1.2 The Binomial Theorem

Note Title 4/25/2004

1. a. Newton's identity

$$\binom{n}{k}\binom{k}{r} = \binom{n}{r}\binom{n-r}{k-r} \qquad n \ge k \ge r \ge 0$$

$$\frac{n!}{k!(n-k)!}\,\frac{k!}{r!(k-r!)} = \frac{n!}{r!}\cdot\frac{1}{(n-k)!(k-r!)}$$

$$= \frac{n!}{r!}\cdot\frac{(n-r)!}{(n-r)!}\cdot\frac{1}{(n-k)!(k-r)!}$$

$$= \frac{n!}{r!(n-r)!}\cdot\frac{(n-r)!}{(k-r)!(n-k)!}$$

$$= \binom{n}{r}\cdot\frac{(n-r)!}{(k-r)!(n-r-(k-r))!} = \binom{n}{r}\binom{n-r}{k-r}$$

b. $\binom{n}{k} = \frac{n-k+1}{k}\binom{n}{k-1} \qquad n \ge k \ge 1$

Without using part (a),

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!\cdot(n-k+1)}{k(k-1)!(n-k+1)!}$$

$$= \frac{n!}{(k-1)!\,(n-k+1)!} \cdot \frac{(n-k+1)}{k} = \frac{(n-k+1)}{k}\binom{n}{k-1}$$

To use part (a), Let $r=1$

Then $\binom{n}{k}\binom{k}{1} = \binom{n}{1}\binom{n-1}{k-r}$ $\quad n \geq k \geq r \geq 0$

So, $\binom{n}{k}k = n\binom{n-1}{k-1}$

$$= n \frac{(n-1)!}{(k-1)!\,(n-k)!}$$

$$= \frac{n!}{(k-1)!\,(n-k+1)!} \cdot (n-k+1)$$

So, $\binom{n}{k} = \frac{n-k+1}{k}\binom{n}{k-1}$

2. If $2 \leq k \leq n-2$, and $n \geq 4$

$$\binom{n}{k} = \binom{n-2}{k-2} + 2\binom{n-2}{k-1} + \binom{n-2}{k}$$

Working from the right hand side,

$$\frac{(n-2)!}{(k-2)!\,(n-k)!} + \frac{2\,(n-2)!}{(k-1)!\,(n-k-1)!} + \frac{(n-2)!}{k!\,(n-k-2)!}$$

$$= \frac{k\cdot(k-1)\,(n-2)!}{k!\,(n-k)!} + \frac{2k\,(n-k)\,(n-2)!}{k!\,(n-k)!} + \frac{(n-k)(n-k-1)(n-2)!}{k!\,(n-k)!}$$

$$= \frac{(n-2)!\,[k^2-k+2kn-2k^2+n^2-nk-n-kn+k^2+k]}{k!\,(n-k)!}$$

$$= \frac{(n-2)!\,[n^2-n]}{k!\,(n-k)!} = \frac{n(n-1)(n-2)!}{k!\,(n-k)!} = \binom{n}{k}$$

$2 \leq k$ for $(k-2)!$ in denominator to work
$n-k-2 \geq 0$, or $n-2 \geq k \geq 2$, so $n \geq 4$ for
$(n-k-2)!$ in denominator to work.

3. a. From binomial Theorem, letting $a=b=1$,

$$(a+b)^n = 2^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 1^k$$

So, $2^n = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}$

b. From binomial Theorem, let $a=1$, $b=-1$

$0^n = 0 = \binom{n}{0} - \binom{n}{1} + \cdots + (-1)^n \binom{n}{n}$

C. $\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n} = n2^{n-1}$

In binomial Theorem, let $a=1$

Then $(1+b)^{n-1} = \sum_{k=0}^{n-1} \binom{n-1}{k} b^k$

So, $n(1+b)^{n-1} = n\left[\binom{n-1}{0} + \binom{n-1}{1} b + \cdots + \binom{n-1}{n-1} b^{k-1}\right]$

