4.3 Special Divisibility Tests

Note Title 2/18/2005

1. (a). For any integer a, The units digit of a^2 is 0, 1, 4, 5, 6, or 9

Pf: Let $a = a_n 10^n + \cdots + a_1 10 + a_0$, $0 \le a_0 < 10$

$\therefore a - a_0 = 10(a_n 10^{n-1} + \cdots + a_1)$

$\therefore a \equiv a_0 \pmod{10}$ $\therefore a^2 \equiv a_0^2 \pmod{10}$

Note That all The other a_i of a are associated with a factor of 10 in a^2, and so don't contribute to units digit.

$\therefore$ only a_0^2 contributes to units digit of a^2.

$a_0^2 = 0, 1, 4, 9, 16, 25, 36, 49, 64, 81$

$\therefore a_0^2 \equiv 0, 1, 4, 5, 6,$ or $9 \pmod{10}$

$\therefore a^2 \equiv 0, 1, 4, 5, 6,$ or $9 \pmod{10}$

(b). Any integer a_0, $0 \le a_0 \le 9$, can occur in units digit of a^3

Pf: as in (a), Let $a = a_n 10^n + \cdots + a_0$, $0 \le a_0 < 10$

$\therefore a - a_0 = 10(a_n 10^{n-1} + \cdots + a_1)$

$\therefore a \equiv a_0 \pmod{10}$ $\therefore a^3 \equiv a_0^3 \pmod{10}$

$a_0^3 = 0, 1, 8, 27, 64, 125, 216, 343, 512, 729$

$\therefore a_0^3 \equiv 0, 1, 2, 3, 4, 5, 6, 7, 8,$ or $9 \pmod{10}$

(c). For any a, the units digit of a^4 is 0, 1, 5, or 6.

Pf: As in (a), the only contributor to units digit in a^4 is a_0^4. ∴ Look at all possibilities of a_0^4, $0 \le a_0 \le 9$.

From (a), $a_0^2 \equiv 0, 1, 4, 5, 6, 9 \pmod{10}$

$$\therefore a_0^4 \equiv 0, 1, 16, 25, 36, 81 \pmod{10}$$
$$\therefore a_0^4 \equiv 0, 1, 5, \text{ or } 6 \pmod{10}$$

(d). The units digit of a triangular number is 0, 1, 3, 5, 6, or 8.

Pf: A number a is triangular $\Leftrightarrow$ there is a number n, $n \ge 1$, s.t. $a = \frac{n(n+1)}{2}$ (Problems 1.3, 1(a)).

Let $n = a_m 10^m + \cdots + a_0$, $\therefore n \equiv a_0 \pmod{10}$
$\therefore n+1 \equiv a_0 + 1 \pmod{10}$. Either n or $n+1$ is even. $\therefore \frac{n}{2}$ or $\frac{n+1}{2}$ is an integer. Similarly for $\frac{a_0}{2}$ or $(a_0+1)/2$.

$$\therefore \frac{n(n+1)}{2} \equiv \frac{a_0(a_0+1)}{2} \pmod{10}$$

Consider all possibilities for a_0

a_0	$\frac{a_0(a_0+1)}{2}$	mod 10	+5 (mod 10)
0	0	0	5
1	1	1	6
2	3	3	8
3	6	6	1
4	10	0	5
5	15	5	0
6	21	1	6
7	28	8	3
8	36	6	1
9	45	5	0

Note that for the other a_i, if associated with a factor of 10, then $\frac{a_i 10^i}{2} \equiv 0$ or $5 \pmod{10}$

Thus, the column [+5 (mod 10)] shows possibilities of other factors contributing to units digit if divided by 2.

$\therefore a \equiv 0, 1, 3, 5, 6,$ or $8 \pmod{10}$ if a is triangular

2. Find the last two digits of 9^{9^9}.

$9^3 - 9 = 9(9^2-1) = 9(80)$. $\therefore 9^3 \equiv 9 \pmod{10}$

$\therefore 9^9 \equiv 9^3 \equiv 9 \pmod{10}$

$\therefore\ 9^9 - 9 = 10K$, some K, or $9^9 = 9 + 10K$

$\therefore\ 9^{9^9} = 9^{9+10K} = 9^9 \cdot 9^{10K}$

Look at the last 2 digits of 9^9 and 9^{10} yields:

9^1 : 9	9^4 : 61	9^7 : 69	9^{10} : 01
9^2 : 81	9^5 : 49	9^8 : 21	
9^3 : 29	9^6 : 41	9^9 : 89	

$\therefore\ 9^9 \equiv 89 \pmod{100}$ and $9^{10} \equiv 1 \pmod{100}$

$\therefore\ 9^{10K} \equiv 1^K \equiv 1 \pmod{100}$

$\therefore\ 9^9 \cdot 9^{10K} \equiv 89 \cdot 1 \pmod{100}$

But $9^{9^9} = 9^9 \cdot 9^{10K}$. $\therefore\ 9^{9^9} \equiv 89 \pmod{100}$.

$\therefore$ Last two digits of 9^{9^9} are 89.

