6.1 The Functions tau and sigma

Note Title 6/3/2005

1. Let m, n be positive integers, and $p_1 \ldots p_r$ be the distinct primes that divide at least one of m or n.
Let $m = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$, $k_i \geq 0$ for $i = 1, 2, 3, \ldots, r$

$$n = p_1^{j_1} p_2^{j_2} \cdots p_r^{j_r}, \quad j_i \geq 0 \text{ for } i = 1, 2, 3, \ldots, r$$

Prove $\gcd(m,n) = p_1^{u_1} p_2^{u_2} \cdots p_r^{u_r}$, $\operatorname{lcm}(m,n) = p_1^{v_1} \cdots p_r^{v_r}$

where $u_i = \min\{k_i, j_i\}$, $v_i = \max\{k_i, j_i\}$

Pf: (a) Consider the integer $a = p_1^{u_1} p_2^{u_2} \cdots p_r^{u_r}$

(1) $u_i = \min\{k_i, j_i\} \Rightarrow 0 \leq u_i \leq k_i \therefore$ By Th. 6.1, $a \mid m$
$u_i = \min\{k_i, j_i\} \Rightarrow 0 \leq u_i \leq j_i \therefore$ By Th. 6.1, $a \mid n$

(2) Let d be any other divisor of m and n
By Th. 6.1, $d = p_1^{s_1} p_2^{s_2} \cdots p_r^{s_r}$, where

$0 \leq s_i \leq k_i$ since $d \mid m$ and
$0 \leq s_i \leq j_i$ since $d \mid n$.

If $k_i = j_i$, then by def of u_i, $u_i = k_i = j_i$
$\therefore s_i \leq u_i$

If $k_i < j_i$, Then $u_i = k_i$. $s_i \le k_i \Rightarrow s_i \le u_i$

If $j_i < k_i$, Then $u_i = j_i$. $s_i \le j_i \Rightarrow s_i \le u_i$

$\therefore s_i \le u_i$ so That $d \mid a$.

$\therefore a = \gcd(m,n)$ by (1) + (2)

(b) Let $a = p_1^{v_1} p_2^{v_2} \cdots p_r^{v_r}$.

(1) $v_i = \max\{k_i, j_i\} \Rightarrow k_i \le v_i$ and $j_i \le v_i$

$\therefore p_i^{k_i} \mid p_i^{v_i}$ and $p_i^{j_i} \mid p_i^{v_i}$

$\therefore m \mid a$ and $n \mid a$, so a is a multiple of m and n.

(2) Suppose $m \mid b$ and $n \mid b$ (b any other multiple)

$\therefore m \mid b \Rightarrow p_i^{k_i} \mid b$ $\quad n \mid b \Rightarrow p_i^{j_i} \mid b$

If $k_i = j_i$, Then $v_i = \max\{k_i, j_i\} \Rightarrow v_i = k_i = j_i$

$\therefore p_i^{v_i} \mid b$, and $\therefore$ by corollary 2, Sec. 2.2, $p_1^{v_1} p_2^{v_2} \cdots p_r^{v_r} \mid b$, so $a \mid b$

If $k_i < j_i$, Then $v_i = \max\{k_i, j_i\} \Rightarrow v_i = j_i$

$\therefore p_i^{j_i} \mid b \Rightarrow p_i^{v_i} \mid b$, so $p_1^{v_1} p_2^{v_2} \cdots p_r^{v_r} \mid b$

by corollary 2, Sec. 2.2. $\therefore a \mid b$

If $j_i < k_i$, Then $v_i = \max\{k_i, j_i\} \Rightarrow v_i = k_i$

$\therefore p_i^{k_i} \mid b \Rightarrow p_i^{v_i} \mid b$, so $p_1^{v_1} p_2^{v_2} \cdots p_r^{v_r} \mid b$

by corollary 2, Sec. 2.2. $\therefore a \mid b$

$\therefore$ By (1) & (2), $a = \operatorname{lcm}(m,n)$

2. Calculate $\gcd(12378, 3054)$ and $\operatorname{lcm}(12378, 3054)$

$12378 = 2 \cdot 6189 = 2 \cdot 3 \cdot 2063$, and 2063 is prime
$3054 = 2 \cdot 1527 = 2 \cdot 3 \cdot 509$, and 509 is prime.

$\therefore \gcd(12378, 3054) = 2 \cdot 3 = 6$
$\operatorname{lcm}(12378, 3054) = 2 \cdot 3 \cdot 509 \cdot 2063 = 6300402$

3. Use Problem 1 to show that $\gcd(m,n) \cdot \operatorname{lcm}(m,n) = mn$

Pf: $mn = p_1^{k_1+j_1} p_2^{k_2+j_2} \cdots p_r^{k_r+j_r}$

$\therefore$ For any i, need to show $k_i + j_i = u_i + v_i$,

where $u_i = \min\{k_i + j_i\}$, $v_i = \max\{k_i, j_i\}$

(1) If $k_i = j_i$, then $u_i = k_i = j_i$, $v_i = k_i = j_i$

$\therefore k_i + j_i = u_i + v_i$

(2) If $k_i > j_i$, then $u_i = j_i$, $v_i = k_i$

$\therefore k_i + j_i = v_i + u_i$

(3) If $k_i < j_i$, then $u_i = k_i$, $v_i = j_i$

$\therefore k_i + j_i = u_i + v_i$

$\therefore$ (1), (2), (3) $\Rightarrow mn = \gcd(m,n)\,\mathrm{lcm}(m,n)$

4. Using notation of Problem 1, show $\gcd(m,n) = 1 \iff k_i j_i = 0$ for $i = 1, 2, \ldots, r$

Pf: $\gcd(m,n) = p_1^{u_1} p_2^{u_2} \cdots p_r^{u_r}$, where $u_i = \min\{k_i, j_i\}$

(1) If $\gcd(m,n) = 1$, then $u_1 = 0, u_2 = 0, \ldots, u_r = 0$, so $u_i = 0$ for $i = 1, 2, \ldots, r$

$\therefore 0 = \min\{k_i, j_i\}$, so one of k_i or j_i must be 0, so $k_i j_i = 0$.

