2.4 The Diophantine Equation ax + by = c

Note Title 11/10/2004

1. (a). $6x + 51y = 22$

$\gcd(6, 51) = 3$, and $3 \nmid 22$. $\therefore$ Can't be solved.

(b) $33x + 14y = 115$

$\gcd(33, 14) = 1$, $\therefore$ it can be solved.

(c) $14x + 35y = 93$

$\gcd(14, 35) = 7$, $7 \nmid 93$. $\therefore$ can't be solved.

2. Use Euclidean Alg. to get $d = \gcd(a, b)$, then express d in terms of a, b, then multiply d to get c and x_0, y_0

(a) $56x + 72y = 40$

$72 = 56 + 16$ $\quad$ $8 = 56 - 3 \cdot 16$

$56 = 3 \cdot 16 + 8$ $\quad$ $= 56 - 3(72 - 56)$

$16 = 2 \cdot 8$ $\quad$ $= 4 \cdot 56 - 3 \cdot 72$

$\therefore \gcd = 8$ $\quad$ $\therefore 5 \cdot 8 = 40 = 20 \cdot 56 - 15 \cdot 72$

$\therefore (20, -15)$ a solution

$$\therefore X = 20 + \frac{72}{8}t, \quad y = -15 - \frac{56}{8}t$$

$$\text{or } x = 20 + 9t, \quad y = -15 - 7t$$

(b) $24x + 138y = 18$

$138 = 5 \cdot 24 + 18$
$24 = 18 + 6$
$18 = 3 \cdot 6$
$\therefore \gcd = 6$

$6 = 24 - 18$
$= 24 - (138 - 5 \cdot 24)$
$= 6 \cdot 24 - 138$
$\therefore 18 = 3 \cdot 6 = (18)24 - (3)138$
$\therefore (18, -3)$ is a solution

$\therefore x = 18 + \frac{138}{6}t = 18 + 23t$

$y = -3 - \frac{24}{6}t = -3 - 4t$

(c) $221x + 35y = 11$

$221 = 6 \cdot 35 + 11$
$35 = 3 \cdot 11 + 2$
$11 = 5 \cdot 2 + 1$
$2 = 2 \cdot 1$
$\therefore \gcd = 1$

$\therefore 1 = 11 - 5 \cdot 2$
$= 11 - 5(35 - 3 \cdot 11)$
$= 16 \cdot 11 - 5 \cdot 35$
$= 16(221 - 6 \cdot 35) - 5 \cdot 35$
$= 16 \cdot 221 - 101 \cdot 35$
$\therefore 11 = (11 \cdot 16)(221) - (11 \cdot 101)(35)$
$\therefore (176, -1111)$ a solution

