1.3 Early Number Theory

Note Title 5/10/2004

1. a. A number is triangular $\iff$ it is of the form $\frac{n(n+1)}{2}$ for some $n \geq 1$.

Proof: $1+2+3+\dots+n = \frac{n(n+1)}{2}$

from problem 1(a) of Problem Set 1.1.

So, if a number X is triangular then for some n, $1+2+\dots+n = X$ by definition, and so $X = \frac{n(n+1)}{2}$

If $X = \frac{n(n+1)}{2}$ for some n, then

$X = 1+2+\dots+n$

b. An integer n is triangular $\iff$ $8n+1$ is a perfect square.

Proof: If n is triangular, then there is a k such that $n = \frac{k(k+1)}{2}$

$\therefore 8n = 4k(k+1)$,

$$8n+1 = 4k(k+1)+1$$
$$= 4k^2+4k+1$$
$$= (2k+1)^2$$

$\therefore$ n triangular $\Rightarrow$ $8n+1$ is a perfect square

If $8n+1$ is a perfect square, then there is an integer k such that $k^2 = 8n+1$.

Note that $8n+1$ must be odd.
$\therefore$ k^2 is odd, and so k is odd.
$\therefore$ There is an s such that $2s+1=k$

$$\therefore (2s+1)^2 = 8n+1$$

$$\therefore 4s^2+4s+1 = 8n+1$$

$$\therefore 4s(s+1) = 8n$$

$$\therefore \frac{s(s+1)}{2} = n$$

$\therefore$ $8n+1$ a perfect square $\Rightarrow$ n triangular

c. If a and b are consecutive triangular numbers, then $a+b$ is a perfect square.

Proof: Let $1+2+\cdots+n = a$

Then $1+2+\cdots+n+n+1 = b$

$$\therefore\ a+b = \frac{n(n+1)}{2} + \frac{n(n+1)}{2} + (n+1)$$

$$= n(n+1) + (n+1)$$

$$= (n+1)(n+1)$$

So, $a+b$ is a perfect square.

d. If n is triangular, then so are

$9n+1$, $25n+3$, and $49+6$.

Proof: Let $1+2+\cdots+k = n$

Then $9n+1 = 9\frac{k(k+1)}{2} + 1 = \frac{9k^2+9k+2}{2}$

$$= \frac{(3k+1)(3k+2)}{2} = \frac{s(s+1)}{2}$$

for $s = 3k+1$, and so by 1(a), $9n+1$ is triangular.

$$25n+3 = \frac{25k(k+1)}{2} + 3 = \frac{25k^2 + 25k + 6}{2}$$

$$= \frac{(5k+2)(5k+3)}{2} = \frac{s(s+1)}{2}$$

for $s = 5k+2$, and so by 1(a), $25n+3$ is triangular.

$$49n+6 = \frac{49k(k+1)}{2} + 6 = \frac{49k^2 + 49k + 12}{2}$$

$$= \frac{(7k+3)(7k+4)}{2} = \frac{s(s+1)}{2}$$

for $s = 7k+2$, and so by 1(a), $49+6$ is triangular.

2. $t_n = \binom{n+1}{2}$, $n \geq 1$, t_n the n^{th} triangular.

$$\binom{n+1}{2} = \frac{(n+1)!}{2!\,(n-1)!} = \frac{(n+1)n}{2}, \text{ so by } 1(a),$$

$$t_n = \binom{n+1}{2}$$

3. $$t_1 + t_2 + \dots + t_n = \frac{n(n+1)(n+2)}{6}, \quad n \geq 1$$

Proof: From problem 1(c) of problem set 1.1,

$$1\cdot 2 + 2\cdot 3 + \dots + n(n+1) = \frac{n(n+1)(n+2)}{3}, \quad n \geq 1$$

$$\therefore \frac{1\cdot 2}{2} + \frac{2\cdot 3}{2} + \dots + \frac{n(n+1)}{2} = \frac{n(n+1)(n+2)}{6}$$

Note that each term K can be written as $\frac{K(K+1)}{2} = t_K$

$$\therefore t_1 + t_2 + \dots + t_n = \frac{n(n+1)(n+2)}{6}$$

The hint given: $t_{k-1} + t_k = k^2$

$$t_{k-1} = \frac{(k-1)(k-1+1)}{2} = \frac{k(k-1)}{2}$$

$$\therefore t_{k-1} + t_k = \frac{k(k-1)}{2} + \frac{k(k+1)}{2}$$

$$= \frac{k^2 - k + k^2 + k}{2} = k^2$$

You could prove the statement using

$1^2 + 2^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$ from

5(b) of problem set 1.2, and breaking problem up into even + odd number of terms.

4. $9(2n+1)^2 = t_{9n+4} - t_{3n+1}$

Proof: Since $t_k = \dfrac{k(k+1)}{2}$,

$$t_{9n+4} = \frac{(9n+4)(9n+5)}{2}, \quad t_{3n+1} = \frac{(3n+1)(3n+2)}{2}$$

$$\therefore\ t_{9n+4} - t_{3n+1} = \frac{(81n^2+81n+20)-(9n^2+9n+2)}{2}$$

$$= \frac{72n^2+72n+18}{2} = 36n^2+36n+9$$

$$= 9(4n^2+4n+1) = 9(2n+1)^2$$

5. a. Find two triangular numbers, t_r and t_s, such that $t_r + t_s$ and $t_r - t_s$ are triangular.