(2) If $k_i j_i = 0$ for $i = 1, 2, \ldots, r$, then either $k_i = 0$ or $j_i = 0$. $\therefore \min\{k_i, j_i\} = 0$

$\therefore u_i = 0$ for $i = 1, 2, \ldots, r$

$\therefore p_1^{u_1} p_2^{u_2} \cdots p_r^{u_r} = 1 = \gcd(m, n)$

5. (a). Verify $\tau(n) = \tau(n+1) = \tau(n+2) = \tau(n+3)$ for $n = 3655$ and $n = 4503$

$3655 = 5 \cdot 17 \cdot 43 \quad \therefore \tau(n) = 2 \cdot 2 \cdot 2 = 8$
$3656 = 2^3 \cdot 457 \quad \therefore \tau(n+1) = 4 \cdot 2 = 8$
$3657 = 3 \cdot 23 \cdot 53 \quad \therefore \tau(n+2) = 2 \cdot 2 \cdot 2 = 8$
$3658 = 2 \cdot 31 \cdot 59 \quad \therefore \tau(n+3) = 2 \cdot 2 \cdot 2 = 8$

$4503 = 3 \cdot 19 \cdot 79 \quad \therefore \tau(n) = 2 \cdot 2 \cdot 2 = 8$
$4504 = 2^3 \cdot 563 \quad \therefore \tau(n+1) = 4 \cdot 2 = 8$
$4505 = 5 \cdot 17 \cdot 53 \quad \therefore \tau(n+2) = 2 \cdot 2 \cdot 2 = 8$
$4506 = 2 \cdot 3 \cdot 751 \quad \therefore \tau(n+3) = 2 \cdot 2 \cdot 2 = 8$

(b) Show $\sigma(n) = \sigma(n+1)$ when $n = 14, 206$, or 957

$\sigma(14) = \sigma(2\cdot 7) = \sigma(2)\,\sigma(7) = 3\cdot 8 = 24$
$\sigma(15) = \sigma(3\cdot 5) = \sigma(3)\cdot\sigma(5) = 4\cdot 6 = 24$

$\sigma(206) = \sigma(2\cdot 103) = \sigma(2)\sigma(103) = 3\cdot 104 = 312$
$\sigma(207) = \sigma(9\cdot 23) = \sigma(3^2)\,\sigma(23) = \frac{27-1}{2}\cdot 24 = 312$

$\sigma(957) = \sigma(3\cdot 11\cdot 29) = \sigma(3)\,\sigma(11)\,\sigma(29) = 4\cdot 12\cdot 30 = 1440$
$\sigma(958) = \sigma(2\cdot 479) = \sigma(2)\sigma(479) = 3\cdot 480 = 1440$

6. For any $n \geq 1$, show $\tau(n) \leq 2\sqrt{n}$

Pf: Note if $d \mid n$, then either $d \leq \sqrt{n}$ or $n/d \leq \sqrt{n}$
For if both $d > \sqrt{n}$ and $n/d > \sqrt{n}$, then
$d\cdot\frac{n}{d} = n > \sqrt{n}\cdot\sqrt{n} = n$, a contradiction.

Let $d_1, d_2, \ldots, d_k$ be the divisors of n,

where $d_1 < d_2 < \cdots < d_k$. Clearly, $d_1 = 1$, $d_k = n$.

Since whenever d_i is a divisor of n, so is
n/d_i, and so n/d_i must be one of the
d_i.

Pair up the divisors so that $d_i d_j = n$, where
$d_j = n/d_i$. Either $d_i \leq d_j$ or $d_j \leq d_i$ for each.

(1) If K is even, we have $\frac{K}{2}$ unique pairs $(d_i \neq d_j)$ $\{d_i, d_j\}$ s.t. $d_i d_j = n$. For each pair, arrange them so that $d_i < d_j$

Let $d_{k'}$ be the largest of the d_i

It must be that $\frac{K}{2} \leq d_{K'}$ since there are $\frac{K}{2}$ unique pairs.

But $\tau(n) = k$, and from above, $d_{K'} \leq \sqrt{n}$

$\therefore \frac{\tau(n)}{2} \leq \sqrt{n}$, so $\tau(n) \leq 2\sqrt{n}$

(2) If K is odd, we have $\frac{k-1}{2}$ unique pairs $\{d_i, d_j\}$ s.t. $d_i d_j = n$ and one pair $\{d_r, d_r\}$ where $d_r \cdot d_r = n$.

For the unique $\frac{k-1}{2}$ unique pairs, arrange them so that $d_i < d_j$.

Let $d_{k'}$ be the largest of the d_i.

$\therefore d_{K'} < d_r$, for if $d_r < d_{K'}$, let d_j be

The associated pair with d_k'. By def. of d_k', $d_k' < d_j$ and $d_k' d_j = n$

$\therefore d_r < d_k'$ and $d_r < d_j$, so $d_r \cdot d_r < d_k' \cdot d_j$,

or $n < n$. $\therefore d_k' < d_r$

But $d_r^2 = n$, so $d_r = \sqrt{n}$

As in (1), $\frac{k-1}{2} \leq d_k'$ and $k = \tau(n)$

$\therefore \frac{\tau(n)-1}{2} \leq d_k' < d_r = \sqrt{n}$

$\therefore \frac{\tau(n)-1}{2} < \sqrt{n}$, $\tau(n)-1 < 2\sqrt{n}$, so

$\tau(n) \leq 2\sqrt{n}$.

