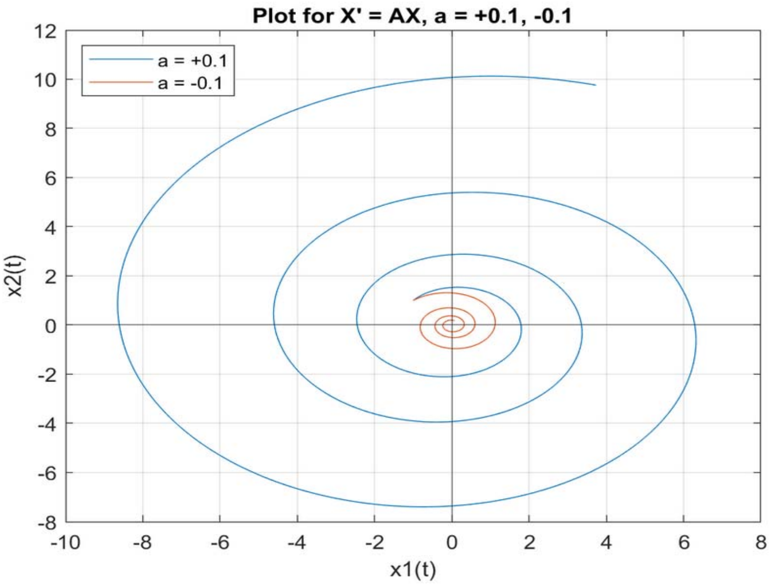
$$\therefore \vec{x}(t) = c_1 e^{\alpha t}\begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix} + c_2 e^{\alpha t}\begin{bmatrix} -\cos(t) \\ \sin(t) \end{bmatrix}$$

$$\therefore \quad x_1(t) = e^{at}\left(\sin(t) - \cos(t)\right) \qquad \text{Using } c_1 = c_2 = 1$$

$$x_2(t) = e^{\alpha t}\left(\cos(t) + \sin(t)\right)$$

Using MATLAB,

```
t = 0:0.01:20;
x1 = exp(a*t).*(sin(t) - cos(t));
x2 = exp(a*t).*(cos(t) + sin(t));
plot(subs(x1,a,0.1),subs(x2,a,0.1))
hold on
plot(subs(x1,a,-0.1),subs(x2,a,-0.1))
yline(0);  % plot axes
xline(0);
grid on
xlabel 'x1(t)', ylabel 'x2(t)'
title 'Plot for X'' = AX, a = +0.1, -0.1'
legend('a = +0.1', 'a = -0.1', 'location','northwest')
```



$\alpha = +0.1$ spirals outward, $\alpha = -0.1$ spirals inward

Note $x_1(0) = -1$, $x_2(0) = 1$

12.

(a)

Using MATLAB,

```
clear
syms a
A = [0, -5;
     1, a];
[V,D] = eig(A)
```

V =

$$\begin{pmatrix} -\frac{a}{2}-\frac{\sqrt{a^2-20}}{2} & \frac{\sqrt{a^2-20}}{2}-\frac{a}{2} \\ 1 & 1 \end{pmatrix}$$

D =

$$\begin{pmatrix} \frac{a}{2}-\frac{\sqrt{a^2-20}}{2} & 0 \\ 0 & \frac{a}{2}+\frac{\sqrt{a^2-20}}{2} \end{pmatrix}$$

$\therefore$ Eigenvalues: $\underline{\frac{\alpha}{2} \pm \frac{\sqrt{\alpha^2-20}}{2}}$

(b)

If $\alpha^2 > 20$, or $|\alpha| \geq \sqrt{20}$, values are real.

Note $\sqrt{a^2-20} < |\alpha|$, so α dominates $\sqrt{\alpha^2-20}$

$\therefore$ For real values, $\underline{\alpha = \pm\sqrt{20}}$ are critical points

For $\lambda = \alpha \geq \sqrt{20}$, $e^{\lambda t} \to \infty$ as $t \to \infty$

For $\lambda = \alpha \leq -\sqrt{20}$, $e^{\lambda t} \to 0$ as $t \to \infty$

If $|\alpha| < \sqrt{20}$, λ is complex, and the nature of $\vec{x}(t)$ depends on $e^{\lambda t}$.

If $0 < \alpha < \sqrt{20}$, then $e^{\lambda t} \to \infty$ as $t \to \infty$

so $\vec{x}(t)$ will spiral outward.

If $-\sqrt{20} < \alpha < 0$, $e^{\lambda t} \to 0$ as $t \to \infty$, so

$\vec{x}(t)$ will spiral toward $(0,0)$.

$\therefore$ $\alpha = 0$ is a critical value.

$\therefore$ Bifurcation values are $\alpha = -\sqrt{20}, 0, \sqrt{20}$

(c)

Use $\alpha = -\sqrt{20} \pm 0.1$, ± 0.1, $\sqrt{20} \pm 0.1$

$\alpha = -\sqrt{20} - 0.1$

$$\therefore \lambda_1 = \frac{\alpha}{2} - \frac{\sqrt{\alpha^2 - 20}}{2}$$

$$\lambda_2 = \frac{\alpha}{2} + \frac{\sqrt{\alpha^2 - 20}}{2}$$

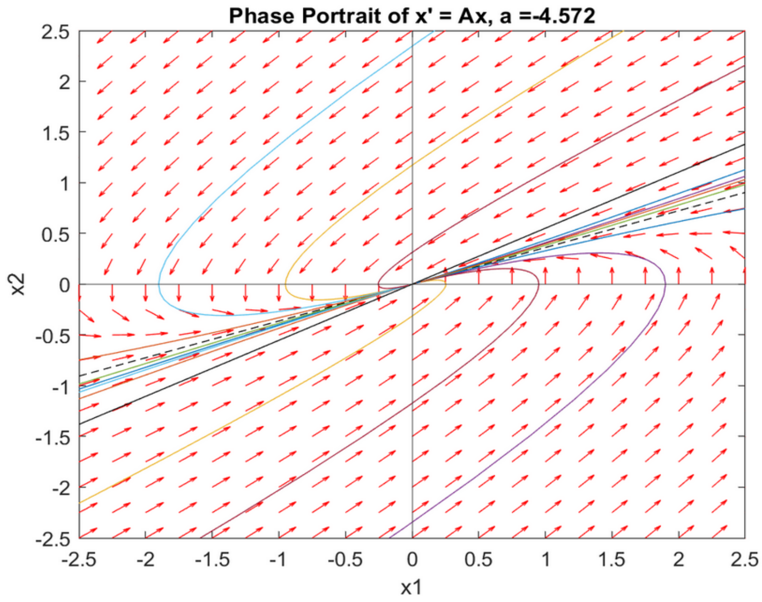
Both are negative and real

$$\vec{x}(t) = c_1 e^{\lambda_1 t} \begin{bmatrix} -\frac{\alpha}{2} - \frac{\sqrt{\alpha^2-20}}{2} \\ 1 \end{bmatrix} + c_2 e^{\lambda_2 t} \begin{bmatrix} -\frac{\alpha}{2} + \frac{\sqrt{\alpha^2-20}}{2} \\ 1 \end{bmatrix}$$

Plot sample trajectories and direction field.

MATLAB code on next page, in two colums

```
clear
syms a
M = [0, -5;
     1, a];
nv = -sqrt(20) - 0.1;  % new value
A = subs(M,a,nv);
[V,D] = eig(A);

% set plot boundaries
Xmin = -2.5; Xmax = 2.5;
Ymin = -2.5; Ymax = 2.5;

% Direction grid
% set how fine to make the grid of values
Step = 0.5;
% total # points on x and y axes
Nx = round((Xmax-Xmin)/Step, 0) + 1;
Ny = round((Ymax-Ymin)/Step, 0) + 1;
% set the actual x and y coords
x = linspace(Xmin, Xmax, Nx);  % a row vector
y = linspace(Ymin, Ymax, Ny);  % a row vector
% preallocate memory for coord, slope arrays
%  to be used in quiver, must be same dimension
xcoord = zeros(Nx, Ny);
ycoord = zeros(Nx, Ny);
dx = zeros(Nx, Ny);
dy = zeros(Nx, Ny);
for i = 1:Nx
    for j = 1:Ny
        xcoord(i,j) = x(i);
        ycoord(i,j) = y(j);
        v = A*[x(i);y(j)];  % get the slope
        s = norm(v);  % length of v
        % make slope vector a unit vector
        dx(i,j) = v(1)/s;
        dy(i,j) = v(2)/s;
    end
end
% plot the slope vectors at (xcoord, ycoord)
quiver(xcoord, ycoord, dx, dy, 0.5, 'r')

% Trajectories

ev1 = D(1,1);    % eigenvalues
ev2 = D(2,2);
evec1 = V(:,1);  % eigenvectors
evec2 = V(:,2);
% compute slopes of asymptotic lines
m1 = evec1(2)/evec1(1);
m2 = evec2(2)/evec2(1);
t = -2.5:0.05:2.5;
%figure
xlim([Xmin,Xmax])
ylim([Ymin,Ymax])
hold on
xline(0); yline(0);  % axes
plot(t,m1*t, 'k')  % asymptotic lines
plot(t,m2*t, '--k')
s1 = evec1*exp(ev1*t);
s2 = evec2*exp(ev2*t);
% plot trajectories for various c1,c2
for c1 = [-2,-1,1,2]
    for c2 = [-2,-1,1,2]
        x = c1*s1 + c2*s2;
        x1 = x(1,:);
        x2 = x(2,:);
        plot(x1,x2)
    end
end
xlabel 'x1', ylabel 'x2'
str = sprintf('%.3f',nv);
s = strcat('Phase Portrait of x'' = Ax, a = ',str);
title(s)
```



$$a = -\sqrt{20} - 0.1 \approx -4.572$$

Trajectories spiral toward origin as $t \to \infty$.

$\alpha = -\sqrt{20} + 0.1$

Use $\lambda_1 = \frac{\alpha}{2} - \frac{\sqrt{\alpha^2-20}}{2}$, which is imaginary

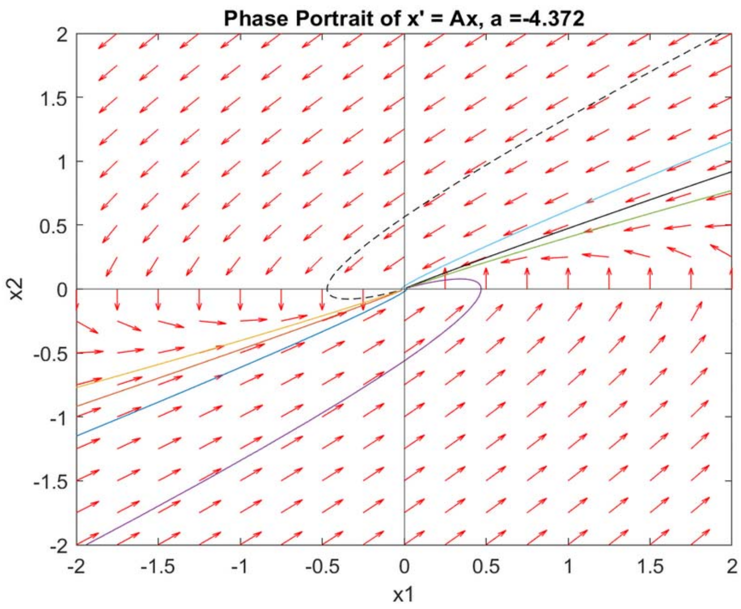
eigenvector: $\begin{bmatrix} -\frac{\alpha}{2} - \frac{\sqrt{\alpha^2-20}}{2} \\ 1 \end{bmatrix} = \vec{ev}$

$\therefore \vec{x}^{(1)}(t) = e^{\lambda_1} \vec{ev}$. MATLAB codes breaks up into real and imaginary parts.

```
clear
syms a
M = [0, -5;
     1, a];
nv = -sqrt(20) + 0.1;  % new value
A = subs(M,a,nv);
[V,D] = eig(A);

% set plot boundaries
Xmin = -2.0; Xmax = 2.0;
Ymin = -2.0; Ymax = 2.0;

% Direction grid
% set how fine to make the grid of values
Step = 0.25;
% total # points on x and y axes
Nx = round((Xmax-Xmin)/Step, 0) + 1;
Ny = round((Ymax-Ymin)/Step, 0) + 1;
% set the actual x and y coords
x = linspace(Xmin, Xmax, Nx);  % a row vector
y = linspace(Ymin, Ymax, Ny);  % a row vector
% preallocate memory for coord, slope arrays
%  to be used in quiver, must be same dimension
xcoord = zeros(Nx, Ny);
ycoord = zeros(Nx, Ny);
dx = zeros(Nx, Ny);
dy = zeros(Nx, Ny);
for i = 1:Nx
    for j = 1:Ny
        xcoord(i,j) = x(i);
        ycoord(i,j) = y(j);
        v = A*[x(i);y(j)];  % get the slope
        s = norm(v);  % length of v
        % make slope vector a unit vector
        dx(i,j) = v(1)/s;
        dy(i,j) = v(2)/s;
    end
end
% plot the slope vectors at (xcoord, ycoord)
quiver(xcoord, ycoord, dx, dy, 0.5, 'r')
xline(0);  % show x and y axes
yline(0);
axis([Xmin, Xmax, Ymin, Ymax])  % plot bounds
xlabel 'x1', ylabel 'x2'
str = sprintf('%.3f',nv);  % display value of alpha
s = strcat('Phase Portrait of x'' = Ax, a =  ',str);
title(s)
```

```
%  Trajectories

ev1 = D(1,1);    % eigenvalues
ev2 = D(2,2);
evec1 = V(:,1);  % eigenvectors
evec2 = V(:,2);
hold on
% size interval to exceed plot bounds
t = -2:0.05:2;
% just use ev1, evec1
%   as real,imag parts are independent
x = evec1*exp(ev1*t);
x1 = real(x);
x2 = imag(x);

% create array of various coeff values
c = [1, 0, -1, -1, -1,  0,  1, 1;
     0, 1, -1,  0,  1, -1, -1, 1];
for n = 1:size(c,2)  % up to # columns of c[]
    p = c(1,n)*x1 + c(2,n)*x2;
    u = p(1,:);  % top row
    v = p(2,:);  % bottom row
    if (c(1,n)==1) && (c(2,n)==0)
        plot(u,v, '-k')
    elseif (c(1,n)==0) && (c(2,n)==1)
        plot(u,v, '--k')
    else
        plot(u,v)
    end
end
```

Plot on next page.



Trajectories spiral toward $(0,0)$ as $\text{real}(\lambda_1) < 0$.

$$\alpha = -\sqrt{20} + 0.1 \approx -4.372$$

$\alpha = 0 - 0.1$

Use $\lambda_1 = \frac{\alpha}{2} - \frac{\sqrt{\alpha^2 - 20}}{2}$, which is imaginary

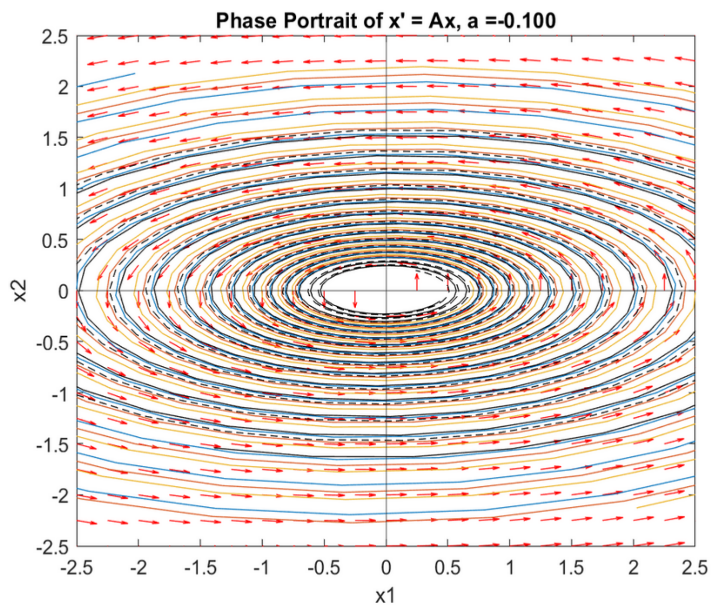
eigenvector: $\begin{bmatrix} -\frac{\alpha}{2} - \frac{\sqrt{\alpha^2-20}}{2} \\ 1 \end{bmatrix} = \vec{ev}$

$\therefore \vec{x}^{(1)}(t) = e^{\lambda_1} \vec{ev}$. MATLAB codes breaks up into real and imaginary parts.

MATLAB code on next page.

```
clear
syms a
M = [0, -5;
     1, a];
nv = 0 - 0.1;  % new value
A = subs(M,a,nv);
[V,D] = eig(A);

% set plot boundaries
Xmin = -2.5; Xmax = 2.5;
Ymin = -2.5; Ymax = 2.5;

% Direction grid
% set how fine to make the grid of values
Step = 0.25;
% total # points on x and y axes
Nx = round((Xmax-Xmin)/Step, 0) + 1;
Ny = round((Ymax-Ymin)/Step, 0) + 1;
% set the actual x and y coords
x = linspace(Xmin, Xmax, Nx);  % a row vector
y = linspace(Ymin, Ymax, Ny);  % a row vector
% preallocate memory for coord, slope arrays
%  to be used in quiver, must be same dimension
xcoord = zeros(Nx, Ny);
ycoord = zeros(Nx, Ny);
dx = zeros(Nx, Ny);
dy = zeros(Nx, Ny);
for i = 1:Nx
    for j = 1:Ny
        xcoord(i,j) = x(i);
        ycoord(i,j) = y(j);
        v = A*[x(i);y(j)];  % get the slope
        s = norm(v);  % length of v
        % make slope vector a unit vector
        dx(i,j) = v(1)/s;
        dy(i,j) = v(2)/s;
    end
end
% plot the slope vectors at (xcoord, ycoord)
quiver(xcoord, ycoord, dx, dy, 0.5, 'r')
xline(0);  % show x and y axes
yline(0);
axis([Xmin, Xmax, Ymin, Ymax])  % plot bounds
xlabel 'x1', ylabel 'x2'
str = sprintf('%.3f',nv);  % display value of alpha
s = strcat('Phase Portrait of x'' = Ax, a =  ',str);
title(s)
```

```
%  Trajectories

ev1 = D(1,1);    % eigenvalues
ev2 = D(2,2);
evec1 = V(:,1);  % eigenvectors
evec2 = V(:,2);
hold on
% size interval to exceed plot bounds
t = -10:0.1:30;
% just use ev1, evec1
%   as real,imag parts are independent
x = evec1*exp(ev1*t);
x1 = real(x);
x2 = imag(x);

% create array of various coeff values
c = [1, 0, -1,  1, 1;
     0, 1, -1, -1, 1];
for n = 1:size(c,2)  % up to # columns of c[]
    p = c(1,n)*x1 + c(2,n)*x2;
    u = p(1,:);  % top row
    v = p(2,:);  % bottom row
    if (c(1,n)==1) && (c(2,n)==0)
        plot(u,v, '-k')
    elseif (c(1,n)==0) && (c(2,n)==1)
        plot(u,v, '--k')
    else
        plot(u,v)
    end
end
```



$\alpha = -0.1$

$real(\lambda_1) < 0$

plot shows $-10 \le t \le 30$

Trajectories spiral toward $(0,0)$ as $t \to \infty$

$\alpha = 0 + 0.1$

Use $\lambda_1 = \frac{\alpha}{2} - \frac{\sqrt{\alpha^2-20}}{2}$, which is imaginary

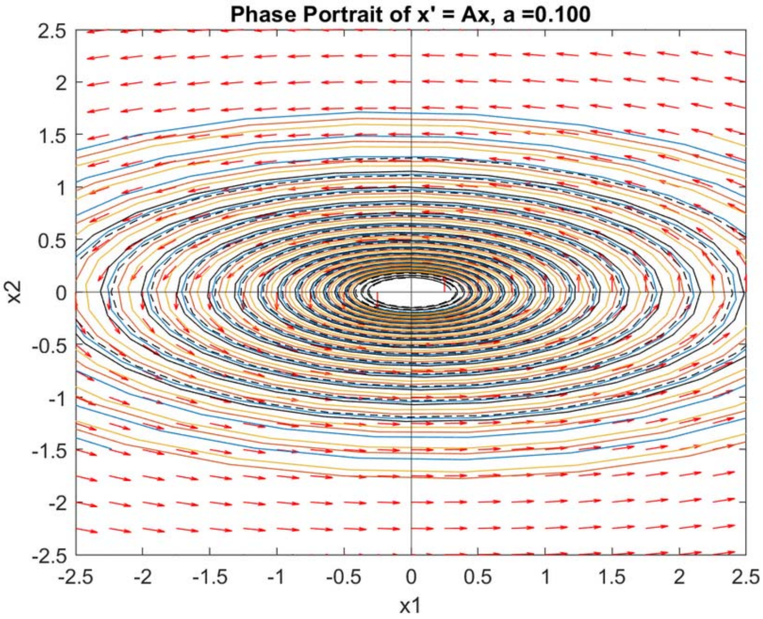
eigenvector: $\begin{bmatrix} -\frac{\alpha}{2} - \frac{\sqrt{\alpha^2-20}}{2} \\ 1 \end{bmatrix} = \vec{ev}$

$\therefore \vec{X}^{(1)}(t) = e^{\lambda_1} \vec{ev}$. MATLAB codes breaks up into real and imaginary parts.

```
clear
syms a
M = [0, -5;
     1, a];
nv = 0 + 0.1;  % new value
A = subs(M,a,nv);
[V,D] = eig(A);

% set plot boundaries
Xmin = -2.5; Xmax = 2.5;
Ymin = -2.5; Ymax = 2.5;

% Direction grid
% set how fine to make the grid of values
Step = 0.25;
% total # points on x and y axes
Nx = round((Xmax-Xmin)/Step, 0) + 1;
Ny = round((Ymax-Ymin)/Step, 0) + 1;
% set the actual x and y coords
x = linspace(Xmin, Xmax, Nx);  % a row vector
y = linspace(Ymin, Ymax, Ny);  % a row vector
% preallocate memory for coord, slope arrays
%  to be used in quiver, must be same dimension
xcoord = zeros(Nx, Ny);
ycoord = zeros(Nx, Ny);
dx = zeros(Nx, Ny);
dy = zeros(Nx, Ny);
for i = 1:Nx
    for j = 1:Ny
        xcoord(i,j) = x(i);
        ycoord(i,j) = y(j);
        v = A*[x(i);y(j)];  % get the slope
        s = norm(v);  % length of v
        % make slope vector a unit vector
        dx(i,j) = v(1)/s;
        dy(i,j) = v(2)/s;
    end
end
% plot the slope vectors at (xcoord, ycoord)
quiver(xcoord, ycoord, dx, dy, 0.5, 'r')
xline(0);  % show x and y axes
yline(0);
axis([Xmin, Xmax, Ymin, Ymax])  % plot bounds
xlabel 'x1', ylabel 'x2'
str = sprintf('%.3f',nv);  % display value of alpha
s = strcat('Phase Portrait of x'' = Ax, a =  ',str);
title(s)
```

```
%  Trajectories

ev1 = D(1,1);    % eigenvalues
ev2 = D(2,2);
evec1 = V(:,1);  % eigenvectors
evec2 = V(:,2);
hold on
% size interval to exceed plot bounds
t = -40:0.1:5;
% just use ev1, evec1
%   as real,imag parts are independent
x = evec1*exp(ev1*t);
x1 = real(x);
x2 = imag(x);

% create array of various coeff values
c = [1, 0, -1,  1, 1;
     0, 1, -1, -1, 1];
for n = 1:size(c,2)  % up to # columns of c[]
    p = c(1,n)*x1 + c(2,n)*x2;
    u = p(1,:);  % top row
    v = p(2,:);  % bottom row
    if (c(1,n)==1) && (c(2,n)==0)
        plot(u,v, '-k')
    elseif (c(1,n)==0) && (c(2,n)==1)
        plot(u,v, '--k')
    else
        plot(u,v)
    end
end
```

Plot on next page.



$\alpha = +0.1$

$\text{real}(\lambda_1) > 0$

Plot shows $-40 \leq t \leq 5$

Trajectories spiral away from $(0,0)$ as $t \to \infty$

$\alpha = \sqrt{20} - 0.1$

Use $\lambda_1 = \frac{\alpha}{2} - \frac{\sqrt{\alpha^2 - 20}}{2}$, which is imaginary

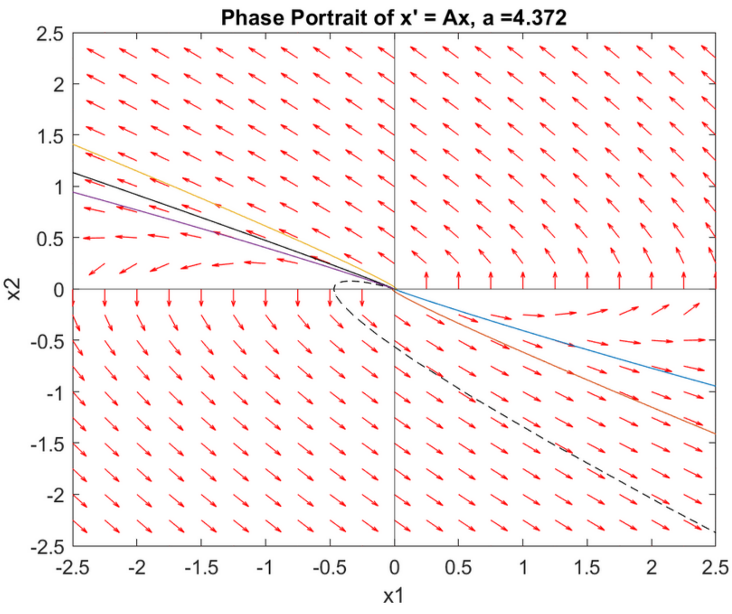
eigenvector: $\begin{bmatrix} -\frac{\alpha}{2} - \frac{\sqrt{\alpha^2-20}}{2} \\ 1 \end{bmatrix} = \vec{ev}$

$\therefore \vec{x}^{(1)}(t) = e^{\lambda_1} \vec{ev}$. MATLAB codes breaks up into real and imaginary parts.

Code on next page.

```
clear
syms a
M = [0, -5;
     1, a];
nv = sqrt(20) - 0.1;  % new value
A = subs(M,a,nv);
[V,D] = eig(A);

% set plot boundaries
Xmin = -2.5; Xmax = 2.5;
Ymin = -2.5; Ymax = 2.5;

% Direction grid
% set how fine to make the grid of values
Step = 0.25;
% total # points on x and y axes
Nx = round((Xmax-Xmin)/Step, 0) + 1;
Ny = round((Ymax-Ymin)/Step, 0) + 1;
% set the actual x and y coords
x = linspace(Xmin, Xmax, Nx);  % a row vector
y = linspace(Ymin, Ymax, Ny);  % a row vector
% preallocate memory for coord, slope arrays
%  to be used in quiver, must be same dimension
xcoord = zeros(Nx, Ny);
ycoord = zeros(Nx, Ny);
dx = zeros(Nx, Ny);
dy = zeros(Nx, Ny);
for i = 1:Nx
    for j = 1:Ny
        xcoord(i,j) = x(i);
        ycoord(i,j) = y(j);
        v = A*[x(i);y(j)];  % get the slope
        s = norm(v);  % length of v
        % make slope vector a unit vector
        dx(i,j) = v(1)/s;
        dy(i,j) = v(2)/s;
    end
end
% plot the slope vectors at (xcoord, ycoord)
quiver(xcoord, ycoord, dx, dy, 0.5, 'r')
xline(0);  % show x and y axes
yline(0);
axis([Xmin, Xmax, Ymin, Ymax])  % plot bounds
xlabel 'x1', ylabel 'x2'
str = sprintf('%.3f',nv);  % display value of alpha
s = strcat('Phase Portrait of x'' = Ax, a =  ',str);
title(s)
```

```
%  Trajectories

ev1 = D(1,1);    % eigenvalues
ev2 = D(2,2);
evec1 = V(:,1);  % eigenvectors
evec2 = V(:,2);
hold on
% size interval to exceed plot bounds
t = -5:0.1:5;
% just use ev1, evec1
%   as real,imag parts are independent
x = evec1*exp(ev1*t);
x1 = real(x);
x2 = imag(x);

% create array of various coeff values
c = [1, 0, -1, -1,  1, 1;
     0, 1, -1,  1, -1, 1];
for n = 1:size(c,2)  % up to # columns of c[]
    p = c(1,n)*x1 + c(2,n)*x2;
    u = p(1,:);  % top row
    v = p(2,:);  % bottom row
    if (c(1,n)==1) && (c(2,n)==0)
        plot(u,v, '-k')
    elseif (c(1,n)==0) && (c(2,n)==1)
        plot(u,v, '--k')
    else
        plot(u,v)
    end
end
```



$\alpha = \sqrt{20} - 0.1$

$\text{real}(\lambda_1) > 0$

Trajectories spiral away from (0,0) as $t \to \infty$

$$\alpha = \sqrt{20} + 0.1$$

$$\therefore \lambda_1 = \frac{\alpha}{2} - \frac{\sqrt{\alpha^2 - 20}}{2}$$

$$\lambda_2 = \frac{\alpha}{2} + \frac{\sqrt{\alpha^2 - 20}}{2}$$

Both are positive and real!

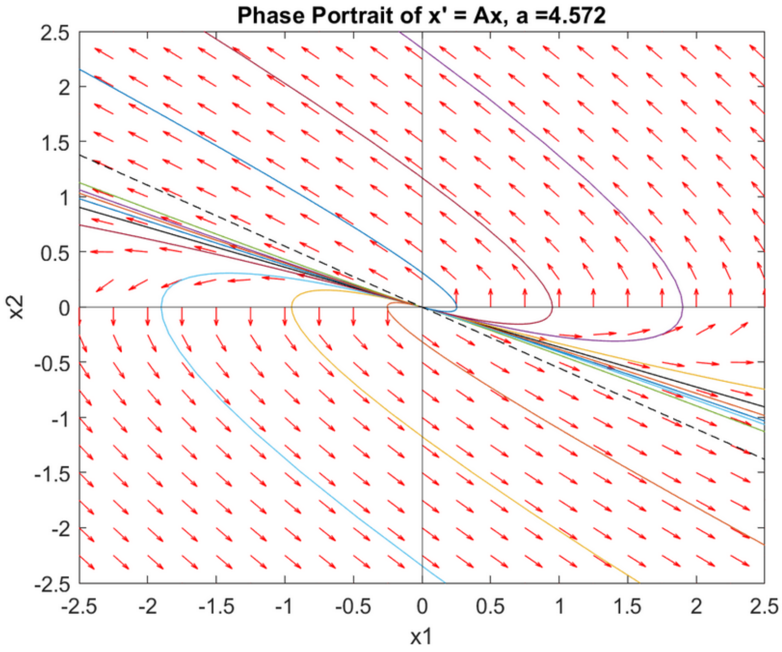
$$\vec{x}(t) = c_1 e^{\lambda_1 t} \begin{bmatrix} -\frac{\alpha}{2} - \frac{\sqrt{\alpha^2-20}}{2} \\ 1 \end{bmatrix} + c_2 e^{\lambda_2 t} \begin{bmatrix} -\frac{\alpha}{2} + \frac{\sqrt{\alpha^2-20}}{2} \\ 1 \end{bmatrix}$$

MATLAB code below.

```
clear
syms a
M = [0, -5;
     1, a];
nv = sqrt(20) + 0.1;  % new value
A = subs(M,a,nv);
[V,D] = eig(A);

% set plot boundaries
Xmin = -2.5; Xmax = 2.5;
Ymin = -2.5; Ymax = 2.5;

% Direction grid
% set how fine to make the grid of values
Step = 0.25;
% total # points on x and y axes
Nx = round((Xmax-Xmin)/Step, 0) + 1;
Ny = round((Ymax-Ymin)/Step, 0) + 1;
% set the actual x and y coords
x = linspace(Xmin, Xmax, Nx);  % a row vector
y = linspace(Ymin, Ymax, Ny);  % a row vector
% preallocate memory for coord, slope arrays
%  to be used in quiver, must be same dimension
xcoord = zeros(Nx, Ny);
ycoord = zeros(Nx, Ny);
dx = zeros(Nx, Ny);
dy = zeros(Nx, Ny);
for i = 1:Nx
    for j = 1:Ny
        xcoord(i,j) = x(i);
        ycoord(i,j) = y(j);
        v = A*[x(i);y(j)];  % get the slope
        s = norm(v);  % length of v
        % make slope vector a unit vector
        dx(i,j) = v(1)/s;
        dy(i,j) = v(2)/s;
    end
end
% plot the slope vectors at (xcoord, ycoord)
quiver(xcoord, ycoord, dx, dy, 0.5, 'r')
```

```
% Trajectories

ev1 = D(1,1);    % eigenvalues
ev2 = D(2,2);
evec1 = V(:,1);  % eigenvectors
evec2 = V(:,2);
% compute slopes of asymptotic lines
m1 = evec1(2)/evec1(1);
m2 = evec2(2)/evec2(1);
t = -2.5:0.05:2.5;
%figure
xlim([Xmin,Xmax])
ylim([Ymin,Ymax])
hold on
xline(0); yline(0);  % axes
plot(t,m1*t, 'k')  % asymptotic lines
plot(t,m2*t, '--k')
s1 = evec1*exp(ev1*t);
s2 = evec2*exp(ev2*t);
% plot trajectories for various c1,c2
for c1 = [-2,-1,1,2]
    for c2 = [-2,-1,1,2]
        x = c1*s1 + c2*s2;
        x1 = x(1,:);
        x2 = x(2,:);
        plot(x1,x2)
    end
end
xlabel 'x1', ylabel 'x2'
str = sprintf('%.3f',nv);
s = strcat('Phase Portrait of x'' = Ax, a = ',str);
title(s)
```

Plot on next page.



$\alpha = \sqrt{20} + 0.1$

$\lambda_1, \lambda_2 > 0$

Trajectories spiral to ∞ as $t \to \infty$.

13.

(a)

Using MATLAB,

```
clear
syms a
A = [5/4, 3/4;
     a, 5/4];
[V,D] = eig(A)
```

V =

$$\begin{pmatrix} -\frac{5}{4a} - \frac{\frac{\sqrt{3}\sqrt{a}}{2} - \frac{5}{4}}{a} & \frac{\frac{\sqrt{3}\sqrt{a}}{2} + \frac{5}{4}}{a} - \frac{5}{4a} \\ 1 & 1 \end{pmatrix}$$

D =

$$\begin{pmatrix} \frac{5}{4} - \frac{\sqrt{3}\sqrt{a}}{2} & 0 \\ 0 & \frac{\sqrt{3}\sqrt{a}}{2} + \frac{5}{4} \end{pmatrix}$$

$$\therefore \underline{\lambda = \frac{5}{4} \pm \frac{\sqrt{3\alpha}}{2}}$$

(b)

If $\alpha > 0$, λ is real, two different values

If $\alpha < 0$, λ is complex

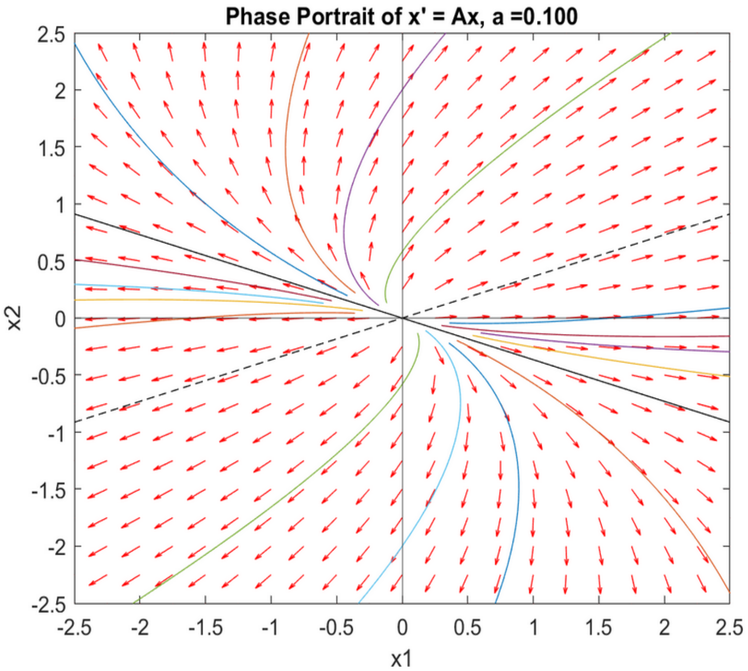
If $\frac{\sqrt{3\alpha}}{2} = \frac{5}{4}$, then $\lambda = \frac{5}{2}, 0 \Rightarrow$ one exponential solution, one linear solution.

$$\frac{\sqrt{3\alpha}}{2} = \frac{5}{4} \Rightarrow 3\alpha = \frac{25}{4}, \quad \alpha = \frac{25}{12}$$

$\therefore$ $\underline{\alpha = 0, \frac{25}{12}}$ are bifurcation points

(c)

$\alpha = +0.1$ MATLAB code on next page



$\lambda_1, \lambda_2 > 0$, and real

Trajectories $\to \pm\infty$ as $t \to +\infty$

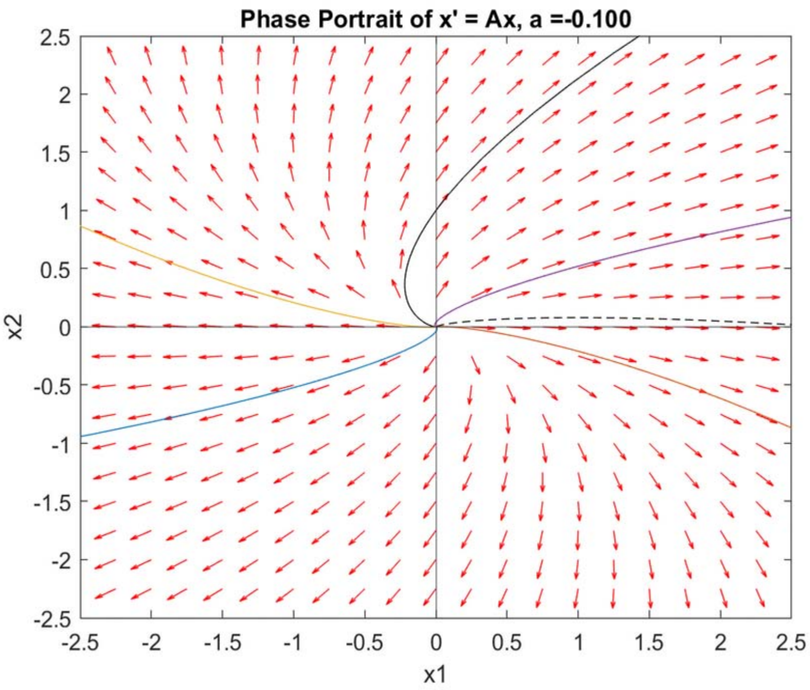
Unstable node

```
clear
syms a
M = [5/4, 3/4;
     a, 5/4];
nv = 0 + 0.1;  % new value
A = subs(M,a,nv);
[V,D] = eig(A);

% set plot boundaries
Xmin = -2.5; Xmax = 2.5;
Ymin = -2.5; Ymax = 2.5;

% Direction grid
% set how fine to make the grid of values
Step = 0.25;
% total # points on x and y axes
Nx = round((Xmax-Xmin)/Step, 0) + 1;
Ny = round((Ymax-Ymin)/Step, 0) + 1;
% set the actual x and y coords
x = linspace(Xmin, Xmax, Nx);  % a row vector
y = linspace(Ymin, Ymax, Ny);  % a row vector
% preallocate memory for coord, slope arrays
%  to be used in quiver, must be same dimension
xcoord = zeros(Nx, Ny);
ycoord = zeros(Nx, Ny);
dx = zeros(Nx, Ny);
dy = zeros(Nx, Ny);
for i = 1:Nx
    for j = 1:Ny
        xcoord(i,j) = x(i);
        ycoord(i,j) = y(j);
        v = A*[x(i);y(j)];  % get the slope
        s = norm(v);  % length of v
        % make slope vector a unit vector
        dx(i,j) = v(1)/s;
        dy(i,j) = v(2)/s;
    end
end
% plot the slope vectors at (xcoord, ycoord)
quiver(xcoord, ycoord, dx, dy, 0.5, 'r')
```

```
% Trajectories

ev1 = D(1,1);    % eigenvalues
ev2 = D(2,2);
evec1 = V(:,1);  % eigenvectors
evec2 = V(:,2);
% compute slopes of asymptotic lines
m1 = evec1(2)/evec1(1);
m2 = evec2(2)/evec2(1);
t = -2.5:0.05:2.5;
%figure
xlim([Xmin,Xmax])
ylim([Ymin,Ymax])
hold on
xline(0); yline(0);  % axes
plot(t,m1*t, 'k')  % asymptotic lines
plot(t,m2*t, '--k')
s1 = evec1*exp(ev1*t);
s2 = evec2*exp(ev2*t);
% plot trajectories for various c1,c2
for c1 = [-2,-1,1,2]
    for c2 = [-2,-1,1,2]
        x = c1*s1 + c2*s2;
        x1 = x(1,:);
        x2 = x(2,:);
        plot(x1,x2)
    end
end
xlabel 'x1', ylabel 'x2'
str = sprintf('%.3f',nv);
s = strcat('Phase Portrait of x'' = Ax, a = ',str);
title(s)
```

$\alpha = -0.1$

MATLAB code on next page

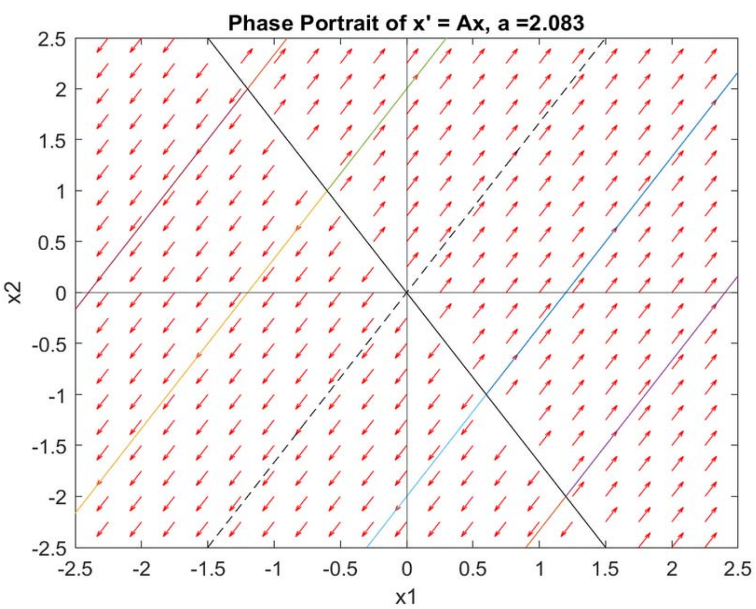


λ is complex

Trajectories spiral away from (0,0) as $t \to \infty$

```
clear
syms a
M = [5/4, 3/4;
     a, 5/4];
nv = 0 - 0.1;  % new value
A = subs(M,a,nv);
[V,D] = eig(A);

% set plot boundaries
Xmin = -2.5; Xmax = 2.5;
Ymin = -2.5; Ymax = 2.5;

% Direction grid
% set how fine to make the grid of values
Step = 0.25;
% total # points on x and y axes
Nx = round((Xmax-Xmin)/Step, 0) + 1;
Ny = round((Ymax-Ymin)/Step, 0) + 1;
% set the actual x and y coords
x = linspace(Xmin, Xmax, Nx);  % a row vector
y = linspace(Ymin, Ymax, Ny);  % a row vector
% preallocate memory for coord, slope arrays
%  to be used in quiver, must be same dimension
xcoord = zeros(Nx, Ny);
ycoord = zeros(Nx, Ny);
dx = zeros(Nx, Ny);
dy = zeros(Nx, Ny);
for i = 1:Nx
    for j = 1:Ny
        xcoord(i,j) = x(i);
        ycoord(i,j) = y(j);
        v = A*[x(i);y(j)];  % get the slope
        s = norm(v);  % length of v
        % make slope vector a unit vector
        dx(i,j) = v(1)/s;
        dy(i,j) = v(2)/s;
    end
end
% plot the slope vectors at (xcoord, ycoord)
quiver(xcoord, ycoord, dx, dy, 0.5, 'r')

%  Trajectories

ev1 = D(1,1);    % eigenvalues
ev2 = D(2,2);
evec1 = V(:,1);  % eigenvectors
evec2 = V(:,2);
hold on
% size interval to exceed plot bounds
t = -5:0.1:5;
% just use ev1, evec1
%   as real,imag parts are independent
x = evec1*exp(ev1*t);
x1 = real(x);
x2 = imag(x);

% create array of various coeff values
c = [1, 0, -1, -1,  1, 1;
     0, 1, -1,  1, -1, 1];
for n = 1:size(c,2)  % up to # columns of c[]
    p = c(1,n)*x1 + c(2,n)*x2;
    u = p(1,:);  % top row
    v = p(2,:);  % bottom row
    if (c(1,n)==1) && (c(2,n)==0)
        plot(u,v, '-k')
    elseif (c(1,n)==0) && (c(2,n)==1)
        plot(u,v, '--k')
    else
        plot(u,v)
    end
end

xline(0);  % show x and y axes
yline(0);
axis([Xmin, Xmax, Ymin, Ymax])  % plot bounds
xlabel 'x1', ylabel 'x2'
str = sprintf('%.3f',nv);  % display value of alpha
s = strcat('Phase Portrait of x'' = Ax, a =  ',str);
title(s)
```

$\alpha = \frac{25}{12}$ MATLAB code on next page.