Now let $b=1$. Then

$$n2^{n-1} = n\binom{n-1}{0} + n\binom{n-1}{1} + \cdots n\binom{n-1}{n-1}$$

$$= \sum_{k=0}^{n-1} n\binom{n-1}{k}$$

But $n\binom{n-1}{k} = \dfrac{n(n-1)!}{k!(n-k-1)!} = \dfrac{n!}{k!(n-(k+1))!} \cdot \dfrac{(k+1)}{(k+1)}$

$$= (k+1)\frac{n!}{(k+1)!\,(n-(k+1))!}$$

$$= (k+1)\binom{n}{k+1}$$

$$\therefore n2^{n-1} = \sum_{k=0}^{n-1} n\binom{n-1}{k} = \sum_{k=0}^{n-1}(k+1)\binom{n}{k+1}$$

$$= \binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \dots + n\binom{n}{n}$$

d. $\binom{n}{0} + 2\binom{n}{1} + 2^2\binom{n}{2} + \dots + 2^n\binom{n}{n} = 3^n$

In binomial Theorem, let $a=1$, $b=2$

$$(a+b)^n = 3^n = \binom{n}{0}1^n + \binom{n}{1}1^{n-1}2 + \dots + \binom{n}{n}2^n$$

$$= \binom{n}{0} + 2\binom{n}{1} + \dots + 2^n\binom{n}{n}$$

e. $\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \dots = 2^{n-1}$

$\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \dots = 2^{n-1}$

Proof: Add / Subtract results of (a) & (b)

If n is even, Then last term is positive

$$\binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n} = 2^n$$
$$+\left[\binom{n}{0} - \binom{n}{1} + \dots + \binom{n}{n} = 0\right]$$
$$2\left[\binom{n}{0} + \binom{n}{2} + \dots + \binom{n}{n}\right] = 2^n$$

If n is odd, last term is $-\binom{n}{n}$

$$\binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n} = 2^n$$
$$+\left[\binom{n}{0} - \binom{n}{1} + \dots - \binom{n}{n} = 0\right]$$
$$2\left[\binom{n}{0} + \binom{n}{2} + \dots + \binom{n}{n-1}\right] = 2^n$$

So, $\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \dots = 2^{n-1}$

If n is even, Then last term is positive

$$\binom{n}{0}+\binom{n}{1}+\cdots+\binom{n}{n}=2^n$$

$$-\left[\binom{n}{0}-\binom{n}{1}+\cdots+\binom{n}{n}=0\right]$$

$$2\left[\binom{n}{1}+\binom{n}{3}+\cdots+\binom{n}{n-1}\right]=2^n$$

If n is odd, last term is $-\binom{n}{n}$

$$\binom{n}{0}+\binom{n}{1}+\cdots+\binom{n}{n}=2^n$$

$$-\left[\binom{n}{0}-\binom{n}{1}+\cdots-\binom{n}{n}=0\right]$$

$$2\left[\binom{n}{1}+\binom{n}{3}+\cdots+\binom{n}{n}\right]=2^n$$

So, $\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\cdots=2^{n-1}$

f. $\binom{n}{0}-\frac{1}{2}\binom{n}{1}+\frac{1}{3}\binom{n}{2}-\cdots+\frac{(-1)^n}{n+1}\binom{n}{n}=\frac{1}{n+1}$

The terms look like the terms in (b) with coefficients. So, need a relation with coefficient in front of binomial term.