3. 176,521,221 : $1+7+6+5+2+1+2+2+1 = 27$

$\therefore$ divisible by 9

$1-2+2-1+2-5+6-7+1 = -3$

$\therefore$ not divisible by 11

149,235,678 : $1+4+9+2+3+5+6+7+8 = 36$

$\therefore$ divisible by 9

$8-7+6-5+3-2+9-4+1 = 9$

$\therefore$ not divisible by 11

4. (a) Prove: If N is represented in the base b by

$$N = a_m b^m + \cdots + a_1 b + a_0, \quad 0 \le a_k \le b-1$$

Then $(b-1) \mid N \Leftrightarrow (b-1) \mid (a_m + a_{m-1} + \cdots + a_1 + a_0)$

Pf: Consider $P(x) = \sum_{k=0}^{m} a_k x^k$, a polynomial with integer coefficients.

Note that $b \equiv 1 \pmod{b-1}$

$\therefore P(b) \equiv P(1) \pmod{b-1}$ by Th. 4.4.

But $P(b) = N$, and

$P(1) = a_m + \cdots + a_1 + a_0$

$\therefore N \equiv a_m + \cdots + a_1 + a_0 \pmod{b-1}$

$\therefore N \equiv 0 \pmod{b-1} \Leftrightarrow a_m + \cdots + a_0 \equiv 0 \pmod{b-1}$

$\therefore (b-1) \mid N \Leftrightarrow (b-1) \mid (a_m + \cdots + a_0)$

Note: $(b-1)$ divides N (base 10) $\Leftrightarrow$ sum of digits (base 10) is divisible by $(b-1)$.

(b) For N written in base 9

(1) N is divisible by 8 $\Leftrightarrow$ sum of digits of N (in base 10) is divisible by 8 (in base 10). This follows from (a).

(2) N is divisible by 3 $\Leftrightarrow$ units digit is divisible by 3 since each term in the polynomial (other than units digit) contain a power of 9.

(c) $(447836)_9$: is divisible by 3 since $3 \mid 6$
$4+4+7+8+3+6 = 32$ (base 10),
so is also divisible by 8.

5. Find the missing digits

(a) $51840 \cdot 273581 = 1418243x040$

$5+1+8+4+0 = 18$, so $9 \mid 51840$,
$\therefore 9 \mid 1418243x040$, so $1+4+1+8+2+4+3+x+4 = x+27$
$\therefore 9 \mid (x+27)$. $\therefore x = 0$ or 9

Since $1-8+5-3+7-2 = 0$, Then $11 \mid 273581$
$\therefore 0-4+0-x+3-4+2-8+1-4+1 = -13-x$
$\therefore 11 \mid (-13-x)$, $\therefore x \neq 0$, and $x = 9$

$\therefore \underline{x = 9}$

(b). $2x99561 = [3(523+x)]^2$

Since 3^2 is on right side, $9 \mid 2x99561$

$\therefore 2+x+9+9+5+6+1 = x+32$, $\therefore \underline{x=4}$

(c) $2784x = x \cdot 5569$

$5+5+6+9 = 25$. From proof of Th. 4.5,

$5569 \equiv 25 \pmod 9$, and $25 \equiv (2+5) \bmod 9$

$\therefore 5569 \equiv 7 \pmod 9$.

$\therefore 5569x \equiv 7x \pmod 9$

$2784x \equiv (2+7+8+4+x) \equiv (3+x) \bmod 9$

$\therefore 7x \equiv (3+x) \pmod 9$, or $6x \equiv 3 \pmod 9$

$\therefore 9 \mid (6x-3)$, so $x = 2, 5, 8$

$5569 \equiv (9-6+5-5) \equiv 3 \pmod{11}$, $5569x \equiv 3x \pmod{11}$

$2784x \equiv (x-4+8-7+2) \equiv (x-1) \pmod{11}$

$\therefore 3x \equiv (x-1) \pmod{11}$, $2x \equiv -1 \equiv 10 \pmod{11}$

$\therefore \underline{x=5}$

(d) $512 \cdot 1x53125 = 1{,}000{,}000{,}000$

$512 \equiv (5+1+2) = 8 \pmod 9$

$1x53125 \equiv (1+x+5+3+1+2+5) \equiv (8+x) \pmod 9$

$\therefore 8 \cdot (8+x) \equiv 1{,}000{,}000{,}000 \equiv 1 \pmod 9$

$\therefore 64 + x \equiv 6+4+x \equiv (1+x) \equiv 1 \pmod 9$

$\therefore x = 0$ or $x = 9$

$512 \equiv (2-1+5) = 6 \pmod{11}$

$1x53125 \equiv (5-2+1-3+5-x+1) \equiv (7-x) \pmod{11}$

$\therefore 6\cdot(7-x) \equiv (0-0+0-0+0-0+0-0+0-1) \equiv -1 \pmod{11}$

$\therefore 42-6x \equiv -1 \pmod{11}$, $43 \equiv 6x \pmod{11}$

$\therefore x \neq 0$, and for $x=9$, $43 \equiv 54 \pmod{11}$.

$\therefore \underline{x=9}$

6. (a). An integer is divisible by 2 $\Leftrightarrow$ its units digit is 0, 2, 4, 6, or 8.

Pf: Since $10 = 5\cdot 2$, in the base 10 representation of an integer $N = a_m 10^m + \cdots + a_1 10 + a_0$, each term, except a_0, contains a power of 10, and so is divisible by 2.
$\therefore$ N is divisible by 2 $\Leftrightarrow$ a_0 is divisible by 2, so $a_0 = 0, 2, 4, 6,$ or 8.

(b) An integer is divisible by 3 $\Leftrightarrow$ The sum of its digits is divisible by 3.

