6.1 The Functions \( \tau \) and \( \sigma 

1. Let \( m, n \) be positive integers, and \( p_1, \ldots, p_r \) be the distinct primes that divide at least one of \( m \) or \( n \). Let \( m = \prod_{i=1}^{r} p_i^{k_i} \), \( n = \prod_{i=1}^{r} p_i^{j_i} \), for \( k_i \geq 0 \) and \( j_i \geq 0 \).

Prove \( \gcd(m, n) = \prod_{i=1}^{r} p_i^{\min\{k_i, j_i\}} \) and \( \operatorname{lcm}(m, n) = \prod_{i=1}^{r} p_i^{\max\{k_i, j_i\}} \n
\text{where} \quad u_i = \min\{k_i, j_i\}, \quad v_i = \max\{k_i, j_i\} \n
\text{Proof: (a) Consider} \ n \text{ integer} \ a = \prod_{i=1}^{r} p_i^{u_i} \n
1. \( u_i = \min\{k_i, j_i\} \Rightarrow 0 \leq u_i \leq k_i \ldots \text{by Th. G.1,} \ a | m \n \quad u_i = \min\{k_i, j_i\} \Rightarrow 0 \leq u_i \leq j_i \ldots \text{by Th. G.1,} \ a | n \n
2. \text{Let} \ d \text{ be any other divisor of} \ m \text{ and} \ n \n \text{By Th. G.1,} \ d = \prod_{i=1}^{r} p_i^{s_i} \text{ where} \n \quad 0 \leq s_i \leq k_i \text{ since} \ d | m \text{ and} \n \quad 0 \leq s_i \leq j_i \text{ since} \ d | n \n \text{If} \ k_i = j_i, \text{ then by def of} \ u_i, \ u_i = k_i = j_i \n \quad \therefore s_i \leq u_i \n
If $k_i < j_i$, Then $u_i = k_i$, $s_i = k_i \Rightarrow s_i \leq u_i$

If $j_i < k_i$, Then $u_i = j_i$, $s_i = j_i \Rightarrow s_i \leq u_i$

\[ \therefore s_i = u_i \text{ so that } d \mid a. \]

\[ \therefore a = \gcd(m, n) \text{ by (1) + (2)} \]

(6) Let $a = p_1^{v_1} p_2^{v_2} \cdots p_r^{v_r}$.

(1) $v_i = \max \left\{ k_i, j_i \right\} \Rightarrow k_i = v_i$ and $j_i = v_i$

\[ \therefore p_i^{k_i} \mid p_i^{v_i} \text{ and } p_i^{j_i} \mid p_i^{v_i} \]

\[ \therefore m \mid a \text{ and } n \mid a, \text{ so } a \text{ is a multiple of } m \text{ and } n. \]

(2) Suppose $m \mid b$ and $n \mid b$ (6 any other multiple)

\[ \therefore m \mid b \Rightarrow p_i^{k_i} \mid b \quad n \mid b \Rightarrow p_i^{j_i} \mid b \]

If $k_i = j_i$, Then $v_i = \max \left\{ k_i, j_i \right\} \Rightarrow v_i = k_i = j_i$

\[ \therefore p_i^{v_i} \mid b \quad \text{and} \quad \therefore \text{by corollary 2, Sec. 2.2}, \]

\[ p_1^{v_1} p_2^{v_2} \cdots p_r^{v_r} \mid b \quad \text{so } a \mid b \]
If \( k_i < j_i \), then \( \nu_i = \max \{ k_i, j_i \} \Rightarrow \nu_i = j_i \).

\[ \therefore \quad p_i^{j_i} \mid b \Rightarrow p_i^{\nu_i} \mid b, \quad \text{so} \quad p_1^{\nu_1} p_2^{\nu_2} \cdots p_r^{\nu_r} \mid b \]

by corollary 2, sec. 2.2. \[ \therefore \quad a \mid b \]

If \( j_i < k_i \), then \( \nu_i = \max \{ k_i, j_i \} \Rightarrow \nu_i = k_i \).

\[ \therefore \quad p_i^{K_i} \mid b \Rightarrow p_i^{\nu_i} \mid b, \quad \text{so} \quad p_1^{\nu_1} p_2^{\nu_2} \cdots p_r^{\nu_r} \mid b \]

by corollary 2, sec. 2.2. \[ \therefore \quad a \mid b \]

\[ \therefore \quad \text{by (1) + (2), } a = \text{lcm}(m, n) \]

2. Calculate \( \text{gcd}(12378, 3054) \) and \( \text{lcm}(12378, 3054) \)

\[ 12378 = 2 \cdot 6189 = 2 \cdot 3 \cdot 2063, \quad \text{and} \quad 2063 \text{ is prime.} \]

\[ 3054 = 2 \cdot 1527 = 2 \cdot 3 \cdot 509, \quad \text{and} \quad 509 \text{ is prime.} \]

\[ \therefore \quad \text{gcd}(12378, 3054) = 2 \cdot 3 = 6 \]

\[ \text{lcm}(12378, 3054) = 2 \cdot 3 \cdot 509 \cdot 2063 = 6300402 \]

3. Use Problem 1 to show that \( \text{gcd}(m, n) \cdot \text{lcm}(m, n) = mn \)

\[ \text{pf.: } mn = p_1^{k_1 + j_1} p_2^{k_2 + j_2} \cdots p_r^{k_r + j_r} \]
For any \(i,j\), need to show \(k_i + j_i = u_i + v_i\),

where \(u_i = \min \{k_i, j_i\}\), \(v_i = \max \{k_i, j_i\}\)

1. If \(k_i = j_i\), then \(u_i = k_i = j_i\), \(v_i = k_i = j_i\)

\[\therefore k_i + j_i = u_i + v_i\]

2. If \(k_i > j_i\), then \(u_i = j_i\), \(v_i = k_i\)

\[\therefore k_i + j_i = v_i + u_i\]

3. If \(k_i < j_i\), then \(u_i = k_i\), \(v_i = j_i\)

\[\therefore k_i + j_i = u_i + v_i\]

\[\therefore (1), (2), (3) \Rightarrow mn = \gcd(m,n) / \text{lcm}(m,n)\]

