7.2 Euler's Phi-Function

Note Title 10/3/2005

1. Calculate $\phi(1001)$, $\phi(5040)$, $\phi(36{,}000)$

$\phi(1001)$: $1001 = 7 \times 11 \times 13$

$$\therefore \phi(1001) = 1001\left(1-\frac{1}{7}\right)\left(1-\frac{1}{11}\right)\left(1-\frac{1}{13}\right)$$

$$= 1001\left(\frac{6}{7}\right)\left(\frac{10}{11}\right)\left(\frac{12}{13}\right)$$

$$= (6)(10)(12) = \underline{720}$$

$\phi(5040)$: $5040 = 2^4 \cdot 3^2 \cdot 5 \cdot 7$

$$\therefore \phi(5040) = 5040\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1-\frac{1}{7}\right)$$

$$= \frac{5040}{210}(2)(4)(6) = \underline{1152}$$

$\phi(36{,}000)$: $36{,}000 = 2^5 \cdot 3^2 \cdot 5^3$

$$\therefore \phi(36{,}000) = 36000\left(\frac{1}{2}\right)\left(\frac{2}{3}\right)\left(\frac{4}{5}\right)$$

$$= 36000\left(\frac{4}{15}\right) = \underline{9600}$$

2. Verify $\phi(n) = \phi(n+1) = \phi(n+2)$ is true for $n = 5186$

$5186 = 2 \cdot 2593 \qquad \phi(5186) = 5186\left(\frac{1}{2}\right)\left(\frac{2592}{2593}\right) = 2592$

$5187 = 3 \cdot 7 \cdot 13 \cdot 19 \qquad \phi(5187) = 5187\left(\frac{2}{3}\right)\left(\frac{6}{7}\right)\left(\frac{12}{13}\right)\left(\frac{18}{19}\right) = 2592$

$5188 = 2^2 \cdot 1297 \qquad \phi(5188) = 5188\left(\frac{1}{2}\right)\left(\frac{1296}{1297}\right) = 2592$

3. Show that $m = 3^k \cdot 568$ and $n = 3^k \cdot 638$, $k \geq 0$ satisfy simultaneously $\tau(m) = \tau(n)$, $\sigma(m) = \sigma(n)$, and $\phi(m) = \phi(n)$

$568 = 2^3 \cdot 71 \qquad 638 = 2 \cdot 11 \cdot 29$

$\therefore \tau(m) = (k+1)(3+1)(1+1) = 6(k+1)$

$\tau(n) = (k+1)(1+1)(1+1)(1+1) = 6(k+1)$

$$\sigma(m) = \frac{(3^{k+1}-1)}{(3-1)} \cdot \frac{(2^4-1)}{(2-1)} \cdot \frac{(71^2-1)}{(71-1)} = \frac{(3^{k+1}-1)(15)(5040)}{(2)(70)}$$

$$= (3^{k+1}-1)(540)$$

$$\sigma(n) = \frac{(3^{k+1}-1)}{(3-1)} \cdot \frac{(2^2-1)}{(2-1)} \cdot \frac{(11^2-1)}{(11-1)} \cdot \frac{(29^2-1)}{(29-1)}$$

$$= \frac{(3^{k+1}-1)}{2} \cdot (3)(12)(30) = (3^{k+1}-1)(540)$$

$$\phi(m) = 3^k \cdot 568 \left(1-\frac{1}{3}\right)\left(1-\frac{1}{2}\right)\left(1-\frac{1}{71}\right)$$

$$= 3^k \cdot 2^3 \cdot 71 \left(\frac{2}{3}\right)\left(\frac{1}{2}\right)\left(\frac{70}{71}\right)$$

$$= 3^{k-1} \cdot 2^3 \cdot 70 = 560 \cdot 3^{k-1}$$

$$\phi(n) = 3^k \cdot 638 \left(1-\frac{1}{3}\right)\left(1-\frac{1}{2}\right)\left(1-\frac{1}{11}\right)\left(1-\frac{1}{29}\right)$$

$$= 3^k \cdot 2 \cdot 11 \cdot 29 \left(\frac{2}{3}\right)\left(\frac{1}{2}\right)\left(\frac{10}{11}\right)\left(\frac{28}{29}\right)$$

$$= 3^{k-1} \cdot 20 \cdot 28 = 560 \cdot 3^{k-1}$$

4. Establish each of the assertions below:

(a) If n is odd, then $\phi(2n) = \phi(n)$

Pf: Let $n = p_1^{k_1} \cdots p_r^{k_r}$. n odd $\Rightarrow p_i \neq 2$

$$\therefore 2n = 2p_1^{k_1} \cdots p_r^{k_r}$$

$$\therefore \phi(2n) = 2n\left(1-\frac{1}{2}\right)\left(1-\frac{1}{p_1}\right)\cdots\left(1-\frac{1}{p_r}\right)$$

$$= n\left(1-\frac{1}{p_1}\right)\cdots\left(1-\frac{1}{p_r}\right)$$

$$= \phi(n)$$

Another proof: n odd $\Rightarrow \gcd(2,n) = 1$
ϕ multiplicative $\Rightarrow \phi(2n) = \phi(2)\phi(n) = \phi(n)$

(b) If n is even, $\phi(2n) = 2\phi(n)$

Pf: n even $\Rightarrow n = 2^{k} p_2^{k_2} \cdots p_r^{k_r}$

$\therefore 2n = 2^{k+1} p_2^{k_2} \cdots p_r^{k_r}$

$\therefore \phi(2n) = 2n\left(1-\frac{1}{2}\right)\left(1-\frac{1}{p_2}\right)\cdots\left(1-\frac{1}{p_r}\right)$

$= n\left(1-\frac{1}{p_2}\right)\cdots\left(1-\frac{1}{p_r}\right)$

$2\phi(n) = 2\cdot n\left(1-\frac{1}{2}\right)\left(1-\frac{1}{p_2}\right)\cdots\left(1-\frac{1}{p_r}\right)$

$= n\left(1-\frac{1}{p_2}\right)\cdots\left(1-\frac{1}{p_r}\right)$

$\therefore \phi(2n) = 2\phi(n)$

(c) $\phi(3n) = 3\phi(n) \Leftrightarrow 3 \mid n$

Let $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$

(1) $3 \mid n \Rightarrow$ one of $p_i = 3$. $\therefore$ Let $n = 3^k q$, where $\gcd(3,q) = 1$

$\therefore\ 3\phi(n) = 3\phi(3^k q) = 3\phi(3^k)\phi(q)$

$= 3\cdot 3^k\left(\left(1-\frac{1}{3}\right)\phi(q)\right) = 2\cdot 3^k\cdot\phi(q)$

$\phi(3n) = \phi(3^{k+1} q) = \phi(3^{k+1})\phi(q)$

$= 3^{k+1}\left(1-\frac{1}{3}\right)\phi(q) = 2\cdot 3^k\cdot\phi(q)$

$\therefore\ \phi(3n) = 3\phi(n)$

(2) Suppose $\phi(3n) = 3\phi(n)$

If $3\nmid n$, Then for $n = p_1^{k_1}\cdots p_r^{k_r}$, $p_i \neq 3$

$\therefore\ \gcd(3,n) = 1 \Rightarrow \phi(3n) = \phi(3)\phi(n)$
$= 2\phi(n)$

This contradicts $\phi(3n) = 3\phi(n)$.

