6.3 The Greatest Integer Function

Note Title 7/21/2005

1. Given integers $a, b > 0$, show there exists a unique integer r with $0 \le r < b$ satisfying $a = [a/b]b + r$!

Pf: By def., $a/b - 1 < [a/b] \le a/b$

$\therefore [a/b]b \le (a/b) \cdot b = a$, $\therefore 0 \le a - [a/b]b$

Let $r = a - [a/b]b$ $\therefore 0 \le r$

Also, $(a/b - 1)b < [a/b]b$, so $a - b < [a/b]b$, or $a - [a/b]b < b$

$\therefore r < b$ $\therefore$ With $r = a - [a/b]b$, $0 \le r < b$

r is unique: let r' be s.t. $a = [a/b]b + r'$

$\therefore r' = a - [a/b]b$, so $r' = r$

2. Let x, y be real numbers. Prove the greatest integer function satisfies the following properties:

(a) $[x+n] = [x] + n$ for any integer n.

By def., $x - 1 < [x] \le x$

$\therefore x - 1 + n < [x] + n \le x + n$ [1]

Also, by def of $[x+n]$,

$$x+n-1<[x+n]\le x+n \qquad [2]$$

$$\therefore x+n<[x+n]+1 \qquad [3]$$

[1] and [3] yield $[x]+n<[x+n]+1$ [4]

From [2], $[x+n]-1\le x+n-1$

and from [1], $x+n-1<[x]+n$

$$\therefore [x+n]-1<[x]+n \qquad [5]$$

[4] and [5] yield $[x+n]-1<[x]+n<[x+n]+1$

$$\therefore -1<[x]+n-([x+n])<1$$

Since all quantities are integers,

$[x]+n-([x+n])=0$, or $[x]+n=[x+n]$

(6) $[x]+[-x]=0$ or -1, according as x is an integer or not.

(1) If x is an integer, $[x] = x$ and $[-x] = -x$

$\therefore [x] + [-x] = 0$

(2) If x is not an integer, $x = [x] + \theta, \ 0 < \theta < 1$

$-x = [-x] + \theta', \ 0 < \theta' < 1$

Adding, $x + (-x) = [x] + [-x] + \theta + \theta'$, or

$$0 = [x] + [-x] + \theta + \theta'$$

But $0 < \theta + \theta' < 2$, and $\theta + \theta' = -([x] + [-x])$

$\therefore 0 < -([x] + [-x]) < 2$, or $-2 < [x] + [-x] < 0$

$[x] + [-x]$ is an integer, so $[x] + [-x] = -1$.

(c) $[x] + [y] \le [x+y]$, and when $x > 0, y > 0$, $[x][y] \le [xy]$

(1) $[x] + [y] \le [x+y]$

If x, y are both integers, $[x] = x$, $[y] = y$,

$[x+y] = x+y$. $\therefore [x] + [y] = [x+y]$

If one of x, y is an integer, say y,

then $[x] + [y] = [x] + y$

By (a), $[x+y] = [x] + y$.

$\therefore [x] + [y] = [x+y]$

Suppose both x and y are not integers.

$\therefore$ By def., $x-1 < [x] < x$
$y-1 < [y] < y$

$\therefore [x] + [y] - 1 < x + y - 1 \qquad [1]$

From the def. of $[x+y]$,

$$x + y - 1 < [x+y] \le x + y \qquad [2]$$

From [1] and [2],

$$[x]+[y]-1 < [x+y] \qquad [3]$$

Since all quantities are integers in [3],

$$[x]+[y] \le [x+y]$$

(2) $[x][y] \le [xy]$ when $x>0, y>0$

Let $x = [x] + \theta \quad , \quad 0 \le \theta < 1$
$y = [y] + \theta' \quad ; \quad 0 \le \theta' < 1$

$$\therefore [xy] = [([x]+\theta)([y]+\theta')]$$

$$= [[x][y] + \theta[y] + \theta'[x] + \theta\cdot\theta']$$

$$= [x][y] + [\theta[y] + \theta'[x] + \theta\theta'],$$ by (a) above since $[x][y]$ is an integer.

All quantities in $[\theta[y] + \theta'[x] + \theta\theta']$ are positive.

$$\therefore [xy] \geq [x][y]$$

(d) $[x/n] = [[x]/n]$ for any positive integer n.