4. Using notation of Problem 1, show \(\gcd(m,n) = 1 \iff k_{i,j_i} = 0 \text{ for } i = 1, 2, \ldots, r\)

\[\text{Proof: } \gcd(m,n) = \rho_1^{u_1} \rho_2^{u_2} \cdots \rho_r^{u_r}, \text{ where } u_i = \min \{k_i, j_i\}\]

1. If \(\gcd(m,n) = 1\), then \(u_1 = 0, u_2 = 0, \ldots, u_r = 0\), so \(u_i = 0 \text{ for } i = 1, 2, \ldots, r\)
\[
\therefore \quad 0 = \min \{ K_i \}, \quad \text{so one of} \quad K_i \quad \text{or} \quad j_i \\
\text{must be} \quad 0, \quad \text{so} \quad K_i j_i = 0.
\]

(2) If \( K_i j_i = 0 \) for \( i = 1, 2, \ldots, r \), then

\[ e_i \text{or} \quad j_i = 0. \]

\[
\therefore \quad U_i = 0 \quad \text{for} \quad i = 1, 2, \ldots, r
\]

\[ e_i = 0. \]

\[
\rho_1, \rho_2, \ldots, \rho_r = 1 = \gcd(m, n)
\]

5. (a) Verify \( \overline{T}(n) = \overline{T}(n+1) = \overline{T}(n+2) = \overline{T}(n+3) \) for \( n = 3655 \) and \( n = 4503 \)

\[ 3655 = 5 \cdot 17 \cdot 43 \quad \therefore \quad \overline{T}(n) = 2 \cdot 2 \cdot 2 = 8 \]

\[ 3656 = 2 \cdot 457 \quad \therefore \quad \overline{T}(n+1) = 4 \cdot 2 = 8 \]

\[ 3657 = 3 \cdot 23 \cdot 53 \quad \therefore \quad \overline{T}(n+2) = 2 \cdot 2 \cdot 2 = 8 \]

\[ 3658 = 2 \cdot 31 \cdot 59 \quad \therefore \quad \overline{T}(n+3) = 2 \cdot 2 \cdot 2 = 8 \]

\[ 4503 = 3 \cdot 19 \cdot 79 \quad \therefore \quad \overline{T}(n) = 2 \cdot 2 \cdot 2 = 8 \]

\[ 4504 = 2 \cdot 563 \quad \therefore \quad \overline{T}(n+1) = 4 \cdot 2 = 8 \]

\[ 4505 = 5 \cdot 17 \cdot 63 \quad \therefore \quad \overline{T}(n+2) = 2 \cdot 2 \cdot 2 = 8 \]

\[ 4506 = 2 \cdot 3 \cdot 751 \quad \therefore \quad \overline{T}(n+3) = 2 \cdot 2 \cdot 2 = 8 \]

(b) Show \( \sigma(n) = 5(n+1) \) when \( n = 14, 205, \) or \( 957 \)
\[\sqrt{14} = \sqrt{2 \cdot 7} = \sqrt{2} \cdot \sqrt{7} = 3.8 = 2.4\]
\[\sqrt{15} = \sqrt{3 \cdot 5} = \sqrt{3} \cdot \sqrt{5} = 4.6 = 2.4\]
\[\sqrt{206} = \sqrt{2 \cdot 103} = \sqrt{2} \cdot \sqrt{103} = 3.104 = 3.12\]
\[\sqrt{202} = \sqrt{9 \cdot 22} = \sqrt{9} \cdot \sqrt{22} = \frac{27-1}{2} \cdot 2.4 = 31.2\]
\[\sqrt{957} = \sqrt{3 \cdot 11 \cdot 29} = \sqrt{3} \cdot \sqrt{11} \cdot \sqrt{29} = 4.12 \cdot 30 = 1440\]
\[\sqrt{958} = \sqrt{2 \cdot 479} = \sqrt{2} \cdot \sqrt{479} = 3.48 \cdot 30 = 1440\]

6. For any \( n \geq 1 \), show \( T(n) \leq 2T(n) \)

**Proof:** Note if \( d \mid n \), then either \( d \leq T(n) \) or \( n/d \leq T(n) \).

For if both \( d > T(n) \) and \( n/d > T(n) \), then
\[d \cdot \frac{n}{d} = n > T(n) \cdot T(n) = n \] a contradiction.

Let \( d_1, d_2, \ldots, d_k \) be the divisors of \( n \),
where \( d_1 < d_2 < \ldots < d_k \). Clearly, \( d_1 = 1 \), \( d_k = n \).

Since whenever \( d_i \) is a divisor of \( n \), so is \( n/d_i \) and so \( n/d_i \) must be one of the \( d_i \).

Pair up the divisors so that \( d_i \cdot d_j = n \), where
\[d_i = n/d_j \]. Either \( d_i \leq d_j \) or \( d_j \leq d_i \) for each.
(1) If \( K \) is even, we have \( \frac{K}{2} \) unique pairs \( \{d_i, d_j\} \) s.t. \( d_i d_j = n \). For each pair, arrange them so that \( d_i \leq d_j \).

Let \( d_k' \) be the largest of the \( d_i \).

It must be that \( \frac{K}{2} \leq d_k' \), since there are \( \frac{K}{2} \) unique pairs.

But \( T(n) = k \), and from above, \( d_k' \leq \sqrt{n} \).

\[ T(n) \leq \frac{n}{2} \text{, so } T(n) \leq 2\sqrt{n} \]

(2) If \( K \) is odd, we have \( \frac{K-1}{2} \) unique pairs \( \{d_i, d_j\} \) s.t. \( d_i d_j = n \) and one pair \( \{d_k, d_r\} \) where \( d_k \cdot d_r = n \).

For the unique \( \frac{K-1}{2} \) unique pairs, arrange them so that \( d_i < d_j \).

Let \( d_k' \) be the largest of the \( d_i \).

\[ \therefore d_k' < d_r \text{, for if } d_r < d_k', \text{ let } d_j \text{ be } \]
The associated pair with $d_k'$. By def. of $d_k'$, $d_k' < d_j$ and $d_k'd_j = n$

$\therefore \ d_r < d_k'$ and $d_r < d_j$, so $d_r > d_k'd_j$

or $n < n$. $\therefore \ d_k' < d_r$

But $d_r^2 = n$, so $d_r = \sqrt{n}$

As in (1), $\frac{k-1}{2} \leq d_k'$ and $k = T(n)$

$\therefore \ T(n) - 1 \leq d_k' < d_r = \sqrt{n}$

$\therefore \ T(n) - 1 < \sqrt{n}$, $T(n) - 1 < 2\sqrt{n}$, so

$T(n) \leq 2\sqrt{n}$.