$\therefore\ 3\mid n$

(d) $\phi(3n) = 2\phi(n) \Leftrightarrow 3\nmid n$

(1) As in (c)(2) above, $3\nmid n \Rightarrow \phi(3n) = 2\phi(n)$

(2) Suppose $\phi(3n) = 2\phi(n)$

From (c) above, if $3 | n$, then

$\phi(3n) = 3\phi(n)$. $\therefore 3 \nmid n$

(e) $\phi(n) = n/2 \Leftrightarrow n = 2^k$ for some $k \geq 1$

(1) If $n = 2^k$, then $\phi(n) = \phi(2^k) = 2^k(1-\frac{1}{2})$

$= 2^k(\frac{1}{2}) = n/2$

(2) If $\phi(n) = n/2$, then for $n/2$ to be an integer, n must be even.

$\therefore$ Let $n = 2^k p_2^{k_2} \cdots p_r^{k_r}$, and assume $k_i \neq 0$

Let $q = p_2^{k_2} \cdots p_r^{k_r}$, so $q > 1$ and $\gcd(2^k, q) = 1$

$\therefore \phi(n) = \phi(2^k q) = \phi(2^k)\phi(q)$

$= 2^k(1-\frac{1}{2})\phi(q) = 2^{k-1}\phi(q)$

$\therefore \phi(n) = n/2 = 2^{k-1}\phi(q)$, $n = 2^k \phi(q)$

$$\therefore p_2^{k_2} \cdots p_r^{k_r} = \phi(q) = \phi(p_2^{k_2} \cdots p_r^{k_r})$$
$$= p_2^{k_2} \cdots p_r^{k_r} \left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_r}\right)$$

$$\therefore p_2 \cdots p_r = (p_2 - 1) \cdots (p_r - 1)$$

$\therefore$ for each p_i, $p_i = (p_j - 1)$ for some j.

This is impossible if $k_i \neq 0$. $\therefore k_i = 0$,

$\therefore$ for $n = 2^k p_2^{k_2} \cdots p_r^{k_r} = 2^k$

5. Prove $\phi(n) = \phi(n+2)$ is satified by $n = 2(2p-1)$ whenever p and $2p-1$ are both odd primes.

Pf: $2p-1$ an odd prime $\Rightarrow \gcd(2, 2p-1) = 1$.

$$\therefore \phi(n) = \phi(2)\,\phi(2p-1) = (2p-1)\left(1 - \frac{1}{2p-1}\right)$$
$$= 2p-2$$

$n+2 = 2(2p-1) + 2 = 4p$, and p odd prime

$\Rightarrow \gcd(4, p) = 1$.

$$\therefore \phi(n+2) = \phi(4)\phi(p) = 2\cdot p\left(1-\frac{1}{p}\right)$$

$$= 2p-2$$

$$\therefore \phi(n) = \phi(n+2)$$

6. Show there are infinitely many integers for which $\phi(n)$ is a perfect square.

Pf: for $k \geq 1$, $\phi(2^k) = 2^k\left(1-\frac{1}{2}\right) = 2^{k-1}$

If k is odd, then $k-1$ is even.
Let $k = 2m+1$, some m

$$\therefore \phi(2^k) = \phi\left(2^{2m+1}\right) = 2^{2m+1-1} = 2^{2m} = (2^m)^2$$

$(2^m)^2$ is a perfect square.

There are infinitely many odd integers,
$\therefore$ infinitely many $n = 2^k$, k odd,
and $\phi(n)$ is a perfect square.

7. Verify the following.

(a) For any positive integer n, $\frac{1}{2}\sqrt{n} \leq \phi(n) \leq n$.

By def., $\phi(n) \le n$

(1) If $n=1$, $\frac{1}{2}\sqrt{1} = \frac{1}{2}$, $\phi(1)=1$, so $\frac{1}{2}\sqrt{n} < \phi(1)$

(2) If $n=2$, Then $\phi(2)=1$, and $\therefore \frac{1}{2}\sqrt{2} < \phi(2)$

If $n=2^k$, for $k>1$, Then $\phi(2^k) = 2^{k-1}$

$\frac{1}{2}\sqrt{2^k} = 2^{-1}\cdot 2^{\frac{k}{2}} = 2^{\frac{k}{2}-1} < 2^{k-1}$, as $\frac{k}{2}<k$

$\therefore \frac{1}{2}\sqrt{n} < \phi(n)$

(3) If $n=p^k$, $p>2$, $k\ge 1$, Then $\phi(n) = p^{k-1}(p-1)$ by Th. 7.1.

But for $p>2$, $p^2 \ge 3p$, so $p^2+1 > 3p$, so

$p^2-2p+1 > p$, $\therefore (p-1)^2 > p$, $p-1 > \sqrt{p}$

$\therefore p^{k-1}(p-1) > p^{k-1}\sqrt{p} \ge p^{\frac{k-1}{2}}\cdot p^{\frac{1}{2}} = p^{\frac{k}{2}}$

$\therefore \phi(p^k) > p^{\frac{k}{2}}$

$\therefore \phi(n) > \sqrt{n}$ if $n=p^k$ and $p\ge 3$

(4) ϕ is multiplicative. Let $n = 2^k p_1^{k_1}\cdots p_r^{k_r}$, $k\ge 0$, $k_i \ge 0$

$$\therefore \phi(n) = \phi(2^k)\phi(p_1^{k_1})\phi(p_2^{k_2})\cdots\phi(p_r^{k_r})$$

$$> \left(\tfrac{1}{2}\sqrt{2^k}\right)\left(\sqrt{p_1^{k_1}}\right)\left(\sqrt{p_2^{k_2}}\right)\cdots\left(\sqrt{p_r^{k_r}}\right) \quad \text{by } (2), (3)$$

$$= \tfrac{1}{2}\sqrt{2^k p_1^{k_1}\cdots p_r^{k_r}}$$

$$= \tfrac{1}{2}\sqrt{n}$$

$$\therefore \tfrac{1}{2}\sqrt{n} < \phi(n)$$

(3) If $n > 1$ has r distinct prime factors, Then $\phi(n) \geq n/2^r$

Let $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$

$$\therefore \phi(n) = n\left(1 - \frac{1}{p_1}\right)\cdots\left(1 - \frac{1}{p_r}\right)$$