Let $x/n = [x/n] + \theta$, $0 \leq \theta < 1$.

$$\therefore x = [x/n]\cdot n + \theta n$$

$$\therefore [x] = [[x/n]\cdot n + \theta n]$$

$$= [x/n]\cdot n + [\theta n],$$ by (a) since $[x/n]\cdot n$ is an integer

$$\therefore [x]/n = [x/n] + [\theta n]/n \qquad [1]$$

But since $0 \le \theta < 1$, Then $0 \le \theta n < n$

$\therefore 0 \le [\theta n] \le \theta n < n$

$\therefore 0 \le [\theta n]/n < 1$, $\therefore [\theta n]/n = 0$

$\therefore$ From [1], $[x]/n = [x/n]$

(e) $[nm/k] \ge n[m/k]$ for positive integers n, m, k.

From (c), $[n\, m/k] \ge [n][m/k]$

But $[n] = n$ for positive integer n.

$\therefore [nm/k] \ge n[m/k]$

(f) $[x] + [y] + [x+y] \le [2x] + [2y]$

Let $x = [x] + \theta$, $0 \le \theta < 1$

$y = [y] + \theta'$, $0 \le \theta' < 1$

$\therefore x + y = [x] + [y] + \theta + \theta'$

$\therefore [x+y] = [[x] + [y] + \theta + \theta']$, and using (a)

$= [x] + [y] + [\theta + \theta']$ [1]

(1) Suppose $0 \le \theta < 0.5$, $0 \le \theta' < 0.5$

$\therefore\ 0 \le \theta + \theta' < 1$, so $[\theta + \theta'] = 0$

$\therefore$ Equation [1] becomes $[x+y] = [x] + [y]$

$\therefore\ [x] + [y] + [x+y] = 2[x] + 2[y]$

But $2[x] = 2x - 2\theta$, so $2x = 2[x] + 2\theta$

$\therefore\ [2x] = [2[x] + 2\theta]$

$= 2[x] + [2\theta]$ by (a)

Since $0 \le \theta < 0.5$, $0 \le 2\theta < 1$.

$\therefore\ [2\theta] = 0$

$\therefore\ [2x] = 2[x]$

Similarly, $[2y] = 2[y]$

$\therefore\ [x] + [y] + [x+y] = 2[x] + 2[y] = [2x] + [2y]$

Or, $[x] + [y] + [x+y] = [2x] + [2y]$

(2) Suppose $0 \le \theta < 0.5$, $0.5 \le \theta' < 1$

$\therefore\ 0 < \theta + \theta' < 1.5$, so $0 \le [\theta + \theta'] \le 1$

$\therefore$ From Equation [1], $[x+y] \le [x] + [y] + 1$

$\therefore [x]+[y]+[x+y] \leq 2[x]+2[y]+1$ [2]

But $2[x] = 2x - 2\theta$, so $2x = 2[x] + 2\theta$

$\therefore [2x] = [2[x] + 2\theta]$

$= 2[x] + [2\theta]$ by (a)

And $0 \leq 2\theta < 1$, $\therefore [2\theta] = 0$

$\therefore [2x] = 2[x]$ [3]

Also $2[y] = 2y - 2\theta'$, so $2y = 2[y] + 2\theta'$

$\therefore [2y] = [2[y] + 2\theta']$

$= 2[y] + [2\theta']$ by (a)

And $1 \leq 2\theta' < 2$

$\therefore [2\theta'] = 1$

$\therefore [2y] = 2[y] + 1$

Or, $2[y] = [2y] - 1$ [4]

Substituting [3] and [4] into [2],

$[x]+[y]+[x+y] \leq [2x] + [2y] - 1 + 1$

$= [2x] + [2y]$

Similarly, if $0.5 \leq \theta < 1$, $0 \leq \theta' < 0.5$,

$[x]+[y]+[x+y] \leq [2x] + [2y]$

(3) Suppose $0.5 \le \theta < 1$, $0.5 \le \theta' < 1$

$\therefore 1 \le \theta + \theta' < 2$, so $[\theta + \theta'] = 1$

$\therefore$ Equation [1] becomes $[x+y] = [x] + [y] + 1$

$\therefore [x] + [y] + [x+y] = 2[x] + 2[y] + 1$ [5]