Pf: Let $N = a_m 10^m + \cdots + a_1 10 + a_0$ be the decimal expansion of N, $0 \le a_k < 10$, and let $S = a_m + \cdots + a_1 + a_0$

Consider $P(x) = \sum_{k=0}^{m} a_k x^k$. Note $P(10) = N$, $P(1) = S$

Note also $10 \equiv 1 \pmod 3$, so $P(10) \equiv P(1) \pmod 3$

$\therefore N \equiv S \pmod 3$.

$\therefore N \equiv 0 \pmod 3 \Leftrightarrow S \equiv 0 \pmod 3$

(c) An integer is divisible by 4 $\Leftrightarrow$ The number formed by its tens and units digits is divisible by 4.

Pf: Let $N = a_m 10^m + \dots + a_2 10^2 + a_1 10 + a_0$,

$0 \le a_k < 10$.

Let $k \ge 2$. Then $10^k = 10^{k-2} \cdot 10^2 = 10^{k-2}(5 \cdot 2)^2$

$= 10^{k-2} \cdot 25 \cdot 4$

$\therefore$ Each term $c_k = a_k 10^k$ is divisible by 4 if $k \ge 2$.

$\therefore N$ is divisible by 4 $\Leftrightarrow a_1 10 + a_0$ is divisible by 4.

(d) An integer is divisible by 5 $\Leftrightarrow$ its units digit is 0 or 5

Pf: Let $N = a_m 10^m + \dots + a_2 10^2 + a_1 10 + a_0$,

$0 \le a_k < 10$.

Let $c_k = a_k 10^k = a_k (5 \cdot 2)^k = a_k 5^k 2^k$

$\therefore$ Each c_k is divisible by 5, if $k \geq 1$.

$\therefore$ N is divisible by 5 $\Leftrightarrow$ a_0 is divisible by 5, and a_0 is divible by 5 $\Leftrightarrow$ $a_0 = 0, 5$.

7. For any integer a, show that $a^2 - a + 7$ ends in one of the digits 3, 7, or 9.

Pf: If $a = a_m 10^m + \cdots + a_1 10 + a_0$, Then $a \equiv a_0 \pmod{10}$

$\therefore$ $a^2 \equiv a_0^2 \pmod{10}$. $\therefore$ $a^2 - a + 7 \equiv a_0^2 - a_0 + 7 \pmod{10}$.

$\therefore$

a_0	$a_0^2 - a_0 + 7$	$a_0^2 - a_0 + 7 - 10k$	
0	7	7	$k=0$
1	7	7	$k=0$
2	9	9	$k=0$
3	13	3	$k=1$
4	19	9	$k=1$
5	27	7	$k=2$
6	37	7	$k=3$
7	49	9	$k=4$
8	63	3	$k=6$
9	79	9	$k=7$

Since $a_0^2 - a_0 + 7 \equiv a_0^2 - a_0 + 7 - 10k \pmod{10}$, Then $a^2 - a + 7 \equiv 3, 7, \text{or } 9 \pmod{10}$

8. Find The remainder when 4444^{4444} is divided by 9.

Note That $4444 \pmod 9) \equiv (4+4+4+4) \equiv 16 \pmod 9)$
$16 = 2^3 \cdot 2$, so $4444 \pmod 9) \equiv 2^3 \cdot 2 \pmod 9)$
Since $2^3 \equiv (-1) \pmod 9$, Then $4444 \equiv (-1) \cdot 2 \pmod 9$

$\therefore$
$$4444^{4444} \equiv (-1)^{4444} \cdot 2^{4444} \equiv 2^{4444} \pmod 9$$

But $4444 = 3 \cdot 1381 + 1$, so

$$2^{4444} = (2^3)^{1381} \cdot 2 \quad \therefore 2^{4444} \equiv (-1)^{1381} \cdot 2 \pmod 9$$

$\therefore$
$$4444^{4444} \equiv 2^{4444} \equiv (-1) \cdot 2 \equiv 7 \pmod 9$$

$\therefore$ remainder is 7

9. Prove that no integer whose digits add up to 15 can be a square or cube.

Pf: Let N be any integer whose digits add up to 15.
$\therefore N \equiv 15 \pmod 9$ (see pf. to Th. 4.5).
But $15 \equiv 6 \pmod 9$.
$\therefore N \equiv 6 \pmod 9$
Consider $a = a_m 9^m + \cdots + a_1 9 + a_0$

$\therefore a \equiv a_0 \pmod 9$, and $\therefore a^2 \equiv a_0^2 \pmod 9$
and $a^3 \equiv a_0^3 \pmod 9$.
Consider all possibilities of a_0:

for $\pmod 9$

a_0	a_0^2	$a_0^2 \pmod 9$	a_0^3	$a_0^3 \pmod 9$
0	0	0	0	0
1	1	1	1	1
2	4	4	8	8
3	9	0	27	0
4	16	7	64	1
5	25	7	125	8
6	36	0	216	0
7	49	4	343	1
8	64	1	512	8

$\therefore a^2 \equiv 0, 1, 4, \text{or } 7 \pmod 9$

$\therefore$ There is no a^2 s.t. $a^2 \equiv 6 \pmod 9$

$a^3 \equiv 0, 1, \text{or } 8 \pmod 9$, and so there is <u>no</u> a^3 s.t. $a^3 \equiv 6 \pmod 9$.