Trajectories parallel the vector $(\frac{3}{5}, 1)$, moving away from line $x_2 = -\frac{5}{3}x$,

$$\vec{x}(t) = c_1 \begin{bmatrix} -3/5 \\ 1 \end{bmatrix} + c_2 e^{5/2\, t} \begin{bmatrix} 3/5 \\ 1 \end{bmatrix}$$

```
clear
syms a
M = [5/4, 3/4;
     a, 5/4];
nv = 25/12;  % new value
A = subs(M,a,nv);
[V,D] = eig(A)

% set plot boundaries
Xmin = -2.5; Xmax = 2.5;
Ymin = -2.5; Ymax = 2.5;

% Direction grid
% set how fine to make the grid of values
Step = 0.25;
% total # points on x and y axes
Nx = round((Xmax-Xmin)/Step, 0) + 1;
Ny = round((Ymax-Ymin)/Step, 0) + 1;
% set the actual x and y coords
x = linspace(Xmin, Xmax, Nx);  % a row vector
y = linspace(Ymin, Ymax, Ny);  % a row vector
% preallocate memory for coord, slope arrays
%  to be used in quiver, must be same dimension
xcoord = zeros(Nx, Ny);
ycoord = zeros(Nx, Ny);
dx = zeros(Nx, Ny);
dy = zeros(Nx, Ny);
for i = 1:Nx
    for j = 1:Ny
        xcoord(i,j) = x(i);
        ycoord(i,j) = y(j);
        v = A*[x(i);y(j)];  % get the slope
        s = norm(v);  % length of v
        % make slope vector a unit vector
        dx(i,j) = v(1)/s;
        dy(i,j) = v(2)/s;
    end
end
% plot the slope vectors at (xcoord, ycoord)
quiver(xcoord, ycoord, dx, dy, 0.5, 'r')
```

```
% Trajectories

ev1 = D(1,1);     % eigenvalues
ev2 = D(2,2);
evec1 = V(:,1);   % eigenvectors
evec2 = V(:,2);
% compute slopes of asymptotic lines
m1 = evec1(2)/evec1(1);
m2 = evec2(2)/evec2(1);
t = -2.5:0.05:2.5;
%figure
xlim([Xmin,Xmax])
ylim([Ymin,Ymax])
hold on
xline(0); yline(0);  % axes
plot(t,m1*t, 'k')  % asymptotic lines
plot(t,m2*t, '--k')
s1 = evec1*exp(ev1*t);
s2 = evec2*exp(ev2*t);
% plot trajectories for various c1,c2
for c1 = [-2,-1,1,2]
    for c2 = [-2,-1,1,2]
        x = c1*s1 + c2*s2;
        x1 = x(1,:);
        x2 = x(2,:);
        plot(x1,x2)
    end
end
xlabel 'x1', ylabel 'x2'
str = sprintf('%.3f',nv);
s = strcat('Phase Portrait of x'' = Ax, a = ',str);
title(s)
```

V =

$$\begin{pmatrix} -\frac{3}{5} & \frac{3}{5} \\ 1 & 1 \end{pmatrix}$$

D =

$$\begin{pmatrix} 0 & 0 \\ 0 & \frac{5}{2} \end{pmatrix}$$

From code above, $\lambda = 0, \frac{5}{2}$,

corresponding eigenvectors: $\begin{bmatrix} -3/5 \\ 1 \end{bmatrix}, \begin{bmatrix} 3/5 \\ 1 \end{bmatrix}$

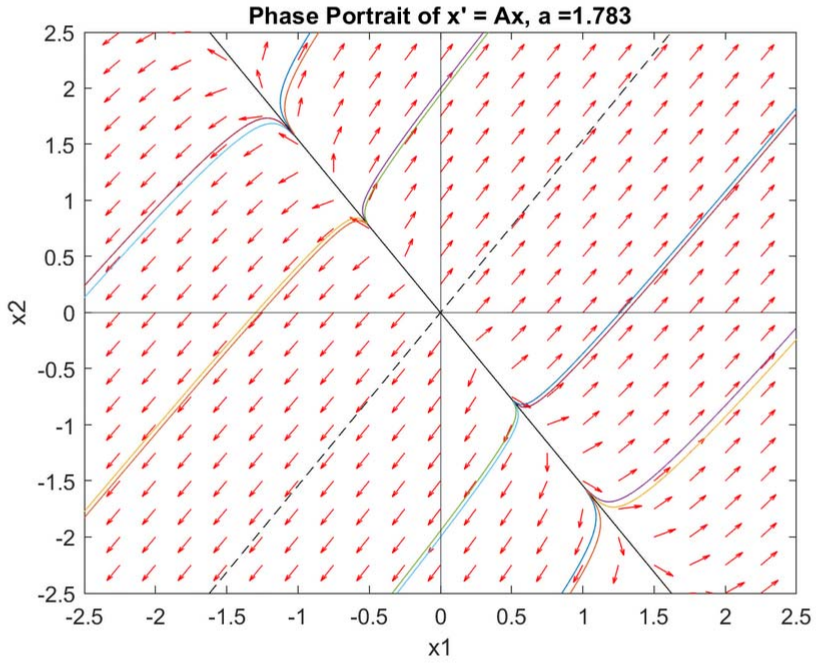
$\alpha = \frac{25}{12} - 0.3$

Here, $\lambda_1 > 0$, $\lambda_2 > 0$, so trajectories $\to \pm\infty$ as $t \to \infty$, just like $\alpha = +0.1$, but phase portrait looks resembles $\alpha = \frac{25}{12}$.

```
clear
syms a
M = [5/4, 3/4;
     a, 5/4];
nv = 25/12 - 0.3;  % new value
A = subs(M,a,nv);
[V,D] = eig(A);

% set plot boundaries
Xmin = -2.5; Xmax = 2.5;
Ymin = -2.5; Ymax = 2.5;

% Direction grid
% set how fine to make the grid of values
Step = 0.25;
% total # points on x and y axes
Nx = round((Xmax-Xmin)/Step, 0) + 1;
Ny = round((Ymax-Ymin)/Step, 0) + 1;
% set the actual x and y coords
x = linspace(Xmin, Xmax, Nx);  % a row vector
y = linspace(Ymin, Ymax, Ny);  % a row vector
% preallocate memory for coord, slope arrays
%  to be used in quiver, must be same dimension
xcoord = zeros(Nx, Ny);
ycoord = zeros(Nx, Ny);
dx = zeros(Nx, Ny);
dy = zeros(Nx, Ny);
for i = 1:Nx
    for j = 1:Ny
        xcoord(i,j) = x(i);
        ycoord(i,j) = y(j);
        v = A*[x(i);y(j)];  % get the slope
        s = norm(v);  % length of v
        % make slope vector a unit vector
        dx(i,j) = v(1)/s;
        dy(i,j) = v(2)/s;
    end
end
% plot the slope vectors at (xcoord, ycoord)
quiver(xcoord, ycoord, dx, dy, 0.5, 'r')
```

```
% Trajectories

ev1 = D(1,1);     % eigenvalues
ev2 = D(2,2);
evec1 = V(:,1);  % eigenvectors
evec2 = V(:,2);
% compute slopes of asymptotic lines
m1 = evec1(2)/evec1(1);
m2 = evec2(2)/evec2(1);
t = -2.5:0.05:2.5;
%figure
xlim([Xmin,Xmax])
ylim([Ymin,Ymax])
hold on
xline(0); yline(0);  % axes
plot(t,m1*t, 'k')  % asymptotic lines
plot(t,m2*t, '--k')
s1 = evec1*exp(ev1*t);
s2 = evec2*exp(ev2*t);
% plot trajectories for various c1,c2
for c1 = [-2,-1,1,2]
    for c2 = [-2,-1,1,2]
        x = c1*s1 + c2*s2;
        x1 = x(1,:);
        x2 = x(2,:);
        plot(x1,x2)
    end
end
xlabel 'x1', ylabel 'x2'
str = sprintf('%.3f',nv);
s = strcat('Phase Portrait of x'' = Ax, a = ',str);
title(s)
```

MATLAB code above for $a = \frac{25}{12} - 0.3$, plot below

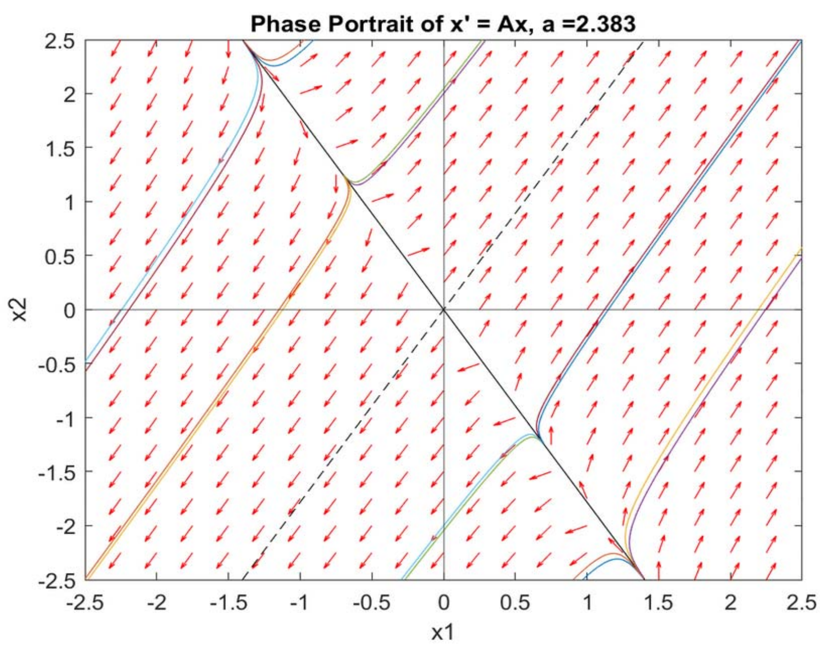


Unstable node

$\alpha = \frac{25}{12} + 0.3$ MATLAB code and plot below

Here, $\lambda_1 > 0$, $\lambda_2 < 0$

```
clear
syms a
M = [5/4, 3/4;
     a, 5/4];
nv = 25/12 + 0.3;  % new value
A = subs(M,a,nv);
[V,D] = eig(A);

% set plot boundaries
Xmin = -2.5; Xmax = 2.5;
Ymin = -2.5; Ymax = 2.5;

% Direction grid
% set how fine to make the grid of values
Step = 0.25;
% total # points on x and y axes
Nx = round((Xmax-Xmin)/Step, 0) + 1;
Ny = round((Ymax-Ymin)/Step, 0) + 1;
% set the actual x and y coords
x = linspace(Xmin, Xmax, Nx);  % a row vector
y = linspace(Ymin, Ymax, Ny);  % a row vector
% preallocate memory for coord, slope arrays
%  to be used in quiver, must be same dimension
xcoord = zeros(Nx, Ny);
ycoord = zeros(Nx, Ny);
dx = zeros(Nx, Ny);
dy = zeros(Nx, Ny);
for i = 1:Nx
    for j = 1:Ny
        xcoord(i,j) = x(i);
        ycoord(i,j) = y(j);
        v = A*[x(i);y(j)];  % get the slope
        s = norm(v);  % length of v
        % make slope vector a unit vector
        dx(i,j) = v(1)/s;
        dy(i,j) = v(2)/s;
    end
end
% plot the slope vectors at (xcoord, ycoord)
quiver(xcoord, ycoord, dx, dy, 0.5, 'r')

% Trajectories

ev1 = D(1,1);    % eigenvalues
ev2 = D(2,2);
evec1 = V(:,1);  % eigenvectors
evec2 = V(:,2);
% compute slopes of asymptotic lines
m1 = evec1(2)/evec1(1);
m2 = evec2(2)/evec2(1);
t = -2.5:0.05:2.5;
%figure
xlim([Xmin,Xmax])
ylim([Ymin,Ymax])
hold on
xline(0); yline(0);  % axes
plot(t,m1*t, 'k')  % asymptotic lines
plot(t,m2*t, '--k')
s1 = evec1*exp(ev1*t);
s2 = evec2*exp(ev2*t);
% plot trajectories for various c1,c2
for c1 = [-2,-1,1,2]
    for c2 = [-2,-1,1,2]
        x = c1*s1 + c2*s2;
        x1 = x(1,:);
        x2 = x(2,:);
        plot(x1,x2)
    end
end
xlabel 'x1', ylabel 'x2'
str = sprintf('%.3f',nv);
s = strcat('Phase Portrait of x'' = Ax, a = ',str);
title(s)
```



Saddle point at origin

14.

(a)

Using MATLAB,

```
clear
syms a
A = [-1, a;
     -1, -1];
[V,D] = eig(A)
```

V =

$$\begin{pmatrix} -\sqrt{-a} & \sqrt{-a} \\ 1 & 1 \end{pmatrix}$$

D =

$$\begin{pmatrix} \sqrt{-a}-1 & 0 \\ 0 & -\sqrt{-a}-1 \end{pmatrix}$$

$\therefore\ \lambda = \underline{-1 \pm \sqrt{-\alpha}}$

(b)

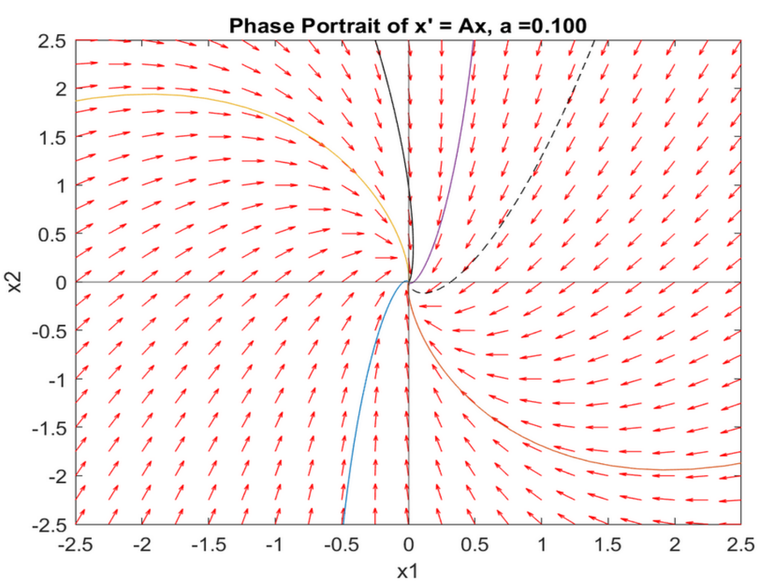
$\alpha > 0$: λ complex

$\alpha < 0$: λ real , $\alpha = -1 \Rightarrow \lambda = -2, 0$

$\therefore$ Bifurcation values : $\underline{\alpha = 0, -1}$

(c)

$\alpha = +0.1$ $\quad\quad$ λ is complex



$\lambda = -1 + \sqrt{-0.1}$

$\Rightarrow e^{-t}$ factor

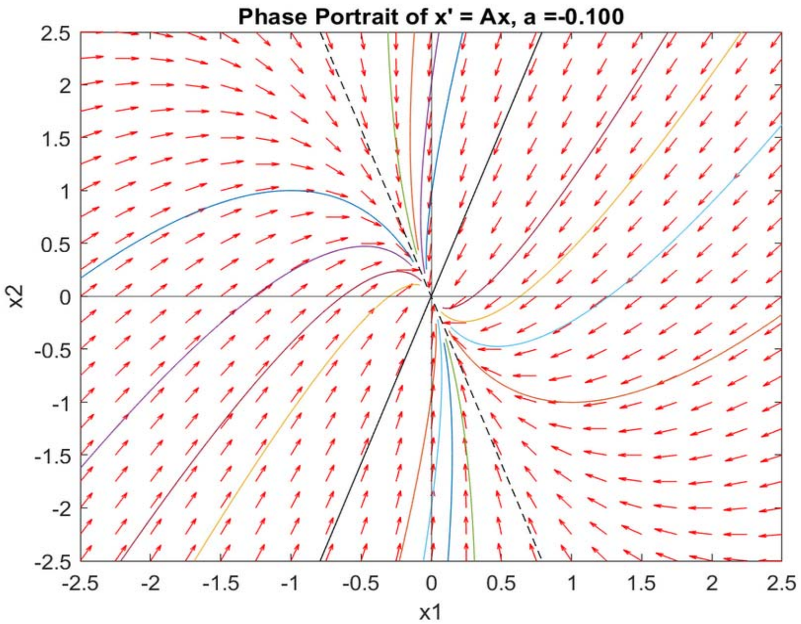
$\Rightarrow$ spiral to $(0,0)$

MATLAB code next page

```
clear
syms a
M = [-1, a;
     -1, -1];
nv = 0 + 0.1;  % new value
A = subs(M,a,nv);
[V,D] = eig(A);

% set plot boundaries
Xmin = -2.5; Xmax = 2.5;
Ymin = -2.5; Ymax = 2.5;

% Direction grid
% set how fine to make the grid of values
Step = 0.25;
% total # points on x and y axes
Nx = round((Xmax-Xmin)/Step, 0) + 1;
Ny = round((Ymax-Ymin)/Step, 0) + 1;
% set the actual x and y coords
x = linspace(Xmin, Xmax, Nx);  % a row vector
y = linspace(Ymin, Ymax, Ny);  % a row vector
% preallocate memory for coord, slope arrays
%  to be used in quiver, must be same dimension
xcoord = zeros(Nx, Ny);
ycoord = zeros(Nx, Ny);
dx = zeros(Nx, Ny);
dy = zeros(Nx, Ny);
for i = 1:Nx
    for j = 1:Ny
        xcoord(i,j) = x(i);
        ycoord(i,j) = y(j);
        v = A*[x(i);y(j)];  % get the slope
        s = norm(v);  % length of v
        % make slope vector a unit vector
        dx(i,j) = v(1)/s;
        dy(i,j) = v(2)/s;
    end
end
% plot the slope vectors at (xcoord, ycoord)
quiver(xcoord, ycoord, dx, dy, 0.5, 'r')
```

```
%  Trajectories

ev1 = D(1,1);    % eigenvalues
ev2 = D(2,2);
evec1 = V(:,1);  % eigenvectors
evec2 = V(:,2);
hold on
% size interval to exceed plot bounds
t = -5:0.1:5;
% just use ev1, evec1
%   as real,imag parts are independent
x = evec1*exp(ev1*t);
x1 = real(x);
x2 = imag(x);

% create array of various coeff values
c = [1, 0, -1, -1,  1, 1;
     0, 1, -1,  1, -1, 1];
for n = 1:size(c,2)  % up to # columns of c[]
    p = c(1,n)*x1 + c(2,n)*x2;
    u = p(1,:);  % top row
    v = p(2,:);  % bottom row
    if (c(1,n)==1) && (c(2,n)==0)
        plot(u,v, '-k')
    elseif (c(1,n)==0) && (c(2,n)==1)
        plot(u,v, '--k')
    else
        plot(u,v)
    end
end

xline(0);  % show x and y axes
yline(0);
axis([Xmin, Xmax, Ymin, Ymax])  % plot bounds
xlabel 'x1', ylabel 'x2'
str = sprintf('%.3f',nv);  % display value of alpha
s = strcat('Phase Portrait of x'' = Ax, a =  ',str);
title(s)
```

$\alpha = -0.1$ $\qquad$ $\lambda_1, \lambda_2 < 0$ and real



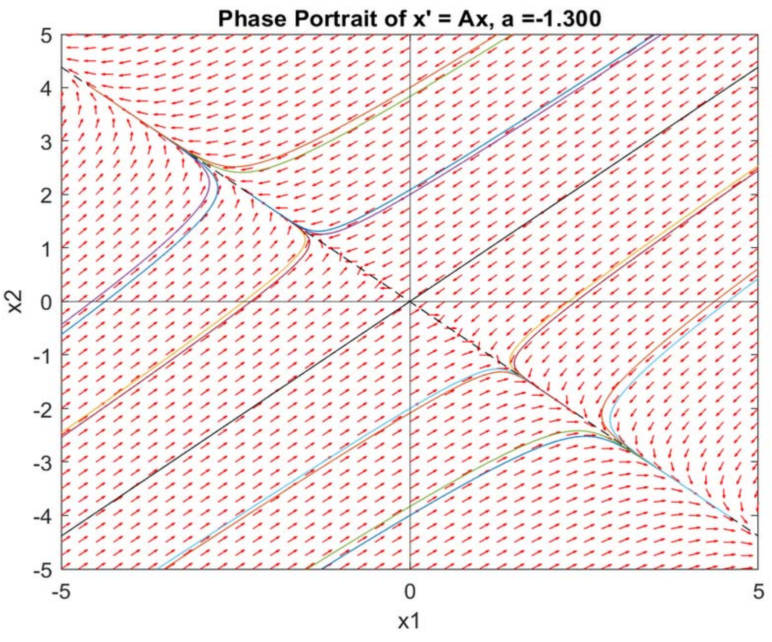
e^{λ} with $\lambda < 0 \Rightarrow$ solutions sink to origin.

MATLAB code on next page

```
clear
syms a
M = [-1, a;
     -1, -1];
nv = 0 - 0.1;  % new value
A = subs(M,a,nv);
[V,D] = eig(A);

% set plot boundaries
Xmin = -2.5; Xmax = 2.5;
Ymin = -2.5; Ymax = 2.5;

% Direction grid
% set how fine to make the grid of values
Step = 0.25;
% total # points on x and y axes
Nx = round((Xmax-Xmin)/Step, 0) + 1;
Ny = round((Ymax-Ymin)/Step, 0) + 1;
% set the actual x and y coords
x = linspace(Xmin, Xmax, Nx);  % a row vector
y = linspace(Ymin, Ymax, Ny);  % a row vector
% preallocate memory for coord, slope arrays
%  to be used in quiver, must be same dimension
xcoord = zeros(Nx, Ny);
ycoord = zeros(Nx, Ny);
dx = zeros(Nx, Ny);
dy = zeros(Nx, Ny);
for i = 1:Nx
    for j = 1:Ny
        xcoord(i,j) = x(i);
        ycoord(i,j) = y(j);
        v = A*[x(i);y(j)];  % get the slope
        s = norm(v);  % length of v
        % make slope vector a unit vector
        dx(i,j) = v(1)/s;
        dy(i,j) = v(2)/s;
    end
end
% plot the slope vectors at (xcoord, ycoord)
quiver(xcoord, ycoord, dx, dy, 0.5, 'r')
```

```
% Trajectories

ev1 = D(1,1);    % eigenvalues
ev2 = D(2,2);
evec1 = V(:,1);  % eigenvectors
evec2 = V(:,2);
% compute slopes of asymptotic lines
m1 = evec1(2)/evec1(1);
m2 = evec2(2)/evec2(1);
t = -2.5:0.05:2.5;
%figure
xlim([Xmin,Xmax])
ylim([Ymin,Ymax])
hold on
xline(0); yline(0);  % axes
plot(t,m1*t, 'k')  % asymptotic lines
plot(t,m2*t, '--k')
s1 = evec1*exp(ev1*t);
s2 = evec2*exp(ev2*t);
% plot trajectories for various c1,c2
for c1 = [-2,-1,1,2]
    for c2 = [-2,-1,1,2]
        x = c1*s1 + c2*s2;
        x1 = x(1,:);
        x2 = x(2,:);
        plot(x1,x2)
    end
end
xlabel 'x1', ylabel 'x2'
str = sprintf('%.3f',nv);
s = strcat('Phase Portrait of x'' = Ax, a = ',str);
title(s)
```

$a = -1 - 0.3$ 0.3 chosen to look more like a saddle



$\lambda_1 > 0, \lambda_2 < 0$

(0,0) is a saddle

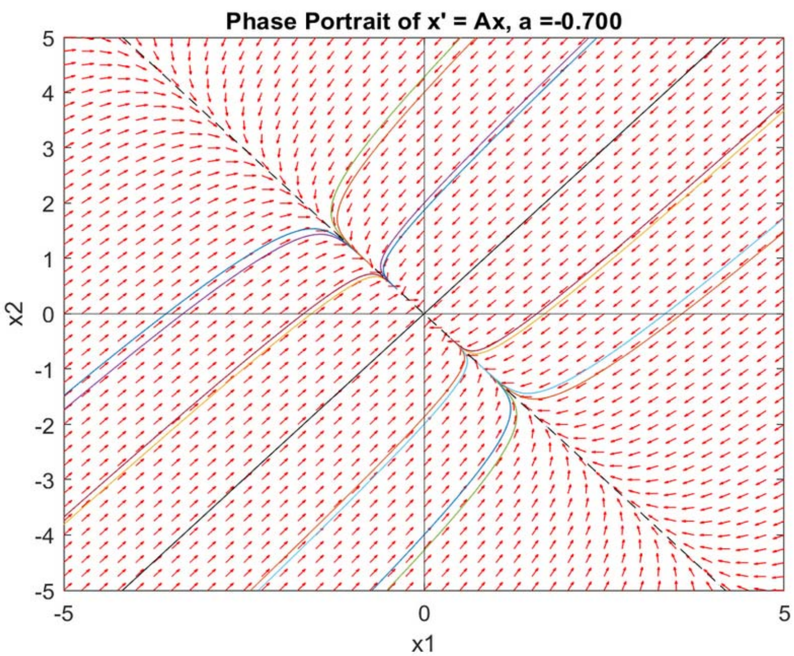
λ's are close to

$\lambda_1 = 0, \lambda_2 = -2$

```
clear
syms a
M = [-1, a;
     -1, -1];
nv = -1 - 0.3;  % new value
A = subs(M,a,nv);
[V,D] = eig(A);

% set plot boundaries
Xmin = -5; Xmax = 5;
Ymin = -5; Ymax = 5;

% Direction grid
% set how fine to make the grid of values
Step = 0.25;
% total # points on x and y axes
Nx = round((Xmax-Xmin)/Step, 0) + 1;
Ny = round((Ymax-Ymin)/Step, 0) + 1;
% set the actual x and y coords
x = linspace(Xmin, Xmax, Nx);  % a row vector
y = linspace(Ymin, Ymax, Ny);  % a row vector
% preallocate memory for coord, slope arrays
%  to be used in quiver, must be same dimension
xcoord = zeros(Nx, Ny);
ycoord = zeros(Nx, Ny);
dx = zeros(Nx, Ny);
dy = zeros(Nx, Ny);
for i = 1:Nx
    for j = 1:Ny
        xcoord(i,j) = x(i);
        ycoord(i,j) = y(j);
        v = A*[x(i);y(j)];  % get the slope
        s = norm(v);  % length of v
        % make slope vector a unit vector
        dx(i,j) = v(1)/s;
        dy(i,j) = v(2)/s;
    end
end
% plot the slope vectors at (xcoord, ycoord)
quiver(xcoord, ycoord, dx, dy, 0.5, 'r')
```

```
% Trajectories

ev1 = D(1,1);    % eigenvalues
ev2 = D(2,2);
evec1 = V(:,1);  % eigenvectors
evec2 = V(:,2);
% compute slopes of asymptotic lines
m1 = evec1(2)/evec1(1);
m2 = evec2(2)/evec2(1);
t = -5:0.05:5;
%figure
xlim([Xmin,Xmax])
ylim([Ymin,Ymax])
hold on
xline(0); yline(0);  % axes
plot(t,m1*t, 'k')  % asymptotic lines
plot(t,m2*t, '--k')
s1 = evec1*exp(ev1*t);
s2 = evec2*exp(ev2*t);
% plot trajectories for various c1,c2
for c1 = [-2,-1,1,2]
    for c2 = [-2,-1,1,2]
        x = c1*s1 + c2*s2;
        x1 = x(1,:);
        x2 = x(2,:);
        plot(x1,x2)
    end
end
xlabel 'x1', ylabel 'x2'
str = sprintf('%.3f',nv);
s = strcat('Phase Portrait of x'' = Ax, a = ',str);
title(s)
```

$a = -1 + 0.3$ $\lambda_1 < 0,\ \lambda_2 < 0$



(0,0) is a nodal sink (stable)

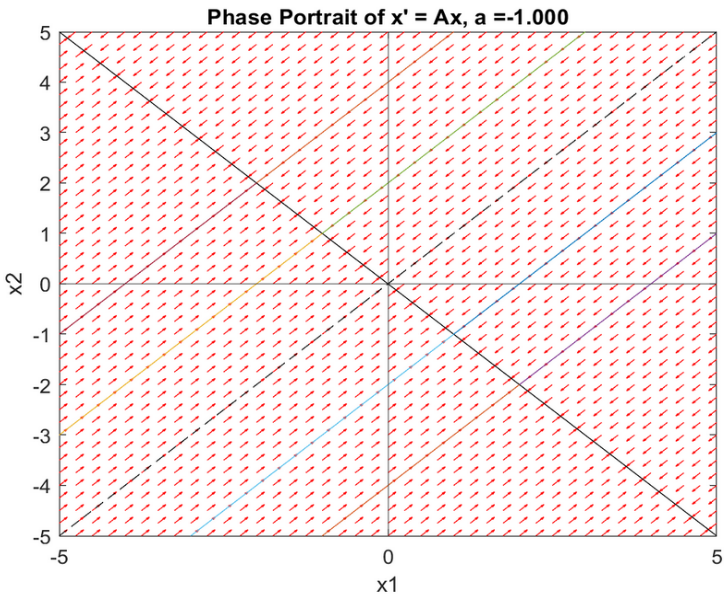
λ_1 is close to 0
λ_2 is close to -2

MATLAB code on next page.

```
clear
syms a
M = [-1, a;
     -1, -1];
nv = -1 + 0.3;  % new value
A = subs(M,a,nv);
[V,D] = eig(A);

% set plot boundaries
Xmin = -5; Xmax = 5;
Ymin = -5; Ymax = 5;

% Direction grid
% set how fine to make the grid of values
Step = 0.25;
% total # points on x and y axes
Nx = round((Xmax-Xmin)/Step, 0) + 1;
Ny = round((Ymax-Ymin)/Step, 0) + 1;
% set the actual x and y coords
x = linspace(Xmin, Xmax, Nx);  % a row vector
y = linspace(Ymin, Ymax, Ny);  % a row vector
% preallocate memory for coord, slope arrays
%  to be used in quiver, must be same dimension
xcoord = zeros(Nx, Ny);
ycoord = zeros(Nx, Ny);
dx = zeros(Nx, Ny);
dy = zeros(Nx, Ny);
for i = 1:Nx
    for j = 1:Ny
        xcoord(i,j) = x(i);
        ycoord(i,j) = y(j);
        v = A*[x(i);y(j)];  % get the slope
        s = norm(v);  % length of v
        % make slope vector a unit vector
        dx(i,j) = v(1)/s;
        dy(i,j) = v(2)/s;
    end
end
% plot the slope vectors at (xcoord, ycoord)
quiver(xcoord, ycoord, dx, dy, 0.5, 'r')
```

```
% Trajectories

ev1 = D(1,1);    % eigenvalues
ev2 = D(2,2);
evec1 = V(:,1);  % eigenvectors
evec2 = V(:,2);
% compute slopes of asymptotic lines
m1 = evec1(2)/evec1(1);
m2 = evec2(2)/evec2(1);
t = -5:0.05:5;
%figure
xlim([Xmin,Xmax])
ylim([Ymin,Ymax])
hold on
xline(0); yline(0);  % axes
plot(t,m1*t, 'k')  % asymptotic lines
plot(t,m2*t, '--k')
s1 = evec1*exp(ev1*t);
s2 = evec2*exp(ev2*t);
% plot trajectories for various c1,c2
for c1 = [-2,-1,1,2]
    for c2 = [-2,-1,1,2]
        x = c1*s1 + c2*s2;
        x1 = x(1,:);
        x2 = x(2,:);
        plot(x1,x2)
    end
end
xlabel 'x1', ylabel 'x2'
str = sprintf('%.3f',nv);
s = strcat('Phase Portrait of x'' = Ax, a = ',str);
title(s)
```

For interest, plot below shows $\alpha = -1$, to compare with above two plots : $\alpha = -1 \pm 0.3$.



Here, $\lambda = 0, -2$.

15.

(a)

Using MATLAB,

```
clear
syms a
A = [4, a;
      8, -6];
[V,D] = eig(A)
```

V =

$$\begin{pmatrix} \frac{5}{8} - \frac{\sqrt{8\,a+25}}{8} & \frac{\sqrt{8\,a+25}}{8} + \frac{5}{8} \\ 1 & 1 \end{pmatrix}$$

D =

$$\begin{pmatrix} -\sqrt{8\,a+25} - 1 & 0 \\ 0 & \sqrt{8\,a+25} - 1 \end{pmatrix}$$

$$\therefore \underline{\lambda = -1 \pm \sqrt{8\alpha + 25}}$$

(b)

If $8\alpha + 25 < 0$, λ is complex, $\alpha < -\frac{25}{8}$

If $8\alpha + 25 > 0$, λ is real, $\alpha > -\frac{25}{8}$

If $8\alpha + 25 = 1$, $\lambda = -2, 0$, $\alpha = -3$

If $8\alpha + 25 < 1$, $\lambda_1 < 0$, $\lambda_2 < 0$, $\alpha < -3$

If $8\alpha + 25 > 1$, $\lambda_1 < 0$, $\lambda_2 > 0$, $\alpha > -3$

$\therefore \alpha = \underline{-\frac{25}{8}, -3}$ are bifurcation points.

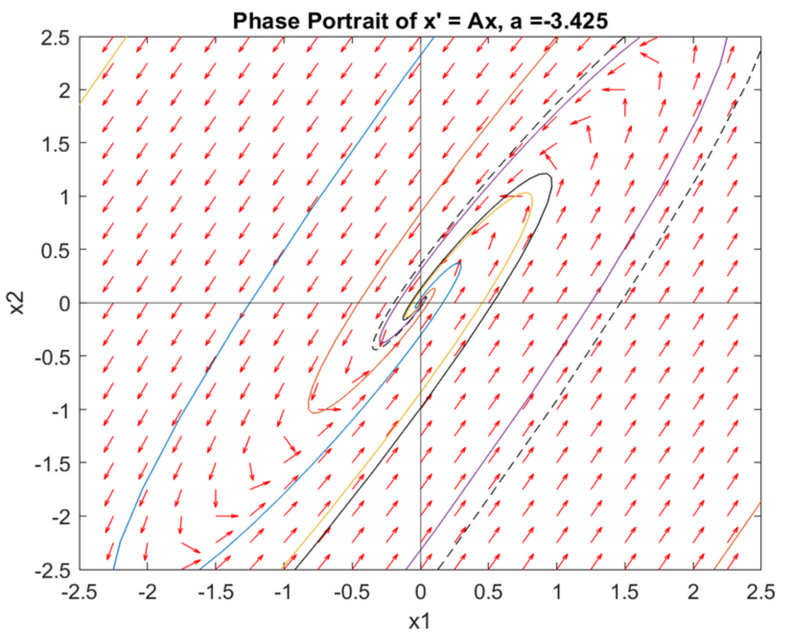
(c)

$\alpha = -\frac{25}{8} - 0.3$ λ is complex

MATLAB code next page.

```
clear
syms a
M = [4, a;
     8, -6];
nv = -25/8 - 0.3;  % new value
A = subs(M,a,nv);
[V,D] = eig(A);

% set plot boundaries
Xmin = -2.5; Xmax = 2.5;
Ymin = -2.5; Ymax = 2.5;

% Direction grid
% set how fine to make the grid of values
Step = 0.25;
% total # points on x and y axes
Nx = round((Xmax-Xmin)/Step, 0) + 1;
Ny = round((Ymax-Ymin)/Step, 0) + 1;
% set the actual x and y coords
x = linspace(Xmin, Xmax, Nx);  % a row vector
y = linspace(Ymin, Ymax, Ny);  % a row vector
% preallocate memory for coord, slope arrays
%  to be used in quiver, must be same dimension
xcoord = zeros(Nx, Ny);
ycoord = zeros(Nx, Ny);
dx = zeros(Nx, Ny);
dy = zeros(Nx, Ny);
for i = 1:Nx
    for j = 1:Ny
        xcoord(i,j) = x(i);
        ycoord(i,j) = y(j);
        v = A*[x(i);y(j)];  % get the slope
        s = norm(v);  % length of v
        % make slope vector a unit vector
        dx(i,j) = v(1)/s;
        dy(i,j) = v(2)/s;
    end
end
% plot the slope vectors at (xcoord, ycoord)
quiver(xcoord, ycoord, dx, dy, 0.5, 'r')
```

```
%  Trajectories - complex eigenvalues

ev1 = D(1,1);    % eigenvalues
ev2 = D(2,2);
evec1 = V(:,1);  % eigenvectors
evec2 = V(:,2);
hold on
% size interval to exceed plot bounds
t = -5:0.1:5;
% just use ev1, evec1
%   as real,imag parts are independent
x = evec1*exp(ev1*t);
x1 = real(x);
x2 = imag(x);

% create array of various coeff values
c = [1, 0, -1, -1,  1, 1;
     0, 1,  1, -1, -1, 1];
for n = 1:size(c,2)  % up to # columns of c[]
    p = c(1,n)*x1 + c(2,n)*x2;
    u = p(1,:);  % top row
    v = p(2,:);  % bottom row
    if (c(1,n)==1) && (c(2,n)==0)
        plot(u,v, '-k')
    elseif (c(1,n)==0) && (c(2,n)==1)
        plot(u,v, '--k')
    else
        plot(u,v)
    end
end

xline(0);  % show x and y axes
yline(0);
axis([Xmin, Xmax, Ymin, Ymax])  % plot bounds
xlabel 'x1', ylabel 'x2'
str = sprintf('%.3f',nv);  % display value of alpha
s = strcat('Phase Portrait of x'' = Ax, a =  ',str);
title(s)
```



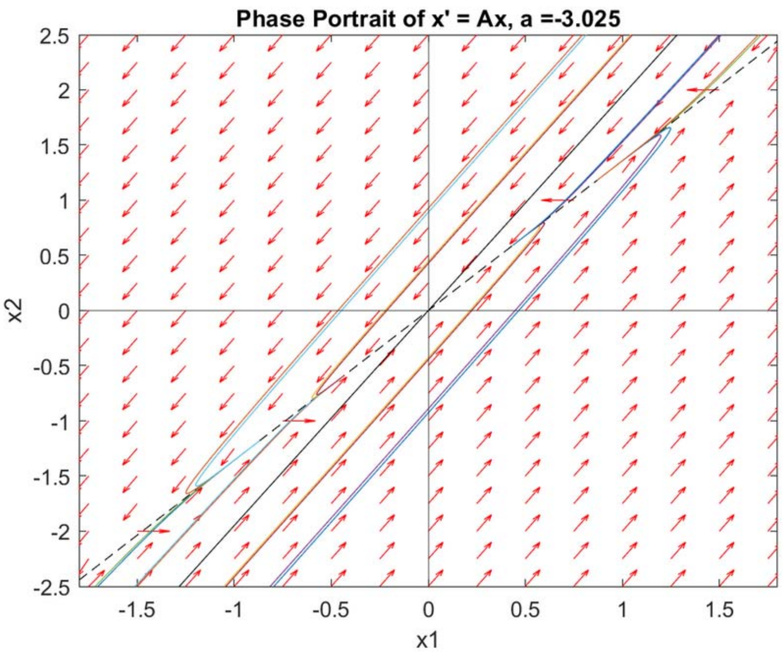
spiral to (0,0)

as real part of λ

is -1, ⇒ e^{-t}

$\alpha = -\frac{25}{8} + 0.1$ λ is real   MATLAB code below

```
clear
syms a
M = [4, a;
     8, -6];
nv = -25/8 + 0.1;  % new value
A = subs(M,a,nv);
[V,D] = eig(A);

% set plot boundaries
Xmin = -2.5; Xmax = 2.5;
Ymin = -2.5; Ymax = 2.5;

% Direction grid
% set how fine to make the grid of values
Step = 0.25;
% total # points on x and y axes
Nx = round((Xmax-Xmin)/Step, 0) + 1;
Ny = round((Ymax-Ymin)/Step, 0) + 1;
% set the actual x and y coords
x = linspace(Xmin, Xmax, Nx);  % a row vector
y = linspace(Ymin, Ymax, Ny);  % a row vector
% preallocate memory for coord, slope arrays
%  to be used in quiver, must be same dimension
xcoord = zeros(Nx, Ny);
ycoord = zeros(Nx, Ny);
dx = zeros(Nx, Ny);
dy = zeros(Nx, Ny);
for i = 1:Nx
    for j = 1:Ny
        xcoord(i,j) = x(i);
        ycoord(i,j) = y(j);
        v = A*[x(i);y(j)];  % get the slope
        s = norm(v);  % length of v
        % make slope vector a unit vector
        dx(i,j) = v(1)/s;
        dy(i,j) = v(2)/s;
    end
end
% plot the slope vectors at (xcoord, ycoord)
quiver(xcoord, ycoord, dx, dy, 0.5, 'r')

% Trajectories - real eigenvalues

ev1 = D(1,1);    % eigenvalues
ev2 = D(2,2);
evec1 = V(:,1);  % eigenvectors
evec2 = V(:,2);
% compute slopes of asymptotic lines
m1 = evec1(2)/evec1(1);
m2 = evec2(2)/evec2(1);
t = -5:0.05:5;

%figure
xlim([Xmin,Xmax])
ylim([Ymin,Ymax])
hold on
xline(0); yline(0);  % axes
plot(t,m1*t, 'k')  % asymptotic lines
plot(t,m2*t, '--k')
s1 = evec1*exp(ev1*t);
s2 = evec2*exp(ev2*t);

% plot trajectories for various c1,c2
for c1 = [-2,-1,1,2]
    for c2 = [-2,-1,1,2]
        x = c1*s1 + c2*s2;
        x1 = x(1,:);
        x2 = x(2,:);
        plot(x1,x2)
    end
end

xlabel 'x1', ylabel 'x2'
str = sprintf('%.3f',nv);
s = strcat('Phase Portrait of x'' = Ax, a = ',str);
title(s)
xlim([-1.8, 1.8])
```



$\lambda_1 < 0$, $\lambda_2 < 0$

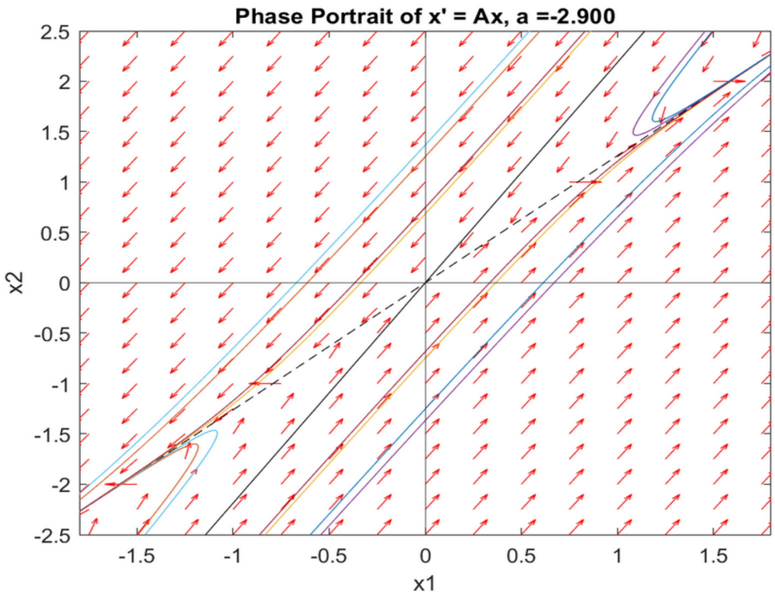
A nodal sink, difficult to see since slope of the two eigenvectors are similar

$\alpha = -3 - 0.01$ is very close to $\alpha = -\frac{25}{8} + 0.1$ and $\lambda_1 < 0$, $\lambda_2 < 0$, so a nodal sink. Since both are close to $\alpha = -3$, the phase portraits look much like $\alpha = -3$, in which the line for the eigenvector for $\lambda_1 = 0$ is a critical point. For $\lambda_2 < 0$, trajectories move toward the critical point.

$\alpha = -3 + 0.3$ $\lambda_1 < 0$, $\lambda_2 < 0$ MATLAB code below

```
clear
syms a
M = [4, a;
     8, -6];
nv = -3 + 0.1;  % new value
A = subs(M,a,nv);
[V,D] = eig(A);

% set plot boundaries
Xmin = -2.5; Xmax = 2.5;
Ymin = -2.5; Ymax = 2.5;

% Direction grid
% set how fine to make the grid of values
Step = 0.25;
% total # points on x and y axes
Nx = round((Xmax-Xmin)/Step, 0) + 1;
Ny = round((Ymax-Ymin)/Step, 0) + 1;
% set the actual x and y coords
x = linspace(Xmin, Xmax, Nx);  % a row vector
y = linspace(Ymin, Ymax, Ny);  % a row vector
% preallocate memory for coord, slope arrays
%  to be used in quiver, must be same dimension
xcoord = zeros(Nx, Ny);
ycoord = zeros(Nx, Ny);
dx = zeros(Nx, Ny);
dy = zeros(Nx, Ny);
for i = 1:Nx
    for j = 1:Ny
        xcoord(i,j) = x(i);
        ycoord(i,j) = y(j);
        v = A*[x(i);y(j)];  % get the slope
        s = norm(v);  % length of v
        % make slope vector a unit vector
        dx(i,j) = v(1)/s;
        dy(i,j) = v(2)/s;
    end
end
% plot the slope vectors at (xcoord, ycoord)
quiver(xcoord, ycoord, dx, dy, 0.5, 'r')
```

```
% Trajectories - real eigenvalues

ev1 = D(1,1);    % eigenvalues
ev2 = D(2,2);
evec1 = V(:,1);  % eigenvectors
evec2 = V(:,2);
% compute slopes of asymptotic lines
m1 = evec1(2)/evec1(1);
m2 = evec2(2)/evec2(1);
t = -5:0.05:5;

%figure
xlim([Xmin,Xmax])
ylim([Ymin,Ymax])
hold on
xline(0); yline(0);  % axes
plot(t,m1*t, 'k')  % asymptotic lines
plot(t,m2*t, '--k')
s1 = evec1*exp(ev1*t);
s2 = evec2*exp(ev2*t);

% plot trajectories for various c1,c2
for c1 = [-2,-1,1,2]
    for c2 = [-2,-1,1,2]
        x = c1*s1 + c2*s2;
        x1 = x(1,:);
        x2 = x(2,:);
        plot(x1,x2)
    end
end

xlabel 'x1', ylabel 'x2'
str = sprintf('%.3f',nv);
s = strcat('Phase Portrait of x'' = Ax, a = ',str);
title(s)
xlim([-1.8, 1.8])
```



Saddle point at $(0,0)$.

16.