But $2x = 2[x] + 2\theta$

$\therefore [2x] = [2[x] + 2\theta]$

$= 2[x] + [2\theta]$ by (a)

And $1 \le 2\theta < 2$, so $[2\theta] = 1$

$\therefore [2x] = 2[x] + 1$

Similarly, $[2y] = 2[y] + 1$

Substituting into [5],

$[x] + [y] + [x+y] = [2x] - 1 + [2y] - 1 + 1$

$= [2x] + [2y] - 1$

$< [2x] + [2y]$

$\therefore [x] + [y] + [x+y] < [2x] + [2y]$

$\therefore$ (1), (2), and (3) yield:

$$[x] + [y] + [x+y] \leq [2x] + [2y]$$

3. Find the highest power of 5 dividing 1000! and the highest power of 7 dividing 2000!

(a). $\left[\frac{1000}{5}\right] + \left[\frac{1000}{5^2}\right] + \left[\frac{1000}{5^3}\right] + \left[\frac{1000}{5^4}\right]$

$= 200 + 40 + 8 + 1 = 249$

$\therefore 5^{249}$ divides $1000!$

(b) $\left[\frac{2000}{7}\right] + \left[\frac{2000}{7^2}\right] + \left[\frac{2000}{7^3}\right]$

$= 285 + 40 + 5 = 330$

$\therefore 7^{330}$ divides $2000!$

4. For an integer $n \geq 0$, show $[n/2] - [-n/2] = n$

Pf: By def., $\frac{n}{2} - 1 < \left[\frac{n}{2}\right] \leq \frac{n}{2}$ [1]

and $-\frac{n}{2} - 1 < \left[-\frac{n}{2}\right] \leq -\frac{n}{2}$ [2]

From [2], $-\left[-\frac{n}{2}\right] < \frac{n}{2} + 1$

Adding to [1], $\left[\frac{n}{2}\right] - \left[-\frac{n}{2}\right] < \frac{n}{2} + \frac{n}{2} + 1 = n + 1$

$\therefore \left[\frac{n}{2}\right] - \left[-\frac{n}{2}\right] \le n$ [3]

Also from [2], $\frac{n}{2} \le -\left[-\frac{n}{2}\right]$

Adding to [1], $\frac{n}{2} + \frac{n}{2} - 1 < \left[\frac{n}{2}\right] - \left[-\frac{n}{2}\right]$,

or, $n - 1 < \left[\frac{n}{2}\right] - \left[-\frac{n}{2}\right]$

$\therefore n \le \left[\frac{n}{2}\right] - \left[-\frac{n}{2}\right]$ [4]

[3] and [4] $\Rightarrow$ $\left[\frac{n}{2}\right] - \left[-\frac{n}{2}\right] = n$

5. (a) Verify that 1000! terminates in 249 zeros.

From #3, 5^{249} divides 1000!, but 5^{250} does not

The greatest power of 2 dividing 1000! is greater than 500 since $\left[\frac{1000}{2}\right] = 500$

$\therefore (2 \cdot 5)^{249}$ divides 1000! but

$(2\cdot 5)^{250}$ does not.

$\therefore 1000! = n \times 10^{249}$, but $1000! \neq n' \times 10^{250}$

$\therefore 1000!$ ends in 249 zeros.

(6) For what value of n does $n!$ terminate in 37 zeros?

Must find n s.t. $2^{37} \mid n!$ and $5^{37} \mid n!$, but either $2^{38} \nmid n!$ or $5^{38} \nmid n!$

$\therefore$ Consider $\left[\frac{n}{5}\right] \leq 37$ since $\left[\frac{n}{5}\right] > \left[\frac{n}{5^2}\right]$

$\therefore n \leq 5 \cdot 37 = 185$

2 is not the limiting factor since $\left[\frac{185}{2}\right] = 92$

Since $5^2 = 25$, $5^3 = 125$, 5^2 will contribute 6 powers of 5 for $n \geq 150$, and 5^3 will contribute at least 1 power.

For $n \geq 150$, 5 contributes $\frac{150}{5} = 30$

For $n \geq 155$, 5 contributes 31 powers.

$\therefore$ For $n \geq 155$, $5, 5^2, 5^3$ will contribute $31 + 6 + 1 = 38$ powers, which is too big.