$\therefore$ Where $\tau(n)$ is even or odd, $\underline{\tau(n) \leq 2\sqrt{n}}$

7. (a) Prove $\tau(n)$ is odd $\Leftrightarrow$ n is a perfect square.

(1) Let $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$

Suppose $\tau(n)$ is odd. Since, by Th. 6.2,

$\tau(n) = (k_1+1)(k_2+1)\cdots(k_r+1)$, Then each

(k_i+1) must be odd, so k_i is even,

$\therefore k_i = 2j_i$, so

$$n = p_1^{2j_1} p_2^{2j_2} \cdots p_r^{2j_r} = \left(p_1^{j_1} p_2^{j_2} \cdots p_r^{j_r}\right)^2,$$

so n is a perfect square.

(2) Suppose n is a perfect square.

$\therefore n = a^2$, some $a = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_1}$

$\therefore n = p_1^{2k_1} p_2^{2k_2} \cdots p_r^{2k_r}$

$\therefore$ By Th. 6.2, $\tau(n) = (2k_1+1)(2k_2+1)\cdots(2k_r+1)$

Since each $2k_i+1$ is odd, $\tau(n)$ is odd.

(3). $\sigma(n)$ is odd $\Leftrightarrow$ n is a perfect square or twice a perfect square.

(1) Suppose $\sigma(n)$ is odd.

Let $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$. As discussed in the proof to Th. 6.2,

$$\sigma(n) = (1 + p_1 + p_1^2 + \cdots + p_1^{k_1}) \cdots (1 + p_r + p_r^2 + \cdots p_r^{k_r})$$

$\therefore$ Each term $(1 + p_i + p_i^2 + \cdots + p_i^{k_i})$ must be odd.

If $p_i = 2$, Then each p_i^n is even, so $1 + 2 + 2^2 + \cdots 2^n$ is odd.

$\therefore$ consider p_i to be odd. $\therefore p_i^n$ is odd.

If k_i is odd, Then you have an odd number of terms: $p_i + p_i^2 + \cdots + p_i^{k_i}$, which must be odd. $\therefore 1 + p_i + \cdots + p_i^{k_i}$ must be even.

$\therefore k_i$ must be even for $1 + p_i + \cdots + p_i^{k_i}$ to be odd.

$$\therefore p_i^{k_i} = p_i^{2j_i} = (p_i^{j_i})^2$$

$\therefore$ If 2 is a factor of n,

$$n = 2^k (p_2^{j_2} p_3^{j_3} \cdots p_r^{j_r})^2, \quad k = 1, 2, \ldots$$

$$= 2^k a^2, \quad k = 1, 2, \ldots \quad a = p_2^{j_2} \cdots p_r^{j_r}$$

For k even, $k = 2s$, so $n = (2^s a)^2$, so n is a perfect square.

For k odd, $k = 2s+1$, so $n = 2^{2s+1} a^2 = 2 \cdot 2^{2s} a^2 = 2(2^s a)^2$, so n is twice a perfect square.

If 2 is not a factor of n, then

$n = (p_1^{j_1} p_2^{j_2} \cdots p_r^{j_r})^2$, so n is a perfect square.

(2) Suppose n is a perfect square or twice a perfect square.

(a) Let $n = a^2$, let $a = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$.

$\therefore n = p_1^{2k_1} \cdots p_r^{2k_r}$. As in the proof to

Th. 6.2, $\sigma(n) = (1+p_1+\cdots+p_1^{2k_1})\cdots(1+p_r+\cdots+p_r^{2k_r})$

If $p_1 = 2$, Then $(1+p_1+p_1^{2k_1})$ is odd.

If p_i is odd, p_i^n is odd. There are an even number of terms in $p_i+\cdots+p_i^{2k_i}$, which means The sum is even (odd times even is even). $\therefore 1+p_i+\cdots+p_i^{2k_i}$ is odd.

$\therefore \sigma(n)$ is odd

(b) suppose $n = 2a^2$

Let $a = p_1^{k_1}p_2^{k_2}\cdots p_r^{k_r}$. $\therefore n = 2(p_1^{k_1}\cdots p_r^{k_r})^2 = 2p_1^{2k_1}\cdots p_r^{2k_r}$, $p_1 < p_2 < \cdots p_r$ (canonical form).

If $p_1 = 2$, Then $n = 2^k p_2^{2k_2}\cdots p_r^{2k_r}$, and all of $2, p_i$ are relatively prime.

$\therefore \sigma(n) = \frac{2^{k+1}-1}{2-1} \cdot \sigma(p_2^{2k_2})\cdots\sigma(p_r^{2k_r})$

But $\sigma(p_i^{2k_i}) = 1+p_i+p_i^2+\cdots+p_i^{2k_i}$. As

p_i^s is odd for any s (p_i is odd).

$\therefore p_i + p_i^2 + \cdots + p_i^{2k_i}$ is even as there are an even # of terms.

$\therefore \sigma(p_i^{k_i}) = 1 + p_i + \cdots + p_i^{2k_i}$ is odd.

$2^{k+1} - 1$ is odd, so $\sigma(n)$ is odd.

If $p_1 \neq 2$, then $n = 2p_1^{2k_1} \cdots p_r^{2k_r}$.

As all $2, p_i$ are relatively prime,

$\sigma(n) = \sigma(2)\,\sigma(p_1^{2k_1}) \cdots \sigma(p_r^{2k_r})$

But $\sigma(2) = 1 + 2 = 3$.

As in (a) above, $\sigma(p_i^{2k_i})$ is odd.

$\therefore \sigma(n)$ is odd.

8. Show that $\sum_{d|n} \frac{1}{d} = \sigma(n)/n$ for all $n > 0$

Pf: Note that d is a divisor of $n \Leftrightarrow \frac{n}{d}$ is a divisor of n, since $d \cdot \frac{n}{d} = n$.