10. Assuming 495 divides $273x49y5$, find x, y.

Let $495 \cdot N = 273x49y5$

$495 \equiv 0 \pmod 9$, $\therefore 495 \cdot N \equiv 0 \cdot N \equiv 0 \pmod 9$

$\therefore 273x49y5 \equiv 0 \pmod 9$

$\therefore (2+7+3+x+4+9+y+5) = 30+x+y \equiv 3+x+y \pmod 9$

so $3+x+y \equiv 0 \pmod 9$, or $x+y \equiv 6 \pmod 9$

and $x+y \equiv 15 \pmod 9$

Also, $5-9+4=0$, so $495 \equiv 0 \pmod{11}$,

$\therefore 495 \cdot N \equiv 0 \pmod{11}$

$\therefore 273x49y5 \equiv 0 \pmod{11}$

$\therefore 5-y+9-4+x-3+7-2 = x-y+12 \equiv 0 \pmod{11}$

$\therefore x-y \equiv -1 \pmod{11}$, or $y-x \equiv 1 \pmod{11}$

$\therefore$
$$\begin{array}{r} x+y=6 \\ \underline{y-x=1} \\ 2y=7 \end{array}$$, no integer

$$\begin{array}{r} x+y=15 \\ \underline{y-x=1} \\ 2y=16 \end{array}$$, $\underline{y=8,\ x=7}$

11. Determine the last 3 digits of 7^{999}

Need to use mod 1000 since want last 3 digits.

Also, $7^4 = 2401 = (1+6 \cdot 400)$

Since $400^2 = 160000$, Then $400^n \equiv 0 \pmod{100}$ for $n \geq 2$.

$\therefore 7^{4n} = (1 + 6 \cdot 400)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} (6 \cdot 400)^k$

So for $k \geq 2$, $\binom{n}{k}(6 \cdot 400)^k \equiv 0 \pmod{1000}$

$\therefore 7^{4n} \equiv 1 + \binom{n}{1} 6 \cdot 400 \pmod{1000}$

Now $999 = 4 \cdot 249 + 3$

$\therefore 7^{4 \cdot 249} \equiv 1 + \binom{249}{1}(6 \cdot 400) \equiv 1 + 249 \cdot 6 \cdot 400 \pmod{1000}$

$249 \cdot 6 \cdot 400 = (49 + 200)(400)(6) = (6)(49)(400) + 6 \cdot 400 \cdot 200$

$\therefore 249 \cdot 6 \cdot 400 \equiv 6 \cdot 49 \cdot 400 \pmod{1000}$

$\therefore 7^{4 \cdot 249} \equiv 1 + 6 \cdot 49 \cdot 400 \pmod{1000}$

$\equiv 1 + 6 \cdot 9 \cdot 400 \pmod{1000}$

$6 \cdot 9 \cdot 4 = 216$, $\therefore 1 + 6 \cdot 9 \cdot 400 = 21601$

$\therefore 7^{996} \equiv 601 \pmod{1000}$

$\therefore 7^{999} \equiv 601 \cdot 7^3 \pmod{1000}$

$7^3 = 343$, $(601)(343) = 206143$

$\therefore 7^{999} \equiv 206143 \equiv 143 \pmod{1000}$.

$\therefore$ <u>143</u> are the last 3 digits.

12. If t_n is The nth triangular number, show That $t_{n+2k} \equiv t_n \pmod{k}$. $\therefore t_n, t_{n+20}$ have same last digit.

Pf: $t_{n+2k} = \dfrac{(n+2k)(n+2k+1)}{2}$

$$= \frac{n^2 + 2kn + n + 2kn + 4k^2 + 2k}{2}$$

$$= \frac{n^2 + n + 4kn + 4k^2 + 2k}{2}$$

$$\therefore t_{n+2k} - t_n = \frac{n^2 + n + 4kn + 4k^2 + 2k}{2} - \frac{n(n+1)}{2}$$

$$= \frac{4kn + 4k^2 + 2k}{2}$$

$$= k(2n + 2k + 1)$$

$$\therefore t_{n+2k} \equiv t_n \pmod{k}$$

$\therefore t_{n+20} \equiv t_n \pmod{10}$, or

$t_{n+20} = t_n + 10k$, some k

$\therefore$ if $t_n = (a_m a_{m-1} \dots a_2 a_1 a_0)_{10}$, Then

adding $10K$ not affect a_0, so
t_{n+20} and t_n have same units digit.

13. For any $n \geq 1$, prove there exists a prime with at least n of its digits equal to 0.

Pf: This follows from Dirichlet's Theorem (p.56).
From problem #12 of section 2.2,
$\gcd(a, a+1) = 1$. $\therefore$ consider $9, 10$ and
arithmetic progressions with powers of 10.
$\therefore$ $9 + 10K$, $K = 1, 2, 3, \ldots$ contains infinitely many primes.
$10^{n+1} + 9$ has n zeroes, and there only a finite number less than $10^{n+1} + 9$.
By Dirichlet's Theorem, there must be a prime in the series $K \cdot 10^{n+1} + 9$, and each has n zeroes.

14. Find the values of $n \geq 1$ for which $1! + 2! + \ldots + n!$ is a perfect square.

$1! = 1$ $\quad$ $3! = 6$ $\qquad$ Note that for $n \geq 5$, $\sum_{K=1}^{n} n!$
$2! = 2$ $\quad$ $4! = 24$ $\qquad$ ends in 0.

$\therefore$ $1! = 1^2$
$1! + 2! = 3$

$1!+2!+3! = 9 = 3^2$

$1!+2!+3!+4! = 33$

$\therefore$ The units digits of $\sum_{k=1}^{n} n!$ will be 3 for $n \geq 4$

By problem 1(a), a perfect square can't end in 3. $\therefore$ There is no perfect square for $n \geq 4$

$\therefore$ $n = 1, 3$ are the only values.

