3.3 The Goldbach Conjecture

Note Title 12/27/2004

1. Verify 1949 and 1951 are twin primes.

From table of primes, $p_{296} = 1949$, $p_{297} = 1951$.

Also, $\sqrt{1951} = 44.2$, and neither divisible by primes ≤ 43.

2. (a). p_1, p_2 twin primes, show $n^2 = p_1 p_2 + 1$ for some n.

Pf: $p_2 = p_1 + 2$

$$\therefore p_1 p_2 + 1 = p_1(p_1 + 2) + 1$$

$$= p_1^2 + 2p_1 + 1 = (p_1 + 1)^2$$

$\therefore$ Let $n = p_1 + 1$

(b) The sum of twin primes $p, p+2$ is divisible by 12, assuming $p > 3$.

Pf: Let $N = p + p + 2 = 2p + 2 = 2(p+1)$

Since $p+1$ is even, $p+1 = 2m$, some m.

$\therefore N = 4m$, $\therefore 4 \mid N$.

Now let $p = 3q + r$, $r = 0, 1, 2$ by Div. Alg.

$r \neq 0$ since p is prime

If $r = 1$, then $p + 2 = 3q + 3$, so $3 \mid p+2$. Since $p+2$ is prime, $r \neq 1$

$\therefore r = 2$, and $p = 3q + 2$

$\therefore p + 2 = 3q + 4 = 3(q+1) + 1$

$\therefore N = p + p + 2 = 3q + 2 + 3(q+1) + 1$

$= 3(2q+1) + 3$

$\therefore 3 \mid N$

$\therefore 3 \mid N$, $4 \mid N$, and since $\gcd(3,4) = 1$, $3 \cdot 4 = 12 \mid N$ (corollary 2, p. 24).

$\therefore 12 \mid p + (p+2)$

3. Find all pairs of primes s.t. $p - q = 3$.

Pf: Since $p = q + 3$, if q is odd, p is even. And $p > 3$. But there is no even prime > 3.
$\therefore q$ is even, and $\therefore q = 2$. $\therefore p = 5$.

4. Every even integer $2n > 4$ is the sum of two primes, one $> n/2$, the other $< 3n/2$. Verify for integers $6 \le 2n \le 26$

Test for $3 \le n \le 38$

$2n$	n	$>$ $n/2$	$<$ $3n/2$
$6 = 3+3$	3	1.5	4.5
$8 = 3+5$	4	2	6
$10 = 3+7$	5	2.5	7.5
$12 = 5+7$	6	3	9
$14 = 7+7$	7	3.5	10.5
$16 = 5+11$	8	4	12
$18 = 7+11$	9	4.5	13.5
$20 = 7+13$	10	5	15
$22 = 11+11$	11	5.5	16.5
$24 = 11+13$	12	6	18
$26 = 13+13$	13	6.5	19.5
$28 = 11+17$	14	7	21
$30 = 11+19$	15	7.5	22.5
$32 = 13+19$	16	8	24
$34 = 11+23$	17	8.5	25.5
$36 = 13+23$	18	9	27
$38 = 19+19$	19	9.5	28.5
$40 = 17+23$	20	10	30

42 =	19+23	21	10.5	31.5
44 =	13+31	22	11	33
46 =	17+29	23	11.5	34.5
48 =	19+29	24	12	36
50 =	19+31	25	12.5	37.5
52 =	23+29	26	13	39
54 =	23+31	27	13.5	40.5
56 =	19+37	28	14	42
58 =	29+29	29	14.5	43.5
60 =	29+31	30	15	45
62 =	31+31	31	15.5	46.5
64 =	23+41	32	16	48
66 =	23+43	33	16.5	49.5
68 =	31+37	34	17	51
70 =	23+47	35	17.5	52.5
72 =	29+43	36	18	54
74 =	31+43	37	18.5	55.5
76 =	29+47	38	19	57

5. Every odd integer can be written as $p+2a^2$, p is prime or 1, $a \geq 0$. Show not true for 5777.

$5777 = p + 2a^2$, $a = \sqrt{\frac{5777-p}{2}}$

Minimum of p would be $p=2$.

$\therefore$ Largest a would be $\sqrt{\frac{5775}{2}} = 53.7$
Smallest a would be 0.

$\therefore$ Test $0 \leq a \leq 53$, or

test $5777 - 2a^2$ for $0 \leq a \leq 53$ and see if it is prime.

From spreadsheet, left column is a, middle column is $5777-2a^2$, and right colum is a factor of $5777-2a^2$, showing that the numbers are not primes.