$\therefore$ The set of divisors of $n = \{d_1, \ldots, d_k\}$

can also be written as $\left\{\frac{n}{d_1}, \ldots, \frac{n}{d_k}\right\}$

$\therefore \sigma(n) = d_1 + d_2 + \cdots d_k = \frac{n}{d_1} + \frac{n}{d_2} + \cdots + \frac{n}{d_k}$

$= n\left(\frac{1}{d_1} + \frac{1}{d_2} + \cdots + \frac{1}{d_k}\right)$

$\therefore \frac{\sigma(n)}{n} = \frac{1}{d_1} + \frac{1}{d_2} + \cdots + \frac{1}{d_k} = \sum_{d \mid n} \frac{1}{d}$

9. If n is a square free integer, prove $\tau(n) = 2^r$, where r is the number of prime divisors of n.

Pf: n square-free $\Rightarrow n = p_1 p_2 \cdots p_r$, where each p_i is distinct and of exponent power 1 (see problem 16(a), section 3.1).

$\therefore$ By Th. 6.2, $\tau(n) = (k_1+1)\cdots(k_r+1)$, and here, each $k_i = 1$.

$\therefore \tau(n) = (1+1)\cdots(1+1) = (2)\cdots(2)$. As there are r terms, $\underline{\tau(n) = 2^r}$

10. Establish the assertions below:

(a) If $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ for $n > 1$, then

$$1 > \frac{n}{\sigma(n)} > \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_r}\right)$$

Since the divisors of n include 1 and n, $\sigma(n) \geq n + 1 > n$, so $\sigma(n) > n$, $\therefore$

$$1 > \frac{n}{\sigma(n)}$$

By Th. 6.2, $\sigma(n) = \dfrac{p_1^{k_1+1} - 1}{p_1 - 1} \cdots \dfrac{p_r^{k_r+1} - 1}{p_r - 1}$

$$\therefore \frac{n}{\sigma(n)} = \frac{p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}}{\dfrac{(p_1^{k_1+1} - 1) \cdots (p_r^{k_r+1} - 1)}{(p_1 - 1) \cdots (p_r - 1)}}$$

$$= \frac{p_1^{k_1} \cdots p_r^{k_1} (p_1 - 1) \cdots (p_r - 1)}{(p_1^{k_1+1} - 1) \cdots (p_r^{k_r+1} - 1)}$$

$$= \frac{(p_1-1)\cdots(p_r-1)}{\dfrac{(p_1^{k_1+1}-1)\cdots(p_r^{k_r+1}-1)}{p_1^{k_1}\cdots p_r^{k_r}}}$$

$$= \frac{(p_1-1)\cdots(p_r-1)}{\left(p_1-\frac{1}{p_1^{k_1}}\right)\cdots\left(p_r-\frac{1}{p_r^{k_r}}\right)} \qquad [1]$$

But $p_i > p_i - \frac{1}{p_i^{k_i}}$, $\therefore \dfrac{1}{p_i - \frac{1}{p_i^{k_i}}} > \dfrac{1}{p_i}$

$\therefore$ from [1], $\dfrac{n}{\sigma(n)} = \dfrac{(p_1-1)\cdots(p_r-1)}{\left(p_1-\frac{1}{p_1^{k_1}}\right)\cdots\left(p_r-\frac{1}{p_r^{k_r}}\right)} > \dfrac{(p_1-1)\cdots(p_r-1)}{p_1\cdots p_r}$

$$= \left(1-\frac{1}{p_1}\right)\cdots\left(1-\frac{1}{p_r}\right)$$

$$\therefore 1 > \frac{n}{\sigma(n)} > \left(1-\frac{1}{p_1}\right)\cdots\left(1-\frac{1}{p_r}\right)$$

(b) For any positive integer n,

$$\frac{\sigma(n!)}{n!} \geq 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

Clearly, all $1, 2, 3, 4, \ldots, n$ are divisors for $n!$
There are more divisors of $n!$ (including $n!$)
since $n! > n$ for $n \geq 3$, and $n! \geq n$ for $n = 2$.

From prob. 8, $\dfrac{\sigma(n!)}{n!} = \displaystyle\sum_{d \mid n!} \frac{1}{d} = \frac{1}{1} + \frac{1}{2} + \cdots \frac{1}{n} + \cdots + \frac{1}{n!}$

$$\geq 1 + \frac{1}{2} + \cdots \frac{1}{n}$$

(c) If $n > 1$ is a composite number, then
$\sigma(n) > n + \sqrt{n}$

Since $\sigma(n) = 1 + d_1 + \cdots + d_k + n$, it
suffices to show $1 + d_1 + \cdots + d_k > \sqrt{n}$

Since n is composite, there is a d_i s.t.
$1 < d_i < n$ and $d_i \mid n$. $\therefore \dfrac{n}{d_i} \mid n$, and

$d_i < n \Rightarrow 1 < \dfrac{n}{d_i}$, and $1 < d_i \Rightarrow \dfrac{1}{d_i} < 1 \Rightarrow$

$\dfrac{n}{d_i} < n$. $\therefore 1 < \dfrac{n}{d_i} < n$ [1]

(a) If $d_i > \sqrt{n}$, then clearly $1 + d_i > \sqrt{n}$, so

$\sigma(n) = 1 + d_1 + \cdots + n > n + \sqrt{n}$

(b) Suppose $d_i \le \sqrt{n}$. $\therefore \frac{1}{\sqrt{n}} \le \frac{1}{d_i} \Rightarrow$

$\sqrt{n} = \frac{n}{\sqrt{n}} \le \frac{n}{d_i}$. Let $d_j = \frac{n}{d_i}$, and

d_j is a divisor of n and $\therefore d_j \ge \sqrt{n}$

$\therefore 1 + d_j + n > n + \sqrt{n}$

$\therefore$ From $\sigma(n) = 1 + d_1 + d_2 + \cdots + n$,

$\sigma(n) > n + \sqrt{n}$

$\therefore$ From (a) + (b), $\sigma(n) > n + \sqrt{n}$

11. Given integer $K > 1$, show there are infinitely many integers n for which $\tau(n) = K$, but at most finitely many n with $\sigma(n) = K$.

