1.1 Vectors in Two- and Three-Dimensional Space

Note Title 11/16/2015

1.

$(-21, 23) - (\underline{4}, 6) = (-25, \underline{17})$

2.

$(399, -0.99, 0) + (-399, 0.99, 0) = \underline{(0, 0, 0)}$

3.

$8a = 52 + \frac{1}{2}x, \quad 16a - 104 = x$

$-26 = 12 + \frac{1}{2}y, \quad -46 - 24 = y$

$13c = 11 + \frac{1}{2}z, \quad 26c - 22 = z$

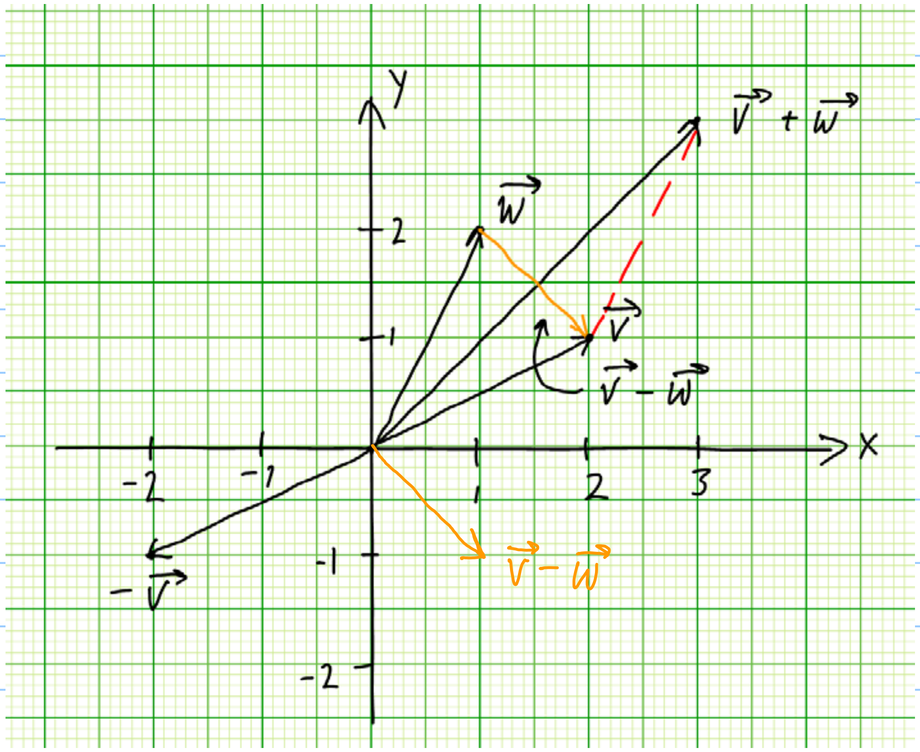
$\therefore \underline{(16a - 104, -46 - 24, 26c - 22)}$

4.

$(2, 3, 5) - 4\hat{i} + 3\hat{j} = (2, 3, 5) - (4, 0, 0) + (0, 3, 0)$

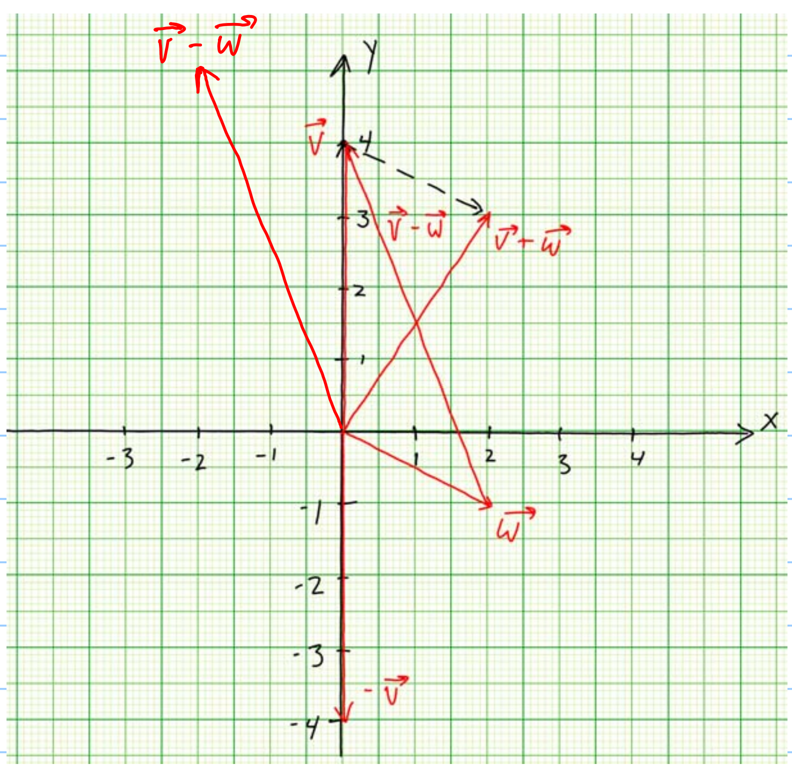
$= \underline{(-2, 6, 5)}$

5.



y
x
$\vec{v} + \vec{w}$
$\vec{w}$
$\vec{v}$
$\vec{v} - \vec{w}$
$-\vec{v}$
-2
-1
1
2
3

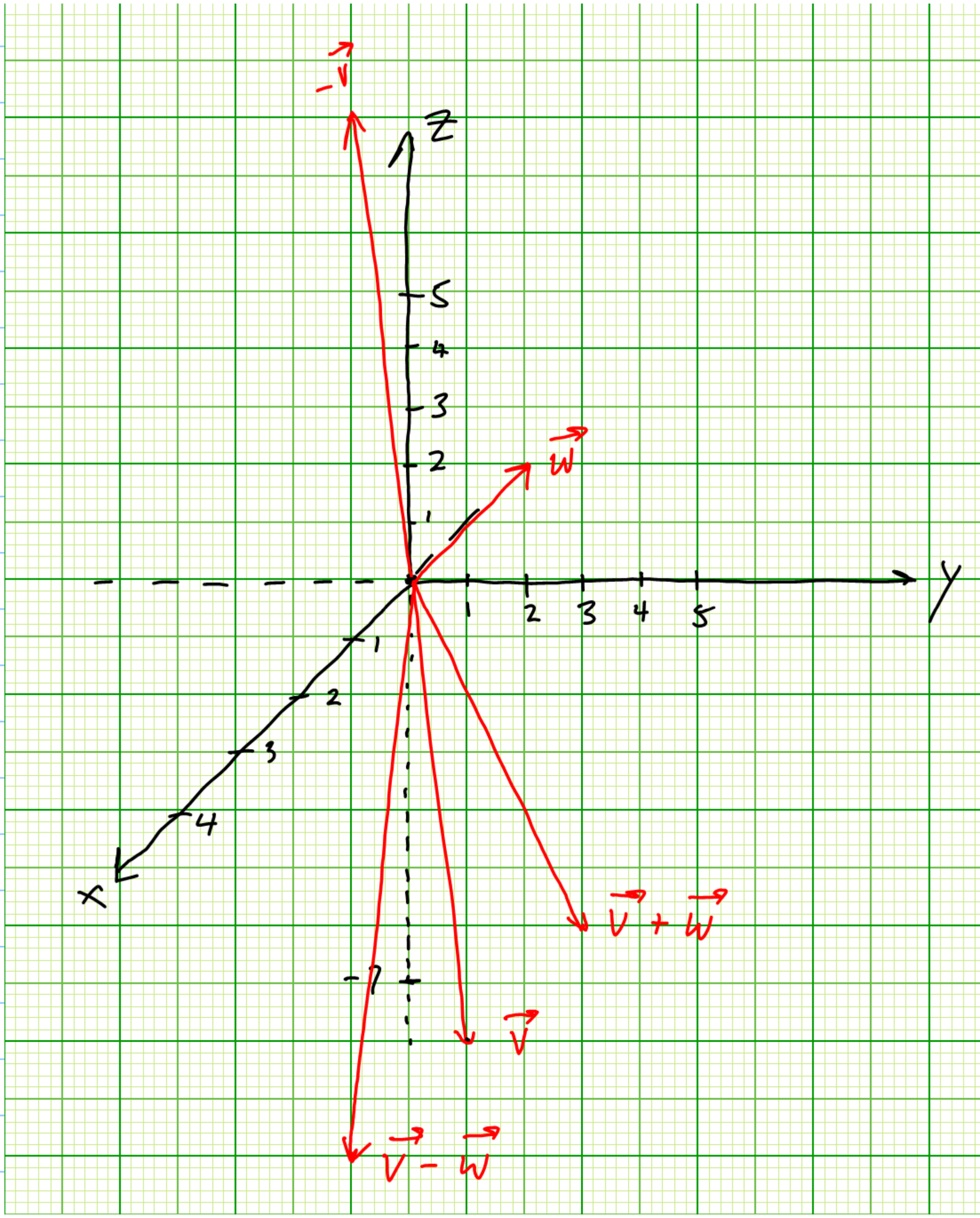
6.



$\vec{v} - \vec{w}$
y
x
$\vec{v}$
$\vec{v} - \vec{w}$
$\vec{v} + \vec{w}$
$\vec{w}$
$-\vec{v}$
-3
-2
-1
1
2
3
4
-4

7.

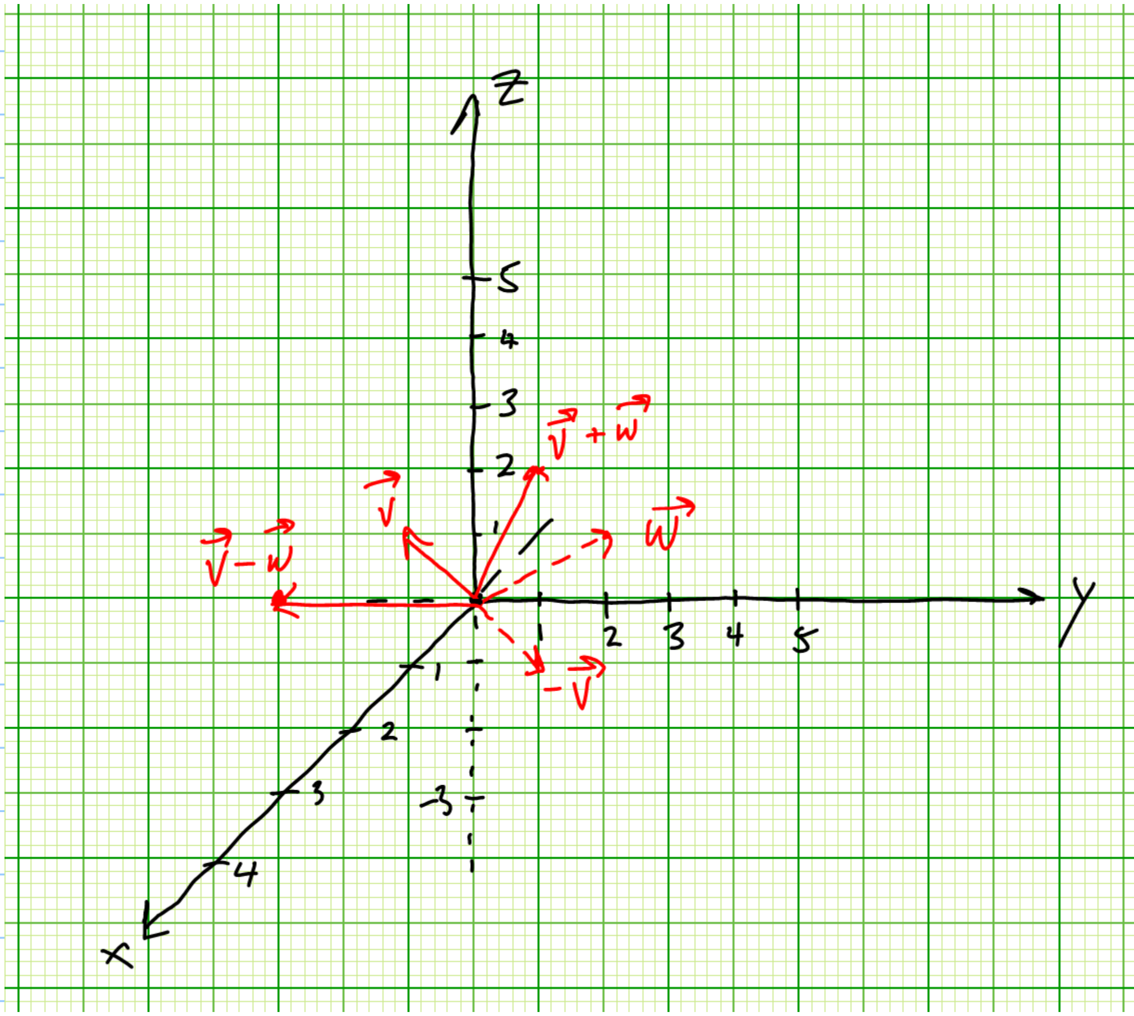
$-V = (-2, -3, 6)$ $\quad V + W = (1, 4, -5)$ $\quad V - W = (3, 2, -7)$



The x-axis is a little "stretched" compared to the y-axis and z-axis.