15. Show that 2^n divides an integer $N \Leftrightarrow 2^n$ divides the number made up of the last n digits of N.

Pf: Let $N = a_{n+j}10^{n+j} + \dots + a_n 10^n + a_{n-1}10^{n-1} + \dots + a_1 10 + a_0$

be the decimal representation for N, $n \geq 1, j \geq 0$.

(a) If 2^n divides the last n digits of N, then

$$2^n \mid (a_{n-1}10^{n-1} + \dots + a_1 10 + a_0) \qquad [1]$$

But $a_{n+j}10^{n+j} + \dots + a_n 10^n = 10^n(a_{n+j}10^j + \dots + a_n)$

$$= 2^n 5^n (a_{n+j}10^j + \dots + a_n)$$

$\therefore 2^n \mid (a_{n+j} + 10^{n+j} + \ldots a_n 10^n)$ [2]

From [1] and [2],

$2^n \mid (a_{n+j} \cdot 10^{n+j} + \ldots + a_n 10^n + a_{n-1} 10^{n-1} + \cdots + a_0)$, so $2^n \mid N$

(b) Suppose $2^n \mid N$. Since $2^n \mid 2^n 5^n$, Then

$2^n \mid 10^n (a_{n+j} \cdot 10^j + \ldots + a_n)$,

$\therefore 2^n \mid (a_{n+j} 10^{n+j} + \ldots + a_n 10^n)$

$\therefore 2^n \mid N - (a_{n+j} \cdot 10^{n+j} + \cdots + a_n 10^n)$

$\Rightarrow 2^n \mid (a_{n-1} 10^{n-1} + \ldots + a_1 10 + a_0)$

$\therefore 2^n$ divides The last n digits of N.

1c. Let $N = a_m 10^m + \ldots + a_1 10 + a_0$, $0 \le a_k \le 9$, be The decimal expansion of a positive integer N.

(a) Prove 7, 11, and 13 all divide $N \Leftrightarrow$ 7, 11, and 13 divide The integer

$$M = (a_0 + 10a_1 + 100a_2) - (a_3 + 10a_4 + 100a_5) + (a_6 + 10a_7 + 100a_8) - \ldots$$

Pf: Since $1001 = 1000+1 = 1000-(-1)$.

$\therefore 10^3 \equiv -1 \pmod{1001}$

Also, $10^6 - 1 = (10^3-1)(10^3+1) \equiv 999(1001)$

$\therefore 10^6 \equiv 1 \pmod{1001}$

Also, $1001 = 7 \cdot 11 \cdot 13$

Consider 10^{3n}

If n is odd, $n = 2k+1$ for some k

$\therefore 10^{3n} = 10^{3(2k+1)} = 10^{6k+3} = 10^{6k} \cdot 10^3$

But $10^6 \equiv 1 \pmod{1001}$, so

$10^{6k} \equiv 1^k = 1 \pmod{1001}$

$\therefore 10^{6k} \cdot 10^3 \equiv 1(-1) = -1 \pmod{1001}$

$\therefore n$ odd $\Rightarrow 10^{3n} \equiv (-1) \pmod{1001}$ [1]

If n is even, then $n = 2k$, some k.

$\therefore 10^{3n} = 10^{6k} \equiv 1^k = 1 \pmod{1001}$

$\therefore n$ even $\Rightarrow 10^{3n} \equiv 1 \pmod{1001}$ [2]

Note $N = (a_0 + 10a_1 + 100a_2) - (-a_3 10^3 - a_4 10^4 - a_5 10^5) + \ldots$

$$= \underbrace{10^0(a_0 + a_1 10 + a_2 100)}_{\text{even}} \underbrace{- 10^3(-a_3 - a_4 10 - a_5 100)}_{\text{odd}} + \ldots$$

From [2], j even: $a_{3j} 10^{3j} \equiv a_{3j} \pmod{1001}$ [3]

$a_{3j+1} 10^{3j+1} \equiv a_{3j+1} 10 \pmod{1001}$ [4]

$$a_{3j+2}10^{3j+2} \equiv a_{3j+2}100 \pmod{1001} \quad [5]$$

From [1], k odd: $a_{3k}10^{3k} \equiv -a_{3k} \pmod{1001}$ [6]

$$a_{3k+1}10^{3k+1} \equiv -a_{3k+1}10 \pmod{1001} \quad [7]$$

$$a_{3k+2}10^{3k+2} \equiv -a_{3k+2}100 \pmod{1001} \quad [8]$$

Adding [3] + [4] + [5], j even:

$$a_{3j}10^{3j} + a_{3j+1}10^{3j+1} + a_{3j+2}10^{3j+2} \equiv a_{3j} + a_{3j+1}10 + a_{3j+2}100 \pmod{1001} \quad [9]$$

Adding [6] + [7] + [8], k odd:

$$a_{3k}10^{3k} + a_{3k+1}10^{3k+1} + a_{3k+2}10^{3k+2} \equiv -a_{3k} - a_{3k+1}10 - a_{3k+2}100 \pmod{1001} [10]$$

Now let $3k = 3j + 3$. $\therefore$ $3k$ is odd, as k is odd.