Using MATLAB,

```
clear
% sym gives nice output

A = sym([-1, -1;
          2, -1]);
[V,D] = eig(A)
```

V =

$$\begin{pmatrix} -\frac{\sqrt{2}\,i}{2} & \frac{\sqrt{2}\,i}{2} \\ 1 & 1 \end{pmatrix}$$

D =

$$\begin{pmatrix} -1-\sqrt{2}\,i & 0 \\ 0 & -1+\sqrt{2}\,i \end{pmatrix}$$

$\therefore \lambda = -1 \pm \sqrt{2}\, i$

$\therefore t^{-1 \pm \sqrt{2}i} = t^{-1} t^{\pm \sqrt{2} i}$

Consider $t^{\sqrt{2}i} = \exp\left[\ln\left(t^{\sqrt{2}i}\right)\right] = \exp\left[\sqrt{2}\, i \ln(t)\right]$

$= \cos\left(\sqrt{2}\ln(t)\right) + i \sin\left(\sqrt{2}\ln(t)\right)$

$\therefore \vec{x}^{2}(t) = \begin{bmatrix} \frac{\sqrt{2}}{2} i \\ 1 \end{bmatrix} t^{-1} \left[\cos\left(\sqrt{2}\ln(t)\right) + i \sin\left(\sqrt{2}\ln(t)\right)\right]$

$$= t^{-1}\begin{bmatrix} -\frac{\sqrt{2}}{2}\sin(\sqrt{2}\ln(t)) \\ \cos(\sqrt{2}\ln(t)) \end{bmatrix} + i\,t^{-1}\begin{bmatrix} \frac{\sqrt{2}}{2}\cos(\sqrt{2}\ln(t)) \\ \sin(\sqrt{2}\ln(t)) \end{bmatrix}$$

$$\det\begin{bmatrix} -t^{-1}\frac{\sqrt{2}}{2}\sin(\sqrt{2}\ln(t)) & t^{-1}\frac{\sqrt{2}}{2}\cos(\sqrt{2}\ln(t)) \\ t^{-1}\cos(\sqrt{2}\ln(t)) & t^{-1}\sin(\sqrt{2}\ln(t)) \end{bmatrix}$$

$$= -t^{-2}\frac{\sqrt{2}}{2} \neq 0 \text{ for } t > 0.$$

$\therefore$ Real, imaginary parts of $\vec{x}^{\,2}(t)$ are independent. Also, to simplify, multiply the first vector by $-\sqrt{2}$, the second by $\sqrt{2}$ to get the answer in the back of the book.

$$\therefore \boxed{\vec{x}(t) = c_1 t^{-1}\begin{bmatrix} \sin(\sqrt{2}\ln(t)) \\ -\sqrt{2}\cos(\sqrt{2}\ln(t)) \end{bmatrix} + c_2 t^{-1}\begin{bmatrix} \cos(\sqrt{2}\ln(t)) \\ \sqrt{2}\sin(\sqrt{2}\ln(t)) \end{bmatrix}}$$

17.

Using MATLAB,

```
clear
% sym gives nice output
A = sym([2, -5;
         1, -2]);
[V,D] = eig(A)
```

V =

$$\begin{pmatrix} 2-i & 2+i \\ 1 & 1 \end{pmatrix}$$

D =

$$\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

$\therefore \lambda = \pm i \quad \therefore$ Consider t^i and $\begin{bmatrix} 2+i \\ 1 \end{bmatrix}$

$$t^i = e^{\ln(t^i)} = e^{i\ln(t)} = \cos(\ln(t)) + i\sin(\ln(t))$$

$$\therefore \vec{x}^{(2)} = \begin{bmatrix} 2+i \\ 1 \end{bmatrix} \left[\cos(\ln(t)) + i\sin(\ln(t))\right]$$

$$= \begin{bmatrix} 2\cos(\ln(t)) - \sin(\ln(t)) \\ \cos(\ln(t)) \end{bmatrix} + i \begin{bmatrix} \cos(\ln(t)) + 2\sin(\ln(t)) \\ \sin(\ln(t)) \end{bmatrix}$$

$\det \begin{bmatrix} \; \end{bmatrix} = -1 \neq 0$ for all t.

$\therefore$ Real, imaginary parts independent.

$$\boxed{\vec{x}(t) = c_1 \begin{bmatrix} 2\cos(\ln(t)) - \sin(\ln(t)) \\ \cos(\ln(t)) \end{bmatrix} + c_2 \begin{bmatrix} \cos(\ln(t)) + 2\sin(\ln(t)) \\ \sin(\ln(t)) \end{bmatrix}}$$

To get the answer in the back of the book,

$$2\begin{bmatrix} 2\cos() - \sin() \\ \cos() \end{bmatrix} + \begin{bmatrix} \cos() + 2\sin() \\ \sin() \end{bmatrix} = \begin{bmatrix} 5\cos() \\ 2\cos() + \sin() \end{bmatrix}$$

and $-1\begin{bmatrix} 2\cos() - \sin() \\ \cos() \end{bmatrix} + 2\begin{bmatrix} \cos() + 2\sin() \\ \sin() \end{bmatrix} = \begin{bmatrix} 5\sin() \\ -\cos() + 2\sin() \end{bmatrix}$

18.

(a) Using MATLAB,

```
clear
A = sym([-1/4, 1,    0;
         -1, -1/4,   0;
          0,    0, -1/4]);
[V,D] = eig(A)
```

V =

$$\begin{pmatrix} 0 & i & -i \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

D =

$$\begin{pmatrix} -\frac{1}{4} & 0 & 0 \\ 0 & -\frac{1}{4} - i & 0 \\ 0 & 0 & -\frac{1}{4} + i \end{pmatrix}$$

$\therefore \lambda = -\frac{1}{4}, -\frac{1}{4} \pm i$

(b) From (a), one solution is $\vec{x}(t) = c_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{-t/4}$

Using $\begin{bmatrix} i \\ 1 \\ 0 \end{bmatrix} e^{(-\frac{1}{4} - i)t} = \begin{bmatrix} i \\ 1 \\ 0 \end{bmatrix} e^{-t/4} \left[\cos(t) - i \sin(t) \right]$

$$= e^{-t/4} \begin{bmatrix} \sin(t) \\ \cos(t) \\ 0 \end{bmatrix} + e^{-t/4} i \begin{bmatrix} \cos(t) \\ -\sin(t) \\ 0 \end{bmatrix}$$

and $\det \begin{bmatrix} 0 & e^{-t/4}\sin(t) & e^{-t/4}\cos(t) \\ 0 & e^{-t/4}\cos(t) & -e^{-t/4}\sin(t) \\ e^{-t/4} & 0 & 0 \end{bmatrix} = -e^{-\frac{3t}{4}} \neq 0$

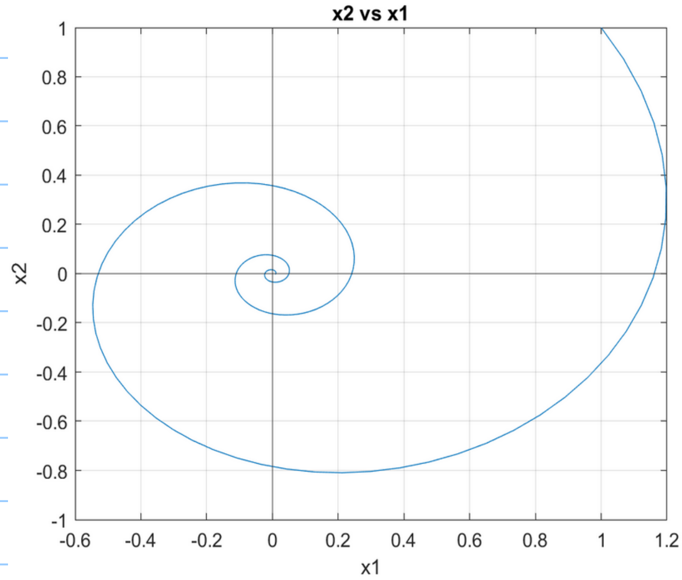
the general solution is:

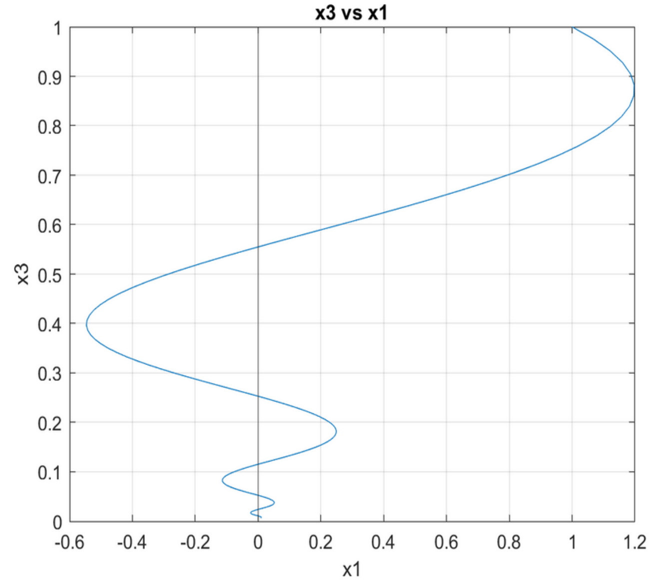
$$\vec{x}(t) = c_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{-t/4} + c_2 \begin{bmatrix} \sin(t) \\ \cos(t) \\ 0 \end{bmatrix} e^{-t/4} + c_3 \begin{bmatrix} \cos(t) \\ -\sin(t) \\ 0 \end{bmatrix} e^{-t/4}$$

Choose $c_1 = 1$, $c_2 = 1$, $c_3 = 1$, so that $\vec{x}(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

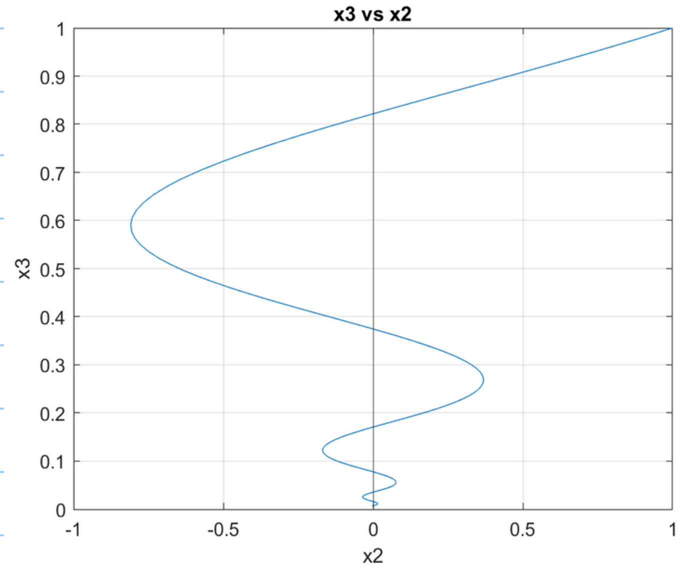
Using MATLAB,

```
clear
t = 0:0.1:20;
% from 1st evector
A1 = [0;0;1]*exp(-t/4);
% real of 2nd evector
A2 = [sin(t);cos(t);0*t].*exp(-t/4);
% imag of 2nd evector
A3 = [cos(t);-sin(t);0*t].*exp(-t/4);
X = A1 + A2 + A3;
x1 = X(1,:);  % 1st row, all cols of X
x2 = X(2,:);  % 2nd row, all cols of X
x3 = X(3,:);  % 3rd row, all cols of X

plot(x1,x2)
grid on
xline(0);  % show x and y axes
yline(0);
xlabel 'x1', ylabel 'x2'
title('x2 vs x1')

plot(x1,x3)
grid on
xline(0);  % show x and y axes
yline(0);
xlabel 'x1', ylabel 'x3'
title('x3 vs x1')

plot(x2,x3)
grid on
xline(0);  % show x and y axes
yline(0);
xlabel 'x2', ylabel 'x3'
title('x3 vs x2')
```





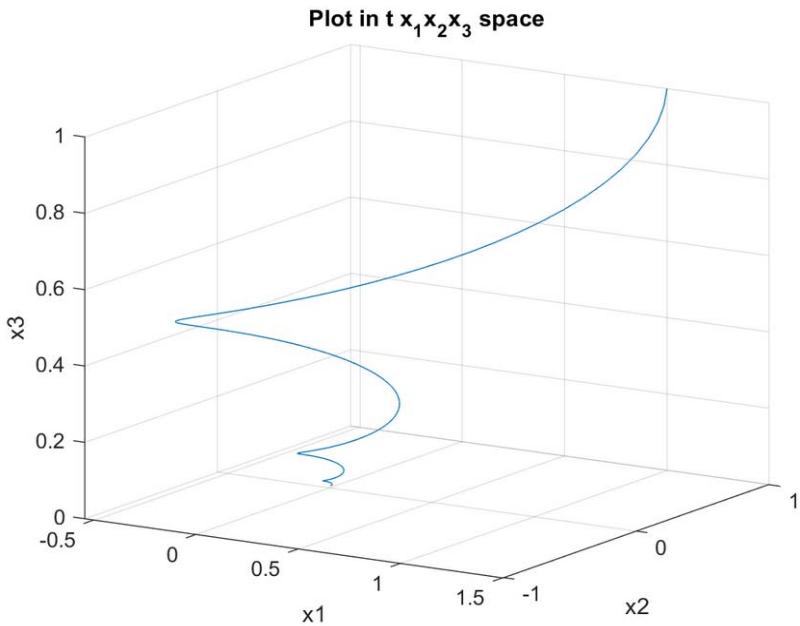
All three plots spiral toward origin.



(c)

Using MATLAB, code continuing from (b),

```
plot3(x1,x2,x3)
grid on
xlabel 'x1', ylabel 'x2', zlabel 'x3'
title 'Plot in t x_1x_2x_3 space'
% MATLAB supplied code after rotating plot
view([32.408 16.061])
```



From (1,1,1) spirals to (0,0,0)

19.

(a)

Using MATLAB,

```
clear
A = sym([-1/4, 1,    0;
          -1, -1/4,  0;
           0,    0, 1/10]);
[V,D] = eig(A)
```

V =

$$\begin{pmatrix} 0 & i & -i \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

D =

$$\begin{pmatrix} \frac{1}{10} & 0 & 0 \\ 0 & -\frac{1}{4} - i & 0 \\ 0 & 0 & -\frac{1}{4} + i \end{pmatrix}$$

$$\therefore \underline{\lambda = \frac{1}{10}, -\frac{1}{4} \pm i}$$

(b), (c)

From (a), one solution is $\vec{x}(t) = c_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{t/10}$

Using $\begin{bmatrix} i \\ 1 \\ 0 \end{bmatrix} e^{(-\frac{1}{4} - i)t} = \begin{bmatrix} i \\ 1 \\ 0 \end{bmatrix} e^{-t/4} \left[\cos(t) - i \sin(t) \right]$

$$= e^{-t/4} \begin{bmatrix} \sin(t) \\ \cos(t) \\ 0 \end{bmatrix} + e^{-t/4} i \begin{bmatrix} \cos(t) \\ -\sin(t) \\ 0 \end{bmatrix}$$

$$\det\begin{bmatrix} 0 & e^{-t/4}\sin(t) & e^{-t/4}\cos(t) \\ 0 & e^{-t/4}\cos(t) & -e^{-t/4}\sin(t) \\ e^{t/10} & 0 & 0 \end{bmatrix} = e^{-\frac{2}{5}t} \neq 0$$

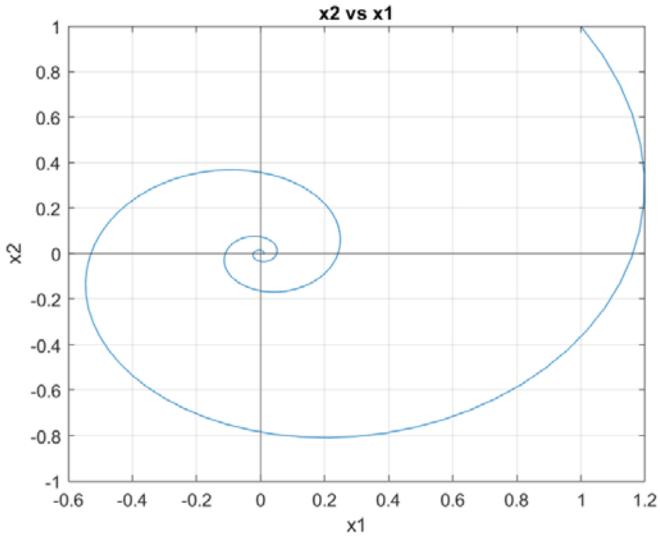
$\therefore$ General solution is:

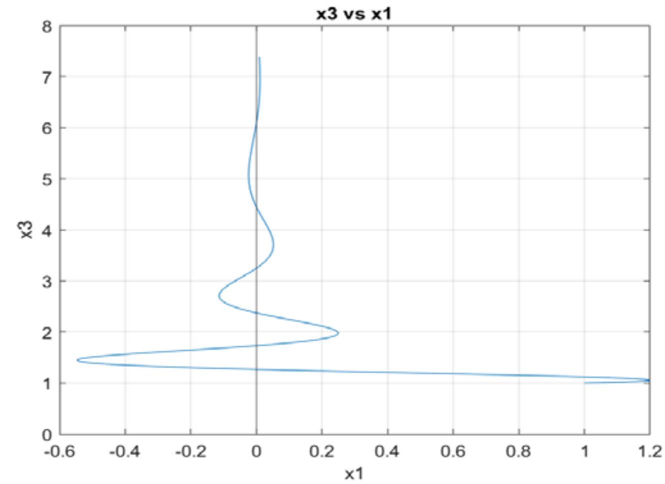
$$\vec{x}(t) = c_1\begin{bmatrix}0\\0\\1\end{bmatrix}e^{t/10} + c_2\begin{bmatrix}\sin(t)\\\cos(t)\\0\end{bmatrix}e^{-t/4} + c_3\begin{bmatrix}\cos(t)\\-\sin(t)\\0\end{bmatrix}e^{-t/4}$$

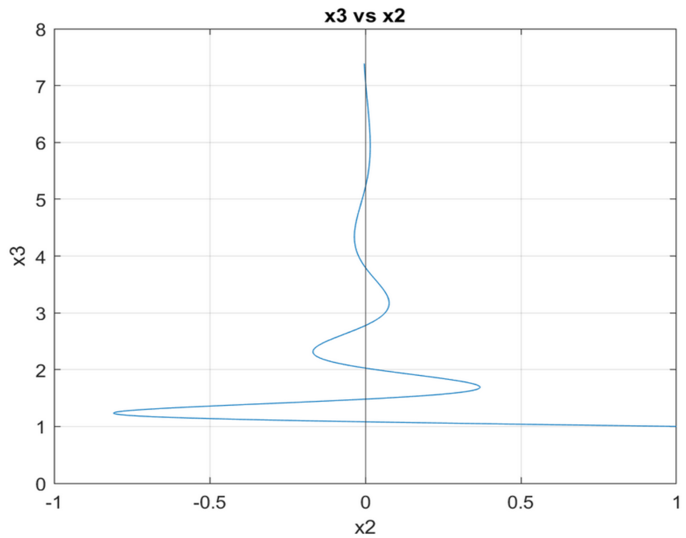
Choose $c_1 = c_2 = c_3 = 1$, $\therefore$ $\vec{x}(0) = (1,1,1)$

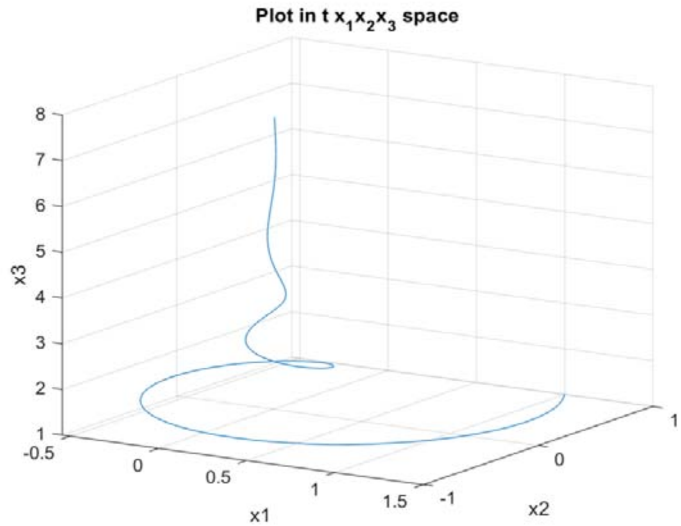
Using MATLAB,

```
clear
t = 0:0.1:20;
% from 1st evector
A1 = [0;0;1]*exp(t/10);
% real of 2nd evector
A2 = [sin(t);cos(t);0*t].*exp(-t/4);
% imag of 2nd evector
A3 = [cos(t);-sin(t);0*t].*exp(-t/4);
X = A1 + A2 + A3;
x1 = X(1,:);  % 1st row, all cols of X
x2 = X(2,:);  % 2nd row, all cols of X
x3 = X(3,:);  % 3rd row, all cols of X

plot(x1,x2)
grid on
xline(0);  % show x and y axes
yline(0);
xlabel 'x1', ylabel 'x2'
title('x2 vs x1')

plot(x1,x3)
grid on
xline(0);  % show x and y axes
yline(0);
xlabel 'x1', ylabel 'x3'
title('x3 vs x1')

plot(x2,x3)
grid on
xline(0);  % show x and y axes
yline(0);
xlabel 'x2', ylabel 'x3'
title('x3 vs x2')

plot3(x1,x2,x3)
grid on
xlabel 'x1', ylabel 'x2', zlabel 'x3'
title 'Plot in t x_1x_2x_3 space'
% MATLAB supplied code after rotating plot
view([32.408 16.061])
```









Plot spirals from (1,1,1) to $(0,0,\infty)$

20.

(a)

Using the analysis in #18 of section 7.1, which is the same circuit as in Figure 7.6.6,

$L \dfrac{dI}{dt} = -R_1 I - V$, $C \dfrac{dV}{dt} = I - \dfrac{V}{R_2}$, or

$$\frac{d}{dt}\begin{bmatrix} I \\ V \end{bmatrix} = \begin{bmatrix} -R_1/L & -1/L \\ 1/C & -1/CR_2 \end{bmatrix}\begin{bmatrix} I \\ V \end{bmatrix}$$

Using $R_1/L = 4/8 = \frac{1}{2}$, $1/L = \frac{1}{8}$, $\frac{1}{C} = 2$, $\frac{1}{CR_2} = \frac{1}{\frac{1}{2}(4)}$

$$\therefore \boxed{\frac{d}{dt}\begin{bmatrix} I \\ V \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{8} \\ 2 & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} I \\ V \end{bmatrix}}$$

(3)

Using MATLAB,

```
clear
A = sym([-1/2, -1/8,;
            2,   -1/2,]);
[V,D] = eig(A)
```

V =

$$\begin{pmatrix} -\frac{1}{4}\mathrm{i} & \frac{1}{4}\mathrm{i} \\ 1 & 1 \end{pmatrix}$$

D =

$$\begin{pmatrix} -\frac{1}{2}-\frac{1}{2}\mathrm{i} & 0 \\ 0 & -\frac{1}{2}+\frac{1}{2}\mathrm{i} \end{pmatrix}$$

$\therefore$ Eigenvalues : $-\frac{1}{2} \pm \frac{1}{2} i$

$$\therefore \vec{x}^{\,2}(t) = \begin{bmatrix} \frac{1}{4} i \\ 1 \end{bmatrix} e^{-t/2}\left(\cos\left(\frac{t}{2}\right) + i \sin\left(\frac{t}{2}\right)\right)$$

$$= \begin{bmatrix} -\frac{1}{4}\sin\left(\frac{t}{2}\right) \\ \cos(t/2) \end{bmatrix} e^{-t/2} + i \begin{bmatrix} \frac{1}{4}\cos(t/2) \\ \sin(t/2) \end{bmatrix} e^{-t/2}$$

$$\det\begin{bmatrix} -e^{-t/2}\,\frac{1}{4}\sin(t/2) & e^{-t/2}\,\frac{1}{4}\cos(t/2) \\ e^{-t/2}\cos(t/2) & e^{-t/2}\sin(t/2) \end{bmatrix} = -\frac{e^{-t}}{4} \neq 0$$

$\therefore$ Real, imaginary parts form fundamental set.

$$\therefore \boxed{\begin{bmatrix} I(t) \\ V(t) \end{bmatrix} = C_1 \begin{bmatrix} -\frac{1}{4}\sin(t/2) \\ \cos(t/2) \end{bmatrix} e^{-t/2} + C_2 \begin{bmatrix} \frac{1}{4}\cos(t/2) \\ \sin(t/2) \end{bmatrix} e^{-t/2}}$$

Multiplying first vector by -4, second vector by 4 gives answer in back of book.

(c)

From (b), $\begin{bmatrix} I(0) \\ V(0) \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} \frac{1}{4} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{4} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

$\therefore \begin{bmatrix} 0 & \frac{1}{4} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $c_1 = 3$, $c_2 = 8$

$$\therefore \begin{bmatrix} I(t) \\ V(t) \end{bmatrix} = \begin{bmatrix} -\frac{3}{4}\sin(t/2) \\ 3\cos(t/2) \end{bmatrix} e^{-t/2} + \begin{bmatrix} 2\cos(t/2) \\ 8\sin(t/2) \end{bmatrix} e^{-t/2}$$

$$\boxed{= e^{-t/2} \begin{bmatrix} 2\cos(t/2) - \frac{3}{4}\sin(t/2) \\ 3\cos(t/2) + 8\sin(t/2) \end{bmatrix}}$$

(d)

Because of the $e^{-t/2}$ factor, as $t \to \infty$, $\underline{I \to 0, V \to 0}$

These values do <u>not</u> depend on intial conditions.

Initial conditions do not affect the $e^{-t/2}$ factor.

21.

(a)

Using MATLAB,

```
clear
syms I V L R C
A = [0,      1/L;
     -1/C, -1/(R*C)];
[V,D] = eig(A)
```

V =

$$\begin{pmatrix} \frac{L+\sqrt{L\,(L-4\,C\,R^2)}}{2\,L\,R}-\frac{1}{R} & \frac{L-\sqrt{L\,(L-4\,C\,R^2)}}{2\,L\,R}-\frac{1}{R} \\ 1 & 1 \end{pmatrix}$$

D =

$$\begin{pmatrix} -\frac{L+\sqrt{L\,(L-4\,C\,R^2)}}{2\,C\,L\,R} & 0 \\ 0 & -\frac{L-\sqrt{L\,(L-4\,C\,R^2)}}{2\,C\,L\,R} \end{pmatrix}$$

$\therefore \lambda = \dfrac{-L \pm \sqrt{L^2 - 4LCR^2}}{2CLR}$ Note: L, R, C always >0

$\therefore$ If $L^2 - 4LCR^2 > 0$, or $L - 4CR^2 > 0$,

then two real, different values for λ

If $L^2 - 4LCR^2 < 0$, or $L - 4CR^2 < 0$,

then λ is complex.

(b)

From (a), $\frac{d}{dt}\begin{bmatrix} I \\ V \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} I \\ V \end{bmatrix}$

Using MATLAB,

```
clear
A = sym([0,  1;
        -2, -2]);
[V,D] = eig(A)
```

V =

$$\begin{pmatrix} -\frac{1}{2}+\frac{1}{2}\mathrm{i} & -\frac{1}{2}-\frac{1}{2}\mathrm{i} \\ 1 & 1 \end{pmatrix}$$

D =

$$\begin{pmatrix} -1-\mathrm{i} & 0 \\ 0 & -1+\mathrm{i} \end{pmatrix}$$

$\therefore \lambda = -1 \pm i$

Using $\lambda = -1-i$, $\vec{x}^{(1)}(t) = \begin{bmatrix} -\frac{1}{2}+\frac{1}{2}i \\ 1 \end{bmatrix} e^{-t}(\cos(t) - i\sin(t))$

$$= \begin{bmatrix} -\frac{1}{2}e^{-t}\cos(t) + \frac{1}{2}e^{-t}\sin(t) \\ e^{-t}\cos(t) \end{bmatrix} + i\begin{bmatrix} \frac{1}{2}e^{-t}\cos(t) + \frac{1}{2}e^{-t}\sin(t) \\ -e^{-t}\sin(t) \end{bmatrix} \quad [1]$$

$\det\begin{bmatrix} \end{bmatrix} = -\frac{1}{2}e^{-2t} \neq 0$, so real, imaginary parts form a fundamental set.

$\therefore$ multiplying components by 2 to simplify,

$$\boxed{\begin{bmatrix} I \\ V \end{bmatrix} = c_1 e^{-t}\begin{bmatrix} -\cos(t) + \sin(t) \\ 2\cos(t) \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} \cos(t) + \sin(t) \\ -2\sin(t) \end{bmatrix}}$$

Note: By adding and subtracting real and imaginary component vectors in [1], the answer in the back of the book is obtained.

(c)

From (b), $\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} I(0) \\ V(0) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

Using MATLAB,

```
clear
% use sym to get rational answers
A = sym([-1,  1;
          2,  0]);
B = [2; 1];
linsolve(A, B)
```

```
ans =
```

$$\begin{pmatrix} \frac{1}{2} \\ \frac{5}{2} \end{pmatrix}$$

$\therefore\ c_1 = \frac{1}{2},\ c_2 = \frac{5}{2}$

$\therefore$

$$\boxed{\begin{aligned} \begin{bmatrix} I \\ V \end{bmatrix} &= \frac{1}{2} e^{-t} \begin{bmatrix} -\cos(t) + \sin(t) \\ 2\cos(t) \end{bmatrix} + \frac{5}{2} e^{-t} \begin{bmatrix} \cos(t) + \sin(t) \\ -2\sin(t) \end{bmatrix} \\ &= e^{-t} \begin{bmatrix} 2\cos(t) + 3\sin(t) \\ \cos(t) - 5\sin(t) \end{bmatrix} \end{aligned}}$$

(d)

$$\lim_{t\to\infty} \begin{bmatrix} I(t) \\ V(t) \end{bmatrix} = \lim_{t\to\infty} c_1 e^{-t} \begin{bmatrix} -\cos(t) + \sin(t) \\ 2\cos(t) \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} \cos(t) + \sin(t) \\ -2\sin(t) \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{due to the } e^{-t} \text{ factor}$$

$\therefore$ $\underline{I(t) \to 0}$, $\underline{V(t) \to 0}$

Due to e^{-t}, the limits do not depend on c_1, c_2.

22.

(a)

$$\left.\begin{aligned} \vec{\epsilon}^{(1)} &= \vec{a} + i\vec{b} \\ \overline{\vec{\epsilon}^{(1)}} &= \vec{a} - i\vec{b} \end{aligned}\right\} \quad \text{Adding, } \vec{\epsilon}^{(1)} + \overline{\vec{\epsilon}^{(1)}} = 2\vec{a}$$

$$\therefore \vec{a} = \frac{1}{2}\left(\vec{\epsilon}^{(1)} + \overline{\vec{\epsilon}^{(1)}}\right)$$

Subtracting, $2i\vec{b} = \vec{\epsilon}^{(1)} - \overline{\vec{\epsilon}^{(1)}}$, $\vec{b} = \frac{1}{2i}\left(\vec{\epsilon}^{(1)} - \overline{\vec{\epsilon}^{(1)}}\right)$

$\therefore$ From $c_1\vec{a} + c_2\vec{b} = \vec{0}$,

$$\frac{c_1}{2}\left(\vec{\epsilon}^{(1)} + \overline{\vec{\epsilon}^{(1)}}\right) + \frac{c_2}{2i}\left(\vec{\epsilon}^{(1)} - \overline{\vec{\epsilon}^{(1)}}\right) = \vec{0}$$

$$\therefore \left(\frac{c_1}{2} + \frac{c_2}{2i}\right)\vec{\epsilon}^{(1)} + \left(\frac{c_1}{2} - \frac{c_2}{2i}\right)\overline{\vec{\epsilon}^{(1)}} = \vec{0},$$

Multiplying by 2

$$\left(c_1 + \frac{c_2}{i}\right)\vec{E}^{(1)} + \left(c_1 - \frac{c_2}{i}\right)\overline{\vec{E}^{(1)}} = \vec{0},$$

Or, $$\boxed{(c_1 - i c_2)\vec{E}^{(1)} + (c_1 + i c_2)\overline{\vec{E}^{(1)}} = \vec{0}}$$

(b)

The assumption $r_1 \neq \bar{r}_1$, or $\lambda + i\mu \neq \lambda - i\mu$ means $\vec{E}^{(1)}$ and $\overline{\vec{E}^{(1)}}$ are independent since they are the corresponding eigenvectors for r_1 and $\bar{r}_1$.

$\therefore$ From (a), $(c_1 - i c_2)\vec{E}^{(1)} + (c_1 + i c_2)\overline{\vec{E}^{(1)}} = \vec{0}$

$$\Rightarrow c_1 - i c_2 = 0 \text{ and } c_1 + i c_2 = 0$$

Adding, $2c_1 = 0 \Rightarrow c_1 = 0$

Subtracting, $-2i c_2 = 0 \Rightarrow c_2 = 0$

$\therefore$ $c_1 \vec{a} + c_2 \vec{b} = \vec{0} \Rightarrow c_1 = 0,\ c_2 = 0$

$\therefore$ $\vec{a}, \vec{b}$ are independent.

(c)

Since $\vec{u}(t) = e^{\lambda t}[\vec{a}\cos(\mu t) - \vec{b}\sin(\mu t)]$

and $\vec{v}(t) = e^{\lambda(t)}[\vec{a}\sin(\mu t) + \vec{b}\cos(\mu t)]$,

then $c_1\vec{u}(t_0) + c_2\vec{v}(t_0) = \vec{0} \Rightarrow$

$$e^{\lambda t_0}[c_1\cos(\mu t_0) + c_2\sin(\mu t_0)]\vec{a} +$$

$$e^{\lambda t_0}[-c_1\sin(\mu t_0) + c_2\cos(\mu t_0)]\vec{b} = \vec{0}$$

$\vec{a}, \vec{b}$ independent $\Rightarrow$

$$e^{\lambda t_0}[c_1\cos(\mu t_0) + c_2\sin(\mu t_0)] = 0$$

and $e^{\lambda t_0}[-c_1\sin(\mu t_0) + c_2\cos(\mu t_0)] = 0$

Or, $c_1\cos(\mu t_0) + c_2\sin(\mu t_0) = 0$ [1]

and $-c_1\sin(\mu t_0) + c_2\cos(\mu t_0) = 0$ [2]

Multiplying [1] by $\sin(\mu t_0)$ and [2] by $\cos(\mu t_0)$, and adding: $c_2[\cos^2(\mu t_0) + \sin^2(\mu t_0)] = 0$

$\Rightarrow c_2 = 0$

Multiplying [1] by $\cos(\mu t_0)$ and [2] by $\sin(\mu t_0)$, and subtracting, $c_1[\cos^2(\mu t_0) + \sin^2(\mu t_0)] = 0$

$\Rightarrow c_1 = 0$

$\therefore\ c_1\vec{u}(t_0) + c_2\vec{v}(t_0) = 0 \Rightarrow c_1 = 0,\ c_2 = 0$

$\therefore\ \vec{u}(t_0), \vec{v}(t_0)$ are linearly independent, and since t_0 was arbitrary, $\vec{u}(t), \vec{v}(t)$ are linearly independent.

23.

(a)

Note that $x_1' = u' = x_2$

$$x_2' = u'' = -\frac{k}{m}u = -\frac{k}{m}x_1$$

$$\therefore\ \left.\begin{aligned} x_1' &= x_2 \\ x_2' &= -\frac{k}{m}x_1 \end{aligned}\right\} \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\therefore$ Letting $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$, $\underline{\vec{x}' = \begin{bmatrix} 0 & 1 \\ -k/m & 0 \end{bmatrix}\vec{x}}$

(b)

$$\begin{bmatrix} 0 & 1 \\ -k/m & 0 \end{bmatrix} - \lambda I = \begin{bmatrix} -\lambda & 1 \\ -k/m & -\lambda \end{bmatrix}$$

$$\det \begin{bmatrix} -\lambda & 1 \\ -k/m & -\lambda \end{bmatrix} = \lambda^2 + \frac{k}{m} = 0, \quad \underline{\lambda = \pm i\sqrt{\frac{k}{m}}}$$

(c)

Computing for eigenvectors, using $\lambda = i\sqrt{\frac{k}{m}}$,

$$\begin{bmatrix} -i\sqrt{\frac{k}{m}} & 1 \\ -\frac{k}{m} & -i\sqrt{\frac{k}{m}} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ i\sqrt{\frac{k}{m}} \end{bmatrix}$$

$$\therefore \vec{x}^{(1)}(t) = \begin{bmatrix} 1 \\ i\sqrt{k/m} \end{bmatrix} \left[\cos\left(\sqrt{\tfrac{k}{m}}\, t\right) + i \sin\left(\sqrt{\tfrac{k}{m}}\, t\right)\right]$$

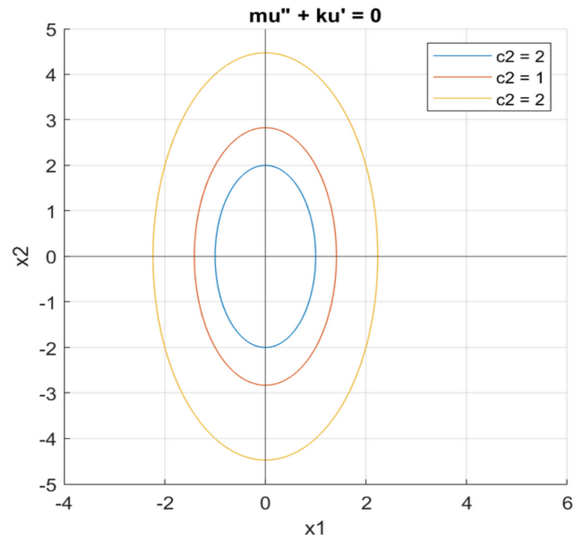
Using real and imaginary parts, and using $k = 4$ and $m = 1$ for arbitrary values,

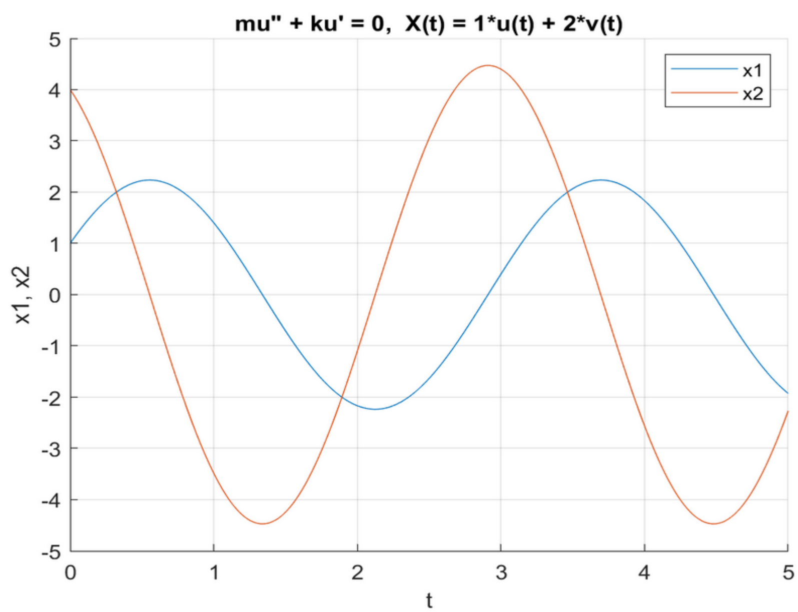
$$\vec{x}(t) = c_1 \begin{bmatrix} \cos(2t) \\ -2\sin(2t) \end{bmatrix} + c_2 \begin{bmatrix} \sin(2t) \\ 2\cos(2t) \end{bmatrix}$$

Trajectories will be based on these vectors.

Using MATLAB,

```
clear
t = 0:0.01:5;
u = [cos(2*t);
     -2*sin(2*t)];
v = [sin(2*t);
     2*cos(2*t)];

c = [1, 1, 1;  % arbitrary coeffs
     0, 1, 2];

figure
axis equal  % so ellipses look proportional
hold on
grid on
axis([-4, 6, -5, 5])  % plot bounds
for n = 1:size(c,2)  % up to # columns of c[]
    X = c(1,n)*u + c(2,n)*v;
    x1 = X(1,:);  % top row
    x2 = X(2,:);  % bottom row
    plot(x1,x2)
end
xline(0);  % show x and y axes
yline(0);
xlabel 'x1', ylabel 'x2'
title('mu" + ku'' = 0')
legend('c2 = 2','c2 = 1', 'c2 = 2')

figure  % now plot x1 vs t, x2 vs t
hold on
grid on
plot(t,x1)
plot(t,x2)
xline(0);
xlabel 't', ylabel 'x1, x2'
title('mu" + ku'' = 0,  X(t) = 1*u(t) + 2*v(t)')
legend('x1', 'x2')
```





(d)

The natural frequency is $\sqrt{\frac{K}{m}} = \omega$

$\therefore$ Eigenvalues $= \pm i\omega$

24.

(a)

Equation (22) is:

$$m_1 \frac{d^2 x_1}{dt^2} = -(k_1 + k_2)x_1 + k_2 x_2,$$
$$m_2 \frac{d^2 x_2}{dt^2} = k_2 x_1 - (k_2 + k_3)x_2. \tag{22}$$

This can be written as:

$$X_1'' = -\left(\frac{K_1 + K_2}{m_1}\right) X_1 + \left(\frac{K_2}{m_1}\right) X_2$$

$$X_2'' = \left(\frac{K_2}{m_2}\right) X_1 - \left(\frac{K_2 + K_3}{m_2}\right) X_2$$

Or,
$$\begin{bmatrix} X_1'' \\ X_2'' \end{bmatrix} = \begin{bmatrix} -\left(\frac{K_1 + K_2}{m_1}\right) & \frac{K_2}{m_1} \\ \frac{K_2}{m_2} & -\left(\frac{K_2 + K_3}{m_2}\right) \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

Substituting $m_1 = 2$, $m_2 = \frac{9}{4}$, $k_1 = 1$, $k_2 = 3$, $k_3 = \frac{15}{4}$

$$\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = \begin{bmatrix} -2 & \frac{3}{2} \\ \frac{4}{3} & -3 \end{bmatrix} \begin{bmatrix} x \\ x_2 \end{bmatrix}$$

Or,

$$\vec{x''} = \underline{\begin{bmatrix} -2 & \frac{3}{2} \\ \frac{4}{3} & -3 \end{bmatrix}} \vec{x} = A\vec{x}$$

(b)

From $\vec{x''} = A\vec{x}$, $A\vec{x} - \vec{x''} = A\vec{x} - I\vec{x''} = \vec{0}$

Letting $\vec{x}(t) = \vec{\epsilon} e^{rt}$, $\vec{x''} = r^2 \vec{\epsilon} e^{rt}$

$\therefore A(\vec{\epsilon} e^{rt}) - I(r^2 \vec{\epsilon} e^{rt}) = \vec{0}$

Or, $(A - r^2 I)\vec{\epsilon} e^{rt} = \vec{0}$,

Or, $\underline{(A - r^2 I)\vec{\epsilon} = \vec{0}}$ since $e^{rt} \neq 0$

(c)

Using MATLAB,

```
clear
A = sym([-2, 3/2;
         4/3, -3]);
[V,D] = eig(A)
```

V =

$$\begin{pmatrix} -\frac{3}{4} & \frac{3}{2} \\ 1 & 1 \end{pmatrix}$$

D =

$$\begin{pmatrix} -4 & 0 \\ 0 & -1 \end{pmatrix}$$

$\therefore$ Eigenvalues: $r^2 = -4, -1$

Eigenvectors: $\begin{bmatrix} -\frac{3}{4} \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$

Or, scaling the eigenvectors, so they resemble solutions in Example 3 in text,

$$\boxed{r_1^2 = -4, \begin{bmatrix} 3 \\ -4 \end{bmatrix} \qquad r_2^2 = -1, \begin{bmatrix} 3 \\ 2 \end{bmatrix}}$$

(d)

From $\vec{x}(t) = \vec{\xi}\, e^{rt}$, and using $r_1^2 = -4$, $r_2^2 = -1$,

then $r_1 = \pm 2i$, $r_2 = \pm i$

$\therefore$ Four solutions are:

$$\begin{bmatrix} 3 \\ -4 \end{bmatrix} e^{2it}, \begin{bmatrix} 3 \\ -4 \end{bmatrix} e^{-2it}, \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{it}, \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{-it}$$

Obtaining 4 real-valued solutions from

$\begin{bmatrix} 3 \\ -4 \end{bmatrix} e^{2it}$ and $\begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{it}$,

$$\begin{bmatrix} 3 \\ -4 \end{bmatrix} e^{2it} = \begin{bmatrix} 3 \\ -4 \end{bmatrix} [\cos(2t) + i\sin(2t)]$$

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{it} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} [\cos(t) + i\sin(t)]$$

$$\therefore \underbrace{\begin{bmatrix} 3 \\ -4 \end{bmatrix} \cos(2t)}_{= \vec{x}^{(1)}}, \underbrace{\begin{bmatrix} 3 \\ -4 \end{bmatrix} \sin(2t)}_{= \vec{x}^{(2)}}, \underbrace{\begin{bmatrix} 3 \\ 2 \end{bmatrix} \cos(t)}_{= \vec{x}^{(3)}}, \underbrace{\begin{bmatrix} 3 \\ 2 \end{bmatrix} \sin(t)}_{= \vec{x}^{(4)}}$$

$$\therefore \vec{x}(t) = c_1 \vec{x}^{(1)}(t) + c_2 \vec{x}^{2}(t) + c_3 \vec{x}^{(3)}(t) + c_4 \vec{x}^{(4)}(t)$$

Or,

$$\boxed{\begin{aligned} x_1(t) &= 3c_1\cos(2t) + 3c_2\sin(2t) + 3c_3\cos(t) + 3c_4\sin(t) \\ x_2(t) &= -4c_1\cos(2t) - 4c_2\sin(2t) + 2c_3\cos(t) + 2c_4\sin(t) \end{aligned}}$$

where $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$

Each of the $\vec{x}^{(i)}$ are clearly independent.

(e)

$$\boxed{\begin{aligned} x_1'(t) &= -6c_1\sin(2t) + 6c_2\cos(2t) - 3c_3\sin(t) + 3c_4\cos(t) \\ x_2'(t) &= 8c_1\sin(2t) - 8c_2\cos(2t) - 2c_3\sin(t) + 2c_4\cos(t) \end{aligned}}$$

25.

(a)

As shown on p. 324 of the text,

$$y' = \begin{bmatrix} y_1' \\ y_2' \\ y_3' \\ y_4' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -(k_1+k_2)/m_1 & k_2/m_1 & 0 & 0 \\ k_2/m_2 & -(k_2+k_3)/m_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

$$\therefore \quad y' = \begin{bmatrix} y_1' \\ y_2' \\ y_3' \\ y_4' \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & 3 & 0 & 0 \\ 9/4 & -\frac{13}{4} & 0 & 0 \end{bmatrix}}_{= A} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

(b)

Using MATLAB,

```
clear
A = sym([ 0,      0,  1, 0;
          0,      0,  0, 1;
         -4,      3,  0, 0;
        9/4, -13/4, 0, 0]);
[V,D] = eig(A)
```

V =

$$\begin{pmatrix} -\frac{8}{15}i & \frac{8}{15}i & i & -i \\ \frac{2}{5}i & -\frac{2}{5}i & i & -i \\ -\frac{4}{3} & -\frac{4}{3} & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

D =

$$\begin{pmatrix} -\frac{5}{2}i & 0 & 0 & 0 \\ 0 & \frac{5}{2}i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \end{pmatrix}$$

Scaling each eigenvector to resemble the answer in the back of the book:

```
V*[ 15i/2,  0,     0,  0;
        0, -15i/2, 0,  0;
        0,     0, -1i,  0;
        0,     0,   0, 1i]
```

ans =

$$\begin{pmatrix} 4 & 4 & 1 & 1 \\ -3 & -3 & 1 & 1 \\ -10\,i & 10\,i & -i & i \\ \frac{15}{2}\,i & -\frac{15}{2}\,i & -i & i \end{pmatrix}$$

$\therefore$ Eigenvalues and corresponding eigenvectors:

$$\boxed{-\frac{5}{2}i,\ \begin{bmatrix} 4 \\ -3 \\ -10i \\ \frac{15}{2}i \end{bmatrix} \qquad \frac{5}{2}i,\ \begin{bmatrix} 4 \\ -3 \\ 10i \\ -\frac{15}{2}i \end{bmatrix} \qquad -i,\ \begin{bmatrix} 1 \\ 1 \\ -i \\ -i \end{bmatrix} \qquad i,\ \begin{bmatrix} 1 \\ 1 \\ i \\ i \end{bmatrix}}$$

(c)

Convert one of the complex conjugate eigenvalue/eigenvector pairs to independent real-valued functions.