8.

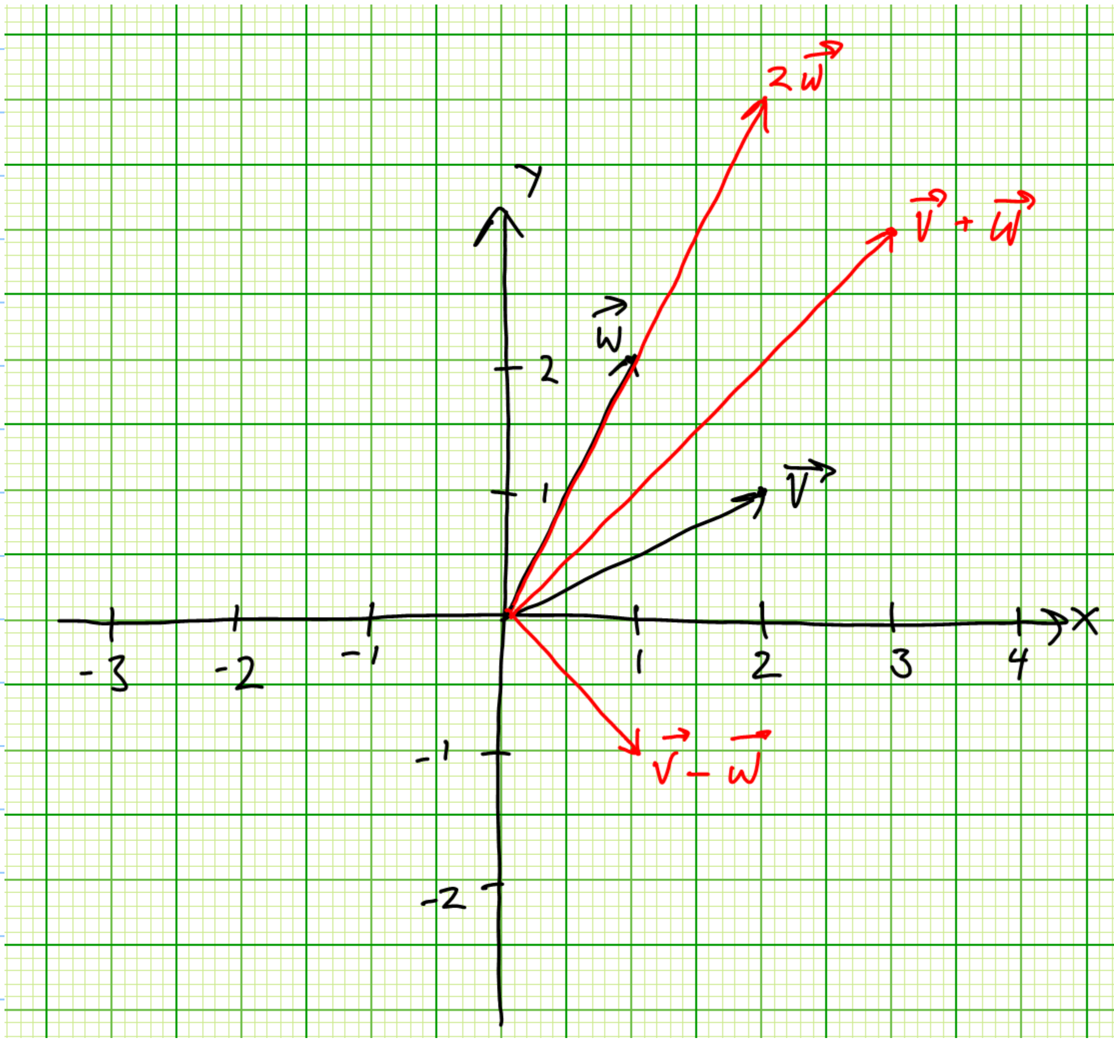
$-v = (-2, -1, -3) \quad v + w = (0, 1, 2) \quad v - w = (4, 1, 4)$



The x-axis is slightly "stretched" compared to the y-axis and z-axis.

9.

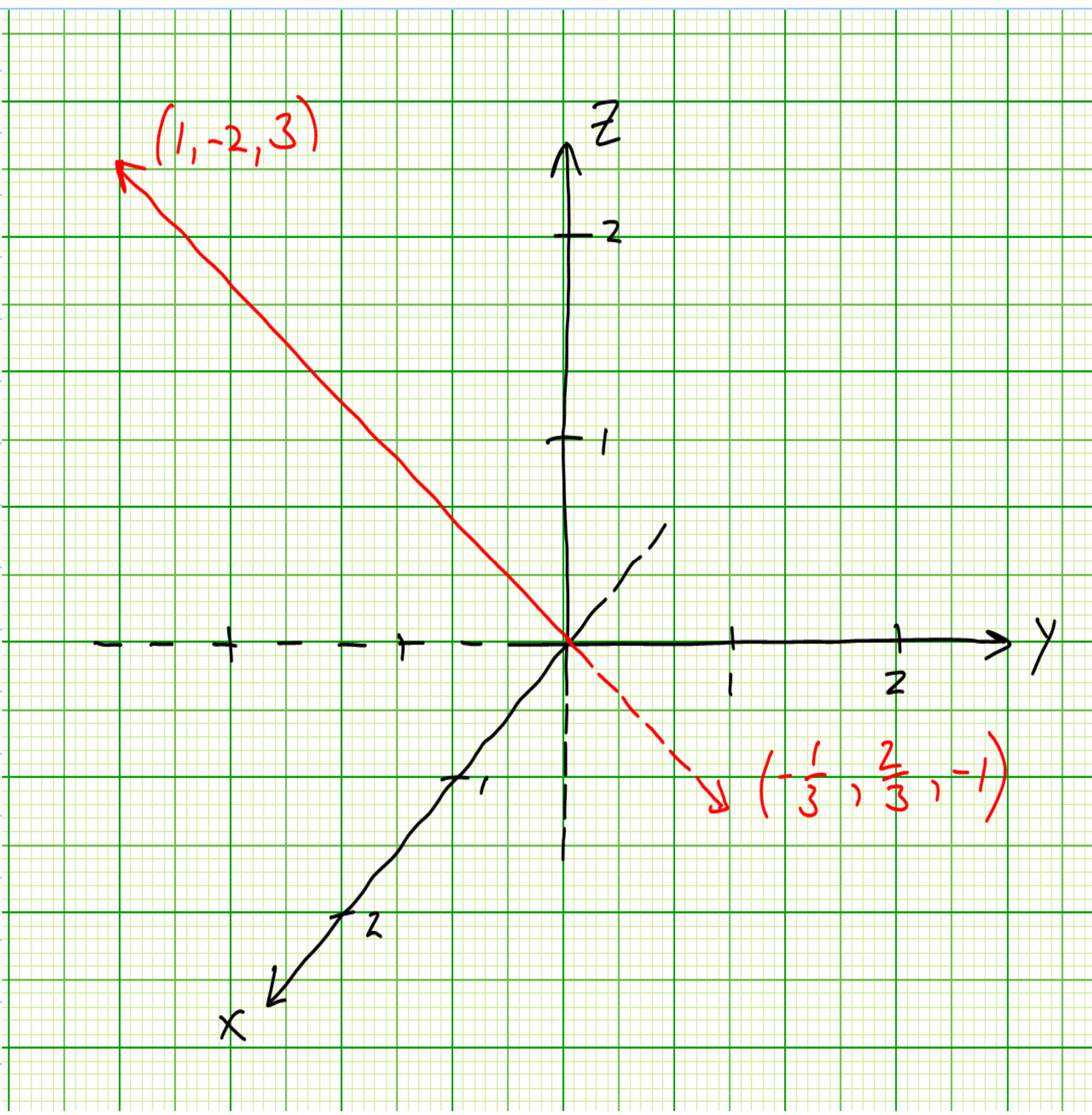
$v + w = (3,3) \quad 2w = (2,4) \quad v - w = (1,-1)$



10.

$$(1,-2,3) = -3\left(-\frac{1}{3}, \frac{2}{3}, -1\right)$$

They point in opposite directions because the second vector is a negative multiple of the first vector.



11.

(a) For (x,y,z) to be on y-axis, $x=0$, $z=0$

$\Rightarrow (0,y,0)$ is a point on the y-axis

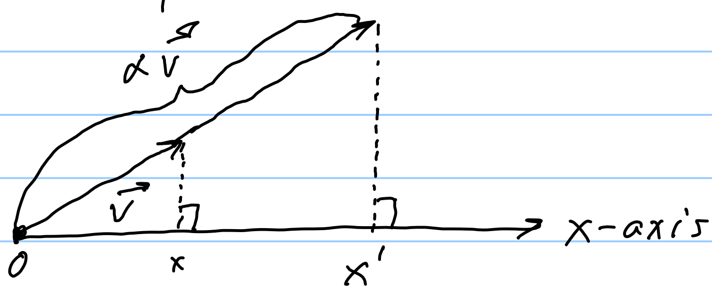
(b) $(0,0,z)$ is on the z-axis. $\therefore x=0, y=0$

(c) $y=0 \Rightarrow (x,0,z) \in xz$ plane.

(d) $x=0 \Rightarrow (0,y,z) \in yz$ plane.

12.6.

Consider just the vectors $\vec{v}$, $\alpha\vec{v}$ and the x-axis. Drop perpendiculars from $\vec{v}$ to the x-axis, and from $\alpha\vec{v}$ to the x-axis:



Assume $\alpha > 0$

x = 1st component of $\vec{v}$

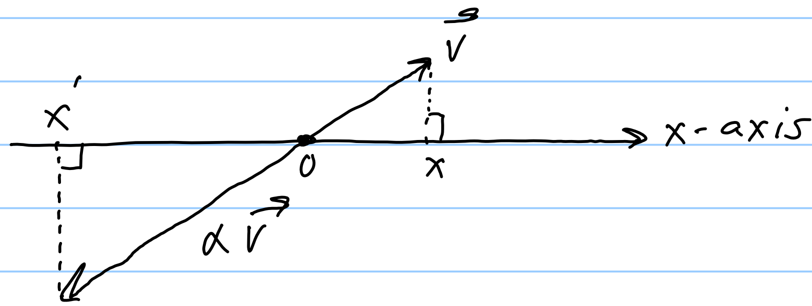
x' = 1st component of $\alpha\vec{v}$

By similar triangles $\dfrac{\text{length } \alpha\vec{v}}{\text{length } \vec{v}} = \dfrac{x'}{x}$

But length $\alpha\vec{v} = \alpha(\text{length } \vec{v})$

$$\therefore \frac{\text{length } \alpha\vec{v}}{\text{length } \vec{v}} = \alpha = \frac{x'}{x}, \quad \therefore x' = \alpha x$$

$\alpha < 0$:



By similar triangles $\dfrac{\text{length } \alpha\vec{v}}{\text{length } \vec{v}} = |\alpha| = \dfrac{|x'|}{x}$

$\therefore |x'| = |\alpha| x = -\alpha x$

Since $x' < 0$, $-x' = |x'|$. $\therefore -x' = -\alpha x$,

$\therefore x' = \alpha x$

The analysis is similar for the y and z components.

$\therefore$ For $\alpha \neq 0$, $\alpha\vec{v} = (\alpha x, \alpha y, \alpha z)$

For $\alpha = 0$, $\alpha\vec{v}$ is the point vector $(0,0,0)$

and $(\alpha x, \alpha y, \alpha z) = (0,0,0)$.

19.

One side: $r(\hat{i} + 3\hat{k})$, $0 \le r \le 1$

Other side: $s(-2\hat{j})$, $0 \le s \le 1$

$$\therefore r(\hat{i} + 3\hat{k}) + s(-2\hat{j})$$

$$= r\hat{i} - 2s\hat{j} + 3r\hat{k}$$

$$= (r, -2s, 3r), \quad 0 \le r \le 1, \ 0 \le s \le 1$$

20.

It is sufficient to show that a point $\vec{p}$ is a point on $l_1 \Leftrightarrow$ it is a point on l_2!

i.e., for every t_1 for l_1, there is a t_2 for l_2, and for every t_2 for l_2, there is a t_1 for l_1

$$\therefore (1,2,3) + t_1(1,0,-2) = (2,2,1) + t_2(-2,0,4)$$

$$\therefore t_1(1,0,-2) - t_2(-2,0,4) = (1,0,-2)$$

or
$$t_1 + 2t_2 = 1 \quad [1]$$
$$t_1(0) - t_2(0) = 0 \quad (\text{true for all } t_1, t_2) \quad [2]$$
$$-2t_1 - 4t_2 = -2 \quad [3]$$

$\therefore \quad t_1 + 2t_2 = 1 \qquad [1]$

$2t_1 + 4t_2 = 2 \iff t_1 + 2t_2 = 1 \quad [3]$

$\therefore$ [1] and [3] are equivalent.