$\therefore$ For $150 \leq n \leq 154$, 5 contributes exactly 37 powers of 5.

For $n = 149$, 5 contributes 29 powers,
5^2 contributes 5
5^3 contributes 1

$\therefore n = 149 \to 35$ powers of 5, too small.

For $150 \le n \le 154$, 2 easily contributes at least 37 powers as $150/2 = 75$.

$\therefore$ For $150 \le n \le 154$, $n!$ will terminate in 37 zeros.

6. If $n \ge 1$ and p is a prime, prove that
(a) $(2n)!/(n!)^2$ is an even integer

Pf: From Th. 6.10, letting $n = 2n$ and $r = n$ in the formula,
Then $\binom{2n}{n} = \dfrac{(2n)!}{n!(2n-n)!} = \dfrac{(2n)!}{(n!)^2}$ is an integer

But $$\frac{(2n)!}{(n!)^2} = \frac{2n\cdot(2n-1)\cdots(n+1)\cdot n!}{(n!)(n!)}$$

$$= \frac{2n(2n-1)\cdots(n+1)}{n!}$$

This is an integer containing 2 as a factor, and so it is even.

(b) The exponent of the highest power of p that divides $\frac{(2n)!}{(n!)^2}$ is $\sum_{k=1}^{\infty}\left(\left[\frac{2n}{p^k}\right]-2\left[\frac{n}{p^k}\right]\right)$

Pf: For any prime p, let s be the highest power of p that divides $(2n)!$

If p also divides $n!$, let k be the highest power of p dividing $n!$.

$\therefore \frac{p^s}{p^k} = p^{s-k}$, so $s-k$ is the highest power of p dividing $\frac{(2n)!}{n!}$.

$\therefore \frac{p^s}{p^k \cdot p^k} = p^{s-2k}$, so $s-2k$ is the highest power of p dividing $\frac{(2n)!}{(n!)^2}$

By Th. 6.9, the highest power of p dividing $(2n)!$ is: $\sum_{k=1}^{\infty}\left[\frac{2n}{p^k}\right]$

and the highest power of p dividing $n!$ is:

$$\sum_{K=1}^{\infty}\left[\frac{n}{p^k}\right]$$

$\therefore$ The highest power of p dividing $\frac{(2n)!}{(n!)^2}$

is: $\displaystyle\sum_{K=1}^{\infty}\left[\frac{2n}{p^k}\right] - 2\sum_{K=1}^{\infty}\left[\frac{n}{p^k}\right]$

$$= \sum_{K=1}^{\infty}\left(\left[\frac{2n}{p^k}\right] - 2\left[\frac{n}{p^k}\right]\right)$$

(c) In the prime factorization of $\frac{(2n)!}{(n!)^2}$, The exponent of any prime p s.t. $n < p < 2n$ is 1.

pf: Since $n < p$, Then $\frac{n}{p} < 1$, so $\left[\frac{n}{p}\right] = 0$.
$\therefore$ For any $k > 0$, $\left[\frac{n}{p^k}\right] = 0$.

$\therefore$ From (b), The highest power of p is: $\displaystyle\sum_{K=1}^{\infty}\left[\frac{2n}{p^k}\right]$ since The contribution by $2\left[\frac{n}{p^k}\right] = 0$.

But since $n < p$, $\frac{n}{p} < 1$, so $\frac{2n}{p} < 2$
As $p < 2n$, $1 < \frac{2n}{p}$. $\therefore \left[\frac{2n}{p}\right] = 1$ and
$\frac{2n}{p \cdot p^k} < \frac{2}{p^k} < 1$ if $p \geq 2$. $\therefore \left[\frac{2n}{p^k}\right] = 0$ for $k > 1$

$$\therefore \sum_{k=1}^{\infty} \left[\frac{2n}{p^k}\right] = 1 \text{ for any } n < p < 2n,$$

so the highest power of p is 1.