Select $\frac{5}{2}i,\ \begin{bmatrix} 4 \\ -3 \\ 10i \\ -\frac{15}{2}i \end{bmatrix}$ and $i,\ \begin{bmatrix} 1 \\ 1 \\ i \\ i \end{bmatrix}$

$$\therefore \begin{bmatrix} 4 \\ -3 \\ 10i \\ -\frac{15}{2}i \end{bmatrix} e^{\frac{5}{2}it} = \begin{bmatrix} 4 \\ -3 \\ 10i \\ -\frac{15}{2}i \end{bmatrix} \left[\cos\left(\tfrac{5}{2}t\right) + i\sin\left(\tfrac{5}{2}t\right)\right]$$

$$= \underbrace{\begin{bmatrix} 4\cos(\frac{5}{2}t) \\ -3\cos(\frac{5}{2}t) \\ -10\sin(\frac{5}{2}t) \\ \frac{15}{2}\sin(\frac{5}{2}t) \end{bmatrix}}_{\text{"}\vec{r}(t)\text{"}} + i \underbrace{\begin{bmatrix} 4\sin(\frac{5}{2}t) \\ -3\sin(\frac{5}{2}t) \\ 10\cos(\frac{5}{2}t) \\ -\frac{15}{2}\cos(\frac{5}{2}t) \end{bmatrix}}_{\text{"}\vec{s}(t)\text{"}}$$

$$\begin{bmatrix} 1 \\ 1 \\ i \\ i \end{bmatrix} e^{it} = \begin{bmatrix} 1 \\ 1 \\ i \\ i \end{bmatrix} \left[\cos(t) + i\sin(t)\right]$$

$$= \underbrace{\begin{bmatrix} \cos(t) \\ \cos(t) \\ -\sin(t) \\ -\sin(t) \end{bmatrix}}_{\text{"}\vec{u}(t)\text{"}} + i \underbrace{\begin{bmatrix} \sin(t) \\ \sin(t) \\ \cos(t) \\ \cos(t) \end{bmatrix}}_{\text{"}\vec{v}(t)\text{"}}$$

$$\boxed{\begin{aligned} \vec{y}(t) &= C_1 \vec{r}(t) + C_2 \vec{s}(t) + C_3 \vec{u}(t) + C_4 \vec{v}(t) \\ &= C_1 \begin{bmatrix} 4\cos(\frac{5}{2}t) \\ -3\cos(\frac{5}{2}t) \\ -10\sin(\frac{5}{2}t) \\ \frac{15}{2}\sin(\frac{5}{2}t) \end{bmatrix} + C_2 \begin{bmatrix} 4\sin(\frac{5}{2}t) \\ -3\sin(\frac{5}{2}t) \\ 10\cos(\frac{5}{2}t) \\ -\frac{15}{2}\cos(\frac{5}{2}t) \end{bmatrix} + C_3 \begin{bmatrix} \cos(t) \\ \cos(t) \\ -\sin(t) \\ -\sin(t) \end{bmatrix} + C_4 \begin{bmatrix} \sin(t) \\ \sin(t) \\ \cos(t) \\ \cos(t) \end{bmatrix} \end{aligned}}$$

(d)

The first two terms $(\vec{r}(t), \vec{s}(t))$ have frequency $\frac{5}{2}$ and period $\frac{2\pi}{5/2} = \frac{4\pi}{5}$. $y_2 = -\frac{3}{4} y_1$, so that m_1 and m_2 move together, but in opposite directions, m_2 moving $\frac{3}{4}$ as much as m_1.

In the $y_1 y_3$ plane and $y_2 y_4$ plane, both masses move in ellipses, clockwise for m_1 and m_2 for $\vec{r}(t)$, clockwise for m_1 and m_2 for $\vec{s}(t)$.

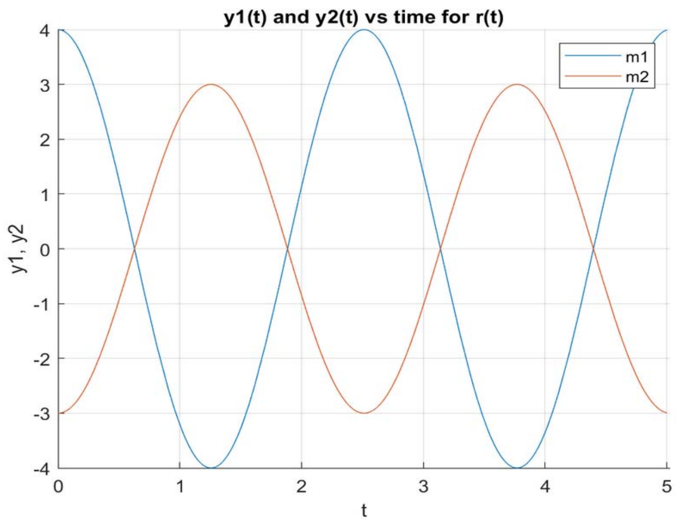
The last two terms $(\vec{u}(t), \vec{v}(t))$ have frequency 1 and period 2π. $y_2 = y_1$, so that m_1 and m_2 move together in the same direction with the same amplitude. In the $y_1 y_3$ and $y_2 y_4$ planes, both m_1 and m_2 move in circles.

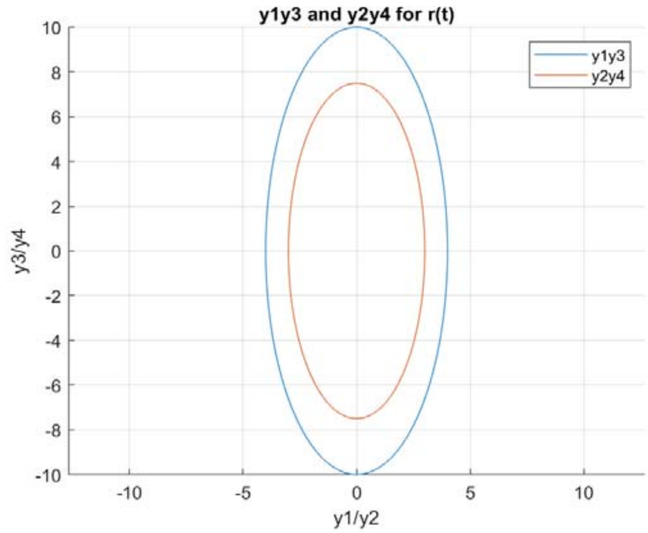
Using MATLAB, plots for y_1 vs t, y_2 vs t, and $y_1 y_3$ plots and $y_2 y_4$ plots are shown for

$\vec{r}(t)$, $\vec{s}(t)$, $\vec{u}(t)$, and $\vec{v}(t)$.

$\vec{r}(t)$:

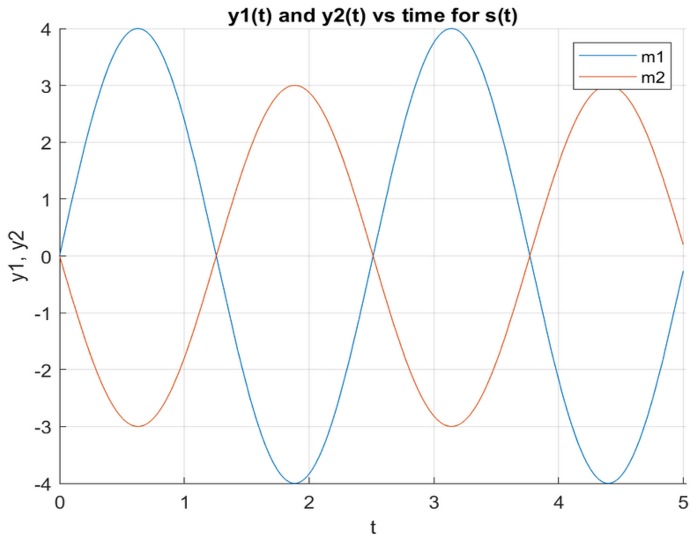
```
clear
syms t
w = 5/2; a = 15/2;
r = [4*cos(w*t);
    -3*cos(w*t);
    -10*sin(w*t);
     a*sin(w*t)];
% y1 vs t, y2 vs t
figure
hold on
grid on
fplot(t,r(1))  % y1 for m1
fplot(t,r(2))  % y2 for m2
% 2 periods
xlim([0,2*(4*pi/5)])
xlabel 't', ylabel 'y1, y2'
legend('m1', 'm2')
title 'y1(t) and y2(t) vs time for r(t)'

% y1y3 and y2y4 planes
figure
hold on
grid on
fplot(r(1), r(3))
fplot(r(2), r(4))
axis equal
xlabel 'y1/y2', ylabel 'y3/y4'
legend('y1y3', 'y2y4')
title 'y1y3 and y2y4 for r(t)'
```

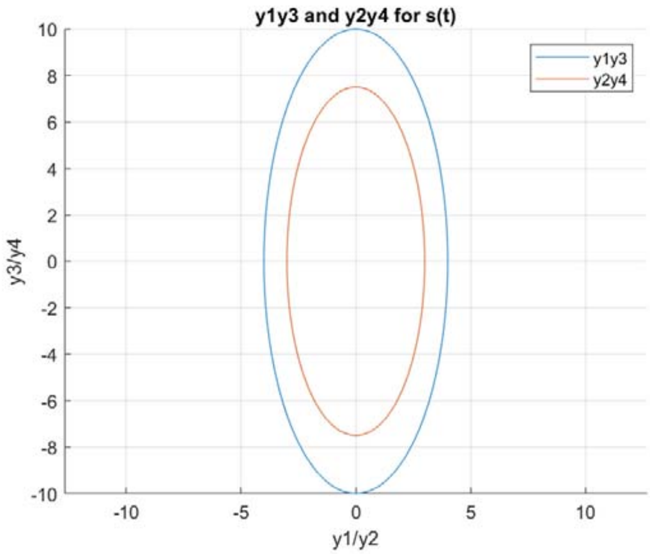




$\vec{s}(t)$:

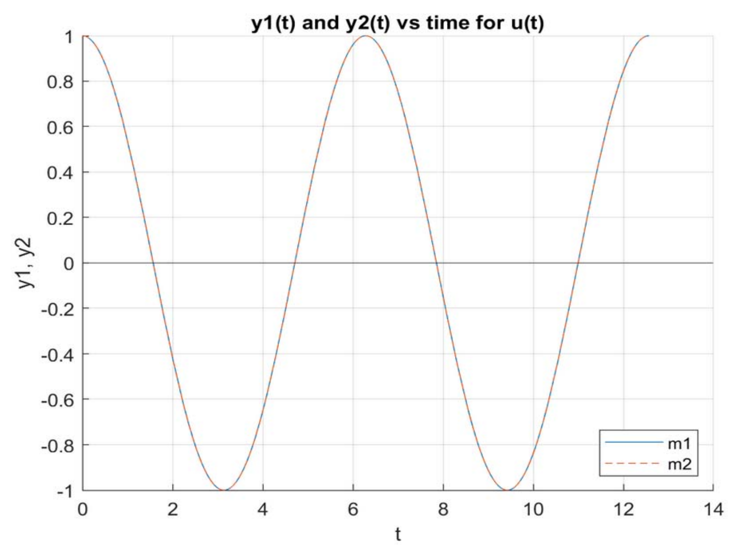
```
clear
syms t
w = 5/2; a = 15/2;
s = [4*sin(w*t);
    -3*sin(w*t);
     10*cos(w*t);
     -a*cos(w*t)];
% y1 vs t, y2 vs t
figure
hold on
grid on
fplot(t,s(1))  % y1 for m1
fplot(t,s(2))  % y2 for m2
% 2 periods
xlim([0,2*(4*pi/5)])
xlabel 't', ylabel 'y1, y2'
legend('m1', 'm2')
title 'y1(t) and y2(t) vs time for s(t)'

% y1y3 and y2y4 planes
figure
hold on
grid on
fplot(s(1), s(3))
fplot(s(2), s(4))
axis equal
xlabel 'y1/y2', ylabel 'y3/y4'
legend('y1y3', 'y2y4')
title 'y1y3 and y2y4 for s(t)'
```

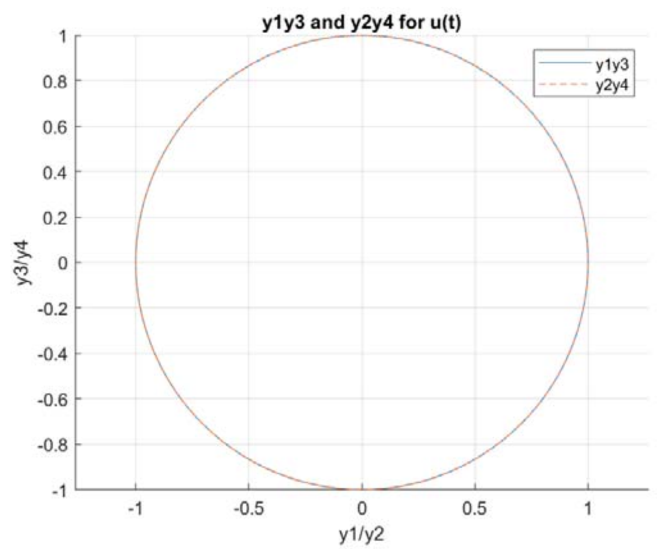




$\vec{u}(t)$:

```
clear
syms t
u = [cos(t);
     cos(t);
     -sin(t);
     -sin(t)];
% y1 vs t, y2 vs t
figure
hold on
grid on
% plot over 2 periods
fplot(t,u(1),[0,4*pi])  % y1 for m1
fplot(t,u(2),[0,4*pi],'--')  % y2 for m2
yline(0);      % x-axis
xlabel 't', ylabel 'y1, y2'
legend('m1', 'm2', 'location', 'southeast')
title 'y1(t) and y2(t) vs time for u(t)'

% y1y3 and y2y4 planes
figure
hold on
grid on
fplot(u(1), u(3))
fplot(u(2), u(4),'--')
axis equal
xlabel 'y1/y2', ylabel 'y3/y4'
legend('y1y3', 'y2y4')
title 'y1y3 and y2y4 for u(t)'
```

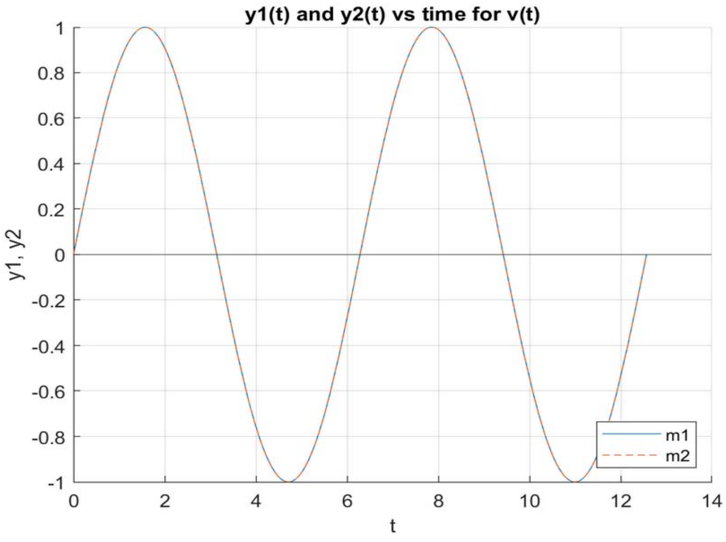


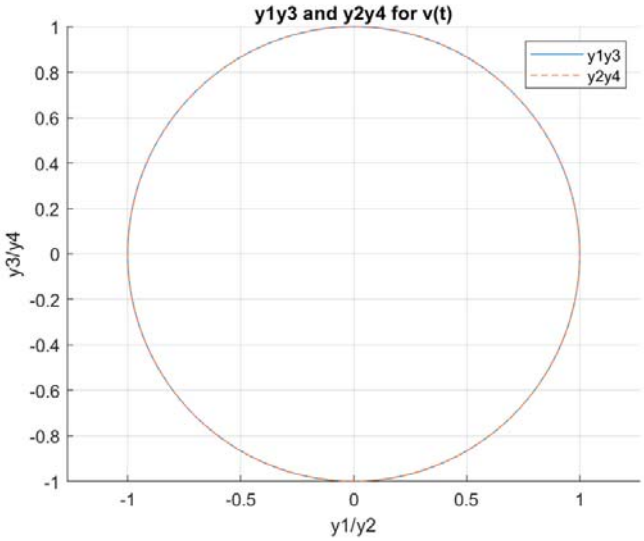


the plots exactly overlie each other

$\vec{v(t)}$:

```
clear
syms t
v = [sin(t);
     sin(t);
     cos(t);
     cos(t)];
% y1 vs t, y2 vs t
figure
hold on
grid on
% plot over 2 periods
fplot(t,v(1),[0,4*pi])  % y1 for m1
fplot(t,v(2),[0,4*pi],'--')  % y2 for m2
yline(0);      % x-axis
xlabel 't', ylabel 'y1, y2'
legend('m1', 'm2', 'location', 'southeast')
title 'y1(t) and y2(t) vs time for v(t)'

% y1y3 and y2y4 planes
figure
hold on
grid on
fplot(v(1), v(3))
fplot(v(2), v(4),'--')
axis equal
xlabel 'y1/y2', ylabel 'y3/y4'
legend('y1y3', 'y2y4')
title 'y1y3 and y2y4 for v(t)'
```





plots overlie each other

(e)

From (c), $\vec{y}(0) = \begin{bmatrix} 4 & 0 & 1 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & 10 & 0 & 1 \\ 0 & -\frac{15}{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

Using MATLAB,

```
clear
A = sym([4, 0, 1, 0;
        -3, 0, 1, 0;
         0, 10, 0, 1;
         0, -15/2, 0, 1]);
B = [2, 1, 0, 0]';
linsolve(A,B)  % solve Ax = B
```

ans =

$$\begin{pmatrix} \frac{1}{7} \\ 0 \\ \frac{10}{7} \\ 0 \end{pmatrix}$$

$$\therefore \vec{y}(t) = \frac{1}{7}\begin{bmatrix} 4\cos(\frac{5}{2}t) \\ -3\cos(\frac{5}{2}t) \\ -10\sin(\frac{5}{2}t) \\ \frac{15}{2}\sin(\frac{5}{2}t) \end{bmatrix} + \frac{10}{7}\begin{bmatrix} \cos(t) \\ \cos(t) \\ -\sin(t) \\ -\sin(t) \end{bmatrix}$$

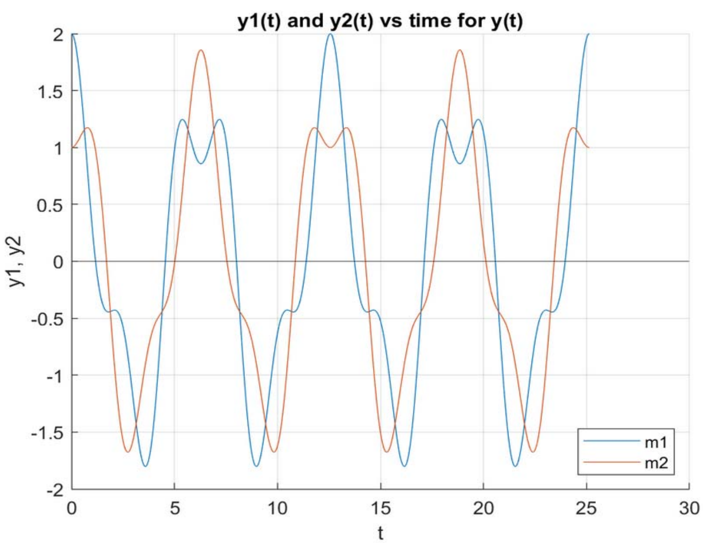
The first vector has period $\frac{2\pi}{5/2} = \frac{4}{5}\pi$

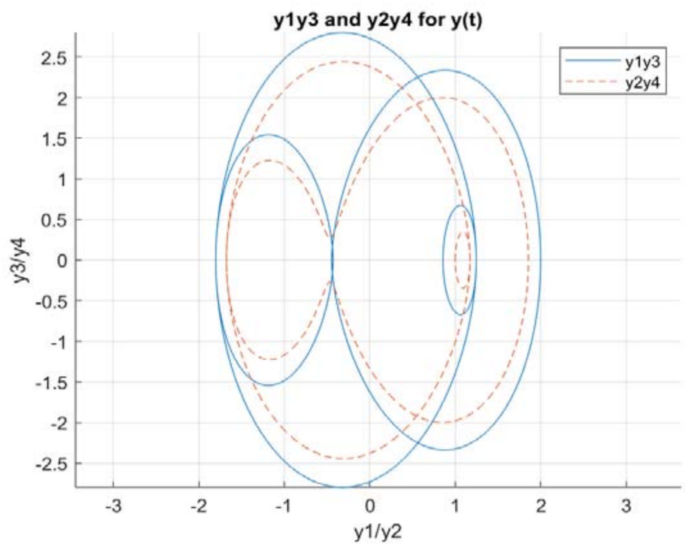
The second vector has a period of 2π.

5 times first period $= 4\pi$.

$\therefore$ $\underline{4\pi}$ is period of motion

```
clear
syms t
w = 5/2; a = 15/2;
f1 = (1/7)*[4*cos(w*t);
           -3*cos(w*t);
           -10*sin(w*t);
            a*sin(w*t)];
f3 = (10/7)*[cos(t);
             cos(t);
            -sin(t);
            -sin(t)];
y = f1 + f3;
% y1 vs t, y2 vs t
figure
hold on
grid on
% plot over 2 periods
fplot(t,y(1),[0,8*pi])  % y1 for m1
fplot(t,y(2),[0,8*pi])  % y2 for m2
yline(0);      % x-axis
xlabel 't', ylabel 'y1, y2'
legend('m1', 'm2', 'location', 'southeast')
title 'y1(t) and y2(t) vs time for y(t)'

% y1y3 and y2y4 planes
figure
hold on
grid on
fplot(y(1), y(3), [0,4*pi])
fplot(y(2), y(4), [0,4*pi], '--')
axis equal
xlabel 'y1/y2', ylabel 'y3/y4'
legend('y1y3', 'y2y4')
title 'y1y3 and y2y4 for y(t)'
```

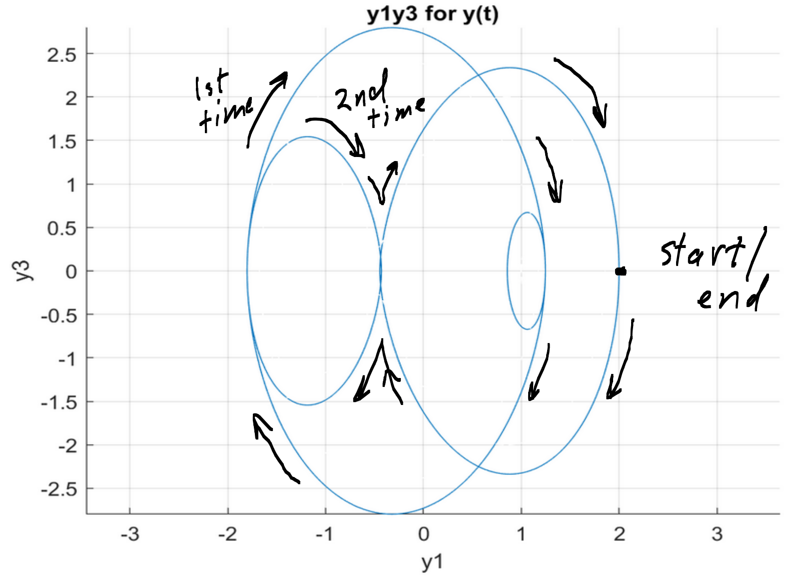




Just one period is drawn for $y_1 y_3$ and $y_2 y_4$

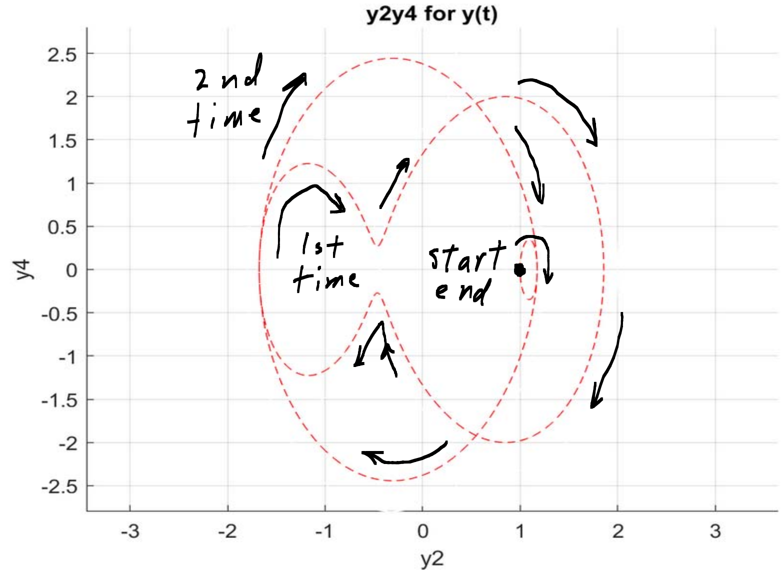
Plots of $y_1 y_3$ and $y_2 y_4$ shown separately.

```
% y1y3 plane
figure
hold on
grid on
fplot(y(1), y(3), [0,4*pi])
% hide y2y4 with white color
fplot(y(2), y(4), [0,4*pi], '--w')
axis equal
xlabel 'y1', ylabel 'y3'
title 'y1y3 for y(t)'
```



Plot starts at (2,0), $y_3(t)$ starts in negative direction.

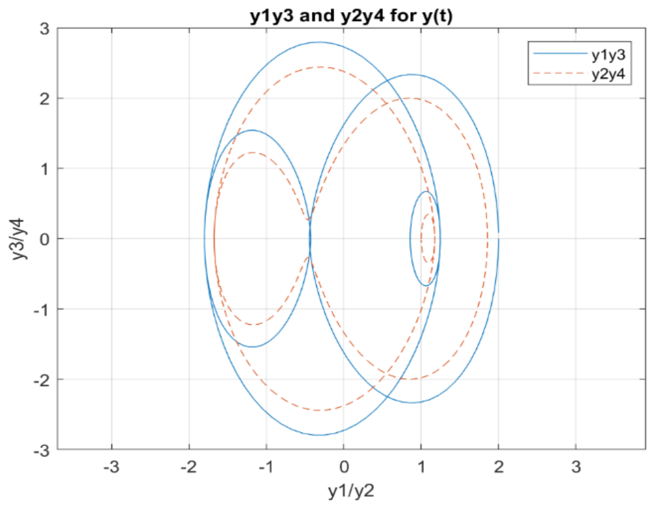
```
% y2y4 plane
figure
hold on
grid on
% hide y1y3 with white color
fplot(y(1), y(3), [0,4*pi], '-w')
fplot(y(2), y(4), [0,4*pi], '--r')
axis equal
xlabel 'y2', ylabel 'y4'
title 'y2y4 for y(t)'
```

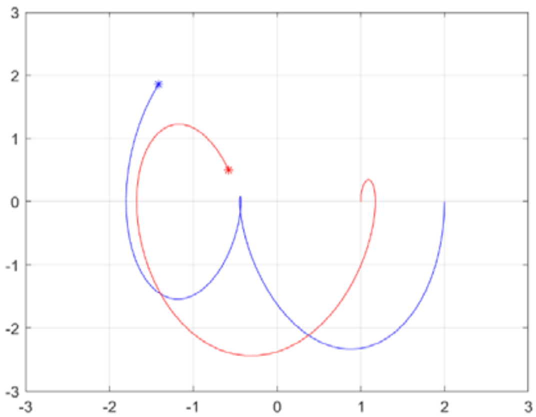


Plot starts out at (1,0) and immediately changes direction.

For interest, an animation plot is shown on the next page. "plot" is used instead of "fplot" in order to show animation.

```
clear
% play with this value for acceptable
%   resolution and animation speed
step = 0.05;
t = 0:step:4*pi; % one period
w = 5/2; a = 15/2;
f1 = (1/7)*[4*cos(w*t);  % from c1 value
           -3*cos(w*t);
           -10*sin(w*t);
            a*sin(w*t)];
f3 = (10/7)*[cos(t);      % from c3 value
             cos(t);
            -sin(t);
            -sin(t)];
y = f1 + f3;
y1 = y(1,:);  % position array of m1
y2 = y(2,:);  % position array of m2
y3 = y(3,:);  % velocity array of m1
y4 = y(4,:);  % velocity array of m2

% y1y3 and y2y4 planes
plot(y1, y3)
hold on
grid on
plot(y2, y4, '--')
xlim([-3 3])
ylim([-3 3])
axis equal
xlabel 'y1/y2', ylabel 'y3/y4'
legend('y1y3', 'y2y4')
title 'y1y3 and y2y4 for y(t)'
hold off

% Animation of y1y3 and y2y4
% R2019a doesn't enable animation by default,
% so enable animation with these lines of code
s = settings;
s.matlab.editor.AllowFigureAnimation.TemporaryValue = 1;

for i = 1:length(t)
    plot(y1(i), y3(i),'*b')
    axis([-3,3,-3,3])
    hold on
    grid on
    plot(y2(i), y4(i),'*r')
    plot(y1(1:i),y3(1:i),'b')
    plot(y2(1:i),y4(1:i),'r')
    pause(0.1)
    drawnow
    % clear previously drawn asterisks
    % so only leading asterisk shows
    if i ~= length(t)
        clf
    end
end
```





Animation captured midway.

(f)

Suppose $\vec{y}(0) = (2,1,1,1)^T$, similar to (e), except each mass starts with a velocity of 1 m/sec to the right. Using MATLAB,

```
clear
A = sym([4, 0, 1, 0;
        -3, 0, 1, 0;
         0, 10, 0, 1;
         0, -15/2, 0, 1]);
B = [2, 1, 1, 1]';
linsolve(A,B)  % solve Ax = B
```

ans =

$$\begin{pmatrix} \frac{1}{7} \\ 0 \\ \frac{10}{7} \\ 1 \end{pmatrix}$$

$\therefore c_1 = \frac{1}{7}$, $c_2 = 0$

$c_3 = \frac{10}{7}$, $c_4 = 1$

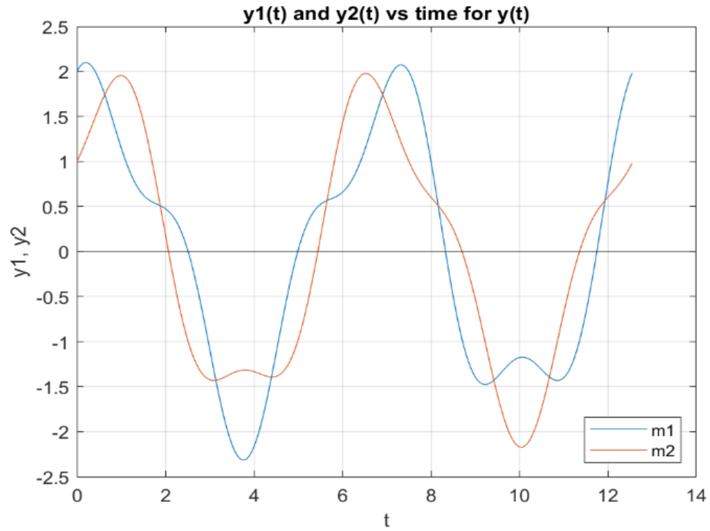
$$\therefore \vec{y}(t) = \frac{1}{7}\begin{bmatrix} 4\cos(\frac{5}{2}t) \\ -3\cos(\frac{5}{2}t) \\ -10\sin(\frac{5}{2}t) \\ \frac{15}{2}\sin(\frac{5}{2}t) \end{bmatrix} + \frac{10}{7}\begin{bmatrix} \cos(t) \\ \cos(t) \\ -\sin(t) \\ -\sin(t) \end{bmatrix} + \begin{bmatrix} \sin(t) \\ \sin(t) \\ \cos(t) \\ \cos(t) \end{bmatrix}$$

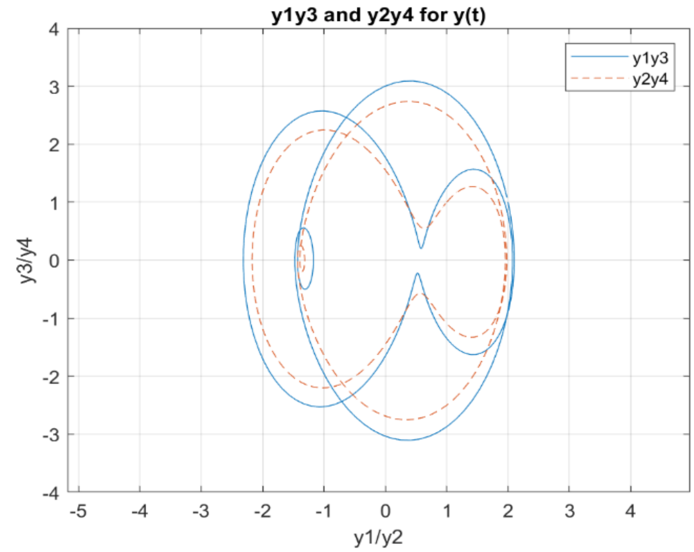
Using the results in (e).

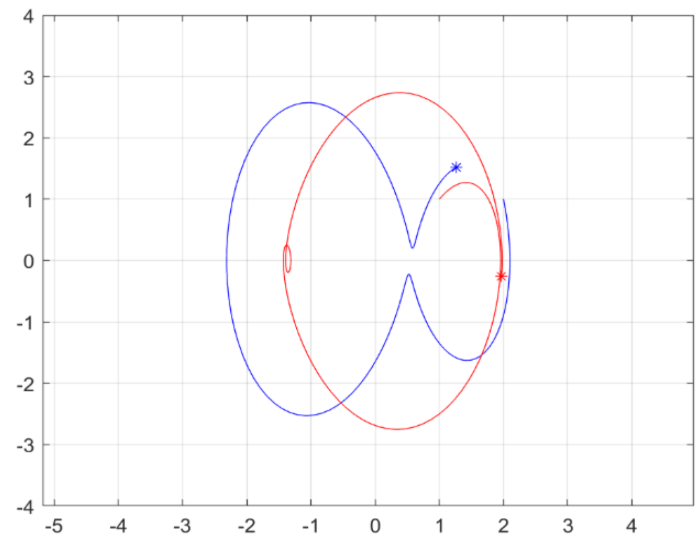
Once again the period for $\vec{y}(t)$ is 4π.

Using similar MATLAB code as in (e), and using the code for animation, code and plots shown on next page.

```
clear
% play with this value for acceptable
%   resolution and animation speed
step = 0.05;
t = 0:step:4*pi; % one period
w = 5/2; a = 15/2;
f1 = (1/7)*[4*cos(w*t);  % from c1 value
           -3*cos(w*t);
           -10*sin(w*t);
            a*sin(w*t)];
f3 = (10/7)*[cos(t);     % from c3 value
             cos(t);
            -sin(t);
            -sin(t)];
f4 = [sin(t);            % from c4 value
      sin(t);
      cos(t);
      cos(t)];
y = f1 + f3 + f4;
y1 = y(1,:);  % position array of m1
y2 = y(2,:);  % position array of m2
y3 = y(3,:);  % velocity array of m1
y4 = y(4,:);  % velocity array of m2

% y1 vs t, y2 vs t
% plot over 1 period
plot(t,y1)  % y1 for m1
hold on
grid on
plot(t,y2)  % y2 for m2
yline(0);      % x-axis
xlabel 't', ylabel 'y1, y2'
legend('m1', 'm2', 'location', 'southeast')
title 'y1(t) and y2(t) vs time for y(t)'
hold off


% y1y3 and y2y4 planes
plot(y1, y3)
hold on
grid on
plot(y2, y4, '--')
xlim([-3 3])
ylim([-4 4])
axis equal
xlabel 'y1/y2', ylabel 'y3/y4'
legend('y1y3', 'y2y4')
title 'y1y3 and y2y4 for y(t)'
hold off

% Animation of y1y3 and y2y4

% R2019a doesn't enable animation by default,
% so enable animation with these lines of code
s = settings;
s.matlab.editor.AllowFigureAnimation.TemporaryValue = 1;

for i = 1:length(t)
    plot(y1(i), y3(i),'*b')
    axis([-3,3,-4,4])
    hold on
    grid on
    axis equal
    plot(y2(i), y4(i),'*r')
    plot(y1(1:i),y3(1:i),'b')
    plot(y2(1:i),y4(1:i),'r')
    pause(0.1)
    drawnow
    % clear previously drawn asterisks
    % so only leading asterisk shows
    if i ~= length(t)
        clf
    end
end
```







picture of animation →
in midcycle.

7.7 Fundamental Matrices

Note Title 2/14/2020

1.

(a) Using MATLAB,

```
clear
A = sym([3, -2;
         2, -2]);
[V,D] = eig(A)
```

V =

$$\begin{pmatrix} \frac{1}{2} & 2 \\ 1 & 1 \end{pmatrix}$$

D =

$$\begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$$

Scaling the first eigenvector, the eigenvectors of A are $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\therefore$ Independent solutions are $\begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t}$, $\begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t}$

$\therefore$ A Fundamental matrix is: $\underline{\begin{bmatrix} e^{-t} & 2e^{t} \\ 2e^{-t} & e^{t} \end{bmatrix}}$

(b) $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}^{-1}$

$$\therefore \quad \phi(t) = \begin{bmatrix} e^{-t} & 2e^{t} \\ 2e^{-t} & e^{t} \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}^{-1}$$

Since $\phi(0) = \begin{bmatrix} e^{-0} & 2e^{0} \\ 2e^{-0} & e^{0} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Using MATLAB, with code continuing from (a),

```
% scale V(:,1) - make all integers
V = [2*V(:,1), V(:,2)]

syms t
% the actual fundamental matrix
Ft = [V(:,1)*exp(D(1,1)*t), ...
          V(:,2)*exp(D(2,2)*t)]

% evaluate at t = 0
Ft0 = subs(Ft,t,0);

% efficient MATLAB code for
%  Ft*(inverse of Ft0)
Phi = Ft/Ft0
```

V =

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

Ft =

$$\begin{pmatrix} e^{-t} & 2\,e^{2\,t} \\ 2\,e^{-t} & e^{2\,t} \end{pmatrix}$$

Phi =

$$\begin{pmatrix} \frac{e^{-t}\,(4\,e^{3\,t}-1)}{3} & -\frac{2\,e^{-t}\,(e^{3\,t}-1)}{3} \\ \frac{2\,e^{-t}\,(e^{3\,t}-1)}{3} & -\frac{e^{-t}\,(e^{3\,t}-4)}{3} \end{pmatrix}$$

$$\therefore \quad \boxed{\phi(t) = \begin{matrix} -\frac{1}{3}e^{-t} + \frac{4}{3}e^{2t} & \frac{2}{3}e^{-t} - \frac{2}{3}e^{2t} \\ -\frac{2}{3}e^{-t} + \frac{2}{3}e^{2t} & \frac{4}{3}e^{-t} - \frac{1}{3}e^{2t} \end{matrix}}$$

2.

(a)

Using MATLAB,

```
clear
A = sym([-3/4, 1/2;
         1/8, -3/4]);
[V,D] = eig(A)

syms t
% the actual fundamental matrix
Ft = [V(:,1)*exp(D(1,1)*t), ...
          V(:,2)*exp(D(2,2)*t)]
```

V =

$$\begin{pmatrix} -2 & 2 \\ 1 & 1 \end{pmatrix}$$

D =

$$\begin{pmatrix} -1 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

Ft =

$$\begin{pmatrix} -2\,e^{-t} & 2\,e^{-\frac{t}{2}} \\ e^{-t} & e^{-\frac{t}{2}} \end{pmatrix}$$

$\therefore \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-t}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-\frac{1}{2}t}$

$\therefore$ Fundamental matrix: $\underline{\begin{bmatrix} -2e^{-t} & 2e^{-t/2} \\ e^{-t} & e^{-t/2} \end{bmatrix}}$

(b)

Using MATLAB, continuation from above,

```
% evaluate at t = 0
Ft0 = subs(Ft,t,0);

% efficient MATLAB code for
%  Ft*(inverse of Ft0)
Phi = Ft/Ft0
```

Ft0 =

$$\begin{pmatrix} -2 & 2 \\ 1 & 1 \end{pmatrix}$$

Phi =

$$\begin{pmatrix} \frac{e^{-t}\,(e^{t/2}+1)}{2} & e^{-t}\,(e^{t/2}-1) \\ \frac{e^{-t}\,(e^{t/2}-1)}{4} & \frac{e^{-t}\,(e^{t/2}+1)}{2} \end{pmatrix}$$

$$\therefore \boxed{\phi(t) = \begin{bmatrix} \frac{1}{2}e^{-t} + \frac{1}{2}e^{-t/2} & -e^{-t} + e^{-t/2} \\ -\frac{1}{4}e^{-t} + \frac{1}{4}e^{-t/2} & \frac{1}{2}e^{-t} + \frac{1}{2}e^{-t/2} \end{bmatrix}}$$

3.

(a)

Using MATLAB,

```
clear
A = sym([2, -5;
         1, -2]);
[V,D] = eig(A)
```

V =

$$\begin{pmatrix} 2-i & 2+i \\ 1 & 1 \end{pmatrix}$$

D =

$$\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

$$\therefore \begin{bmatrix} 2-i \\ 1 \end{bmatrix} e^{-it}, \quad \begin{bmatrix} 2+i \\ 1 \end{bmatrix} e^{it}$$

Taking real and imaginary components of $\vec{x}^{(1)}$,

$$\begin{bmatrix} 2-i \\ 1 \end{bmatrix} [\cos(t) - i\sin(t)] = \begin{bmatrix} 2\cos(t) - \sin(t) \\ \cos(t) \end{bmatrix} + i \begin{bmatrix} -\cos(t) - 2\sin(t) \\ -\sin(t) \end{bmatrix}$$

$\therefore$ Two independent solutions are:

$$\begin{bmatrix} 2\cos(t) - \sin(t) \\ \cos(t) \end{bmatrix} \text{ and } \begin{bmatrix} \cos(t) + 2\sin(t) \\ \sin(t) \end{bmatrix}$$

$\therefore$ A fundamental matrix is:

$$\boxed{\begin{bmatrix} 2\cos(t) - \sin(t) & \cos(t) + 2\sin(t) \\ \cos(t) & \sin(t) \end{bmatrix}}$$

(b)

Using MATLAB, continuing from above,

```
syms t
% the actual fundamental matrix
Ft = [2*cos(t) - sin(t), cos(t) + 2*sin(t);
            cos(t),           sin(t)];

% evaluate at t = 0
Ft0 = subs(Ft,t,0);

% efficient MATLAB code for
%  Ft*(inverse of Ft0)
Phi = Ft/Ft0
```

Phi =

$$\begin{pmatrix} \cos(t) + 2\sin(t) & -5\sin(t) \\ \sin(t) & \cos(t) - 2\sin(t) \end{pmatrix}$$

$$\therefore \boxed{\phi(t) = \begin{bmatrix} \cos(t) + 2\sin(t) & -5\sin(t) \\ \sin(t) & \cos(t) - 2\sin(t) \end{bmatrix}}$$

4.

(a)

Using MATLAB,

```
clear
A = sym([-1, -4;
          1, -1]);
[V,D] = eig(A)
```

V =

$$\begin{pmatrix} -2\,\mathrm{i} & 2\,\mathrm{i} \\ 1 & 1 \end{pmatrix}$$

D =

$$\begin{pmatrix} -1 - 2\,\mathrm{i} & 0 \\ 0 & -1 + 2\,\mathrm{i} \end{pmatrix}$$

$$\therefore \begin{bmatrix} -2i \\ 1 \end{bmatrix} e^{-t-2it}, \quad \begin{bmatrix} 2i \\ 1 \end{bmatrix} e^{-t+2it}$$

Taking real, imaginary parts of $\vec{x}^{(1)}$,

$$\begin{bmatrix} -2i \\ 1 \end{bmatrix} e^{-t}(\cos(2t) - i\sin(2t)) = \begin{bmatrix} -2e^{-t}\sin(2t) \\ e^{-t}\cos(2t) \end{bmatrix} + i\begin{bmatrix} -2e^{-t}\cos(2t) \\ -e^{-t}\sin(2t) \end{bmatrix}$$

$\therefore$ A fundamental matrix is:

$$\boxed{\begin{bmatrix} -2e^{-t}\sin(2t) & 2e^{-t}\cos(2t) \\ e^{-t}\cos(2t) & e^{-t}\sin(2t) \end{bmatrix}}$$

(b)

Using MATLAB, continuing from above,

```
syms t
% the actual fundamental matrix
Ft = exp(-t)*[-2*sin(2*t), 2*cos(2*t);
                 cos(2*t),   sin(2*t)];

% evaluate at t = 0
Ft0 = subs(Ft,t,0);

% efficient MATLAB code for
%  Ft*(inverse of Ft0)
Phi = Ft/Ft0
```

Phi =

$$\begin{pmatrix} \cos(2\,t)\,e^{-t} & -2\sin(2\,t)\,e^{-t} \\ \frac{\sin(2\,t)\,e^{-t}}{2} & \cos(2\,t)\,e^{-t} \end{pmatrix}$$

$$\therefore \boxed{\phi(t) = \begin{bmatrix} e^{-t}\cos(2t) & -2e^{-t}\sin(2t) \\ \frac{1}{2}e^{-t}\sin(2t) & e^{-t}\cos(2t) \end{bmatrix}}$$

5.