$\therefore \text{Where } T(n) \text{ is even or odd, } T(n) \leq 2\sqrt{n}$

7. (a) Prove $T(n)$ is odd $\iff$ $n$ is a perfect square.

(i) Let $n = p_1^{k_1}p_2^{k_2}...p_r^{k_r}$
Suppose \( T(n) \) is odd. Since, by Th. 6.2,
\[
T(n) = (k_1 + 1)(k_2 + 1) \cdots (k_r + 1),
\]
then each \((k_i + 1)\) must be odd, so \( k_i \) is even.

\[ \therefore k_i = 2j_i, \quad \text{so} \]
\[
n = p_1^{2j_1} p_2^{2j_2} \cdots p_r^{2j_r} = (p_1^{j_1} p_2^{j_2} \cdots p_r^{j_r})^2,
\]
so \( n \) is a perfect square.

(2) Suppose \( n \) is a perfect square.

\[ \therefore n = a^2, \quad \text{some} \ a = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}. \]

\[ \therefore n = p_1^{2k_1} p_2^{2k_2} \cdots p_r^{2k_r} \]

\[ \therefore \text{By Th. 6.2, } T(n) = (2k_1 + 1)(2k_2 + 1) \cdots (2k_r + 1) \]

Since each \( 2k_i + 1 \) is odd, \( T(n) \) is odd.

(3) \( T(n) \) is odd \( \Leftrightarrow \) \( n \) is a perfect square or twice a perfect square.
(1) Suppose $\sigma(n)$ is odd.

Let $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$. As discussed in the proof to Th. 6.2,

$$\sigma(n) = \left(1 + p_1 + p_1^2 + \cdots + p_1^{k_1}\right) \cdots \left(1 + p_r + p_r^2 + \cdots + p_r^{k_r}\right)$$

$\therefore$ Each term $(1 + p_i + p_i^2 + \cdots + p_i^{k_i})$ must be odd.

If $p_i = 2$, then each $p_i^{k_i}$ is even, so $1 + 2 + 2^2 + \cdots + 2^{k_i}$ is odd.

$\therefore$ Consider $p_i$ to be odd. $\therefore p_i^{k_i}$ is odd.

If $k_i$ is odd, then you have an odd number of terms $p_i + p_i^2 + \cdots + p_i^{k_i}$, which must be odd $\therefore 1 + p_i + \cdots + p_i^{k_i}$ must be even.

$\therefore k_i$ must be even for $1 + p_i + \cdots + p_i^{k_i}$ to be odd.

$\therefore p_i^{k_i} = p_i^{2u_i} = \left(p_i^{u_i}\right)^2$
- If 2 is a factor of \( n \),
  \[
  n = 2^k \left( \frac{p_1^{i_1} p_2^{i_2} \cdots p_r^{i_r}}{a} \right)^2, \quad k=1,2,\ldots
  \]
  \[
  = 2^k a^2, \quad k=1,2,\ldots \quad a = \frac{p_2^{i_2} \cdots p_r^{i_r}}{p_r}
  \]
  For \( k \) even, \( k = 2s \), so \( n = (2^s a)^2 \), so \( n \) is a perfect square.
  For \( k \) odd, \( k = 2s + 1 \), so \( n = 2 \cdot 2^{2s} a^2 = 2 (2^s a)^2 \), so \( n \) is twice a perfect square.

- If 2 is not a factor of \( n \), then
  \[
  n = \left( \frac{p_1^{i_1} p_2^{i_2} \cdots p_r^{i_r}}{a} \right)^2, \quad \text{so} \quad n \text{ is a perfect square.}
  \]

(2) Suppose \( n \) is a perfect square or twice a perfect square.

(a) Let \( n = a^2 \), let \( a = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r} \).
  \[
  \therefore n = p_1^{2k_1} \cdots p_r^{2k_r}. \quad \text{As in The proof to}
\[ \sigma(n) = (1 + p_1 + \cdots + p_1^{2k_1}) \cdots (1 + p_r + \cdots + p_r^{2k_r}) \]

If \( p_i = 2 \), then \( (1 + p_i + p_i^{2k_i}) \) is odd.

If \( p_i \) is odd, \( p_i^n \) is odd. There are an even number of terms in \( p_i + \cdots + p_i^{2k_i} \), which means the sum is even (odd times even is even). \( \equiv 1 + p_i + \cdots + p_i^{2k_i} \) is odd.

\( \sigma(n) \) is odd.

(6) Suppose \( n = 2^a \)

Let \( a = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r} \), \( n = 2 \( p_1^{k_1} \cdots p_r^{k_r} \) \) = 

\( 2^{2k_1} \cdots 2^{2k_r} \), \( p_1 < p_2 < \cdots < p_r \) (canonical form).

If \( p = 2 \), then \( n = 2^{k_1 + \cdots + k_r} \) and all of \( 2, p_i \) are relatively prime.

\[ \sigma(n) = 2^{k_1 + \cdots + k_r} - 1 \]

\( \sigma(p_i^{2k_i}) = \tau(p_i^{2k_i}) \cdot \sigma(p_i^{2k_i}) \)

But \( \sigma(p_i^{2k_i}) = 1 + p_i + p_i^2 + \cdots + p_i^{2k_i} \). As
\[ p_i^s \text{ is odd for any } s \text{ ( } p_i \text{ is odd),} \]
\[ \therefore p_1 + p_1^2 + \cdots + p_1^{2k_i} \text{ is even as} \]

There are an even # of terms.
\[ \therefore \sigma(p_i^{2k_i}) = 1 + p_1 + \cdots + p_1^{2k_i} \text{ is odd.} \]

\[ 2^{k+1} - 1 \text{ is odd, so } \sigma(n) \text{ is odd.} \]

If \( p_i \neq 2 \), then \( n = 2^{2k_1} \cdot \cdots \cdot 2^{2k_r} \).

As all 2, \( p_i \) are relatively prime,
\[ \sigma(n) = \sigma(2) \sigma(p_i^{2k_1}) \cdots \sigma(p_i^{2k_r}) \]

But \( \sigma(2) = 1 + 2 = 3 \).