(a) Let p be any prime, let $n = p^{K-1}$

$\therefore \tau(n) = K$ by Th. 6.2. Since there are infinitely many primes, there are infinitely many n s.t. $n = p^{K-1}$ and $\tau(n) = K$.

(b) By problem 10(a), $1 > \frac{n}{\sigma(n)}$ for any n.

$\therefore \sigma(n) > n$

$\therefore$ Given any k, if $\sigma(n)$ is to equal k, Then k serves as an upper bound to n.

In fact, for any $n \geq k$, $\sigma(n) > k$ by problem 10(a). $\therefore$ There are at most k positive integers s.t. $\sigma(n) \leq k$.

12. (a). Find The form of all positive integers n satisfying $\tau(n) = 10$. What is The smallest positive integer for which This is true?

Since $\tau(n) = (k_1+1)(k_2+1)\cdots(k_r+1)$ if $n = p_1^{k_1}\cdots p_r^{k_r}$, Then $10 = (k_1+1)\cdots(k_r+1)$

Possibilities include $10 = 10$ or $10 = 5 \cdot 2$

$\therefore (k_1+1) = 10$ or $(k_1+1)(k_2+1) = 5 \cdot 2$

$\therefore n = p_1^9$ or $n = p_1^4 p_2$, p_1, p_2 distinct primes.

The smallest such integer would be 2^9 or $2^4 \cdot 3$ or $3^4 \cdot 2$. Of the three, the smallest is $2^4 \cdot 3 = \underline{48}$

(b) Show there are no positive integers n satisfying $\sigma(n) = 10$!

From proof of problem 11, the only possible integers under consideration would be $1, 2, 3, \ldots, 9$. $\sigma(1) = 1$. For $2, 3, 5, 7$, $\sigma(n) = 2$

$\sigma(4) = 7 \quad (1+2+4)$
$\sigma(6) = 12 \quad (1+2+3+6)$
$\sigma(8) = 15 \quad (1+2+4+8)$
$\sigma(9) = 13 \quad (1+3+9)$

$\therefore$ No positive integers n s.t. $\sigma(n) = 10$

13. Prove there are infinitely many pairs of integers m, n s.t. $\sigma(m^2) = \sigma(n^2)$

Pf: There are infinitely many k s.t. $\gcd(k, 10) = 1$. Consider $m = 5k$, $n = 4k$. $\therefore$ There are infinitely many such m, n. Let $k = p$, a prime s.t. $p \neq 2$ or 5.

$m^2 = 5^2 p^2$ and $n^2 = 4^2 p^2 = 2^4 p^2$

$\therefore$ By Th. 6.2,

$$\sigma(m^2) = \frac{5^3-1}{5-1} \cdot \frac{p^3-1}{p-1} = \frac{124}{4} \cdot \frac{p^3-1}{p-1} = 31\left(\frac{p^3-1}{p-1}\right)$$

$$\sigma(n^2) = \frac{2^5-1}{2-1} \cdot \frac{p^3-1}{p-1} = 31\left(\frac{p^3-1}{p-1}\right)$$

$\therefore$ There are infinitely many m, n, s.t. $\sigma(m^2) = \sigma(n^2)$

14. For $K \geq 2$, show each of the following:

(a) $n = 2^{K-1}$ satisfies $\sigma(n) = 2n-1$

By Th. 6.2, $\sigma(n) = \frac{2^{K-1+1}-1}{2-1} = 2^K - 1$

But $2n-1 = 2(2^{K-1}) - 1 = 2^K - 1$. $\therefore \sigma(n) = 2n-1$

(b) If $2^K - 1$ is prime, then $n = 2^{K-1}(2^K-1)$ satisfies $\sigma(n) = 2n$

If $2^K - 1$ is prime, then $2^K - 1 \neq 2$ since

$k \geq 2$. $\therefore$ let $p = 2^k - 1$.

By Th. 6.2,

$$\sigma(n) = \frac{2^{k-1+1}-1}{2-1} \cdot \frac{p^2-1}{p-1} = (2^k-1)(p+1)$$

$$= (2^k-1)(2^k)$$

But $2n = 2(2^{k-1})(2^k-1) = 2^k(2^k-1)$

$\therefore \sigma(n) = 2n$

(c) If 2^k-3 is prime, then $n = 2^{k-1}(2^k-3)$ satisfies $\sigma(n) = 2n+2$

Let $p = 2^k-3$. Since p is prime, $k \geq 3$, so $p \neq 2$.

$\therefore n = 2^{k-1} \cdot p$, and by Th. 6.2,

$$\sigma(n) = \frac{2^{k-1+1}-1}{2-1} \cdot \frac{p^2-1}{p-1} = (2^k-1)(p+1)$$

$$= (2^k-1)(2^k-3+1) = (2^k-1)(2^k-2)$$

$$= 2^{2k} - 3 \cdot 2^k + 2$$

But $2n+2 = 2(2^{k-1})(2^k - 3) + 2$

$$= 2^k(2^k - 3) + 2 = 2^{2k} - 3\cdot 2^k + 2$$

$\therefore\ \sigma(n) = 2n+2$

15. If n and $n+2$ are a pair of twin primes, show that $\sigma(n+2) = \sigma(n) + 2$.

Pf: For any prime, p, The only divisors are 1 and p! $\therefore\ \sigma(p) = p+1$

$$\therefore\ \sigma(n+2) = (n+2) + 1 = n+3$$
$$\sigma(n) + 2 = (n+1) + 2 = n+3$$

$\therefore$ n and $n+2$ prime $\Rightarrow \sigma(n+2) = \sigma(n) + 2$

16. (a) For any integer $n > 1$, prove There exists integers n_1 and n_2 s.t. $\tau(n_1) + \tau(n_2) = n$

Pf: If n is prime, $\tau(n) = 2$. Since $\tau(1) = 1$, Then let $n_1 = n_2 = 1$. $\therefore\ \tau(n_1) + \tau(n_2) = \tau(n)$

If n is composite, let $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ be The canonical prime factorization.