$\therefore$ No prime p exists s.t.

$$5777 = p + 2a^2$$

53	159	3
52	369	3
51	575	5
50	777	3
49	975	3
48	1169	7
47	1359	3
46	1545	3
45	1727	11
44	1905	3
43	2079	3
42	2249	13
41	2415	3
40	2577	3
39	2735	5
38	2889	3
37	3039	3
36	3185	5
35	3327	3
34	3465	3
33	3599	59
32	3729	3
31	3855	3
30	3977	41
29	4095	3
28	4209	3
27	4319	7
26	4425	3
25	4527	3
24	4625	5
23	4719	3
22	4809	3
21	4895	5
20	4977	3
19	5055	3
18	5129	23
17	5199	3
16	5265	3
15	5327	7
14	5385	3
13	5439	3
12	5489	11
11	5535	3
10	5577	3
9	5615	5
8	5649	3
7	5679	3
6	5705	5
5	5727	3
4	5745	3
3	5759	13
2	5769	3
1	5775	3
0	5777	53

6. Prove: (a) Every even integer >2 is the sum of two primes

$\Leftrightarrow$ (b) Every integer >5 is the sum of three primes

Pf: (a) $\Rightarrow$ (b) Let N be any integer >5.

If N is even, so is $N-2$, and $N-2>3$. $\therefore$ by (a), $N-2 = p_1+p_2$, $\therefore N = 2+p_1+p_2$

If N is odd, $N-3$ is even, and $N-3>2$. $\therefore$ by (a), $N-3 = p_1+p_2$, $\therefore N = 3+p_1+p_2$.

$\therefore N$ is the sum of three primes.

(b) $\Rightarrow$ (a) Let N be any even integer >2. Since $4=2+2$, let N be ≥ 6. Consider $N+2$. From (b), $N+2>5$, $N+2 = p_1+p_2+p_3$. Since $N+2$ is even, not all of p_1, p_2, p_3 is odd. One of p_1, p_2, p_3 must be even, and so one of p_1, p_2, p_3 must be 2, the only even prime. Let it be p_1.

$\therefore N+2 = 2+p_2+p_3$, $N = p_2+p_3$

7. Every odd integer ≥ 5 can be written as $p_1 + 2p_2$
Confirm for all odd integers ≤ 75.

Pf:

$7 = 3 + 2\cdot 2$	$41 = 37 + 2\cdot 2$
$9 = 3 + 2\cdot 3$	$43 = 29 + 2\cdot 7$
$11 = 5 + 2\cdot 3$	$45 = 41 + 2\cdot 2$
$13 = 7 + 2\cdot 3$	$47 = 37 + 2\cdot 5$
$15 = 11 + 2\cdot 2$	$49 = 23 + 2\cdot 13$
$17 = 11 + 2\cdot 3$	$51 = 29 + 2\cdot 11$
$19 = 13 + 2\cdot 3$	$53 = 43 + 2\cdot 5$
$21 = 17 + 2\cdot 2$	$55 = 29 + 2\cdot 13$
$23 = 17 + 2\cdot 3$	$57 = 43 + 2\cdot 7$
$25 = 19 + 2\cdot 3$	$59 = 37 + 2\cdot 11$
$27 = 23 + 2\cdot 2$	$61 = 47 + 2\cdot 7$
$29 = 23 + 2\cdot 3$	$63 = 59 + 2\cdot 2$
$31 = 17 + 2\cdot 7$	$65 = 59 + 2\cdot 3$
$33 = 29 + 2\cdot 2$	$67 = 53 + 2\cdot 7$
$35 = 29 + 2\cdot 3$	$69 = 59 + 2\cdot 5$
$37 = 31 + 2\cdot 3$	$71 = 67 + 2\cdot 2$
$39 = 29 + 2\cdot 5$	$73 = 59 + 2\cdot 7$
	$75 = 53 + 2\cdot 11$

8. $60 = p_1 + p_2$ in 6 ways

$60 = 53 + 7$ $\quad 60 = 43 + 17$ $\quad 60 = 37 + 23$

$60 = 47 + 13$ $\quad 60 = 41 + 19$ $\quad 60 = 31 + 29$

$78 = p_1 + p_2$ in 7 ways

$78 = 73 + 5$

$78 = 71 + 7$

$78 = 67 + 11$

$78 = 61 + 17$

$78 = 59 + 19$

$78 = 47 + 31$

$78 = 41 + 37$

$84 = p_1 + p_2$ in 8 ways

$84 = 79 + 5$

$84 = 73 + 11$

$84 = 71 + 13$

$84 = 53 + 31$

$84 = 47 + 37$

$84 = 43 + 41$

$84 = 67 + 17$

$84 = 61 + 23$

9. (a) For $n > 3$, $n, n+2, n+4$ cannot all be prime.

Pf: By Division Alg., n can be expressed as $6q + r$, $0 \le r \le 5$

$r \ne 0, 2, 4$, for then n would be even.

$\therefore r = 1, 3, 5$

$r = 1$: $n = 6q + 1$, so $n + 2 = 6q + 3$, which is divisible by 3.

$\therefore r \ne 1$

$r = 3$: $n = 6q + 3$, but then $3 \mid n$.

so $r \ne 3$.

$r = 5$: $n = 6q + 5$, then $n + 4 = 6q + 9$, so $3 \mid n + 4$. $\therefore r \ne 5$.