But $p_i \geq 2$, so $\frac{1}{2} \geq \frac{1}{p_i}$, $-\frac{1}{p_i} \geq -\frac{1}{2}$,

$$\therefore 1 - \frac{1}{p_i} \geq 1 - \frac{1}{2} = \frac{1}{2}$$

$$\therefore \left(1 - \frac{1}{p_1}\right)\cdots\left(1 - \frac{1}{p_r}\right) \geq \left(\frac{1}{2}\right)\cdots\left(\frac{1}{2}\right) = \frac{1}{2^r}$$

$$\therefore \phi(n) \geq n \cdot \frac{1}{2^r} = n/2^r$$

(c) If $n>1$ is composite, then $\phi(n) \le n-\sqrt{n}$

Let $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$, $p_1 < p_2 < \cdots < p_r$, $k_i \ge 1$.

$= p_1(b)$, and $p_1 \le b \Rightarrow p_1^2 \le p_1 b \Rightarrow p_1 \le \sqrt{n}$

$\therefore \frac{1}{\sqrt{n}} \le \frac{1}{p_1}$, or $\frac{\sqrt{n}}{n} \le \frac{1}{p_1}$, so $\sqrt{n} \le \frac{n}{p_1}$

$\therefore -\frac{n}{p_1} \le -\sqrt{n}$

$\phi(n) = n\left(1-\frac{1}{p_1}\right)\cdots\left(1-\frac{1}{p_r}\right)$

$\le n\left(1-\frac{1}{p_1}\right)$, since $1-\frac{1}{p_i} < 1$

$= n - \frac{n}{p_1} \le n - \sqrt{n}$

8. Prove if n has r distinct odd prime factors, then $2^r \mid \phi(n)$

Pf: Let $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$, $p_i > 2$

$\therefore \phi(n) = p_1^{k_1-1}(p_1-1)\,p_2^{k_2-1}(p_2-1)\cdots p_r^{k_r-1}(p_r-1)$

As each p_i is odd, let $p_i = 2s_i + 1$, some s_i.

$$\therefore \phi(n) = p_1^{k_1-1} p_2^{k_2-1} \cdots p_r^{k_r-1} (2s_1)(2s_2)\cdots(2s_r)$$
$$= 2^r p_1^{k_1-1} p_2^{k_2-1} \cdots p_r^{k_r-1} s_1 s_2 \cdots s_r$$
$$\therefore 2^r \mid \phi(n)$$

9. Prove The following:

(a) If n and $n+2$ are twin primes, Then $\phi(n+2) = \phi(n) + 2$

Pf: For any prime p, $\phi(p) = p-1$.

$$\therefore \phi(n+2) = (n+2) - 1 = n+1$$
$$\phi(n) = n-1$$

$$\therefore \phi(n) + 2 = n-1+2 = n+1 = \phi(n+2)$$

(b) If p and $2p+1$ are both odd primes, Then $n = 4p$ satifies $\phi(n+2) = \phi(n) + 2$

Pf: Since p is odd, Then $\gcd(4,p) = 1$, so $\phi(n) = \phi(4p) = \phi(4)\cdot\phi(p) = 2\cdot(p-1)$

$$\therefore \phi(n) + 2 = 2\cdot(p-1) + 2 = 2p$$

Since $2p+1$ is prime, $\phi(2p+1) = (2p+1)-1$
$= 2p$

$\therefore \phi(n)+2 = \phi(n+2)$ for $n = 4p$

10. If every prime that divides n also divides m, establish that $\phi(n \cdot m) = n\,\phi(m)$.

Pf: Let $p_1, p_2, \dots, p_r$ be all the primes of n that divide m.

Let $n = p_1^{k_1} \cdots p_r^{k_r}$

$m = p_1^{j_1} \cdots p_r^{j_r} q_1^{m_1} \cdots q_s^{m_s}$, q_i prime so that $q_i \neq p_j$.

$$\therefore nm = p_1^{k_1+j_1} \cdots p_r^{k_r+j_r} q_1^{m_1} \cdots q_s^{m_s}$$

$$\phi(nm) = p_1^{k_1+j_1} \cdots p_r^{k_r+j_r} q_1^{m_1} \cdots q_s^{m_s} \left(1-\frac{1}{p_1}\right)\cdots\left(1-\frac{1}{p_r}\right)\left(1-\frac{1}{q_1}\right)\cdots\left(1-\frac{1}{q_s}\right)$$

$$= p_1^{j_1} \cdots p_r^{j_r}\left(1-\frac{1}{p_1}\right)\cdots\left(1-\frac{1}{p_r}\right)\left(1-\frac{1}{q_1}\right)\cdots\left(1-\frac{1}{q_s}\right) p_1^{k_1} \cdots p_r^{k_r}$$

$$= \phi(m) \cdot p_1^{k_1} \cdots p_r^{k_r} = \phi(m) \cdot n$$

11. (a) If $\phi(n) \mid n-1$, prove n is square-free.

Pf: Let $n = p_1^{k_1} \cdots p_r^{k_r}$, and assume n is not square-free, so that $k_i \geq 2$ for some i.

$$\phi(n) = (p_1^{k_1} - p_1^{k_1-1}) \cdots (p_i^{k_i} - p_i^{k_i-1}) \cdots (p_r^{k_r} - p_r^{k_r-1})$$

Since $k_i \geq 2$, $k_i - 1 \geq 1$, so $p_i \mid p_i^{k_i-1}(p_i - 1)$

$\therefore p_i \mid (p_i^{k_i} - p_i^{k_i-1}) \Rightarrow p_i \mid \phi(n)$

By assumption, $\phi(n) \mid n-1$, so that $p_i \mid n-1$. Clearly $p_i \mid n$,

$\therefore p_i \mid n-(n-1) \Rightarrow p_i \mid 1$, a contradiction.

$\therefore k_i = 1$ for all $i \Rightarrow n$ is square-free.

(b) Show that if $n = 2^k$ or $2^k 3^j$, k, j positive, then $\phi(n) \mid n$

Pf: If $n = 2^k$, $\phi(n) = 2^{k-1}$, and $k-1 \geq 0$ since $k > 0$. $\therefore \phi(n) \mid n$

If $n = 2^k 3^j$, Then $\phi(n) = 2^k 3^j (1-\frac{1}{2})(1-\frac{1}{3})$

$= 2^k 3^j (\frac{1}{2})(\frac{2}{3}) = 2^k 3^{j-1}$. Since $j > 0$,

$j-1 \geq 0$. $\therefore$ $\phi(n) \mid n$.