As in (a) above, \( \sigma(p_i^{2k_i}) \) is odd.
\[ \therefore \sigma(n) \text{ is odd.} \]

8. Show that \( \sum_{d | n} \frac{d}{n} = \sigma(n) / n \) for all \( n > 0 \).

PF: Note that \( d \) is a divisor of \( n \Rightarrow \frac{n}{d} \) is a
divisor of \( n \) since \( d \cdot \frac{n}{d} = n \).

\[ \therefore \text{The set of divisors of } n = \{d_1, \ldots, d_k\} \]

can also be written as \( \left\{ \frac{n}{d_1}, \ldots, \frac{n}{d_k} \right\} \).

\[ \therefore \sigma(n) = d_1 + d_2 + \cdots + d_k = \frac{n}{d_1} + \frac{n}{d_2} + \cdots + \frac{n}{d_k} \]

\[ = n \left( \frac{1}{d_1} + \frac{1}{d_2} + \cdots + \frac{1}{d_k} \right) \]

\[ \therefore \frac{\sigma(n)}{n} = \frac{1}{d_1} + \frac{1}{d_2} + \cdots + \frac{1}{d_k} = \sum_{d \mid n} \frac{1}{d} \]

9. If \( n \) is a square free integer, prove \( \tau(n) = 2^r \), where \( r \) is the number of prime divisors of \( n \).

**PF:** \( n \) square free \( \Rightarrow n = p_1 p_2 \cdots p_r \), where each \( p_i \) is distinct and of exponent power \( 1 \) (see problem 6 (a), section 3.1).

\[ \therefore \text{By Th. 6.2, } \tau(n) = (k_1 + 1) \cdots (k_r + 1), \text{ and here, each } k_i = 1. \]

\[ \therefore \tau(n) = (1+1) \cdots (1+1) = 2 \cdots 2. \text{ As there are } r \text{ terms, } \tau(n) = 2^r. \]
10. Establish the assertions below:

(a) If \( n = p_1^{k_1} p_2^{k_2} \ldots p_r^{k_r} \), for \( n > 1 \), then

\[
1 > \frac{n}{\sigma(n)} > (1 - \frac{1}{p_1})(1 - \frac{1}{p_2}) \ldots (1 - \frac{1}{p_r})
\]

Since the divisors of \( n \) include 1 and \( n \),

\( \sigma(n) \geq n+1 > n \), so \( \sigma(n) > n \).

\[
1 > \frac{n}{\sigma(n)}
\]

By Th. 6.2, \( \sigma(n) = \frac{p_1^{k_1+1} - 1}{p_1 - 1} \ldots \frac{p_r^{k_r+1} - 1}{p_r - 1} \)

\[
\ldots \frac{n}{\sigma(n)} = \frac{p_1^{k_1} \ldots p_r^{k_r}}{(p_1^{k_1+1} - 1) \ldots (p_r^{k_r+1} - 1)} \frac{(p_1 - 1) \ldots (p_r - 1)}{(p_1 - 1) \ldots (p_r - 1)}
\]

\[
= \frac{p_1^{k_1} \ldots p_r^{k_r} (p_1 - 1) \ldots (p_r - 1)}{(p_1^{k_1+1} - 1) \ldots (p_r^{k_r+1} - 1)}
\]
\[
\begin{align*}
&= \frac{(p_i - 1) \cdots (p_r - 1)}{(p_i - \frac{1}{p_i}) \cdots (p_r - \frac{1}{p_r})} \\
&= \frac{(p_i - 1) \cdots (p_r - 1)}{(p_i - \frac{1}{p_i}) \cdots (p_r - \frac{1}{p_r})}
\end{align*}
\]

\[\text{But } p_i > p_i - \frac{1}{p_i}, \quad \therefore \quad \frac{1}{p_i} > \frac{1}{p_i^2} \]

\[
\therefore \text{ from } \Sigma \beta, \quad \frac{n}{\sigma(n)} = \frac{(p_i - 1) \cdots (p_r - 1)}{(p_i - \frac{1}{p_i}) \cdots (p_r - \frac{1}{p_r})} > \frac{(p_i - 1) \cdots (p_r - 1)}{p_i \cdots p_r} = (1 - \frac{1}{p_i}) \cdots (1 - \frac{1}{p_r})
\]

\[
\therefore 1 > \frac{n}{\sigma(n)} > (1 - \frac{1}{p_i}) \cdots (1 - \frac{1}{p_r})
\]

(6) For any positive integer \(n\),

\[
\frac{\sigma(n+1)}{n!} \geq 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}
\]
Clearly, all 1, 2, 3, 4, ..., n are divisors for \( n! \). There are more divisors of \( n! \) (including \( n! \)) since \( n! > n \) for \( n \geq 3 \), and \( n! \) is n for \( n = 2 \).

From prob. 8, \( \frac{\sigma(n!)}{n!} = \sum_{d|n!} \frac{1}{d} = \frac{1}{1} + \frac{1}{2} + \cdots \frac{1}{n} + \cdots + \frac{1}{n!} \geq 1 + \frac{1}{2} + \cdots + \frac{1}{n} \)

(C) If \( n > 1 \) is a composite number, then \( \sigma(n) > n + \sqrt{n} \)

Since \( \sigma(n) = 1 + d_1 + \cdots + d_k + n \), it suffices to show \( 1 + d_1 + \cdots + d_k > \sqrt{n} \)

Since \( n \) is composite, there is a \( d_i \) s.t. \( 1 < d_i < n \) and \( d_i | n \) \( \Rightarrow \frac{n}{d_i} < n \) \( \Rightarrow \frac{n}{d_i} < 1 \Rightarrow d_i > \sqrt{n} \)

\( \frac{n}{d_i} < n \Rightarrow 1 < \frac{n}{d_i} < n \) \[1\]

(1) If \( d_i > \sqrt{n} \), then clearly \( 1 + d_i > \sqrt{n} \), so

\( \sigma(n) = 1 + d_1 + \cdots + n > n + \sqrt{n} \)
(6) Suppose \( d_i \leq \sqrt{n} \). \( \therefore \frac{1}{d_i} \leq \frac{1}{\sqrt{n}} \Rightarrow \)

\[
\frac{\sqrt{n}}{d_i} = \frac{n}{d_i} \leq \frac{n}{d_i} \leq \frac{n}{d_i}, \text{ and}
\]

\( d_j \) is a divisor of \( n \) and \( \therefore d_j \geq \sqrt{n} \)

\( \therefore 1 + d_j + n > n + \sqrt{n} \)

\( \therefore \text{From } \sigma(n) = 1 + d_1 + d_2 + \ldots + n, \)

\( \sigma(n) \geq n + \sqrt{n} \)

\( \therefore \text{From (a) + (b), } \sigma(n) > n + \sqrt{n} \)

11. Given integer \( K \geq 1 \), show there are infinitely many integers \( n \) for which \( T(n) = K \) but at most finitely many \( n \) with \( \sigma(n) = K \).