$\therefore$ given any t_1, $t_2 = \frac{1-t_1}{2}$

$\therefore$ given any point on l_1, there is a unique t_2 s.t. the point is on l_2

given any t_2, $t_1 = 1 - 2t_2$

so given any point on l_2, there is a unique t_1 s.t. the point is on l_1.

$\therefore$ the lines are the same.

21.

Line connecting $(2,3,-4)$ and $(2,1,-1)$ is:

$(2,3,-4) + t\,[(2,3,-4) - (2,1,-1)] =$

$(2,3,-4) + t\,(0,2,-3)$

Is there a value of t s.t. $(2,7,-10) = (2,3,-4) + t(0,2,-3)$?

$(2,7,-10)-(2,3,-4)=(0,4,-6)=2(0,2,-3)$.

$\therefore$ If $t=2$, then $(2,7,-10)$ is on the same line.

$\therefore$ **All 3 points lie on the same line.**

25.

Normal to the plane $2x-3y+z-2=0$ is $(2,-3,1)$.

The normal is perpendicular to the vector parallel to the line as $(2,-3,1)\cdot(1,1,1)=0$.

$\therefore$ The line is parallel to the plane

Also, a point of the line, $(2,-2,1)$ is not in the plane since it doesn't satify

$$2x-3y+z-2=0 : 2(2)-3(-2)+(1)-2=9\neq 0$$

$\therefore$ line is not in plane and is parallel to it.

Alternative method:

$$\vec{v}=(2,-2,-1)+t(1,1,1)=(2+t,-2+t,-1+t)$$

$\therefore$ $x = 2+t$, $y = -2+t$, $z = -1+t$

$\therefore$ Substituting into $2x - 3y + z - 2 = 0$,

$2(2+t) - 3(-2+t) + (-1+t) - 2 = 0$,

or $4 + 2t + 6 - 3t + (-1) + t - 2 = 0$,

or $7 = 0$, i.e., there is no value of t to make a point on the line lie in the plane.

26.

$\vec{v} = (1+2t, -1+3t, 2+t)$, or

$x = 1+2t$, $y = -1+3t$, $z = 2+t$

$\therefore$ Substituting into $5x - 3y - z - 6 = 0$,

$5(1+2t) - 3(-1+3t) - (2+t) - 6 = 0$, or

$5 + 10t + 3 - 9t - 2 - t - 6 = 0$, or

$8 - 8 + 10t - 10t = 0$, or $0 = 0$ for all t

$\therefore$ For all t, the points $\vec{v}$ satisfy $5x-3y-z-6=0$,

28.

Solve: $t+4 = 2s+3$ [x]
$4t+5 = s+1$ [y]
$t-2 = 2s-3$ [z]

$\therefore$ $t = 2s-1$ [x]
$t = \frac{1}{4}s-1$ [y]
$t = 2s-1$ [z]

From [x], [y], $2s-1 = \frac{1}{4}s-1$, $s=0$
$\therefore t=-1$

$\therefore$ For $s=0$ and $t=-1$, the lines intersect at the point: $(3, 1, -3)$

31.

Let $\vec{v} = (x_1, y_1, z_1) - (x_0, y_0, z_0)$
$\vec{w} = (x_2, y_2, z_2) - (x_0, y_0, z_0)$

Assuming points are not colinear (i.e.,

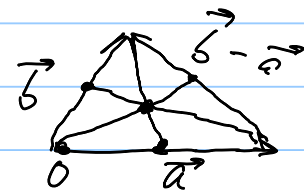
There is no value c s.t. $\vec{v} = c\vec{w}$),

The plane is described by:

$(x_0, y_0, z_0) + r\vec{v} + s\vec{w}$, for all real values r, s.

34.

Let the sides of a triangle be: $\vec{a}, \vec{b}, \vec{b} - \vec{a}$



O is the origin.

Median from $\vec{b}$: $\vec{a} - \frac{1}{2}\vec{b}$ [1]

Median from $\vec{a}$: $\vec{b} - \frac{1}{2}\vec{a}$ [2]

Median from $\vec{b} - \vec{a}$: $\vec{a} + \frac{1}{2}(\vec{b} - \vec{a}) - (0,0) =$
$\frac{1}{2}\vec{a} + \frac{1}{2}\vec{b}$ [3]

Consider median from $\vec{a}$:

$$\frac{1}{2}\vec{a} + \frac{1}{3}\left(\vec{b} - \frac{1}{2}\vec{a}\right) = \frac{1}{3}\vec{a} + \frac{1}{3}\vec{b}$$

Consider median from $\vec{b}$:

$$\frac{1}{2}\vec{b} + \frac{1}{3}\left(\vec{a} - \frac{1}{2}\vec{b}\right) = \frac{1}{3}\vec{b} + \frac{1}{3}\vec{a}$$

$\therefore$ The above two medians meet at the same point, $\frac{1}{3}\vec{a}+\frac{1}{3}\vec{b}$, which was obtained by going $\frac{1}{3}$ of the way from the origin of the medians (i.e., 2:1 ratio).

Consider median from $\vec{b}-\vec{a}$:

This vector goes from the origin to the point $\frac{1}{2}$ along $(\vec{b}-\vec{a})$, or: $\frac{1}{2}\vec{a}+\frac{1}{2}\vec{b}$ from [3].
2/3 along this vector is $\frac{2}{3}\left(\frac{1}{2}\vec{a}+\frac{1}{2}\vec{b}\right)=$
$\frac{1}{3}\vec{a}+\frac{1}{3}\vec{b}$.

$\therefore$ The point of intersection of the first two medians is on the third median, and the point is $\frac{1}{3}$ from the median point (or, $\frac{2}{3}$ from the origin).

$\therefore$ All 3 median intersect at one point, and that point is $\frac{1}{3}$ along the segment from the base (2:1 ratio).

37.

For different values of z, $x^2+y^2 = 1+z^2$, or circles of radii $\sqrt{1+z^2}$, parallel to xy-plane.

For $x=0$, $y^2-z^2=1$, an hyperbola.

$\therefore$ Line can't be parallel to a coordinate plane, so it must be oblique.

Consider the point $(1,0,0)$. This is on the surface, since $1^2+0^2-0^2=1$

Consider the vector $t(0,1,1)=(0,t,t)$

For all values t, $(1,0,0)+t(0,1,1)=(1,t,t)$

and $(1,t,t)$ is on the surface, since $1^2+t^2-t^2=1$.

$\therefore$ All points on the line $\underline{(1,0,0)+t(0,1,1)}$ lie on the surface $x^2+y^2-z^2=1$

5.

Since $\vec{a}\cdot\vec{b} = \|a\|\,\|b\|\cos\theta$,

Let $\vec{a} = (8\hat{i} - 12\hat{k})$, $\vec{b} = (6\hat{j} + \hat{k})$

$\therefore \|\vec{a}\|\cdot\|\vec{b}\| - |\vec{a}\cdot\vec{b}| = \|\vec{a}\|\cdot\|\vec{b}\| - \left|\|\vec{a}\|\cdot\|\vec{b}\|\cos\theta\right|$

$= \|\vec{a}\|\,\|\vec{b}\|\,(1 - |\cos\theta|)$

This will only be 0 when $\cos\theta = \pm 1$,

or $\theta = 0, \pi$, since $\vec{a} \neq 0$, $\vec{b} \neq 0$.

$\therefore$ for $\theta = 0, \pi$, $\vec{a} = r\vec{b}$, r some real $\neq 0$.

Since $8\hat{i} - 12\hat{k} \neq r(6\hat{j} + \hat{k})$, Then

<u>The value is not zero.</u>

12.

$(-3,2) \perp (2,3)$ since $(-3,2)\cdot(2,3) = -6+6 = 0$

$\|(-3,2)\| = \sqrt{13}$. $\therefore \pm\frac{5}{\sqrt{13}}(-3,2)$

15.

$\vec{v}\cdot\vec{w} = \|\vec{v}\|\,\|\vec{w}\|\cos\theta = -\|\vec{v}\|\,\|\vec{w}\| \Rightarrow \cos\theta = -1.$

$\therefore\ \theta = \pi$, so $\vec{v}, \vec{w}$ are parallel, and point in opposite directions.

21.

Let θ = angle $(0 \le \theta \le \pi)$ between $\vec{v}, \vec{u}$.

$\therefore\ \|\vec{v}\|\cos\theta$ = length and direction of projection of $\vec{v}$ onto $\vec{u}$.

$\vec{v}\cdot\vec{u} = |\vec{v}|\,|\vec{u}|\cos\theta$, $\therefore\ \|\vec{v}\|\cos\theta = \dfrac{\vec{v}\cdot\vec{u}}{\|\vec{u}\|}$

$\therefore$ Projection is $\dfrac{\vec{v}\cdot\vec{u}}{\|\vec{u}\|}\cdot\dfrac{\vec{u}}{\|\vec{u}\|} = \dfrac{\vec{v}\cdot\vec{u}}{\|\vec{u}\|^2}\vec{u}$

$\therefore\ \vec{v}\cdot\vec{u} = (2,1,-3)\cdot(-1,1,1) = -2+1-3 = -4$

$\|\vec{u}\|^2 = 3$. $\therefore$ Projection $\vec{v} = \underline{\underline{-\frac{4}{3}(-1,1,1)}}$

$= \underline{\underline{-\frac{4}{3}\hat{i} + \frac{4}{3}\hat{j} + \frac{4}{3}\hat{k}}}$

24.

(a) $(1,-1,0)$ is on the plane

$(1,1,-1)$ is on the plane.

$(1,-1,0)\cdot(1,1,-1)=0$, so vectors are $\perp$.

$\therefore$ Let $\underline{\vec{v}_1=(1,-1,0)}$, $\underline{\vec{v}_2=(1,1,-1)}$

(b) $$\text{Proj}_{v_1} b = \frac{\vec{b}\cdot\vec{v}_1}{\|\vec{v}_1\|^2}\vec{v}_1 = \frac{(3,1,1)\cdot(1,-1,0)}{2}(1,-1,0)$$

$$= (1,-1,0)$$

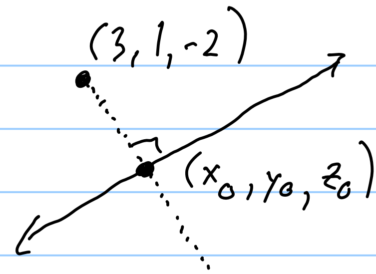
$$\text{Proj}_{v_2} b = \frac{\vec{b}\cdot\vec{v}_2}{\|\vec{v}_2\|^2}\vec{v}_2 = \frac{(3,1,1)\cdot(1,1,-1)}{3}(1,1,-1)$$

$$= (1,1,-1)$$

$\therefore$ Projection $\vec{b}$ onto plane $= (1,-1,0)+(1,1,-1)$

$$= \underline{(2,0,-1)}$$

26.



Let (x_0, y_0, z_0) be point of intersection.

$\therefore (x_0, y_0, z_0)$ is on line $l_1 = (-1,-2,-1) + t(1,1,1)$

The projection of $(3,1,-2) - (x_0, y_0, z_0) = \vec{w}$ onto $\vec{v} = (1,1,1)$ must be zero.

$\therefore \vec{w} \cdot \vec{v} = 0.$

$\therefore \quad (3 - x_0, 1 - y_0, -2 - z_0) \cdot (1,1,1) = 0$, or

$$3 - x_0 + 1 - y_0 - 2 - z_0 = 0, \text{ or}$$

$$x_0 + y_0 + z_0 = 2 \qquad [1]$$

But $(x_0, y_0, z_0) = (-1,-2,-1) + t(1,1,1)$, some t

$$\therefore \; x_0 = -1 + t, \; y_0 = -2 + t, \; z_0 = -1 + t \quad [2]$$

Substituting [2] into [1],

$$(-1 + t) + (-2 + t) + (-1 + t) = 2, \text{ or}$$

$$-4 + 3t = 2, \quad t = 2$$

$$\therefore (x_0, y_0, z_0) = (-1,-2,-1) + 2(1,1,1) = (1,0,1)$$

$$\therefore l_2 = (3,1,-2) + s\left[(3,1,-2) - (x_0, y_0, z_0)\right]$$

$$= (3,1,-2) + s\left[(3,1,-2) - (1,0,1)\right]$$

$$\therefore \underline{l_2 = (3,1,-2) + s(2,1,-3)}$$

28.

$$\vec{v} \cdot \hat{i} = \|\vec{v}\| \cdot \|\hat{i}\| \cos\alpha \quad \therefore v_1 = \sqrt{v_1^2 + v_2^2 + v_3^2}\cos\alpha$$

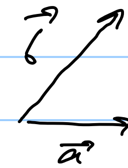
$$\text{or} \quad \cos\alpha = \frac{v_1}{\sqrt{v_1^2 + v_2^2 + v_3^2}}$$

$$\vec{v} \cdot \hat{j} = v_2 . \quad \therefore \cos\beta = \frac{v_2}{\sqrt{v_1^2 + v_2^2 + v_3^2}}$$

$$\vec{v} \cdot \hat{k} = v_3 \quad \therefore \cos\gamma = \frac{v_3}{\sqrt{v_1^2 + v_2^2 + v_3^2}}$$

$$\therefore \cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{v_1^2 + v_2^2 + v_3^2}{\left(\sqrt{v_1^2 + v_2^2 + v_3^2}\right)^2} = 1$$

38.