Adding [9] + [10],

$$a_{3j}10^{3j} + a_{3j+1}10^{3j+1} + a_{3j+2}10^{3j+2} + a_{3j+3}10^{3j+3} + a_{3j+4}10^{3j+4} + a_{3j+5}10^{3j+5}$$

$$\equiv (a_{3j} + 10a_{3j+1} + 100a_{3j+2}) - (a_{3j+3} + 10a_{3j+4} + 100a_{3j+5}) \pmod{1001}$$

$\therefore$ $N \equiv M \pmod{1001}$, $\therefore$ $N \equiv M \pmod{7.11.13}$

$\therefore N \equiv 0 \pmod{7\cdot 11\cdot 13} \Leftrightarrow M \equiv 0 \pmod{7\cdot 11 \cdot 13}$

(b). Prove $6 \mid N \Leftrightarrow 6$ divides The integer

$$M = a_0 + 4a_1 + \dots + 4a_m$$

Pf: $10 \equiv 4 \pmod 6$, and $40 \equiv 4 \pmod 6$

Lemma: $10^n \equiv 4 \pmod 6$, $n \geq 1$

Pf: True for $n=1$, since $10 \equiv 4 \pmod 6$

Suppose true for k: $10^k \equiv 4 \pmod 6$

$\therefore 10^{k+1} = 10^k \cdot 10 \equiv 4 \cdot 10 = 40 \equiv 4 \pmod 6$

$\therefore$ true for $k+1$.

$\therefore a_k 10^k \equiv 4a_k$, $k \geq 1$

$\therefore N = a_0 + a_1 10 + \dots a_m 10^m \equiv a_0 + 4a_1 + \dots + 4a_m \pmod 6$

$\therefore N \equiv M \pmod 6$

$\therefore N \equiv 0 \pmod 6 \Leftrightarrow M \equiv 0 \pmod 6$

17. Is 1,010,908,899 divisible by 7, 11, and 13?

$9 + 9\cdot 10 + 8 \cdot 100 - (8 + 0\cdot 10 + 9\cdot 100) + (0 + 1\cdot 10 + 0 \cdot 100) - (1)$

$= 1 + 90 - 100 + 10 - 1$

$= 0$

$\therefore N \equiv 0 \pmod{1001}$, $\therefore N$ divisible by 7, 11, 13

18. (a) Given integer N, let M be the integer formed by reversing the order of digits of N.
Show $N-M$ is divisible by 9.

Pf: Let $N = a_m 10^m + \cdots + 10a_1 + a_0$

$$\therefore M = a_0 10^m + \cdots + a_k 10^{m-k} + \cdots + a_{m-1}10 + a_m$$

$$\therefore N - M = (a_m - a_0)10^m + \cdots + (a_1 - a_{m-1})10 + (a_0 - a_m)$$

The sum of the coefficients of $N-M$ is:
$(a_m + a_{m-1} + \cdots + a_0) - (a_0 + \cdots + a_{m-1} + a_m) = 0$.

$\therefore$ By Th. 4.5, since $9|0$, $9|(N-M)$

(b) Prove any palindrome with an even number of digits is divisible by 11.

Pf: Let $N = a_m 10^m + \cdots + a_0$ have even number of digits. $\therefore$ m is odd.
Also, $N = a_0 10^m + \cdots + a_{m-1}10 + a_m$

$\therefore$ $a_m = a_0$, $a_{m-1} = a_1$, and in general,

$$a_k = a_{m-k}, \quad 0 \le k \le m$$

Look at $T = (a_0 - a_1) + (a_2 - a_3) + \dots + (a_{m-1} - a_m)$

Since m is odd, There is no coefficient That is ungrouped.

Rearranging terms of T, by reversing The order of The negative coefficients,

$$T = (a_0 - a_m) + (a_2 - a_{m-1}) + \dots + (a_{m-1} - a_1)$$

$$= 0 + 0 + \dots + 0$$
$$= 0$$

$\therefore$ By Th. 4.6, $11 | N$.

19. Given repunit R_n, prove:

(a) $9 | R_n \Leftrightarrow 9 | n$

Pf: Note for R_n, sum of digits, S, is n since $R_n = 11\dots1$ (n digits of 1).
$\therefore$ since $R_n \equiv S \pmod 9$ by Th. 4.5,
$\therefore R_n \equiv n \pmod 9$.

$\therefore R_n \equiv 0 \pmod 9 \Leftrightarrow n \equiv 0 \pmod 9$

(b) $11 \mid R_n \Leftrightarrow n$ is even

Pf: Let $R_n = 1 \cdot 10^m + \dots + 1 \cdot 10 + 1$

Look at $T = (a_0 - a_1) + (a_2 - a_3) + \dots + (-1)^m a_m$

$= (1-1) + (1-1) + \dots + (-1)^m a_m$

T will be $0 \Leftrightarrow$ can group terms, which means m is odd

$\therefore T = 0 \Leftrightarrow$ number terms is even.

$\therefore$ By Th. 4.6, $11 \mid R_n \Leftrightarrow T = 0$

$\therefore 11 \mid R_n \Leftrightarrow n$ is even.

20. Factor $R_6 = 111{,}111$ into a product of primes

Since R_6 has an even number of groups of three coefficients, and

$m = (1 + 1 \cdot 10 + 1 \cdot 100) - (1 + 1 \cdot 10 + 1 \cdot 100)$
$= 0$,

Then $R_6 \equiv 0 \pmod{1001}$ by prob. 16.(a).