12. If $n = p_1^{k_1} \cdots p_r^{k_r}$, derive The following inequalities:

(a) $\sigma(n)\phi(n) \geq n^2 (1-\frac{1}{p_1^2}) \cdots (1-\frac{1}{p_r^2})$

Pf: $\sigma(n) = \dfrac{p_1^{k_1+1}-1}{p_1-1} \cdots \dfrac{p_r^{k_r+1}-1}{p_r-1}$

$\phi(n) = p_1^{k_1-1}(p_1-1) \cdots p_r^{k_r-1}(p_r-1)$

$\therefore$ $\sigma(n)\phi(n) = (p_1^{k_1+1}-1)p_1^{k_1-1} \cdots (p_r^{k_r+1}-1)p_r^{k_r-1}$

$= (p_1^{2k_1} - p_1^{k_1-1}) \cdots (p_r^{2k_r} - p_r^{k_r-1})$

But $p_i^{2k_i} - p_i^{k_i-1} = p_i^{2k_i}\left(1 - \dfrac{p_i^{k_i-1}}{p_i^{2k_i}}\right)$

$= (p_i^{k_i})^2 \left(1 - \dfrac{1}{p_i^{k_i+1}}\right)$

For $k_i \geq 1$, $p_i^{k_i+1} \geq p_i^2$, so $\frac{1}{p_i^2} \geq \frac{1}{p_i^{k_i+1}}$

$\therefore -\frac{1}{p_i^{k_i+1}} \geq -\frac{1}{p_i^2}$, so $1-\frac{1}{p_i^{k_i+1}} \geq 1-\frac{1}{p_i^2}$

$\therefore p_i^{2k_i} - p_i^{k_i-1} \geq (p_i^{k_i})^2 \left(1-\frac{1}{p_i^2}\right)$

$$\therefore \sigma(n)\phi(n) \geq (p_1^{k_1})^2\left(1-\frac{1}{p_1^2}\right)\cdots(p_r^{k_r})^2\left(1-\frac{1}{p_r^2}\right)$$

$$= (p_1^{k_1}\cdots p_r^{k_r})^2\left(1-\frac{1}{p_1^2}\right)\cdots\left(1-\frac{1}{p_r^2}\right)$$

$$= n^2\left(1-\frac{1}{p_1^2}\right)\cdots\left(1-\frac{1}{p_r^2}\right)$$

$$\therefore \sigma(n)\phi(n) \geq n^2\left(1-\frac{1}{p_1^2}\right)\cdots\left(1-\frac{1}{p_r^2}\right)$$

(b) $\tau(n)\phi(n) \geq n$

Pf: If $2 \leq p$, then $\frac{1}{p} \leq \frac{1}{2}$, $-\frac{1}{2} \leq -\frac{1}{p}$, $\frac{1}{2} \leq 1-\frac{1}{p}$

If $1 \leq k$, then $2 \leq k+1$, so $2 \cdot \frac{1}{2} \leq (k+1)\left(1-\frac{1}{p}\right)$, or $1 \leq (k+1)\left(1-\frac{1}{p}\right)$

$\therefore$ Let $n = p_1^{k_1}\cdots p_r^{k_r}$, $k_i \geq 1$

$$\therefore \tau(n)\phi(n) = (k_1+1)\cdots(k_r+1)\cdot n \cdot \left(1-\frac{1}{p_1}\right)\cdots\left(1-\frac{1}{p_r}\right)$$

$$= n \cdot \left(1-\frac{1}{p_1}\right)(k_1+1) \cdots \left(1-\frac{1}{p_r}\right)(k_r+1)$$

$$\geq n \cdot 1 \cdots 1 = n$$

$$\therefore \tau(n)\phi(n) \geq n$$

13. Assuming that $d \mid n$, prove $\phi(d) \mid \phi(n)$

Pf: Let $n = p_1^{k_1} \cdots p_r^{k_r}$. Then, by Th. 6.1,

$d = p_1^{a_1} \cdots p_r^{a_r}$, where $0 \leq a_i \leq k_i$

Let $d = q_1^{b_1} \cdots q_s^{b_s}$, where $q_i \in \{p_1, \ldots, p_r\}$

and $1 \leq b_i$

$$\therefore \phi(d) = d\left(1-\frac{1}{q_1}\right) \cdots \left(1-\frac{1}{q_s}\right)$$

Since each $q_i = p_j$, some j s.t. $1 \leq j \leq r$,

Then $1-\frac{1}{q_i} = 1-\frac{1}{p_j}$. Name this p_j

p_{j_i}. $\therefore 1-\frac{1}{q_i} = 1-\frac{1}{p_{j_i}}$

$$\therefore \phi(d) = d\left(1-\frac{1}{p_{j_1}}\right) \cdots \left(1-\frac{1}{p_{j_s}}\right)$$

As each $p_{j_i} \in \{p_1, \dots, p_r\}$, Then

$$\left(1-\frac{1}{p_{j_1}}\right)\cdots\left(1-\frac{1}{p_{j_s}}\right) \mid \left(1-\frac{1}{p_1}\right)\cdots\left(1-\frac{1}{p_r}\right)$$

Since $d \mid n$, Then

$$d\left(1-\frac{1}{p_{j_1}}\right)\cdots\left(1-\frac{1}{p_{j_s}}\right) \mid n\left(1-\frac{1}{p_1}\right)\cdots\left(1-\frac{1}{p_r}\right)$$

$\therefore \phi(d) \mid \phi(n)$

14. Obtain The following two generalizations of Th. 7.2:

(a) For positive integers m and n, $d = \gcd(m,n)$,

$$\phi(m)\phi(n) = \phi(mn)\frac{\phi(d)}{d}$$

Pf: (1) If $m=1$ or $n=1$, Then $d=1$, $\phi(d)=1$.
and clearly $\phi(m)\phi(n) = \phi(mn)\frac{\phi(d)}{d}$

(2) $\therefore$ Assume both $m, n > 1$.