Let $\vec{a}, \vec{b}$ be two non-parallel vectors

The diagonals are $\vec{a} + \vec{b}$ and $\vec{b} - \vec{a}$

$\therefore \|\vec{a} + \vec{b}\|^2 + \|\vec{b} - \vec{a}\|^2 =$

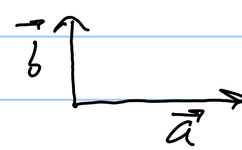
$$(\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) + (\vec{b} - \vec{a}) \cdot (\vec{b} - \vec{a}) =$$

$$[\vec{a} \cdot \vec{a} + 2\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{b}] + [\vec{b} \cdot \vec{b} - 2\vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{a}]$$

$$= 2\,\vec{a} \cdot \vec{a} + 2\,\vec{b} \cdot \vec{b} = 2\|\vec{a}\|^2 + 2\|\vec{b}\|^2$$

$$= \|\vec{a}\|^2 + \|\vec{a}\|^2 + \|\vec{b}\|^2 + \|\vec{b}\|^2$$

39.



Let $\vec{a}, \vec{b}$ be non-parallel vectors representing the sides of a rectangle. $\therefore \vec{a} \cdot \vec{b} = 0$

Diagonals are: $\vec{a} + \vec{b}$, $\vec{b} - \vec{a}$

$\therefore (\vec{a} + \vec{b}) \cdot (\vec{b} - \vec{a}) = 0 \iff$

$\vec{a} \cdot \vec{b} - \vec{a} \cdot \vec{a} + \vec{b} \cdot \vec{b} - \vec{b} \cdot \vec{a} = \vec{b} \cdot \vec{b} - \vec{a} \cdot \vec{a} = 0$

since $\vec{a} \cdot \vec{b} = 0$ (a rectangle)

$$\therefore (\vec{a}+\vec{b})\cdot(\vec{b}-\vec{a}) = 0 \iff \vec{b}\cdot\vec{b} = \vec{a}\cdot\vec{a}$$

$$\iff \|\vec{b}\|^2 = \|\vec{a}\|^2$$

i.e., diagonals perpendicular $\iff$ sides of rectangle are equal

i.e., diagonals perpendicular $\iff$ rectangle is a square

1.3 Matrices, Determinants, and the Cross Product

Note Title 11/23/2015

3.

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 1 \\ 2 & 1 & 1 \end{vmatrix} = (-2-1,\ -(1-2),\ 1-(-4))$$

$$= \underline{(-3,\ 1,\ 5)}$$

6.

$$\vec{a} = (1,1,1) - (0,0,0) = (1,1,1)$$

$$\vec{b} = (0,-2,3) - (0,0,0) = (0,-2,3)$$

$$\text{Area} = \tfrac{1}{2}\|\vec{a}\times\vec{b}\|,\quad \vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 0 & -2 & 3 \end{vmatrix} = (5,-3,-2)$$

$$\therefore \|(5,-3,-2)\| = \sqrt{5^2+3^2+2^2} = \sqrt{38}$$

$$\therefore \underline{\text{Area} = \tfrac{1}{2}\sqrt{38}}$$

8.

$$\text{Volume} = \left|\hat{i}\cdot\left[(3\hat{j}-\hat{k})\times(4\hat{i}+2\hat{j}-\hat{k})\right]\right|$$

$$= \mathrm{abs}\left(\begin{vmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 4 & 2 & -1 \end{vmatrix}\right) = |-3-(-2)| = 1$$

$\therefore$ $\underline{\underline{\text{Volume} = 1}}$

15.

(a) $(1,1,1)\cdot(x,y,z) - (1,1,1)\cdot(1,0,0) =$

$$x + y + z - 1 = 0$$

(b) $(1,2,3)\cdot(x,y,z) - (1,2,3)\cdot(1,1,1) =$

$$x + 2y + 3z - 6 = 0$$

(c) $l(t) = \vec{a} + t\vec{b}$. Perpendicular to $l(t)$ means $\perp \vec{b}$. $\therefore (5,0,2)\cdot(x,y,z) - (5,0,2)\cdot(5,-1,0)$

$$= 5x + 2z - 25 = 0$$

(d) $\ell(t) = \vec{a} + t\vec{b}$, $\vec{b} = (-1,-2,3)$.

$\therefore (-1,-2,3)\cdot(x,y,z) - (-1,-2,3)\cdot(2,4,-1) =$

$-x - 2y + 3z + 13 = 0$

18.

The points must be colinear

$(0,-2,-1) - (1,4,0) = (-1,-6,-1)$

$(0,-2,-1) - (2,10,1) = (-2,-12,-2)$

$(-2,-12,-2) = 2(-1,-6,-1)$

$\therefore$ All 3 points are <u>colinear</u>

19.

(a) 2 parallel planes have equations:

$P_1: Ax + By + Cz + D_1 = 0 \quad [1]$

$P_2: Ax + By + Cz + D_2 = 0 \quad [2]$

where (A,B,C) is the normal to each plane

If $D_1 = D_2$, then the planes are identical,

for $(x,y,z) \in P_1 \iff (A,B,C)\cdot(x,y,z) = -D_1 = -D_2$

$\iff (x,y,z) \in P_2$

If $D_1 \neq D_2$, then

if $(x,y,z) \in P_1$, then $(A,B,C)\cdot(x,y,z) = -D_1 \neq D_2$

$\therefore (x,y,z) \notin P_2$

and if $(x,y,z) \in P_2$, then $(A,B,C)\cdot(x,y,z) = -D_2 \neq -D_1$

$\therefore (x,y,z) \notin P_2$

$\therefore$ Planes never intersect.

(3) Two nonparallel planes intersect in a line.

If $\vec{n_1}$ is a normal for P_1, $\vec{p_1}$ any point of P_1
$\vec{n_2}$ is a normal for P_2, $\vec{p_2}$ any point of P_2

Let $\vec{x}$ be any point of the intersection.

Consider $\vec{x} + t(\vec{n_1} \times \vec{n_2})$, the equation of a line containg $\vec{x}$ and parallel to $\vec{n_1} \times \vec{n_2}$, t any real number.

$\therefore$ If $\vec{x} \in P_1$ and $\vec{x} \in P_2$, then so is

$\vec{x} + t(\vec{n}_1 \times \vec{n}_2)$, for any t, for

$[\vec{x} + t(\vec{n}_1 \times \vec{n}_2) - \vec{p}_1] \cdot \vec{n}_1 =$

$(\vec{x} - \vec{p}_1) \cdot \vec{n}_1 + t[(\vec{n}_1 \times \vec{n}_2)] \cdot \vec{n}_1 =$

$0 \quad + \quad 0 = 0$

since $(\vec{x} - \vec{p}_1) \cdot \vec{n}_1 = 0$ as $\vec{x}, \vec{p}_1 \in P_1$

and $(\vec{n}_1 \times \vec{n}_2) \cdot \vec{n}_1 = 0$ since $\vec{n}_1 \times \vec{n}_2$ is $\perp$ to both $\vec{n}_1, \vec{n}_2$

$\therefore$ Whenever $\vec{x} \in P_1 \cap P_2$, then

$\vec{x} + t(\vec{n}_1 \times \vec{n}_2) \in P_1$

Similarly, $\vec{x} + t(\vec{n}_1 \times \vec{n}_2) \in P_2$

$\therefore \vec{x} + t(\vec{n}_1 \times \vec{n}_2) \in P_1 \cap P_2$

But this line must be all of $P_1 \cap P_2$,

for if $\vec{y}$ is any other intersection point,

then since $\vec{x}, \vec{y} \in P_1 \cap P_2$, then

$(\vec{y} - \vec{x}) \cdot \vec{n}_1 = 0$ and $(\vec{y} - \vec{x}) \cdot \vec{n}_2 = 0$

$\therefore \vec{y} - \vec{x}$ is parallel to $\vec{n}_1 \times \vec{n}_2 \Rightarrow$

$\vec{y} - \vec{x} = t(\vec{n}_1 \times \vec{n}_2)$ some real t.

$\therefore \vec{y} = \vec{x} + t(\vec{n}_1 \times \vec{n}_2)$ and so $\vec{y}$ is on the same line of $\vec{x} + t(\vec{n}_1 \times \vec{n}_2)$

20.

For $x + 2y + z = 0$, $\vec{n}_1 = (1, 2, 1)$

$x - 3y - z = 0$, $\vec{n}_2 = (1, -3, -1)$

$$\therefore \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 1 \\ 1 & -3 & -1 \end{vmatrix} = (1, 2, -5)$$

A common point on each plane is $(0,0,0)$

$$\therefore \ell(t) = (0,0,0) + t(1,2,-5)$$

$$\therefore \underline{\underline{\ell(t) = t(1,2,-5)}}$$

23.

(a) Let $a = (a_1, a_2, a_3)$

$b = (b_1, b_2, b_3)$

$c = (c_1, c_2, c_3)$

$$a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2 b_3 - a_3 b_2, \; a_3 b_1 - a_1 b_3, \; a_1 b_2 - a_2 b_1)$$

$$\therefore (a \times b) \times c = \begin{vmatrix} i & j & k \\ a_2 b_3 - a_3 b_2 & a_3 b_1 - a_1 b_3 & a_1 b_2 - a_2 b_1 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= [(a_3 b_1 - a_1 b_3) c_3 - (a_1 b_2 - a_2 b_1) c_2,$$
$$(a_1 b_2 - a_2 b_1) c_1 - (a_2 b_3 - a_3 b_2) c_3,$$
$$(a_2 b_3 - a_3 b_2) c_2 - (a_3 b_1 - a_1 b_3) c_1]$$

$$= [a_3 c_3 b_1 - a_1 c_3 b_3 - a_1 c_2 b_2 + a_2 c_2 b_1,$$
$$a_1 c_1 b_2 - a_2 c_1 b_1 - a_2 c_3 b_3 + a_3 c_3 b_2,$$
$$a_2 c_2 b_3 - a_3 c_2 b_2 - a_3 c_1 b_1 + a_1 c_1 b_3]$$

$$= [(a_2 c_2 + a_3 c_3) b_1 - (c_2 b_2 + c_3 b_3) a_1,$$
$$(a_1 c_1 + a_3 c_3) b_2 - (c_1 b_1 + c_3 b_3) a_2,$$
$$(a_1 c_1 + a_2 c_2) b_3 - (c_1 b_1 + c_2 b_2) a_3]$$

$$= [a_1c_1b_1 + (a_2c_2 + a_3c_3)b_1 - c_1b_1a_1 - (c_2b_2 + c_3b_3)a_1,$$
$$a_2c_2b_2 + (a_1c_1 + a_3c_3)b_2 - c_2b_2a_2 - (c_1b_1 + c_3b_3)a_2,$$
$$a_3c_3b_3 + (a_1c_1 + a_2c_2)b_3 - c_3b_3a_3 - (c_1b_1 + c_2b_2)a_3]$$

$$= [(a\cdot c)b_1 - (c\cdot b)a_1,$$
$$(a\cdot c)b_2 - (c\cdot b)a_2,$$
$$(a\cdot c)b_3 - (c\cdot b)a_3]$$

$$= (a\cdot c)(b_1, b_2, b_3) - (c\cdot b)(a_1, a_2, a_3)$$

$$= (a\cdot c)b - (b\cdot c)a$$

$$\therefore \underline{(a\times b)\times c = (a\cdot c)b - (b\cdot c)a}$$

Similarly, $a\times(b\times c) = -(b\times c)\times a$

$$= -[(a\cdot b)c - (a\cdot c)b]$$

$$= (a\cdot c)b - (a\cdot b)c,$$

since $u\times v = -v\times u$

(b) Assume $u, v, w \neq 0$, since if any one is 0, then statement clearly true.

(1) By (a), $(u \times v) \times w = u \times (v \times w) \Rightarrow$

$$(u \cdot w)v - (v \cdot w)u = (u \cdot w)v - (u \cdot v)w \Rightarrow$$

$$(v \cdot w)u = (u \cdot v)w \qquad [a]$$

If $(v \cdot w) \neq 0$, then $u = \frac{u \cdot v}{v \cdot w} w$

$$\therefore (u \times w) \times v = \left(\frac{u \cdot v}{v \cdot w} w \times w\right) \times v = 0,$$

since $(\alpha w \times w) = \alpha (w \times w) = \alpha \cdot 0 = 0$.