The triangular numbers are:

1, 3, 6, 10, 15, 21, 28, 36, 45, 55, ...

$15 + 21 = 36$, $21 - 15 = 6$

6. Three successive triangular numbers whose product is a perfect square.

$$\frac{n(n+1)}{2} \cdot \frac{(n+1)(n+2)}{2} \cdot \frac{(n+2)(n+3)}{2} = k^2$$

$$(n+1)^2(n+2)^2 \cdot \frac{n \cdot (n+3)}{2} = 4k^2 = (2k)^2$$

So, if can find an n such that $\frac{n(n+3)}{2}$ is a perfect square, problem would be solved. The perfect squares are 1, 4, 9, 16, 25, 36, 49, ...

By trial and error, if $n=3$, Then $\frac{3(3+3)}{2} = 9$

So, $t_3 \cdot t_4 \cdot t_5 = 6 \cdot 10 \cdot 15 = 900 = 30^2$

c. Three successive triangular numbers whose sum is a perfect square.

Trial & error works faster than trying to figure this out.

$$t_5 + T_6 + t_7 = 15 + 21 + 28 = 64 = 8^2$$

6.a. If t_n is a perfect square, then $t_{4n(n+1)}$ is also a perfect square.

Proof: Assume $t_n = k^2 = \frac{n(n+1)}{2}$

Then $2k^2 = n(n+1)$

$$t_{4n(n+1)} = \frac{4n(n+1)\left(4n(n+1)+1\right)}{2}$$

$$= \frac{4 \cdot 2k^2 \cdot \left[4n^2 + 4n + 1\right]}{2}$$

$$= 4k^2 (2n+1)^2$$

$$= \left[2k(2n+1)\right]^2$$, and so is a square.

6. $t_1 = 1$ is a perfect square

$t_{4 \cdot 1(1+1)} = t_8 = 36$ is a perfect square

$$t_{4 \cdot 8(8+1)} = t_{288} = \frac{288(288+1)}{2} = 41,616 = 204^2$$

7. $t_{n+1}^2 - t_n^2 = k^3$, for some integer k

Proof: $t_{n+1} = \dfrac{(n+1)(n+2)}{2}$, $t_n = \dfrac{n(n+1)}{2}$

$$\therefore t_{n+1}^2 - t_n^2 = \frac{(n+1)^2(n+2)^2 - (n+1)^2 n^2}{4}$$

$$= \frac{(n+1)^2\left[n^2+4n+4-n^2\right]}{4}$$

$$= \frac{(n+1)^2 \cdot (4n+4)}{4} = (n+1)^3, \text{ for } n \geq 1$$

8. $\frac{1}{1} + \frac{1}{3} + \frac{1}{6} + \dots + \frac{1}{t_n} < 2$

Proof: Each term can be written as

$$\frac{1}{\frac{k(k+1)}{2}} = \frac{2}{k(k+1)} = 2\left[\frac{1}{k} - \frac{1}{k+1}\right]$$

$$\therefore \frac{1}{1} + \frac{1}{3} + \dots + \frac{1}{t_n} = 2\left[\frac{1}{1} - \frac{1}{2}\right] + 2\left[\frac{1}{2} - \frac{1}{3}\right] + \dots + 2\left[\frac{1}{n} - \frac{1}{n+1}\right]$$

$$= 2\left[\frac{1}{1} - \frac{1}{n+1}\right] = 2\left(1 - \frac{1}{n+1}\right)$$

Since $n > 0$, Then $n+1 > 0$, so $\frac{1}{n+1} > 0$,

and $-\frac{1}{n+1} < 0$, so $1 - \frac{1}{n+1} < 1$,

so $2\left(1 - \frac{1}{n+1}\right) < 2$.

$\therefore \frac{1}{1} + \frac{1}{3} + \dots + \frac{1}{t_n} < 2$

9. a. $t_x = t_y + t_z$, $x = \frac{n(n+3)}{2} + 1$, $y = n+1$, $z = \frac{n(n+3)}{2}$

Proof: $t_y + t_z = \frac{(n+1)(n+2)}{2} + \frac{\frac{n(n+3)}{2}\left[\frac{n(n+3)}{2}+1\right]}{2}$

$$= \frac{2\,\frac{(n+1)(n+2)}{2} + \frac{n(n+3)}{2}\left[\frac{n(n+3)}{2}+1\right]}{2}$$

$$= \frac{2\,\frac{\left[n^2+3n+2\right]}{2} + \frac{n(n+3)}{2}\left[\frac{n(n+3)}{2}+1\right]}{2}$$

$$= \frac{2\left[\frac{n(n+3)}{2}+1\right] + \frac{n(n+3)}{2}\left[\frac{n(n+3)}{2}+1\right]}{2}$$

$$= \frac{\left[\frac{n(n+3)}{2}+1\right]\left[\frac{n(n+3)}{2}+1+1\right]}{2}$$

$= t_x$, for $n \geq 1$ (if $n=0$, then $z=0$)

b. $n=1$: $t_3 = t_2 + t_2$, or $6 = 3+3$

$n=2$: $t_6 = t_3 + t_5$, or $21 = 6 + 15$

$n = 3$; $t_{10} = t_4 + t_9$, or $55 = 10 + 45$