$\therefore x = 176 + 35t$
$y = -1111 - 221t$

3. (a) $18x + 5y = 48$

$18 = 3 \cdot 5 + 3$
$5 = 3 + 2$

$1 = 3 - 2$
$= 3 - (5 - 3)$

$3 = 2 + 1$ $\quad = 2\cdot 3 - 5$

$2 = 2\cdot 1$ $\quad = 2(18 - 3\cdot 5) - 5$

$\therefore \gcd = 1$ $\quad = 2\cdot 18 - 7\cdot 5$

$\therefore 48 = 96\cdot 18 - (48\cdot 7)\cdot 5$

$\therefore (96, -336)$ a solution

$\therefore x = 96 + 5t$

$y = -336 - 18t$

Since $x, y > 0$, $96 + 5t > 0 \Rightarrow t > -19.2$

$-336 - 18t > 0 \Rightarrow t < -18.7$

$\therefore t = 19$

$\therefore x = 96 + 5(-19) = 1$

$y = -336 - 18(-19) = 6$

(6) $54x + 21y = 906$

$54 = 2\cdot 21 + 12$ $\quad \therefore 3 = 12 - 9$

$21 = 12 + 9$ $\quad = 12 - (21 - 12) = 2\cdot 12 - 21$

$12 = 9 + 3$ $\quad = 2(54 - 2\cdot 21) - 21$

$9 = 3\cdot 3$ $\quad = 2\cdot 54 - 5\cdot 21$

$\gcd = 3$ $\quad \therefore 906 = (302\cdot 2)(54 - (302\cdot 5)(21)$

$\therefore (604, -1510)$ a solution

$\therefore x = 604 + 7t > 0 \Rightarrow t > -86.3$

$y = -1510 - 18t > 0 \Rightarrow t < -83.9$

$\therefore t = -84, -85, -86$

$\therefore (x,y) = (16,2), (9,20), (2,38)$

(c) $123x + 360y = 99$

$360 = 3\cdot 123 - 9$ $\quad \therefore 3 = 14\cdot 9 - 123$

$123 = 14\cdot 9 - 3$ $\quad = 14(3\cdot 123 - 360) - 123$

$9 = 3\cdot 3$ $\quad = 41\cdot 123 - 14\cdot 360$

$\therefore \gcd = 3$ $\quad \therefore 99 = (33\cdot 41)123 - (33\cdot 14)360$

$\therefore (1353, -462)$ a solution

$\therefore x = 1353 + 120t > 0 \Rightarrow t > -11.275$

$y = -462 - 41t > 0 \Rightarrow t < -11.3$

$\therefore$ no t exists, so no positive solutions

(d) $158x - 57y = 7$

$158 = 3\cdot 57 - 13$ $\quad 1 = 3 - 2 = 3 - (5-3)$

$57 = 4\cdot 13 + 5$ $\quad = 2\cdot 3 - 5 = 2(13 - 2\cdot 5) - 5$

$13 = 2\cdot 5 + 3$ $\quad = 2\cdot 13 - 5\cdot 5$

$5 = 3 + 2$ $\quad = 2\cdot 13 - 5(57 - 4\cdot 13)$

$3 = 2 + 1$ $\quad = 22\cdot 13 - 5\cdot 57$

$2 = 2\cdot 1$ $\quad = 22(3\cdot 57 - 158) - 5\cdot 57$

$\therefore \gcd = 1$ $\quad = 61(57) - 22\cdot 158$

$\therefore 7 = 427 \cdot 57 - 154 \cdot 158$

$\therefore (-154, -427)$ a solution

$\therefore x = -154 - 57t > 0 \Rightarrow t < -2.7 \Rightarrow t \leq -3$

$y = -427 - 158t > 0 \Rightarrow t < -2.7 \Rightarrow t \leq -3$

4. $\gcd(a, b) = 1$, then $ax - by = c$ has infinitely many positive solutions.

Pf: Assume $a, b > 0$.
Since $1 | c$, a solution exists. Let x_0, y_0 be a solution. $\therefore ax_0 - by_0 = c$. By corollary on p.36, all solutions are given by:

$$x = x_0 - bt \qquad y = y_0 - at$$

For $x, y > 0$, $\quad x_0 - bt > 0, \quad t < \frac{x_0}{b}$
$\quad y_0 - at > 0 \quad t < \frac{y_0}{a}$

$\therefore$ if $t < \min\left(\frac{x_0}{b}, \frac{y_0}{a}\right)$, Then

$$t < \frac{x_0}{b} \Rightarrow bt < x_0 \Rightarrow x_0 - bt > 0$$

$$t < \frac{y_0}{a} \Rightarrow at < y_0 \Rightarrow y_0 - at > 0$$

There are infinitely many t s.t. $t < \min\left(\frac{x_0}{b}, \frac{y_0}{a}\right)$

5. (a) $ax + by + cz = d$ is solvable in integers $\Leftrightarrow$ $\gcd(a,b,c) \mid d$

Pf: (1) Let $g = \gcd(a,b,c)$. $\therefore \exists\, p, q, r$ s.t. $gp = a$, $gq = b$, $gr = c$.

$\therefore gpx + gqy + grz = g(px + qy + rz) = d$

$\therefore g \mid d$

(2) Let $g = \gcd(a,b,c)$ and suppose $g \mid d$

By Lemma, $\exists\, x_0, y_0, z_0$ s.t. $g = ax_0 + by_0 + cz_0$
Let t be s.t. $gt = d$.
$\therefore d = gt = ax_0t + by_0t + cz_0t$
$\therefore$ a solution is (x_0t, y_0t, z_0t)

Lemma: Given a, b, c not all of which are zero. There exist integers x, y, z s.t.

$$\gcd(a,b,c) = ax + by + cz$$

Pf: Analagous to proof of Th. 2.3.

Let $S = \{au + bv + cz \mid au + bv + cw > 0,$
$u, v, w \text{ integers}\}$

S is non-empty: Suppose $a \neq 0$.