(a)

Using MATLAB,

```
clear
A = sym([5, -1;
         3,  1]);
[V,D] = eig(A)

% scale V(:,1) - make all integers
V = [3*V(:,1), V(:,2)]

syms t
% the actual fundamental matrix
Ft = [V(:,1)*exp(D(1,1)*t), ...
          V(:,2)*exp(D(2,2)*t)]
```

V =

$$\begin{pmatrix} \frac{1}{3} & 1 \\ 1 & 1 \end{pmatrix}$$

D =

$$\begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$$

V =

$$\begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix}$$

Ft =

$$\begin{pmatrix} e^{2t} & e^{4t} \\ 3e^{2t} & e^{4t} \end{pmatrix}$$

$\therefore$ A fundamental matrix is: $\underline{\begin{bmatrix} e^{2t} & e^{4t} \\ 3e^{2t} & e^{4t} \end{bmatrix}}$

(b)

Using MATLAB, continuing from above,

```
% evaluate at t = 0
Ft0 = subs(Ft,t,0);

% efficient MATLAB code for
%  Ft*(inverse of Ft0)
Phi = Ft/Ft0
```

Phi =

$$\begin{pmatrix} \frac{3e^{4t}}{2} - \frac{e^{2t}}{2} & \frac{e^{2t}}{2} - \frac{e^{4t}}{2} \\ \frac{3e^{4t}}{2} - \frac{3e^{2t}}{2} & \frac{3e^{2t}}{2} - \frac{e^{4t}}{2} \end{pmatrix}$$

$$\therefore \quad \boxed{\phi(t) = \begin{bmatrix} -\frac{1}{2}e^{2t} + \frac{3}{2}e^{4t} & \frac{1}{2}e^{2t} - \frac{1}{2}e^{4t} \\ -\frac{3}{2}e^{2t} + \frac{3}{2}e^{4t} & \frac{3}{2}e^{2t} - \frac{1}{2}e^{4t} \end{bmatrix}}$$

6.

(a)

Using MATLAB,

```
clear
A = sym([1, -1;
         5, -3]);
[V,D] = eig(A)
```

V =

$$\begin{pmatrix} \frac{2}{5}-\frac{1}{5}i & \frac{2}{5}+\frac{1}{5}i \\ 1 & 1 \end{pmatrix}$$

D =

$$\begin{pmatrix} -1-i & 0 \\ 0 & -1+i \end{pmatrix}$$

$$\therefore \vec{x}^{(1)}(t) = \begin{bmatrix} \frac{2}{5}-\frac{1}{5}i \\ 1 \end{bmatrix} e^{-t}\left[\cos(t) - i\sin(t)\right]$$

Taking real and imaginary parts of $\vec{x}^{(1)}$ to form fundamental matrix, after scaling $\times 5$:

$$\begin{bmatrix} 2e^{-t}\cos(t) - e^{-t}\sin(t) & -e^{-t}\cos(t) - 2e^{-t}\sin(t) \\ 5e^{-t}\cos(t) & -5e^{-t}\sin(t) \end{bmatrix}$$

(b)

Using MATLAB, continuing from above,

```
syms t
% the actual fundamental matrix
Ft = exp(-t)*[2*cos(t)-sin(t), -cos(t)-2*sin(t);
                  5*cos(t),        -5*sin(t)];

% evaluate at t = 0
Ft0 = subs(Ft,t,0);

% efficient MATLAB code for
%  Ft*(inverse of Ft0)
Phi = Ft/Ft0
```

Phi =

$$\begin{pmatrix} e^{-t}\,(\cos(t)+2\sin(t)) & -e^{-t}\sin(t) \\ 5\,e^{-t}\sin(t) & e^{-t}\,(\cos(t)-2\sin(t)) \end{pmatrix}$$

$$\therefore \quad \boxed{\phi(t) = \begin{bmatrix} e^{-t}\cos(t) + 2e^{-t}\sin(t) & -e^{-t}\sin(t) \\ 5e^{-t}\sin(t) & e^{-t}\cos(t) - 2e^{-t}\sin(t) \end{bmatrix}}$$

7.

(a)

Using MATLAB,

```
clear
A = sym([1, 1, 1;
         2, 1, -1;
         -8, -5, -3]);
[V,D] = eig(A)

% scale V(:,2), V(:,3) - make all integers
V = [V(:,1), 7*V(:,2), 2*V(:,3)]

syms t
% the actual fundamental matrix
Ft = [V(:,1)*exp(D(1,1)*t), ...
      V(:,2)*exp(D(2,2)*t), ...
      V(:,3)*exp(D(3,3)*t)]
```

V =

$$\begin{pmatrix} 0 & -\frac{4}{7} & -\frac{3}{2} \\ -1 & \frac{5}{7} & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

D =

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

V =

$$\begin{pmatrix} 0 & -4 & -3 \\ -1 & 5 & 4 \\ 1 & 7 & 2 \end{pmatrix}$$

Ft =

$$\begin{pmatrix} 0 & -4\,e^{-2t} & -3\,e^{-t} \\ -e^{2t} & 5\,e^{-2t} & 4\,e^{-t} \\ e^{2t} & 7\,e^{-2t} & 2\,e^{-t} \end{pmatrix}$$

$\therefore$ A fundamental matrix: $\begin{bmatrix} 0 & -4e^{-2t} & -3e^{-t} \\ -e^{2t} & 5e^{-2t} & 4e^{-t} \\ e^{2t} & 7e^{-2t} & 2e^{-t} \end{bmatrix}$

(b)

Using MATLAB, continuing from (a),

```
% evaluate at t = 0
Ft0 = subs(Ft,t,0);

% efficient MATLAB code for
%  Ft*(inverse of Ft0)
Phi = Ft/Ft0
```

Phi =

$$\begin{pmatrix} e^{-2t}\,(3\,e^{t}-2) & \sigma_1 & \sigma_1 \\ \frac{e^{-2t}\,(\sigma_3-8\,e^{t}+5)}{2} & \frac{e^{-2t}\,(\sigma_2-16\,e^{t}+15)}{12} & \frac{e^{-2t}\,(e^{4t}-16\,e^{t}+15)}{12} \\ -\frac{e^{-2t}\,(\sigma_3+4\,e^{t}-7)}{2} & -\frac{e^{-2t}\,(\sigma_2+8\,e^{t}-21)}{12} & -\frac{e^{-2t}\,(e^{4t}+8\,e^{t}-21)}{12} \end{pmatrix}$$

where

$$\sigma_1 = e^{-2t}\,(e^{t}-1)$$

$$\sigma_2 = 13\,e^{4t}$$

$$\sigma_3 = 3\,e^{4t}$$

$\therefore\ \phi(t) =$

$$\begin{bmatrix} 3e^{-t}-2e^{-2t} & e^{-t}-e^{-2t} & e^{-t}-e^{-2t} \\ -4e^{-t}+\frac{5}{2}e^{-2t}+\frac{3}{2}e^{2t} & -\frac{4}{3}e^{-t}+\frac{5}{4}e^{-2t}+\frac{13}{12}e^{2t} & -\frac{4}{3}e^{-t}+\frac{5}{4}e^{-2t}+\frac{1}{12}e^{2t} \\ -2e^{-t}+\frac{7}{2}e^{-2t}-\frac{3}{2}e^{2t} & -\frac{2}{3}e^{-t}+\frac{7}{4}e^{-2t}-\frac{13}{12}e^{2t} & -\frac{2}{3}e^{-t}+\frac{7}{4}e^{-2t}-\frac{1}{12}e^{2t} \end{bmatrix}$$

8.

(a)

Using MATLAB,

```
clear
A = sym([1, -1, 4;
         3, 2, -1;
         2, 1, -1]);
[V,D] = eig(A)

syms t
% the actual fundamental matrix
Ft = [V(:,1)*exp(D(1,1)*t), ...
      V(:,2)*exp(D(2,2)*t), ...
        V(:,3)*exp(D(3,3)*t)]
```

V =

$$\begin{pmatrix} -1 & 1 & -1 \\ 4 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

D =

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Ft =

$$\begin{pmatrix} -e^{t} & e^{3t} & -e^{-2t} \\ 4\,e^{t} & 2\,e^{3t} & e^{-2t} \\ e^{t} & e^{3t} & e^{-2t} \end{pmatrix}$$

$\therefore$ A fundamental matrix: $\begin{bmatrix} -e^{t} & e^{3t} & -e^{-2t} \\ 4e^{t} & 2e^{3t} & e^{-2t} \\ e^{t} & e^{3t} & e^{-2t} \end{bmatrix}$

(b)

Using MATLAB, continuing from (a),

```
% evaluate at t = 0
Ft0 = subs(Ft,t,0);

% efficient MATLAB code for
%  Ft*(inverse of Ft0)
Phi = Ft/Ft0
```

Phi =

$$\begin{pmatrix} \frac{e^{-2t}\,(e^{3t}+\sigma_2+2)}{6} & -\sigma_1 & \frac{e^{-2t}\,(e^{3t}+e^{5t}-2)}{2} \\ -\frac{e^{-2t}\,(\sigma_3-\sigma_2+1)}{3} & \frac{e^{-2t}\,(4\,e^{3t}-1)}{3} & e^{-2t}\,(e^{5t}-\sigma_3+1) \\ -\frac{e^{-2t}\,(e^{3t}-\sigma_2+2)}{6} & \sigma_1 & \frac{e^{-2t}\,(e^{5t}-e^{3t}+2)}{2} \end{pmatrix}$$

where

$$\sigma_1 = \frac{e^{-2t}\,(e^{3t}-1)}{3}$$

$$\sigma_2 = 3\,e^{5t}$$

$$\sigma_3 = 2\,e^{3t}$$

$\therefore \quad \phi(t) =$

$$\begin{bmatrix} \frac{1}{6}e^{t}+\frac{1}{2}e^{3t}+\frac{1}{3}e^{-2t} & -\frac{1}{3}e^{t}+\frac{1}{3}e^{-2t} & \frac{1}{2}e^{t}+\frac{1}{2}e^{3t}-e^{-2t} \\ -\frac{2}{3}e^{t}+e^{3t}-\frac{1}{3}e^{-2t} & \frac{4}{3}e^{t}-\frac{1}{3}e^{-2t} & -2e^{t}+e^{3t}+e^{-2t} \\ -\frac{1}{6}e^{t}+\frac{1}{2}e^{3t}-\frac{1}{3}e^{-2t} & \frac{1}{3}e^{t}-\frac{1}{3}e^{-2t} & -\frac{1}{2}e^{t}+\frac{1}{2}e^{3t}+e^{-2t} \end{bmatrix}$$

9.

From #4, $\phi(t) = \begin{bmatrix} e^{-t}\cos(2t) & -2e^{-t}\sin(2t) \\ \frac{1}{2}e^{-t}\sin(2t) & e^{-t}\cos(2t) \end{bmatrix}$

$\therefore \quad x(t) = \phi(t)\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$$= \begin{bmatrix} 3e^{-t}\cos(2t) - 2e^{-t}\sin(t) \\ e^{-t}\cos(2t) + \frac{3}{2}e^{-t}\sin(t) \end{bmatrix}$$

10.

Given that $\vec{x}^{(1)}(t), \ldots, \vec{x}^{(n)}(t)$ are independent solutions to $\vec{x}'(t) = P(t)\vec{x}(t)$, where $P(t)$ is $n \times n$, $\Psi(t)$ is defined as $\Psi(t) = [\vec{x}^{(1)}(t), \ldots, \vec{x}^{(n)}(t)]$, and $\vec{x}(t)$ is $n \times 1$. $\therefore$ $\Psi(t)$ is $n \times n$.

$\therefore \quad \Psi(t)\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \Psi(t)\vec{c}$ is the general solution to: $\vec{x}'(t) = P(t)\vec{x}(t)$

By Theorem 7.4.2, there is a unique solution to $\vec{x}'(t) = P(t)\vec{x}(t)$ s.t. $\vec{x}(t_0) = \vec{e}_i$, so that

$\vec{c}_i$ exists s.t. $\Psi(t_0)\vec{c}_i = \vec{e}_i$.

$\therefore \Psi(t_0)[\vec{c}_1, \ldots, \vec{c}_n] = I$

Since $\vec{x}^{(i)}(t)$ are independent at each point, then $\Psi(t_0)$ is invertible. $\therefore [\vec{c}_1, \ldots, \vec{c}_n] = [\Psi(t_0)]^{-1}$

$\therefore$ Let $\vec{\phi}_i(t) = \Psi(t)\vec{c}_i$

$$\therefore \phi(t) = [\vec{\phi}_1(t), \ldots, \vec{\phi}_n(t)] = \Psi(t)[\vec{c}_1, \ldots, \vec{c}_n]$$

$$= \Psi(t)[\Psi(t_0)]^{-1}$$

$$\therefore \underline{\phi(t) = \Psi(t)[\Psi(t_0)]^{-1}}$$

$$\text{and } \phi(t_0) = \Psi(t_0)[\Psi(t_0)]^{-1} = I$$

11.

From Ex. 2,

$$\phi(t) = \begin{bmatrix} \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} & \frac{1}{4}e^{3t} - \frac{1}{4}e^{-t} \\ e^{3t} - e^{-t} & \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} \end{bmatrix}$$

Using MATLAB,

```
clear
syms t s
at = exp(3*t); as = exp(3*s);
bt = exp(-t); bs = exp(-s);
A = [(1/2)*at + (1/2)*bt, (1/4)*at - (1/4)*bt;
     at - bt, (1/2)*at + (1/2)*bt];
B = [(1/2)*as + (1/2)*bs, (1/4)*as - (1/4)*bs;
     as - bs, (1/2)*as + (1/2)*bs];
simplify(A*B)
```

ans =

$$\begin{pmatrix} \frac{e^{-s-t}\,(e^{4s+4t}+1)}{2} & \frac{e^{-s-t}\,(e^{4s+4t}-1)}{4} \\ e^{3s+3t}-e^{-s-t} & \frac{e^{-s-t}\,(e^{4s+4t}+1)}{2} \end{pmatrix}$$

$$\therefore \phi(t)\phi(s) = \begin{bmatrix} \frac{1}{2}e^{3(t+s)} + \frac{1}{2}e^{-(t+s)} & \frac{1}{4}e^{3(t+s)} - \frac{1}{4}e^{-(t+s)} \\ e^{3(t+s)} - e^{-(t+s)} & \frac{1}{2}e^{3(t+s)} + \frac{1}{2}e^{-(t+s)} \end{bmatrix}$$

$$= \underline{\underline{\phi(t+s)}}$$

12.

(a)

(1) Let s be any fixed number, and consider

$$Y(t) = \phi(t)\phi(s) \quad \therefore \frac{d}{dt}Y(t) = \phi'(t)\phi(s)$$

Since $Z = \phi(t)$ satisfies $Z'(t) = AZ(t)$, $Z(0) = I$

because $\phi'(t) = A\phi(t)$ and $\phi(0) = I$,

then $\phi'(t)\phi(s) = A\phi(t)\phi(s)$ and $\phi(0)\phi(s) = \phi(s)$

$\therefore$ $Y(t)$ satisfies $z'(t) = Az(t)$, $z(0) = \phi(s)$

since $Y(0) = \phi(0)\phi(s) = I\,\phi(s) = \phi(s)$

i.e., $\phi(t)\phi(s)$ satisfies $z'(t) = Az(t)$, $z(0) = \phi(s)$

(2) Consider $\phi(t+s)$. For each element,

$$\frac{d}{dt}\left[\phi(t+s)\right]_{ij} = \left[\frac{d}{dt}\phi(t+s)\right]_{ij} \cdot \frac{d}{dt}(t+s) =$$

$$= \left[\frac{d}{dt}\phi(t+s)\right]_{ij}$$

Since $z = \phi(x)$ satisfies $z' = Az$,

then $z = \phi(t+s)$ satisfies $z' = Az$ since

$\frac{d}{dt}(t+s) = 1$. In addition, $\phi(t+s)\big|_{t=0} = \phi(s)$

$\therefore$ $z = \phi(t+s)$ satisfies $z' = Az$, $z(0) = \phi(s)$.

(3) By uniqueness, $\phi(t)\phi(s) = \phi(t+s)$ since

both satisfy $z' = Az$, $z(0) = \phi(s)$

(b)

(1) From (a), let $s = -t$.

$\therefore \phi(t)\phi(-t) = \phi(t+(-t)) = \phi(0) = I$

(2) Similar to (1), $\phi(-t)\phi(t) = \phi(-t+t) = \phi(0) = I$

$\therefore \phi(t)\phi(-t) = \phi(-t)\phi(t) = I$, so that

$\phi(-t)$ is the inverse matrix for $\phi(t)$.

$\therefore \phi(t)^{-1} = \phi(-t)$, for all t.

(c)

$\phi(t-s) = \phi(t+(-s)) = \phi(t)\phi(-s) = \phi(t)\phi^{-1}(s)$

by (b).

13.

If A is a square diagonal matrix, then

$A^2, A^3, \dots, A^n, \dots$ are all diagonal matrices.

The entries for A^2 are $a_1^2, a_2^2, \dots, a_n^2$. In

general, the entries for A^k are $a_1^n, a_2^n, \dots, a_n^k$.

$\therefore \frac{A^k t^k}{k!}$ is a diagonal matrix with entries

$$\frac{a_1^k t^k}{k!}, \frac{a_2^k t^k}{k!}, \ldots, \frac{a_n^k t^k}{k!}.$$

The sum of diagonal matrices is a diagonal matrix.

$\therefore$ The sum of $I + At + \frac{A^2 t^2}{2!} + \ldots + \frac{A^k t^k}{k!} + \ldots$ is a diagonal matrix, and its entries are, for the i-th row: $1 + a_i t + \frac{a_i^2 t^2}{2!} + \ldots + \frac{a_i^k t^k}{k!} + \ldots$ and this sum equals $e^{(a_i t)}$.

$\therefore$ $\exp(At)$ is diagonal with entries

$$e^{a_1 t}, e^{a_2 t}, \ldots, e^{a_n t}.$$

14.

(a) $\leftarrow$ should be (50)

$$x_1' = u' = x_2, \quad x_2' = u'' = -\omega^2 u = -\omega^2 x_1$$

$$\therefore \begin{aligned} x_1' &= x_2 \\ x_2' &= -\omega^2 x_1 \end{aligned} \qquad \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\therefore \quad \boxed{\begin{aligned} \vec{x}\,'(t) &= \begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = A\,\vec{x}(t) \\ \text{and } \vec{x}(0) &= \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \end{aligned}}$$

(b)

$$A^1 = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix}, \quad A^2 = \begin{bmatrix} -\omega^2 & 0 \\ 0 & -\omega^2 \end{bmatrix}, \quad A^3 = \begin{bmatrix} 0 & -\omega^2 \\ \omega^4 & 0 \end{bmatrix}, \quad A^4 = \begin{bmatrix} \omega^4 & 0 \\ 0 & \omega^4 \end{bmatrix}$$

$\therefore \exp(At) =$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & t \\ -\omega^2 t & 0 \end{bmatrix} + \begin{bmatrix} -\frac{\omega^2 t^2}{2} & 0 \\ 0 & -\frac{\omega^2 t^2}{2} \end{bmatrix} + \begin{bmatrix} 0 & -\frac{\omega^2 t^3}{3!} \\ \frac{\omega^4 t^3}{3!} & 0 \end{bmatrix} + \begin{bmatrix} \frac{\omega^4 t^4}{4!} & 0 \\ 0 & \frac{\omega^4 t^4}{4!} \end{bmatrix} + \cdots$$

$$\begin{bmatrix} 1 - \frac{\omega^2 t^2}{2!} + \frac{\omega^4 t^4}{4!} + \cdots & t - \frac{\omega^2 t^3}{3!} + \cdots \\ -\omega^2 t + \frac{\omega^4 t^3}{3!} + \cdots & 1 - \frac{\omega^2 t^2}{2!} + \frac{\omega^4 t^4}{4!} + \cdots \end{bmatrix}$$

Noting

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^{n+1}\frac{x^{2n-1}}{(2n-1)!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^{n+1}x^{2n-2}}{(2n-2)!} + \cdots,$$

Then $\exp(At) = \begin{bmatrix} \cos(\omega t) & t - \frac{\omega^2 t^3}{3!} + \dots \\ -\omega^2 t + \frac{\omega^4 t^3}{3!} + \dots & \cos(\omega t) \end{bmatrix}$

$$= \begin{bmatrix} \cos(\omega t) & \frac{1}{\omega}\left(\omega t - \frac{\omega^3 t^3}{3!} + \dots\right) \\ -\omega\left(\omega t - \frac{\omega^3 t^3}{3!} + \dots\right) & \cos(\omega t) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\omega t) & \frac{1}{\omega}\sin(\omega t) \\ -\omega \sin(\omega t) & \cos(\omega t) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\omega t) & 0 \\ 0 & \cos(\omega t) \end{bmatrix} + \begin{bmatrix} 0 & \frac{1}{\omega}\sin(\omega t) \\ -\omega \sin(\omega t) & 0 \end{bmatrix}$$

$$= I\cos(\omega t) + \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} \frac{1}{\omega}\sin(\omega t)$$

$$\therefore \boxed{\exp(At) = I\cos(\omega t) + A\frac{1}{\omega}\sin(\omega t)}$$

(c)

Eigenvalues, eigenvectors of $\begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix}$ are:

Using MATLAB:

```
clear
syms w
A = sym([0, 1;
         -w^2, 0]);
[V,D] = eig(A)
```

V =

$$\begin{pmatrix} \frac{i}{w} & -\frac{i}{w} \\ 1 & 1 \end{pmatrix}$$

D =

$$\begin{pmatrix} -w\,i & 0 \\ 0 & w\,i \end{pmatrix}$$

$\therefore$ Eigenvalues: $\pm i\omega$

Eigenvectors: $\begin{bmatrix} \frac{i}{\omega} \\ 1 \end{bmatrix} e^{-i\omega t}$, $\begin{bmatrix} \frac{-i}{\omega} \\ 1 \end{bmatrix} e^{i\omega t}$

Or, looking just at $\begin{bmatrix} i/\omega \\ 1 \end{bmatrix} [\cos(\omega t) - i\sin(\omega t)]$

and just its real and imaginary parts,

$$\begin{bmatrix} \frac{1}{\omega}\sin(\omega t) \\ \cos(\omega t) \end{bmatrix} \text{ and } i \begin{bmatrix} \frac{1}{\omega}\cos(\omega t) \\ -\sin(\omega t) \end{bmatrix}$$

$$\therefore \vec{X}(t) = c_1 \begin{bmatrix} \frac{1}{\omega}\sin(\omega t) \\ \cos(\omega t) \end{bmatrix} + c_2 \begin{bmatrix} \frac{1}{\omega}\cos(\omega t) \\ -\sin(\omega t) \end{bmatrix}$$

$$= c_1 \vec{X}^{(1)}(t) + c_2 \vec{X}^{(2)}(t)$$

$$= c_1 \begin{bmatrix} X^{(1)}_1(t) \\ X^{(1)}_2(t) \end{bmatrix} + c_2 \begin{bmatrix} X^{2}_1(t) \\ X^{2}_2(t) \end{bmatrix}$$

$$X(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} \frac{1}{\omega} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{\omega} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\therefore c_1 = v_0, \quad c_2 = \omega u_0$$

$$\therefore \vec{X}(t) = \begin{bmatrix} \frac{v_0}{\omega}\sin(\omega t) + u_0\cos(\omega t) \\ v_0\cos(\omega t) - \omega u_0\sin(\omega t) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{v_0}{\omega} \\ -\omega u_0 \end{bmatrix} \sin(\omega t) + \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \cos(\omega t)$$

$$= \begin{bmatrix} v_0 \\ -\omega^2 u_0 \end{bmatrix} \frac{\sin(\omega t)}{\omega} + \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \cos(\omega t)$$

With this $\vec{x}(t)$, $\vec{x}'(t) = A\vec{x}(t)$

Also, $u(t) = x_1(t) = \frac{v_0}{\omega}\sin(\omega t) + u_0 \cos(\omega t)$

15.

(a)

$$\frac{d}{dt}\vec{\phi}(t) = \underbrace{\frac{d}{dt}\vec{x}^0}_{=\vec{0}} + \frac{d}{dt}\int_0^t A\vec{\phi}(s)\,ds$$

$\therefore \vec{\phi}'(t) = \vec{0} + A\vec{\phi}(t)$ by Fundamental Theorem of Calculus

$\therefore \vec{\phi}'(t) = A\vec{\phi}(t)$

Also, $\vec{\phi}(0) = \vec{x}^0 + \int_0^0 A\vec{\phi}(s)\,ds = \vec{x}^0 + \vec{0} = \vec{x}^0$

$\therefore$ The $\vec{\phi}(t)$ defined by this integral satisfies $\vec{x}' = Ax$ and $\vec{x}(0) = \vec{0}$.

By the uniqueness theorem, any other solution must therefor be equal to $\vec{\phi}(t)$, and so satisfies the integral equation.

(b)

$\leftarrow$ equation (54)

$$\vec{\phi}^{(1)}(t) = \vec{x}^{\,0} + \int_0^t A\vec{x}^{\,0}\,ds = \vec{x}^{\,0} + tA\vec{x}^{\,0}$$

since $A\vec{x}^{\,0}$ is a constant matrix

and $\int_0^t ds = t$

$$\therefore \vec{\phi}^{(1)}(t) = \vec{x}^{\,0} + tA\vec{x}^{\,0} = I\vec{x}^{\,0} + tA\vec{x}^{\,0}$$

$$= \underline{\underline{(I + tA)}}\,\vec{x}^{\,0}$$

(c)

$$\vec{\phi}^{(2)}(t) = \vec{x}^{\,0} + \int_0^t A\vec{\phi}^{(1)}(s)\,ds$$

$$= \vec{x}^0 + \int_0^t A(I + sA)\vec{x}^0 \, ds$$

$$= \vec{x}^0 + \int_0^t \left(A\vec{x}^0 + sA^2\vec{x}^0\right) ds$$

$$= \vec{x}^0 + \left(sA\vec{x}^0 + \frac{s^2}{2}A^2\vec{x}^0\right)\Big|_{s=0}^{s=t}$$

$$= \vec{x}^0 + tA\vec{x}^0 + \frac{t^2}{2}A^2\vec{x}^0$$

$$= \left(I + tA + \frac{t^2}{2}A^2\right)\vec{x}^0$$

$$\vec{\phi}^{(3)}(t) = \vec{x}^0 + \int_0^t A\vec{\phi}^{(2)}(s)\, ds$$

$$= \vec{x}^0 + \int_0^t A\left(I + sA + \frac{s^2}{2}A^2\right)\vec{x}^0 \, ds$$

$$= \vec{x}^0 + tA\vec{x}^0 + \frac{t^2}{2}A^2\vec{x}^0 + \frac{t^3}{3!}A^3\vec{x}^0$$

$$= \left(I + \frac{tA}{1!} + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3\right)\vec{x}^0$$

$\therefore$ Formula is true for $n = 1, 2, 3$. Assume true for $n = K$. $\therefore$

$$\vec{\phi}^{(K+1)} = \vec{x}^0 + \int_0^t A\vec{\phi}^{(K)}(s)\, ds$$

$$= \vec{x}^{\,0} + \int_0^t A\left(I + \frac{sA}{1!} + \dots + \frac{s^K A^K}{K!}\right)\vec{x}^{\,0}\, ds$$

$$= \vec{x}^{\,0} + tA\vec{x}^{\,0} + \frac{t^2A^2}{2!}\vec{x}^{\,0} + \dots + \frac{s^{K+1}A^{K+1}}{(K+1)!}\vec{x}^{\,0}$$

$$= \left(I + tA + \frac{t^2}{2!}A^2 + \dots + \frac{t^{K+1}A^{K+1}}{(K+1)!}\right)\vec{x}^{\,0}$$

and so is true for $n = K+1$.

$\therefore$ True for all $n = 1, 2, \dots$

(d)

$$\lim_{n\to\infty} \vec{\phi}^{(n)}(t) = \lim_{n\to\infty}\left(I + \sum_{K=1}^{n} \frac{t^K A^K}{K!}\right)\vec{x}^{\,0}$$

$$= [\exp(At)]\,\vec{x}^{\,0} \quad \text{by def. of } \exp(At)$$

$$\therefore\ \vec{\phi}(t) = \lim_{n\to\infty}\vec{\phi}^{(n)}(t) = [\exp(At)]\,\vec{x}^{\,0}$$

Note that this shows $\exp(At)\vec{x}^{\,0}$ is the solution, without calculating eigenvectors, so that n-independent eigenvectors aren't needed.

7.8 Repeated Eigenvalues

Note Title 2/25/2020

1.

Using MATLAB,

```
clear

A = sym([3, -4;
         1, -1]);

% get eigenvalue, eigenvector
[V,D] = eig(A);
ev1 = D(1,1)    % eigenvalues
ev2 = D(2,2)    % show evalue is repeated
if V(2,1) ~= 0  % normalize the eigenvector
    evec1 = V(:,1)/V(2,1)
else
    evec1 = V(:,1)/V(1,1)
end
```

ev1 = 1

ev2 = 1

evec1 =

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$\therefore \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^t = \vec{x}^{(1)}$ is a solution

Assume 2nd solution $\vec{x}^{(2)} = \vec{\epsilon}\, t e^t + \vec{n} e^t$ [1]

$\therefore \vec{x}^{(2)\prime} = \vec{\epsilon}(t e^t + e^t) + \vec{n} e^t = A(\vec{\epsilon}\, t e^t + \vec{n} e^t)$

$\therefore \vec{\epsilon}\, t e^t + (\vec{\epsilon} + \vec{n}) e^t = A\vec{\epsilon}\, t e^t + A\vec{n} e^t$

$\therefore \vec{\epsilon} = A\vec{\epsilon}$ and $\vec{\epsilon} + \vec{n} = A\vec{n}$

Or, $(A-I)\vec{\epsilon} = \vec{0}$ and $(A-I)\vec{n} = \vec{\epsilon}$

$(A-I)\vec{\epsilon} = \vec{0} \Rightarrow \vec{\epsilon} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ as in MATLAB code

$\therefore$ To solve $(A-I)\vec{n} = \vec{e}$, or $\begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} \vec{n} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

If $\vec{n} = \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}$, then $n_1 - 2n_2 = 1$

Let $n_2 = K$, then $n_1 = 1 + 2K$

$$\therefore \vec{n} = \begin{bmatrix} 1+2k \\ k \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} k$$

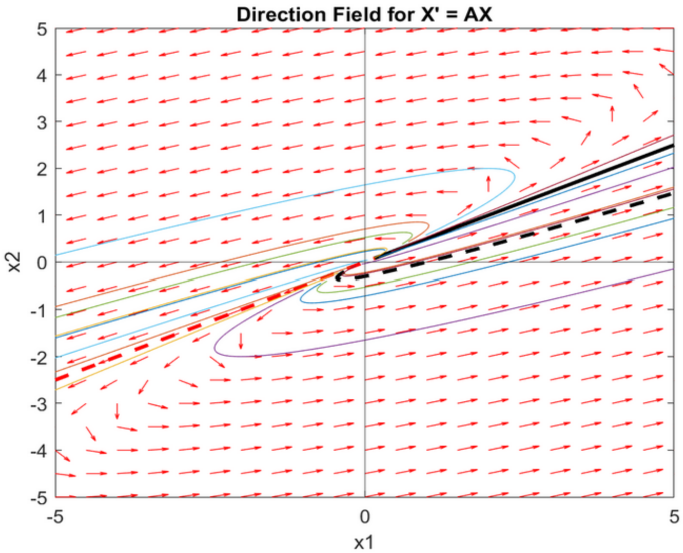
$\therefore$ [1] becomes $\vec{X}^{(2)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} te^t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + \begin{bmatrix} 2 \\ 1 \end{bmatrix} ke^t$

The last term is just a multiple of $\vec{X}^{(1)}$

$$\therefore \vec{X}^{(2)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} te^t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t$$

$$\therefore \boxed{\vec{X}(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^t + c_2 \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} te^t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t \right)}$$

(a)



MATLAB code on next page.

```
% Draw direction field

% set plot boundaries
Xmin = -5; Xmax = 5;
Ymin = -5; Ymax = 5;
% set how fine to make the grid of values
Step = 0.5;
% total # points on x and y axes
Nx = round((Xmax-Xmin)/Step, 0) + 1;
Ny = round((Ymax-Ymin)/Step, 0) + 1;
% set the actual x and y coords
x = linspace(Xmin, Xmax, Nx);  % a row vector
y = linspace(Ymin, Ymax, Ny);  % a row vector
% preallocate memory for coord, slope arrays
%  to be used in quiver, must be same dimension
xcoord = zeros(Nx, Ny);
ycoord = zeros(Nx, Ny);
dx = zeros(Nx, Ny);
dy = zeros(Nx, Ny);
for i = 1:Nx
    for j = 1:Ny
        xcoord(i,j) = x(i);
        ycoord(i,j) = y(j);
        v = A*[x(i);y(j)];  % get the slope
        s = norm(v);  % length of v
        % make slope vector a unit vector
        dx(i,j) = v(1)/s;
        dy(i,j) = v(2)/s;
    end
end
% plot the slope vectors at (xcoord, ycoord)
quiver(xcoord, ycoord, dx, dy, 0.5, 'r')
xline(0);  % show x and y axes
yline(0);
axis([Xmin, Xmax, Ymin, Ymax])
xlabel 'x1', ylabel 'x2'
title 'Direction Field for X'' = AX'

% Draw a few phase portrait trajectories

% x_1 from above, x_2 from calculations
t = -2.5:0.05:2.5;
x_1 = evec1*exp(ev1*t);
x_2 = evec1*t.*exp(ev1*t) + [1,0]'*exp(ev1*t);

hold on
m1 = evec1(2)/evec1(1);  % asymptotic line
plot(x,m1*x, '--r','LineWidth',2)

c1 = 1; c2 = 0;  % just x_1
    X = c1*x_1 + c2*x_2;
    x1 = X(1,:);
    x2 = X(2,:);
    plot(x1,x2, '-k', 'LineWidth',2)
c1 = 0; c2 = 1;  % just x_2
    X = c1*x_1 + c2*x_2;
    x1 = X(1,:);
    x2 = X(2,:);
    plot(x1,x2, '--k', 'LineWidth',2)
for c1 = [-2,-0.3,0.3,2]
    for c2 = [-2,-0.3,0.3,2]
        X = c1*x_1 + c2*x_2;
        x1 = X(1,:);
        x2 = X(2,:);
        plot(x1,x2)
    end
end
```

Code a continuation

From above.

(b)

Solutions head toward ∞, parallel to the $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ vector, and for small t, flee from origin.

(c)

As above, $\boxed{\vec{x}(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{t} + c_2 \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t e^{t} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{t} \right)}$

2.

Using MATLAB,

```
clear

A = sym([4, -2;
         8, -4]);

% get eigenvalue, eigenvector
[V,D] = eig(A);
ev1 = D(1,1)    % eigenvalues
ev2 = D(2,2)    % show evalue is repeated
if V(1,1) ~= 0  % normalize the eigenvector
    evec1 = V(:,1)/V(1,1)
else
    evec1 = V(:,1)/V(2,1)
end
```

ev1 = 0

ev2 = 0

evec1 =

$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\therefore \ \underline{\vec{x}^{(1)}(t)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is a solution $(e^{0t} = 1)$

Assume $\vec{x}^{(2)}(t) = \vec{\epsilon}\, t + \vec{\eta}$ [1]

$\therefore \ \vec{x}^{(2)\prime} = \vec{\epsilon} = A(\vec{\epsilon}\, t + \vec{\eta})$

$\therefore\ \vec{\epsilon} = A\vec{\epsilon}t + A\vec{\eta}$

Equating like terms, $\vec{\epsilon} = A\vec{\eta}$ and $A\vec{\epsilon} = \vec{0}$

From $A\vec{\epsilon} = \vec{0}$, $\vec{\epsilon} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ from above.

$\therefore$ with $\vec{\eta} = \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}$, $\begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix}\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\therefore\ 4\eta_1 - 2\eta_2 = 1$. Let $\eta_1 = K$, $4K - 1 = 2\eta_2$,

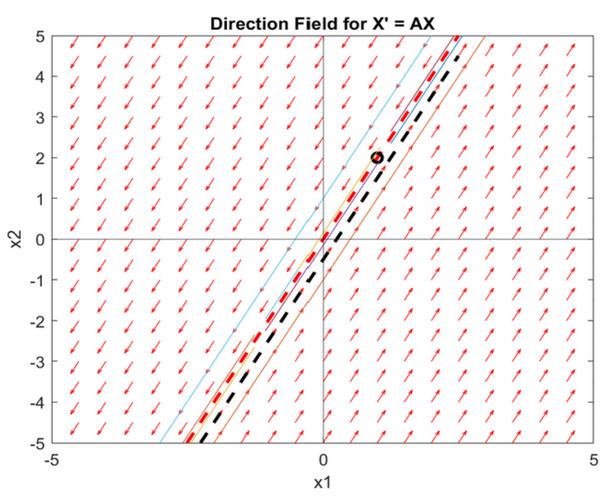
$\eta_2 = 2K - \frac{1}{2}$. $\therefore\ \vec{\eta} = \begin{bmatrix} K \\ 2K - \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ -1/2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} K$

$\therefore$ [1] becomes $\vec{x}^{(2)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} t + \begin{bmatrix} 0 \\ -1/2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} K$

The last term is a multiple of $\vec{x}^{(1)}$

$$\therefore\ \vec{x}^{(2)}(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} t + \begin{bmatrix} 0 \\ -1/2 \end{bmatrix}$$

(a)



MATLAB code next page.

Note $\vec{x}^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is just a point.

Code a continuation

from above.

```
% Draw direction field

% set plot boundaries
Xmin = -5; Xmax = 5;
Ymin = -5; Ymax = 5;
% set how fine to make the grid of values
Step = 0.5;
% total # points on x and y axes
Nx = round((Xmax-Xmin)/Step, 0) + 1;
Ny = round((Ymax-Ymin)/Step, 0) + 1;
% set the actual x and y coords
x = linspace(Xmin, Xmax, Nx);  % a row vector
y = linspace(Ymin, Ymax, Ny);  % a row vector
% preallocate memory for coord, slope arrays
%  to be used in quiver, must be same dimension
xcoord = zeros(Nx, Ny);
ycoord = zeros(Nx, Ny);
dx = zeros(Nx, Ny);
dy = zeros(Nx, Ny);
for i = 1:Nx
    for j = 1:Ny
        xcoord(i,j) = x(i);
        ycoord(i,j) = y(j);
        v = A*[x(i);y(j)];  % get the slope
        s = norm(v);  % length of v
        % make slope vector a unit vector
        dx(i,j) = v(1)/s;
        dy(i,j) = v(2)/s;
    end
end
% plot the slope vectors at (xcoord, ycoord)
quiver(xcoord, ycoord, dx, dy, 0.5, 'r')
xline(0);  % show x and y axes
yline(0);
axis([Xmin, Xmax, Ymin, Ymax])
xlabel 'x1', ylabel 'x2'
title 'Direction Field for X'' = AX'

% Draw a few phase portrait trajectories

% x_1 from above, x_2 from calculations
t = -2.5:0.05:2.5;
x_1 = evec1*exp(ev1*t);
x_2 = evec1*t.*exp(ev1*t) + [0,-1/2]'*exp(ev1*t);

hold on
m1 = evec1(2)/evec1(1);  % asymptotic line
plot(x,m1*x, '--r','LineWidth',2)

c1 = 1; c2 = 0;  % just x_1
    X = c1*x_1 + c2*x_2;
    x1 = X(1,:);
    x2 = X(2,:);
    plot(x1,x2, 'ok', 'LineWidth',2)
c1 = 0; c2 = 1;  % just x_2
    X = c1*x_1 + c2*x_2;
    x1 = X(1,:);
    x2 = X(2,:);
    plot(x1,x2, '--k', 'LineWidth',2)
for c1 = [-2,-0.3,0.3,2]
    for c2 = [-2,-0.3,0.3,2]
        X = c1*x_1 + c2*x_2;
        x1 = X(1,:);
        x2 = X(2,:);
        plot(x1,x2)
    end
end
```

(b)

Assuming c_2 for $\vec{x}^{(2)}$ is not 0, $\vec{x} \to \pm\infty$

If $c_2 = 0$ for $\vec{x}^{(2)}$ and $c_1 \neq 0$ for $\vec{x}^{(1)}$, $\vec{x}$ stays at $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

(c)

From above, $\boxed{\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} t + \begin{bmatrix} 0 \\ -1/2 \end{bmatrix} \right)}$

3.

Using MATLAB,

```
clear

A = sym([-3/2,    1;
         -1/4, -1/2]);

% get eigenvalue, eigenvector
[V,D] = eig(A);
ev1 = D(1,1)    % eigenvalues
ev2 = D(2,2)    % show evalue is repeated
if V(2,1) ~= 0  % normalize the eigenvector
    evec1 = V(:,1)/V(2,1)
else
    evec1 = V(:,1)/V(1,1)
end
```

ev1 = -1

ev2 = -1

evec1 =

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$\therefore\ \vec{x}^{(1)}(t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t}$ is a solution

Assume $\vec{X}^{(2)}(t) = \vec{\epsilon}\, t e^{-t} + \vec{\eta} e^{-t}$

$\therefore \vec{X}^{(2)\prime} = \vec{\epsilon} e^{-t} - \vec{\epsilon}\, t e^{-t} - \vec{\eta} e^{-t} = A(\vec{\epsilon}\, t e^{-t} + \vec{\eta} e^{-t})$

Equating like terms, $-\vec{\epsilon}\, t e^{-t} = A\vec{\epsilon}\, t e^{-t}$

$$(\vec{\epsilon} - \vec{\eta}) e^{-t} = A\vec{\eta} e^{-t}$$

$\therefore (A+I)\vec{\epsilon}\, t e^{-t} = \vec{0}$ and $(A+I)\vec{\eta} e^{-t} = \vec{\epsilon} e^{-t}$

From above, $\vec{\epsilon} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\therefore (A+I)\vec{\eta} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\therefore \begin{bmatrix} -1/2 & 1 \\ -1/4 & 1/2 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, letting $\vec{\eta} = \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}$

$\therefore -\frac{1}{2} n_1 + n_2 = 2$, or $n_2 = 2 + \frac{1}{2} n_1$

Let $n_1 = K$, $\therefore \vec{\eta} = \begin{bmatrix} K \\ 2 + \frac{1}{2}K \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} K$

$\therefore \vec{X}^{(2)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} t e^{-t} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} e^{-t} + K \begin{bmatrix} 1 \\ 1/2 \end{bmatrix} e^{-t}$

The last term is just a multiple of $\vec{x}^{(1)}$, so drop it.

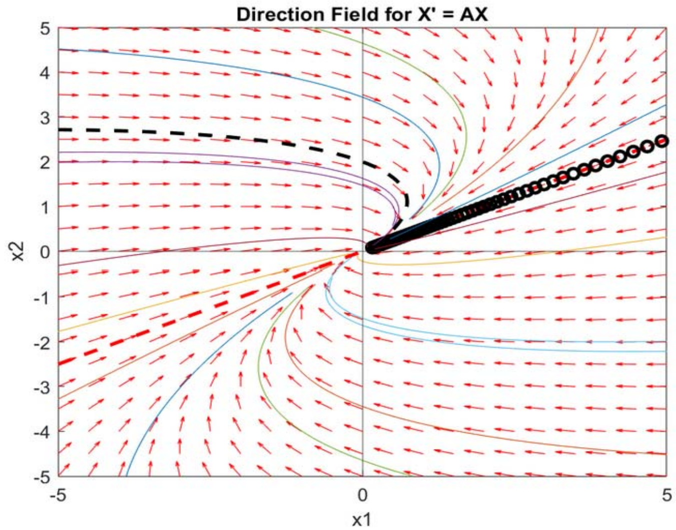
$$\therefore \vec{X}^{(2)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} t e^{-t} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} e^{-t}$$

(a)

Continuation of above MATLAB code on next page.

```
% Draw direction field

% set plot boundaries
Xmin = -5; Xmax = 5;
Ymin = -5; Ymax = 5;
% set how fine to make the grid of values
Step = 0.5;
% total # points on x and y axes
Nx = round((Xmax-Xmin)/Step, 0) + 1;
Ny = round((Ymax-Ymin)/Step, 0) + 1;
% set the actual x and y coords
x = linspace(Xmin, Xmax, Nx);  % a row vector
y = linspace(Ymin, Ymax, Ny);  % a row vector
% preallocate memory for coord, slope arrays
%  to be used in quiver, must be same dimension
xcoord = zeros(Nx, Ny);
ycoord = zeros(Nx, Ny);
dx = zeros(Nx, Ny);
dy = zeros(Nx, Ny);
for i = 1:Nx
    for j = 1:Ny
        xcoord(i,j) = x(i);
        ycoord(i,j) = y(j);
        v = A*[x(i);y(j)];  % get the slope
        s = norm(v);  % length of v
        % make slope vector a unit vector
        dx(i,j) = v(1)/s;
        dy(i,j) = v(2)/s;
    end
end
% plot the slope vectors at (xcoord, ycoord)
quiver(xcoord, ycoord, dx, dy, 0.5, 'r')
xline(0);  % show x and y axes
yline(0);
axis([Xmin, Xmax, Ymin, Ymax])
xlabel 'x1', ylabel 'x2'
title 'Direction Field for X'' = AX'
```

```
% Draw a few phase portrait trajectories

% x_1 from above, x_2 from calculations
t = -2.5:0.05:2.5;
x_1 = evec1*exp(ev1*t);
x_2 = evec1*t.*exp(ev1*t) + [0,2]'*exp(ev1*t);

hold on
m1 = evec1(2)/evec1(1);  % asymptotic line
plot(x,m1*x, '--r','LineWidth',2)

c1 = 1; c2 = 0;  % just x_1
    X = c1*x_1 + c2*x_2;
    x1 = X(1,:);
    x2 = X(2,:);
    plot(x1,x2, 'ok', 'LineWidth',2)
c1 = 0; c2 = 1;  % just x_2
    X = c1*x_1 + c2*x_2;
    x1 = X(1,:);
    x2 = X(2,:);
    plot(x1,x2, '--k', 'LineWidth',2)
for c1 = [-2,-0.3,0.3,2]
    for c2 = [-2,-0.3,0.3,2]
        X = c1*x_1 + c2*x_2;
        x1 = X(1,:);
        x2 = X(2,:);
        plot(x1,x2)
    end
end
```



(b)

As $t \to \infty$, $\vec{x} \to \vec{0}$, along vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

(c)

From above,

$$\boxed{\vec{X}(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t} + c_2 \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t e^{-t} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} e^{-t} \right)}$$

4.