Since n is composite, at least one of the $k_i \geq 2$. Let $p_j^{k_j}$ be that factor.

$$\therefore \tau(n) = (k_1+1)(k_2+1)\cdots(k_j+1)\cdots(k_r+1)$$

$$= k_j\,(k_1+1)(k_2+1)\cdots(k_r+1)$$

$$+ (k_1+1)(k_2+1)\cdots(k_r+1)$$

$$= (k_j - 1 + 1)(k_1+1)(k_2+1)\cdots(k_r+1)$$

$$+ (k_1+1)(k_2+1)\cdots(k_r+1)$$

$\therefore$ Let $n_1 = p_1^{k_1} p_2^{k_2} \cdots p_j^{k_j-1} \cdots p_r^{k_r}$ and

let $n_2 = n / p_j^{k_j} = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ $(p_i \neq p_j)$

$$\therefore \tau(n_1) = (k_1+1)(k_2+2)\cdots(k_j-1+1)\cdots(k_r+1)$$
$$= k_j\,(k_1+1)\cdots(k_r+1)$$

$$\tau(n_2) = (k_1+1)(k_2+1)\cdots(k_r+1)$$

$$\therefore \tau(n) = \tau(n_1) + \tau(n_2)$$

(b) Prove the Goldbach conjecture implies that for each even integer $2n$, There exist integers n_1 and n_2 with $\sigma(n_1) + \sigma(n_2) = 2n$

Pf: Since $2n$ is even, so is $2n-2$

Assume $2n-2 > 4$. Goldbach conjecture states There exist odd primes, p_1 and p_2, s.t. $p_1 + p_2 = 2n-2$

$\therefore$ Let $n_1 = p_1$, $n_2 = p_2$

$\therefore \sigma(n_1) + \sigma(n_2) = (p_1+1) + (p_2+1) = p_1 + p_2 + 2$

$= 2n-2+2 = 2n$

17. For a fixed integer k, show The function f defined by $f(n) = n^k$ is multiplicative.

Pf: $f(mn) = (mn)^k = m^k n^k = f(m) \cdot f(n)$

m and n don't even have to be relatively prime.

18. Let f, g be multiplicative, not identically zero, and s.t. $f(p^k) = g(p^k)$ for each prime p and $k \geq 1$. Prove $f = g$

Pf: Let n be any positive integer >1

Let $n = p^{k_1} p_2^{k_2} \cdots p_r^{k}$ be the prime factorization

$$\therefore f(n) = f(p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}) = f(p_1^{k_1}) \cdots f(p_r^{k_r})$$
$$= g(p_1^{k_1}) \cdots g(p_r^{k_r}) = g(p_1^{k_1} \cdots p_r^{k_r})$$
$$= g(n)$$

If $n=1$, then $f(1) = g(1) = 1$ by discussion on p. 107.

$\therefore f = g$

19. Prove if F and g are multiplicative functions, then so is $f \cdot g$ and f/g (whenever f/g is defined).

Pf: Let m, n be integers s.t. $\gcd(m,n) = 1$.

$$f \cdot g(mn) = f(mn) \cdot g(mn)$$
$$= f(m) \cdot f(n) \cdot g(m) \cdot g(n)$$
$$= f(m) \cdot g(m) \cdot f(n) \cdot g(n)$$
$$= f \cdot g(m) \cdot f \cdot g(n)$$

$$f/g\,(mn) = f(mn)\,/\,g(mn)$$
$$= f(m)\cdot f(n)\,/\,g(m)\cdot g(n)$$
$$= \frac{f(m)}{g(m)} \cdot \frac{f(n)}{g(n)}$$
$$= f/g\,(m) \cdot f/g\,(n)$$

20. Let $\omega(n)$ denote the number of distinct prime divisors of $n>1$, with $\omega(1)=0$. For example, $\omega(360) = \omega(2^3\cdot 3^2\cdot 5) = 3$.

(a) Show $2^{\omega(n)}$ is a multiplicative function

Let m, n be integers s.t. $\gcd(m,n)=1$, and let $f(n) = 2^{\omega(n)}$

Let $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ and $m = q_1^{j_1} q_2^{j_2} \cdots q_s^{j_s}$

Since $\gcd(m,n)=1$, then $p_u \neq q_v$ for $1 \le u \le r$, $1 \le v \le s$.

$\therefore$ n has r distinct primes, m has s distinct primes.

$\therefore$ $nm = p_1^{k_1}\cdots p_r^{k_s} q_1^{j_1}\cdots q_s^{j_s}$ has $r+s$ distinct primes (Fund. Th. of Arith. says this is unique).

$\therefore \omega(n) = r, \ \omega(m) = s,$ and $\omega(nm) = r+s$

$\therefore f(nm) = 2^{\omega(nm)} = 2^{\omega(n)+\omega(m)} = 2^{\omega(n)} \cdot 2^{\omega(m)}$

$= f(n) \cdot f(m)$

$\therefore 2^{\omega(n)}$ is multiplicative.