If $v \cdot w = 0$, then $u \cdot v = 0$ from [a].

$$\therefore v \perp w, \ v \perp u, \ \therefore v = \alpha (u \times w)$$

$$\therefore (u \times w) \times v = (u \times w) \times [\alpha (u \times w)]$$
$$= \alpha [(u \times w) \times (u \times w)] = 0$$

$$\therefore (u \times v) w = u \times (v \times w) \Rightarrow \underline{(u \times w) \times v = 0}$$

(2) Assume $(u \times w) \times v = 0$. $\therefore$ By (a),

$$(u \cdot v) w - (w \cdot v) u = 0 \Rightarrow$$

$$-(v \cdot w) u = -(u \cdot v) w \Rightarrow$$

$$(u\cdot w)v-(v\cdot w)u=(u\cdot w)v-(u\cdot v)w \Rightarrow$$

$$(u\times v)\times w = u\times(v\times w)$$

$$\therefore (u\times w)\times v=0 \Rightarrow \underline{(u\times v)\times w = u\times(v\times w)}$$

(c) $(u\times v)\times w = (u\cdot w)v-(v\cdot w)u$ [1]

$(v\times w)\times u = (u\cdot v)w-(u\cdot w)v$ [2]

$(w\times u)\times v = (v\cdot w)u-(u\cdot v)w$ [3]

Adding the right sides of [1], [2], [3],

$$[(u\cdot w)v-(v\cdot w)u]+[(u\cdot v)w-(u\cdot w)v]+[(v\cdot w)u-(u\cdot v)w]$$

$$=[(u\cdot w)v-(u\cdot w)v]+[(u\cdot v)w-(u\cdot v)w]+[(v\cdot w)u-(v\cdot w)u]$$

$$=0+0+0=0$$

$$\therefore (u\times v)\times w+(v\times w)\times u+(w\times u)\times v=0$$

24.

(a) $u\cdot(v\times w) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$, if $u = (u_1, u_2, u_3)$
$v = (v_1, v_2, v_3$
$w = (w_1, w_2, w_3)$

$$\parallel$$

$$-\begin{vmatrix} v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \begin{vmatrix} v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \end{vmatrix} = -\begin{vmatrix} w_1 & w_2 & w_3 \\ v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \end{vmatrix}$$

using single row exchanges.

$\therefore u\cdot(v\times w) = -v\cdot(u\times w) = v\cdot(w\times u) = -w\cdot(v\times u)$

Using $a\times b = -b\times a$, we also get,

$-w\cdot(v\times u) = w\cdot(u\times v)$, $u\cdot(v\times w) = -u\cdot(w\times v)$

$\therefore u\cdot(v\times w) = v\cdot(w\times u) = w\cdot(u\times v) =$
$-u\cdot(w\times v) = -v\cdot(u\times w) = -w\cdot(v\times u)$

(b) From (a) above, $(u\times v)\cdot(u'\times v') =$

$u'\cdot[v'\times(u\times v)] =$

$$u' \cdot [(v' \cdot v)u - (v' \cdot u)v] \quad \text{, by 23(a)}$$

$$= (v' \cdot v)(u' \cdot u) - (v' \cdot u)(u' \cdot v)$$

$$= \begin{vmatrix} u \cdot u' & u \cdot v' \\ u' \cdot v & v \cdot v' \end{vmatrix}$$

28.

The vector $(3,-2,4)$, parallel to the line, is perpendicular to the plane.

$\therefore \; 3x - 2y + 4z + D = 0$

$3(2) - 2(-1) + 4(3) = 20 = -D$

$\therefore \; \underline{3x - 2y + 4z - 20 = 0}$

30.

$(3,-1,-2)$ is normal to the plane, and so is parallel to line in question.

$\therefore$ line : $\underline{(1,-2,-3) + t(3,-1,-2)}$

31.

The lines are parallel as they have the same generating vector, $(2,3,-1)$, but go through different points.

The vector, $(0,1,-2)-(2,-1,0)=(-2,2,-2)$ is parallel to the plane, as is the line generating vector, $(2,3,-1)$.

$\therefore$ $(-2,2,-2)\times(2,3,-1)$ is perpendicular to the plane, & is $\therefore$ a normal for the plane.

$$\therefore \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 2 & -2 \\ 2 & 3 & -1 \end{vmatrix} = (4,-6,-10) = 2(2,-3,-5)$$

$\therefore$ $(2,-3,-5)$ serves as a normal to the plane.

$\therefore$ $2x-3y-5z+D=0$.

Since $(0,1,-2)$, from the first line, is in the plane,

$2(0)-3(1)-5(-2)+D=0$, $D=-7$

$\therefore$ $\underline{2x-3y-5z-7=0}$

34.

A point in the plane is $(-5,0,0)$

$\therefore$ Find projection of $(-5,0,0)-(2,1,-1) = (-7,-1,1)$ onto the unit normal to the plane, $\frac{(1,-2,2)}{\sqrt{1+2^2+2^2}} =$

$\frac{1}{3}(1,-2,2) = \vec{n}$.

The projection is: $\|(-7,-1,1)\| \cos\theta$, θ = angle between $(-7,-1,1)$ and $\vec{n}$.

Since $\|(-7,-1,1)\| \, \|\vec{n}\| \cos\theta = (-7,-1,1)\cdot\vec{n}$

$\therefore \|(-7,-1,1)\| \cos\theta = \frac{(-7,-1,1)\cdot\vec{n}}{\|n\|} = \frac{(-7,-1,1)\cdot\vec{n}}{1}$

$= (-7,-1,1)\cdot\left(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right) = -\frac{7}{3} + \frac{2}{3} + \frac{2}{3} = -1$

$\therefore$ Distance equals length of projection of $(-7,-1,1)$ onto $\vec{n}$ $= \left|\,\|(-7,-1,1)\| \cos\theta\,\right| = |-1| = \underline{\underline{1}}$

35.

Note the normal to plane $2x+y-3z+4=0$ is parallel to the plane in question, as is the vector $(3,2,4)$ of the line.

$\therefore (2,1,-3) \times (3,2,4)$ is perpendicular to these two vectors, and $\therefore$ is a normal to the plane in question.

$$\therefore \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -3 \\ 3 & 2 & 4 \end{vmatrix} = (10, -17, 1) = \text{normal.}$$

$$\therefore 10x - 17y + z + D = 0$$

The plane contains the point $(-1, 1, 2)$ of the line.

$$\therefore 10(-1) - 17(1) + 1(2) + D = 0, \quad D = 25$$

$$\therefore \underline{10x - 17y + z + 25 = 0}$$

38.

Let $x = (x_1, x_2, x_3)$, $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3)$

Analytically:

$$\vec{x} \times \vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_1 & x_2 & x_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = \left(a_3 x_2 - a_2 x_3,\ a_1 x_3 - a_3 x_1,\ a_2 x_1 - a_1 x_2 \right)$$

$$\therefore \quad a_3 x_2 - a_2 x_3 = b_1 \qquad [1]$$

$$-a_3 x_1 + a_1 x_3 = b_2 \qquad [2]$$

$$a_2 x_1 - a_1 x_2 = b_3 \qquad [3]$$

Multiplying [1] by a_1, [2] by a_2 and adding,

$$-a_2 a_3 x_1 + a_1 a_3 x_2 = a_1 b_1 + a_2 b_2 \qquad [4]$$

$$a_2 x_1 - a_1 x_2 = b_3 \qquad [3]$$

If multiply [3] by a_3 and add to [4],

$$0 = a_1 b_1 + a_2 b_2 + a_3 b_3$$

$\therefore$ $\vec{x} \times \vec{a} = \vec{b}$ is indeterminate for $\vec{x}$, and [1], [2], and [3] alone cannot solve for x_1, x_2, x_3

$$\det \begin{bmatrix} 0 & a_3 & -a_2 \\ -a_3 & 0 & a_1 \\ a_2 & -a_1 & 0 \end{bmatrix} = 0,$$ so can't solve

$$\begin{bmatrix} 0 & a_3 & -a_2 \\ -a_3 & 0 & a_1 \\ a_2 & -a_1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Using $\vec{x} \cdot \vec{a} = \|\vec{a}\|$, $a_1 x_1 + a_2 x_2 + a_3 x_3 = \sqrt{a_1^2 + a_2^2 + a_3^2}$,
and for simplicity, let $K = \|\vec{a}\|$, a constant.

Look at $\begin{bmatrix} 0 & a_3 & -a_2 \\ -a_3 & 0 & a_1 \\ a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ K \end{bmatrix}$

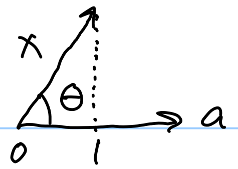
$$\det \begin{bmatrix} 0 & a_3 & -a_2 \\ -a_3 & 0 & a_1 \\ a_1 & a_2 & a_3 \end{bmatrix} = -a_3(-a_3^2 - a_1^2) - a_2(-a_2 a_3)$$
$$= a_3^3 + a_1^2 a_3 + a_2^2 a_3$$
$$= a_3(a_1^2 + a_2^2 + a_3^2)$$

Assuming $a_3 \neq 0$ and $\|a\| \neq 0$, then the solution is determined, and is unique (using Cramer's rule).

Geometrically,

Assume $\vec{a}, \vec{b} \neq 0$.

Since $x \cdot a = \|a\| = \|x\| \|a\| \cos\theta$, then
$\|x\| \cos\theta = 1$, so the projection of $\vec{x}$ onto $\vec{a}$ has length 1



$\therefore$ Assume $\vec{a}$ is along the x-axis.

If P is a plane perpendicular to $\vec{a}$, intersecting the x-axis at $(1,0,0)$, then $\vec{x}$ could be any vector from the origin to any point on P.

Using $\vec{x} \times \vec{a} = \vec{b}$, mean $\|x\| \|a\| \sin\theta = \|b\|$, so $\|x\| \sin\theta = \frac{\|b\|}{\|a\|}$. Assuming $\|a\| \neq 0$, then $\|x\| \sin\theta$ is fixed, meaning $\vec{x}$ describes a fixed circle on plane P, of radius $\|x\| \sin\theta$.

Since $\vec{x} \times \vec{a} = \vec{b}$, there is only one possibility of $\vec{x} \times \vec{a}$ such that $\vec{x} \times \vec{a}$ is in the direction of $\vec{b}$.

$\therefore$ $\vec{x}$ is uniquely determined.

39.

$(12, 13, 5)$ is a normal to the plane, and

$$\vec{n} = \frac{(12, 13, 5)}{\sqrt{12^2+13^2+5^2}} = \frac{1}{13\sqrt{2}}(12,13,5) \text{ is a unit normal.}$$

The point $(0,0,-\frac{2}{5})$ is in the plane.

$\therefore$ Project $(1,1,-5) - (0,0,-\frac{2}{5}) = (1,1,-\frac{23}{5}) = \vec{p}$

onto $\vec{n}$

$$\therefore \vec{n}\cdot\vec{p} = \|\vec{n}\|\,\|\vec{p}\|\cos\theta, \quad \therefore \|\vec{p}\|\cos\theta = \frac{\vec{n}\cdot\vec{p}}{\|\vec{n}\|} = \vec{n}\cdot\vec{p}$$

$$= \left(1,1,-\frac{23}{5}\right)\cdot\left(\frac{12}{13\sqrt{2}}, \frac{13}{13\sqrt{2}}, \frac{5}{13\sqrt{2}}\right)$$

$$= \frac{12}{13\sqrt{2}} + \frac{13}{13\sqrt{2}} - \frac{23}{5}\cdot\frac{5}{13\sqrt{2}} = \frac{12+13-23}{13\sqrt{2}}$$

$$= \frac{2}{13\sqrt{2}} = \underline{\underline{\frac{\sqrt{2}}{13}}}$$

40.

$(1,-2,1)$ is a normal to the plane.

$\therefore x - 2y + z + D = 0$. Since $(0,0,0)$ is in the plane, then $D = 0$.

$\therefore x - 2y + z = 0$

$\therefore$ Project $(6,1,0) - (0,0,0) = (6,1,0)$ onto $(1,-2,1)$

$\therefore \vec{n}\cdot\vec{p} = |\vec{n}||\vec{p}|\cos\theta, \quad |\vec{p}|\cos\theta = \dfrac{\vec{n}\cdot\vec{p}}{|\vec{n}|}$

$$= \frac{(1,-2,1)\cdot(6,1,0)}{\sqrt{1^2+(-2)^2+1^2}} = \frac{6-2+0}{\sqrt{6}} = \underline{\underline{\frac{4}{\sqrt{6}}}}$$

42.

All 4 points, A, B, C, P, are in the plane.