$\therefore$ for no value of r can all three numbers be prime.

(b) prime triplets : $p, p+2, p+6$

$5, 7, 11$ $\quad$ $41, 43, 47$

$11, 13, 17$ $\quad$ $101, 103, 107$

$17, 19, 23$

10. $(n+1)!-2, (n+1)!-3, \ldots, (n+1)!-(n+1)$ produces n consecutive composite numbers.

pf: For each $k \leq n+1$, k is in the term $(n+1)!$, so that $k \mid [(n+1)! - k]$

11. $f(n) = n^2 + n + 17$
$g(n) = n^2 + 21n + 1$
$h(n) = 3n^2 + 3n + 23$

Find smallest n for each function that makes value a composite.

$f(16) = 289 = 17^2$
$g(18) = 703 = 19 \times 37$
$h(22) = 1541 = 23 \times 67$

12. Let p_n be n^{th} prime number. For $n \geq 3$, prove $p_{n+3}^2 < p_n \, p_{n+1} \, p_{n+2}$

Pf: From section 3.2, $p_{n+1} < 2p_n$

$\therefore p_{n+3} < 2p_{n+2}$

so $p_{n+3}^2 < 4p_{n+2}^2 < 4p_{n+2}(2p_{n+1}) = 8p_{n+2}p_{n+1}$

Since $p_5 = 11$, $8p_{n+2}p_{n+1} < p_5 p_{n+2} p_{n+1}$

$\therefore p_{n+3}^2 < p_n p_{n+1} p_{n+2}$ if $n \geq 5$

For $n = 4$, $p_7^2 = 17^2 = 289 < p_4 p_5 p_6 = 7 \cdot 11 \cdot 13 = 1001$

$n = 3$: $p_6^2 = 13^2 = 169 < p_3 p_4 p_5 = 5 \cdot 7 \cdot 11 = 385$

$n = 2$: $p_5^2 = 11^2 = 121 \overset{?}{<} p_2 p_3 p_4 = 3 \cdot 5 \cdot 7 = 105$

$\therefore$ For $n \geq 3$, $p_{n+3}^2 < p_n p_{n+1} p_{n+2}$

13. There are infinitely many primes of form: $6n+5$

Pf: Assume only finite number of primes of form $6n+5$. Let these be $q_1, q_2, q_3, \ldots q_s$

Consider $N = 6q_1 q_2 \cdots q_s - 1 = 6(q_1 q_2 \cdots q_s - 1) + 5$

Let $N = r_1 r_2 \cdots r_t$ be the prime factorization. Since N is odd, $r_i \neq 2$, so each r_i can only be of form $6n+1$, $6n+3$, or $6n+5$.

Since $(6n+1)(6m+1) = 36nm + 6m + 6n + 1$
$= 6(6nm + m + n) + 1$
product of two integers of $6n+1$ form is same form.

Since $(6n+3)(6m+3) = 36nm + 18m + 18n + 9$
$= 6(6nm + 3m + 3n + 3) + 3$
product of two integers of $6n+3$ form is same form.

Since $(6n+1)(6m+3) = 36nm + 6m + 18n + 3$
$= 6(6nm + m + 3n) + 3$
product of two integers of $6n+1$ form and $6n+3$ form is of $6n+3$ form.

So, the only way for N to be of form $6n+5$, of which it is, N must contain at least one factor r_i of form $6n+5$. But can't find such a prime among the $q_1, q_2, \ldots, q_s$. If such a prime existed, then from construction of N,

$$N - 6q_1q_2\cdots q_s = -1,$$

both terms on left side would be divisible by this prime of $6n+5$ form, so -1, and thus, 1, would be divisible by this prime, a contradiction.

$\therefore$ Can't be finite # of primes of $6n+5$ form.

14. $4(3\cdot 7\cdot 11)-1 = 13\times 71$, 71 is of form $4n+3$

$4(3\cdot 7\cdot 11\cdot 15)-1 = 13{,}859$, a prime of form $4n+3$

15. Five consecutive odd integers, 4 of which are prime.

3, 5, 7, 9, 11   11, 13, 15, 17, 19   191, 193, 195, 197, 199

5, 7, 9, 11, 13   101, 103, 105, 107, 109

16. $23 = p_9 = p_{2\cdot 4+1} = 2p_{2\cdot 4} + \sum_{k=0}^{2\cdot 4-1} \epsilon_k p_k$

$$= 2p_8 + \sum_{K_0}^{7} \epsilon_K p_K$$

$$\therefore 23 = 2\cdot 19 + \epsilon_0 + 2\epsilon_1 + 3\epsilon_2 + 5\epsilon_3 + 7\epsilon_4 + 11\epsilon_5 + 13\epsilon_6 + 17\epsilon_7$$

