2.1 The Division Algorithm

Note Title 9/13/2004

1. a, b integers, $b > 0$, $\exists$ unique q, r s.t.
$a = qb + r$, $2b \le r < 3b$

Pf: By Division Alg., $\exists$ unique q', r', s.t.

$$a = q'b + r', \quad 0 \le r' < b$$

$$\therefore a = q'b + r' + 2b - 2b = (q'-2)b + r' + 2b$$

Let $q = q' - 2$, $r = r' + 2b$. $\therefore r, q$ unique

Since $0 \le r' < b$, Then

$$2b \le r' + 2b < b + 2b, \text{ or } 2b \le r < 3b$$

2. If $a = 6k+5$, then for some j, $a = 3j + 2$

Pf: $a = 6k + 5 = 3 \cdot 2k + 3 + 2 = 3(2k+1) + 2$
Let $j = 2k+1$. Conversely, if $a = 8 = 3(2) + 2$,
$8 = 6(1) + 2$, 1 and 2 are unique, so $8 \ne 6k+5$.

3. a. If a is an integer, Then $a^2 = 3k$ or $a^2 = 3k+1$

Pf: By Division Algorithm, $\exists$ a q s.t.

$a = 3q$ or $a = 3q+1$ or $a = 3q+2$

$a = 3q$: $\therefore a^2 = 9q^2 = 3(3q^2)$. Let $k = 3q^2$

$a = 3q+1$: $\therefore a^2 = 9q^2 + 6q + 1 = 3(3q^2 + 2q) + 1$

Let $k = 3q^2 + 2q$

$a = 3q+2$: $\therefore a^2 = 9q^2 + 6q + 4 = 9q^2 + 6q + 3 + 1$

$= 3(3q^2 + 2q + 1) + 1$

Let $k = 3q^2 + 2q + 1$

6. If a an integer, Then $a^3 = 9k$, or $9k+1$, or $9k+8$

Pf: Let $a = 3q + r$, $r = 0, 1, 2$

$$(3q)^3 = 27q^3 = 9(3q^3) = 9k$$

$$(3q+1)^3 = \binom{3}{0}(3q)^3 + \binom{3}{1}(3q)^2 + \binom{3}{2}3q + \binom{3}{3}$$
$$= 27q^3 + 27q^2 + 9q + 1$$
$$= 9(3q^3 + 3q^2 + q) + 1 = 9k+1$$

$$(3q+2)^3 = \binom{3}{0}(3q)^3 + \binom{3}{1}(3q)^2 \cdot 2 + \binom{3}{2}(3q)2^2 + \binom{3}{3}2^3$$
$$= 27q^3 + 54q^2 + 36q + 8$$
$$= 9(3q^3 + 6q^2 + 4q) + 8 = 9k+8$$

c. If n an integer, Then $n^4 = 5K$ or $5K+1$

Pf: Let $n = 5q + r$, $0 \le r < 5$

Consider $n^4 = (5q + r)^4$

From binomial expansion, each term is a factor of 5 except last term:

$$\binom{4}{0}(5q)^4 + \binom{4}{1}(5q)^3 r + \binom{4}{2}(5q)^2 r^2 + \binom{4}{3}(5q) r^3 + r^4$$

$r = 0$, Then $r^4 = 0$, and $n^4 = 5K$ as all other terms have 5 as a factor

$r = 1$, Then clearly $n^4 = 5K + 1$

$r = 2$, Then $r^4 = 16 = 15 + 1$, so all terms and 15 have 5 as a factor, so again, $n = 5K + 1$

$r = 3$, Then $r^4 = 81 = 80 + 1$, and $80 = 5 \cdot 16$, so, again, $n^4 = 5K + 1$

4. Prove $3a^2 - 1$ is never a perfect square.

Pf: Suppose $3a^2 - 1 = n^2$, some n. By 3(a), $3a^2 - 1 = 3K + 1$

or $3a^2 - 1 = 3k$. $\therefore 3(a^2-k) = 2$ or $3(a^2-k) = 1$, each impossible, since by Div. Alg., $2 = 3\cdot 0 + 2$ and $1 = 3\cdot 0 + 1$.

5. For $n \geq 1$, prove $\dfrac{n(n+1)(2n+1)}{6}$ is an integer.

Pf: $n = 6k + r$, $0 \leq r < 5$. Let $A = \dfrac{n(n+1)(2n+1)}{6}$

$r = 0$: Then $A = k(6k+1)(12k+1)$, an integer

$r = 1$: $A = \dfrac{(6k+1)(6k+2)(12k+3)}{6}$

$= \dfrac{(6k+1)(72k^2 + 42k + 6)}{6}$

$= (6k+1)(12k^2 + 7k + 1)$, an integer

$r = 2$: $A = \dfrac{(6k+2)(6k+3)(12k+5)}{6}$

$= \dfrac{(36k^2 + 30k + 6)(12k+5)}{6}$

$= (6k^2 + 5k + 1)(12k+5)$, an integer

$r = 3$: $A = \dfrac{(6k+3)(6k+4)(12k+7)}{6}$

$= \dfrac{(36k^2 + 42k + 12)(12k+7)}{6}$

$= (6k^2 + 7k + 2)(12k+7)$, an int.