If $m = n$, Then $m = \gcd(m,n)$

Let $m = n = p_1^{k_1} \cdots p_r^{k_r}$, $mn = p_1^{2k_1} \cdots p_r^{2k_r}$

$$\therefore \phi(mn)\frac{\phi(d)}{d} = mn\left(1-\frac{1}{p_1}\right)\cdots\left(1-\frac{1}{p_r}\right)\frac{m\left(1-\frac{1}{p_1}\right)\cdots\left(1-\frac{1}{p_r}\right)}{m}$$

$$= mn\left(1-\frac{1}{p_1}\right)^2\cdots\left(1-\frac{1}{p_r}\right)^2$$

$$= m\left(1-\frac{1}{p_1}\right)\cdots\left(1-\frac{1}{p_r}\right) n\left(1-\frac{1}{p_1}\right)\cdots\left(1-\frac{1}{p_r}\right)$$

$$= \phi(m)\,\phi(n)$$

(3) Now assume $m \neq n$, $m, n > 1$.

If $\gcd(m,n) = 1$, Then $d = 1$, $\phi(d) = 1$, and $\phi(m)\phi(n) = \phi(mn)\frac{\phi(d)}{d}$ by Th. 7.2.

(4) Assume $m \neq n$, $m, n > 1$, $\gcd(m,n) > 1$.

Let $d = p_1^{k_1}\cdots p_r^{k_n}$, $k_i \geq 1$.

$m = p_1^{a_1}\cdots p_r^{a_r} q_1^{u_1}\cdots q_s^{u_s}$, $a_i \geq k_i$, $u_i \geq 0$

$n = p_1^{b_1}\cdots p_r^{b_r} w_1^{v_1}\cdots w_t^{v_t}$, $b_i \geq k_i$, $v_i \geq 0$

where p_i, q_i, w_i are prime, and

$q_i \neq w_j$, $p_i \neq q_j$, $p_i \neq w_j$, for any i, j.

$$\therefore mn = p_1^{a_1+b_1} \cdots p_r^{a_r+b_r} q_1^{u_1} \cdots q_s^{u_s} w_1^{v_1} \cdots w_t^{v_t}$$

$$\therefore \phi(mn)\frac{\phi(d)}{d} = \left[p_1^{a_1+b_1} \cdots p_r^{a_r+b_r} q_1^{u_1} \cdots q_s^{u_s} w_1^{v_1} \cdots w_t^{v_t}\right] \cdot$$

$$\left[\left(1-\frac{1}{p_1}\right)\cdots\left(1-\frac{1}{p_r}\right)\left(1-\frac{1}{q_1}\right)\cdots\left(1-\frac{1}{q_s}\right)\left(1-\frac{1}{w_1}\right)\cdots\left(1-\frac{1}{w_t}\right)\right] \cdot$$

$$\left[\frac{d}{d}\left(1-\frac{1}{p_1}\right)\cdots\left(1-\frac{1}{p_r}\right)\right]$$

$$= \left[p_1^{a_1} \cdots p_r^{a_r} p_1^{b_1} \cdots p_r^{b_r} q_1^{u_1} \cdots q_s^{u_s} w_1^{v_1} \cdots w_t^{v_t}\right] \cdot$$

$$\left[\left(1-\frac{1}{p_1}\right)^2\cdots\left(1-\frac{1}{p_r}\right)^2\left(1-\frac{1}{q_1}\right)\cdots\left(1-\frac{1}{q_s}\right)\left(1-\frac{1}{w_1}\right)\cdots\left(1-\frac{1}{w_t}\right)\right]$$

$$= \left[p_1^{a_1} \cdots p_r^{a_r}\left(1-\frac{1}{p_1}\right)\cdots\left(1-\frac{1}{p_r}\right)\left(1-\frac{1}{q_1}\right)\cdots\left(1-\frac{1}{q_s}\right)\right] \cdot$$

$$\left[p_1^{b_1} \cdots p_r^{b_r}\left(1-\frac{1}{p_1}\right)\cdots\left(1-\frac{1}{p_r}\right)\left(1-\frac{1}{w_1}\right)\cdots\left(1-\frac{1}{w_t}\right)\right]$$

$$= \phi(m)\phi(n)$$

Note if any $u_i = 0$ or $v_j = 0$, some i, some j, Then The corresponding term $\left(1-\frac{1}{u_i}\right)$ or $\left(1-\frac{1}{v_j}\right)$ is not present.

(b) For positive integers m, n,

$$\phi(m)\phi(n) = \phi(\gcd(m,n))\,\phi(\operatorname{lcm}(m,n))$$

Pf: If $m=1$, then $\gcd(m,n)=1$ and $\operatorname{lcm}(m,n)=n$

$\therefore$ Clearly $\phi(m)\phi(n) = \phi(\gcd(m,n))\cdot\phi(\operatorname{lcm}(m,n))$

Similar reasoning applies for $n=1$.

If $\gcd(m,n)=1$, then $\operatorname{lcm}(m,n)=mn$, and so the relation holds.

If $m=n$, $\gcd(m,n)=m=\operatorname{lcm}(m,n)$, so the relation holds.

If $\gcd(m,n)=m$, then $\operatorname{lcm}(m,n)=n$, so the relation holds, and similarly for $\gcd(m,n)=n$.

$\therefore$ Assume $m\neq n$, $\gcd(m,n)>1$, and $\gcd(m,n)\neq m$ or n.

$\therefore$ Let $d=\gcd(m,n)=p_1^{k_1}\cdots p_r^{k_r}$, $k_i\geq 1$

$$m = p_1^{a_1}\cdots p_r^{a_r}\,q_1^{u_1}\cdots q_s^{u_s},\quad a_i\geq k_i,\ u_i\geq 0$$

$n = p_1^{b_1} \cdots p_r^{b_r} w_1^{v_1} \cdots w_t^{v_t}$, $b_i \geq k_i$, $v_i \geq 0$

with p_i, q_i, w_i prime,

$p_i \neq q_j$, $1 \leq i \leq r$, $1 \leq j \leq s$

$p_i \neq w_j$, $1 \leq i \leq r$, $1 \leq j \leq t$

$q_i \neq w_j$, $1 \leq i \leq r$, $1 \leq j \leq t$

and $u_i \neq 0$ some i, $v_j \neq 0$ some j since $\gcd(m,n) \neq m$ or n.