$$= 38 + 1 + 2 + 3 + 5 - 7 + 11 - 13 - 17$$

$$= 38 + (2-17) + (1+3+5) + (-7+11-13)$$

$$= 38 - 15 + 9 - 9$$

$$29 = p_{10} = p_{2\cdot 5} = p_{2\cdot 5-1} + \sum_{K=0}^{2\cdot 5-2} = 23 + \sum_{K=0}^{8} \epsilon_K p_K$$

$$= 23 + \epsilon_0 + 2\epsilon_1 + 3\epsilon_2 + 5\epsilon_3 + 7\epsilon_4 + 11\epsilon_5 + 13\epsilon_6$$
$$+ 17\epsilon_7 + 19\epsilon_8$$

$$= 23 + 6 + 12 + 6 - 6$$

$$= 23 + (1+2+3) + (5+7) + (-11+17) + (13-19)$$

$$= 23 + 1 + 2 + 3 + 5 + 7 - 11 + 17 + 13 - 19$$

$$31 = p_{11} = p_{2\cdot 5+1} = 2p_{2\cdot 5} + \sum_{K=0}^{2\cdot 5-1} \epsilon_K p_K = 2\cdot 29 + \sum_{K=0}^{9} \epsilon_K p_K$$

$= 2\cdot 29 + \epsilon_0 + 2\epsilon_1 + 3\epsilon_2 + 5\epsilon_3 + 7\epsilon_4 + 11\epsilon_5 + 13\epsilon_6 + 17\epsilon_7 +$

$19\epsilon_8 + 23\epsilon_9$

$31 = 58 - 27$, so find -27

$$-27 = 0 - 8 + (-11 - 4 - 4)$$
$$= (2+3-5) + (-1-7) + (-11-4-4)$$

$\therefore 31 = 2\cdot 29 - 1 + 2 + 3 - 5 - 7 - 11 + 13 - 17 + 19 - 23$

$$37 = p_{12} = p_{2\cdot 6} = p_{2\cdot 6 - 1} + \sum_{k=0}^{2\cdot 6 - 2} \epsilon_k p_k = 31 + \sum_{k=0}^{10} \epsilon_k p_k$$

$= 31 + \epsilon_0 + 2\epsilon_1 + 3\epsilon_2 + 5\epsilon_3 + 7\epsilon_4 + 11\epsilon_5 + 13\epsilon_6 + 17\epsilon_7 +$

$19\epsilon_8 + 23\epsilon_9 + 29\epsilon_{10}$

$$37 = 31 + 6 = 31 + 6 + (-2 + -2 + -42) + 46$$
$$= 31\ (1+2+3) + (5-7+11-13-19-23) + (17+29)$$

$\therefore 37 = 31 + 1 + 2 + 3 + 5 - 7 + 11 - 13 + 17 - 19 - 23 + 29$

17. Show 509 and 877 can't be the sum of a prime and power of 2.

From spreadsheet,
1st column is n,
2nd column is 2^n,
3rd column is $509 - 2^n$
4th column is $877 - 2^n$
None of the positive entries in 3rd or 4th cols. is prime.

0	1	508	876
1	2	507	875
2	4	505	873
3	8	501	869
4	16	493	861
5	32	477	845
6	64	445	813
7	128	381	749
8	256	253	621
9	512	-3	365
10	1024	-515	-147

18. (a). p prime, $p \nmid b$, show every pth term in $a, a+b, a+2b, \ldots$ is divisible by p.

Better restatement: there is a term within the first p terms that is divisible by p, and every pth term thereafter is divisible by p. (because the pth term from the beginning is not always divisible by p).

Pf: Since $p \nmid b$, and p is prime, $\gcd(p, b) = 1$.

$\therefore$ There exist integers r, s s.t. $pr + bs = 1$ [1]

Consider $n_k = kp - as$, $k = 1, 2, 3, \ldots$

For $k = 1$, $n = p - as$, and clearly $n_1 < p$.
Note that n_2 is the pth term after n_1, n_3 the pth term after n_2, etc.

$$a + n_k b = a + (kp - as)b = a + kpb - abs$$
$$= a(1 - bs) + kpb$$
$$= a(pr) + kpb \quad (\text{using } [1])$$

$\therefore p \mid a + n_k b$, so there is a term within the first p terms that is divisible by p, and every pth term after that is divisible by p.

(b) if b is odd in $a, a+b, a+2b, \ldots$

Then since $2 \nmid b$ and 2 is prime, by (a) either a or $a+b$ is divisible by 2, and every 2nd term is also. So every other term is even.

19. $25 = 5 + 7 + 13$ $\qquad 81 = 3 + 5 + 73$
$69 = 3 + 5 + 61$ $\qquad 125 = 5 + 7 + 113$

20. If p and $p^2 + 8$ are both prime, then $p^3 + 4$ is prime.

Pf: As in prob. # 4 of Problem 3.1, if $p > 3$ is prime it is of form $6k+1$ or $6k+5$.

$\therefore\ p^2+8 = 36k^2+12k+9$, or $36k^2+60k+33$

But $3 \mid (36k^2+12k+9)$

And $3 \mid (36k^2+60k+33)$

So p^2+8 is not prime if $p>3$.