The "k"th term can be written as:

$$(-1)^{k-1} \cdot \frac{1}{k}\binom{n}{k-1}$$

Note that $\binom{n}{k-1} = \dfrac{n!}{(k-1)!\,(n-k+1)!}$

$$= \frac{k}{n+1} \cdot \frac{(n+1)!}{k!\,(n-k+1)!}$$

Thus, $\dfrac{1}{k}\binom{n}{k-1} = \dfrac{1}{n+1}\binom{n+1}{k}$

So, problem is equivalent to:

$$\underset{(k=1)}{\binom{n}{0}} - \underset{(k=2)}{\frac{1}{2}\binom{n}{1}} + \dots + \underset{(k=n+1)}{\frac{(-1)^n}{n+1}\binom{n}{n}}$$

$$= \frac{1}{n+1}\binom{n+1}{1} - \frac{1}{n+1}\binom{n+1}{2} + \dots + \frac{(-1)^n}{n+1}\binom{n+1}{n+1}$$

$$= \frac{1}{n+1}\left[\binom{n+1}{1} - \binom{n+1}{2} + \binom{n+1}{3} - \dots (-1)^n\binom{n+1}{n+1}\right]$$

From (6), $\binom{n}{0} = \binom{n}{1} - \binom{n}{2} + \dots - (-1)^n \binom{n}{n}$

Substituting $n = s+1$,

$$\binom{s+1}{0} = \binom{s+1}{1} - \binom{s+1}{2} + \dots - (-1)^{s+1} \binom{s+1}{s+1}$$

$$1 = \binom{s+1}{1} - \binom{s+1}{2} + \dots + (-1)^{s} \binom{s+1}{s+1}$$

$$\therefore \binom{n}{0} - \frac{1}{2}\binom{n}{1} + \dots + \frac{(-1)^n}{n+1}\binom{n+1}{n+1}$$

$$= \frac{1}{n+1}\left[\binom{n+1}{1} - \binom{n+1}{2} + \dots + (-1)^n \binom{n+1}{n+1}\right]$$

$$= \frac{1}{n+1}\left[1\right] = \frac{1}{n+1}$$

4. a. For $n \geq 1$, $\binom{n}{r} < \binom{n}{r+1} \iff 0 \leq r < \frac{1}{2}(n-1)$

Proof: $\binom{n}{r} < \binom{n}{r+1}$

$$\iff \frac{n!}{r!(n-r)!} < \frac{n!}{(r+1)!(n-r-1)!}, \quad 0 \leq r,\ 0 \leq n-r-1$$

$$\iff \frac{(r+1)!}{r!} < \frac{(n-r)!}{(n-r-1)!}, \quad 0 \le r \le n-1$$

$$\iff r+1 < n-r, \quad 0 \le r \le n-1$$

$$\iff 0 \le 2r < n-1$$

$$\iff 0 \le r < \frac{1}{2}(n-1)$$

6. $\binom{n}{r} > \binom{n}{r+1} \iff n-1 \ge r > \frac{1}{2}(n-1)$

Proof: $\binom{n}{r} > \binom{n}{r+1}$

$$\iff \frac{n!}{r!(n-r)!} > \frac{n!}{(r+1)!(n-r-1)!}, \quad r \ge 0, n-r-1 \ge 0$$

$$\iff \frac{(r+1)!}{r!} > \frac{(n-r)!}{(n-r-1)!}, \quad r \ge 0, n-r-1 \ge 0$$

$$\iff r+1 > n-r, \quad n-1 \ge r \ge 0$$

$$\iff 2r > n-1, \quad n-1 \ge r \ge 0$$

$$\iff n-1 \ge r > \frac{1}{2}(n-1) \ge 0$$

c. $\binom{n}{r} = \binom{n}{r+1} \iff r = \frac{1}{2}(n-1)$

Proof: From the steps in (a) + (b),

$$\binom{n}{r} = \binom{n}{r+1} \iff r+1 = n-r, \quad n-1 \geq r \geq 0$$

$$\iff 2r = n-1, \quad n-1 \geq r \geq 0.$$

$$\iff r = \tfrac{1}{2}(n-1), \quad n-1 \geq r \geq 0$$

5. a. For $n \geq 2$, $\binom{2}{2} + \binom{3}{2} + \dots + \binom{n}{2} = \binom{n+1}{3}$