Since $mn = \gcd(m,n) \cdot \operatorname{lcm}(m,n)$,

$$\operatorname{lcm}(m,n) = mn / \gcd(m,n)$$

$$= p_1^{a_1+b_1-k_1} \cdots p_r^{a_r+b_r-k_r} q_1^{u_1} \cdots q_s^{u_s} w_1^{v_1} \cdots w_t^{v_t},$$

where $a_i + b_i - k_i \geq 0$ since $a_i \geq k_i$, $b_i \geq k_i$

$$\therefore \phi(\gcd(m,n)) \cdot \phi(\operatorname{lcm}(m,n)) = \left[p_1^{k_1} \cdots p_r^{k_1}\left(1-\frac{1}{p_1}\right) \cdots \left(1-\frac{1}{p_r}\right)\right] \cdot$$

$$\left[p_1^{a_1+b_1-k_1} \cdots p_r^{a_r+b_r-k_r} q_1^{u_1} \cdots q_s^{u_s} w_1^{v_1} \cdots w_t^{v_t}\right] \cdot$$

$$\left[\left(1-\frac{1}{p_1}\right)\cdots\left(1-\frac{1}{p_r}\right)\left(1-\frac{1}{q_1}\right)\cdots\left(1-\frac{1}{q_s}\right)\left(1-\frac{1}{w_1}\right)\left(1-\frac{1}{w_t}\right)\right]$$

$$= \left[p_1^{a_1+b_1}\cdots p_r^{a_r+b_r}\, q_1^{u_1}\cdots q_s^{u_s}\, w_1^{v_1}\cdots w_t^{v_t}\right]\cdot$$

$$\left[\left(1-\frac{1}{p_1}\right)^2\cdots\left(1-\frac{1}{p_r}\right)^2\left(1-\frac{1}{q_1}\right)\cdots\left(1-\frac{1}{q_s}\right)\left(1-\frac{1}{w_1}\right)\cdots\left(1-\frac{1}{w_t}\right)\right]$$

$$= \left[p_1^{a_1}\cdots p_r^{a_r}\, q_1^{u_1}\cdots q_s^{u_s}\left(1-\frac{1}{p_1}\right)\cdots\left(1-\frac{1}{p_r}\right)\left(1-\frac{1}{q_1}\right)\cdots\left(1-\frac{1}{q_s}\right)\right]\cdot$$

$$\left[p_1^{b_1}\cdots p_r^{b_r}\, w_1^{v_1}\cdots w_s^{v_s}\left(1-\frac{1}{p_1}\right)\cdots\left(1-\frac{1}{p_r}\right)\left(1-\frac{1}{w_1}\right)\cdots\left(1-\frac{1}{w_t}\right)\right]$$

$$\phi(m)\cdot\phi(n)$$

15. Prove the following:

(a) There are infinitely many n for which $\phi(n) = n/3$.

Pf: consider $n = 2^i 3^j$, $i, j \geq 1$

$$\therefore \phi(n) = n\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right) = n\left(\frac{1}{2}\right)\left(\frac{2}{3}\right) = n/3$$

(b) There are no integers n for which $\phi(n) = n/4$.

Pf: $\phi(1) = 1$, $\phi(2) = 1$, $\phi(3) = 2$, $\phi(4) = 2$.
So for $n = 1, 2, 3, 4$, $\phi(n) \neq n/4$.

Assume $n > 4$ and $\phi(n) = n/4$.

Let $n = p_1^{k_1} \cdots p_r^{k_r}$, $k_i \geq 1$

$\therefore \phi(n) = n\left(1 - \frac{1}{p_1}\right) \cdots \left(1 - \frac{1}{p_r}\right) = n/4$

$\therefore \frac{(p_1 - 1) \cdots (p_r - 1)}{p_1 \cdots p_r} = \frac{1}{4}$, or

$4(p_1 - 1) \cdots (p_r - 1) = p_1 \cdots p_r$

If $p_1 = 2$, then $2(p_2 - 1) \cdots (p_r - 1) = p_2 \cdots p_r$

But $p_2 \cdots p_r$ is odd since $p_2, \ldots, p_r$ are all odd.

And $2(p_2 - 1) \cdots (p_r - 1)$ is even.

$\therefore$ Can't work for $p_1 = 2$.

And if all $p_1, \ldots, p_r$ are odd, so is $p_1 \cdots p_r$ and $4(p_1 - 1) \cdots (p_r - 1)$ is even.

$\therefore$ No such n exists.

16. Show that the Goldbach conjecture implies that for each even integer $2n$ there exists integers n_1 and n_2 with $\phi(n_1) + \phi(n_2) = 2n$

Pf: Goldbach conjective says for any even integer greater than 4, there are two odd primes, n_1 and n_2, s.t. $n_1 + n_2 =$ the even integer.

Let $2n+2$ be such an even integer, so that $n_1 + n_2 = 2n+2$, $n_1, n_2 =$ odd primes.

If n_1 and n_2 are both prime, then $\phi(n_1) = n_1 - 1$, $\phi(n_2) = n_2 - 1$

$$\therefore \phi(n_1) + \phi(n_2) = n_1 + n_2 - 2 = 2n + 2 - 2 = 2n$$

And if $2n = 4$, choose $n_1 = n_2 = 4$, so that $\phi(4) + \phi(4) = 2 + 2 = 4 = 2n$

If $2n = 2$, choose $n_1 = n_2 = 1$, so that $\phi(1) + \phi(1) = 2$

17. Given a positive integer K, show:

(a) There are at most a finite number of integers n for which $\phi(n) = K$.

Pf: Need to find an integer, z, s.t. whenever $n \geq z$, $\phi(n) > k$. Thus, there are at most a finite # of integers, $1, 2, \ldots, z-1$, for which $\phi(n)$ may be equal to k.

By problem 7(a) above, it was proved that $\phi(n) > \frac{1}{2}\sqrt{n}$, for all n.

$\therefore$ choose $z = 4k^2$. $\therefore \phi(z) > \frac{1}{2}\sqrt{4k^2} = k$

$\therefore$ For all integers $n > 4k^2$, $\phi(n) > k$

$\therefore$ There are at most $n = 4k^2$ integers for which $\phi(n)$ may be k.

Note: it was important in 7(a) to prove $\phi(n) > \frac{1}{2}\sqrt{n}$, not just $\phi(n) \geq \frac{1}{2}\sqrt{n}$
"$\geq$" does not give proper bound.

(b) If the equation $\phi(n) = k$ has a unique solution, say $n = n_0$, then $4 \mid n_0$.

Pf: Suppose $\phi(n_0) = k$, and n_0 is unique

If n_0 is odd, then by problem 4(a),

$\phi(2n_0) = \phi(n_0) = k$, so that n_0 is not unique.

$\therefore$ n_0 is even, so $n_0 = 2r$, some r.

If r is odd, then $\gcd(2, r) = 1$, so $\phi(n_0) = \phi(2r) = \phi(2)\phi(r) = \phi(r) = k$, so, again, uniqueness of $n_0 \Rightarrow r$ is even $\Rightarrow r = 2s$, some s.

$\therefore$ $n_0 = 2(2s) = 4s \Rightarrow 4 \mid n_0$.