Using MATLAB,

```
clear

A = sym([1, 1, 1;
         2, 1, -1;
         0, -1, 1]);

[V,D] = eig(A)
```

V =

$$\begin{pmatrix} -\frac{3}{2} & 0 \\ 2 & -1 \\ 1 & 1 \end{pmatrix}$$

D =

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$\therefore \lambda_1 = -1, \begin{bmatrix} -3 \\ 4 \\ 2 \end{bmatrix} \quad \lambda_2 = 2, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

$\therefore \vec{X}^{(1)} = \underline{\begin{bmatrix} -3 \\ 4 \\ 2 \end{bmatrix} e^{-t}}, \quad \vec{X}^{(2)} = \underline{\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} e^{2t}}$

Assume $\vec{X}^{(3)} = \vec{\epsilon}\, t e^{2t} + \vec{n}\, e^{2t}$, where $\vec{\epsilon} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

$\therefore \vec{X}^{(3)\prime} = \vec{\epsilon} e^{2t} + 2\vec{\epsilon} t e^{2t} + 2\vec{n} e^{2t}$

$= A\left(\vec{\epsilon}\, t e^{2t} + \vec{n}\, e^{2t}\right)$

Equating like terms,

$(A-2I)\vec{\epsilon}\, t e^{2t} = \vec{0}$, true given $\vec{\epsilon} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ from above MATLAB code

$$(A-2I)\vec{n}\, e^{t} = \vec{\epsilon}\, e^{2t}$$

$$\therefore \begin{bmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$\therefore n_1 = -1. \therefore 1 + n_2 + n_3 = 0$

Let $n_3 = K$. $\therefore n_2 = -1 - K$

$$\therefore \vec{n} = \begin{bmatrix} -1 \\ -1-K \\ K \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} + K \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\therefore \vec{X}^{(3)} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} t e^{2t} + \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} e^{2t} + K \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} e^{2t}$$

The last term is a multiple of $\vec{X}^{(2)}$, $\therefore$ drop it.

$$\therefore \vec{X}^{(3)} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} t e^{2t} + \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} e^{2t}$$

$$\therefore \quad \boxed{\vec{x}(t) = c_1\begin{bmatrix}-3\\4\\2\end{bmatrix}e^{-t} + c_2\begin{bmatrix}0\\-1\\1\end{bmatrix}e^{2t} + c_3\left(\begin{bmatrix}0\\-1\\1\end{bmatrix}te^{2t} + \begin{bmatrix}-1\\-1\\0\end{bmatrix}e^{2t}\right)}$$

Note, to get then answer in the back of the book, let $k_2 = -c_2$, $k_3 = -c_3$ to get

$$k_2\begin{bmatrix}0\\1\\-1\end{bmatrix}e^{2t} + k_3\left(\begin{bmatrix}0\\1\\-1\end{bmatrix}te^{2t} + \begin{bmatrix}1\\1\\0\end{bmatrix}e^{2t}\right)$$

and note if $k_2 = -1$ and $k_3 = 1$, then you get the term $\begin{bmatrix}1\\0\\1\end{bmatrix}e^{2t}$

Also, instead of letting $n_3 = k$ above, could let $n_2 = k$ and $\therefore n_3 = -1 - k$. Then also get answer in back of book.

5.

Using MATLAB,

```
clear

A = sym([0, 1, 1;
         1, 0, 1;
         1, 1, 0]);

[V,D] = eig(A)
```

V =

$$\begin{pmatrix}1 & -1 & -1\\1 & 1 & 0\\1 & 0 & 1\end{pmatrix}$$

D =

$$\begin{pmatrix}2 & 0 & 0\\0 & -1 & 0\\0 & 0 & -1\end{pmatrix}$$

$\therefore \lambda_1 = 2, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \lambda_2 = -1, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \lambda_3 = -1, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

A is symmetric

$$\therefore \boxed{\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{-t} + c_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{-t}}$$

To get the answer in the back of the book, note if $\vec{x_2}$ and $\vec{x_3}$ are eigenvectors for λ, then $\vec{x_2} - \vec{x_3}$ is also an eigenvector since

$A(x_2 - x_3) = Ax_2 - Ax_3 = \lambda x_2 - \lambda x_3 = \lambda(x_2 - x_3)$

Here, $\vec{x_2} - \vec{x_3} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, and

$\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ are independent.

6.

(a) Using MATLAB

```
clear
% solve Ax = B, given x(0) = B
A = sym([1, -4;
         4, -7]);
B = [3, 2]';

% get eigenvalue, eigenvector
[V,D] = eig(A)
% script only works for repeated eigenvalues
if D(1,1) ~= D(2,2)
    exit
end

ev1 = D(1,1);
evec1 = V;
% n = (A-ev1*I)n = evec1
n = linsolve((A-ev1*eye(2)),evec1)

syms t c1 c2
x_1 = evec1*exp(ev1*t);
x_2 = evec1*t*exp(ev1*t) + n*exp(ev1*t);
x = [x_1, x_2];
c = linsolve(subs(x,t,0), B);
x = c(1)*x_1 + c(2)*x_2
```

V =

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

D =

$$\begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix}$$

n =

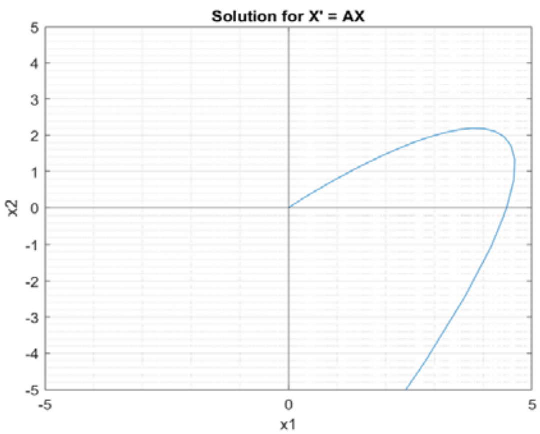
$$\begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix}$$

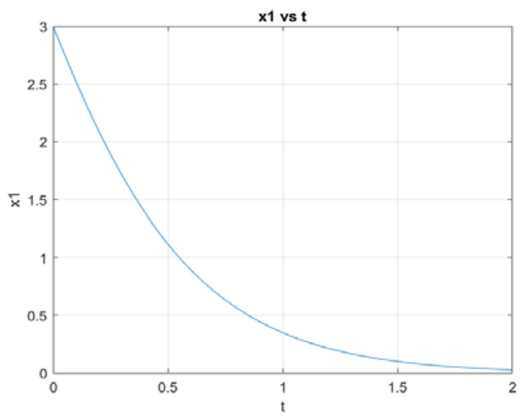
x =

$$\begin{pmatrix} 3\,e^{-3\,t} + 4\,t\,e^{-3\,t} \\ 2\,e^{-3\,t} + 4\,t\,e^{-3\,t} \end{pmatrix}$$

$$\therefore \vec{X}(t) = \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{-3t} + \begin{bmatrix} 4 \\ 4 \end{bmatrix} t e^{-3t}$$

(b)





code on next page

This is a continuation of the code in (a)

```
% plot solution in x1x2 plane

Xmin = -5; Xmax = 5;  % set plot boundaries
Ymin = -5; Ymax = 5;
t = -2.5:0.05:2.5;
% rewrite x_1, x_2 for plot
x_1 = evec1*exp(ev1*t);
x_2 = evec1*t.*exp(ev1*t) + n*exp(ev1*t);
x = c(1)*x_1 + c(2)*x_2;
x1 = x(1,:);
x2 = x(2,:);
plot(x1,x2)
xline(0);  % show x and y axes
yline(0);
axis([Xmin, Xmax, Ymin, Ymax])
grid on, grid minor
xlabel 'x1', ylabel 'x2'
title 'Solution for X'' = AX'

% plot x1 vs t

figure
plot(t,x1)
xlim([0,2])  % confine plot to t >= 0
grid on
xlabel 't', ylabel 'x1'
title 'x1 vs t'
```

7.

(a)

Using Matlab,

```
clear
% solve Ax = B, given x(0) = B
A = sym([-5/2, 3/2;
         -3/2, 1/2]);
B = [3, -1]';

% get eigenvalue, eigenvector
[V,D] = eig(A)
% script only works for repeated eigenvalues
if D(1,1) ~= D(2,2)
    exit
end

ev1 = D(1,1);
evec1 = V;
% n = (A-ev1*I)n = evec1
n = linsolve((A-ev1*eye(2)),evec1)

syms t c1 c2
x_1 = evec1*exp(ev1*t);
x_2 = evec1*t*exp(ev1*t) + n*exp(ev1*t);
x = [x_1, x_2];
c = linsolve(subs(x,t,0), B)
x = c(1)*x_1 + c(2)*x_2
```

V =

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

D =

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

n =

$$\begin{pmatrix} -\frac{2}{3} \\ 0 \end{pmatrix}$$

c =

$$\begin{pmatrix} -1 \\ -6 \end{pmatrix}$$

x =

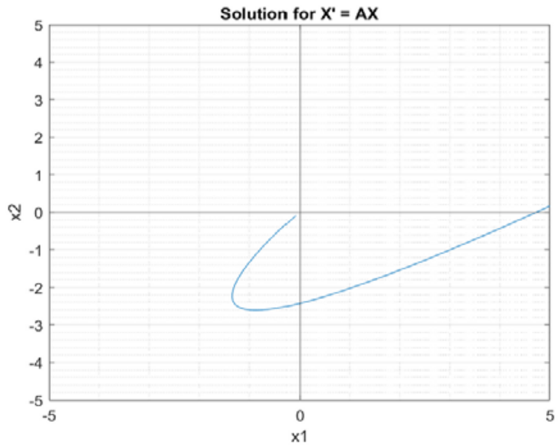
$$\begin{pmatrix} 3\,e^{-t} - 6\,t\,e^{-t} \\ -e^{-t} - 6\,t\,e^{-t} \end{pmatrix}$$

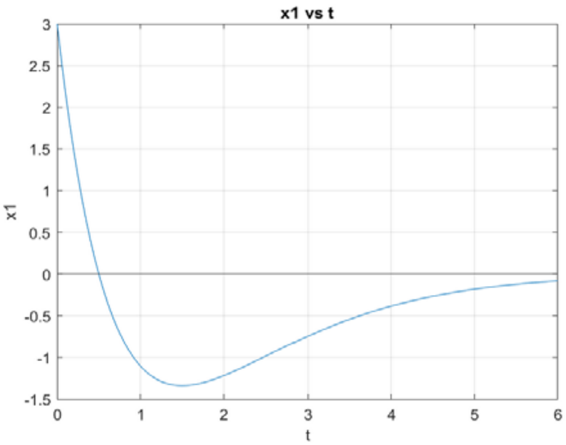
$$\therefore \vec{x}(t) = \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{-t} + \begin{bmatrix} -6 \\ -6 \end{bmatrix} t e^{-t}$$

(b)

Continuation of code in (a)

```
% plot solution in x1x2 plane

Xmin = -5; Xmax = 5;  % set plot boundaries
Ymin = -5; Ymax = 5;
t = -6:0.05:6;
% rewrite x_1, x_2 for plot
x_1 = evec1*exp(ev1*t);
x_2 = evec1*t.*exp(ev1*t) + n*exp(ev1*t);
x = c(1)*x_1 + c(2)*x_2;
x1 = x(1,:);
x2 = x(2,:);
plot(x1,x2)
xline(0);  % show x and y axes
yline(0);
axis([Xmin, Xmax, Ymin, Ymax])
grid on, grid minor
xlabel 'x1', ylabel 'x2'
title 'Solution for X'' = AX'

% plot x1 vs t

figure
plot(t,x1)
xlim([0,6])  % confine plot to t >= 0
yline(0);    % show x-axis
grid on
xlabel 't', ylabel 'x1'
title 'x1 vs t'
```





8.

(a)

$$\vec{x}(t) = \underline{\begin{bmatrix} 2 \\ 4 \end{bmatrix}} + \begin{bmatrix} 42 \\ -14 \end{bmatrix} t$$

Matlab code on next page

```
clear
% solve Ax = B, given x(0) = B
A = sym([3, 9;
        -1, -3]);
B = [2, 4]';

% get eigenvalue, eigenvector
[V,D] = eig(A)
% script only works for repeated eigenvalues
if D(1,1) ~= D(2,2)
    exit
end

ev1 = D(1,1);
evec1 = V;
% n = (A-ev1*I)n = evec1
n = linsolve((A-ev1*eye(2)),evec1)

syms t c1 c2
x_1 = evec1*exp(ev1*t);
x_2 = evec1*t*exp(ev1*t) + n*exp(ev1*t);
x = [x_1, x_2];
c = linsolve(subs(x,t,0), B)
x = c(1)*x_1 + c(2)*x_2
```

V =

$$\begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

D =

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

n =

$$\begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

c =

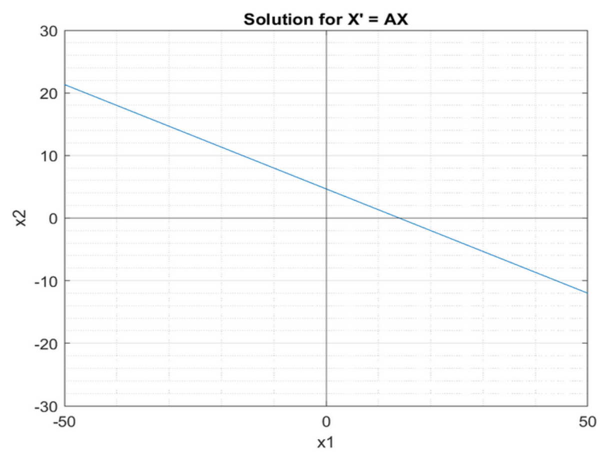
$$\begin{pmatrix} 4 \\ -14 \end{pmatrix}$$

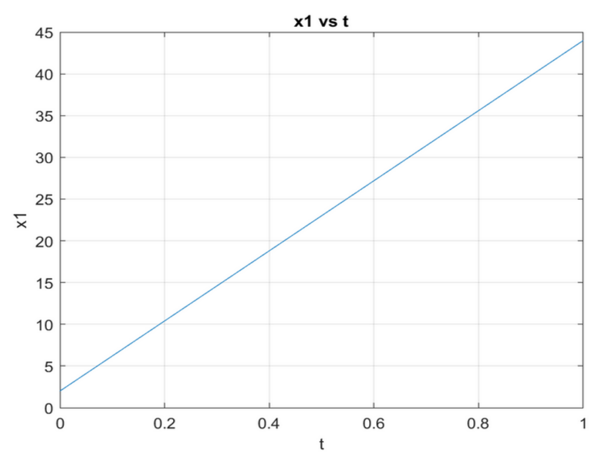
x =

$$\begin{pmatrix} 42\,t + 2 \\ 4 - 14\,t \end{pmatrix}$$

(b)

```
% plot solution in x1x2 plane

Xmin = -50; Xmax = 50;  % set plot boundaries
Ymin = -30; Ymax = 30;
t = -6:0.05:6;
% rewrite x_1, x_2 for plot
x_1 = evec1*exp(ev1*t);
x_2 = evec1*t.*exp(ev1*t) + n*exp(ev1*t);
x = c(1)*x_1 + c(2)*x_2;
x1 = x(1,:);
x2 = x(2,:);
plot(x1,x2)
xline(0);  % show x and y axes
yline(0);
axis([Xmin, Xmax, Ymin, Ymax])
grid on, grid minor
xlabel 'x1', ylabel 'x2'
title 'Solution for X'' = AX'

% plot x1 vs t

figure
plot(t,x1)
xlim([0,1])  % confine plot to t >= 0
yline(0);    % show x-axis
grid on
xlabel 't', ylabel 'x1'
title 'x1 vs t'
```





Code a continuation from (a)

9.

(a) Using MATLAB,

```
clear
% solve Ax = B, given x(0) = B
A = sym([ 1, 0, 0;
         -4, 1, 0;
          3, 6, 2]);
B = [-1, 2, -30]';

% get eigenvalue, eigenvector
[V,D] = eig(A)

% script only works for insufficient eigenvectors
if size(V,2) == size(D,2)
    exit
end

ev1 = D(1,1);
evec1 = V(:,1);
ev2 = D(2,2);
evec2 = V(:,2);

% n = (A-ev2*I)n = evec2
n = linsolve((A-ev2*eye(3)),evec2)

syms t
x_1 = evec1*exp(ev1*t);
x_2 = evec2*exp(ev2*t);
x_3 = evec2*t*exp(ev2*t) + n*exp(ev2*t);
x = [x_1, x_2, x_3];
c = linsolve(subs(x,t,0), B)
x = c(1)*x_1 + c(2)*x_2 + c(3)*x_3
```

V =

$$\begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{6} \\ 1 & 1 \end{pmatrix}$$

D =

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

n =

$$\begin{pmatrix} \frac{1}{24} \\ \frac{7}{48} \\ 0 \end{pmatrix}$$

c =

$$\begin{pmatrix} 3 \\ -33 \\ -24 \end{pmatrix}$$

x =

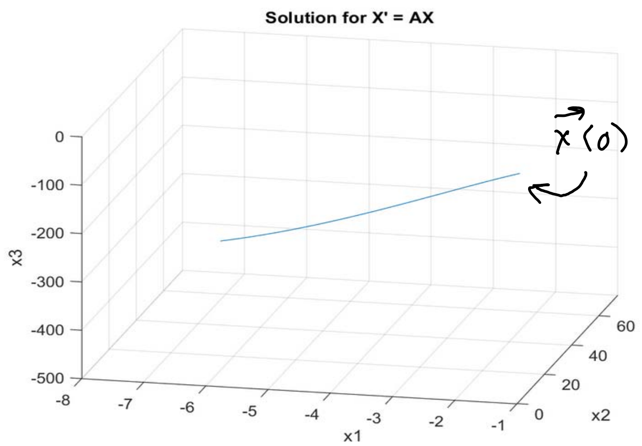
$$\begin{pmatrix} -e^t \\ 2\,e^t + 4\,t\,e^t \\ 3\,e^{2\,t} - 33\,e^t - 24\,t\,e^t \end{pmatrix}$$

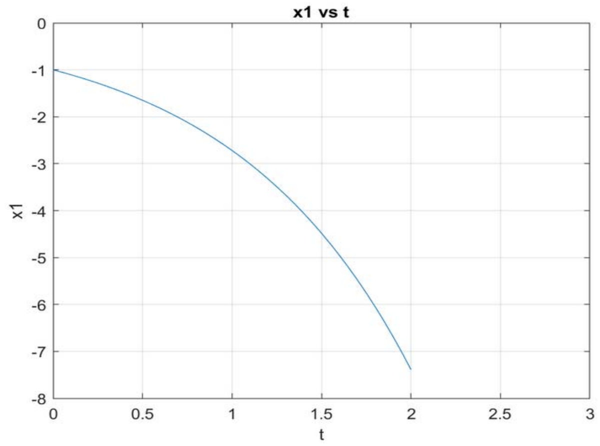
$$\therefore \vec{x}(t) = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} e^{2t} + \begin{bmatrix} -1 \\ 2 \\ -33 \end{bmatrix} e^{t} + \begin{bmatrix} 0 \\ 4 \\ -24 \end{bmatrix} t e^{t}$$

(b) + (c)

Continuation of above code

```
% plot solution in x1x2x3 space

t = 0:0.05:2;
% rewrite x_1, x_2, x_3 for plot
x_1 = evec1*exp(ev1*t);
x_2 = evec2*exp(ev2*t);
x_3 = evec2*t.*exp(ev2*t) + n*exp(ev2*t);
x = c(1)*x_1 + c(2)*x_2 + c(3)*x_3;
x1 = x(1,:);
x2 = x(2,:);
x3 = x(3,:);
plot3(x1,x2,x3)
grid on
xlabel 'x1', ylabel 'x2', zlabel 'x3'
title 'Solution for X'' = AX'
view([13 25])  % supplied by MATLAB

% plot x1 vs t

figure
plot(t,x1)
xlim([0,3])  % confine plot to t >= 0
yline(0);    % show x-axis
grid on
xlabel 't', ylabel 'x1'
title 'x1 vs t'
```





Note: $x_1(t) = -e^t$ from (a)

10.

(a)

This is a symmetric matrix, so there should be 3 independent eigenvectors.

Using MATLAB,

```
clear
% solve Ax = B, given x(0) = B
A = sym([-5/2,  1,    1;
           1, -5/2,   1;
           1,    1, -5/2]);
B = [2, 3, -1]';

% get eigenvalue, eigenvector
[V,D] = eig(A)
ev1 = D(1,1);
evec1 = V(:,1);
ev2 = D(2,2);
evec2 = V(:,2);
ev3 = D(3,3);
evec3 = V(:,3);

syms t
x_1 = evec1*exp(ev1*t);
x_2 = evec2*exp(ev2*t);
x_3 = evec3*exp(ev3*t);
x = [x_1, x_2, x_3];
c = linsolve(subs(x,t,0), B)
x = c(1)*x_1 + c(2)*x_2 + c(3)*x_3
```

V =

$$\begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

D =

$$\begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{7}{2} & 0 \\ 0 & 0 & -\frac{7}{2} \end{pmatrix}$$

c =

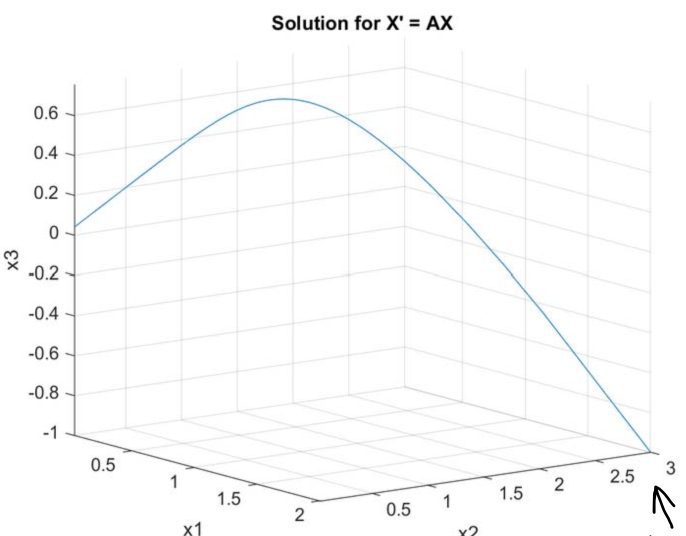
$$\begin{pmatrix} \frac{4}{3} \\ \frac{5}{3} \\ -\frac{7}{3} \end{pmatrix}$$

x =

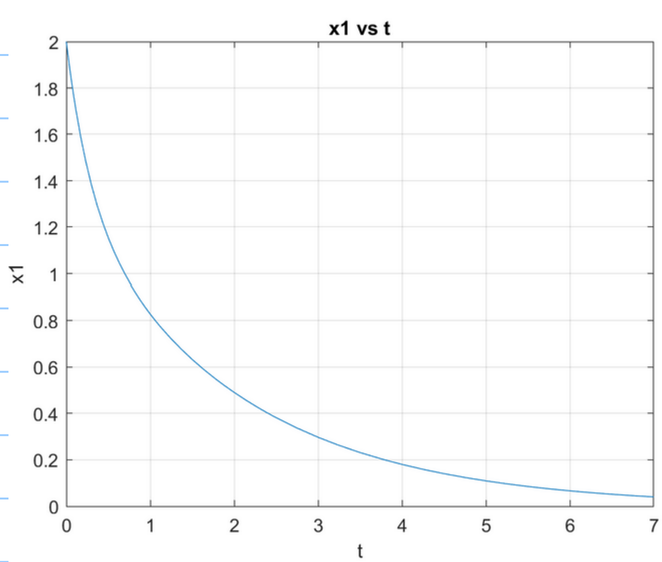
$$\begin{pmatrix} \frac{4\,e^{-\frac{t}{2}}}{3} + \frac{2\,e^{-\frac{7t}{2}}}{3} \\ \frac{4\,e^{-\frac{t}{2}}}{3} + \frac{5\,e^{-\frac{7t}{2}}}{3} \\ \frac{4\,e^{-\frac{t}{2}}}{3} - \frac{7\,e^{-\frac{7t}{2}}}{3} \end{pmatrix}$$

$$\therefore \vec{x}(t) = \begin{bmatrix} 4/3 \\ 4/3 \\ 4/3 \end{bmatrix} e^{-t/2} + \begin{bmatrix} 2/3 \\ 5/3 \\ -7/3 \end{bmatrix} e^{-\frac{7}{2}t}$$

(b) & (c)



start



```
% plot solution in x1x2x3 space

x1 = x(1,:);
x2 = x(2,:);
x3 = x(3,:);
fplot3(x1,x2,x3,[0,7])
grid on
xlabel 'x1', ylabel 'x2', zlabel 'x3'
title 'Solution for X'' = AX'
view([53.37 13.12])  % supplied by MATLAB

% plot x1 vs t

figure
fplot(t,x1,[0,7]) % confine plot to t >= 0
yline(0);     % show x-axis
grid on
xlabel 't', ylabel 'x1'
title 'x1 vs t'
```

Code for above plots, a continuation of code in (a), using "fplot" for symbolic math instead of "plot" used in #9.

Note $\lim_{t\to\infty} \vec{x}(t) = \vec{0}$

11.

Let $\vec{x} = \vec{\epsilon}\, t^r$. $\vec{x}$ is a solution $\Leftrightarrow (A - rI)\vec{\epsilon} = \vec{0}$

$\therefore$ $\vec{\epsilon}$ nonzero $\Leftrightarrow \det(A - rI) = 0$. $\therefore$ Get eigenvalues and eigenvectors to A.

Using MATLAB,

```
clear
A = sym([3, -4;
         1, -1]);
[V,D] = eig(A)
```

V =

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

D =

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Repeated eigenvalue.

One solution is

$$\vec{x}^{(1)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} t$$

Use method of repeated eigenvalues for $x' = Ax$.

$\therefore$ Consider $\vec{x}(t) = \vec{\epsilon}\, f(t)\, t^r + \vec{n}\, t^r$

$\therefore A\vec{x} = A\vec{\epsilon}\, f(t)\, t^r + A\vec{n}\, t^r$

$t\vec{x}' = t\left[\vec{\epsilon}\, f'(t)\, t^r + \vec{\epsilon}\, f(t)\, r t^{r-1} + \vec{n}\, r t^{r-1}\right]$

$= \vec{\epsilon}\, f'(t)\, t^{r+1} + \vec{\epsilon}\, f(t)\, r t^r + \vec{n}\, r t^r$

$\therefore A\vec{x} - t\vec{x}' =$

$(A - rI)\vec{\epsilon}\, f(t)\, t^r + (A - rI)\vec{n}\, t^r - \vec{\epsilon}\, f'(t)\, t^{r+1} = \vec{0}$

$(A - rI)\vec{\epsilon} = \vec{0}$ since $r, \vec{\epsilon}$ is an eigenvalue, eigenvector from above.

$\therefore (A - rI)\vec{n}\, t^r = \vec{\epsilon}\, f'(t)\, t^{r+1}$, and since $t > 0$,

$(A - rI)\vec{n} = \vec{\epsilon}\, f'(t)\, t$

Multiplying by $(A - rI)$,

$(A - rI)^2 \vec{n} = (A - rI)\vec{\epsilon}\, f'(t)\, t = \vec{0}$, since $(A - rI)\vec{\epsilon} = \vec{0}$

Such a non-zero $\vec{n}$ exists since $\det(A - rI)^2 = \left[\det(A - rI)\right]^2 = 0$ since $\det(A - rI) = 0$.

In this specific case, $(A-rI)^2 = \vec{0}$, which is not helpful. $\therefore$ Look at $(A-rI)\vec{n} = \vec{\epsilon} f'(t) t$

If $f(t) = \ln(t)$, then $f'(t)t = 1$, so then the problem becomes $(A-rI)\vec{n} = \vec{\epsilon}$, and we're assuming $\vec{x}^{(2)} = \vec{\epsilon} \ln(t) t^r + \vec{n} t^r$

Note it was never required that $f(t) \neq 0$

$\therefore$ In this case, solve $\begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\therefore \vec{n} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad \therefore \vec{x}^{(2)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \ln(t) t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} t$ $\quad r=1$

$\therefore \vec{x}(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} t + c_2 \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t \ln(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} t \right)$

12.

Using MATLAB and method in #11,

```
clear
A = sym([1, -4;
         4, -7]);
[V,D] = eig(A)

syms t
ev1 = D(1,1);
evec1 = V(:,1);

n = linsolve(A - ev1*eye(2), evec1)
```

V =

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

D =

$$\begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix}$$

n =

$$\begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix}$$

$$\therefore \vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} t^{-3} + c_2 \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} t^{-3} \ln(t) + \begin{bmatrix} 1/4 \\ 0 \end{bmatrix} t^{-3} \right)$$

Note, the answer in the back of the book can be obtained by substituting for $\vec{x}$ above.

$C_1 = c_1 - \frac{1}{4} c_2$.

Note that $(A - rI)\vec{n} = \vec{\epsilon}$ becomes, with $r = -3$,

$$\begin{bmatrix} 4 & -4 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$ so can choose $\vec{n}$ to be $\vec{n} = \begin{bmatrix} 1/4 \\ 0 \end{bmatrix}$ or $\vec{n} = \begin{bmatrix} 0 \\ -1/4 \end{bmatrix}$

13.

Let λ be an eigenvalue.

$$\therefore \det \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} = (\lambda - a)(\lambda - d) - bc$$

$$= \lambda^2 - (a+d)\lambda + ad - bc = 0 \quad [1]$$

$$\therefore \lambda = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$$

Let $\vec{v}_1, \vec{v}_2$ be corresponding eigenvectors.

(1) Suppose $(a+d) < 0$ and $(ad-bc) > 0$

(a) if $(a+d)^2 - 4(ad-bc) > 0$, then λ_1, λ_2 are real,

$$\lambda_1 = \frac{a+d}{2} - \frac{\sqrt{(a+d)^2 - 4(ac-bd)}}{2} < 0 \quad \text{and}$$

$$\lambda_2 = \frac{a+d}{2} + \frac{\sqrt{(a+d)^2 - 4(ac-bd)}}{2} < 0 \quad \text{since}$$

$$(a+d)^2 - 4(ac-bd) < (a+d)^2$$

$$\therefore \frac{a+d}{2} + \frac{\sqrt{(a+d)^2 - 4(ac-bd)}}{2} < \frac{a+d}{2} + \frac{|a+d|}{2} = 0$$

$$\therefore \lim_{t\to\infty} e^{\lambda_1 t} = \lim_{t\to\infty} e^{\lambda_2 t} = 0$$

$$\therefore \lim_{t\to\infty} \vec{x} = \lim_{t\to\infty} c_1 \vec{v}_1 e^{\lambda_1 t} + c_2 \vec{v}_2 e^{\lambda_2 t} = \vec{0}$$

(b) if $(a+d)^2 - 4(ad-bc) = 0$, then $\lambda_1 = \lambda_2 = \frac{a+d}{2} < 0$

$$\therefore \vec{x} = c_1 \vec{v}_1 e^{\lambda_1 t} + c_2 \vec{v}_2 t e^{\lambda_2 t}$$

$$\lim_{t\to\infty} e^{\lambda_1 t} = 0 \text{ so } \lim_{t\to\infty} c_1 \vec{v}_1 e^{\lambda_1 t} = \vec{0}$$

$$\lim_{t\to\infty} te^{\lambda_2 t} = \lim_{t\to\infty} \frac{t}{e^{-\lambda_2 t}} = 0 \quad \therefore \lim_{t\to\infty} c_2 \vec{v}_2 te^{\lambda_2 t} = \vec{0}$$

$$\therefore \lim_{t\to\infty} \vec{x} = \vec{0}$$

(c) if $(a+d)^2 - 4(ad-bc) < 0$, then λ_1, λ_2 are complex conjugates of form $\frac{a+d}{2} \pm i\theta$,

$$\theta = \sqrt{\frac{4(ad-bc)-(a+d)^2}{2}}$$

$\therefore$ Using $\vec{v}_1$, $\vec{x} = e^{\left(\frac{a+d}{2}\right)t}\left[c_1 \vec{v}_1 \cos(\theta t) + c_2 \sin(\theta t)\right]$

Since $\lim_{t\to\infty} e^{\left(\frac{a+d}{2}\right)t} = 0$,

then $\lim_{t\to\infty} \vec{x} = \vec{0}$

$\therefore$ (a), (b), (c) $\Rightarrow$ if $(a+d) < 0$ and $(ad-bc) > 0$,

then $\lim_{t\to\infty} \vec{x} = \vec{0}$

(2) Suppose $(a+d) \geq 0$ or $(ad-bc) \leq 0$

(a) If $(ad-bc) \leq 0$, then $(a+d)^2 - 4(ad-bc) \geq 0$,

so $\lambda = \left[(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}\right] / 2$

has two real values, one non-negative, say λ_1

$\therefore \vec{x}$ has the form $c_1 \vec{v}_1 e^{\lambda_1 t} + c_2 \vec{v}_2 e^{\lambda_2 t}$

for $\lambda_1 > 0$, or $c_1 \vec{v}_1 + c_2 \vec{v}_2 e^{\lambda_2 t}$ for $\lambda_1 = 0$

or $c_1 \vec{v}_1 e^{\lambda_1 t} + c_2 \vec{v}_2 t e^{\lambda_1 t}$ for $\lambda_1 = \lambda_2$

In any case, $\lim\limits_{t \to \infty} \vec{x} \neq \vec{0}$

(b) If $(a+d) \geq 0$, then

(1) $(a+d)^2 - 4(ad-bc) > 0 \Rightarrow \lambda_1, \lambda_2$ real, $\lambda_1 \neq \lambda_2$

at least one λ (say λ_1) is positive.

$\therefore \vec{x} = c_1 \vec{v}_1 e^{\lambda_1 t} + c_2 \vec{v}_2 e^{\lambda_2 t}$

Since $\lambda_1 > 0$, $\lim\limits_{t \to \infty} e^{\lambda_1 t} = \infty$, $\therefore \lim\limits_{t \to \infty} \vec{x} \neq \vec{0}$

(2) $(a+d)^2 - 4(ad-bc) = 0 \Rightarrow \lambda_1 = \lambda_2 = \frac{a+d}{2} \geq 0$

$\therefore \vec{x} = c_1 \vec{v}_1 e^{\lambda_1 t} + c_2 \vec{v}_2 t e^{\lambda_2 t}$

Since $\lim\limits_{t \to \infty} t e^{\lambda_2 t} = \infty$, $\lim\limits_{t \to 0} \vec{x} \neq \vec{0}$

(3) $(a+d)^2 - 4(ad-bc) < 0 \Rightarrow \lambda_1, \lambda_2$ complex

conjugates. $\therefore \lambda = \frac{a+d}{2} \pm i\theta$

where $\theta = \frac{\sqrt{4(ad-bc) - (a+d)^2}}{2}$

Using $\vec{v}_1$,

$$\vec{x} = c_1 \vec{v}_1 e^{\left(\frac{a+d}{2}\right)t} \cos(\theta t) + c_2 \vec{v}_1 e^{\left(\frac{a+d}{2}\right)t} \sin(\theta t)$$

Since $\frac{a+d}{2} \geq 0$, $\lim_{t \to \infty} e^{\left(\frac{a+d}{2}\right)t} \neq 0$

$\therefore \lim_{t \to \infty} \vec{x} \neq \vec{0}$

$\therefore$ (a) + (b) above mean $\lim_{t \to \infty} \vec{x} = \vec{0} \iff (a+d) < 0$ and $(ad-bc) > 0$

14.

(a)

Let $A = \begin{bmatrix} 0 & 1/L \\ -1/C & -1/RC \end{bmatrix}$. λ is an eigenvalue $\iff$

$\det(A - \lambda I) = 0$

$$\therefore \det \begin{bmatrix} -\lambda & 1/L \\ -1/C & -1/RC - \lambda \end{bmatrix} = \lambda^2 + \left(\frac{1}{RC}\right)\lambda + \frac{1}{LC} = 0$$

$$\Leftrightarrow \lambda = \frac{-\frac{1}{RC} \pm \sqrt{\frac{1}{(RC)^2} - \frac{4}{LC}}}{2}$$

If $\frac{1}{(RC)^2} - \frac{4}{LC} = 0$, then $\lambda = -\frac{1}{2RC}$ is a real double root.

$$\frac{1}{(RC)^2} - \frac{4}{LC} = 0 \Leftrightarrow \frac{1}{(RC)^2} = \frac{4}{LC}, \text{ or}$$

$$LC = 4R^2C^2, \text{ or } L = 4R^2C$$

$\therefore$ If $\underline{L = 4R^2C}$, then λ is a real double root.

(b)

Since $4 = 4(1)^2(1)$, then $L = 4R^2C$, so $\lambda = -\frac{1}{2RC}$ is a real double root. $\therefore \lambda = -\frac{1}{2(1)(1)} = -\frac{1}{2}$

$\therefore$ Find eigenvectors for $\lambda = -\frac{1}{2}$.

Using MATLAB, using $B = \begin{bmatrix} I(0) \\ V(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and using A from (a),

```
clear
% solve Ax = B, given x(0) = B
A = sym([0, 1/4;
         -1, -1]);
B = [1, 2]';

% get eigenvalue, eigenvector
[evec1,eval] = eig(A)

ev1 = eval(1,1);
% n = (A-ev1*I)n = evec1
n = linsolve((A-ev1*eye(2)),evec1)

syms t
x_1 = evec1*exp(ev1*t);
x_2 = evec1*t*exp(ev1*t) + n*exp(ev1*t);
x = [x_1, x_2];
c = linsolve(subs(x,t,0), B)
x = c(1)*x_1 + c(2)*x_2
```

evec1 =

$$\begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix}$$

eval =

$$\begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

n =

$$\begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

c =

$$\begin{pmatrix} 2 \\ -2 \end{pmatrix}$$

x =

$$\begin{pmatrix} e^{-\frac{t}{2}} + t\,e^{-\frac{t}{2}} \\ 2\,e^{-\frac{t}{2}} - 2\,t\,e^{-\frac{t}{2}} \end{pmatrix}$$

$$\therefore \vec{x}(t) = \begin{bmatrix} I(t) \\ V(t) \end{bmatrix} = 2\,\vec{x}^{(1)} - 2\,\vec{x}^{2}$$

$$= 2\begin{bmatrix} -1/2 \\ 1 \end{bmatrix} e^{-t/2} - 2\left(\begin{bmatrix} -1/2 \\ 1 \end{bmatrix} t e^{-t/2} + \underbrace{\begin{bmatrix} -1 \\ 0 \end{bmatrix}}_{=\vec{n}} e^{-t/2}\right)$$

$$= \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{-t/2} + \left(\begin{bmatrix} 1 \\ -2 \end{bmatrix} t e^{-t/2} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} e^{-t/2}\right)$$

$$\therefore \begin{bmatrix} I(t) \\ V(t) \end{bmatrix} = \underline{\begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t/2} + \begin{bmatrix} 1 \\ -2 \end{bmatrix} t e^{-t/2}}$$

15.

(a)

Since $(A-2I)\vec{n} = \vec{\epsilon}$, then

$$(A-2I)(A-2I)\vec{n} = (A-2I)\vec{\epsilon} = \vec{0}$$

$$\therefore (A-2I)^2\vec{n} = \vec{0}$$

(b)

$$A-2I = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$\therefore (A-2I)^2 = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \vec{0}$$

(c)

$$(A-2I)\vec{n} = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \vec{\epsilon}$$

(d)

$$(A-2I)\vec{\eta} = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \vec{\epsilon} = -\vec{\epsilon}^{(1)}$$

(e)

$$(A-2I)\vec{\eta} = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} = \begin{bmatrix} -K_1 - K_2 \\ K_1 + K_2 \end{bmatrix} = -(K_1+K_2)\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\therefore \vec{\epsilon} = -(K_1 + K_2)\vec{\epsilon}^{(1)}$

$\therefore$ Want $\underline{K_1 + K_2 \neq 0}$. $\therefore \vec{\epsilon}$ will be a non-zero multiple of $\vec{\epsilon}^{(1)}$, and so will be an eigenvector of A with eigenvalue $\lambda = 2$, and $\vec{\eta} = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix}$ will not be a multiple of $\vec{\epsilon}$, since a multiple would look like $\begin{bmatrix} K_1 \\ -K_1 \end{bmatrix}$ or $\begin{bmatrix} -K_2 \\ K_2 \end{bmatrix}$.

16.

(a)

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}, \quad A^* = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$$

Using MATLAB, "ctranspose" = complex conjugate

```
clear
A = sym([1, -1;
         1, 3]);
[V,D] = eig(ctranspose(A))
```

V =

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

D =

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$\therefore \lambda = 2$ (repeated), and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ = eigenvector

(b)

Eigenvector of $A = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\therefore$ Inner product of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0$

$\therefore$ The eigenvectors are orthogonal.

(c)

Using Problem #25 of section 7.3,

$A\vec{x} = \vec{b}$ has solutions if $(\vec{b}, \vec{y}) = 0$ for every $\vec{y}$, where $\det(A) = 0$ and $\vec{y}$ is a solution of $A^*\vec{y} = \vec{0}$.

Here, $A = (A - 2I)$, and $\det(A - 2I) = 0$.

Also, $(A-2I)^* = A^* - 2I$, and $\vec{y} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ since $\vec{y}$ is an eigenvector, so

$$(A^* - 2I)\vec{y} = (A - 2I)^*\vec{y} = \vec{0}.$$

Since $(\vec{b}, \vec{y}) = 0$, where $\vec{b} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Then $(A - 2I)\vec{x} = \vec{b}$ has solutions.

That is,
$$(\mathbf{A} - 2\mathbf{I})\eta = \xi, \qquad (16)$$
has solutions for $\vec{n}$.

Briefly, the reasoning is as follows.

$\det(A - 2I) = 0 \Rightarrow$ the row reduced form of $A - 2I$ has some zero rows. Let E be the product of elementary matrices producing this row reduced form. $\therefore E(A-2I)$ has at least one zero row. Let E_i be a row of E s.t. $E_i(A-2I)=0$. For $(A-2I)\vec{x}=\vec{\epsilon}$ to have a solution $\vec{x}$, $E_i\vec{\epsilon}$ must be 0 for $E(A-2I)$ and $E\vec{\epsilon}$ to have corresponding zero elements.

Since $(\vec{\epsilon},\vec{y})=0$, the $\vec{y}$ is effectively E_i. $\therefore \vec{y}^T(A-2I)=0$, or $(A-2I)^T\vec{y}=0$.

But here $(A-2I)^T=(A-2I)^*$, so $\vec{y}$ is an eigenvector of $(A-2I)^T$. $\vec{\epsilon}$ is an eigenvector for $(A-2I)$. $\therefore (\vec{\epsilon},\vec{y})=0$ means the row reduced form of the the augmented matrix $[A-2I,\vec{\epsilon}]$ will have rows with corresponding zeros.

$\therefore E(A-2I)\vec{n}=E\vec{\epsilon}$ is solvable for $\vec{n}$.

17.

(a)

Using MATLAB,

```
clear
syms r
A = sym([1, 1, 1;
         2, 1, -1;
         -3, 2, 4]);
p = det(A - r*eye(3))
expand(-(r - 2)^3)
% show eigenvectors and values
[V,D] = eig(A)
```

p = $-r^3 + 6r^2 - 12r + 8$

ans = $-r^3 + 6r^2 - 12r + 8$

V =

$$\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

D =

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$\therefore$ Solving $\det(A - rI) = 0$ means $-(r-2)^3 = 0$,

So, $r = 2$ is a triple root.

$\therefore$ $\underline{r = 2}$ is an eigenvalue of algebraic multiplicity $\underline{3}$

From above, eigenvector $= \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$, or any scalar multipe, such as $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

(b)

$$\vec{x}^{(1)}(t) = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{2t}$$

(c)

$$\frac{d}{dt}\vec{x} = \frac{d}{dt}\left[\vec{\epsilon}\, t e^{2t} + \vec{n} e^{2t}\right]$$

$$= \vec{\epsilon} e^{2t} + \vec{\epsilon}\, 2t e^{2t} + \vec{n}\, 2 e^{2t}$$

$$= (\vec{\epsilon} + 2\vec{n}) e^{2t} + 2\vec{\epsilon}\, t e^{2t}$$

$$A\vec{x} = A\left(\vec{\epsilon}\, t e^{2t} + \vec{n} e^{2t}\right) = A\vec{\epsilon}\, t e^{2t} + A\vec{n} e^{2t}$$

$\therefore$ Equating $\frac{d}{dt}\vec{x} = A\vec{x}$,

$$(\vec{\epsilon} + 2\vec{n}) e^{2t} + 2\vec{\epsilon}\, t e^{2t} = A\vec{n} e^{2t} + A\vec{\epsilon}\, t e^{2t}$$

Equating like coefficients of the vectors,

$$\vec{\epsilon} + 2\vec{n} = A\vec{n} \quad , \quad 2\vec{\epsilon} = A\vec{\epsilon}$$

Or, $\vec{\epsilon} = A\vec{n} - 2\vec{n}$, $A\vec{\epsilon} - 2\vec{\epsilon} = \vec{0}$

$\therefore \; A\vec{n} - 2I\vec{n} = \vec{\epsilon} \;, \quad A\vec{\epsilon} - 2I\vec{\epsilon} = \vec{0}$

Or, $\underline{(A-2I)\vec{n} = \vec{\epsilon}} \;, \quad \underline{(A-2I)\vec{\epsilon} = \vec{0}}$

Use MATLAB to solve $(A-2I)\vec{n} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

Code is continuation of above.

```
% solve (A-2I)n = -V
n = linsolve(A-2*eye(3), -V)
```

n =

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\therefore \; \vec{x}^{(2)}(t) = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} t e^{2t} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{2t}$$

Or, more laboriously,

```
Aug = [A-2*eye(3),-V]
rref(Aug)
```

Aug =

$$\begin{pmatrix} -1 & 1 & 1 & 0 \\ 2 & -1 & -1 & 1 \\ -3 & 2 & 2 & -1 \end{pmatrix}$$

ans =

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\therefore \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow n_1 = 1, \; n_2 + n_3 = 1 \quad \therefore \vec{n} = \begin{bmatrix} 1 \\ 1-K \\ K \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + K \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\therefore \vec{x}^{(2)} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} te^{2t} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{2t} + k \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} e^{2t}$$

Drop the last term as it is a multiple of $\vec{x}^{(1)}$

$$\therefore \vec{x}^{(2)}(t) = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} te^{2t} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{2t}, \text{ as above}$$

(d)

$$\vec{x}\,' = \vec{\epsilon}\, te^{2t} + \vec{\epsilon}\, t^2 e^{2t} + \vec{n} e^{2t} + \vec{n}\, 2te^{2t} + \vec{c}\, 2e^{2t}$$

$$A\vec{x} = A\vec{\epsilon}\, \frac{t^2}{2} e^{2t} + A\vec{n}\, te^{2t} + A\vec{c}\, e^{2t}$$

Equating like terms in $A\vec{x} = \vec{x}\,'$,

$$A\vec{\epsilon}\, \frac{t^2}{2} e^{2t} = \vec{\epsilon}\, t^2 e^{2t} \qquad [1]$$

$$A\vec{n}\, te^{2t} = \vec{\epsilon}\, te^{2t} + \vec{n}\, 2te^{2t} \qquad [2]$$

$$A\vec{c}\, e^{2t} = \vec{n}\, e^{2t} + \vec{c}\, 2e^{t} \qquad [3]$$

Since must be true for all t, divide by t^2e^{2t}, te^{2t}, e^{2t}

$$A\vec{\epsilon} = 2\vec{\epsilon} \qquad [1']$$

$$A\vec{n} = \vec{\epsilon} + 2\vec{n} \qquad [2']$$

$$A\vec{c} = \vec{n} + 2\vec{c} \qquad [3']$$

$$\therefore \boxed{\begin{aligned}(A-2I)\vec{\epsilon} &= \vec{0}\\ (A-2I)\vec{n} &= \vec{\epsilon}\\ (A-2I)\vec{c} &= \vec{n}\end{aligned}}$$

Since $\vec{n} = \begin{bmatrix}1\\1\\0\end{bmatrix}$ from (c) above, use MATLAB to solve for $\vec{c}$. Code is continuation of above.

```
% solve (A-2I)c = n
c_Aug = [A-2*eye(3), n]
rref(c_Aug)
```

c_Aug =

$$\begin{pmatrix}-1 & 1 & 1 & 1\\ 2 & -1 & -1 & 1\\ -3 & 2 & 2 & 0\end{pmatrix}$$

ans =

$$\begin{pmatrix}1 & 0 & 0 & 2\\ 0 & 1 & 1 & 3\\ 0 & 0 & 0 & 0\end{pmatrix}$$

$$\therefore \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 1\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}2\\3\\0\end{bmatrix} \Rightarrow \begin{aligned}c_1 &= 2\\ c_2 + c_3 &= 3\end{aligned}$$

Let $c_2 = K$, $\therefore c_3 = 3-K$ $\quad \therefore \vec{c} = \begin{bmatrix}2\\K\\3-K\end{bmatrix} = \begin{bmatrix}2\\0\\3\end{bmatrix} + K\begin{bmatrix}0\\1\\-1\end{bmatrix}$

The last term is a multiple of $\vec{\epsilon}$, $\therefore$ drop.

$$\therefore \vec{X}^{(3)}(t) = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \frac{t^2}{2} e^{2t} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} te^{t} + \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} e^{2t}$$

(e)

$$\Psi(t) = \left[\vec{X}^{(1)}, \vec{X}^{(2)}, \vec{X}^{(3)} \right]$$

$$= \left[\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{2t}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} te^{2t} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{2t}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \frac{t^2}{2} e^{2t} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} te^{2t} + \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} e^{2t} \right]$$

$$= e^{2t} \begin{bmatrix} 0 & 1 & t+2 \\ 1 & t+1 & t^2/2 + t \\ -1 & -t & -t^2/2 + 3 \end{bmatrix}$$

(f)

$$T = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ -1 & 0 & 3 \end{bmatrix}$$

Use MATLAB to find T^{-1}, code continuation from above using $\vec{C} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$

```
T = [-V, n, [2,0,3]']
Tinv = rref([T,eye(3)])
T_1 = Tinv(:,4:6)
```

T =
$$\begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ -1 & 0 & 3 \end{pmatrix}$$

Tinv =
$$\begin{pmatrix} 1 & 0 & 0 & -3 & 3 & 2 \\ 0 & 1 & 0 & 3 & -2 & -2 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{pmatrix}$$

T_1 =
$$\begin{pmatrix} -3 & 3 & 2 \\ 3 & -2 & -2 \\ -1 & 1 & 1 \end{pmatrix}$$

$$\therefore\ T^{-1} = \begin{bmatrix} -3 & 3 & 2 \\ 3 & -2 & -2 \\ -1 & 1 & 1 \end{bmatrix}$$

```
J = T_1*A*T
```

J =
$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\therefore\ J = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

18.

