1.1 Mathematical Induction

Note Title 4/21/2004

1. a. $1+2+3\ldots+n = \dfrac{n(n+1)}{2}$ for all $n \geq 1$.

$$1 = \frac{1\cdot(1+1)}{2} = 1$$

Suppose $1+2+\cdots+k = \dfrac{k(k+1)}{2}$

Then $1+2+\cdots+k+(k+1) = \dfrac{k(k+1)}{2} + (k+1)$

$$= \frac{k(k+1)}{2} + \frac{2(k+1)}{2}$$

$$= \frac{(k+1)(k+2)}{2}$$

So, $k \Rightarrow k+1$

b. $1+3+5+\ldots+(2n-1) = n^2$ for all $n \geq 1$.

$1 = 1$

Suppose $1+3+\cdots+(2k-1) = k^2$

Then $1+3+\cdots+(2k-1)+(2k+1) = k^2+2k+1$

$$= (k+1)^2$$

So, $k \Rightarrow k+1$

c. $1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \cdots + n(n+1) = \dfrac{n(n+1)(n+2)}{3}$, all $n \geq 1$

$k=1$: $1\cdot 2 = \dfrac{1(1+1)(1+2)}{3} = \dfrac{1\cdot 2\cdot 3}{3} = 2$

Suppose statement is true k. Then,

$1\cdot 2 + \cdots + k(k+1) + (k+1)(k+2) = \dfrac{k(k+1)(k+2)}{3} + (k+1)(k+2)$

$= \dfrac{k(k+1)(k+2)}{3} + \dfrac{3(k+1)(k+2)}{3}$

$= \dfrac{(k+1)\left[k(k+2) + 3k + 6\right]}{3} = \dfrac{(k+1)\left[k^2+5k+6\right]}{3}$

$= \dfrac{(k+1)\left[(k+2)(k+3)\right]}{3}$ So, $k \Rightarrow k+1$

d. $1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \dfrac{n(2n-1)(2n+1)}{3}$, all $n \geq 1$

$k=1$: $1^2 = \dfrac{1(2\cdot 1-1)(2\cdot 1+1)}{3} = \dfrac{1\cdot 1\cdot 3}{3} = 1$

$k \Rightarrow k+1$:

$1^2 + 3^2 + \cdots (2k-1)^2 + (2k+1)^2 =$

$$\frac{k(2k-1)(2k+1)}{3} + (2k+1)^2$$

$$= \frac{k(2k-1)(2k+1)}{3} + \frac{3(2k+1)^2}{3}$$

$$= \frac{(2k+1)\left[k(2k-1) + 3(2k+1)\right]}{3}$$

$$= \frac{(2k+1)\left[2k^2 - k + 6k + 3\right]}{3}$$

$$= \frac{(2k+1)\left[2k^2 + 5k + 3\right]}{3} = \frac{(2k+1)(2k+3)(k+1)}{3}$$

$$= \frac{(k+1)(2k+1)(2k+3)}{3}.$$ So true for $k+1$.

e. $1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$, all $n \geq 1$

$k=1$: $1^3 = \left[\frac{1\cdot 2}{2}\right]^2 = 1^2 = 1$

$k \Rightarrow k+1$:

$$1^3 + 2^3 + \ldots + k^3 + (k+1)^3$$

$$= \left[\frac{k(k+1)}{2}\right]^2 + (k+1)^3$$

$$= \frac{k^2(k+1)^2}{4} + \frac{4(k+1)^2(k+1)}{4}$$

$$= \frac{(k+1)^2\left[k^2+4k+4\right]}{4} = \frac{(k+1)^2(k+2)^2}{4}$$

$$= \left[\frac{(k+1)(k+2)}{2}\right]^2$$ So, true for $k+1$

2. If $r \neq 1$, Then $a + ar + \cdots + ar^n = \frac{a(r^{n+1}-1)}{r-1}$, $n \geq 1$

For $k=1$: $a + ar^1 = a(1+r) = \frac{a(r^2-1)}{r-1} = a(r+1)$

$k \Rightarrow k+1$:

$$a + ar + \cdots + ar^n + ar^{n+1} = \frac{a(r^{n+1}-1)}{r-1} + ar^{n+1}$$

$$= \frac{a(r^{n+1}-1)}{r-1} + \frac{ar^{n+1}(r-1)}{r-1}$$

$$= \frac{ar^{n+1} - a + ar^{n+2} - ar^{n+1}}{r-1}$$

$$= \frac{ar^{n+2} - a}{r-1} = \frac{a(r^{n+2} - 1)}{r-1}$$

So, true for $K+1$

3. $a^n - 1 = (a-1)(a^{n-1} + a^{n-2} + \dots + a + 1)$, for $n \geq 1$

For $K=1$: $a^1 - 1 = a - 1 = (a-1)(a^0) = a-1$

$K \Rightarrow K+1$: $a^{K+1} - 1 = a^{K+1} - a^K - a + a^K + a - 1$

$$= a^{K+1} - a + a^K - 1 - a^K + a$$

$$= a(a^K - 1) + a^K - 1 - a(a^{K-1} - 1)$$

$$= (a+1)(a^K - 1) - a(a^{K-1} - 1)$$

Use 2nd principle of finite induction for $K, K-1$

$$= (a+1)\left[(a-1)(a^{K-1} + a^{K-2} + \dots + a + 1)\right]$$

$$- a\left[(a-1)(a^{K-2} + a^{K-3} + \dots + a + 1)\right]$$

$$= (a-1)\Big[a(a^{K-1} + a^{K-2} + \dots + a + 1)$$

$$+ (a^{K-1} + a^{K-2} + \dots + a + 1)$$

$$- a(a^{K-2} + a^{K-3} + \dots + a + 1)\Big]$$

combine these two

$$= (a-1)\Big[a(a^{k-1}+a^{k-2}+\dots+a+1)$$
$$+(a^{k-1}+a^{k-2}+\dots+a+1)$$
$$-(a^{k-1}+a^{k-2}+\dots+a^2+a)\Big]$$

$$= (a-1)\left[(a^k+a^{k-1}+\dots+a^2+a)+1\right]$$

and so, works for $k+1$

4. Cube of any integer can be written as the difference of two squares.

Proof: $n^3 = (1^3+2^3+\dots+n^3)-(1^3+2^3+\dots+(n-1)^3)$, all n

From 1(e), $1^3+2^3+\dots+n^3 = \left[\frac{n(n+1)}{2}\right]^2$, $n \geq 1$

So, $n^3 = \left[\frac{n(n+1)}{2}\right]^2 - \left[\frac{(n-1)n}{2}\right]^2$

If n is even, then $\frac{n}{2}$ is an integer.
If n is odd, then $n+1$ and $n-1$ are even,
so $\frac{n+1}{2}$ and $\frac{n-1}{2}$ are integers.

$\therefore$ n^3 is the difference between squares.

5. (a). For $n=4$, $n!+1 = 25 = 5^2$

$n=5$, $n!+1 = 121 = 11^2$

$n=7$, $n!+1 = 5041 = 71^2$

(b). False

$(3\cdot 2)! = 720 \neq 3!2! = 6\cdot 2 = 12$

$(2+3)! = 120 \neq 2!+3! = 2+6$

6. a. $n! > n^2$ for $n \geq 4$

Proof: $4! = 24 > 16 > 4^2$

Suppose $k! > k^2$, for $k > 4$

$(k+1)! = (k+1)\cdot k! > (k+1)\cdot k^2$

$= k^3 + k^2$

Since $k > 4$, then $k > 2$, so $k^2 > 2k$.