$\therefore |a| = au + b \cdot 0 + c \cdot 0 > 0$, where
$u = 1 \cdot \text{sign}(a)$

By Well ordering Principal, S has a minimum value, d

By def. of S, $\exists$ integers x, y, z s.t.
$d = ax + by + cz$

Let q, r be s.t. (by Division Alg).
$a = qd + r, \quad 0 \leq r < d$

$\therefore r = a - qd = a - q(ax + by + cz)$
$= a(1 - qa) - bqy - cqz$

if $r > 0$, Then $r \in S \Rightarrow r > d$
But d is smallest element. $\therefore r = 0$

$\therefore a = qd \Rightarrow d \mid a$.
Similarly, $d \mid b$, $d \mid c$

$\therefore d$ is a common divisor.

Let e be any other common divisor of a, b, c. Let $eh = a$, $ej = b$, $ek = c$

$\therefore d = ax + by + cz = ehx + ejy + ekz$
$= e(hx + jy + kz) \Rightarrow e \mid d$

By Th. 2.2 p. 21, $|e| < |d|$,

$\therefore d = \gcd(a, b, c)$

Alternate Lemma: By #11, p. 32,
$\gcd(a,b,c) = \gcd(\gcd(a,b), c)$.
Let $u = \gcd(a,b)$. $\therefore \exists\, x, y$ s.t.
$u = ax + by$. Also, $\exists\, h, k$ s.t.

$$d = uh + ck = (ax+by)h + ck$$
$$= axh + byh + ck$$

$\therefore \exists$ integers p, q, r s.t. $d = ap + bq + cr$

(b) Find all integer solutions of $15x + 12y + 30z = 24$

$\gcd(15, 12, 30) = 3$, and $3 \mid 24$, so integer solutions exist by (a) above.
Now divide by gcd to simplify.

$$15x + 12y + 30z = 24 \iff 5x + 4y + 10z = 8$$

$\therefore 5x + 10z = 8 - 4y$. Since $\gcd(5,10) = 5$,

$5x + 10z = 5n$, some n. $\therefore 5n = 8 - 4y$

$x = n$, $z = 0$ is a solution, $\therefore$ by Th. 2.9,
$x = n + 2t$, $z = -t$, gives all
solutions for $5x + 10z = 5n$

$\therefore (x, y, z)$ is a solution $\iff$

$x = n + 2t$ for some n, t
$z = -t$
$8 - 4y = 5n$, or $4y = 8 - 5n$

$\therefore y = 2 - \frac{5n}{4}$, which means $4 | n$,
so n must be divisible by 4.
$\therefore$ Let $n = 4k$. $\therefore y = 2 - 5k$. $\therefore x = 4k + 2t$

$\therefore$ if k and t are any integers,

$x = 4k + 2t$
$y = 2 - 5k$
$z = -t$

6. (a) \$4.55 in dimes and quarters.
(1) Determine max + min. # of coins
(2) Can # dimes = # quarters?

(1) $10d + 25q = 455$, $d \geq 0$, $q \geq 0$
$\gcd(10, 25) = 5$
Equation $\equiv 2d + 5q = 91$ $(3, 17)$ a solution.
$\therefore$ All solutions of form
$d = 3 + 5t$ $\quad d \geq 0 \Rightarrow 3 + 5t \geq 0$, $t \geq \frac{-3}{5}$, $t \geq 0$
$q = 17 - 2t$ $\quad q \geq 0 \Rightarrow 17 - 2t \geq 0$, $t \leq \frac{17}{2}$, $t \leq 8$

$\therefore 0 \leq t \leq 8$

Max # coins is when $d+q$ is a max.

$d+q = 20+3t$, so when $t=8$, you will have 44 coins (43d, 1q)

Min # coins : $t=0$, or 20 coins (3d, 17q)

(2) For $d=q$, $3+5t = 17-2t$, $7t=14$, $t=2$

$\therefore$ 13 dimes, 13 quarters is a solution.