7. Let $n = a_k p^k + a_{k-1} p^{k-1} + \dots + a_1 p + a_0$, where $0 \le a_i < p$.

Show the exponent of highest power of p appearing in the prime factorization of $n!$ is:

$$\frac{n - (a_k + \dots + a_2 + a_1 + a_0)}{p-1}$$

Pf: The exponent of the highest power of p, by Th. 6.9, is:

$$\sum_{i=1}^{\infty}\left[\frac{n}{p^i}\right] = \left[a_k p^{k-1} + \dots + a_1 + \frac{a_0}{p}\right]$$

$$+ \left[a_k p^{k-2} + \dots + a_2 + \frac{a_1}{p} + \frac{a_0}{p^2}\right]$$

$$+$$

$$\vdots$$

$$+ \left[a_k + \dots + \frac{a_1}{p^{k-1}} + \frac{a_0}{p^k}\right]$$

$$+ \left[\frac{a_k}{p} + \dots + \frac{a_1}{p^k} + \frac{a_0}{p^{k+1}}\right] \qquad [1]$$

Lemma: for $p > 1$, $n > 1$, $(p-1)\left(\frac{1}{p} + \frac{1}{p^2} + \cdots \frac{1}{p^n}\right) < 1$

By induction, let $k = 1$.

$(p-1)\left(\frac{1}{p}\right) = 1 - \frac{1}{p} < 1$

Suppose true for k

$\therefore (p-1)\left(\frac{1}{p} + \cdots + \frac{1}{p^k} + \frac{1}{p^{k+1}}\right) =$

$(p-1)\left(\frac{1}{p} + \cdots + \frac{1}{p^k}\right) + \frac{p-1}{p^{k+1}} =$

$p\left(\frac{1}{p} + \cdots + \frac{1}{p^k}\right) - \frac{1}{p} - \cdots - \frac{1}{p^k} + \frac{1}{p^k} - \frac{1}{p^{k+1}}$

But by assumption,

$p\left(\frac{1}{p} + \cdots + \frac{1}{p^k}\right) - \frac{1}{p} - \cdots - \frac{1}{p^k} < 1$

$\therefore p\left(\frac{1}{p} + \cdots + \frac{1}{p^k}\right) - \frac{1}{p} - \cdots - \frac{1}{p^k} + \frac{1}{p^k} - \frac{1}{p^{k+1}} < 1 + \frac{1}{p^k} - \frac{1}{p^{k+1}}$

or, $p\left(\frac{1}{p} + \cdots + \frac{1}{p^k}\right) - \frac{1}{p} - \cdots - \frac{1}{p^k} - \frac{1}{p^{k+1}} < 1 - \frac{1}{p^{k+1}} < 1$

$\therefore$ True for $k+1$ also.

$\therefore$ True for all $n > 1$.

In [1], all $0 \le a_i \le p-1$

$\therefore$ By the lemma, all terms in [1] divided by p^k add up to less than 1, [1] becomes:

$$\left[\frac{n}{p}\right] = a_k p^{k-1} + \dots + a_2 p + a_1 \qquad [a_1]$$

$$\left[\frac{n}{p^2}\right] = a_k p^{k-2} + \dots + a_2 \qquad [a_2]$$

$$\dots$$

$$\left[\frac{n}{p^{k-1}}\right] = a_k p + a_{k-1} \qquad [a_{k-1}]$$

$$\left[\frac{n}{p^k}\right] = a_k \qquad [a_k]$$

Note that $\left[\frac{n}{p^k}\right] p = a_k p = \left[\frac{n}{p^{k-1}}\right] - a_{k-1}$

$$\therefore \left[\frac{n}{p}\right] p = n - a_0$$

$$\left[\frac{n}{p^2}\right] p = \left[\frac{n}{p}\right] - a_1$$

$$\vdots$$

$$\left[\frac{n}{p^{k-1}}\right] p = \left[\frac{n}{p^{k-2}}\right] - a_{k-2}$$

$$\left[\frac{n}{p^k}\right] p = \left[\frac{n}{p^{k-1}}\right] - a_{k-1}$$

$$0 = \left[\frac{n}{p^k}\right] - a_k$$

Adding the left column entries and right column entries, you get:

$$\left(\left[\frac{n}{p}\right]+\left[\frac{n}{p^2}\right]+\cdots+\left[\frac{n}{p^k}\right]\right)p=\left[\frac{n}{p}\right]+\left[\frac{n}{p^2}\right]+\cdots+\left[\frac{n}{p^k}\right]+n-(a_0+\cdots+a_k)$$

$$\therefore \left(\left[\frac{n}{p}\right]+\left[\frac{n}{p^2}\right]+\cdots+\left[\frac{n}{p^k}\right]\right)(p-1)=n-(a_0+\cdots+a_k)$$

$$\text{or} \quad \sum_{k=1}^{\infty}\left[\frac{n}{p^k}\right]=\frac{n-(a_0+a_1+\cdots+a_k)}{p-1}$$

8. (a). Using #7, show that the exponent of highest power of p dividing $(p^k-1)!$ is: $\dfrac{p^k-(p-1)k-1}{p-1}$

Strategy: find coefficients for the p-based expansion of p^k-1 $(0\le a_i<p)$.