(a) Let \( p \) be any prime, let \( n = p^{K-1} \)

\( \therefore T(n) = K \) by Th. 6.2. Since there are infinitely many primes, there are infinitely many \( n \) s.t. \( n = p^{K-1} \) and \( T(n) = K \).
(6) By problem 10(a), \(1 > \frac{n}{\sigma(n)}\) for any \(n\).

\[\therefore \sigma(n) > n\]

\[\therefore \text{Given any } k, \text{ if } \sigma(n) \text{ is to equal } k, \text{ then } k \text{ serves as an upper bound to } n.\]

In fact, for any \(n \geq k\), \(\sigma(n) > k\) by problem 10(a). \(\therefore\) There are at most \(k\) positive integers \(s.t.\) \(\sigma(n) = k\).

12. (a) Find the form of all positive integers \(n\) satisfying \(T(n) = 10\). What is the smallest positive integer for which this is true?

Since \(T(n) = (k_1 + 1)(k_2 + 1)\ldots(k_r + 1)\) if \(n = p_1^{k_1} \ldots p_r^{k_r}\), then \(10 = (k_1 + 1)\ldots(k_r + 1)\)

Possibilities include \(10 = 10\) or \(10 = 5 \cdot 2\)

\[\therefore (k_1 + 1) = 10 \text{ or } (k_1 + 1)(k_2 + 1) = 5 \cdot 2\]

\[\therefore n = p_1^9 \text{ or } n = p_1^{4}p_2^2 \text{, } p_1, p_2 \text{ distinct primes.}\]
The smallest such integer would be $2^9$ or $2^4 \cdot 3$ or $3^4 \cdot 2$. Of these three, the smallest is $2^4 \cdot 3 = 48$.

(b) Show there are no positive integers $n$ satisfying $\sigma(n) = 10$.

From proof of problem 11, the only possible integers under consideration would be $1, 2, 3, \ldots, 9$. $\sigma(1) = 1$. For $2, 3, 5, 7$, $\sigma(n) = 2$.

$\sigma(4) = 7$ \hspace{1em} (1+2+4)
\sigma(6) = 12 \hspace{1em} (1+2+3+6)
\sigma(8) = 15 \hspace{1em} (1+2+4+8)
\sigma(9) = 13 \hspace{1em} (1+3+9)

"No positive integers $n$ s.t. $\sigma(n) = 10$"

13. Prove there are infinitely many pairs of integers $m, n$ s.t. $\sigma(m^2) = \sigma(n^2)$

Pf: There are infinitely many $k$ s.t. $\gcd(k, 6) = 1$; consider $m = 5k$, $n = 4k$. \(\therefore\) There are infinitely many such $m, n$. Let $\rho$ be a prime s.t. $\rho \neq 2$ or 5.
\[ m^2 = 5^2 \rho^2 \quad \text{and} \quad n^2 = 4^2 \rho^2 = 2^4 \rho^2 \]

- By Th. C.2, \[ \sigma(m^2) = \frac{5^3 - 1}{5 - 1} \cdot \frac{\rho^3 - 1}{\rho - 1} = \frac{124}{4} \cdot \frac{\rho^3 - 1}{\rho - 1} = 31 \left( \frac{\rho^3 - 1}{\rho - 1} \right) \]

\[ \sigma(n^2) = \frac{2^5 - 1}{2 - 1} \cdot \frac{\rho^3 - 1}{\rho - 1} = 31 \left( \frac{\rho^3 - 1}{\rho - 1} \right) \]

Therefore, there are infinitely many \( m, n \), s.t. \[ \sigma(m^2) = \sigma(n^2) \]

14. For \( K \geq 2 \), show each of the following:

(a) \( n = 2^{K-1} \) satisfies \( \sigma(n) = 2n - 1 \)

By Th. C.2, \( \sigma(n) = \frac{2^{K+1} - 1}{2 - 1} = 2^K - 1 \)

But, \( 2n - 1 = 2(2^{K-1}) - 1 = 2^K - 1 \). \( \therefore \sigma(n) = 2n - 1 \)

(b) If \( 2^{K-1} \) is prime, then \( n = 2^{K-1} (2^{K-1}) \) satisfies \( \sigma(n) = 2n \)

If \( 2^{K-1} \) is prime, then \( 2^{K-1} \neq 2 \) since
\( k = 2 \), \( \therefore \) let \( \rho = 2^k - 1 \).

By Th. 6.2,

\[
\sigma(n) = \frac{2^{k-1} + 1}{2 - 1} \cdot \frac{\rho^2 - 1}{\rho - 1} = (2^k - 1)(\rho + 1)
\]

\[
= (2^k - 1)(2^k)
\]

\[\text{But } 2n = 2(2^k - 1)(2^k - 1) = 2^k(2^k - 1)\]

\[\therefore \sigma(n) = 2n\]

(c) If \( 2^k - 3 \) is prime, then \( n = 2^{k-1}(2^k - 3) \) satisfies \( \sigma(n) = 2n + 2 \)

Let \( \rho = 2^k - 3 \). Since \( \rho \) is prime, \( k \geq 3 \), so \( \rho \neq 2 \).

\[\therefore n = 2^{k-1} \cdot \rho, \text{ and by Th. 6.2,}\]

\[
\sigma(n) = \frac{2^{k-1} + 1}{2 - 1} \cdot \frac{\rho^2 - 1}{\rho - 1} = (2^{k-1})(\rho + 1)
\]

\[
= (2^{k-1})(2^k - 3 + 1) = (2^{k-1})(2^k - 2)
\]

\[= 2^{2k} - 3 \cdot 2^k + 2\]
But \(2n+2 = 2 \left(2^{k-1}\right) \left(2^k - 3\right) + 2\)

\[= 2^k \left(2^k - 3\right) + 2 = 2^{2k} - 3 \cdot 2^k + 2\]

\[\therefore \ \sigma(n) = 2n + 2\]