$\therefore$ The parallelpiped of sides $A-P$, $B-P$, $C-P$ must have a zero volume.

$$\therefore (\overrightarrow{A-P})\cdot[(\overrightarrow{B-P})\times(\overrightarrow{C-P})] = 0$$

$$\therefore \begin{vmatrix} a_1-x & a_2-y & a_3-z \\ b_1-x & b_2-y & b_3-z \\ c_1-x & c_2-y & c_3-z \end{vmatrix} = 0$$

46.

Since $V \perp W$, $V\cdot W = 0$ Now use #23 above

(1) $\therefore\ u \times v = (v \times w) \times v = (v \cdot v)w - (w \cdot v)v$

$= \|v\|^2 w - (0)v$

$= 1 \cdot w - 0 = w$

$\therefore\ w = u \times v$

(2) $w \times u = w \times (v \times w) = (w \cdot w)v - (w \cdot v)w$

$= 1 \cdot v - 0$

$= v$

$\therefore\ v = w \times u$

3.

(a) $(1, 45^\circ, 1)$: $r\cos\theta = x = \sqrt{2}/2$

$r\sin\theta = y = \sqrt{2}/2$

$z = 1$

$\therefore$ rectangular: $\underline{(\sqrt{2}/2, \sqrt{2}/2, 1)}$

spherical: $\rho = \sqrt{r^2+z^2} = \sqrt{2}$

$\theta = 45^\circ$

$\phi = \arccos\left(\frac{z}{\rho}\right) = \arccos\left(\frac{1}{\sqrt{2}}\right) = 45^\circ$

$\therefore$ spherical: $\underline{(\sqrt{2}, 45^\circ, 45^\circ)}$

$(2, \frac{\pi}{2}, -4)$: $2\cos\frac{\pi}{2} = x = 0$

$2\sin\pi/2 = y = 2$

$z = -4$

$\therefore$ rectangular: $\underline{(0, 2, -4)}$

spherical: $\rho = \sqrt{r^2+z^2} = \sqrt{4+16} = 2\sqrt{5}$

$\theta = \frac{\pi}{2} = 90^\circ$

$\phi = \arccos\left(\frac{-4}{2\sqrt{5}}\right) = 153.4^\circ = 2.68 \text{ rad}$

$\therefore$ $\underline{(2\sqrt{5}, 90^\circ, 153.4^\circ)}$

$(0, 45^\circ, 10)$:

Rectangular: $0 \cdot \cos 45^\circ = x = 0$

$0 \cdot \sin 45^\circ = y = 0$

$z = 10$

$\therefore \underline{(0, 0, 10)}$

Spherical: $\rho = \sqrt{0^2 + 10^2} = 10$

$\theta = 45^\circ$

$\phi = \arccos\left(\frac{10}{10}\right) = 0^\circ$

$\therefore \underline{(10, 45^\circ, 0^\circ)}$

$(3, \frac{\pi}{6}, 4)$: Rectangular: $x = 3 \cos \frac{\pi}{6} = \frac{3\sqrt{3}}{2}$

$y = 3 \sin \frac{\pi}{6} = \frac{3}{2}$

$z = 4$

$\therefore \underline{\left(\frac{3\sqrt{3}}{2}, \frac{3}{2}, 4\right)}$

Spherical: $\rho = \sqrt{r^2 + z^2} = \sqrt{9 + 16} = 5$

$\theta = \frac{\pi}{6} = 30^\circ$

$\phi = \arccos\left(\frac{4}{5}\right) = 36.9^\circ$

$\therefore \underline{(5, 30^\circ, 36.9^\circ)}$

$(1, \frac{\pi}{6}, 0)$: Rectangular: $x = 1 \cdot \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

$y = 1 \cdot \sin\frac{\pi}{6} = \frac{1}{2}$

$z = 0$

$\therefore \underline{(\sqrt{3}/2, \frac{1}{2}, 0)}$

Spherical: $\rho = \sqrt{r^2+z^2} = 1$

$\theta = \frac{\pi}{6} = 30^\circ$

$\phi = \arccos\left(\frac{0}{1}\right) = 90^\circ$

$\therefore \underline{(1, 30^\circ, 90^\circ)}$

$(2, \frac{3}{4}\pi, -2)$: Rectangular: $x = 2\cos\frac{3}{4}\pi = -\sqrt{2}$

$y = 2\sin\frac{3}{4}\pi = \sqrt{2}$

$z = -2$

$\therefore \underline{(-\sqrt{2}, \sqrt{2}, -2)}$

Spherical: $\rho = \sqrt{r^2+z^2} = 2\sqrt{2}$

$\theta = \frac{3}{4}\pi = 135^\circ$

$\phi = \arccos\left(\frac{-2}{2\sqrt{2}}\right) = 135^\circ$

$\therefore \underline{(2\sqrt{2}, 135^\circ, 135^\circ)}$

Summary:

Cylindrical	Rectangular	Spherical
$(1, 45^\circ, 1)$	$(\sqrt{2}/2, \sqrt{2}/2, 1)$	$(\sqrt{2}, 45^\circ, 45^\circ)$
$(2, \frac{\pi}{2}, -4)$	$(0, 2, -4)$	$(2\sqrt{5}, 90^\circ, 153.4^\circ)$
$(0, 45^\circ, 10)$	$(0, 0, 10)$	$(10, 45^\circ, 0^\circ)$
$(3, \frac{\pi}{6}, 4)$	$(\frac{3\sqrt{3}}{2}, \frac{3}{2}, 4)$	$(5, 30^\circ, 36.9^\circ)$
$(1, \frac{\pi}{6}, 0)$	$(\sqrt{3}/2, \frac{1}{2}, 0)$	$(1, 30^\circ, 90^\circ)$
$(2, \frac{3}{4}\pi, -2)$	$(-\sqrt{2}, \sqrt{2}, -2)$	$(2\sqrt{2}, 135^\circ, 135^\circ)$

(6)

$(2, 1, -2)$: Spherical: $\rho = \sqrt{4+1+4} = 3$

$$\theta = \arctan\left(\frac{1}{2}\right) = 26.6^\circ$$

$$\phi = \arccos\left(-\frac{2}{3}\right) = 131.8^\circ$$

$$\therefore (3, \underline{26.6^\circ}, 131.8^\circ)$$

Cylindrical: $r = \sqrt{4+1} = \sqrt{5}$

$\theta = \arctan\left(\frac{1}{2}\right) = 26.6^\circ$

$z = -2$

$\therefore \underline{\underline{(\sqrt{5}, 26.6^\circ, -2)}}$

$(0,3,4)$: Spherical: $\rho = \sqrt{0+9+16} = 5$

$\theta = \frac{\pi}{2} = 90^\circ$

$\phi = \arccos\left(\frac{4}{5}\right) = 36.9^\circ$

$\therefore \underline{\underline{(5, 90^\circ, 36.9^\circ)}}$

Cylindrical: $r = \sqrt{0+9} = 3$

$\theta = 90^\circ$

$z = 4$

$\therefore \underline{\underline{(3, 90^\circ, 4)}}$

$(\sqrt{2},1,1)$: Spherical: $\rho = \sqrt{2+1+1} = 2$

$\theta = \arctan\left(\frac{1}{\sqrt{2}}\right) = 35.3^\circ$

$\phi = \arccos\left(\frac{1}{2}\right) = 60^\circ$

$\therefore \underline{\underline{(2, 35.3^\circ, 60^\circ)}}$

Cylindrical: $r = \sqrt{2+1} = \sqrt{3}$

$$\theta = 35.3^\circ$$

$$z = 1$$

$$\therefore \underline{(\sqrt{3}, 35.3^\circ, 1)}$$

$(-2\sqrt{3}, -2, 3)$: Spherical: $\rho = \sqrt{12+4+9} = 5$

$$\theta = \arctan\left(\frac{-2}{-2\sqrt{3}}\right) = 210^\circ$$

$$\phi = \arccos\left(\frac{3}{5}\right) = 53.1^\circ$$

$$\therefore \underline{(5, 210^\circ, 53.1^\circ)}$$

Cylindrical: $r = \sqrt{12+4} = 4$

$$\theta = 210^\circ$$

$$z = 3$$

$$\therefore \underline{(4, 210^\circ, 3)}$$

Summary:

Rectangular	Spherical	Cylindrical
$(2, 1, -2)$	$(3, 26.6^\circ, 131.8^\circ)$	$(\sqrt{5}, 26.6^\circ, -2)$
$(0, 3, 4)$	$(5, 90^\circ, 36.9^\circ)$	$(3, 90^\circ, 4)$
$(\sqrt{2}, 1, 1)$	$(2, 35.3^\circ, 60^\circ)$	$(\sqrt{3}, 35.3^\circ, 1)$
$(-2\sqrt{3}, -2, 3)$	$(5, 210^\circ, 53.1^\circ)$	$(4, 210^\circ, 3)$

4.

(a) Reflection with respect to xy-plane

(b) Rotation counterclockwise by 180° about the z-axis, with reflection about xy-plane

(c) Reflection about the z-axis, with clockwise rotation by 45° about z-axis.

5.

(a) Counterclockwise rotation by 180° about z-axis.

(b) Reflection about the xy-plane

(c) Counterclockwise rotation about z-axis by 90°, with radial expansion (zooming) by factor of 2.

12.

(a) $\hat{e}_r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$

$$= \frac{x}{\sqrt{x^2+y^2}}\hat{i} + \frac{y}{\sqrt{x^2+y^2}}\hat{j}$$

$$\hat{e}_\theta = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$$

$$= -\frac{y}{\sqrt{x^2+y^2}}\hat{i} + \frac{x}{\sqrt{x^2+y^2}}\hat{j}$$

$$\hat{e}_z = \hat{k}$$

(b) $\hat{e}_\theta \times \hat{j} = (-\sin\theta\,\hat{i} + \cos\theta\,\hat{j}) \times \hat{j}$

$$= -\sin\theta\,(\hat{i}\times\hat{j}) + \cos\theta\,(\hat{j}\times\hat{j})$$

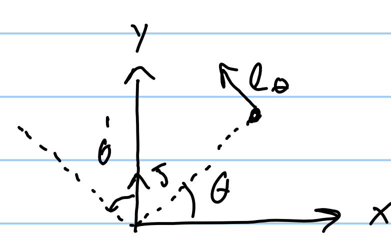
$$= -\sin\theta\,\hat{k} = -\frac{y}{\sqrt{x^2+y^2}}\hat{k}$$

Geometrically: $\|\hat{e}_\theta \times \hat{j}\| = \|\hat{e}_\theta\|\,\|\hat{j}\|\,|\sin\theta'|$

$= |\sin\theta'|$,

where θ' = angle between $\hat{e}_\theta$ and $\hat{j}$.

But $\theta' = \theta$



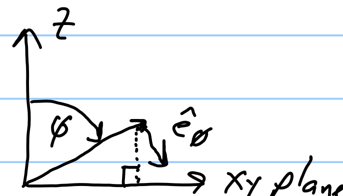
and $\hat{e}_\theta \times \hat{j}$ points toward $-\hat{k}$.

$\therefore \hat{e}_\theta \times \hat{j}$ points toward the negative z-axis and has magnitude $|\sin\theta|$.

13.

(a) $$\hat{e}_\rho = \frac{x}{\sqrt{x^2+y^2+z^2}}\hat{i} + \frac{y}{\sqrt{x^2+y^2+z^2}}\hat{j} + \frac{z}{\sqrt{x^2+y^2+z^2}}\hat{k}$$

$$\hat{e}_\theta = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$$
$$= \frac{-y}{\sqrt{x^2+y^2}}\hat{i} + \frac{x}{\sqrt{x^2+y^2}}\hat{j}$$



$\hat{e}_\phi$: k-component is

$$-|\hat{e}_\phi|\cos\left(\frac{\pi}{2}-\phi\right) = -\sin\phi$$

xy-plane component is $|\hat{e}_\phi|\sin\left(\frac{\pi}{2}-\phi\right) = \cos\phi$

$$\therefore \hat{e}_\phi = \cos\phi\cos\theta\,\hat{i} + \cos\phi\sin\theta\,\hat{j} - \sin\phi\,\hat{k}$$

$$\therefore \hat{e}_\phi = \frac{z}{\sqrt{x^2+y^2+z^2}} \cdot \frac{x}{\sqrt{x^2+y^2}}\hat{i} \quad +$$

$$\frac{z}{\sqrt{x^2+y^2+z^2}} \cdot \frac{y}{\sqrt{x^2+y^2}}\,\hat{j} \; -$$

$$\frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}\,\hat{k}$$

(b) $\hat{e}_\theta \times \hat{j} = (-\sin\theta\,\hat{i} + \cos\theta\,\hat{j}) \times \hat{j}$

$= -\sin\theta\,(\hat{i}\times\hat{j}) + \cos\theta\,(\hat{j}\times\hat{j})$

$= -\sin\theta\,\hat{k} = \underline{-\frac{y}{\sqrt{x^2+y^2}}\,\hat{k}}$

$\hat{e}_\phi \times \hat{j} = \left(\frac{zx}{\rho r}\,\hat{i} + \frac{zy}{\rho r}\,\hat{j} - \frac{r}{\rho}\,\hat{k}\right) \times \hat{j}$

$= \underline{\frac{zx}{\rho r}\,\hat{k} + \frac{r}{\rho}\,\hat{i}}$

Geometrically, $\hat{e}_\theta \times \hat{j}$, as in #12 above, points fowards $-\hat{k}$ and has magnitude $|\sin\theta|$

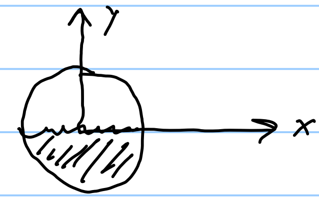
$\hat{e}_\phi \times \hat{j}$ is in the xz plane

18.