First note $p^k-1=(p-1)(p^{k-1}+p^{k-2}+\cdots+p+1)$

$$=(p-1)p^{k-1}+(p-1)p^{k-2}+\cdots+(p-1)p+(p-1)$$

Since p is prime, $0\le p-1<p$, so

$a_{k-1}=p-1 \quad a_{k-2}=p-1, \ldots, a_1=p-1, a_0=p-1$

$\therefore$ Using $n = p^k - 1$, The formula in #7 becomes:

$$\frac{(p^k - 1) - [\overset{a_0}{(p-1)} + \overset{a_1}{(p-1)} + \dots + \overset{a_{k-1}}{(p-1)}]}{p-1}$$

$$= \frac{(p^k - 1) - [k(p-1)]}{p-1}$$

(b). Determine the highest power of 3 dividing 80! and the highest power of 7 dividing 2400!.

(1) $80 = 81 - 1 = 3^4 - 1$, here $k = 4$, $p = 3$

Using the formula in (a), $\dfrac{(3^4 - 1) - [4(3-1)]}{3-1}$

$= \dfrac{80 - 8}{2} = 36$

$\therefore 3^{36} \mid 80!$, 36 is the highest power of 3.

(2) $2400 = 7^4 - 1$, here $k = 4$, $p = 7$

Using the formula in (a), $\dfrac{(7^4 - 1) - [4(7-1)]}{7-1}$

$= \dfrac{2400 - 24}{6} = \quad 400 - 4 = 396$

$\therefore 7^{396} \mid 2400!$, 396 is the highest power of 7.

9. Find an integer $n \geq 1$ s.t. the highest power of 5 contained in $n!$ is 100.

Using problem #7, express n as a p-based number and use formula: $\frac{n-r}{p-1}$, where

$r = \sum a_i$ coefficients, $r > 0$.

$\therefore 100 = \frac{n-r}{5-1} = \frac{n-r}{4}$. $\therefore 400 = n - r$

If $r = 1$, $n = 401$. $401 = 3 \cdot 5^3 + 1 \cdot 5^2 + 1$,
$r = 3 + 1 + 1 = 5$

Because n must be > 400, for $a_3 \cdot 5^3$, $a_3 = 3$.
$\therefore$ r must be at least 3.

Try $n = 404$, $404 = 3 \cdot 5^3 + 1 \cdot 5^2 + 4$,
$r = 3 + 1 + 4 = 8$

Try $n = 405$, $405 = 3 \cdot 5^3 + 1 \cdot 5^2 + 1 \cdot 5 + 0$
$r = 3 + 1 + 1 = 5$

$\therefore$ When $\underline{n = 405}$, highest power of 5 dividing $405!$ is 100.

10. Given a positive integer N, show the following:

(a) $\sum_{n=1}^{N} \mu(n) \left[\frac{N}{n}\right] = 1$

Pf: Let $F(n) = \sum_{d|n} \mu(d)$

By Th. 6.6, $F(n) = 1$ if $n=1$, $F(n) = 0$, $n > 1$.

By Th. 6.11, $\sum_{k=1}^{N} F(k) = \sum_{n=1}^{N} \mu(n) \left[\frac{N}{n}\right]$

But $\sum_{k=1}^{N} F(k) = F(1) + F(2) + \cdots + F(N)$

$= 1 + 0 + \cdots + 0$

$= 1$

$\therefore \sum_{n=1}^{N} \mu(n) \left[\frac{N}{n}\right] = 1$

(b) $\left| \sum_{n=1}^{N} \frac{\mu(n)}{n} \right| \leq 1$

Pf: From Professor David M. Burton

From (a), $\sum_{n=1}^{N-1} \mu(n) \left[\frac{N}{n}\right] + \mu(N) \left[\frac{N}{N}\right] = 1$