Proof: For $k=2$, $\binom{2}{2} = 1 = \binom{2+1}{3} = 1$

$k \Rightarrow k+1$: Assume $\binom{2}{2} + \dots + \binom{k}{2} = \binom{k+1}{3}$

Then, $\binom{2}{2} + \dots + \binom{k}{2} + \binom{k+1}{2}$

$$= \binom{k+1}{3} + \binom{k+1}{2}$$

$$= \binom{k+2}{3}$$ From Pascal's identity $\binom{r}{s} + \binom{r}{s-1} = \binom{r+1}{s}$, $1 \leq s \leq r$

6. First, $m^2 = 2\binom{m}{2} + m$, $m \geq 2$

$$2\binom{m}{2} + m \iff 2\frac{m!}{2!(m-2)!} + m, \quad m \geq 2$$

$$\iff m(m-1) + m = m^2, \quad m \geq 2$$

Now, $1^2 + 2^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

Proof: $1^2 + 2^2 + \cdots + n^2$

$$= 1 + 2\binom{2}{2} + 2 + 2\binom{3}{2} + 3 + \cdots + 2\binom{n}{2} + n$$

$$= (1 + 2 + \cdots + n) + 2\left[\binom{2}{2} + \binom{3}{2} + \cdots + \binom{n}{2}\right]$$

$$= (1 + 2 + \cdots + n) + 2\binom{n+1}{3}$$

$$= (1 + 2 + \cdots + n) + 2\frac{(n+1)!}{3 \cdot 2 \cdot (n-2)!}$$

$$= \frac{n(n+1)}{2} + \frac{(n+1)(n)(n-1)}{3}$$

$$= \frac{3n(n+1) + 2(n+1)(n)(n-1)}{6}$$

$$= \frac{n(n+1)[3 + 2n - 2]}{6} = \frac{n(n+1)(2n+1)}{6}$$

c. $1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \cdots + n(n+1) = \dfrac{n(n+1)(n+2)}{3}$

Proof: From (b), $m^2 = 2\binom{m}{2} + m$, or

$$m(m-1) = 2\binom{m}{2}, \quad m \geq 2$$

$$\therefore\ 1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \cdots + n(n+1)$$

$$= 2\binom{2}{2} + 2\binom{3}{2} + 2\binom{4}{2} + \cdots + 2\binom{n+1}{2}$$

$$= 2\left[\binom{n+2}{3}\right], \text{ from (a)}$$

$$= 2\,\frac{(n+2)!}{3!\,(n-1)!} = \frac{2\,(n+2)(n+1)(n)}{3\cdot 2\cdot 1}$$

$$= \frac{n(n+1)(n+2)}{3}$$

6. $\dbinom{2}{2} + \dbinom{4}{2} + \cdots + \dbinom{2n}{2} = \dfrac{n(n+1)(4n-1)}{6}, \ n \geq 2$

Proof: First, $\dbinom{2m}{2} = \dfrac{(2m)!}{2!(2m-2)!} = \dfrac{2m(2m-1)}{2}$

$$= 2m^2 - m = m^2 + m^2 - m$$

$$= m^2 + 2\binom{m}{2}, \quad m \geq 2, \text{ from } 5(c)$$

$$\therefore \binom{2}{2} + \binom{4}{2} + \cdots + \binom{2n}{2}$$

$$= 1 + \left[2^2 + 2\binom{2}{2} + \cdots + n^2 + 2\binom{n}{2}\right]$$

$$= (1^2 + 2^2 + \cdots + n^2) + 2\left[\binom{2}{2} + \cdots + \binom{n}{2}\right]$$

$$= \frac{n(n+1)(2n+1)}{6} + 2\binom{n+1}{3}, \quad \text{from } 5(b), 5(a) \quad n \geq 2$$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{2(n+1)!}{3 \cdot 2 \cdot (n-2)!}$$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{2(n+1)(n)(n-1)}{6}$$