(b) For a positive integer n, establish

$$\tau(n^2) = \sum_{d|n} 2^{\omega(d)}$$

From (a), $F(n) = \sum_{d|n} 2^{\omega(d)}$ is multiplicative

$\therefore$ Let $n = p_1^{k_1} \cdots p_r^{k_r}$

$$\therefore F(n) = F(p_1^{k_1} \cdots p_r^{k_r})$$

$$= F(p_1^{k_1}) \cdots F(p_r^{k_r})$$

$$= \sum_{d|p_1^{k_1}} 2^{\omega(d)} \cdots \sum_{d|p_r^{k_r}} 2^{\omega(d)}$$

All the divisors of $p_i^{k_i}$ are $1, p_i, p_i^2, \ldots, p_i^{k_i}$

by Th. 6.1. Also, $\omega(p_i^{a_i}) = 1$ for $1 \le a_i \le k_i$ and $\omega(p_i^0) = 0$.

$$\therefore \sum_{d \mid p_i^{k_i}} 2^{\omega(d)} = 2^0 + \underbrace{2^1 + \cdots + 2^1}_{k_i \text{ terms}}$$

$$= 1 + k_i 2^1 = 1 + 2k_i$$

$$\therefore F(n) = \sum_{d \mid p_1^{k_1}} 2^{\omega(d)} \cdots \sum_{d \mid p_r^{k_r}} 2^{\omega(d)}$$

$$= (1 + 2k_1) \cdots (1 + 2k_r)$$

But $n^2 = p_1^{2k_1} \cdots p_r^{2k_r}$, so by Th. 6.2

$$\tau(n^2) = (2k_1 + 1) \cdots (2k_r + 1)$$

$$\therefore \tau(n^2) = F(n) = \sum_{d \mid n} 2^{\omega(d)}$$

21. For any positive integer n, prove

$$\sum_{d \mid n} \tau(d)^3 = \left(\sum_{d \mid n} \tau(d) \right)^2$$

Pf: $\tau(n)$ is a multiplicative function

Since $[\tau(mn)]^3 = [\tau(m)\tau(n)]^3$

$$= \tau(m)^3 \cdot \tau(n)^3,$$

Then $\tau(n)^3$ is multiplicative.

$\therefore$ By Th. 6.4., $F(n) = \sum_{d|n} \tau(d)^3$ is multiplicative.

Also, $G(n) = \sum_{d|n} \tau(d)$ is multiplicative,

so $H(n) = G^2(n)$ is multiplicative, since

$$H(mn) = G^2(mn) = [G(mn)]^2 = [G(m)G(n)]^2$$
$$= G^2(m) \cdot G^2(n).$$

Let $n = p_1^{k_1} \cdots p_r^{k_r}$.

If $F(n) = H(n)$ for $n = p^k$, then since F and H are multiplicative, the statement will also be true for $n = p_1^{k_1} \cdots p_r^{k_r}$, since

$$F(p_1^{k_1} \cdots p_r^{k_r}) = F(p_1^{k_1}) \cdots F(p_r^{k_r}) = H(p_1^{k_1}) \cdots H(p_r^{k_r})$$
$$= H(p_1^{k_1} \cdots p_r^{k_r})$$

$\therefore$ Consider $n = p^k$. By Th. 6.1, all the divisors of n are $1, p, p^2, \cdots, p^k$

$$\therefore \sum_{d \mid p^k} \tau(d)^3 = \tau(1)^3 + \tau(p)^3 + \tau(p^2)^3 + \cdots + \tau(p^k)^3$$

$$= 1 + (1+1)^3 + (2+1)^3 + \cdots + (k+1)^3$$

$$= 1 + 2^3 + 3^3 + \cdots + (k+1)^3$$

$$= \left[\frac{(k+1)(k+2)}{2}\right]^2 \quad \text{by prob. 1.e, Sec. 1.1.}$$

$$\left[\sum_{d \mid p^k} \tau(d)\right]^2 = \left[\tau(1) + \tau(p) + \cdots + \tau(p^k)\right]^2$$

$$= \left[1 + 2 + \cdots + (k+1)\right]^2$$

$$= \left[\frac{(k+1)(k+2)}{2}\right]^2 \quad \text{by prob. 1.a, Sec. 1.1}$$

$$\therefore \sum_{d \mid p^k} \tau(d)^3 = \left[\sum_{d \mid p^k} \tau(d)\right]^2,$$ so $F(n) = H(n)$ for $n = p^k$

22. Given $n \geq 1$, let $\sigma_s(n)$ denote the sum of the sth powers of the positive divisors of n; that is,

$$\sigma_s(n) = \sum_{d|n} d^s$$

Verify the following:

(a) $\sigma_0 = \tau$ and $\sigma_1 = \sigma$

(1) since $d^0 = 1$ for all $d \geq 1$, $\sigma_0(n) = \sum_{d|n} 1 = \tau(n)$ by definition.

(2) since $d^1 = d$ for all $d \geq 1$, $\sigma_1(n) = \sum_{d|n} d = \sigma(n)$ by definition.

(b) σ_s is a multiplicative function.

Pf: If $f(n) = n^s$ can be shown to be multiplicative, then by Th. 6.4,

$$\sum_{d|n} f(d) = \sum_{d|n} d^s = \sigma_s(n)$$ will be multiplicative.

$\therefore$ Consider $f(n) = n^s$

$f(mn) = (mn)^s = m^s n^s = f(m) \cdot f(n)$,

so $f(n) = n^s$ is multiplicative, and
$\therefore$ so is $\sigma_s(n)$.

(c) If $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ is the prime factorization,

then $\sigma_s(n) = \left(\dfrac{p_1^{s(k+1)} - 1}{p_1^s - 1} \right) \cdots \left(\dfrac{p_r^{s(k+1)} - 1}{p_r^s - 1} \right)$

Pf: By Th. 6.1, all positive divisors of n are of the form $p_1^{a_1} \cdots p_r^{a_r}$, $0 \le a_i \le k_i$.