But $\left[\frac{N}{N}\right] = 1$, so $\sum_{n=1}^{N-1} \mu(n) \left[\frac{N}{n}\right] + \mu(N) = 1$

Dividing by N,

$$\frac{\mu(N)}{N} = \frac{1}{N} - \frac{1}{N}\sum_{n=1}^{N-1}\mu(n)\left[\frac{N}{n}\right] \qquad [1]$$

Now, $$\sum_{n=1}^{N}\mu(n)/n = \sum_{n=1}^{N-1}\frac{\mu(n)}{n} + \frac{\mu(N)}{N}$$

$$= \frac{1}{N}\sum_{n=1}^{N-1}\mu(n)\frac{N}{n} + \frac{\mu(N)}{N} \qquad [2]$$

Substituting [1] into [2],

$$\sum_{n=1}^{N}\frac{\mu(n)}{n} = \frac{1}{N}\sum_{n=1}^{N-1}\mu(n)\frac{N}{n} + \frac{1}{N} - \frac{1}{N}\sum_{n=1}^{N-1}\mu(n)\left[\frac{N}{n}\right]$$

$$= \frac{1}{N}\sum_{n=1}^{N-1}\mu(n)\left(\frac{N}{n} - \left[\frac{N}{n}\right]\right) + \frac{1}{N}$$

Since $|a+b| \le |a| + |b|$, and $0 \le \left|\frac{N}{n} - \left[\frac{N}{n}\right]\right| < 1$, and $\left|\frac{1}{N}\right| = \frac{1}{N}$, and $|a \cdot b| = |a| \cdot |b|$,

$$\left|\sum_{n=1}^{N}\frac{\mu(n)}{n}\right| \le \frac{1}{N}\sum_{n=1}^{N-1}|\mu(n)|\left|\frac{N}{n} - \left[\frac{N}{n}\right]\right| + \frac{1}{N}$$

$$\le \frac{1}{N}\sum_{n=1}^{N-1}|\mu(n)| + \frac{1}{N} \qquad (\text{and } |\mu(n)| \le 1)$$

$$\le \frac{1}{N}(N-1) + \frac{1}{N} = 1$$

11. Illustrate problem 10 when $N=6$

(a) $\displaystyle\sum_{n=1}^{6} \mu(n)\left[\frac{6}{n}\right] = \mu(1)\left[\frac{6}{1}\right] + \mu(2)\left[\frac{6}{2}\right] +$

$\mu(3)\left[\frac{6}{3}\right] + \mu(4)\left[\frac{6}{4}\right] + \mu(5)\left[\frac{6}{5}\right] + \mu(6)\left[\frac{6}{6}\right]$

$= 1\cdot 6 + (-1)\cdot 3 + (-1)\cdot 2 + 0\cdot 1 + (-1)\cdot 1 + 1\cdot 1$

$= 6 - 3 - 2 + 0 - 1 + 1$

$= 1$

(b) $\displaystyle\left|\sum_{n=1}^{6} \frac{\mu(n)}{n}\right| = \left|\frac{\mu(1)}{1} + \frac{\mu(2)}{2} + \frac{\mu(3)}{3} + \frac{\mu(4)}{4} + \frac{\mu(5)}{5} + \frac{\mu(6)}{6}\right|$

$= \left| 1 + \left(-\frac{1}{2}\right) + \left(-\frac{1}{3}\right) + \frac{0}{4} + \left(\frac{-1}{5}\right) + \frac{1}{6}\right|$

$= \left| 1 + \left(-\frac{5}{6}\right) + \left(-\frac{1}{5}\right) + \frac{1}{6}\right|$

$= \left| 1 + \left(-\frac{2}{3}\right) + \left(-\frac{1}{5}\right)\right|$

$= \left| 1 + \left(-\frac{13}{15}\right)\right| = \left|\frac{2}{15}\right| < 1$

12. Verify that the formula $\sum_{n=1}^{N} \lambda(n)\left[\frac{N}{n}\right] = [\sqrt{N}]$ hold for any positive integer N.

Pf: Let $F(n) = \sum_{d|n} \lambda(d)$ (λ-function defined on page 116, # 7, Sec. 6.2).