Choose z-axis along height, going through center of cylinder.

$\therefore \quad 0 \leq z \leq 16$ ft

Look at cylinder from base



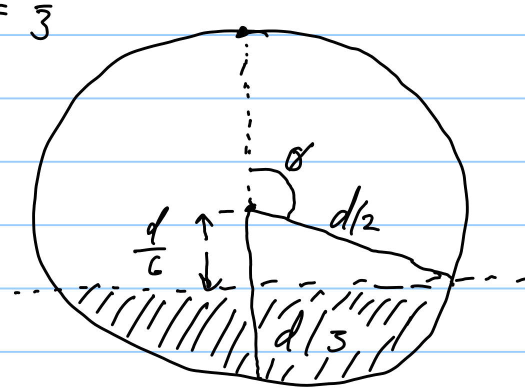
$\therefore \; 0 \leq r \leq 10$ ft, $\; -\pi \leq \theta \leq 0$

19.

Look at cross-section of sphere:

Let surface of earth be parallel to xy-plane. Describe "volume" under surface.

$\frac{d}{2} - \frac{d}{3} = \frac{d}{6} \qquad \cos(\pi - \phi) = \frac{d/6}{d/2} = \frac{1}{3}$



Choose coordinate system w/ center of sphere as origin.

$\therefore \; \pi - \arccos\left(\frac{1}{3}\right) \leq \phi \leq \pi$

$0 \le \theta \le 2\pi$ for θ.

For ρ, since $\phi > \frac{\pi}{2}$, then $\cos\phi \le 0$.

$\therefore$ Minimum of $-\rho\cos\phi$ is $\frac{d}{6}$

and the maximum of ρ is $\frac{d}{2}$

$\therefore -\frac{d}{6\cos\phi} \le \rho \le \frac{d}{2}$

Altogether, $-\frac{d}{6\cos\phi} \le \rho \le \frac{d}{2}$, $0 \le \theta \le 2\pi$,

$\pi - \arccos\left(\frac{1}{3}\right) \le \phi \le \pi$

20.

Let z-axis be along the drilled hole.

$\therefore \frac{5}{8}'' \le r \le 4.5''$, $0 \le \theta \le 2\pi$, $0 \le z \le 5.6''$

21.

$0 \le \cos 2\theta \le 1$, and $\rho = \sqrt{x^2+y^2+z^2}$.

Consider $z=0$. $r = \cos 2\theta$ is a 4-leaf rose contained in a unit circle ($x^2+y^2=1$)

ρ doesn't depend on ϕ. $\therefore$ For any θ,

$\rho = \cos 2\theta$ describes a semicircle from $\phi = 0$ to $\phi = \pi$, a semicircle of radius $\cos 2\theta$.

These semicircles "fan" out from tiny (radius near 0 when $\cos 2\theta$ is near 0) to a radius near 1 when $\cos 2\theta$ is near 1.

Essentially 3-D dumbbells that shrink to zero in size near origin.

22.

(a) Since $(x, y, z) = (\rho, \theta, \phi)$, then

$x = \rho = \sqrt{x^2+y^2+z^2} \Rightarrow y^2+z^2 = 0 \Rightarrow y = 0, z = 0$

$\therefore (x, 0, 0) = (\rho, \theta, \phi) \rightarrow \theta = 0,\ \phi = 0$

Since $x = \rho \sin\phi \cos\theta$, then $x = \rho \sin(0)\cos(0)$
$\Rightarrow x = 0.$

$\therefore \underline{(0,0,0)}$ is the only point

(b) $(x, y, z) = (r, \theta, z)$

$\therefore\ x = \sqrt{x^2 + y^2} \Rightarrow y = 0 \Rightarrow \theta = 0$

$\therefore$ All points of form $(x, 0, z)$, or

all points on $\underline{\underline{xz\text{-plane}}}$

2.

(a) $\|x+y\|^2 + \|x-y\|^2 = (x+y)\cdot(x+y) + (x-y)\cdot(x-y)$

$$= x\cdot x + 2x\cdot y + y\cdot y + x\cdot x - 2x\cdot y + y\cdot y$$

$$= 2\|x\|^2 + 2\|y\|^2$$

(b) $\|x-y\|^2\|x+y\|^2 =$

$$\left[\|x\|^2 - 2x\cdot y + \|y\|^2\right]\left[\|x\|^2 + 2x\cdot y + \|y\|^2\right]$$

$$= \|x\|^4 - 2x\cdot y\|x\|^2 + \|x\|^2\|y\|^2 + 2x\cdot y\|x\|^2 - 4(x\cdot y)^2 + 2x\cdot y\|y\|^2 + \|x\|^2\|y\|^2 - 2x\cdot y\|y\|^2 + \|y\|^4$$

$$= \|x\|^4 + 2\|x\|^2\|y\|^2 + \|y\|^4 - 4(x\cdot y)^2$$

$$= \left[\|x\|^2 + \|y\|^2\right]^2 - 4(x\cdot y)^2$$

$$\leq \left[\|x\|^2 + \|y\|^2\right]^2, \quad \text{since } -4(x\cdot y)^2 \leq 0$$

$$\therefore \|x-y\|^2\|x+y\|^2 \leq \left[\|x\|^2 + \|y\|^2\right]^2$$

$\therefore \|x-y\|\,\|x+y\| \leq \|x\|^2 + \|y\|^2$, as both sides are positive.

(c) $\|x+y\|^2 - \|x-y\|^2 =$

$$(x+y)\cdot(x+y) - (x-y)\cdot(x-y)$$

$$= \|x\|^2 + 2x\cdot y + \|y\|^2 - [\|x\|^2 - 2x\cdot y + \|y\|^2]$$

$$= 4x\cdot y = 4\langle x,y\rangle$$

$$\therefore 4\langle x,y\rangle = \|x+y\|^2 - \|x-y\|^2$$

(a) The sum of the squares of the diagonals of a parallelogram = the sum of the squares of all sides of the parallelogram

(b) The product of the diagonals of a parallelogram is $\leq$ the sum of the squares of two adjacent sides.

(c) The difference between the squares of the diagonals of a parallelogram equals four times the inner product of the sides.

3.

Note: $y = 2x$

(a) $|x \cdot y| = |2 \cdot 4 + 0 \cdot 0 + (-1)(-2)| = 10$

$\|x\| = \sqrt{2^2 + 0^2 + (-1)^2} = \sqrt{5}$ $\|y\| = \sqrt{4^2 + 0^2 + (-2)^2} = \sqrt{20}$

$\|x\|\,\|y\| = \sqrt{5} \cdot \sqrt{20} = 10$

$\therefore |x \cdot y| \leq \|x\|\,\|y\|$

(b) $\|x + y\| = \|(6, 0, -3)\| = \sqrt{6^2 + 0^2 + (-3)^2} = \sqrt{45} = 3\sqrt{5}$

$\|x\| + \|y\| = \sqrt{5} + \sqrt{20} = 3\sqrt{5}$

$\therefore \|x + y\| \leq \|x\| + \|y\|$

7.

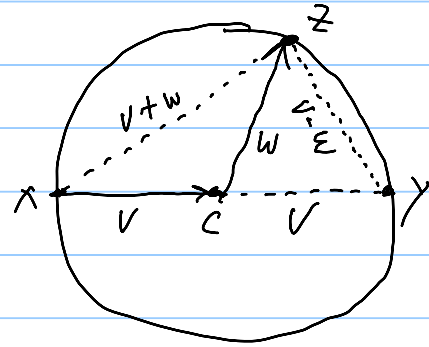
$(v + w) \cdot (v - w) = v \cdot v - v \cdot w + w \cdot v - w \cdot w$

$= -v \cdot w + w \cdot v$, since $v \cdot v = w \cdot w$

$= 0$, since $v \cdot w = w \cdot v$

$\therefore$ Dot product $= 0 \Rightarrow (v + w) \perp (v - w)$

8.



Let the points be x, y, z, where x, y line on the circle's diameter. Let C be the center of the circle.

Let $\vec{v} = \vec{XC}$, $\vec{w} = \vec{CZ}$. Then $\vec{CY}$ also $= \vec{v}$

and $\|\vec{v}\| = \|\vec{w}\|$ since these are all radii.

$\therefore \vec{XZ} = \vec{v} + \vec{w}$, and $\vec{ZY} = \vec{v} - \vec{w}$

By #7, $\vec{XZ} \perp \vec{ZY}$. $\therefore \Delta XYZ$ is a right triangle.

12.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 3 & 3 \end{bmatrix} \quad \text{Let } x = \underline{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}}$$

$$\therefore Ax = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

13.

(I) $k=1$: Clearly, $\|x_1\| \leq \|x_1\|$

(2) $K=2$: $\|x_1 + x_2\| \leq \|x_1\| + \|x_2\|$

This is true by the triangle inequality.

(3) True for $K>2 \Rightarrow$ true for $K+1$

Suppose $\|x_1 + \ldots + x_K\| \leq \|x_1\| + \ldots + \|x_K\|$

$\therefore \|x_1 + \ldots + x_{K-1} + x_K + x_{K+1}\| =$

$\|x_1 + \ldots + x_{K-1} + (x_K + x_{K+1})\|$

$\leq \|x_1\| + \ldots + \|x_{K-1}\| + \|x_K + x_{K+1}\|$, by assumption

$\leq \|x_1\| + \ldots + \|x_{K-1}\| + \|x_K\| + \|x_{K+1}\|$, since

$\|x_K + x_{K+1}\| \leq \|x_K\| + \|x_{K+1}\|$, by (2)

$\therefore$ When true for $K>2$, true for $K+1$.

$\therefore$ By (1), (2), & (3), true for all K

14.

First, clarify term $\sum_{i<j} (x_i y_j - x_j y_i)^2$

If $n=2$, $\sum\limits_{i<j} (x_i y_j - x_j y_i)^2 = (x_1 y_2 - x_2 y_1)^2$

If $n=3$, it's $\underset{i=1,\, j=2}{(x_1 y_2 - x_2 y_1)^2} + \underset{i=1,\, j=3}{(x_1 y_3 - x_3 y_1)^2} + \underset{i=2,\, j=3}{(x_2 y_3 - x_3 y_2)^2}$

If instead of using $\sum\limits_{i<j}$ we used $\sum\limits_{i,j=1}$,

Then for $n=2$, $\sum\limits_{i,j=1} (x_i y_j - x_j y_i)^2 =$

$$\sum_{i=1}^{2} \left[\sum_{j=1}^{2} (x_i y_j - x_j y_i)^2 \right] =$$

$$\underset{i=1,\, j=1}{(x_1 y_1 - x_1 y_1)^2} + \underset{i=1,\, j=2}{(x_1 y_2 - x_2 y_1)^2} + \underset{i=2,\, j=1}{(x_2 y_1 - x_1 y_2)^2} + \underset{i=2,\, j=2}{(x_2 y_2 - x_2 y_2)^2}$$

$$= 0 + 2(x_1 y_2 - x_2 y_1)^2 + 0$$

$$= 2 \sum_{i<j}^{2} (x_i y_j - x_j y_i)^2$$

Thus, in general, $\sum\limits_{i,j=1}^{n} (x_i y_j - x_j y_i)^2 = 2 \sum\limits_{i<j}^{n} (x_i y_j - x_j y_i)^2$

because when $i=j$, $(x_i y_j - x_j y_i)^2 = 0$, and when

$i \neq j$, $(x_i y_j - x_j y_i)^2 = (x_j y_i - x_i y_j)^2$

and so this latter quantity is counted twice (e.g., when $i=2, j=5$ and $i=5, j=2$)

$\therefore$ Need to prove $\frac{1}{2}\sum_{i,j=1}^{n}(x_i y_j - x_j y_i)^2 = \left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right) - \left(\sum_{i=1}^{n} x_i y_i\right)^2$

or, $$\sum_{i,j=1}^{n}(x_i y_j - x_j y_i)^2 = 2\left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right) - 2\left(\sum_{i=1}^{n} x_i y_i\right)^2$$

The $\sum_{i,j=1}^{n}$ seems conceptually easier to understand.