18. Find all solutions $\phi(n) = 16$ and $\phi(n) = 24$

Note: if $n = p_1^{k_1} \cdots p_r^{k_r}$, then $\phi(n) = (p_1^{k_1} - p_1^{k_1-1}) \cdots (p_r^{k_r} - p_r^{k_r-1})$

Since $(p_i - 1) \mid p_i^{k_i} - p_i^{k_i-1}$, then $(p_i - 1) \mid \phi(n)$,

(a) For $\phi(n) = 16$, $(p_i - 1) \mid 2^4$

$\therefore$ $p_i - 1 = 1, 2, 4, 8,$ or 16

$\therefore$ $p_i = 2, 3, 5, 9,$ or 17, and 9 not prime, so $p_i = 2, 3, 5,$ or 17

$\therefore$ $n = 2^{k_1} 3^{k_2} 5^{k_3} 17^{k_4}$

$$16 = (2^{k_1} - 2^{k_1-1})(3^{k_2} - 3^{k_2-1})(5^{k_3} - 5^{k_3-1})(17^{k_4} - 17^{k_4-1})$$
$$= 2^{k_1-1} \cdot (3^{k_2-1} \cdot 2) \cdot (5^{k_3-1} \cdot 4) \cdot (17^{k_4-1} \cdot 16)$$

16 clearly has an upper bound effect.

$\therefore k_4 \le 1,\ k_3 \le 1,\ k_2 \le 3,\ k_1 \le 5$

If $k_4 = 1$, $17^{k_4-1} \cdot 16 = 16$

$\therefore k_2 = k_3 = 0$. $k_1 = 0 \quad n = 17$

or $k_1 = 1 \quad n = 34$

$\therefore$ Consider cases for $k_4 = 0$.

$\therefore 16 = 2^{k_1-1} \cdot (3^{k_2-1} \cdot 2)(5^{k_3-1} \cdot 4)$

If $k_3 = 1$, Then $k_2 = 1$, $k_1 = 2$, $n = 60$

or $k_2 = 0$, $k_1 = 3 \quad n = 40$

If $k_3 = 0$, Then $k_2 = 1$, $k_1 = 4 \quad n = 48$

or $k_2 = 0$, $k_1 = 5 \quad n = 32$

$\therefore$ For $\phi(n) = 16$, $n = 17, 34, 40, 60, 32, 48$

(b) For $\phi(n) = 24$, $(p_i - 1) \mid 2^3 \cdot 3$

$\therefore (p_i - 1) \mid 2^3$ or $(p_i - 1) \mid 3$

$\therefore p_i - 1 = 1, 2, 4, 8$ or $p_i - 1 = 3, 6, 12, 24$

$\therefore p_i = 2, 3, 5$ or $p_i = 7, 13$

$\therefore n = 2^{k_1} 3^{k_2} 5^{k_3} 7^{k_4} 13^{k_5}$

$\therefore 24 = (2^{k_1 - 1})(3^{k_2 - 1} \cdot 2)(5^{k_3 - 1} \cdot 4)(7^{k_4 - 1} \cdot 6)(13^{k_5 - 1} \cdot 12)$

$\therefore k_5 \leq 1, k_4 \leq 1, k_3 \leq 2, k_2 \leq 3, k_1 \leq 5$

For $k_5 = 1$, $k_3 = 0$, $k_4 = 0$. $k_2 = 1, k_1 = 0$ $n = 39$
$k_2 = 1, k_1 = 1$ $n = 78$
$k_2 = 0, k_1 = 2$ $n = 52$

$\therefore$ Now assume $k_5 = 0$ $(n = 2^{k_1} 3^{k_2} 5^{k_3} 7^{k_4})$

$\therefore 24 = (2^{k_1 - 1})(3^{k_2 - 1} \cdot 2)(5^{k_3 - 1} \cdot 4)(7^{k_4 - 1} \cdot 6)$

If $k_4 = 1$, $4 = (2^{k_1 - 1})(3^{k_2 - 1} \cdot 2)(5^{k_3 - 1} \cdot 4)$
$k_3 = 1, k_2 = 0, k_1 = 0$ $n = 35$
$k_3 = 1, k_2 = 0, k_1 = 1$ $n = 70$
$k_3 = 0, k_2 = 0, k_1 = 3$ $n = 56$
$k_3 = 0, k_2 = 1, k_1 = 2$ $n = 84$

Now assume $k_5 = 0, k_4 = 0$ $(n = 2^{k_1} 3^{k_2} 5^{k_3})$

$\therefore 24 = (2^{k_1-1})(3^{k_2-1}\cdot 2)(5^{k_3-1}\cdot 4)$. $\therefore k_3 \le 1$

$\therefore$ If $k_3 = 1$, $6 = (2^{k_1-1})(3^{k_2-1}\cdot 2)$

$k_2 \ne 0$, $k_2 \ne 1$

$k_2 = 2$, $k_1 = 0$ $\quad n = 45$

$k_2 = 2$, $k_1 = 1$ $\quad n = 90$

$\therefore$ Assume $k_5 = k_4 = k_3 = 0$ $\quad (n = 2^{k_1}3^{k_2})$

$\therefore 24 = (2^{k_1-1})(3^{k_2-1}\cdot 2)$.

If $k_2 = 0$, no solution

If $k_2 = 1$, no solution

If $k_2 = 2$, $k_1 = 3$ $\quad n = 72$

If $k_3 = 3$, no solution

$\therefore$ For $\phi(n) = 24$,

$n = 39, 78, 52, 35, 70, 56, 84, 45, 90, 72$

19. (a). Prove that the equation $\phi(n) = 2p$, where p is prime and $2p+1$ is composite, is not solvable.

Pf: Let $n = p_1^{k_1}\cdots p_r^{k_r}$.

$\therefore \phi(n) = (p_1^{k_1} - p_1^{k_1-1})\cdots(p_r^{k_r} - p_r^{k_r-1})$

$$= p_1^{k_1-1} \cdots p_r^{k_r-1} (p_1-1) \cdots (p_r-1)$$

Suppose $\phi(n) = 2p$ ($2p+1$ is composite).
Then $p \neq 2$ as $2p+1 = 5$. Also, $n \neq 1$.

(a) Suppose n consists of more than one odd prime factor, p_j, p_k.

$\therefore$ In $\phi(n)$, $(p_j-1)(p_k-1) = 2a_j \cdot 2a_k$

$\therefore$ $\phi(n) = 2a_j \cdot 2a_k \cdot Q = 2p$, where Q = other factors in $\phi(n)$

$\therefore$ $a_j \cdot 2a_k \cdot Q = p$, so p is even $\Rightarrow p = 2$. But $2p+1 = 5$ is not composite.

$\therefore$ n can consist of at most one odd prime factor.