(a)

Using MATLAB,

```
clear
syms r
A = sym([5, -3, -2;
         8, -5, -4;
         -4, 3, 3]);
p = det(A - r*eye(3))
factor(p)
```

p = $-r^3 + 3\,r^2 - 3\,r + 1$

ans =

$$\begin{pmatrix} -1 & r-1 & r-1 & r-1 \end{pmatrix}$$

$\therefore$ From the characteristic polynomial, $(r-1)^3 = 0$ and so $\underline{r=1}$ is a $\underline{\text{triple}}$ eigenvalue

Solving for $(A - I)\vec{c} = \vec{0}$,

```
rref(A - eye(3))
```

ans =

$$\begin{pmatrix} 1 & -\frac{3}{4} & -\frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\therefore \begin{bmatrix} 1 & -\frac{3}{4} & -\frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ 3x3 matrix of rank 1, can choose 2 independent vectors. $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}$ solve the system and are independent.

$\therefore$ Let $\vec{\epsilon}^{(1)} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ and $\epsilon^{(2)} = \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}$

$\therefore \vec{X}^{(1)} = \vec{\epsilon}^{(1)} e^{rt}, \ \vec{x}^{(2)} = \vec{\epsilon}^{(2)} e^{rt}$

Or, $\vec{x}^{(1)}(t) = \underline{\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} e^{t}}$ $\quad x^{(2)}(t) = \underline{\begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix} e^{t}}$

(b)

$$\vec{x}\,' = \vec{\epsilon}\, e^t + \vec{\epsilon}\, t e^t + \vec{n}\, e^t$$

$$A\vec{x} = A\vec{\epsilon}\, t e^t + A\vec{n}\, e^t$$

From $A\vec{x} = \vec{x}\,'$, equate like terms

$$\therefore \; A\vec{\epsilon}\, t e^t = \vec{\epsilon}\, t e^t$$

$$A\vec{n}\, e^t = (\vec{\epsilon} + \vec{n})\, e^t$$

Since must be true for all t, divide by te^t and e^t

$$\therefore \; A\vec{\epsilon} = \vec{\epsilon} \quad \text{and} \quad A\vec{n} = \vec{\epsilon} + \vec{n}$$

Or, $\underline{(A - I)\vec{\epsilon} = \vec{0}}$ and $\underline{(A - I)\vec{n} = \vec{\epsilon}}$

(c)

(1) $(A-I)^2 \vec{n} = (A-I)\left[(A-I)\vec{n}\right] = (A-I)\vec{\epsilon} = \vec{0}$

(2) $(A - I) = \begin{bmatrix} 5 & -3 & -2 \\ 8 & -5 & -4 \\ -4 & 3 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -3 & -2 \\ 8 & -6 & -4 \\ -4 & 3 & 2 \end{bmatrix}$

Using MATLAB,

```
clear
A = sym([5, -3, -2;
         8, -5, -4;
        -4, 3, 3]);
(A-eye(3))^2
```

ans =

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\therefore \underline{(A-I)^2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(d)

$$(A-I)\vec{n} = \vec{\epsilon} : \begin{bmatrix} 4 & -3 & -2 \\ 8 & -6 & -4 \\ -4 & 3 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \underline{\begin{bmatrix} -2 \\ -4 \\ 2 \end{bmatrix}} = \vec{\epsilon}$$

Using MATLAB,

```
clear
A = sym([5, -3, -2;
         8, -5, -4;
        -4, 3, 3]);

n = [0,0,1]';
e = (A-eye(3))*n

A*e
```

e =

$$\begin{pmatrix} -2 \\ -4 \\ 2 \end{pmatrix}$$

ans =

$$\begin{pmatrix} -2 \\ -4 \\ 2 \end{pmatrix}$$

$$\therefore A\vec{\epsilon} = \vec{\epsilon}$$

$$\therefore \underline{\vec{x}^{(3)}} = \vec{\epsilon}te^t + \vec{n}e^t = \begin{bmatrix} -2 \\ -4 \\ 2 \end{bmatrix} te^t + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^t$$

(e)

$$\Psi(t) = \left[\vec{x}^{(1)}, \vec{x}^{(2)}, \vec{x}^{(3)}\right]$$

$$= \begin{bmatrix} e^t & 0 & -2te^t \\ 0 & 2e^t & -4te^t \\ 2e^t & -3e^t & 2te^t + e^t \end{bmatrix}$$

$$= e^t \begin{bmatrix} 1 & 0 & -2t \\ 0 & 2 & -4t \\ 2 & -3 & 2t+1 \end{bmatrix}$$

(f)

$$T = \begin{bmatrix} 1 & -2 & 0 \\ 0 & -4 & 0 \\ 2 & 2 & 1 \end{bmatrix}$$

Using MATLAB,

```
clear
A = sym([ 5, -3, -2;
          8, -5, -4;
         -4,  3,  3]);

T = sym([1, -2, 0;
         0, -4, 0;
         2,  2, 1]);
Tinv = rref([T,eye(3)])
T_1 = Tinv(:,4:6)

J = T_1*A*T
```

Tinv =

$$\begin{pmatrix} 1 & 0 & 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{4} & 0 \\ 0 & 0 & 1 & -2 & \frac{3}{2} & 1 \end{pmatrix}$$

T_1 =

$$\begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{4} & 0 \\ -2 & \frac{3}{2} & 1 \end{pmatrix}$$

J =

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\therefore T^{-1} = \begin{bmatrix} 1 & -1/2 & 0 \\ 0 & -1/4 & 0 \\ -2 & 3/2 & 1 \end{bmatrix} \qquad J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

19.

(a)

$$J^2 = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} \lambda^2 & 2\lambda \\ 0 & \lambda^2 \end{bmatrix}$$

$$J^3 = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}\begin{bmatrix} \lambda^2 & 2\lambda \\ 0 & \lambda^2 \end{bmatrix} = \begin{bmatrix} \lambda^3 & 3\lambda^2 \\ 0 & \lambda^3 \end{bmatrix}$$

$$J^4 = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}\begin{bmatrix} \lambda^3 & 3\lambda^2 \\ 0 & \lambda^3 \end{bmatrix} = \begin{bmatrix} \lambda^4 & 4\lambda^3 \\ 0 & \lambda^4 \end{bmatrix}$$

(b)

(a) shows true for $n = 1, 2, 3$

(b) Suppose true for k : $J^k = \begin{bmatrix} \lambda^k & k\lambda^{k-1} \\ 0 & \lambda^k \end{bmatrix}$

$$\therefore \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}\begin{bmatrix} \lambda^k & k\lambda^{k-1} \\ 0 & \lambda^k \end{bmatrix} = \begin{bmatrix} \lambda^{k+1} & k\lambda^k + \lambda^k \\ 0 & \lambda^{k+1} \end{bmatrix}$$

$$= \begin{bmatrix} \lambda^{k+1} & (k+1)\lambda^k \\ 0 & \lambda^{k+1} \end{bmatrix} = \begin{bmatrix} \lambda^{k+1} & (k+1)\lambda^{(k+1)-1} \\ 0 & \lambda^{k+1} \end{bmatrix}$$

$\therefore$ True for $K \Rightarrow$ true for $K+1$

$\therefore$ True for all positive integers n.

(c)

$$\exp(Jt) = I + \frac{Jt}{1} + \frac{J^2 t^2}{2!} + \frac{J^3 t^3}{3!} + \cdots$$

$$= I + \sum_{K=1}^{\infty} \frac{J^K t^K}{K!}$$

$$= I + \sum_{K=1}^{\infty} \begin{bmatrix} \lambda^K & K\lambda^{K-1} \\ 0 & \lambda^K \end{bmatrix} \frac{t^K}{K!}$$

$$= I + \sum_{K=1}^{\infty} \begin{bmatrix} \frac{(\lambda t)^K}{K!} & t\,\frac{(\lambda t)^{K-1}}{(K-1)!} \\ 0 & \frac{(\lambda t)^K}{K!} \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{K=0}^{\infty} \frac{(\lambda t)^K}{K!} & t \sum_{K=1}^{\infty} \frac{(\lambda t)^{K-1}}{(K-1)!} \\ 0 & \sum_{K=0}^{\infty} \frac{(\lambda t)^K}{K!} \end{bmatrix}$$

$$= \begin{bmatrix} e^{\lambda t} & t \sum_{K=0}^{\infty} \frac{(\lambda t)^K}{K!} \\ 0 & e^{\lambda t} \end{bmatrix}$$

$$\therefore \exp(Jt) = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix} = e^{\lambda t}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

(d)

As was shown on p. 332 of the text, equation (27), the solution to $\phi' = A\phi$, $\phi(0) = I$ is $\exp(At)$

$\therefore$ $X(t) = \exp(Jt)$ solves $X' = JX$, $X(0) = I$

By linearity, $x(t) = \exp(Jt)\vec{x}^{(0)}$ satisfies $x' = Jx$ and $x(0) = \exp(J(0))\vec{x}^{(0)} = I\vec{x}^{(0)} = \vec{x}^{0}$

$\therefore$ By uniqueness $\exp(Jt)\vec{x}^{(0)}$ is the solution to $x' = Jx$, $x(0) = \vec{x}^{(0)}$

20.

(a)

Using MATLAB,

```
clear
syms r
J = [r, 0, 0;
     0, r, 1;
     0, 0, r];
J2 = J^2
J3 = J^3
J4 = J^4
```

J2 =

$$\begin{pmatrix} r^2 & 0 & 0 \\ 0 & r^2 & 2\,r \\ 0 & 0 & r^2 \end{pmatrix}$$

J3 =

$$\begin{pmatrix} r^3 & 0 & 0 \\ 0 & r^3 & 3\,r^2 \\ 0 & 0 & r^3 \end{pmatrix}$$

J4 =

$$\begin{pmatrix} r^4 & 0 & 0 \\ 0 & r^4 & 4\,r^3 \\ 0 & 0 & r^4 \end{pmatrix}$$

$$\therefore J^2 = \begin{bmatrix} \lambda^2 & 0 & 0 \\ 0 & \lambda^2 & 2\lambda \\ 0 & 0 & \lambda^2 \end{bmatrix} \quad J^3 = \begin{bmatrix} \lambda^3 & 0 & 0 \\ 0 & \lambda^3 & 3\lambda^2 \\ 0 & 0 & \lambda^3 \end{bmatrix} \quad J^4 = \begin{bmatrix} \lambda^4 & 0 & 0 \\ 0 & \lambda^4 & 4\lambda^3 \\ 0 & 0 & \lambda^4 \end{bmatrix}$$

(b)

(a) showed true for $n = 1, 2, 3, 4$

Suppose true for $n = k$.

$$\therefore J^{k+1} = J \cdot J^k$$

Using MATLAB,

```
clear
syms r k
J = [r, 0, 0;
     0, r, 1;
     0, 0, r];
J_k = [r^k, 0,      0;
       0,   r^k, k*r^(k-1);
       0,   0,     r^k];
J_k1 = J*J_k
simplify(J_k1)
```

J_k1 =

$$\begin{pmatrix} r\,r^k & 0 & 0 \\ 0 & r\,r^k & r^k + k\,r\,r^{k-1} \\ 0 & 0 & r\,r^k \end{pmatrix}$$

ans =

$$\begin{pmatrix} r^{k+1} & 0 & 0 \\ 0 & r^{k+1} & r^k\,(k+1) \\ 0 & 0 & r^{k+1} \end{pmatrix}$$

$$\therefore J^{k+1} = \begin{bmatrix} \lambda^{k+1} & 0 & 0 \\ 0 & \lambda^{k+1} & (k+1)\lambda^{(k+1)-1} \\ 0 & 0 & \lambda^{k+1} \end{bmatrix}$$

$\therefore$ When true for $n=k$, true for $n=k+1$

$\therefore$ True <u>for all</u> positive integers n.

(c)

$$\exp(Jt) = I + \sum_{k=1}^{\infty} \frac{(Jt)^k}{k!} = I + \sum_{k=1}^{\infty} \frac{J^k t^k}{k!}$$

$$= I + \sum_{k=1}^{\infty} \begin{bmatrix} \lambda^k & 0 & 0 \\ 0 & \lambda^k & k\lambda^{k-1} \\ 0 & 0 & \lambda^k \end{bmatrix} \frac{t^k}{k!}$$

$$= I + \begin{bmatrix} \sum_{k=1}^{\infty} \frac{\lambda^k t^k}{k!} & 0 & 0 \\ 0 & \sum_{k=1}^{\infty} \frac{\lambda^k t^k}{k!} & t\sum_{k=1}^{\infty} \frac{\lambda^{k-1} t^{k-1}}{(k-1)!} \\ 0 & 0 & \sum_{k=1}^{\infty} \frac{\lambda^k t^k}{k!} \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} & 0 & 0 \\ 0 & \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} & t \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} \\ 0 & 0 & \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} \end{bmatrix}$$

$$\therefore \underline{\exp(Jt) = \begin{bmatrix} e^{\lambda t} & 0 & 0 \\ 0 & e^{\lambda t} & t e^{\lambda t} \\ 0 & 0 & e^{\lambda t} \end{bmatrix}} = \underline{e^{\lambda t} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}}$$

(d)

$$T = \begin{bmatrix} 1 & -2 & 0 \\ 0 & -4 & 0 \\ 2 & 2 & 1 \end{bmatrix} \qquad \exp(Jt) = e^{t} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \underline{T \exp(Jt) = e^{t} \begin{bmatrix} 1 & -2 & -2t \\ 0 & -4 & -4t \\ 2 & 2 & 2t+1 \end{bmatrix}}$$

(e)

No. From 18(e), $\Psi(t) = e^{t}\begin{bmatrix} 1 & 0 & -2t \\ 0 & 2 & -4t \\ 2 & -3 & 2t+1 \end{bmatrix}$

The middle column is different from $T\exp(Jt)$

$\Psi(t)$ in #18 is from:

$$\Psi(t) = \left[\vec{\epsilon}^{(1)} e^{t},\ \vec{\epsilon}^{(2)} e^{t},\ \vec{\epsilon}^{(3)} te^{t} + \vec{n} e^{t}\right]$$

Looking at (d) above, the solution set is

$$\left[\vec{\epsilon}^{(1)} e^{t},\ \vec{\epsilon}^{(3)} e^{t},\ \vec{\epsilon}^{(3)} te^{t} + \vec{n} e^{t}\right]$$

using the eigenvectors from #18.

$\therefore$ The solution sets are different.

Fundamental solution sets are not unique, just as a set of basis vectors is not unique.

But one set can be obtained from the other.

Note: $-2\vec{\epsilon}^{(1)} - 2\vec{\epsilon}^{(2)} = \vec{\epsilon}^{(3)}$

21.

(a)

Using MATLAB,

```
clear
syms r k
J = [r, 1, 0;
     0, r, 1;
     0, 0, r];
J2 = J^2
J3 = J^3
J4 = J^4
```

J2 =

$$\begin{pmatrix} r^2 & 2r & 1 \\ 0 & r^2 & 2r \\ 0 & 0 & r^2 \end{pmatrix}$$

J3 =

$$\begin{pmatrix} r^3 & 3r^2 & 3r \\ 0 & r^3 & 3r^2 \\ 0 & 0 & r^3 \end{pmatrix}$$

J4 =

$$\begin{pmatrix} r^4 & 4r^3 & 6r^2 \\ 0 & r^4 & 4r^3 \\ 0 & 0 & r^4 \end{pmatrix}$$

$$\therefore \underline{J^2 = \begin{bmatrix} \lambda^2 & 2\lambda & 1 \\ 0 & \lambda^2 & 2\lambda \\ 0 & 0 & \lambda^2 \end{bmatrix}} \quad \underline{J^3 = \begin{bmatrix} \lambda^3 & 3\lambda^2 & 3\lambda \\ 0 & \lambda^3 & 3\lambda^2 \\ 0 & 0 & \lambda^3 \end{bmatrix}} \quad \underline{J^4 = \begin{bmatrix} \lambda^4 & 4\lambda^3 & 6\lambda^2 \\ 0 & \lambda^4 & 4\lambda^3 \\ 0 & 0 & \lambda^4 \end{bmatrix}}$$

(b)

(a) showed true for $n = 1, 2, 3, 4$

Assume true for $n = k$

$$\therefore J^{k+1} = J \cdot J^k = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} \begin{bmatrix} \lambda^k & k\lambda^{k-1} & \frac{k(k-1)}{2}\lambda^{k-2} \\ 0 & \lambda^k & k\lambda^{k-1} \\ 0 & 0 & \lambda^k \end{bmatrix}$$

Using MATLAB,

```
clear
syms r k
J = [r, 1, 0;
     0, r, 1;
     0, 0, r];

J_k = [r^k, k*r^(k-1), 1/2*k*(k-1)*r^(k-2);
        0,     r^k,     k*r^(k-1);
        0,      0,          r^k];

J_k1 = J*J_k
simplify(J_k1)
```

J_k1 =

$$\begin{pmatrix} r\,r^k & r^k + k\,r\,r^{k-1} & k\,r^{k-1} + \frac{k\,r\,r^{k-2}\,(k-1)}{2} \\ 0 & r\,r^k & r^k + k\,r\,r^{k-1} \\ 0 & 0 & r\,r^k \end{pmatrix}$$

ans =

$$\begin{pmatrix} r^{k+1} & r^k\,(k+1) & \frac{k\,r^{k-1}\,(k+1)}{2} \\ 0 & r^{k+1} & r^k\,(k+1) \\ 0 & 0 & r^{k+1} \end{pmatrix}$$

$$\therefore J^{k+1} = \begin{bmatrix} \lambda^{k+1} & (k+1)\lambda^{(k+1)-1} & \frac{1}{2}(k+1)(k+1-1)\lambda^{(k+1)-2} \\ 0 & \lambda^{k+1} & (k+1)\lambda^{(k+1)-1} \\ 0 & 0 & \lambda^{k+1} \end{bmatrix}$$

$\therefore$ When true for $n=k$, true for $n=k+1$

$\therefore$ True for all positive integers.

(c)

$$\exp(Jt) = I + \sum_{k=1}^{\infty} \frac{(Jt)^k}{k!} = I + \sum_{k=1}^{\infty} J^k \frac{t^k}{k!}$$

$$= I + \sum_{k=1}^{\infty} \begin{bmatrix} \lambda^k & k\lambda^{k-1} & \frac{1}{2}k(k-1)\lambda^{k-2} \\ 0 & \lambda^k & k\lambda^{k-1} \\ 0 & 0 & \lambda^k \end{bmatrix} \frac{t^k}{k!}$$

$$= I + \begin{bmatrix} \sum_{K=1}^{\infty} \lambda^K \frac{t^K}{K!} & \sum_{K=1}^{\infty} K \lambda^{K-1} \frac{t^K}{K!} & \sum_{K=1}^{\infty} \frac{1}{2} K(K-1) \lambda^{K-2} \frac{t^K}{K!} \\ 0 & \sum_{K=1}^{\infty} \lambda^K \frac{t^K}{K!} & \sum_{K=1}^{\infty} K \lambda^{K-1} \frac{t^K}{K!} \\ 0 & 0 & \sum_{K=1}^{\infty} \lambda^K \frac{t^K}{K!} \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{K=0}^{\infty} \frac{(\lambda t)^K}{K!} & t \sum_{K=1}^{\infty} \frac{(\lambda t)^{K-1}}{(K-1)!} & \frac{t^2}{2} \sum_{K=2}^{\infty} \frac{(\lambda t)^{K-2}}{(K-2)!} \\ 0 & \sum_{K=0}^{\infty} \frac{(\lambda t)^K}{K!} & t \sum_{K=1}^{\infty} \frac{(\lambda t)^{K-1}}{(K-1)!} \\ 0 & 0 & \sum_{K=0}^{\infty} \frac{(\lambda t)^K}{K!} \end{bmatrix}$$

$$= \begin{bmatrix} e^{\lambda t} & t \sum_{K=0}^{\infty} \frac{(\lambda t)^K}{K!} & \frac{t^2}{2} \sum_{K=0}^{\infty} \frac{(\lambda t)^K}{K!} \\ 0 & e^{\lambda t} & t \sum_{K=0}^{\infty} \frac{(\lambda t)^K}{K!} \\ 0 & 0 & e^{\lambda t} \end{bmatrix}$$

$$= \begin{bmatrix} e^{\lambda t} & t e^{\lambda t} & \frac{t^2}{2} e^{\lambda t} \\ 0 & e^{\lambda t} & t e^{\lambda t} \\ 0 & 0 & e^{\lambda t} \end{bmatrix}$$

$$\therefore \exp(Jt) = e^{\lambda t}\begin{bmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

(d)

From #17(f), $T = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ -1 & 0 & 3 \end{bmatrix}$

$$T\exp(Jt) = T e^{2t}\begin{bmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

$$= e^{2t}\begin{bmatrix} 0 & 1 & (t+2) \\ 1 & (t+1) & \left(\frac{t^2}{2}+t\right) \\ -1 & -t & \left(-\frac{t^2}{2}+3\right) \end{bmatrix}$$

From #17(e), $\Psi(t) = e^{2t}\begin{bmatrix} 0 & 1 & t+2 \\ 1 & t+1 & t^2/2+t \\ -1 & -t & -t^2/2+3 \end{bmatrix}$

The two matrices are the same.

7.9 Nonhomogeneous Linear Systems

Note Title 3/13/2020

1.

Using MATLAB,

```
clear
A = sym([2, -1;
          3, -2]);
[T,D] = eig(A)
```

T =

$$\begin{pmatrix} 1 & \frac{1}{3} \\ 1 & 1 \end{pmatrix}$$

D =

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$\therefore$ Two different eigenvalues & eigenvectors

Homogeneous solution: $\vec{x}_c(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-t}$

Try variation of parameters

$$\Psi(t) = \begin{bmatrix} e^{t} & e^{-t} \\ e^{t} & 3e^{-t} \end{bmatrix}, \quad \vec{g}(t) = \begin{bmatrix} e^{t} \\ t \end{bmatrix}$$

Assume $\vec{v}(t) = \Psi(t)\vec{u}(t)$, $\vec{v}$ nonhomogeneous solution.

$$\therefore \Psi(t)\vec{u}'(t) = \vec{g}(t) \qquad \Psi^{-1}(t) = \frac{1}{2}\begin{bmatrix} 3e^{-t} & -e^{-t} \\ -e^{t} & e^{t} \end{bmatrix}$$

$$\therefore \Psi^{-1}(t)\vec{g}(t) = \frac{1}{2}\begin{bmatrix} 3e^{-t} & -e^{-t} \\ -e^{t} & e^{t} \end{bmatrix}\begin{bmatrix} e^{t} \\ t \end{bmatrix} = \begin{bmatrix} \frac{3}{2} - \frac{1}{2}te^{-t} \\ -\frac{1}{2}e^{2t} + \frac{1}{2}te^{t} \end{bmatrix}$$

$$\therefore u'(t) = \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} \frac{3}{2} - \frac{1}{2}te^{-t} \\ -\frac{1}{2}e^{2t} + \frac{1}{2}te^{t} \end{bmatrix}$$

$$\therefore \quad u_1 = \frac{3}{2}t + \frac{1}{2}te^{-t} + \frac{1}{2}e^{-t}$$

using

$$u_2 = -\frac{1}{4}e^{2t} + \frac{1}{2}te^{t} - \frac{1}{2}e^{t}$$

$$\int ue^{au}\,du = \frac{1}{a^2}(au-1)e^{au} + C$$

$$\therefore \quad \vec{v}(t) = \Psi(t)\vec{u}(t) = \begin{bmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{bmatrix} \begin{bmatrix} \frac{3}{2}t + \frac{1}{2}te^{-t} + \frac{1}{2}e^{-t} \\ -\frac{1}{4}e^{2t} + \frac{1}{2}te^{t} - \frac{1}{2}e^{t} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{2}te^t + \frac{1}{2}t + \frac{1}{2} - \frac{1}{4}e^t + \frac{1}{2}t - \frac{1}{2} \\ \frac{3}{2}te^t + \frac{1}{2}t + \frac{1}{2} - \frac{3}{4}e^t + \frac{3}{2}t - \frac{3}{2} \end{bmatrix}$$

$$= \frac{3}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} te^t + \begin{bmatrix} 1 \\ 2 \end{bmatrix} t - \frac{1}{4}\begin{bmatrix} 1 \\ 3 \end{bmatrix} e^t - \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$\therefore$

$$\boxed{\begin{aligned} \vec{x}(t) &= \vec{x}_c(t) + \vec{v}(t) \\ &= c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t + c_2\begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-t} + \frac{3}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} te^t + \begin{bmatrix} 1 \\ 2 \end{bmatrix} t - \frac{1}{4}\begin{bmatrix} 1 \\ 3 \end{bmatrix} e^t - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{aligned}}$$

2.

Using MATLAB,

```
clear
Ay = sym([2, -5;
          1, -2]);
[V,De] = eig(Ay)
```

V =

$$\begin{pmatrix} 2-i & 2+i \\ 1 & 1 \end{pmatrix}$$

De =

$$\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

Using the first eigenvector, using $e^{-it} = \cos(t) - i\sin(t)$

$$\begin{bmatrix} 2-i \\ 1 \end{bmatrix}(\cos(t) - i\sin(t)) = \begin{bmatrix} 2\cos(t) - \sin(t) \\ \cos(t) \end{bmatrix} - i\begin{bmatrix} \cos(t) + 2\sin(t) \\ \sin(t) \end{bmatrix}$$

$\therefore$ Homogeneous solution:

$$\vec{X}_c(t) = c_1\begin{bmatrix} 2\cos(t) - \sin(t) \\ \cos(t) \end{bmatrix} + c_2\begin{bmatrix} \cos(t) + 2\sin(t) \\ \sin(t) \end{bmatrix}$$

Assume particular solution involving similar terms, but note $\cos(t)$ and $\sin(t)$ are solutions in the second row of $\vec{X}_c(t)$. $\therefore$ Let $\vec{g} = \begin{bmatrix} -\cos(t) \\ \sin(t) \end{bmatrix}$

and $\vec{v}(t) = \begin{bmatrix} At\cos(t) + B\cos(t) + Ct\sin(t) + D\sin(t) \\ Et\cos(t) + F\cos(t) + Gt\sin(t) + H\sin(t) \end{bmatrix}$

Using MATLAB to compute $\vec{v}' = A\vec{v} + \vec{g}$

code is continuation from above.

```
syms A B C D E F G H t
v = [A*t*cos(t)+B*cos(t)+C*t*sin(t)+D*sin(t);
     E*t*cos(t)+F*cos(t)+G*t*sin(t)+H*sin(t)];
g = [-cos(t), sin(t)]';
Eq = diff(v,t,1) - Ay*v - g;

collect(Eq,[cos(t),sin(t)])
```

So $v = \begin{bmatrix} A \\ E \end{bmatrix} t\cos(t) + \begin{bmatrix} B \\ F \end{bmatrix}\cos(t) + \begin{bmatrix} C \\ G \end{bmatrix} t\sin(t) + \begin{bmatrix} D \\ H \end{bmatrix}\sin(t)$

ans =

$$\begin{pmatrix} (A - 2B + D + 5F - 2At + Ct + 5Et)\cos(t) + (C - B - 2D + 5H - At - 2Ct + 5Gt)\sin(t) + \cos(\overline{t}) \\ (E - B + 2F + H - At + 2Et + Gt)\cos(t) + (G - F - D + 2H - Ct - Et + 2Gt)\sin(t) - \sin(\overline{t}) \end{pmatrix}$$

$$\therefore \quad \begin{array}{lll} A - 2B + D + 5F + 1 = 0 & & \cos(t) \text{ row } 1 \\ -2A + C + 5E = 0 & & t\cos(t) \text{ row } 1 \\ C - B - 2D + 5H = 0 & & \sin(t) \text{ row } 1 \\ -A - 2C + 5G = 0 & & t\sin(t) \text{ row } 1 \\ E - B + 2F + H = 0 & & \cos(t) \text{ row } 2 \\ -A + 2E + G = 0 & & t\cos(t) \text{ row } 2 \\ G - F - D + 2H - 1 = 0 & & \sin(t) \text{ row } 2 \\ -C - E + 2G = 0 & & t\sin(t) \text{ row } 2 \end{array}$$

In matrix form,

$$\begin{bmatrix} 1 & -2 & 0 & 1 & 0 & 5 & 0 & 0 \\ -2 & 0 & 1 & 0 & 5 & 0 & 0 & 0 \\ 0 & -1 & 1 & -2 & 0 & 0 & 0 & 5 \\ -1 & 0 & -2 & 0 & 0 & 0 & 5 & 0 \\ 0 & -1 & 0 & 0 & 1 & 2 & 0 & 1 \\ -1 & 0 & 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 & 1 & 2 \\ 0 & 0 & -1 & 0 & -1 & 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \\ E \\ F \\ G \\ H \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Using MATLAB,

```
S = [1, -2, 0, 1, 0, 5, 0, 0;
     -2, 0, 1, 0, 5, 0, 0, 0;
     0, -1, 1, -2, 0, 0, 0, 5;
     -1, 0, -2, 0, 0, 0, 5, 0;
     0, -1, 0, 0, 1, 2, 0, 1;
     -1, 0, 0, 0, 2, 0, 1, 0;
     0, 0, 0, -1, 0, -1, 1, 2;
     0, 0, -1, 0, -1, 0, 2, 0];
W = [-1, 0, 0, 0, 0, 0, 1, 0]';
rref([S,W])
```

ans = 8×9

1	0	0	0	0	0	0	0	2
0	1	0	0	0	-2	0	-1	1
0	0	1	0	0	0	0	0	-1
0	0	0	1	0	1	0	-2	-1
0	0	0	0	1	0	0	0	1
0	0	0	0	0	0	1	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
A	B	C	D	E	F	G	H	

$\therefore A = 2 \quad C = -1 \quad E = 1 \quad G = 0$

$$B - 2F - H = 1$$
$$D + F - 2H = -1$$

$\therefore$ System indeterminate. For simplicity, set $D = 0$, $H = 0$ which means $\begin{bmatrix} D \\ H \end{bmatrix} \sin(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore F = -1,\ B = -1 \quad \therefore \begin{bmatrix} B \\ F \end{bmatrix} \cos(t) = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \cos(t)$

$\therefore \ v(t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} t\cos(t) - \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cos(t) - \begin{bmatrix} 1 \\ 0 \end{bmatrix} t\sin(t)$ is a

particular solution. Could also have set $B = 0$, $F = 0$, so $H = -1$, $D = -3$ so $\begin{bmatrix} -3 \\ -1 \end{bmatrix} \sin(t)$ would be in the particular solution.

These variations are included in the general solution with appropriate choice of c_1 and c_2.

Note the general solution in the back of the book can be obtained by multiplying the above first eigenvector $\begin{bmatrix} 2-i \\ 1 \end{bmatrix}$ by $2+i$

$$\boxed{\begin{aligned} \therefore \quad \vec{x}(t) &= \vec{x}_c(t) + \vec{v}(t) \\ &= C_1 \begin{bmatrix} 2\cos(t) - \sin(t) \\ \cos(t) \end{bmatrix} + C_2 \begin{bmatrix} \cos(t) + 2\sin(t) \\ \sin(t) \end{bmatrix} + \\ &\quad \begin{bmatrix} 2 \\ 1 \end{bmatrix} t\cos(t) - \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cos(t) - \begin{bmatrix} 1 \\ 0 \end{bmatrix} t\sin(t) \end{aligned}}$$

3.

Homogeneous solution:

Using MATLAB

```
clear
A = sym([1, 1;
         4, -2]);
[T,D] = eig(A)
```

T =

$$\begin{pmatrix} -\frac{1}{4} & 1 \\ 1 & 1 \end{pmatrix}$$

D =

$$\begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\therefore \quad \vec{x}_c(t) = c_1 \begin{bmatrix} -1 \\ 4 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}$$

Try variation of parameters: assume $\vec{x}_p(t) = \Psi(t)\,\vec{u}(t)$

where $\Psi(t) = \begin{bmatrix} -e^{-3t} & e^{2t} \\ 4e^{-3t} & e^{2t} \end{bmatrix}$. $\therefore \vec{u}'(t) = \Psi^{-1}(t)\,\vec{g}(t)$

where $\vec{g}(t) = \begin{bmatrix} e^{-2t} \\ -2e^{t} \end{bmatrix}$, $\Psi^{-1}(t) = -\frac{1}{5} e^{t} \begin{bmatrix} e^{2t} & -e^{2t} \\ -4e^{-3t} & -e^{-3t} \end{bmatrix}$

Using MATLAB to compute $\Psi^{-1}(t)\,\vec{g}(t)$

```
syms t
Y = [4*T(:,1)*exp(D(1,1)*t),...
     T(:,2)*exp(D(2,2)*t)]
Y_1 = inv(Y)
g = [exp(-2*t); -2*exp(t)]
du = Y_1*g
```

Y =

$$\begin{pmatrix} -e^{-3t} & e^{2t} \\ 4\,e^{-3t} & e^{2t} \end{pmatrix}$$

g =

$$\begin{pmatrix} e^{-2t} \\ -2\,e^{t} \end{pmatrix}$$

Y_1 =

$$\begin{pmatrix} -\frac{e^{3t}}{5} & \frac{e^{3t}}{5} \\ \frac{4\,e^{-2t}}{5} & \frac{e^{-2t}}{5} \end{pmatrix}$$

du =

$$\begin{pmatrix} -\frac{2\,e^{4t}}{5} - \frac{e^{t}}{5} \\ \frac{4\,e^{-4t}}{5} - \frac{2\,e^{-t}}{5} \end{pmatrix}$$

$$\therefore\ \vec{u}'(t) = \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} -\frac{2}{5}e^{4t} - \frac{1}{5}e^{t} \\ \frac{4}{5}e^{-4t} - \frac{2}{5}e^{-t} \end{bmatrix}$$

$$\therefore\ \vec{u}(t) = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{10}e^{4t} - \frac{1}{5}e^{t} \\ -\frac{1}{5}e^{-4t} + \frac{2}{5}e^{-t} \end{bmatrix}$$

$\therefore\ \vec{x}_p(t) = \Psi(t)\,\vec{u}(t)$ Continue using MATLAB,

```
u = int(du)
xp = simplify(Y*u)
```

u =

$$\begin{pmatrix} -\frac{e^{t}\,(e^{3t}+2)}{10} \\ \frac{2\,e^{-t}}{5} - \frac{e^{-4t}}{5} \end{pmatrix}$$

xp =

$$\begin{pmatrix} \frac{e^{t}}{2} \\ -e^{-2t} \end{pmatrix}$$

$$\therefore\ \vec{x}_p(t) = \begin{bmatrix} \frac{1}{2}e^{t} \\ -e^{-2t} \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^{-2t} + \frac{1}{2}\begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{t}$$

$$\therefore\ \boxed{\vec{x}(t) = \vec{x}_c + \vec{x}_p = c_1\begin{bmatrix} -1 \\ 4 \end{bmatrix} e^{-3t} + c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^{-2t} + \frac{1}{2}\begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{t}}$$

4.

Homogeneous solution, using MATLAB

```
clear
A = sym([4, -2;
         8, -4]);
[T,D] = eig(A)
```

T =

$$\begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}$$

D =

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$\therefore$ Repeated eigenvalue. $\vec{x}^{(1)}(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{0t} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Find second generalized eigenvector.

$\therefore$ Let $\vec{\epsilon} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. $\vec{x}^{(2)} = \vec{\epsilon}t + \vec{n}$

$\therefore$ $(A - \lambda I)\vec{n} = \vec{\epsilon}$ with $\lambda = 0$ becomes $A\vec{n} = \vec{\epsilon}$

or $\begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix}\begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 4 & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\therefore$ $4n_1 - 2n_2 = 1$. Let $n_2 = k$, $\therefore$ $n_1 = \frac{1}{4} + \frac{1}{2}k$

$\therefore$ $\vec{n} = \begin{bmatrix} \frac{1}{4} + \frac{1}{2}k \\ k \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ 0 \end{bmatrix} + k\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$

$\therefore$ $\vec{x}^{(2)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}t + \begin{bmatrix} 1/4 \\ 0 \end{bmatrix} + k\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$

The last term is just a multiple of $\vec{x}^{(1)}$, $\therefore$ drop

$\therefore$ $\vec{x}_c(t) = c_1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}t + \begin{bmatrix} 1/4 \\ 0 \end{bmatrix}\right)$

Note: if chose $n_1 = k$ above, then $n_2 = 2k - \frac{1}{2}$,

so $\vec{n} = \begin{bmatrix} k \\ 2k-\frac{1}{2} \end{bmatrix} = k\begin{bmatrix} 1 \\ 2 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, or $\vec{n} = -\frac{1}{2}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

which gives answer in back of book.

Particular solution: try variation of parameters

using $\Psi(t) = \begin{bmatrix} 1 & t+\frac{1}{4} \\ 2 & 2t \end{bmatrix}$, $\vec{g} = \begin{bmatrix} t^{-3} \\ -t^{-2} \end{bmatrix}$

Assume $\vec{x_p} = \Psi(t)\,\vec{u}(t)$ $\therefore$ $\vec{u}'(t) = \Psi^{-1}(t)\,\vec{g}(t)$

Using MATLAB,

```
syms t
Y = [1, t+1/4;...
     2, 2*t]
Y_1 = inv(Y)
g = [t^(-3); -t^(-2)]
du = Y_1*g

u = int(du)
xp = Y*u
```

Y =

$$\begin{pmatrix} 1 & t+\frac{1}{4} \\ 2 & 2\,t \end{pmatrix}$$

Y_1 =

$$\begin{pmatrix} -4\,t & 2\,t+\frac{1}{2} \\ 4 & -2 \end{pmatrix}$$

g =

$$\begin{pmatrix} \frac{1}{t^3} \\ -\frac{1}{t^2} \end{pmatrix}$$

du =

$$\begin{pmatrix} -\frac{2\,t+\frac{1}{2}}{t^2} - \frac{4}{t^2} \\ \frac{2}{t^2} + \frac{4}{t^3} \end{pmatrix}$$

u =

$$\begin{pmatrix} \frac{9}{2\,t} - 2\log(t) \\ -\frac{2\,(t+1)}{t^2} \end{pmatrix}$$

xp =

$$\begin{pmatrix} \frac{9}{2\,t} - 2\log(t) - \frac{2\,(t+1)\,\left(t+\frac{1}{4}\right)}{t^2} \\ \frac{9}{t} - \frac{4\,(t+1)}{t} - 4\log(t) \end{pmatrix}$$

$$\therefore \vec{x_p}(t) = \begin{bmatrix} \frac{9}{2}t^{-1} - 2\ln(t) - 2 - \frac{5}{2}t^{-1} - \frac{1}{2}t^{-2} \\ 9t^{-1} - 4 - 4t^{-1} - 4\ln(t) \end{bmatrix}$$

$$= \begin{bmatrix} 2t^{-1} - 2\ln(t) - \frac{1}{2}t^{-2} - 2 \\ 5t^{-1} - 4\ln(t) - 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 5 \end{bmatrix} \frac{1}{t} - \begin{bmatrix} 2 \\ 4 \end{bmatrix} \ln(t) - \frac{1}{2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} t^{-2} - \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

The last term can be incorporated into the $c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ term of $\vec{x}_c(t)$. $\therefore$ drop

$$\therefore \boxed{\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} t + \begin{bmatrix} 1/4 \\ 0 \end{bmatrix} \right) + \begin{bmatrix} 2 \\ 5 \end{bmatrix} \frac{1}{t} - \frac{1}{2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} t^{-2} - \begin{bmatrix} 2 \\ 4 \end{bmatrix} \ln(t)}$$

5.

Homogeneous solution, using MATLAB,

```
clear
A = sym([1, 1;
         4, 1]);
[T,D] = eig(A)
```

T =

$$\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ 1 & 1 \end{pmatrix}$$

D =

$$\begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}$$

$$\therefore \vec{x}_c(t) = c_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t}$$

Particular solution: try undetermined coefficients

Let $\vec{g} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} e^{t}$. Since e^t is not a component of the homogeneous solution, try $\vec{v} = \begin{bmatrix} c \\ d \end{bmatrix} e^t \quad \therefore \vec{v}\,' = A\vec{v} + \vec{g}$

$$\therefore \begin{bmatrix} c \\ d \end{bmatrix} e^t = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} e^t + \begin{bmatrix} 2 \\ -1 \end{bmatrix} e^t$$

Or, $(I - A) \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

Using MATLAB, continuing from above,

```
g = [2; -1];
v = linsolve(eye(2)-A, g)
```

v =

$$\begin{pmatrix} \frac{1}{4} \\ -2 \end{pmatrix}$$

$$\therefore \vec{v} = \begin{bmatrix} 1/4 \\ -2 \end{bmatrix} e^t$$

$$\therefore \boxed{\begin{aligned} \vec{x}(t) &= \vec{x}_c(t) + \vec{v}(t) \\ &= c_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + \begin{bmatrix} 1/4 \\ -2 \end{bmatrix} e^t \end{aligned}}$$

6.

Homogeneous solution, using MATLAB.

```
clear
A = sym([2, -1;
         3, -2]);
[T,D] = eig(A)
```

T =

$$\begin{pmatrix} 1 & \frac{1}{3} \\ 1 & 1 \end{pmatrix}$$

D =

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\therefore \vec{x}_c(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-t}$$

Particular solution: try undetermined coefficients

Let $\vec{g} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{t}$ Since e^{t} is part of $\vec{x}_c(t)$,

try $\vec{x}_p(t) = \vec{a}\, t e^{t} + \vec{b}\, e^{t}$, $\vec{a}, \vec{b}$ to be determined.

$$\therefore \vec{x}_p{}' = \vec{a} e^{t} + \vec{a}\, t e^{t} + \vec{b} e^{t}$$

$$A\vec{x}_p + \vec{g} = A\vec{a}\, t e^{t} + A\vec{b}\, e^{t} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{t}$$

$\therefore$ From $\vec{x}_p{}' = A\vec{x}_p + \vec{g}$, and dividing by e^{t},

$$\vec{a} + \vec{a}\, t + \vec{b} = A\vec{a}\, t + A\vec{b} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Equating like terms,

$$A\vec{a} = \vec{a} \quad , \quad (A - I)\vec{b} = \vec{a} - \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\therefore$ $\vec{a}$ is an eigenvector with eigenvalue $\lambda = 1$.

As above, set $\vec{a} = \begin{bmatrix} K \\ K \end{bmatrix}$, K a constant

$\therefore$ Solve $(A - I)\vec{b} = \begin{bmatrix} K \\ K \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} K-1 \\ K+1 \end{bmatrix}$

$$A - I = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix}$$

Let $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ $\therefore$ $\begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} k-1 \\ k+1 \end{bmatrix}$

Since $A-I$ has rank 1 (row 2 = 3 times row 1), to get a consistent solution, $3(k-1) = k+1$, or $\underline{k=2}$. $\therefore$ $\vec{a} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

$\therefore$ $\begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ $\therefore$ Set $b_1 = 1$, $b_2 = 0$

$\therefore$ $\vec{x_p} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} te^t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t$

$$\therefore \boxed{\begin{aligned} \vec{x}(t) &= \vec{x_c}(t) + \vec{x_p}(t) \\ &= c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-t} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} te^t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t \end{aligned}}$$

7.

Homogeneous solution, using MATLAB,

```
clear
A = sym([-3, sqrt(2);
          sqrt(2), -2]);
[T,D] = eig(A)
```

T =

$$\begin{pmatrix} -\sqrt{2} & \frac{\sqrt{2}}{2} \\ 1 & 1 \end{pmatrix}$$

D =

$$\begin{pmatrix} -4 & 0 \\ 0 & -1 \end{pmatrix}$$

Multiplying 2nd eigenvector by $\sqrt{2}$ to get "pretty" numbers,

$$\therefore \vec{x}_c(t) = c_1 \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix} e^{-4t} + c_2 \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} e^{-t}$$

Try diagonalization to get complete solution.