(3) $180a + 75c = 9000$, $a > c$ Also, $a \geq 0$, $c \geq 0$

$\gcd(75,180) = 15$

Reduce equation to $12a + 5c = 600$

One solution is $(50,0)$

$\therefore$ All solutions of form $a = 50 + 5t$

$c = -12t$

$a > c \Rightarrow 50 + 5t > -12t$, $17t > -50$, $t \geq -2$

$a \geq 0 \Rightarrow 5t \geq -50$, $t \geq -50$

$c \geq 0 \Rightarrow -12t \geq 0$, $t \leq 0$

$\therefore -2 \leq t \leq 0$

$\therefore t = 0, -1, -2$

So, 50 adults, 0 children

45 adults, 12 children

40 adults, 24 children

(c) $6x + 9y = 126$ $\qquad 36x + 54y = 756$

$6y + 9x = 114$ $\qquad 36x + 24y = 456$

$\therefore 30y = 300, \ y = 10$

$\therefore 6x + 9(10) = 126$

$6x = 36, \quad x = 6$

$\therefore$ 6 sixes, 10 nines

7. $c + l + p = 100$ $\qquad c, l, p \geq 1$

$120c + 50l + 25p = 4000$

$\gcd(120, 50, 25) = 5$

$\therefore$ Reduce to $24c + 10l + 5p = 800$

$p = 100 - c - l$

$\therefore 24c + 10l + 5(100 - c - l) = 800$

$19c + 5l = 300$ $\quad (0, 60)$ a solution

$\therefore c = 5t$

$l = 60 - 19t$ $\qquad \therefore p = 100 - 5t - (60 - 19t)$

$p = 40 + 14t$

Now, $5t \geq 1 \Rightarrow t \geq 1$

$60 - 19t \geq 1 \Rightarrow 19t \leq 59, \ t \leq 3$

$40 + 14t \geq 1 \Rightarrow 14t \geq -39, \ t \geq -2$

$\therefore 1 \leq t \leq 3$

$\therefore$ 5 calves, 41 lambs, 54 piglets

10 calves, 22 lambs, 68 piglets

15 calves, 3 lambs, 82 piglets

8. Let original check be d dollars and c cents.
So, Mr. Smith was given $100c + d$ cents.
Find smallest value of $100d + C$.

$\therefore 100c + d - 68 = 2(100d + c)$, $d, c \geq 0$

$\therefore 98c - 199d = 68$ $\quad gcd(98, 199) = 1$

$199 = 2 \cdot 98 + 3$ $\quad \therefore 1 = 3 - 2$

$98 = 32 \cdot 3 + 2$ $\quad = 3 - (98 - 32 \cdot 3) = 33 \cdot 3 - 98$

$3 = 2 + 1$ $\quad = 33(199 - 2 \cdot 98) - 98$

$2 = 2 \cdot 1$ $\quad = 33 \cdot 199 - 67 \cdot 98$

$\therefore 68 = (68 \cdot 33)199 - (68 \cdot 67)98$

$\therefore (-4556, -2244)$ is a solution.

$\therefore$ All solutions are of form:

$c = -4556 - 199t$ $\quad c \geq 0 \Rightarrow 199t \leq -4556$, $t \leq -22.9$

$d = -2244 - 98t$ $\quad d \geq 0 \Rightarrow 98t \leq -2244$, $t \leq -22.9$

$\therefore t \leq -23$

$100d + c = -228956 - 9999t$

This is smallest when t is biggest, so $t = -23$

$\therefore 100d + c = -228956 - 9999(-23) \overset{!!}{=} 1021$ cents

or <u>10 dollars 21 cents</u>

Check: 21 dollars 10 cents $-$ 68 cents $=$ 20 d 42 cents

9. (a) $m + w + c = 100$, $m, w, c > 0$

$3m + 2w + \frac{1}{2}c = 100$, or $6m + 4w + c = 200$

$C = 100 - m - w$

$\therefore 5m + 3w = 100$

One solution is $(14, 10)$

$\therefore$ All solutions of form:

$m = 14 + 3t \qquad m > 0 \Rightarrow t > -\frac{14}{3}, \ t \geq -4$

$w = 10 - 5t \qquad w > 0 \Rightarrow 5t < 10, \ t \leq 1$

$c = 76 + 2t \qquad c > 0 \Rightarrow t > -38, \ t \geq -37$

$\therefore -4 \leq t \leq 1$

$\therefore t = -4$: 2 men, 30 women, 68 children

$t = -3$: 5 men, 25 women, 70 children

$t = -2$: 8 men, 20 women, 72 children

$t = -1$: 11 men, 15 women, 74 children

$t = 0$: 14 men, 10 women, 76 children

$t = 1$: 17 men, 5 women, 78 children

(b) Let x = # plantain fruit in each of the 63 piles.

$\therefore 63x + 7$ = total # fruit , $x > 0$

Let y = # fruit to each of the 23 travelers.

$\therefore 23y$ = total # fruit, $y > 0$

$\therefore 63x + 7 = 23y$, or $63x - 23y = -7$

$63 = 3 \cdot 23 - 6 \quad \therefore 1 = 4 \cdot 6 - 23$

$23 = 4 \cdot 6 - 1 \quad\quad = 4(3 \cdot 23 - 63) - 23$

$6 = 6 \cdot 1 \quad\quad = 11 \cdot 23 - 4 \cdot 63$

$\therefore -7 = -77 \cdot 23 + 28 \cdot 63$

$\therefore (28, 77)$ a solution.