$\therefore\ p=3$

$p^2+8=17$

$p^3+4=31$

21. (a). Let $k>0$ be any integer, and let a, b be integers with $\gcd(a,b)=1$. Prove the series,

$a+b,\ a+2b,\ a+3b, \ldots$ contains k consecutive composite terms.

Pf: Let k be any integer, and let n be the integer formed by:

$$n=(a+b)(a+2b)\cdots(a+kb)$$

Consider the series of k terms:

$$a+(n+1)b,\ a+(n+2)b, \ldots, a+(n+k)b$$

For the "i"th term, $(1 \le i \le k)$

$$a+(n+i)b =$$

$a + nb + ib = a + ib + nb$

But n contains $(a+ib)$ as one of its terms by definition of n.

$\therefore (a+ib) \mid (a+(n+i)b)$ for all i

For $K \geq 2$, $a+ib < a+(n+i)b$

For $K=1$, $n = a+b$, and the "i"th term of our series is $a+(a+b+1)b =$

$a + ab + b^2 + b = a(1+b) + b(b+1)$

$= (a+b)(b+1)$

$\therefore a+b < a+(a+b+1)b$

$\therefore$ for $K \geq 1$, $a+ib < a+(n+i)b$

Also, $1 < a+ib$ for all i.

$\therefore$ all K terms of $a+(n+1)b, \ldots, a+(n+K)b$ are divisible by an integer that is >1 and $<$ the term.

$\therefore$ all K terms are composite.

Note: proof doesn't use $\gcd(a,b)=1$. It does assume $a \neq b \neq 0$.

(b) From our construction, let

$$n = (6+5)(6+2\cdot5)(6+3\cdot5)(6+4\cdot5)(6+5\cdot5)$$
$$= 2{,}978{,}976$$

$\therefore\ 6+(n+1)5 = 14{,}894{,}891 \quad \div\ 6+5 = 11$
$6+(n+2)5 = 14{,}894{,}896 \quad \div\ 6+2\cdot5 = 16$
$6+(n+3)5 = 14{,}894{,}901 \quad \div\ 21$
$6+(n+4)5 = 14{,}894{,}906 \quad \div\ 26$
$6+(n+5)5 = 14{,}894{,}911 \quad \div\ 31$

$\therefore$ The above 5 consecutive terms are composite.

22. Show 13 is largest prime that can divide two successive integers of form n^2+3

Pf: First, look at first possibilities for n

n	n^2+3	prime fac.	n	n^2+3	prime fac
0	3	3	9	84	$2^2\times3\times7$
1	4	2^2	10	103	103
2	7	7	11	124	$2^2\times31$
3	12	$2^2\times3$	12	147	3×7^2
4	19	19	13	172	$2^2\times43$
5	28	$2^2\times7$	14	199	199
6	39	3×13	15	228	$2^2\times3\times19$
7	52	$2^2\times13$	16	259	7×37
8	67	67			

It seems that after $n \geq 8$, there are <u>no</u> common factors for adjacent terms.

Adjacent terms are n^2+3

$(n+1)^2+3 = n^2+2n+4$

Use Euclid's algorithm to find gcd for $n \geq 8$

$\therefore$ Suppose 1st term is even, i.e., $n = 2s$, and $s \geq 4$

$\therefore$ terms are $(2s)^2+3 = 4s^2+3$

$(2s+1)^2+3 = 4s^2+4s+4$

$4s^2+4s+4 = 1\cdot(4s^2+3) + 4s+1$ $\quad 4s+1 < 4s^2+3$

$4s^2+3 = s(4s+1) - s+3$ $\quad$ but $-s+3 < 0$ for $s \geq 4$

$\therefore 4s^2+3 = (s-1)(4s+1) + 3s+4$ and for $s \geq 4$, $4s+1 > 3s+4$

$4s+1 = 1\cdot(3s+4) + s-3$ $\quad 3s+4 > s-3$, for $s \geq 4$

$3s+4 = 3\cdot(s-3) + 13$ $\quad (*)$

$\gcd(s-3, 13) = 13$ or 1

So $\gcd = 1, 13$ if $s-3 > 13$, or $s > 16$

So, must prove $\gcd = 1, 13$ for $4 \leq s \leq 16$ for $(*)$

So $(*)$ becomes for each s:

$3s+4 = a(s-3) + r$

$\therefore s=4$: $16 = 16\cdot 1$ $\quad \gcd = 1$

$s=5$: $19 = 9\cdot 2 + 1$, $2 = 2\cdot 1$, $\gcd = 1$

$s=6$: $22 = 7\cdot 3+1$, $3=3\cdot 1$, $\gcd=1$

$s=7$: $25 = 6\cdot 4+1$, $4=4\cdot 1$, $\gcd=1$

$s=8$: $28 = 5\cdot 5+3$, $5=1\cdot 3+2$, $3=1\cdot 2+1$, $2=2\cdot 1$, $\gcd=1$

$s=9$: $31 = 5\cdot 6+1$, $6=6\cdot 1$, $\gcd=1$

$s=10$: $34 = 4\cdot 7+6$, $7=1\cdot 6+1$, $6=6\cdot 1$, $\gcd=1$

$s=11$: $37 = 4\cdot 8+5$, $8=1\cdot 5+3$, $\gcd=1$

$s=12$: $40 = 4\cdot 9+4$, $9=2\cdot 4+1$, $\gcd=1$

$s=13$: $43 = 4\cdot 10+3$, $10=3\cdot 3+1$, $\gcd=1$

$s=14$: $46 = 4\cdot 11+2$, $11=5\cdot 2+1$, $\gcd=1$

$s=15$: $49 = 4\cdot 12+1$, $12=12\cdot 1$, $\gcd=1$

$s=16$: $52 = 4\cdot 13$ $\gcd=13$

$\therefore$ Examples show $\gcd=1$ or $\gcd=13$ for adjacent terms $0\le n\le 16$, and above shows $\gcd=1$ or $\gcd=13$ for all n if 1st term is even.

Now suppose first term is odd, i.e., $n=2s+1$, and $s\ge 4$

$$\therefore (2s+1)^2+3 = 4s^2+4s+4$$
$$(2s+2)^2+3 = 4s^2+8s+7$$

$4s^2+8s+7 = 1\cdot(4s^2+4s+4)+4s+3$

$4s^2+4s+4 = s(4s+3)+s+4$ $\qquad 4s+3 > s+4$ for $s\ge 4$

$4s+3 = 3(s+4) + s-9 \quad (*) \qquad 0 < s-9 < s+4 \text{ if } s>9$

$s+4 = 1\cdot(s-9) + 13 \qquad 13 < s-9, \text{ if } 22 < s$

$\gcd(s-9, 13) = 1 \text{ or } 13$

$\therefore \gcd = 1$ if $s > 22$, and so must test $(*)$ for $4 \le s \le 22$. $(*)$ becomes

$4s+3 = a(s+4) + r$

$s=4: \ 19 = 2\cdot 8 + 3, \ 8 = 2\cdot 3 + 2, \ 3 = 1\cdot 2 + 1 \qquad \gcd = 1$

$s=5: \ 23 = 2\cdot 9 + 5, \ 9 = 1\cdot 5 + 4, \ 5 = 4+1, \ \gcd = 1$

$s=6: \ 27 = 2\cdot 10 + 7 \qquad \gcd = 1$

$s=7: \ 31 = 2\cdot 11 + 9 \qquad \gcd = 1$

$s=8: \ 35 = 2\cdot 12 + 11 \qquad \gcd = 1$

$s=9: \ 39 = 3\cdot 13 \qquad \gcd = 13$

$s=10: \ 43 = 3\cdot 14 + 1 \qquad \gcd = 1$

$s=11: \ 47 = 3\cdot 15 + 2 \qquad \gcd = 1$

$s=12: \ 51 = 3\cdot 16 + 3 \qquad \gcd = 1$

$s=13: \ 55 = 3\cdot 17 + 4 \qquad \gcd = 1$

$s=14: \ 59 = 3\cdot 18 + 5 \qquad \gcd = 1$

$s=15: \ 63 = 3\cdot 19 + 6 \qquad \gcd = 1$

$s=16: \ 67 = 3\cdot 20 + 7 \qquad \gcd = 1$

$s=17: \ 71 = 3\cdot 21 + 8 \qquad \gcd = 1$

$s=18: \ 75 = 3\cdot 22 + 9, \qquad \gcd = 1$

$s=19: \ 79 = 3\cdot 23 + 10, \qquad \gcd = 1$

$s=20: \ 83 = 3\cdot 24 + 11 \qquad \gcd = 1$

$s=21: \ 87 = 3\cdot 25 + 12, \ 25 = 2\cdot 12 + 1, \ \gcd = 1$

$s=22: \ 91 = 3\cdot 26 + 13, \qquad \gcd = 13$

$\therefore$ for $4 \le s \le 22$, gcd $= 1$ or gcd $= 13$
for $s > 22$, gcd $= 1$ or 13
$\therefore$ for all $s \ge 4$, gcd $= 1$ or 13
$\therefore$ for all terms of n^2+3, $(n+1)^2+3$ beginning with n odd and $n \ge 9$, gcd $= 1$ or 13.

$\therefore$ for all $n \ge 0$, adjacent terms have gcd of 1 or 13

Note: it would have been difficult to start with

$n^2+2n+4 = 1(n^2+3) + 2n+1$

$n^2+3 = n(2n+1) - n^2 - 2n + 3$

This approach is not fruitful.

23. (a) Twin primes with a triangular mean

Some triangular numbers: $1+2=3$, $1+2+3=6$, $1+2+3+4=10$, $10+5=15$, $15+6=21$, $21+7=28$
19, 23 are adjacent but not twin primes.