$$r=4: A = \frac{(6k+4)(6k+5)(12k+9)}{6}$$

$$= \frac{(72k^2+102k+36)(6k+5)}{6}$$

$$= (12k^2+17k+6)(6k+5), \text{ an int.}$$

$$r=5: A = \frac{(6k+5)(6k+6)(12k+11)}{6}$$

$$= \frac{(36k^2+66k+30)(12k+11)}{6}$$

$$= (6k^2+11k+5)(12k+11), \text{ an int.}$$

6. If A an integer, then $A^3 = 7k$ or $7k \pm 1$, some k.

Pf: $A = 7q + r, \quad 0 \leq r < 7$

$$r=0: A^3 = 7^3q^3 = 7(7^2q^3). \text{ Let } k = 7^2q^3$$

$$r=1: A^3 = (7q+1)^3 = \binom{3}{0}(7q)^3 + \binom{3}{1}(7q)^2 + \binom{3}{2}(7q) + 1$$

$$= 7(7^2q^3 + 3\cdot 7q^2 + 3q) + 1$$

$$r=2: A^3 = (7q+2)^3 = \binom{3}{0}(7q)^3 + \binom{3}{1}(7q)^2 2 + \binom{3}{2}(7q)2^2 + 2^3$$

$$= 7[\ldots] + 8 = 7[\ldots] + 7 + 1$$

$$= 7[\ldots + 1] + 1$$

$r=3: A^3=(7q+3)^3 = \binom{3}{0}(7q)^3 + \binom{3}{1}(7q)^2 3 + \binom{3}{2}(7q)3^2 + 3^3$

$= 7[\ldots] + 28 - 1 = 7[\ldots + 4] - 1$

$r=4: A^3=(7q+4)^3$. Last term is $4^3 = 64 = 7\cdot 9 + 1$

So, $A^3 = 7[\ldots + 9] + 1$

$r=5: A^3 = (7q+5)^3$. Last term $= 5^3 = 125 = 7\cdot 18 - 1$

So, $A^3 = 7[\ldots + 18] - 1$

$r=6: A^3 = (7q+6)^3$. Last term $= 6^3 = 216 = 31\cdot 7 - 1$

$\therefore A^3 = 7[\ldots + 31] - 1$

7. For a, b s.t. $b \neq 0$, $\exists$ unique q, r s.t.
$a = qb + r$ and $-\frac{1}{2}|b| < r \leq \frac{1}{2}|b|$

Pf.: Break up $0 < |b|$ into $0 < \frac{1}{2}|b|$ and $\frac{1}{2}|b| < |b|$

$\exists$ unique q', r' s.t. $a = bq' + r'$, $0 \leq r' < |b|$

If $0 \leq r' \leq \frac{1}{2}|b|$, let $r = r'$, $q = q'$

If $\frac{1}{2}|b| < r' < |b|$, then $-\frac{1}{2}|b| < r' - |b| < 0$

$\therefore a = bq' + r' - |b| + |b|$

If $b \geq 0$, then $a = b(q'+1) + r' - |b|$, so let $r = r' - |b|$, $q = q'+1$

If $b < 0$, then $|b| = -b$, so $a = bq' + r' - |b| - b$
$a = b(q'-1) + r' - |b|$, so
let $q = q'-1$, $r = r' - |b|$

8. None of integers in below sequence is a perfect square:

$11, 111, 1111, 11111, \ldots$

Pf: Any number in sequence can be written as

$$A = 11 + 100 + 1000 + \ldots = 11 + \sum_{i=2}^{n} 10^i$$

Each term of $\sum_{i=2}^{n} 10^i$ is divisible by 4.

So, $A_i = 11 + 4r_i = 4(r+2) + 3$, for certain r_i.

e.g., $11 = 4(2+2) + 3$, $111 = 4(25+2) + 3$

So $A_i = 4r_i' + 3$. By Div. Alg., r_i' and 3 are unique.

Suppose $A_1 = s^2$. Let $s = 4q + r$

$r \neq 0$, as $s^2 = 16q^2 = 4(4q^2)$, which is not of $4r_i' + 3$ form.

$r \neq 1$: $s^2 = 16q^2 + 8q + 1 = 4(4q^2 + 2q) + 1$ and so not of $4r_i' + 3$ form

$r \neq 2$: $s^2 = 16q^2 + 16q + 4 = 4(4q^2 + 4q + 1)$, and so not of $4r_i' + 3$ form.

$r \neq 3$: $s^2 = 16q^2 + 24q + 9 = 4(4q^2 + 6q + 2) + 1$, and so not of form $4r_i' + 3$ form.

$\therefore$ There is no s s.t. $s^2 = 4r_i' + 3$.
$\therefore$ All A_i are not perfect squares.