15. If \(n\) and \(n+2\) are a pair of twin primes, show that \(\sigma(n+2) > \sigma(n) + 2\).

**Pf:** For any prime \(p\), the only divisors are \(1\) and \(p\). \(\therefore \sigma(p) = p + 1\)

\[\therefore \sigma(n+2) = (n+2) + 1 = n + 3\]

\[\sigma(n) + 2 = (n+1) + 2 = n + 3\]

\[\therefore n \ and \ n+2 \ prime \Rightarrow \sigma(n+2) = \sigma(n) + 2\]

16.(a) For any integer \(n > 1\), prove there exists integers \(n_1\) and \(n_2\) such that \(\tau(n_1) + \tau(n_2) = n\)

**Pf:** If \(n\) is prime, \(\tau(n) = 2\). Since \(\tau(1) = 1\), then let \(n_1 = n_2 = 1\). \(\therefore \tau(n_1) + \tau(n_2) = \tau(n)\)

If \(n\) is composite, let \(n = p_1^{k_1} p_2^{k_2} \ldots p_r^{k_r}\) be the canonical prime factorization.
Since \( \frac{n}{k} \) is composite, at least one of the \( k_i \geq 2 \). Let \( p_j \) be that factor.

\[ \therefore \tau(n) = (k_1 + 1)(k_2 + 1) \cdots (k_j + 1) \cdots (k_r + 1) \]

\[ = k_j (k_1 + 1)(k_2 + 1) \cdots (k_r + 1) \]

\[ + (k_1 + 1)(k_2 + 1) \cdots (k_r + 1) \]

\[ = (k_{j-1} + 1)(k_1 + 1)(k_2 + 1) \cdots (k_r + 1) \]

\[ + (k_1 + 1)(k_2 + 1) \cdots (k_r + 1) \]

\[ = \tau(n_1) + \tau(n_2) \]

(6) Prove the Goldbach conjecture implies that
For each even integer $2n$, there exist integers $n_1$ and $n_2$ with $\sigma(n_1) + \sigma(n_2) = 2n$.

**Proof:** Since $2n$ is even, so is $2n - 2$.

Assume $2n - 2 > 4$. Goldbach conjecture states there exist odd primes $\rho_1$ and $\rho_2$, s.t. $\rho_1 + \rho_2 = 2n - 2$.

Therefore, let $n_1 = \rho_1$, $n_2 = \rho_2$.

Then $\sigma(n_1) + \sigma(n_2) = (\rho_1 + 1) + (\rho_2 + 1) = \rho_1 + \rho_2 + 2 = 2n - 2 + 2 = 2n$.

17. For a fixed integer $K$, show the function $f$ defined by $f(n) = n^K$ is multiplicative.

**Proof:** $f(mn) = (mn)^K = m^K n^K = f(m) \cdot f(n)$.

$m$ and $n$ don't even have to be relatively prime.

18. Let $f$, $g$ be multiplicative, not identically zero, and s.t. $f(p^k) = g(p^k)$ for each prime $p$ and $k \geq 1$. Prove $f = g$. 
**Proof:** Let \( n \) be any positive integer \( > 1 \)

Let \( n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r} \) be the prime factorization.

\[
\therefore f(n) = f(p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}) = f(p_1^{k_1}) \cdots f(p_r^{k_r})
\]

\[
= g(p_1^{k_1}) \cdots g(p_r^{k_r}) = g(p_1^{k_1} \cdots p_r^{k_r})
\]

\[
= g(n)
\]

If \( n = 1 \), then \( f(1) = g(1) = 1 \) by definition.

\[\therefore f = g\]

19. **Proof:** If \( f \) and \( g \) are multiplicative functions, then so is \( f \cdot g \) and \( f/g \) (whenever \( f/g \) is defined).

**Proof:** Let \( m, n \) be integers s.t. \( \gcd(m, n) = 1 \).

\[
f \cdot g (mn) = f(mn) \cdot g (mn)
\]

\[
= f(m) \cdot f(n) \cdot g(m) \cdot g(n)
\]

\[
= f(m) \cdot g(m) \cdot f(n) \cdot g(n)
\]

\[
= f \cdot g (m) \cdot f \cdot g (n)
\]
\[
\frac{f}{g}(mn) = \frac{f(mn)}{g(mn)}
\]
\[
= \frac{f(m)}{g(m)} \cdot \frac{f(n)}{g(n)}
\]
\[
= \frac{f(m)}{g(m)} \cdot f/g(n)
\]

20. Let \( w(n) \) denote the number of distinct prime divisors of \( n \geq 1 \), with \( w(1) = 0 \). For example, \( w(360) = w(2^3 \cdot 3^2 \cdot 5) = 3 \).

(a) Show \( 2^w(n) \) is a multiplicative function.

Let \( m, n \) be integers s.t. \( \text{gcd}(m, n) = 1 \), and let \( f(n) = 2^{w(n)} \).

\( \quad \) Let \( n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r} \) and \( m = q_1^{j_1} q_2^{j_2} \cdots q_s^{j_s} \).

Since \( \text{gcd}(m, n) = 1 \), then \( p_u \neq q_v \) for \( 1 \leq u \leq r, \ 1 \leq v \leq s \).

\( \therefore \) \( n \) has \( r \) distinct primes, \( m \) has \( s \) distinct primes.