$(5+7)/2 = 6$, so 5, 7 work. Suppose $p > 7$.

From problem #1(a), sec. 1.3, a number is triangular $\Leftrightarrow$ it is of form $n(n+1)/2$

$\therefore (p + p+2)/2 = n(n+1)/2$

$\therefore 2p+2 = n^2+n, \; 2p = n^2+n-2 = (n+2)(n-1)$

Since $2p$ is even, one of $n+2$ or $n-1$ must be even.

Suppose $n-1 = 2k$. $\therefore n+2 = 2k+3$

$\therefore 2p = (2k+3)(2k)$

$p = (2k+3)(k)$

For p to be prime, $k=1$, $\therefore p = 5$

Suppose $n+2 = 2k$. $\therefore n-1 = 2k-3$

$\therefore 2p = 2k(2k-3)$

$p = k(2k-3)$

$\therefore 2k-3 = 1, \; k = 2$, or

$k = 1, \; 2k-3 = -1$.

$\therefore n+2 \neq 2k$.

So, only possible twin primes are 5, 7.

(6) Twin primes with square mean.

Suppose: $(p+p+2)/2 = n^2$

$\therefore p+1 = n^2, \; p = n^2-1 = (n+1)(n-1)$

For p to be prime, $n-1=1$, $\therefore n=2$
$\therefore n^2 = 4$

$\therefore$ Only possibility is 3, 5

24. Determine all twin primes p and $q = p+2$ for which $pq-2$ is prime.

Pf: 3, 5 : $3 \cdot 5 - 2 = 13$

Suppose $p > 3$. All primes > 3 are of form $6K+1$ or $6K+5$. But p must be of form $6K+5$ since $6K+1+2 = 6K+3 = 3(2K+1)$.

$\therefore$ Let $p = 6K+5$, $q = 6K+7$

$$\therefore (6K+5)(6K+7) - 2 = 36K^2 + 72K + 35 - 2$$
$$= 36K^2 + 72K + 33$$
$$= 3(12K^2 + 24k + 11)$$

$\therefore$ if $p > 3$, there are no twin primes such that $pq-2$ is prime.
3, 5 is the only pair!

25. Let p_n be nth prime. For $n > 3$, show
$$p_n < p_1 + p_2 + \dots + p_{n-1}$$

Pf: $p_3 = 5 = 2+3 = p_1 + p_2$

$p_4 = 7 < 2+3+5 = p_1 + p_2 + p_3$

$\therefore$ Assume for $k > 4$,

$$p_k < p_1 + p_2 + \dots + p_{k-1}$$

$\therefore 2p_k < p_1 + \dots + p_{k-1} + p_k$

By Bertrand's conjecture, $\exists\, p$ s.t. $p_k < p < 2p_k$

But $p_k < p_{k+1} \leq p$

$\therefore p_{k+1} \leq p < 2p_k < p_1 + \dots + p_{k-1} + p_k$

$\therefore$ true for $k+1$

$\therefore$ True for all $n > 4$

2C. (a) Infinitely many primes ending in 33.

Pf: $100 = 2^2 \times 5^{-2}$ and $33 = 3 \times 11$ are relatively prime.

$\therefore$ By Dirichlet's Theorem, The series
$33, 33+100, 33+2\cdot100, \ldots$
$= 33, 133, 233, \ldots$ contains infinitely
many primes.

(b) Infinitely many primes which do not belong
to any pair of twin primes.

Pf: 5 and $21 = 3 \cdot 7$ are relatively prime.
$\therefore$ By Dirichlet's Theorem, The series,

$5 + 21k$, for $k = 1, 2, 3, \ldots$

contains infinitely many primes.

Let p be one such prime.
$\therefore$ For some k, $p = 5 + 21k$.
$\therefore$ $p+2 = 7 + 21k = 7(1+3k)$
$p-2 = 3 + 21k = 3(1+7k)$.
$\therefore$ $p+2$ and $p-2$ can't be prime.

$\therefore$ all The primes contained in
$5+21k$ cannot be members of
twin primes.

(c) There exists a prime ending in as many consecutive 1's as desired.

Pf: $R_n = (10^n - 1)/9$ by def.

Since $1 \cdot 10^n - 9 \cdot R_n = 1$, $\gcd(10^n, R_n) = 1$

$\therefore$ Using Dirichlet's Theorem, form the series

$$10^n \cdot k + R_n, \quad k = 1, 2, 3, \ldots$$

which is for $n=1$: $11, 21, 31, \ldots$

$n=2$: $111, 211, 311, \ldots$

Each contains infinitely many primes.

$\therefore$ each contains at least one prime ending in n 1's.

(d) There are infinitely many primes that contain but do not end in the block of digits 123456789.

Pf: Consider $10^{11} = 2^{11} \times 5^{11}$

The number 1234567891 is odd, so contains no factor of 2, and does not end in 0 or 5, so contains no factor of 5.