$$= \frac{n(n+1)\left[2n+1 + 2n-2\right]}{6}$$

$$= \frac{n(n+1)(4n-1)}{6}$$

7. For $n \geq 1$, $1^2 + 3^2 + \dots + (2n-1)^2 = \binom{2n+1}{3}$

Proof: $k=1$: $1^2 = 1 = \binom{2\cdot 1+1}{3} = \binom{3}{3} = 1$

$k \Rightarrow k+1$: Assume $1^2 + 3^2 + \dots (2k-1)^2 = \binom{2k+1}{3}$

Then, $1^2 + 3^2 + \dots + (2k-1)^2 + (2(k+1)-1)^2$

$$= 1^2 + \dots + (2k-1)^2 + (2k+1)^2$$

$$= \binom{2k+1}{3} + (2k+1)^2$$

$$= \frac{(2k+1)!}{3!(2k-2)!} + (2k+1)^2$$

$$= \frac{(2k+1)!\,(2k)(2k-1)}{3!(2k-2)!\,(2k)(2k-1)} + \frac{6\cdot(2k+1)^2\cdot(2k)!}{6\cdot(2k)!}$$

$$= \frac{(2k+1)!\,[(2k)(2k-1) + 6(2k+1)]}{3!\,(2k)!}$$

$$= \frac{(2k+1)!\,[4k^2 - 2k + 12k + 6]}{3!\,(2k)!}$$

$$= \frac{(2k+1)!\,[(2k+2)(2k+3)]}{3!\,(2k)!}$$

$$= \frac{(2k+3)!}{3!\,(2k+3-3)!} = \binom{2k+3}{3}$$

So, $k \Rightarrow k+1$

8. For $n \geq 1$, $\binom{2n}{n} = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n} 2^{2n}$

$k=1$: $\binom{2}{1} = \frac{2!}{1!\,1!} = 2$, $\frac{1}{2} 2^2 = \frac{4}{2} = 2$

$k \Rightarrow k+1$: Suppose $\binom{2k}{k} = \frac{1\cdot 3\cdot 5\cdots(2k-1)}{2\cdot 4\cdot 6\cdots 2k} 2^{2k}$

Then $\binom{2k+2}{k+1} = \frac{(2k+2)!}{(k+1)!\,(k+1)!}$

$$= \frac{(2k+2)(2k+1)\,(2k)!}{(k+1)(k+1)\,k!\,k!}$$

$$= \frac{(2k+2)(2k+1)}{(k+1)(k+1)} \binom{2k}{k}$$

$$= \frac{(2k+2)(2k+1)}{(k+1)(k+1)} \cdot \frac{1\cdot 3\cdot 5\cdots(2k-1)}{2\cdot 4\cdot 6\cdots 2k} 2^{2k}$$

$$= \frac{2(k+1)}{(k+1)(k+1)} \cdot \frac{1\cdot 3\cdot 5\cdots(2k-1)(2k+1)}{2\cdot 4\cdot 6\cdots 2k} 2^{2k}$$

$$= \frac{2}{(k+1)} \cdot \frac{1\cdot 3\cdot 5\cdots(2k+1)}{2\cdot 4\cdot 6\cdots 2k} 2^{2k}$$

$$= \frac{4}{(2k+2)} \cdot \frac{1\cdot 3\cdot 5\cdots(2k+1)}{2\cdot 4\cdot 6\cdots 2k} 2^{2k}$$

$$= \frac{1\cdot 3\cdot 5\cdots(2k+1)}{2\cdot 4\cdot 6\cdots(2k)(2k+2)} \cdot 2^{2k+2}$$