9. If integer $A = r^2 = s^3$ for some r, q, then $A = 7k$ or $A = 7k + 1$ for some k.

Pf: Let $S = 7K + 6$, $0 \leq 6 < 7$

From #6 above $s^3 = 7K_i$ if $b = 0$

$s^3 = 7K_i + 1$ if $b = 1, 2, 4$

$s^3 = 7K_i - 1$ if $b = 3, 5, 6$

for some K_i $(i = 0, 1, 2, \ldots, 6)$

Or, $s^3 = 7K_i$ if $b = 0$

$s^3 = 7K_i + 1$ if $b = 1, 2, 4$

$s^3 = 7K_i' + 6$ if $b = 3, 5, 6$,

for some $K_0, K_1, K_2, K_4, K_3', K_5', K_6'$

Now look at $A = r^2$

Let $r = 7c + d$, $0 \le d < 6$

$d = 0$: $r^2 = 7(7c^2) = s^3 = 7K_0$

$d = 1$: $r^2 = 49c^2 + 14c + 1 = 7(7c^2 + 2c) + 1 = s^3 = 7K_1 + 1$

$d = 2$: $r^2 = 49c^2 + 28c + 4 = 7(7c^2 + 4c) + 4$

$d = 3$: r^2 last term $= 9 = 7 + 2$

so $r^2 = 7K + 2$

$d = 4$: r^2 last term $= 16 = 14 + 2$,

so $r^2 = 7K + 2$

$d = 5$: r^2 last term $= 25 = 21 + 4$

so $r^2 = 7K + 4$

$d = 6$: r^2 last term $= 36 = 35 + 1$,

so $r^2 = 7K + 1$

Thus, r^2 of form: $7k$, $7k+1$, $7k+2$, $7k+4$

s^3 of form $7k$, $7k+1$, or $7k+6$

By uniqueness part of Div. Algorithm,

A must be either of form $7k$ or $7k+1$

10. For $n \geq 1$, show $n(7n^2+5)$ is of form $6k$

Pf: Let $n = 6k + r$, $0 \leq r < 6$. Let $A = n(7n^2+5)$

$r=0$: $A = 6k(7(6k)^2+5) = 6[\quad]$

$$
\begin{aligned}
r=1: A &= (6k+1)(7(6k+1)^2+5) \\
&= 7(6k+1)^3 + 30k + 5 \\
&= 7[6(\ldots)+1] + 30k + 5 \\
&= 6\cdot7(\ldots) + 7 + 6\cdot5k + 5 \\
&= 6[7(\ldots)+5k] + 12 \\
&= 6[7(\ldots)+5k+2] = 6k'
\end{aligned}
$$

$$
\begin{aligned}
r=2: A &= (6k+2)(7(6k+2)^2+5) \\
&= 7(6k+2)^3 + 30k + 10 \\
&= 7[6(\ldots)+8] + 30k + 10 \\
&= 6\cdot7(\ldots) + 56 + 6\cdot5k + 10 \\
&= 6[7(\ldots)+5k+11] = 6k'
\end{aligned}
$$

$r=3: A = (6k+3)(7(6k+3)^2+5)$
$= 7(6(\ldots)+27) + 6\cdot 5k + 15$
$= 6\cdot 7(\ldots) + 6\cdot 5k + 7\cdot 27 + 15$
$= 6[7(\ldots) + 5k + 34] = 6k'$

$r=4: A = (6k+4)(7(6k+4)^2+5)$
$= 7[6(\ldots)+64] + 6\cdot 5k + 20$
$= 6\cdot 7(\ldots) + 6\cdot 5k + 7\cdot 64 + 20$
$= 6[7(\ldots) + 5k + 78] = 6k'$

$r=5: A = (6k+5)(7(6k+5)^2+5)$
$= 7[6(\ldots)+125] + 6\cdot 5k + 25$
$= 6\cdot 7(\ldots) + 6\cdot 5k + 7\cdot 125 + 25$
$= 6[7(\ldots) + 5k + 150] = 6k'$

11. If n is odd, show n^4+4n^2+11 is of form $16k$.

Pf: Let $n = 2k+1$

$$n^4+4n^2+11 = (n^2+2)^2+7$$
$$= [(2k+1)^2+2]^2+7$$
$$= [4k^2+4k+1+2]^2+7$$

$$= \left(4k^2 + 4k + 3\right)^2 + 7$$

$$\begin{aligned} = 16k^4 &+ 16k^3 + 12k^2 \\ &+ 16k^3 + 16k^2 + 12k \\ &\qquad + 12k^2 + 12k + 9 + 7 \end{aligned}$$

$$= 16k^4 + 32k^3 + 40k^2 + 24k + 16$$

$k = 2q$ or $2q + 1$

$$k = 2q: \quad 16(2q)^4 + 32(2q)^3 + 40(2q)^2 + 24(2q) + 16$$

$$= 16\left[(2q)^4 + 2(2q)^3 + 10q^2 + 3q + 1\right]$$

$$= 16x$$

$$k = 2q+1: \quad 16(2q+1)^4 + 32(2q+1)^3 + 40(2q+1)^2 + 24(2q+1) + 16$$

$$= 16(\ )^4 + 32(\ )^3 + 160q^2 + 160q + 40 + 48q + 24 + 16$$

$$= 16\left[(\ )^4 + 2(\ )^3 + 10q^2 + 10q + 3q + 4 + 1\right]$$

$$= 16x$$