$\therefore$ All solutions of form:

$x = 28 - 23t \quad x > 0 \Rightarrow 23t < 28, \ t \leq 1$

$y = 77 - 63t \quad y > 0 \Rightarrow 63t < 77, \ t \leq 1$

$\therefore$ Infinitely many solutions

$t = 1: \ x = 5, \ y = 14$ (5 fruits/pile, 14/traveler)

$t = 0: \ x = 28, \ y = 77$ (28 fruits/pile, 77/traveler)

$\vdots$

(c) Let x = # coins on a string when make 77 strings. $\therefore 77x - 50$ = total # coins

Let y = # coins on a string when make 78 strings. $\therefore 78y$ = total # coins.

$\therefore 77x - 50 = 78y$, or $77x - 78y = 50, \ x, y > 0$

$\gcd(77, 78) = 1$

$\therefore 78 = 77 + 1 \quad \therefore 50 = 50(78) - 50(77)$

$77 = 1 \cdot 77 \quad \therefore (-50, -50)$ a solution

$\therefore$ All solutions of form:

$x = -50 - 78t$, $x > 0 \Rightarrow 78t < -50$, $t \le -1$

$y = -50 - 77t$, $y > 0 \Rightarrow 77t < -50$, $t \le -1$

Infinitely many solutions.

One solution is $t = -1$, $\therefore x = 28, y = 27$

$\therefore$ total # coins is $78(27) = 2106$

(d) $m + w + c = 20$, $m, w, c > 0$

$3m + 2w + \frac{1}{2}c = 20$, or $6m + 4w + c = 40$

$c = 20 - m - w$

$\therefore 5m + 3w = 20$

One solution is $(1, 5)$

$\therefore$ All solutions of form:

$m = 1 + 3t > 0 \Rightarrow t \ge -\frac{1}{3}$, $t \ge 0$

$w = 5 - 5t > 0 \Rightarrow 5t < 5$, $t \le 0$

$c = 14 + 2t > 0 \Rightarrow t > -7$, $t \ge -6$

$\therefore t = 0$

$\therefore$ 1 man, 5 women, 14 children

(e) $100 = x + y$, $7 \mid x$, $11 \mid y$, $0 \le x, y \le 100$

Let $7k = x$, $11n = y$

$\therefore 7k + 11n = 100$, $0 \le 7k, 11n \le 100$

$11 = 7 + 4$ $\therefore$ $1 = 4 - 3 = 4 - (7-4) = 2\cdot 4 - 7$

$7 = 4 + 3$ $= 2(11-7) - 7$

$4 = 3 + 1$ $= 2\cdot 11 - 3\cdot 7$

$3 = 3\cdot 1$ $\therefore 100 = 200\cdot 11 - 300\cdot 7$

$\therefore (-300, 200)$ a solution

$\therefore k = -300 + 11t$

$n = 200 - 7t$

Now $0 \le 7k \le 100 \Rightarrow 0 \le k \le \frac{100}{7}$, $0 \le k \le 14$

$0 \le 11n \le 100 \Rightarrow 0 \le n \le \frac{100}{11}$, $0 \le n \le 9$

$\therefore 0 \le -300 + 11t \le 14 \Rightarrow 300 \le 11t \le 314$,

$28 \le t \le 28$. $\therefore t = 28$

$0 \le 200 - 7t \le 9 \Rightarrow -200 \le -7t \le -191$

$28.6 \ge t \ge 27.3$, $t = 28$

$\therefore k = -300 + 11(28) = 8$ $\therefore 7k = 56$

$n = 200 - 7(28) = 4$ $11n = 44$

Note on problem 5(6) - Author's solution

The choice of $y = 3s - 5t$, $z = -s + 2t$

was ad hoc. Other choices for the coefficients of s and t are fine, as long as they are relatively prime. They were chosen so that $12y + 30z$ would eliminate one variable. You would then reduce 2 variables (y, z) to one variable (s), thereby solving an equation in two variables (x, s) in terms of a parameter r using Th. 2.9. By substitution, y and z would then be expressed in terms of r and t.

Any value of y and z can be expressed in terms of s and t, as long as the coefficients are "non-parallel" lines, and are relatively prime. In other words, $y = 3s - 10t$, $z = -5 + 4t$ would not work, as $t = 2.5$ could give a solution in integers, but $t = 2.5$ is not allowed. Relatively prime coefficients precludes this.