$\therefore$ all the sth powers of the divisors of n are of the form $p_1^{a_1 s} p_2^{a_2 s} \cdots p_r^{a_r s}$.

$\therefore$ Consider the sum:

$$(1 + p_1^s + p_1^{2s} + \cdots + p_1^{k_1 s}) \cdots (1 + p_r^s + \cdots + p_r^{k_r s})$$

Each positive divisor to the sth power occurs once and only once as a term in the expansion of the product.

$$\therefore \sigma_s(n) = (1+p_1^s+p_1^{2s}+\cdots+p_1^{k_1 s})\cdots(1+p_r^s+\cdots+p_r^{k_r s})$$

Using the formula for the sum of a finite geometric series,

$$1+p_i^s+p_i^{2s}+\cdots+p_i^{k_i s} = \frac{p_i^{s(k_i+1)}-1}{p_i^s-1},$$

$$\therefore \sigma_s(n) = \left(\frac{p_1^{s(k_1+1)}-1}{p_1^s-1}\right)\cdots\left(\frac{p_r^{s(k_r+1)}-1}{p_r^s-1}\right)$$

23. For any positive integer n, show the following:

(a) $\sum_{d|n} \sigma(d) = \sum_{d|n} \left(\frac{n}{d}\right)\tau(d)$

Pf: (1) First note that since $\tau(n)$ is multiplicative, $H(n) = \sum_{d|n} \frac{n}{d}\tau(d)$ is multiplicative.

Pf: $H(mn) = \sum_{d|mn} \frac{mn}{d}\tau(d) = \sum_{\substack{d_1|m \\ d_2|n}} \frac{mn}{d_1 d_2}\tau(d_1 d_2)$

$$= \sum_{\substack{d_1 | m \\ d_2 | n}} \frac{mn}{d_1 d_2} \tau(d_1)\tau(d_2)$$

$$= \sum_{\substack{d_1 | m \\ d_2 | n}} \frac{m}{d_1}\tau(d_1) \frac{n}{d_2}\tau(d_2)$$

$$= \left(\sum_{d_1 | m} \frac{m}{d_1}\tau(d_1)\right)\left(\sum_{d_2 | n} \frac{n}{d_2}\tau(d_2)\right)$$

$$= H(m) \cdot H(n)$$

$\therefore$ The functions

$$F(n) = \sum_{d|n} \sigma(d) \quad \text{and} \quad G(n) = \sum_{d|n} \frac{n}{d}\tau(d)$$

are multiplicative.

(2) Now, let $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ be the prime factorization of n. If it can be shown that $F(p^k) = G(p^k)$, then $F(n) =$

$$F(p_1^{k_1} \cdots p_r^{k_r}) = F(p_1^{k_1}) \cdots F(p_r^{k_r}) = G(p_1^{k_1}) \cdots G(p_r^{k_r})$$

$$= G(p_1^{k_1} \cdots p_r^{k_r}) = G(n).$$

$$(3) \therefore F(p^k) = \sum_{d \mid p^k} \tau(d) = (p^0) + (p^0 + p^1) + \cdots + (p^0 + p^1 + \cdots + p^k)$$

$$= (k+1)p^0 + (k)p^1 + \cdots + (1)p^k$$

$$= (1) \cdot p^k + \cdots + (k)p + (k+1) \quad [1]$$

$$G(p^k) = \sum_{d \mid p^k} \frac{p^k}{d} \tau(d)$$

$$= \left(\frac{p^k}{p^0} \cdot \tau(p^0)\right) + \left(\frac{p^k}{p^1} \cdot \tau(p^1)\right) + \cdots + \left(\frac{p^k}{p^k} \tau(p^k)\right)$$

$$= (1) \cdot p^k + 2 \cdot p^{k-1} + \cdots + (k)p + (k+1) \quad [2]$$

$\therefore$ Since $[1] = [2]$, Then $F(p^k) = G(p^k)$

$\therefore$ As stated in (2), because of (3), $\sum_{d \mid n} \sigma(d) = \sum_{d \mid n} \frac{n}{d} \tau(d)$

(b) $\sum_{d|n} \left(\frac{n}{d}\right) \sigma(d) = \sum_{d|n} d\, \tau(d)$

Pf: Since $f(n) = n$ is multiplicative, so is $f \cdot \tau$

$\therefore\ G(n) = \sum_{d|n} d\, \tau(d)$ is multiplicative.

As in (a), The proof That $F(n) = \sum_{d|n} \left(\frac{n}{d}\right) \sigma(d)$ is multiplicative is identical to Th. 6.4.

So, as in (a) it suffices to prove

$F(n) = G(n)$ for $n = p^k$.

$$F(p^k) = \sum_{d|p^k} \left(\frac{p^k}{d}\right) \sigma(d)$$

$$= p^k \cdot p^0$$

$$+ p^{k-1}(p^0 + p)$$

$$+ p^{k-2}(p^0 + p + p^2)$$

$$+ \dots$$

$$+ p^0(p^0 + p + \dots + p^k)$$

$$= (k+1)p^k + k \cdot p^{k-1} + (k-1)p^{k-2} + \dots + 1$$

$$= 1 + \sum_{i=1}^{k} (i+1)p^i \qquad [1]$$

$$G(p^k) = \sum_{d \mid p^k} d\,\tau(d)$$

$$= 1 \cdot 1$$

$$+ p \cdot (1+1)$$

$$+ p^2 \cdot (1+1+1)$$

$$+ \dots$$

$$+ p^k(k+1)$$

$$= 1 + \sum_{i=1}^{k} p^i (i+1) \qquad [2]$$

Since [1] = [2], $F(p^k) = G(p^k)$