$\therefore$ By Th. 6.11, $\sum_{n=1}^{N} F(n) = \sum_{n=1}^{N} \lambda(n)\left[\frac{N}{n}\right]$

By Prob. # 7(b), Sec. 6.2,

$$F(n) = \begin{cases} 1 & \text{if } n = m^2 \text{ for some integer } m \\ 0 & \text{otherwise} \end{cases}$$

$\therefore \sum_{n=1}^{N} F(n)$ keeps track of the number of perfect squares $\leq N$, as $F()$ assigns a value of 1 to each n that can be expressed as a perfect square.

$\therefore \sum_{n=1}^{N} \lambda(n)\left[\frac{N}{n}\right] = \#$ perfect squares $\leq N$.

Now consider $[\sqrt{N}]$ and perfect squares. The perfect squares are $1^2, 2^2, 3^2, \ldots$

$\therefore$ For any $N = m^2$, there are exactly m perfect squares (positive integers) less than or equal to N.

Suppose $\sqrt{N}$ is not an integer.

Let m be the largest integer s.t.

$m^2 < N$

$\therefore N < (m+1)^2$

$\therefore m < \sqrt{N} < m+1$

Since $m = [\sqrt{N}]$, then $[\sqrt{N}]$ is the number of perfect squares $\leq N$.

$$\therefore \sum_{n=1}^{N} \lambda(n) \left[\frac{N}{n}\right] = [\sqrt{N}]$$

13. If N is a positive integer, establish the following:

(a) $N = \sum_{n=1}^{2N} \tau(n) - \sum_{n=1}^{N} [2N/n]$

By Corollary 1 to Th. 6.11, $\sum_{n=1}^{N} \tau(n) = \sum_{n=1}^{N} \left[\frac{N}{n}\right]$

$$\therefore \sum_{n=1}^{2N} \tau(n) = \sum_{n=1}^{2N} \left[\frac{2N}{n}\right]$$

$$\therefore \sum_{n=1}^{2N} \tau(n) - \sum_{n=1}^{N} \left[\frac{2N}{n}\right]$$

$$= \sum_{n=1}^{2N} \left[\frac{2N}{n}\right] - \sum_{n=1}^{N} \left[\frac{2N}{n}\right]$$

$$= \sum_{n=N+1}^{2N} \left[\frac{2N}{n}\right]$$

But $1 \le \frac{2N}{N+k} < 2$ for all $1 \le k \le N$

Pf: $k \le N \Rightarrow N+k \le N+N = 2N$

$\therefore 1 \le \frac{2N}{N+k}$

Also, $1 \le k \Rightarrow 0 < k \Rightarrow 0 < 2k$

$\therefore 2N < 2N + 2k = 2(N+k)$

$\therefore \frac{2N}{N+k} < 2$

$\therefore \left[\frac{2N}{n}\right] = 1$ for all $N+1 \le n \le 2N$

$$\therefore \sum_{n=N+1}^{2N} \left[\frac{2N}{n}\right] = N \cdot 1 = N$$

$$\therefore \sum_{n=1}^{2N} \tau(n) - \sum_{n=1}^{N} \left[\frac{2N}{n}\right] = N$$

(b) $\tau(N) = \sum_{n=1}^{N} \left(\left[\frac{N}{n}\right] - \left[\frac{N-1}{n}\right] \right)$

By Corollary 1 to Th. 6.11,

$$\sum_{n=1}^{N} \left[\frac{N}{n}\right] = \sum_{n=1}^{N} \tau(n)$$

$$\therefore \sum_{n=1}^{N} \left[\frac{N-1}{n}\right] = \sum_{n=1}^{N-1} \left[\frac{N-1}{n}\right] + \left[\frac{N-1}{N}\right]$$

$$= \sum_{n=1}^{N-1} \tau(n) + \left[\frac{N-1}{N}\right]$$

But $\frac{N-1}{N} < 1$ for all $N > 0$. $\therefore \left[\frac{N-1}{N}\right] = 0$

$$\therefore \sum_{n=1}^{N} \left[\frac{N-1}{n}\right] = \sum_{n=1}^{N-1} \tau(n)$$

$$\therefore \sum_{n=1}^{N} \left(\left[\frac{N}{n}\right] - \left[\frac{N-1}{n}\right] \right) = \sum_{n=1}^{N} \tau(n) - \sum_{n=1}^{N-1} \tau(n)$$

$$= \tau(N)$$