$\therefore R_6 = 111{,}111 = 7 \cdot 11 \cdot 13 \cdot k = 1001 \cdot k$

Try $k = 111$ since $111 \cdot 1{,}000 = 111{,}000$.
Find that $R_6 = 111 \cdot 1001$, and $111 = 3 \cdot 37$
$\therefore R_6 = 3 \cdot 7 \cdot 11 \cdot 13 \cdot 37$

21. Show why

$$1 \cdot 9 + 2 = 11$$
$$12 \cdot 9 + 3 = 111$$
$$\vdots$$
$$123456789 \cdot 9 + 10 = 1111111111$$

i.e., show that

$$(10^{n-1} + 2 \cdot 10^{n-2} + \cdots + n) \cdot 9 + (n+1) = \frac{10^{n+1} - 1}{9}$$

Since $9 = 10 - 1$, then

$$(10^{n-1} + 2 \cdot 10^{n-2} + \cdots + n) \cdot 9 = (10^{n-1} + 2 \cdot 10^{n-2} + \cdots + n)(10 - 1)$$

$$= (10^{n-1} + 2 \cdot 10^{n-2} + \cdots + (n-1) \cdot 10 + n) \cdot 10 - (10^{n-1} + \cdots + n)$$

$$= 10^n + 2 \cdot 10^{n-1} + 3 \cdot 10^{n-2} + \cdots + (n-1) \cdot 10^2 + n \cdot 10 - (10^{n-1} + \cdots + n)$$

$$= 10^n + 1 \cdot 10^{n-1} + 1 \cdot 10^{n-2} + \cdots + 1 \cdot 10 - n$$

$$\therefore (10^{n-1} + 2 \cdot 10^{n-2} + \cdots + n) \cdot 9 + (n+1) =$$
$$10^n + 1 \cdot 10^{n-1} + 1 \cdot 10^{n-2} + \cdots + 1 \cdot 10 - n + (n+1)$$

$$= 10^n + 1\cdot 10^{n-1} + \dots + 1\cdot 10 + 1 \qquad [1]$$

The latter basically proves the assertion. However, to go further, multiply [1] by 9 and add 1.

$$(10^n + 1\cdot 10^{n-1} + \dots + 1\cdot 10 + 1)\cdot 9 + 1$$

$$= (10^n + 10^{n-1} + \dots + 10 + 1)(10-1) + 1$$

$$= 10^{n+1} + 10^n + \dots + 10^2 + 10$$
$$- \quad 10^n - 10^{n-1} - \dots - 10 - 1 + 1$$

$$= 10^{n+1}$$

$$\therefore 10^n + 1\cdot 10^{n-1} + \dots + 1\cdot 10 + 1 \qquad [1]$$

$$= \frac{10^{n+1}-1}{9}$$

$$\therefore (10^{n-1} + 2\cdot 10^{n-2} + \dots + n)\cdot 9 + (n+1) = \frac{10^{n+1}-1}{9}$$

22. An invoice shows that 72 canned hams were purchased for \$x67.9y. Find the missing digits.

Solution: $72 \cdot N = x679y$, where N = cost in cents of one ham.

Note $72 = 8 \cdot 9 = 2^3 \cdot 9$. $\therefore 2^3 \mid x679y$

By problem 15, $2^3 \mid 79y$ $\therefore y = 2$

since $79y \div 8 = 90 + 7y$, and $\therefore 8 \mid 7y$

$\therefore x679y = x6792$

Since $9 \mid 72$, $9 \mid x6792$, $\therefore 9 \mid x+6+7+9+2$

$\therefore x = 3$

$\therefore x679y = 36792$ $(x=3, y=2)$

23. If 792 divides $13xy45z$, find x, y, z

Solution: Since $792 = 8 \cdot 99$, $8 \mid 792$, so $8 \mid 13xy45z$, and by problem 15, $8 \mid 45z$

$45z = 8 \cdot 50 + 5z$, $\therefore 8 \mid 5z$, $\therefore z = 6$

Since $9 \mid 13xy456$, $9 \mid 1+3+x+y+4+5+6$,

$\therefore 9 \mid 1+x+y$, $\therefore x+y+1 = 9, 18$, $x+y = 8, 17$

Also, $6 \mid 792$, so, by problem 16(b),

$6 \mid 6 + 4 \cdot 5 + 4 \cdot 4 + 4y + 4x + 4 \cdot 3 + 4 \cdot 1$, or

$6 \mid 4y + 4x + 58$, or $6 \mid 4x + 4y + 4$

$\therefore 6 \mid 4(1+x+y)$, so $3 \mid (1+x+y)$

This doesn't help since $9 \mid (1+x+y)$.

Note $11 \mid 792$, so $11 \mid 13xy456$

$\therefore 11 \mid 6-5+4-y+x-3+1$, or $11 \mid 3+x-y$

$\therefore 3+x-y = 11$ (can't $= 22$, since $x,y \le 9$)

$\therefore x-y = 8$

From $9 \mid 1+x+y$, $x+y = 8, 17$

$\therefore x+y=8 \qquad x+y=17$
$\quad\; x-y=8 \qquad x-y=8$

$\therefore x=8, y=0 \qquad 2x=25$, $\therefore$ not a solution

$\therefore 13xy45z = 1380456$

24. For any prime $p>3$, prove $13 \mid 10^{2p}-10^{p}+1$

Pf: Use result of problem 16(a) and look at possible coefficients of the constructed integer M, formed by alternately summing and subtracting blocks of 3 coefficients of the decimal expansion of N.