So, $k \Rightarrow k+1$

9. $2^n < \binom{2n}{n} < 2^{2n}$, for $n > 1$

Proof: $n! < 1\cdot 3\cdot 5\cdots(2n-1)$, for $n > 1$
Since it's true for $k=2$ $(2 < 1\cdot 3)$
and if $k! < 1\cdot 3\cdots(2k-1)$, Then
$(k+1)! = k!(k+1) < 1\cdot 3\cdot 5\cdots(2k-1)(k+1)$
$< 1\cdot 3\cdot 5\cdots(2k-1)(2k+1)$

Also, $2^n n! = 2\cdot4\cdot6\cdots 2n$, since it's true for $k=1$, and $2^{k+1}(k+1)! =$
$2(k+1)2^k k! = 2(k+1)\cdot 2\cdot4\cdot6\cdots 2k$
$= 2\cdot4\cdot6\cdots 2k\cdot 2(k+1)$

So, $2^n n! < 1\cdot3\cdot5\cdots(2n-1)2^n$

$$\Rightarrow \quad 1 < \frac{1\cdot3\cdot5\cdots(2n-1)}{2^n n!}\,2^n$$

$$= \frac{1\cdot3\cdot5\cdots(2n-1)}{2\cdot4\cdots 2n}\,2^n$$

$$\therefore \quad 2^n < \frac{1\cdot3\cdot5\cdots(2n-1)}{2\cdot4\cdot6\cdots 2n}\,2^{2n} = \binom{2n}{n}, \text{ by } (8)$$

Now, since $2k-1 < 2k$ for $k \geq 1$,

Then $1\cdot3\cdot5\cdots(2n-1) < 2\cdot4\cdot6\cdots 2n$,

$$\text{So, } \frac{1\cdot3\cdot5\cdots(2n-1)}{2\cdot4\cdot6\cdots 2n} < 1, \text{ for } n \geq 1$$

$$\therefore \text{ by } (8), \binom{2n}{n} = \frac{1\cdot3\cdot5\cdots(2n-1)}{2\cdot4\cdot6\cdots 2n}\,2^{2n} < 2^{2n}, \; n \geq 1$$

10. Given $C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{n!\,(n+1)!}$, $n \geq 0$

Prove: $C_n = \frac{2(2n-1)}{n+1} C_{n-1}$, $n \geq 1$

Proof: $k=1$: $C_1 = \frac{2!}{1!\,2!} = 1$, $C_0 = \frac{0!}{0!\,1!} = 1$

$$\therefore \frac{2(2-1)}{1+1} = 1,$$

$$\text{So, } C_1 = \frac{2(2-1)}{1+1} C_0$$

$k \Rightarrow k+1$: Suppose $C_k = \frac{2(2k-1)}{k+1} C_{k-1}$

$$\text{Then } C_{k+1} = \frac{(2k+2)!}{(k+1)!\,(k+2)!}$$

$$= \frac{(2k+2)(2k+1)}{(k+1)(k+2)} \cdot \frac{(2k)!}{k!\,(k+1)!}$$

$$= \frac{2(2k+1)}{(k+2)} \cdot C_k = \frac{2(2k+1)}{(k+2)} \cdot \frac{2(2k-1)}{(k+1)} C_{k-1}$$

$$= \frac{2(2k+1)\cdot 2(2k-1)}{(k+2)(k+1)} \cdot \frac{(2k-2)!}{(k-1)!\,k!}$$

$$= \frac{2(2k+1)\cdot 2}{(k+2)} \cdot \frac{(2k-1)!}{(k-1)!\,(k+1)!}$$

$$= \frac{2(2k+1)}{(k+2)} \cdot \frac{2k}{k} \cdot \frac{(2k-1)!}{(k-1)!\,(k+1)!}$$

$$= \frac{2(2k+1)}{(k+2)} \cdot \frac{(2k)!}{k!\,(k+1)!}$$

$$= \frac{2[2(k+1)-1]}{[(k+1)+1]} \, C_{(k+1)-1}$$

So, $k \Rightarrow k+1$