$\therefore$ 10^{11} and 1234567891 are relatively prime.

$\therefore$ By Dirichlet's Theorem, The series

$10^{11} \cdot K + 1234567891$ contains infinitely many primes, and each number in The series contains 123456789 but ends in 1.

A few numbers in The series are:
$11234567891, 21234567891, 31234567891, \ldots$

27. For every $n \geq 2$, There exists a prime $p \leq n < 2p$.

Pf: Suppose n is odd. $\therefore \exists K$ s.t. $n = 2K+1$, and since $n \geq 2$, $K \geq 1$.

By Bertrand's conjecture, There is a prime p s.t. $K < p < 2K$.

$\therefore p < p+1 < 2K+1 = n$, so $p < n$

Also, $2K < 2p$, so $2K+1 \leq 2p$

$\therefore n \leq 2p$. But $2K+1$ is odd, and $2p$ is even. $\therefore n < 2p$

$\therefore \exists$ a p s.t. $p < n < 2p$

Suppose n is even. $\therefore \exists K$ s.t. $n = 2K$, $K \geq 1$

By Bertrand's conjecture, There is a prime p s.t. $K < p < 2K = n$, so $p < n$

$\therefore\ n = 2k < 2p$, so $n < 2p$

$\therefore\ p < n < 2p$

28. (a) If $n > 1$, show that $n!$ is never a perfect square.

Pf: Lemma 1: If $p_1 < p_2$ are adjacent primes, Then if $p_1 < N < p_2$, Then The prime factors of N are less than p_1 (for $p_1 > 3$).

pf: Let $q_1 q_2 \cdots q_r = N$, $r \geq 2$. Suppose $q_i = p_i$, some i.

Since $q_i \geq 2$ for all i, $N = q_1 \cdots q_r \geq p_i 2^{r-1}$

$\therefore\ N = q_1 \cdots q_r \geq p_1 \cdot 2 \cdot 2^{r-2} > p_2$

since $2p_1 > p_2$ (top p. 50, a direct consequence of Bertrand conjecture). $\therefore\ N > p_2$, a contradiction.

Lemma 2: Let $q_1^{k_1} q_2^{k_2} \cdots q_r^{k_r}$ be The prime canonical factorization of $n!$ Then $k_r = 1$ for all $n \geq 2$.

Pf: Clearly true for $n=2$, $n=3$.
Let N be any integer > 3
Suppose true for $N!$
$\therefore$ $N! = q_1^{k_1} \cdots q_{r-1}^{k_{r-1}}\, q_r \quad (q_i < q_r)$

Since each term of $N!$ is $< N$, then the prime factors of each term (which are < each term) are $< N$. $\therefore$ $q_i < N$, and so $q_r < N$.

Consider $(N+1)! = N!\,(N+1)$

If $N+1$ is prime, then $q_r < N+1$.
$\therefore$ $(N+1)! = q_1^{k_1} \cdots q_{r-1}^{k_{r-1}}\, q_r\,(N+1)$

$\therefore$ lemma true

Suppose $N+1$ is not prime.
Then q_r must be largest prime $< N+1$. If a larger prime existed, it would be a term in $(N+1)!$, and $\therefore$ would be represented in the prime factorization: $q_1^{k_1} \cdots q_{r-1}^{k_{r-1}}\, q_r$

By Lemma 1 above, prime factors of $N+1$ are $< q_r$!

$\therefore q_r$ remains largest prime factor and it has exponent 1.

$\therefore$ Lemma true for $N+1$ when true for N.

Back to main problem:

$\therefore$ The prime factorization of $n!$ has exponent 1 for largest factor.

$\therefore$ If $n! = a^2$, some a, all prime factors would have even exponents, as would the last factor.

$\therefore n! \neq a^2$ for any $n \geq 2$.

Note: By Lemma 2, $n!$ can't be any power of any number.

(6). Find values of $n \geq 1$ for which $n! + (n+1)! + (n+2)!$ is a perfect square.

$$n! + (n+1)! + (n+2)! = n!\left[1 + (n+1) + (n+1)(n+2)\right]$$
$$= n!\left[1 + (n+1) + n^2 + 3n + 2\right]$$
$$= n!\left[n^2 + 4n + 4\right]$$
$$= n!\,(n+2)^2$$

$\therefore$ Let $a^2 = n!(n+2)^2$

From (a), all the prime factors of a^2 have even exponents. $\therefore$ prime factors of $n!(n+2)^2$ should have even exponents.
But $n!$ has, for its largest prime factor $(n \geq 2)$, an exponent of 1. (from (a) above). Call this factor p. Even if $(n+2)^2$ had p as a factor, its exponent would be even.
Thus, the exponent of p in the factorization of $n!(n+2)^2$ will be odd. This contradicts expectation of a^2.
$\therefore$ n can't be ≥ 2.

$\therefore$ for $n=1$, $n! + (n+1)! + (n+2)! = 9 = 3^2$.

$\therefore$ Only $n=1$ is statement true.