For an integer N, proof of 16(a) showed $N \equiv M \pmod{7\cdot 11\cdot 13}$. $\therefore N \equiv M \pmod{13}$.

$\therefore$ Goal is to show constructed $M \equiv 0 \pmod{13}$, and so $N \equiv 0 \pmod{13}$.

To analyze M, look at decimal expansion coefficients of $N = 10^{2p} - 10^{p} + 1$.

First consider 10^{p}. Since p is prime, p is odd, so there will be an odd number of zeroes in 10^{p}.

Example: $10^5 = 1 \cdot 10^5 + 0 \cdot 10^4 + \cdots + 0 \cdot 10 + 0$
$= a_5 \cdot 10^5 + a_4 \cdot 10^4 + \cdots + a_1 \cdot 10 + a_0$

Look at M:

Note $p > 3$.
For $p = 5, 11, 17, \ldots$
$M = -100$

For $p = 7, 13, \ldots$
$M = +10$

$p \neq 9, 15, \ldots$ as p is prime.

	$100\times$ a_{k+2}	$10\times$ a_{k+1}	$1\times$ a_k	+/-
value of k	2	1	0	+
	5	4	3	−
	8	7	6	+
	11	10	9	−
	14	13	12	+
	17	16	15	−
		⋮		

From Div. Alg., $p = 3q + r$, and as p is prime, $r \neq 0$.

$\therefore$ Can restrict considerations to $p = 3q+1$ or $p = 3q+2$, as above, and $p > 3$.
For $p = 3q+1$, q must be even for p to be odd. Let $q = 2K$. $\therefore p = 6K+1$ $(K = 1, 2, \ldots)$
For $p = 3q+2$, q must be odd. Let $q = 2K'+1$.
$\therefore p = 6K'+5$ $(K' = 0, 1, \ldots)$

$$\therefore \begin{aligned} p &= 6K+1 \quad (K = 1, 2, \ldots), \text{ or} \\ p &= 6K'+5 \quad (K' = 0, 1, 2, \ldots) \end{aligned}$$

$$\therefore \begin{aligned} 10^p &= 10^{6K} \cdot 10 \quad (K = 1, 2, \ldots), \text{ or} \\ 10^p &= 10^{6K} \cdot 10^5 \quad (K' = 0, 1, 2, \ldots) \end{aligned}$$

From above, since $M = -100$ for 10^5, and since $N \equiv M \pmod{13}$, letting $N = 10^5$, Then $10^5 \equiv -100 \pmod{13}$
Similarly $10 \equiv 10 \pmod{13}$

Since $10^6 \equiv 1 \pmod{13}$, $10^{6K} \equiv 1^K = 1 \pmod{13}$

$$\therefore \begin{aligned} 10^p &= 10^{6K} \cdot 10 \equiv 10 \pmod{13} \quad (K = 1, 2, \ldots) \\ 10^p &= 10^{6K} \cdot 10^5 \equiv 10^5 \pmod{13} \quad (K' = 0, 1, 2 \ldots) \end{aligned}$$
and $10^5 \equiv -100 \pmod{13}$

$\therefore\ 10^{p} \equiv 10 \pmod{13}$ $(k=1,2,\dots)$ [1]
or $10^{p} \equiv -100 \pmod{13}$ $(k'=0,1,2,\dots)$

For 10^{2p}: $2p = 12k+2$ $(k=1,2,\dots)$
$2p = 12k+10$ $(k'=0,1,2,\dots)$

$\therefore\ 10^{2p} = 10^{12k}\cdot 100$ $(k=1,2,\dots)$
$10^{2p} = 10^{12k}\cdot 10^{5}\cdot 10^{5}$ $(k'=0,1,2,\dots)$

Using $N \equiv m \pmod{13}$, $N = 10^{2p}$, and
using $10^{6} \equiv 1 \pmod{13}$, so $10^{12k} \equiv 1 \pmod{13}$,

$\therefore\ 10^{2p} \equiv 100 \pmod{13}$ $(k=1,2,\dots)$ [2]
or $10^{2p} \equiv 10^{5}\cdot 10^{5} \pmod{13}$ $(k'=0,1,2,\dots)$

From above, $10^{5} \equiv -100 \pmod{13}$
$\therefore\ 10^{5}\cdot 10^{5} \equiv (-100)(-100) = 10000 \pmod{13}$
Since $\gcd(10,13)=1$, Then
$10^{5}/10 \equiv -100/10 \pmod{13}$, so
$10000 \equiv -10 \pmod{13}$

$\therefore$ From [2], $10^{2p} \equiv 100 \pmod{13}$ $(k=1,2,3,\dots)$ [2']
$10^{2p} \equiv -10 \pmod{13}$ $(k'=0,1,2,\dots)$

$\therefore\ N = 10^{2p} - 10^{p} + 1 = [2'] - [1] + 1$ becomes

$$10^{2p} - 10^{p} + 1 \equiv 100 - 10 + 1 = 91 \pmod{13} \quad (k = 1, 2, 3, \ldots)$$

$$10^{2p} - 10^{p} + 1 \equiv -10 - (-100) + 1 = 91 \pmod{13} \quad (k' = 0, 1, 2, \ldots)$$

But $91 = 7 \cdot 13$, so $91 \equiv 0 \pmod{13}$

$\therefore\ 10^{2p} - 10^{p} + 1 \equiv 0 \pmod{13}$, so

$$13 \mid (10^{2p} - 10^{p} + 1).$$