Since $k > 1$, then $k^3 > 2k$, so $k^3 \geq 2k+1$

So, $k^3 + k^2 \geq k^2 + 2k + 1 = (k+1)^2$

So $(k+1)! > k^3 + k^2 \geq (k+1)^2$

So $(k+1)! > (k+1)^2$

So, $k \Rightarrow k+1$

6.6. $n! > n^3$, $n \geq 6$

Proof: $6! = 720 > 216 = 6^3$

Suppose $k! > k^3$ for any $k > 6$

$\therefore (k+1)! = (k+1)k! > (k+1)k^3$

$= k^4 + k^3$

Since $k > 6$, Then $k > 3$, so $k^2 > 3k$

Also, $k > 1$, so $k^2 > k$, and $k^2 \geq k+1$

So $k^4 = k^2 \cdot k^2 > 3k(k+1) = 3k^2 + 3k$

$\therefore k^3 + k^4 > k^3 + 3k^2 + 3k$

$\therefore k^4 + k^3 \geq k^3 + 3k^2 + 3k + 1 = (k+1)^3$

$\therefore (k+1)! > k^4 + k^3 \geq (k+1)^3$

$(k+1)! > (k+1)^3$

So, $k \Rightarrow k+1$

7. $1(1!) + 2(2!) + \dots + n(n!) = (n+1)! - 1, \quad n \geq 1$

$K=1: \quad 1(1!) = 1 = (1+1)! - 1 = 2! - 1 = 1$

$K \Rightarrow K+1$: Let $1(1!) + 2(2!) + \dots + k(k!) = (k+1)! - 1$

Then $1(1!) + \dots + (k+1)(k+1)!$

$= (k+1)! - 1 + (k+1)(k+1)!$

$= (k+1)! \left[1 + k+1\right] - 1$

$= (k+1)! (k+2) - 1$

$= (k+2)! - 1$ So, true for $k+1$

8. a. $2 \cdot 6 \cdot 10 \cdot \dots \cdot (4n-2) = \dfrac{(2n)!}{n!}, \quad n \geq 1$

$K=1: \quad 2 = \dfrac{2!}{1!} = 2$

$K \Rightarrow K+1$: Let $2 \cdot 6 \cdot 10 \cdots (4k-2) = \dfrac{(2k)!}{k!}$

Then $2 \cdot 6 \cdot 10 \cdots (4k-2)(4k+2) =$

$$\frac{(2k)!}{k!}(4k+2) = \frac{(2k)!}{k!}(2k+1)\,2$$

$$= \frac{(2k)!}{k!}(2k+1)\,2\,\frac{(k+1)}{(k+1)}$$

$$= \frac{(2k)!\,(2k+1)(2k+2)}{k!\,(k+1)} = \frac{(2k+2)!}{(k+1)!}$$

So, true for $k+1$

6. $2^n(n!)^2 \le (2n)!\ ,\quad n \ge 1$

From (a), $(2n)! = 2\cdot 6\cdot 10\cdots(4n-2)\,(n!)$

So problem reduces to:

$$2^n(n!)^2 \le 2\cdot 6\cdot 10\cdots(4n-2)\,(n!)$$

Or, $2^n(n!) \le 2\cdot 6\cdot 10\cdots(4n-2)$

For $k=1$: $2^1(1!) = 2 \le 2$

$k \Rightarrow k+1$: Let $2^k(k!) \le 2\cdot 6\cdot 10\cdots(4k-2)$

$$2^{k+1}(k+1)! = 2^k(k!)\cdot 2\cdot(k+1)$$

$$= 2^k(k!)(2k+2)$$

$$< 2^k(k!)(4k+2)$$

$$\leq 2\cdot 6\cdot 10\cdots(4k-2)(4k+2)$$

$$= 2\cdot 6\cdot 10\cdot\cdots(4(k+1)-2)$$

So, true for $k+1$

9. If $1+a>0$, Then $(1+a)^n \geq 1+na$, $n \geq 1$

$k=1$: $1+a \geq 1+a$

$k \Rightarrow k+1$: Let $(1+a)^k \geq 1+ka$

$$(1+a)^{k+1} = (1+a)^k(1+a)$$

$$\geq (1+ka)(1+a)$$

$$= 1+ka+a+ka^2$$

$$\geq 1+ka+a \quad (a^2>0,\ \text{so } ka^2>0)$$

$= 1 + (k+1)a$

$\therefore (1+a)^{k+1} \geq 1 + (k+1)a$

So, it's true for $k+1$

10\. a. $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$, $n \geq 1$

$k=1$: $\frac{1}{1^2} = 1 \leq 2 - \frac{1}{1} = 1$

$k \Rightarrow k+1$: Let $\frac{1}{1^2} + \frac{1}{2^2} + \dots + \frac{1}{k^2} \leq 2 - \frac{1}{k}$

Then, $\frac{1}{1^2} + \dots + \frac{1}{k^2} + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2}$

Since $k \geq 1$, then $k^2+2k < k^2+2k+1$,

Or, $\frac{k^2+2k}{(k+1)^2} < 1$, or $\frac{k(k+2)}{(k+1)^2} < 1$

$\therefore \frac{(k+1)+1}{(k+1)^2} < \frac{1}{k} \Rightarrow \frac{1}{k+1} + \frac{1}{(k+1)^2} < \frac{1}{k}$

$\therefore -\frac{1}{k} + \frac{1}{(k+1)^2} < -\frac{1}{k+1}$

$$\therefore\ 2 - \frac{1}{k} + \frac{1}{(k+1)^2} < 2 - \frac{1}{k+1}$$

$$\therefore\ \frac{1}{1^2} + \frac{1}{2^2} + \dots + \frac{1}{(k+1)^2} < 2 - \frac{1}{k+1}$$