Let $\vec{x} = T\vec{y}$, $\vec{g} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}$, where $T = \begin{bmatrix} -\sqrt{2} & \sqrt{2}/2 \\ 1 & 1 \end{bmatrix}$

$$\therefore \vec{x}' = T\vec{y}' = A(T\vec{y}) + \vec{g}$$

Or, $\vec{y}' = (T^{-1}AT)\vec{y} + T^{-1}\vec{g}$

$T^{-1}AT = \begin{bmatrix} -4 & 0 \\ 0 & -1 \end{bmatrix}$, The eigenvalue matrix

Note $\det(T) = -\frac{3}{2}\sqrt{2}$, $\therefore T^{-1} = -\frac{\sqrt{2}}{3}\begin{bmatrix} 1 & -\sqrt{2}/2 \\ -1 & -\sqrt{2} \end{bmatrix}$

Or, $T^{-1} = \frac{1}{3}\begin{bmatrix} -\sqrt{2} & 1 \\ \sqrt{2} & 2 \end{bmatrix}$

$$\therefore T^{-1}\vec{g} = \frac{1}{3}\begin{bmatrix} -\sqrt{2} & 1 \\ \sqrt{2} & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} = \frac{1}{3}\begin{bmatrix} -\sqrt{2} & -1 \\ \sqrt{2} & -2 \end{bmatrix} e^{-t}$$

$$\therefore \quad \vec{y}\,' = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} -\frac{\sqrt{2}}{3} - \frac{1}{3} \\ \frac{\sqrt{2}}{3} - \frac{2}{3} \end{bmatrix} e^{-t}$$

$$\therefore \quad y_1' + 4y_1 = \left(-\frac{\sqrt{2}}{3} - \frac{1}{3}\right) e^{-t}$$

$$y_2' + y_2 = \left(\frac{\sqrt{2}}{3} - \frac{2}{3}\right) e^{-t}$$

$$\therefore \quad \frac{d}{dt}\left(y_1 e^{4t}\right) = \left(-\frac{\sqrt{2}}{3} - \frac{1}{3}\right) e^{3t}$$

$$\frac{d}{dt}\left(y_2 e^{t}\right) = \left(\frac{\sqrt{2}}{3} - \frac{2}{3}\right)$$

$$\therefore \quad y_1 = \left(-\frac{\sqrt{2}}{9} - \frac{1}{9}\right) e^{-t} + c_1 e^{-4t}$$

$$y_2 = \left(\frac{\sqrt{2}}{3} - \frac{2}{3}\right) t e^{-t} + c_2 e^{-t}$$

$$\therefore \quad \vec{x} = T\vec{y} = \begin{bmatrix} -\sqrt{2} & \frac{\sqrt{2}}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \left(-\frac{\sqrt{2}}{9} - \frac{1}{9}\right) e^{-t} + c_1 e^{-4t} \\ \left(\frac{\sqrt{2}}{3} - \frac{2}{3}\right) t e^{-t} + c_2 e^{-t} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{9} e^{-t} + \frac{\sqrt{2}}{9} e^{-t} - c_1 \sqrt{2}\, e^{-4t} + \frac{1}{3} t e^{-t} - \frac{\sqrt{2}}{3} t e^{-t} + c_2 \frac{\sqrt{2}}{2} e^{-t} \\ -\frac{\sqrt{2}}{9} e^{-t} - \frac{1}{9} e^{-t} + c_1 e^{-4t} + \frac{\sqrt{2}}{3} t e^{t} - \frac{2}{3} t e^{-t} + c_2 e^{-t} \end{bmatrix}$$

$$= \boxed{c_1 \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix} e^{-4t} + c_2 \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 1 \end{bmatrix} e^{-t} + \frac{1}{9} \begin{bmatrix} 2+\sqrt{2} \\ -1-\sqrt{2} \end{bmatrix} e^{-t} + \frac{1}{3} \begin{bmatrix} 1-\sqrt{2} \\ \sqrt{2}-2 \end{bmatrix} t e^{-t}}$$

Note the diagonalization includes the complete solution, not just the particular solution.

A complete MATLAB solution is as follows:

```
clear
A = sym([-3, sqrt(2);
         sqrt(2), -2]);
[T,D] = eig(A)

syms y1(t) y2(t)
y = [y1; y2];
g = [1; -1]*exp(-t);
% MATLAB prefers T\A to inv(T)*A
eqn = diff(y,t) == T\A*T*y + T\g
% solve for y1, y2
ySol = dsolve(eqn);
% dsolve gives a MATLAB structure
%   with fields accessed using dot
y = [ySol.y1; ySol.y2]
x = expand(T*y);
% make it somewhat readable
collect(x,exp(-t))
```

T =

$$\begin{pmatrix} -\sqrt{2} & \frac{\sqrt{2}}{2} \\ 1 & 1 \end{pmatrix}$$

D =

$$\begin{pmatrix} -4 & 0 \\ 0 & -1 \end{pmatrix}$$

eqn(t) =

$$\begin{pmatrix} \frac{\partial}{\partial t} y_1(t) = -4\, y_1(t) - \frac{e^{-t}\,(\sqrt{2}+1)}{3} \\ \frac{\partial}{\partial t} y_2(t) = -y_2(t) - \frac{\sqrt{2}\,e^{-t}\,(\sqrt{2}-1)}{3} \end{pmatrix}$$

y =

$$\begin{pmatrix} e^{-4t}\left(C_4 - \frac{e^{3t}\,(\sqrt{2}+1)}{9}\right) \\ e^{-t}\left(C_3 + t\left(\frac{\sqrt{2}}{3} - \frac{2}{3}\right)\right) \end{pmatrix}$$

ans =

$$\begin{pmatrix} \left(\frac{t}{3} + \frac{\sqrt{2}\,C_3}{2} - \frac{\sqrt{2}\,t}{3} + \frac{\sqrt{2}}{9} + \frac{2}{9}\right) e^{-t} - \sqrt{2}\,C_4\,e^{-4t} \\ \left(C_3 - \frac{2t}{3} + \frac{\sqrt{2}\,t}{3} - \frac{\sqrt{2}}{9} - \frac{1}{9}\right) e^{-t} + C_4\,e^{-4t} \end{pmatrix}$$

Separating the components into separate vectors yields the answer above.

8.

Homogeneous solution, using MATLAB.

```
clear
A = sym([2, -5;
         1, -2]);
[T,D] = eig(A)
```

T =

$$\begin{pmatrix} 2-i & 2+i \\ 1 & 1 \end{pmatrix}$$

D =

$$\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

$$\therefore \vec{x}^{(2)}(t) = \begin{bmatrix} 2+i \\ 1 \end{bmatrix} \left(\cos(t) + i \sin(t) \right)$$

$$= \begin{bmatrix} 2\cos(t) - \sin(t) \\ \cos(t) \end{bmatrix} + i \begin{bmatrix} \cos(t) + 2\sin(t) \\ \sin(t) \end{bmatrix}$$

using the second eigenvector/eigenvalue

$$\therefore \vec{x}_c(t) = c_1 \begin{bmatrix} 2\cos(t) - \sin(t) \\ \cos(t) \end{bmatrix} + c_2 \begin{bmatrix} \cos(t) + 2\sin(t) \\ \sin(t) \end{bmatrix}$$

The eigenvalue matrix $= \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$ makes diagonalization seem unattractive.

Try a Laplace transform for a particular solution. Assume $\vec{x}(0) = \vec{0}$, $\vec{g}(t) = \begin{bmatrix} 0 \\ \cos(t) \end{bmatrix}$

$$\therefore \mathcal{L}\{\vec{x}(t)\} = X(s), \quad \mathcal{L}\{\vec{x}'(t)\} = sX(s) - \vec{x}(0)$$

$$\mathcal{L}\{\vec{g}(t)\} = G(s) = \begin{bmatrix} 0 \\ \frac{s}{s^2+1} \end{bmatrix}$$

$$\therefore \; sX(s) = AX(s) + G(s)$$

$$(sI - A)X(s) = G(s), \quad X(s) = (sI - A)^{-1} G(s)$$

$$sI - A = \begin{bmatrix} s-2 & 5 \\ -1 & s+2 \end{bmatrix} \qquad \det(sI - A) = s^2 + 1$$

$$\therefore \; (sI - A)^{-1} = \frac{1}{s^2+1} \begin{bmatrix} s+2 & -5 \\ 1 & s-2 \end{bmatrix}$$

$$\therefore \; (sI - A)^{-1} G(s) = \frac{1}{s^2+1} \begin{bmatrix} \dfrac{-5s}{s^2+1} \\ \dfrac{s(s-2)}{s^2+1} \end{bmatrix} = \begin{bmatrix} \dfrac{-5s}{(s^2+1)^2} \\ \dfrac{s(s-2)}{(s^2+1)^2} \end{bmatrix}$$

Using MATLAB for the inverse transform

```
syms s
x = [ilaplace(-5*s/(s^2 + 1)^2);
     ilaplace(s*(s-2)/(s^2 + 1)^2)]
```

```
x =
```

$$\begin{pmatrix} -\dfrac{5\,t\sin(t)}{2} \\ \dfrac{\sin(t)}{2} + \dfrac{t\cos(t)}{2} - t\sin(t) \end{pmatrix}$$

$$\therefore \; \vec{X}_p(t) = \begin{bmatrix} -5/2 \\ -1 \end{bmatrix} t\sin(t) + \begin{bmatrix} 0 \\ 1/2 \end{bmatrix} t\cos(t) + \begin{bmatrix} 0 \\ 1/2 \end{bmatrix} \sin(t)$$

$$\boxed{\begin{aligned} \vec{X}(t) = \vec{X}_c + \vec{X}_p &= c_1 \begin{bmatrix} 2\cos(t) - \sin(t) \\ \cos(t) \end{bmatrix} + c_2 \begin{bmatrix} \cos(t) + 2\sin(t) \\ \sin(t) \end{bmatrix} \\ &+ \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} t\cos(t) - \begin{bmatrix} 5/2 \\ 1 \end{bmatrix} t\sin(t) + \begin{bmatrix} 0 \\ 1/2 \end{bmatrix} \sin(t) \end{aligned}}$$

Note by choosing $c_1 = 2$, $c_2 = 1$ and then $c_1 = -1$, $c_2 = 2$, the homogeneous solution in the back of the book is obtained (or scale the first eigenvector by $2+i$).

The particular solution in the back of the book, call it $\vec{v}(t)$, can be obtained by choosing $c_1 = -1$, $c_2 = -\frac{1}{2}$. The resulting homogeneous terms, when added to $\begin{bmatrix} 0 \\ 1/2 \end{bmatrix} \sin(t)$, will yield $\begin{bmatrix} -5/2 \\ -1 \end{bmatrix} \cos(t)$ in the last term of $\vec{v}(t)$.

9.

(a)

Using MATLAB,

```
clear
A = sym([-1/2, -1/8;
           2,    -1/2]);
[T,D] = eig(A)
```

T =

$$\begin{pmatrix} -\frac{1}{4}\,\mathrm{i} & \frac{1}{4}\,\mathrm{i} \\ 1 & 1 \end{pmatrix}$$

D =

$$\begin{pmatrix} -\frac{1}{2}-\frac{1}{2}\,\mathrm{i} & 0 \\ 0 & -\frac{1}{2}+\frac{1}{2}\,\mathrm{i} \end{pmatrix}$$

Using the first eigenvector, scaled by 4,

$$\vec{x}^{(1)} = \begin{bmatrix} -i \\ 4 \end{bmatrix} e^{-t/2}\left(\cos(t/2) - i\sin(t/2)\right)$$

$$= \begin{bmatrix} -e^{-t/2}\sin(t/2) \\ 4e^{-t/2}\cos(t/2) \end{bmatrix} + i\begin{bmatrix} -e^{-t/2}\cos(t/2) \\ -4e^{-t/2}\sin(t/2) \end{bmatrix}$$

Using the real and imaginary parts, multiplying by -1 to minimize negative signs,

$$\boxed{\Psi(t) = \begin{bmatrix} e^{-t/2}\sin(t/2) & e^{-t/2}\cos(t/2) \\ -4e^{-t/2}\cos(t/2) & 4e^{-t/2}\sin(t/2) \end{bmatrix}}$$

(b)

Use Laplace transform method.

Here, $\vec{g}(t) = \begin{bmatrix} \frac{1}{2}e^{-t/2} \\ 0 \end{bmatrix}$

$\therefore \mathcal{L}\{\vec{g}(t)\} = G(s) = \begin{bmatrix} \frac{1}{2(s+\frac{1}{2})} \\ 0 \end{bmatrix}$ using Table 6.2.1

Let $\mathcal{L}\{\vec{x}(t)\} = X(s)$

$$\mathcal{L}\{\vec{x}\,'(t)\} = s\,\mathcal{L}\{\vec{x}(t)\} - \vec{x}(0) = sX(s)$$

$$\therefore \quad sX(s) = AX(s) + G(s)$$

$$(sI - A)X(s) = G(s), \quad X(s) = (sI-A)^{-1}G(s)$$

Using MATLAB, continuing code from (a),

```
syms t s
g = [1/2*exp(-t/2); 0]
G = laplace(g)
M = (s*eye(2) - A)
% MATLAB prefers M\G
%   to inv(M)*G
X = M\G
xSol = ilaplace(X)
```

G =

$$\begin{pmatrix} \dfrac{1}{2\left(s+\frac{1}{2}\right)} \\ 0 \end{pmatrix}$$

M =

$$\begin{pmatrix} s+\frac{1}{2} & \frac{1}{8} \\ -2 & s+\frac{1}{2} \end{pmatrix}$$

X =

$$\begin{pmatrix} \dfrac{1}{2s^2+2s+1} \\ \dfrac{4}{(2s+1)\,(2s^2+2s+1)} \end{pmatrix}$$

xSol =

$$\begin{pmatrix} \sin\left(\frac{t}{2}\right)\mathrm{e}^{-\frac{t}{2}} \\ 4\,\mathrm{e}^{-\frac{t}{2}} - 4\cos\left(\frac{t}{2}\right)\mathrm{e}^{-\frac{t}{2}} \end{pmatrix}$$

$$\therefore \quad \vec{x}(t) = e^{-t/2}\begin{bmatrix} \sin(t/2) \\ 4 - 4\cos(t/2) \end{bmatrix}$$

Note: Since $\Psi(t)$ was found in (a), could have used variation of parameters, $\vec{u}\,'(t) = \Psi(t)^{-1}\vec{g}(t)$, then $\vec{u} = \int \Psi^{-1}\vec{g}$, then $\vec{x} = \vec{c}\,\Psi(t) + \Psi(t)\vec{u}(t)$ then use $\vec{x}(0) = \vec{0}$ to determine $\vec{c}$, then simplify.

10.

(a) Using MATLAB,

```
clear
syms c1 c2 t
A = sym([2, -1;
         3, -2]);
Xc = c1*[1;1]*t + c2*[1;3]*(1/t);
test = t*diff(Xc,t) - A*Xc
simplify(test)
```

test =

$$\begin{pmatrix} t\left(c_1 - \frac{c_2}{t^2}\right) - c_1\,t + \frac{c_2}{t} \\ t\left(c_1 - \frac{3\,c_2}{t^2}\right) - c_1\,t + \frac{3\,c_2}{t} \end{pmatrix}$$

ans =

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore \; t\,\vec{x}^{(c)}(t)' = A\,\vec{x}^{(c)}(t)$$

(b) Use variation of parameters

Let $\Psi(t) = \begin{bmatrix} t & 1/t \\ t & 3/t \end{bmatrix}$ from $\vec{x}^{(c)}$ above

Assume $\vec{x}_p(t) = \Psi(t)\,\vec{u}(t)$, $\vec{u}$ a 2x1 vector.

$\therefore \; t(\Psi'\vec{u} + \Psi\vec{u}\,') = A\Psi\vec{u} + \vec{g}$, $\vec{g} = \begin{bmatrix} 1-t^2 \\ 2t \end{bmatrix}$

$\therefore \; t\Psi'\vec{u} + t\Psi\vec{u}\,' = A\Psi\vec{u} + \vec{g}$

From (a), $t\Psi' = A\Psi$, so $t\Psi'\vec{u} = A\Psi\vec{u}$

$\therefore\ t\Psi\vec{u}\,' = \vec{g}\ ,\quad \vec{u}\,' = (t\Psi)^{-1}\vec{g}\ ,$ if $(t\Psi)^{-1}$ exists.

But $t\Psi = \begin{bmatrix} t^2 & 1 \\ t^2 & 3 \end{bmatrix}$, $\therefore (t\Psi)^{-1} = \dfrac{1}{2t^2}\begin{bmatrix} 3 & -1 \\ -t^2 & t^2 \end{bmatrix}$

$$\therefore\ \vec{u}\,' = \frac{1}{2t^2}\begin{bmatrix} 3 & -1 \\ -t^2 & t^2 \end{bmatrix}\begin{bmatrix} 1-t^2 \\ 2t \end{bmatrix}$$

Using MATLAB, code continuation from above.

```
g = [1-t^2; 2*t];
Y = [t, 1/t;
     t, 3/t];
tY = t*Y;
% MATLAB for du = inv(tY)*g
du = tY\g;
u = int(du)
Xp = simplify(Y*u)
```

u =

$$\begin{pmatrix} -\frac{3\,t}{2} - \log(t) - \frac{3}{2\,t} \\ \frac{t\,(t^2+3\,t-3)}{6} \end{pmatrix}$$

Xp =

$$\begin{pmatrix} \frac{t}{2} - t\log(t) - \frac{4\,t^2}{3} - 2 \\ \frac{3\,t}{2} - t\log(t) - t^2 - 3 \end{pmatrix}$$

$$\therefore\ \vec{X}_p = \frac{1}{2}\begin{bmatrix} 1 \\ 3 \end{bmatrix} t - \begin{bmatrix} 1 \\ 1 \end{bmatrix} t\ln(t) - \begin{bmatrix} 4/3 \\ 1 \end{bmatrix} t^2 - \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\therefore\ \boxed{\begin{aligned} \vec{x} &= \vec{x}^{(c)} + \vec{X}_p \\ &= c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} t + c_2\begin{bmatrix} 1 \\ 3 \end{bmatrix} t^{-1} \\ &\quad + \frac{1}{2}\begin{bmatrix} 1 \\ 3 \end{bmatrix} t - \begin{bmatrix} 1 \\ 1 \end{bmatrix} t\ln(t) - \begin{bmatrix} 4/3 \\ 1 \end{bmatrix} t^2 - \begin{bmatrix} 2 \\ 3 \end{bmatrix} \end{aligned}}$$

11.

(a) Using MATLAB,

```
clear
syms c1 c2 t
A = sym([3, -2;
         2, -2]);
% fundamental matrix
Y = [1/t, 2*t^2;
     2/t, t^2];
% homogeneous solution
Xc = c1*Y(:,1) + c2*Y(:,2);
test = t*diff(Xc,t) - A*Xc
simplify(test)
```

test =

$$\begin{pmatrix} \frac{c_1}{t} - 4\,c_2\,t^2 + t\left(4\,c_2\,t - \frac{c_1}{t^2}\right) \\ \frac{2\,c_1}{t} - 2\,c_2\,t^2 + t\left(2\,c_2\,t - \frac{2\,c_1}{t^2}\right) \end{pmatrix}$$

ans =

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore\ t\,\vec{x}^{(c)}(t)' = A\,\vec{x}^{(c)}(t)$$

(b) Use variation of parameters since $\Psi(t)$ is known.

Let $\vec{x}_p(t) = \Psi(t)\,\vec{u}(t)$, $\vec{u}$ a 2×1 vector.

From #10 above, $\vec{u}\,' = (t\Psi)^{-1}\vec{g}$, where

$$\vec{g} = \begin{bmatrix} -2t \\ t^4-1 \end{bmatrix} \text{ and } \Psi(t) = \begin{bmatrix} 1/t & 2t^2 \\ 2/t & t^2 \end{bmatrix}$$

Using MATLAB, code continuation from above,

```
g = [-2*t; t^4-1];
tY = t*Y;
% MATLAB for du = inv(tY)*g
du = tY\g;
u = int(du)
Xp = simplify(Y*u)
```

u =

$$\begin{pmatrix} \frac{t\,(2\,t^4 + 5\,t - 10)}{15} \\ -\frac{t^4 - 8\,t + 1}{6\,t^2} \end{pmatrix}$$

Xp =

$$\begin{pmatrix} -\frac{t^4}{5} + 3\,t - 1 \\ \frac{t^4}{10} + 2\,t - \frac{3}{2} \end{pmatrix}$$

$$\therefore \vec{x}_p(t) = \frac{1}{10}\begin{bmatrix}-2\\1\end{bmatrix}t^4 + \begin{bmatrix}3\\2\end{bmatrix}t - \frac{1}{2}\begin{bmatrix}2\\3\end{bmatrix}$$

$$\therefore \boxed{\begin{aligned}\vec{x}(t) &= \vec{x}^{(c)} + \vec{x}_p \\ &= c_1\begin{bmatrix}1\\2\end{bmatrix}t^{-1} + c_2\begin{bmatrix}2\\1\end{bmatrix}t^2 \\ &\quad + \frac{1}{10}\begin{bmatrix}-2\\1\end{bmatrix}t^4 + \begin{bmatrix}3\\2\end{bmatrix}t - \frac{1}{2}\begin{bmatrix}2\\3\end{bmatrix}\end{aligned}}$$

12.

Given $\vec{\phi}' = P\vec{\phi} + \vec{g}$ and $\vec{v}' = P\vec{v} + \vec{g}$,

then $(\vec{\phi} - \vec{v})' = \vec{\phi}' - \vec{v}' = (P\vec{\phi} + \vec{g}) - (P\vec{v} + \vec{g})$

$\therefore (\vec{\phi} - \vec{v})' = P(\vec{\phi} - \vec{v})$, so $\vec{\phi} - \vec{v}$ solves the homogeneous system. Since $\vec{\phi}$ is a general solution, then $\vec{\phi} - \vec{v}$ must be a general solution to the homogeneous system.

$\therefore \vec{\phi} - \vec{v} = \vec{u} \Rightarrow \vec{\phi} = \vec{u} + \vec{v}$

Perhaps more convincingly, let $\vec{u}$ = the general solution to $\vec{x}' = P\vec{x}$. Consider $\vec{\phi} - \vec{u}$

$$\therefore (\vec{\phi} - \vec{u})' = \vec{\phi}' - \vec{u}' = (P\vec{\phi} + \vec{g}) - (P\vec{u})$$

$$= P(\vec{\phi} - \vec{u}) + \vec{g}$$

$\therefore$ $\vec{\phi} - \vec{u}$ is a particular solution to $\vec{x}' = P\vec{x} + \vec{g}$.

$\therefore$ By the uniqueness theorem, $\vec{\phi} - \vec{u} = \vec{v}$

$\therefore$ $\vec{\phi} = \vec{u} + \vec{v}$

13.

(a)

Let $x_1 = y$, $x_2 = y'$

$$\therefore x_1' = x_2, \quad x_2' = y'' = 5y' - 6y + 2e^t$$

$$= 5x_2 - 6x_1 + 2e^t$$

Or, $x_1' = 0 \cdot x_1 + 1x_2$

$x_2' = 5x_2 - 6x_1 + 2e^t$

Let $\vec{g}(t) = \begin{bmatrix} 0 \\ 2e^t \end{bmatrix}$

$\therefore$ In matrix form,

$$\therefore \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2e^t \end{bmatrix}$$

(b)

$$\vec{x}^{(1)}(t) = \begin{bmatrix} y_1 \\ y_1' \end{bmatrix} = \begin{bmatrix} e^{2t} \\ 2e^{2t} \end{bmatrix}, \quad \vec{x}^{(2)}(t) = \begin{bmatrix} y_2 \\ y_2' \end{bmatrix} = \begin{bmatrix} e^{3t} \\ 3e^{3t} \end{bmatrix}$$

$$\therefore \frac{d}{dt}\vec{x}^{(1)}(t) = \begin{bmatrix} 2e^{2t} \\ 4e^{2t} \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix} \begin{bmatrix} e^{2t} \\ 2e^{2t} \end{bmatrix} = \begin{bmatrix} 2e^{2t} \\ 4e^{2t} \end{bmatrix}$$

$\therefore \vec{x}^{(1)}(t)$ is a solution to $\vec{x}' = A\vec{x}$

$$\frac{d}{dt}\vec{x}^{(2)}(t) = \begin{bmatrix} 3e^{3t} \\ 9e^{3t} \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix} \begin{bmatrix} e^{3t} \\ 3e^{3t} \end{bmatrix} = \begin{bmatrix} 3e^{3t} \\ 9e^{3t} \end{bmatrix}$$

$\therefore \vec{x}^{(2)}(t)$ is a solution to $\vec{x}' = A\vec{x}$

(c)

Note: y_1 and y_2 are solutions to: $y'' - 5y + 6y = 0$

Following the outline of Section 3.6, pp. 144-145,

$$y' = y_1' u_1 + y_1 u_1' + y_2' u_2 + y_2 u_2'$$

To prevent higher order derivatives of the unknown coefficients (u_1 and u_2) which will occur when examining y'', set

$$y_1 u_1' + y_2 u_2' = 0 \qquad [1]$$

$$\therefore\ y' = y_1' u_1 + y_2' u_2$$

$$\therefore\ y'' = y_1'' u_1 + y_1' u_1' + y_2'' u_2 + y_2' u_2'$$

$$\therefore\ y'' - 5y' + 6y$$

$$= y_1'' u_1 + y_1' u_1' + y_2'' u_2 + y_2' u_2'$$

$$-5(y_1' u_1 + y_2' u_2)$$

$$+6(y_1 u_1 + y_2 u_2)$$

$$= u_1 \underbrace{(y_1'' - 5y_1' + 6y_1)}_{=0} + u_2 \underbrace{(y_2'' - 5y_2' + 6y_2)}_{=0}$$

$$+ y_1' u_1' + y_2' u_2' = 2e^t$$

since y_1 and y_2 are solutions to $y'' - 5y + 6y = 0$

$$\therefore\ y_1' u_1' + y_2' u_2' = 2e^t \qquad [2]$$

$\therefore$ From [1], [2]: $\boxed{\begin{array}{l} e^{2t}u_1' + e^{3t}u_2' = 0 \\ 2e^{2t}u_1' + 3e^{3t}u_2' = 2e^{t} \end{array}}$

(d)

Assume $\vec{x} = \Psi(t)\,\vec{u}(t)$. From above, this is, in matrix form:

$$\begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} y_1 u_1 + y_2 u_2 \\ y_1' u_1 + y_2' u_2 \end{bmatrix}$$

As shown in text on p. 349, this leads to

$\Psi(t)\,\vec{u}\,'(t) = \vec{g}(t)$, or $\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$

From above, $\vec{g}(t) = \begin{bmatrix} 0 \\ 2e^{t} \end{bmatrix}$. $\therefore$ The last

equation can be written $\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ g_2 \end{bmatrix}$

$\therefore$ With $\Psi(t) = \begin{bmatrix} e^{2t} & e^{3t} \\ 2e^{2t} & 3e^{3t} \end{bmatrix}$ and $\vec{g}(t) = \begin{bmatrix} 0 \\ 2e^{t} \end{bmatrix}$

$$\therefore \begin{bmatrix} e^{2t} & e^{3t} \\ 2e^{2t} & 3e^{3t} \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 2e^{t} \end{bmatrix}$$

(e)

Expansion of matrix equation in (d)

$$\begin{bmatrix} e^{2t} & e^{3t} \\ 2e^{2t} & 3e^{3t} \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 2e^{t} \end{bmatrix}$$

yields: $e^{2t}u_1' + e^{3t}u_2' = 0$

$$2e^{2t}u_1' + 3e^{3t}u_2' = 2e^{t}$$

which are the same two equations in (c)

14.

(a)

Let $x_1 = y$, $x_2 = y'$

$\therefore x_1' = x_2$

$$x_2' = y'' = \frac{t(t+2)}{t^2} y' - \frac{(t+2)}{t^2} y + 2t$$

$$= \frac{t+2}{t} x_2 - \left(\frac{t+2}{t^2}\right) x_1 + 2t$$

$$\therefore \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{(t+2)}{t^2} & \frac{(t+2)}{t} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2t \end{bmatrix}$$

(b)

$$\vec{x}^{(1)}(t) = \begin{bmatrix} y_1 \\ y_1' \end{bmatrix} = \begin{bmatrix} t \\ 1 \end{bmatrix}, \quad \vec{x}^{(2)}(t) = \begin{bmatrix} y_2 \\ y_2' \end{bmatrix} = \begin{bmatrix} te^t \\ e^t + te^t \end{bmatrix}$$

$$\frac{d}{dt} \vec{x}^{(1)}(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ -\frac{(t+2)}{t^2} & \frac{(t+2)}{t} \end{bmatrix} \begin{bmatrix} t \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{(t+2)}{t} + \frac{(t+2)}{t} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$\therefore \vec{x}^{(1)}(t)$ is a solution to $\vec{x}' = A\vec{x}$

$$\frac{d}{dt} \vec{x}^{(2)}(t) = \begin{bmatrix} te^t + e^t \\ e^t + e^t + te^t \end{bmatrix} = \begin{bmatrix} e^t + te^t \\ 2e^t + te^t \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ -\frac{(t+2)}{t^2} & \frac{(t+2)}{t} \end{bmatrix} \begin{bmatrix} te^t \\ e^t + te^t \end{bmatrix} = \begin{bmatrix} e^t + te^t \\ -\frac{(t+2)}{t}e^t + \frac{(t+2)}{t}e^t + (t+2)e^t \end{bmatrix}$$

$$= \begin{bmatrix} e^t + te^t \\ te^t + 2e^t \end{bmatrix}$$

$\therefore \vec{x}^{(2)}(t)$ is a solution to $\vec{x}' = A\vec{x}$

(c)

Note: y_1 and y_2 are solutions to: $y'' - \frac{(t+2)}{t} y' + \frac{(t+2)}{t^2} y = 0$

Following the outline of Section 3.6, pp. 144-145,

$y' = y_1' u_1 + y_1 u_1' + y_2' u_2 + y_2 u_2'$

To prevent higher order derivatives of the unknown coefficients (u_1 and u_2) which will occur when examining y'', set

$$y_1 u_1' + y_2 u_2' = 0 \qquad [1]$$

$\therefore y' = y_1' u_1 + y_2' u_2$

$\therefore y'' = y_1'' u_1 + y_1' u_1' + y_2'' u_2 + y_2' u_2'$

$\therefore y'' - \frac{(t+2)}{t} y' + \frac{(t+2)}{t^2} y =$

$$y_1'' u_1 + y_1' u_1' + y_2'' u_2 + y_2' u_2'$$
$$- \frac{(t+2)}{t}\left[y_1' u_1 + y_2' u_2 \right]$$

$$+\frac{(t+2)}{t^2}\left[y_1 u_1 + y_2 u_2\right]$$

$$= u_1\Big[\underbrace{y_1'' - \frac{(t+2)}{t} y_1' + \frac{(t+2)}{t^2} y_1}_{=0}\Big]$$

$$+ u_2\Big[\underbrace{y_2'' - \frac{(t+2)}{t} y_2' + \frac{(t+2)}{t^2} y_2}_{=0}\Big]$$

$$+ y_1' u_1' + y_2' u_2' = 2t$$

Since y_1 and y_2 are solutions to: $y'' - \frac{(t+2)}{t} y' + \frac{(t+2)}{t^2} y = 0$

$\therefore\ y_1' u_1' + y_2' u_2' = 2t \qquad [2]$

$\therefore$ From [1], [2]:

$$\boxed{\begin{aligned} t u_1' + t e^t u_2' &= 0 \\ u_1' + (e^t + t e^t) u_2' &= 2t \end{aligned}}$$

(d)

Assume $\vec{x} = \Psi(t)\,\vec{u}(t)$. From above,

this is, in matrix form:

$$\begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} y_1 u_1 + y_2 u_2 \\ y_1' u_1 + y_2' u_2 \end{bmatrix}$$

As shown in text on p. 349, this leads to

$\Psi(t)\vec{u}\,'(t) = \vec{g}(t)$, or $\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$

From above, $\vec{g}(t) = \begin{bmatrix} 0 \\ 2t \end{bmatrix}$. $\therefore$ The last

equation can be written $\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ g_2 \end{bmatrix}$

$\therefore$ with $\Psi(t) = \begin{bmatrix} t & te^t \\ 1 & e^t + te^t \end{bmatrix}$ and $\vec{g}(t) = \begin{bmatrix} 0 \\ 2t \end{bmatrix}$

$$\begin{bmatrix} t & te^t \\ 1 & e^t + te^t \end{bmatrix}\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 2t \end{bmatrix}$$

(e)

Expansion of the matrix in (d)

$$\begin{bmatrix} t & te^t \\ 1 & e^t + te^t \end{bmatrix}\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 2t \end{bmatrix}$$

yields: $t u_1' + te^t u_2' = 0$

$$u_1' + (e^t + te^t)u_2' = 2t$$

which are the same two equations in (c)

15.

(a)

Let $x_1 = y$, $x_2 = y'$

$\therefore x_1' = x_2 \quad x_2' = y_2'' = -py' - qy + g$

$= -p x_2 - q x_1 + g$

$\therefore$ In matrix form,

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -q(t) & -p(t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ g(t) \end{bmatrix}$$

(b)

For the homogeneous system, $y'' + py' + qy = 0$.

or $y'' = -py' - qy$

Let $\vec{x}(t) = \begin{bmatrix} y \\ y' \end{bmatrix}$ [1]

$\therefore \frac{d}{dt} \vec{x}(t) = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} y' \\ -py' - qy \end{bmatrix}$ [2]

Also, $\begin{bmatrix} 0 & 1 \\ -q & -p \end{bmatrix}\begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} y' \\ -qy - py' \end{bmatrix}$ [3]

Let $A = \begin{bmatrix} 0 & 1 \\ -q(t) & -p(t) \end{bmatrix}$

From [2] and [3], $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t)$

so $\vec{x}(t)$ is a solution to $\vec{x}'(t) = A\vec{x}$

$\therefore \vec{x}^{(1)}(t) = \begin{bmatrix} y_1 \\ y_1' \end{bmatrix}$ is a solution to $\frac{d}{dt}\vec{x}^{(1)} = A\vec{x}^{(1)}$

substituting y_1 for y in [1], [2], [3]

and $\vec{x}^{(2)}(t) = \begin{bmatrix} y_2 \\ y_2' \end{bmatrix}$ is a solution to $\frac{d}{dt}\vec{x}^{(2)} = A\vec{x}^{(2)}$

substituting y_2 for y in [1], [2], [3]

(c)

Note: y_1 and y_2 are solutions to: $y'' + py + qy = 0$

Following the outline of Section 3.6, pp. 144-145,

$y' = y_1'u_1 + y_1u_1' + y_2'u_2 + y_2u_2'$

To prevent higher order derivatives of the unknown coefficients (u_1 and u_2) which will occur when examining y'', set

$$y_1 u_1' + y_2 u_2' = 0 \qquad [1]$$

$$\therefore\; y' = y_1' u_1 + y_2' u_2$$

$$\therefore\; y'' = y_1'' u_1 + y_1' u_1' + y_2'' u_2 + y_2' u_2'$$

$$\therefore\; y'' + p y' + q y =$$

$$y_1'' u_1 + y_1' u_1' + y_2'' u_2 + y_2' u_2'$$

$$+ p\,(y_1' u_1 + y_2' u_2)$$

$$+ q\,(y_1 u_1 + y_2 u_2)$$

$$= u_1 \overbrace{(y_1'' + p y_1' + q y)}^{=0} + u_2 \overbrace{(y_2'' + p y_1' + q y)}^{=0}$$

$$+ y_1' u_1' + y_2' u_2' = g(t)$$

since y_1 and y_2 are solutions to $y'' + p y' + q y = 0$

$$\therefore\; y_1' u_1' + y_2' u_2' = g(t) \qquad [2]$$

From [1], [2]

$$\boxed{\begin{aligned} y_1 u_1' + y_2 u_2' &= 0 \\ y_1' u_1' + y_2' u_2' &= g \end{aligned}}$$

(d)

The system in (a) is $\vec{x}'(t) = P(t)\vec{x} + \vec{g}$,

where $P(t) = \begin{bmatrix} 0 & 1 \\ -q(t) & -p(t) \end{bmatrix}$, $\vec{g}(t) = \begin{bmatrix} 0 \\ g(t) \end{bmatrix}$

Assuming $\vec{x} = \Psi(t)\vec{u}(t)$ where $\Psi(t) = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}$

and $\vec{u}(t) = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

and noting $\Psi'(t) = P(t)\Psi(t)$ since y_1, y_2 are homogeneous solutions,

$$\therefore \vec{x}' = \Psi'(t)\vec{u}(t) + \Psi(t)\vec{u}'(t)$$

$$= P(t)\Psi(t)\vec{u}(t) + \Psi(t)\vec{u}'(t)$$

$$= P(t)\vec{x}(t) + \Psi(t)\vec{u}'(t)$$

$$= P(t)\vec{x}(t) + \vec{g}(t)$$

$$\therefore \Psi(t)\vec{u}'(t) = \vec{g}(t)$$

Or, in expanded form, $\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ g(t) \end{bmatrix}$

(e)

Multiplying the matrix in (d),

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ g(t) \end{bmatrix}$$

$$y_1 u_1' + y_2 u_2' = 0$$

$$y_1' u_1' + y_2' u_2' = g(t)$$

which are the same two equations in (c).

16.

(a)

As shown in the discussion of variation of parameters in the text on p. 349, the solution to $\vec{x}' = P(t)\vec{x} + \vec{g}(t)$, $\vec{x}(0) = \vec{x}^0$ is

$$\mathbf{x} = \mathbf{\Phi}(t)\mathbf{x}^0 + \mathbf{\Phi}(t)\int_{t_0}^{t} \mathbf{\Phi}^{-1}(s)\mathbf{g}(s)\,ds. \qquad (34)$$

Here $P(t) = A$.

As shown in problem 12(b), Section 7.7, $\phi(-s) = \phi^{-1}(s)$

$\therefore$ (34) can be written as:

$$\vec{x} = \phi(t)\,\vec{x}^{\,0} + \phi(t)\int_{t_0}^{t} \phi(-s)\,\vec{g}(s)\,ds$$

$$= \phi(t)\,\vec{x}^{\,0} + \int_{t_0}^{t} \phi(t)\phi(-s)\,\vec{g}(s)\,ds \quad [1]$$

$\phi(t)$ acts like a constant in regard to the integral because integration is via the variable s, not t.

From problem 12(b) + 12(c), Section 7.7,

$$\phi(t)\phi(-s) = \phi(t-s)$$

$\therefore$ From [1],

$$\vec{x} = \phi(t)\,\vec{x}^{\,0} + \int_{t_0}^{t} \phi(t-s)\,\vec{g}(s)\,ds$$

(6)

As shown in Section 7.7, pp. 332-333,

$$\phi(t) = \exp(At)$$

$\therefore$ Just substitute $\exp(At)$ for the matrix $\phi(t)$ in (a).

Section 3.6, problem #22, shows the solution to $y'' + by' + cy = g(t)$, $y(t_0) = 0$, $y'(t_0) = 0$ is $y = \phi(t) = \int_{t_0}^{t} K(t-s)\,g(s)\,ds$, where $K(x)$ only depends on the homogeneous solutions y_1 and y_2, (from the roots of the characteristic equation) and is independent of $g(t)$.

The above form is similar: $\exp(At)$ is only dependent on the homogeneous solutions $\vec{x}^{(1)}(t)$ and $\vec{x}^{(2)}(t)$ that comprise the columns of $\phi(t)$, and is independent of $\vec{g}(t)$.

17.

(a) Let $\mathcal{L}\{\vec{x}(t)\} = X(s)$, $\mathcal{L}\{\vec{g}(t)\} = G(s)$

and let $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$

$\therefore \mathcal{L}\{\vec{x}'(t)\} = sX(s) - \vec{x}(0) = AX(s) + G(s)$

$\therefore (sI - A)X(s) = G(s) + \vec{x}(0)$

$X(s) = (sI - A)^{-1}\left[G(s) + \vec{x}(0)\right]$

$sI - A = \begin{bmatrix} s+2 & -1 \\ -1 & s+2 \end{bmatrix}$ $\therefore (sI - A)^{-1} = \dfrac{1}{s^2+4s+3}\begin{bmatrix} s+2 & 1 \\ 1 & s+2 \end{bmatrix}$

$G(s) = \begin{bmatrix} \dfrac{2}{s+1} \\ \dfrac{3}{t^2} \end{bmatrix}$ $\therefore G(s) + \vec{x}(0) = \begin{bmatrix} \dfrac{2}{s+1} + \alpha_1 \\ \dfrac{3}{s^2} + \alpha_2 \end{bmatrix}$

$\therefore (sI - A)^{-1}\left[G(s) + \vec{x}(0)\right] =$ using $s^2 + 4s + 3 = (s+3)(s+1)$

$$\begin{bmatrix} \dfrac{2(s+2)}{(s+1)^2(s+3)} + \dfrac{\alpha_1(s+2)}{(s+3)(s+1)} + \dfrac{3}{s^2(s+3)(s+1)} + \dfrac{\alpha_2}{(s+3)(s+1)} \\ \dfrac{2}{(s+1)^2(s+3)} + \dfrac{\alpha_1}{(s+3)(s+1)} + \dfrac{3(s+2)}{s^2(s+3)(s+1)} + \dfrac{\alpha_2(s+2)}{(s+3)(s+1)} \end{bmatrix}$$

Using partial fraction decomposition and inverse Laplace transforms, using MATLAB,

```
clear
syms s

a = partfrac( 2*(s+2) / ( (s+1)^2*(s+3) ) )
ilaplace(a)

b = partfrac( (s+2) / ((s+3)*(s+1)) )
ilaplace(b)

c = partfrac( 3 / ( s^2*(s+3)*(s+1) ) )
ilaplace(c)

d = partfrac( 1 / ( (s+3)*(s+1) ) )
ilaplace(d)

e = partfrac( 2 / ( (s+1)^2*(s+3) ) )
ilaplace(e)

f = partfrac( 3*(s+2) / ( s^2*(s+3)*(s+1) )
)
ilaplace(f)
```

a =

$$\frac{1}{2\ (s+1)}+\frac{1}{(s+1)^2}-\frac{1}{2\ (s+3)}$$

ans =

$$\frac{e^{-t}}{2}-\frac{e^{-3\,t}}{2}+t\,e^{-t}$$

b =

$$\frac{1}{2\ (s+1)}+\frac{1}{2\ (s+3)}$$

ans =

$$\frac{e^{-t}}{2}+\frac{e^{-3\,t}}{2}$$

c =

$$\frac{3}{2\ (s+1)}-\frac{1}{6\ (s+3)}-\frac{4}{3\,s}+\frac{1}{s^2}$$

ans =

$$t+\frac{3\,e^{-t}}{2}-\frac{e^{-3\,t}}{6}-\frac{4}{3}$$

d =

$$\frac{1}{2\ (s+1)}-\frac{1}{2\ (s+3)}$$

ans =

$$\frac{e^{-t}}{2}-\frac{e^{-3\,t}}{2}$$

e =

$$\frac{1}{(s+1)^2}-\frac{1}{2\ (s+1)}+\frac{1}{2\ (s+3)}$$

ans =

$$\frac{e^{-3\,t}}{2}-\frac{e^{-t}}{2}+t\,e^{-t}$$

f =

$$\frac{3}{2\ (s+1)}+\frac{1}{6\ (s+3)}-\frac{5}{3\,s}+\frac{2}{s^2}$$

ans =

$$2\,t+\frac{3\,e^{-t}}{2}+\frac{e^{-3\,t}}{6}-\frac{5}{3}$$

$$\therefore \mathcal{L}^{-1}\left\{(sI-A)^{-1}\left[G(s)+\vec{x}(0)\right]\right\} =$$

$$\begin{bmatrix} \frac{e^{-t}}{2}-\frac{e^{-3t}}{2}+te^{-t}+\alpha_1\left(\frac{e^{-t}}{2}+\frac{e^{-3t}}{2}\right)+t+\frac{3}{2}e^{-t}-\frac{1}{6}e^{-3t}-\frac{4}{3}+\alpha_2\left(\frac{e^{-t}}{2}-\frac{e^{-3t}}{2}\right) \\ \frac{e^{-3t}}{2}-\frac{e^{-t}}{2}+te^{-t}+\alpha_1\left(\frac{e^{-t}}{2}-\frac{e^{-3t}}{2}\right)+2t+\frac{3e^{-t}}{2}+\frac{e^{-3t}}{6}-\frac{5}{3}+\alpha_2\left(\frac{e^{-t}}{2}+\frac{e^{-3t}}{2}\right) \end{bmatrix}$$

$$= \boxed{\frac{(\alpha_1+\alpha_2)}{2}\begin{bmatrix}1\\1\end{bmatrix}e^{-t}+\frac{(\alpha_1-\alpha_2)}{2}\begin{bmatrix}1\\-1\end{bmatrix}e^{-3t}+\begin{bmatrix}2\\1\end{bmatrix}e^{-t}-\frac{2}{3}\begin{bmatrix}1\\-1\end{bmatrix}e^{-3t}+\begin{bmatrix}1\\1\end{bmatrix}te^{t}+\begin{bmatrix}1\\2\end{bmatrix}t-\frac{1}{3}\begin{bmatrix}4\\5\end{bmatrix}}$$

(b) Equation (38) is:

$$= c_1\begin{pmatrix}1\\-1\end{pmatrix}e^{-3t} + c_2\begin{pmatrix}1\\1\end{pmatrix}e^{-t} + \frac{1}{2}\begin{pmatrix}1\\-1\end{pmatrix}e^{-t} + \begin{pmatrix}1\\1\end{pmatrix}te^{-t} + \begin{pmatrix}1\\2\end{pmatrix}t - \frac{1}{3}\begin{pmatrix}4\\5\end{pmatrix}, \qquad (38)$$

$\therefore$ Coefficient for e^{-t}: $\frac{\alpha_1+\alpha_2}{2} + 2 = c_2 + \frac{1}{2}$

$$\frac{\alpha_1+\alpha_2}{2} + 1 = c_2 - \frac{1}{2}$$

Coefficient for e^{-3t}: $\frac{\alpha_1-\alpha_2}{2} - \frac{2}{3} = c_1$

$$\therefore \boxed{\begin{aligned} c_1 &= \frac{3\alpha_1 - 3\alpha_2 - 4}{6} \\ c_2 &= \frac{\alpha_1 + \alpha_2 + 3}{2} \end{aligned}}$$