\( \therefore \) \( nm = p_1^{k_1} \cdots p_r^{k_r} q_1^{j_1} \cdots q_s^{j_s} \) has \( r + s \) distinct primes (Fundamental Theorem of Arithmetic says this is unique).
\[
\begin{align*}
\therefore w(n) &= r, \quad w(m) = s, \quad \text{and} \quad w(n \cdot m) = r + s \\
\therefore f(n \cdot m) &= 2^{w(n \cdot m)} = 2^{w(n) + w(m)} = 2^{w(n)} \cdot 2^{w(m)} = f(n) \cdot f(m) \\
\therefore 2^{w(n)} \text{ is multiplicative.}
\end{align*}
\]

(5) For a positive integer \( n \), establish

\[
\tau(n) = \sum_{d|n} 2^{w(d)}
\]

From (a), \( F(n) = \sum_{d|n} 2^{w(d)} \text{ is multiplicative.} \)

\[
\therefore \tau(n) = \sum_{d|n} 2^{w(d)}
\]

Let \( n = p_1^{k_1} \ldots p_r^{k_r} \)

\[
\begin{align*}
F(n) &= F(p_1^{k_1} \ldots p_r^{k_r}) \\
&= F(p_1^{k_1}) \cdots F(p_r^{k_r}) \\
&= \sum_{d|p_1^{k_1}} 2^{w(d)} \cdots \sum_{d|p_r^{k_r}} 2^{w(d)}
\end{align*}
\]

All the divisors of \( p_i^{k_i} \) are \( 1, p_i, p_i^2, \ldots, p_i^{k_i} \).
by Th. 6.1. Also, \( w(\rho_i^{a_i}) = 1 \) for \( 1 \leq a_i \leq K_i \) and \( w(\rho_i^0) = 0 \).

\[
\sum_{d | \rho_i^{K_i}} 2^{w(d)} = 2^0 + 2^1 + \cdots + 2^{K_i} \\
= (1 + K_i) 2^{K_i} = 1 + 2K_i
\]

Similarly, \( 2^{w(d)} \)

\[
F(n) = \sum_{d | \rho_i^{K_i}} 2^{w(d)} \cdot \sum_{d | \rho_r^{K_r}} 2^{w(d)} \\
= (1 + 2K_i) \cdots (1 + 2K_r)
\]

But \( n^2 = \rho_1^{2K_1} \cdots \rho_r^{2K_r} \), so by Th. 6.2

\[
\tau(n^2) = (2K_1 + 1) \cdots (2K_r + 1)
\]

\[
\tau(n^2) = F(n) = \sum_{d | n} 2^{w(d)}
\]

21. For any positive integer \( n \), prove

\[
\sum_{d | n} \tau(d)^3 = \left( \sum_{d | n} \tau(d) \right)^2
\]
pf: \( T(n) \) is a multiplicative function

Since \( [T(mn)]^3 = [T(m) T(n)]^3 \)

\[ = T(m)^3 \cdot T(n)^3 \]

Then \( T(n)^3 \) is multiplicative.

\[ \therefore \text{By Th. C.4, } F(n) = \sum \frac{\tau(d)^3}{d|n} \] is multiplicative.

Also, \( G(n) = \sum \tau(d) \) is multiplicative, so \( H(n) = G^2(n) \) is multiplicative. Since

\[ H(mn) = G^2(mn) = [G(mn)]^2 = [G(m) G(n)]^2 \]

\[ = G^2(m) \cdot G^2(n). \]

Let \( n = p_1^{k_1} \cdots p_r^{k_r} \).

If \( F(n) = H(n) \) for \( n = p_k \), then since \( F \) and \( H \) are multiplicative, the statement will also be true for \( n = p_1^{k_1} \cdots p_r^{k_r} \), since

\[ F(p_1^{k_1} \cdots p_r^{k_r}) = F(p_1^{k_1}) \cdots F(p_r^{k_r}) = H(p_1^{k_1}) \cdots H(p_r^{k_r}) \]

\[ = H(p_1^{k_1} \cdots p_r^{k_r}) \]
Consider \( n = p^k \). By \( \text{Th. 6.1} \), all the divisors of \( n \) are \( 1, p, p^2, \ldots, p^k \)

\[
\sum_{d \mid p^k} \tau(d)^3 = \tau(1)^3 + \tau(p)^3 + \tau(p^2)^3 + \cdots + \tau(p^k)^3
\]

\[
= 1 + (p+1)^3 + (p^2+1)^3 + \cdots + (p^k+1)^3
\]

\[
= 1 + 2^3 + 3^3 + \cdots + (k+1)^3
\]

\[
= \left[ \frac{(k+1)(k+2)}{2} \right]^2 \quad \text{by prob. 1.4, Sec. 1.1.}
\]

\[
\left[ \sum_{d \mid p^k} \tau(d) \right]^2 = \left[ \tau(1) + \tau(p) + \cdots + \tau(p^k) \right]^2
\]

\[
= \sum 1 + 2 + \cdots + (k+1) \right]^2
\]

\[
= \left[ \frac{(k+1)(k+2)}{2} \right]^2 \quad \text{by prob. 1.4, Sec. 1.1.}
\]

\[
\therefore \sum_{d \mid p^k} \tau(d)^3 = \left[ \sum_{d \mid p^k} \tau(d) \right]^2, \quad \text{so} \quad F(n) = H(n) \quad \text{for} \quad n = p^k
\]
22. Given \( n \geq 1 \), let \( \sigma_s(n) \) denote the sum of the \( s \)th powers of the positive divisors of \( n \); that is,

\[
\sigma_s(n) = \sum_{d \mid n} d^s
\]

Verify the following:

(a) \( \sigma_0 = \tau \) and \( \sigma_1 = \tau \)

(i) since \( d^0 = 1 \) for all \( d \geq 1 \), \( \sigma_0(n) = \sum_{d \mid n} 1 = \tau(n) \)

by definition.

(ii) since \( d^1 = d \) for all \( d \geq 1 \), \( \sigma_1(n) = \sum_{d \mid n} d = \sigma(n) \)

by definition.

(b) \( \sigma_s \) is a multiplicative function.

\[ \text{pf:} \quad \text{If } f(n) = n^s \text{ can be shown to be multiplicative, then by Th. 6.4,} \]

\[
\sum_{d \mid n} f(d) = \sum_{d \mid n} d^s = \sigma_s(n) \text{ will be multiplicative.} \]
Consider $f(n) = n^s$

$$f(mn) = (mn)^s = m^s n^s = f(m) \cdot f(n),$$

so $f(n) = n^s$ is multiplicative, and so $\sigma_s(n)$.

(C) If $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ is the prime factorization,

Then $\sigma_s(n) = \left(\frac{p_1^{s(k_1+1)} - 1}{p_1^s - 1}\right) \cdots \left(\frac{p_r^{s(k_r+1)} - 1}{p_r^s - 1}\right)$

**Proof:** By Th. 6.1, all positive divisors of $n$ are of the form $p_1^{a_1} \cdots p_r^{a_r}$, $0 \leq a_i \leq k_i$.

:. all the $s$th powers of the divisors of $n$ are of the form $p_1^{as_1} p_2^{as_2} \cdots p_r^{ars}$.