So, $k \Rightarrow k+1$

6. $\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \dots + \frac{n}{2^n} = 2 - \frac{n+2}{2^n}$

$k=1$: $\frac{1}{2} = 2 - \frac{1+2}{2^1} = 2 - \frac{3}{2} = \frac{1}{2}$

$k \Rightarrow k+1$: Let $\frac{1}{2} + \frac{2}{2^2} + \dots + \frac{k}{2^k} = 2 - \frac{k+2}{2^k}$

Then, $\frac{1}{2} + \dots \frac{k}{2^k} + \frac{k+1}{2^{k+1}} = 2 - \frac{k+2}{2^k} + \frac{k+1}{2^{k+1}}$

$$= 2 - \frac{k+2}{2^k} \cdot \frac{2}{2} + \frac{k+1}{2^{k+1}}$$

$$= 2 + \frac{(k+1) - (2k+4)}{2^{k+1}} = 2 + \frac{-k-3}{2^{k+1}}$$

$$= 2 - \frac{(k+1)+2}{2^{k+1}}$$ So, $k \Rightarrow k+1$

11. $(2n)!/2^n n!$ is an integer, $n \ge 0$

$n=0$: $0! = 1$ by definition. So $\frac{0!}{2^0\, 0!} = \frac{1}{1 \cdot 1} = 1$

$k \Rightarrow k+1$: Suppose $\frac{(2k)!}{2^k\, k!}$ is an integer.

$$\frac{[2(k+1)]!}{2^{k+1}(k+1)!} = \frac{(2k)!(2k+1)(2k+2)}{2^k (k!)\, 2(k+1)}$$

$$= \frac{(2k)!}{2^k\, k!} \cdot (2k+1)\frac{(2k+2)}{2k+2}$$

$$= (\text{integer}) \cdot (2k+1) = \text{integer}$$

12. $T(21) = 32$ $\quad$ $T(23) = 35$

$T(T(21)) = 16$ $\quad$ $T(35) = 53$

$T(16) = 8$ $\quad$ $T(53) = 80$

$T(8) = 4$ $\quad$ $T(80) = 40$

$T(4) = 2$ $\quad$ $T(40) = 20$

$T(2) = 1$ $\quad$ $T(20) = 10$

$T(10) = 5$

$T(5) = 8$

$T(8) = 4$

$T(4) = 2$

$T(2) = 1$

13. $a_1 = 1 \quad a_2 = 2 \quad a_3 = 3$

$a_n = a_{n-1} + a_{n-2} + a_{n-3}$, for $n \geq 4$

Prove: $a_n < 2^n$, $n \geq 1$

Proof: $1 < 2^1$, $2 < 2^2$, $3 < 2^3$

Let $k \geq 4$, and assume $a_K < 2^K$, $K = 4, \dots, k$

Then $a_{K+1} = a_k + a_{K-1} + a_{K-2}$

$$< 2^K + 2^{K-1} + 2^{K-2}$$

$$< 2^K + 2^{K-1} + 2^{K-1}$$

$$= 2^K + 2 \cdot 2^{K-1} = 2^K + 2^K$$

$$= 2 \cdot 2^k = 2^{K+1}$$

$\therefore \; a_{K+1} < 2^{K+1}$

14. $a_1 = 11$, $a_2 = 21$, $a_n = 3a_{n-1} - 2a_{n-2}$, $n \geq 3$

Prove: $a_n = 5 \cdot 2^n + 1$, $n \geq 1$

Proof: $a_1 = 5 \cdot 2 + 1 = 11$

$a_2 = 5 \cdot 4 + 1 = 21$

Suppose $a_k = 5 \cdot 2^k + 1$ for $3, 4, \ldots, k$

Then $a_{k+1} = 3a_k - 2a_{k-1}$

$= 3(5 \cdot 2^k + 1) - 2(5 \cdot 2^{k-1} + 1)$

$= 15 \cdot 2^k + 3 - 5 \cdot 2^k - 2$

$= 10 \cdot 2^k + 1$

$= 5 \cdot 2 \cdot 2^k + 1$

$= 5 \cdot 2^{k+1} + 1$

$\therefore$ if works for $3, 4, \ldots, k$, then works for $k+1$.

$\therefore$ Works for all $k \geq 1$