(b) $\therefore$ Let $n = 2^k p_1^{k_1}$, $k \geq 0$, $k_1 \geq 0$

(i) Suppose $k = 0$, so $n = p_1^{k_1}$, $k_1 > 0$

$\therefore$ $\phi(n) = p_1^{k_1-1}(p_1-1) = 2p$

If $k_1 = 1$, Then $p_1 - 1 = 2p$, $p_1 = 2p+1$, and $2p+1$ composite $\Rightarrow p_1$ is not prime.

If $k_1 > 1$, Then let $p_1 - 1 = 2r$

$\therefore \phi(n) = p_1^{k_1 - 1} \cdot 2r = 2p$,

$p_1^{k_1-1} \cdot r = p \Rightarrow r = p_1 = p$

$\therefore p_1^{k_1 - 1} = 1 \Rightarrow k_1 = 1$

$\therefore \phi(n) = (p-1) = 2p$, $0 = p+1$.

$\therefore k \neq 0$, $n = 2^k p_1^{k_1}$, $k > 0$, $k_1 \geq 0$

(ii) Assume $k = 1$

(1) If $k_1 = 0$, Then $n = 2$, $\phi(n) = 1 \neq 2p$

(2) If $k_1 = 1$, Then $n = 2p_1$, $\phi(n) = p_1 - 1$
$\therefore p_1 - 1 = 2p$, $p_1 = 2p+1 \Rightarrow p_1$ composite.

(3) If $k_1 > 1$, Then $n = 2p_1^{k_1}$,

$\phi(n) = p_1^{k_1 - 1}(p_1 - 1) = 2p$

Let $p_1 - 1 = 2r$, $\therefore p_1^{k_1-1} \cdot 2r = 2p$,

$\therefore p_1^{k_1-1} \cdot r = p$. p prime $\Rightarrow r = 1$
and $k_1 = 2 \Rightarrow p_1 = p$.

$\therefore n = 2p^2$, $\phi(n) = p(p-1) = 2p$
$\Rightarrow p - 1 = 2$, $p = 3 \Rightarrow 2p+1 = 7$,
which is not composite.

$\therefore (1), (2), (3) \Rightarrow k \neq 1$

$\therefore (i), (ii) \Rightarrow k > 1$

(iii) $\therefore n = 2^k p_1^{k_1}$, $k > 1$, $k_1 \geq 0$

(1) If $k_1 = 0$, $n = 2^k$, $\phi(n) = 2^{k-1} = 2p$
$\therefore p = 2 \Rightarrow 2p+1 = 5$ is composite.

(2) If $k_1 = 1$, Then $n = 2^k p_1$,

$\therefore \phi(n) = 2^{k-1}(p_1 - 1) = 2p$

$\therefore 2^{k-2}(p_1 - 1) = p$

Only possibility is $p = 2$ or $p_1 - 1 = p$
$p = 2 \Rightarrow 2p+1 = 5$ is composite

$p_1 - 1 = p \Rightarrow p_1 = p + 1 \Rightarrow p_1$ is even and p_1 is supposed to be odd.

(iv) $\therefore n = 2^k p_1^{k_1}$, $k > 1$, $k_1 > 1$

$$\therefore \phi(n) = 2^{k-1} p_1^{k_1 - 1}(p_1 - 1) = 2p$$

$$\Rightarrow p_1^{k_1-1}(p_1 - 1) = p$$

$p \neq 2$ as $2p+1$ must be composite.

$\therefore p$ is odd but $(p_1 - 1)$ is even, so $p_1^{k_1-1}(p_1-1)$ is even.

$\therefore$ (i), (ii), (iii), and (iv) $\Rightarrow$ There is no

k, k_1 s.t. $n = 2^k p_1^{k_1}$, with $\phi(n) = 2p$ and $2p+1$ composite

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(b) Prove there is no solution to the equation $\phi(n) = 14$, and that 14 is the smallest positive even integer with this property.

Pf: From (a) $\phi(n) = 2 \cdot 7$, and

$2(7)+1=15$ is composite.
$\therefore \phi(n)=14$ is not solvable.

$\phi(13)=12$, $\phi(11)=10$, $\phi(7)=6$,
$\phi(5)=4$, $\phi(3)=2$

for $\phi(n)=8$, note if $n=2^k$, $\phi(n)=2^{k-1}$
$\therefore 2^3=8=2^{4-1}$, so $n=16$
$\therefore \phi(16)=8$

20. If p is prime and $k \geq 2$, show that

$$\phi(\phi(p^k)) = p^{k-2}\phi((p-1)^2)$$

Pf: $\phi(p^k) = p^{k-1}(p-1)$

Since $\gcd(p,p-1)=1$, Then $\gcd(p^{k-1},p-1)=1$

ϕ is multiplicative,

$$\therefore \phi(\phi(p^k)) = \phi(p^{k-1}(p-1))$$
$$= \phi(p^{k-1})\phi(p-1)$$
$$= p^{k-2}(p-1)\phi(p-1)$$

From problem 10, $\phi(n^2) = n\phi(n)$ for every positive integer n.

$\therefore (p-1)\phi(p-1) = \phi((p-1)^2)$

$\therefore \phi(\phi(p^k)) = p^{k-2}(p-1)\phi(p-1) = p^{k-2}\phi((p-1)^2)$

21. Verify that $\phi(n)\sigma(n)$ is a perfect square when $n = 63457 = 23 \cdot 31 \cdot 89$.

Pf: If $n = p_1^{k_1} \cdots p_r^{k_r}$, and $k_i = 1$, Then

$$\phi(n) = (p_1 - 1) \cdots (p_r - 1)$$

$$\sigma(n) = \frac{(p_1^{k_1+1} - 1)}{p_1 - 1} \cdots \frac{(p_r^{k_r+1} - 1)}{p_r - 1}$$

$$\therefore \phi(n)\sigma(n) = (p_1^2 - 1) \cdots (p_r^2 - 1) = (p_1 - 1)(p_1 + 1) \cdots (p_r - 1)(p_r + 1)$$

$\therefore$ for $n = 23 \cdot 31 \cdot 89$,

$$\phi(n)\sigma(n) = (23^2 - 1)(31^2 - 1)(89^2 - 1)$$

$$= (22)(24)(30)(32)(88)(90)$$

$$= (2 \cdot 11)(2^3 \cdot 3)(2 \cdot 3 \cdot 5)(2^5)(2^3 \cdot 11)(2 \cdot 3^2 \cdot 5)$$

$$= 2^{14} \cdot 3^{4} \cdot 5^{2} \cdot 11^{2}$$

$$= \left(2^{7} \cdot 3^{2} \cdot 5 \cdot 11\right)^{2}$$