:. Consider the sum:

$$\left(1 + p_1^s + p_1^{2s} + \cdots + p_1^{ks}\right) \cdots \left(1 + p_r^s + \cdots + p_r^{ks}\right)$$

Each positive divisor to the $s$th power occurs once and only once as a term in the expansion of the product.
\[ \sum_{s} (n) = \left(1 + \beta_1^s + \beta_2^s + \cdots + \beta_i^s\right) \cdots \left(1 + \beta_r^s + \cdots + \beta_r^s\right) \]

Using the formula for the sum of a finite geometric series,

\[ 1 + \beta_i^s + \beta_i^{2s} + \cdots + \beta_i^{k_i s} = \frac{\beta_i^{s(k_i+1)} - 1}{\beta_i^s - 1}, \]

\[ \therefore \sum_{s} (n) = \left(\frac{\beta_1^{s(k_1+1)} - 1}{\beta_1^s - 1}\right) \cdots \left(\frac{\beta_r^{s(k_r+1)} - 1}{\beta_r^s - 1}\right) \]

23. For any positive integer \( n \), show the following:

(a) \[ \sum_{d \mid n} \sigma(d) = \sum_{d \mid n} \left(\frac{n}{d}\right) \tau(d) \]

\[ d \mid \ln \]

\[ d \mid \ln \]

Proof: (1) First note that since \( \tau(n) \) is multiplicative, \( H(n) = \sum_{d \mid n} \frac{n}{d} \tau(d) \) is multiplicative.

Proof: \( H(mn) = \sum_{d \mid mn} \frac{mn}{d} \sigma(d) = \sum_{d_1 \mid m} \sum_{d_2 \mid n} \frac{mn}{d_1 d_2} \sigma(d_1 d_2) \)
\[
\sum_{d_1 \mid m} \sum_{d_2 \mid n} \frac{m}{d_1 d_2} \tau(d_1) \tau(d_2)
\]

\[
= \sum_{d_1 \mid m} \frac{m}{d_1} \tau(d_1) \sum_{d_2 \mid n} \frac{n}{d_2} \tau(d_2)
\]

\[
= \left( \sum_{d_1 \mid m} \frac{m}{d_1} \tau(d_1) \right) \left( \sum_{d_2 \mid n} \frac{n}{d_2} \tau(d_2) \right)
\]

\[
= H(m) \cdot H(n)
\]

The functions

\[
F(n) = \sum_{d \mid n} \sigma(d) \quad \text{and} \quad G(n) = \sum_{d \mid n} \frac{n}{d} \tau(d)
\]

are multiplicative.

(2) Now, let \( n = \rho_1^{k_1} \rho_2^{k_2} \ldots \rho_r^{k_r} \) be the prime factorization of \( n \). If it can be

shown that \( F(\rho^k) = G(\rho^k) \), then \( F(n) = \)
\[ F(\rho^1_k \ldots \rho^r_k) = F(\rho^1_k) \ldots F(\rho^r_k) = G(\rho^1_k) \ldots G(\rho^r_k) \]
\[ = G(\rho^1_k \ldots \rho^k_n) = G(n). \]

(3): \[ F(\rho^k) = \sum_{d \mid \rho^k} \tau(\rho^d) = (\rho^0) + \\
\rho^0 + \rho^1 + \ldots + \\
(\rho^0 + \rho^1 + \ldots + \rho^k) \]
\[ = (k + 1) \rho^0 + (k) \rho^1 + \ldots + (1) \rho^k \]
\[ = (1) \cdot \rho^k + \ldots + (k) \rho + (k + 1) \quad [1] \]

\[ G(\rho^k) = \sum_{d \mid \rho^k} \frac{\rho^k}{d} \tau(\rho^d) \]
\[ = \left( \frac{\rho^k}{\rho^0} \cdot \tau(\rho^0) \right) + \left( \frac{\rho^k}{\rho^1} \cdot \tau(\rho^1) \right) + \ldots + \left( \frac{\rho^k}{\rho^k} \cdot \tau(\rho^k) \right) \]
\[ = (1) \cdot \rho^k + 2 \cdot \rho^{k-1} + \ldots + (k) \rho + (k + 1) \quad [2] \]

\[ \therefore \text{Since } [1] = [2], \text{ then } F(\rho^k) = G(\rho^k) \]

\[ \therefore \text{As stated in (2), because of (3), } \sum_{d \mid \rho^k} \frac{\rho^k}{d} \tau(\rho^d) = \sum_{d \mid \rho^k} \tau(\rho^d). \]
(b) \[ \sum_{d \mid n} \left( \frac{n}{d} \right) \tau(d) = \sum_{d \mid n} d \tau(d) \]

**Pf:** Since \( f(n) = n \) is multiplicative, so is \( f \cdot \tau \\
\therefore G(n) = \sum_{d \mid n} d \tau(d) \) is multiplicative.

As in (a), the proof that \( \mathcal{F}(n) = \sum_{d \mid n} \left( \frac{n}{d} \right) \sigma(d) \)

is multiplicative is identical to Th. 6.4.

So, as in (a), it suffices to prove

\[ \mathcal{F}(n) = G(n) \text{ for } n = p^k. \]

\[ \mathcal{F}(p^k) = \sum_{d \mid p^k} \left( \frac{p^k}{d} \right) \tau(d) \]

\[ = p^k \cdot p^0 \]

\[ + p^{k-1} (p^0 + p) \]

\[ + p^{k-2} (p^0 + p + p^2) \]

\[ + \ldots \]

\[ + p^0 (p^0 + p + \ldots + p^k) \]
\[ (K+1) \rho^K + K \cdot \rho^{K-1} + (K-1) \rho^{K-2} + \ldots + 1 \]

\[ = 1 + \sum_{i=1}^{K} (i+1) \rho^i \quad \text{[1]} \]

\[ G(\rho^K) = \sum_{d} d \mathcal{T}(d) \left/ \rho^K \right. \]

\[ = 1 \cdot 1 \]

\[ + \rho \cdot (1+1) \]

\[ + \rho^2 \cdot (1+1+1) \]

\[ + \ldots \]

\[ + \rho^K \cdot (K+1) \]

\[ = 1 + \sum_{i=1}^{K} \rho^i (i+1) \quad \text{[2]} \]

Since [1] = [2], \( F(\rho^K) = G(\rho^K) \)