Now, $(x_i y_j - x_j y_i)^2 = x_i^2 y_j^2 + x_j^2 y_i^2 - 2 x_i x_j y_i y_j$

$$\therefore \sum_{i,j=1}^{n}(x_i y_j - x_j y_i)^2 = \sum_{i,j=1}^{n}\left(x_i^2 y_j^2 + x_j^2 y_i^2 - 2 x_i x_j y_i y_j\right)$$

$$= \sum_{i,j=1}^{n} x_i^2 y_j^2 + \sum_{i,j=1}^{n} x_j^2 y_i^2 - 2\sum_{i,j=1}^{n} x_i x_j y_i y_j$$

But $\sum_{i,j=1}^{n} x_i^2 y_j^2 = \sum_{i,j=1}^{n} x_j^2 y_i^2$ since both are

$$(x_1^2 + \ldots + x_n^2)(y_1^2 + \ldots + y_n^2) = \left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right)$$

$$\therefore \sum_{i,j=1}^{n}(x_i y_j - x_j y_i)^2 = 2\left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right) - 2\sum_{i,j=1}^{n} x_i x_j y_i y_j$$

Now examine $\sum_{i,j=1}^{n} x_i x_j y_i y_j =$

$$x_1 y_1 \underset{i=1,\ j=1\ldots n}{(x_1y_1 + \ldots + x_ny_n)} + x_2y_2 \underset{i=2,\ j=1\ldots n}{(x_1y_1 + \ldots + x_ny_n)} + \ldots +$$

$$x_ny_n \underset{i=n,\ j=1\ldots n}{(x_1y_1 + \ldots + x_ny_n)}$$

$$= (x_1y_1 + \ldots + x_ny_n)(x_1y_1 + \ldots + x_ny_n)$$

$$= \left(\sum_{i=1}^{n} x_i y_i\right)^2$$

$$\therefore \sum_{i,j=1}^{n} (x_i y_j - x_j y_i)^2 = 2\left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right) - 2\left(\sum_{i=1}^{n} x_i y_i\right)^2,$$

$$\text{or } \left(\sum_{i=1}^{n} x_i y_i\right)^2 = \left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right) - \frac{1}{2}\sum_{i,j=1}^{n} (x_i y_j - x_j y_i)^2$$

which was to be proved.

The above term, $\frac{1}{2}\sum_{i,j=1}^{n} (x_i y_j - x_j y_i)^2$ contains all non-negative values.

$$\therefore \left(\sum_{i=1}^{n} x_i y_i\right)^2 \leq \left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right)$$

All terms are non-negative. $\therefore$ Taking square roots, $|\vec{x}\cdot\vec{y}| \leq \|\vec{x}\|\,\|\vec{y}\|$

15.

(a) (1) For a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\lambda A = \begin{bmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{bmatrix}$

$\det A = ad - bc$, $\det(\lambda A) = \lambda^2 ad - \lambda^2 bc$
$= \lambda^2(ad-bc) = \lambda^2 \det A$

(2) Assume true for $n = k \geq 2$
i.e., for $k\times k$ matrix, $\det(\lambda A) = \lambda^k \det(A)$

Consider a $n\times n$ matrix A, where $n = k+1$

By definition $\det(\lambda A) =$

$$\lambda a_{11}\det(\lambda A_1) - \lambda a_{12}\det(\lambda A_2) + \dots + (-1)^{i+1}\lambda a_{1i}\det(\lambda A_i)$$
$$+ \dots + (-1)^{n+1}\lambda a_{1n}\det(\lambda A_n)$$

But all the A_i are $k\times k$ matrices, and by assumption $\det(\lambda A_i) = \lambda^k \det(A_i)$

$$\therefore \det(\lambda A) = \lambda a_{11}\lambda^k \det(A_1) + \dots + (-1)^{i+1}\lambda a_{1i}\lambda^k \det(A_i)$$
$$+ \dots + (-1)^{n+1}\lambda a_{1n}\lambda^k \det(A_n)$$
$$= a_{11}\lambda^{k+1}\det(A_1) + \dots + (-1)^{i+1} a_{1i}\lambda^{k+1}\det(A_i)$$
$$+ \dots + (-1)^{n+1} a_{1n}\lambda^{k+1}\det(A_n)$$
$$= \lambda^{k+1}\det(A)$$

$\therefore$ By (1) + (2), $\det(\lambda A) = \lambda^n \det(A)$

for all $n \geq 2$

(obviously true for $n=1$)

(b) Let B be obtained from A by multiplying the "k"th row of A by λ!

$$\therefore \det B = (-1)^{k+1} b_{k1} \det(B_1) + \ldots + (-1)^{k+n} b_{kn} \det(B_n)$$

$$= (-1)^{k+1} \lambda a_{k1} \det(B_1) + \ldots + (-1)^{k+n} \lambda a_{kn} \det(B_n)$$

since $b_{ki} = \lambda a_{ki}$

$$= (-1)^{k+1} \lambda a_{k1} \det(A_1) + \ldots + (-1)^{k+n} \lambda a_{kn} \det(A_n)$$

since $A_i = B_i$

$$= \lambda \left[(-1)^{k+1} a_{k1} \det(A_1) + \ldots + (-1)^{k+n} a_{kn} \det(A_n) \right]$$

$$= \lambda \det(A)$$

The $A_i = B_i$ since the matrices are otherwise identical except for row k.

The same holds if it column k that is changed, using:

$$\det A = (-1)^{1+k} a_{1k} \det(A_1) + \ldots + (-1)^{n+k} a_{nk} \det(A_n)$$

16.

In general, $\det(A+B) \neq \det A + \det B$

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$\therefore \det(A+B) = \det \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} = 4$

$\det A = 1$, $\det B = 1$. $4 \neq 1+1$

17.

In general, $(A+B)(A-B) \neq A^2 - B^2$

$(A+B)(A-B) = A^2 + BA - AB - B^2$,

and $BA \neq AB$ in general.

18.

$$\det(ABC) = \det(A)\det(BC)$$
$$= \det(A)\left[\det(B)\det(C)\right]$$
$$= \det(A)\det(B)\det(C)$$

19.

(a) Let f be any continuous function $f: [0,1] \to R$, and let V = set of all numbers

$$v = \int_0^1 f(x)\,dx$$

Define addition for V: $v + w = \int_0^1 f(x)\,dx + \int_0^1 g(x)\,dx$,

Define scalar multiplication on V:

$$\alpha v = \alpha \int_0^1 f(x)\,dx$$

where f, g are continuous functions $: [0,1] \to R$ and $\alpha \in R$

(1) ∴ Whenever $v \in V$, so is αv, $\alpha \in R$, since

$$\alpha v = \alpha \int_0^1 f(x)\,dx = \int_0^1 \alpha f(x)\,dx,$$ and since

$\alpha f(x)$ is continuous on $[0,1]$, $\int_0^1 \alpha f(x)\,dx \in V$

(2) Whenever $v, w \in V$, then $v + w \in V$

Let $v = \int_0^1 f(x)\,dx$, $w = \int_0^1 g(x)\,dx$,

f, g some continous functions on $[0,1]$

$$\therefore v + w = \int_0^1 f(x)\,dx + \int_0^1 g(x)\,dx$$

$$= \int_0^1 [f(x) + g(x)]\,dx \in V,$$

since $f + g$ is continuous on $[0,1]$.

(3) Commutivity

Let $v, w \in V$. $\therefore v + w = w + v$

since $\int f + \int g = \int g + \int f$

(4) Associativity (a) Let $u, v, w \in V$, $u = \int_0^1 f$, $v = \int_0^1 g$, $w = \int_0^1 h$, for some continuous $f, g, h : [0,1] \to \mathbb{R}$

$$\therefore (u + v) + w = \left(\int f + \int g\right) + \int h = \int (f + g) + \int h =$$

$$\int [(f+g) + h] = \int [f + (g + h)] = \int f + \int (g + h)$$

$$= \int f + \left(\int g + \int h\right) = u + (v + w)$$

(b) Now let $a, b \in \mathbb{R}$.

$\therefore (ab)v = (ab)\int_0^1 f = a\left(b\int_0^1 f\right) = a(bv)$

(5) Additive Identity

Let $g(x): [0,1] \to \mathbb{R}$ be s.t. $g(x) = 0$. $\therefore \int_0^1 g(x)\,dx = 0$

$\therefore$ Define $0 \in V$ as $0 = \int_0^1 0\,dx$

$$v + \int_0^1 0\,dx = \int_0^1 (f(x) + 0)\,dx = \int_0^1 f(x)\,dx = v,$$

f any continuou function on $[0,1]$

(6) Additive Inverse

Let $v \in V$. $\therefore$ There is a continuous

function $f: [0,1] \to \mathbb{R}$ s.t. $v = \int_0^1 f(x)\,dx$

$\therefore$ Let $w = -\int_0^1 f(x)\,dx$

$$\therefore v + w = \int_0^1 f(x) + \left(-\int_0^1 f(x)\,dx\right) = 0$$

Since $-\int_0^1 f(x)\,dx = \int_0^1 -f(x)\,dx$, and $-f$ is

continuous on $[0,1]$, then $\int_0^1 -f(x)\,dx \in V$,

and $\therefore w \in V$, so an additive inverse exists for every $v \in V$.

(7) Multiplicative Identity

By definition, for any $v \in V$, $1v = 1\int_0^1 f(x)\,dx$

$= \int_0^1 1 \cdot f(x)\,dx = \int_0^1 f(x)\,dx = V$, f a continuous function on $[0,1]$.

(8) Distributive properties

Let $v, w \in V$, $\alpha, \beta \in R$.

$\therefore \exists\ f, g : [0,1] \to R$ s.t. $V = \int_0^1 f(x)\,dx$, $W = \int_0^1 g(x)\,dx$

$\therefore \alpha(V + W) = \alpha\left(\int_0^1 f(x)\,dx + \int_0^1 g(x)\,dx\right)$

$$= \alpha\int_0^1 f(x)\,dx + \alpha\int_0^1 g(x)\,dx$$

$$= \alpha V + \alpha W$$

Also, $(\alpha + \beta)V = (\alpha + \beta)\int_0^1 f(x)\,dx$

$$= \alpha\int_0^1 f(x)\,dx + \beta\int_0^1 f(x)\,dx$$

$$= \alpha V + \beta V$$

$\therefore V$ is a vector space

(b)

Let $u, v, w \in V$. $\therefore \exists$ continuous $f, g, h : [0,1] \to R$

s.t. $u = \int_0^1 f(x)\,dx$, $v = \int_0^1 g(x)\,dx$, $w = \int_0^1 h(x)\,dx$

Define $u \cdot v = \int_0^1 f(x) g(x)\,dx$

Since $f(x)g(x)$ is continuous on $[0,1]$, so

$u \cdot v \in V$.

Let $\alpha, \beta \in R$

(i) Consider $(\alpha u + \beta v) \cdot w$

$$\alpha u + \beta v = \alpha \int_0^1 f + \beta \int_0^1 g = \int_0^1 (\alpha f + \beta g)$$

$$\therefore (\alpha u + \beta v) \cdot w = \int_0^1 (\alpha f + \beta g) \cdot h$$

$$= \int_0^1 \alpha f \cdot h + \beta g \cdot h = \alpha \int_0^1 f \cdot h + \beta \int_0^1 g \cdot h$$

$$= \alpha (u \cdot w) + \beta (v \cdot w)$$

(ii) $u \cdot v = \int_0^1 f \cdot g = \int_0^1 g \cdot f = v \cdot u$

(iii) $u \cdot u = \int_0^1 f(x) \cdot f(x)\,dx \geq 0$ since $f(x)^2 \geq 0$ on $[0,1]$ and $f(x)^2$ is continuous on $[0,1]$

(iv) (a) If $u \cdot u = 0$, then $\int_0^1 f(x)^2\,dx = 0$ by def of $u \cdot u$. Since $f(x)^2 \geq 0$ on $[0,1]$, and since $f(x)^2$ is continuous, then $\int_0^1 f(x)^2\,dx = 0 \Rightarrow f(x) = 0$ on $[0,1]$. $\therefore u = \int_0^1 f(x)\,dx = \int_0^1 0\,dx = 0$

$\therefore u \cdot u = 0 \Rightarrow u = 0$

(b) If $u = 0$, then $u = \int_0^1 0\,dx$, from above definition of additive identity vector.

$\therefore u \cdot u = \int_0^1 0 \cdot 0\,dx = 0$

Note: proof of $|u \cdot v| \leq \|u\|\,\|v\|$ depends on (iva) above, not (ivb).

=

The proof on pp. 61-62 of the text shows that given any vector space V, with an inner product defined in such a way that properties (i)–(iv a) are satisfied, then if $x, y \in V$, then

$$(x \cdot y)^2 \leq (x \cdot x) \cdot (y \cdot y)$$

Proof: Let $x = \int_0^1 f(x)\,dx$, $y = \int_0^1 g(x)\,dx$

(a) If $y \cdot y = 0$, then by (iv a), $y = 0$

$$\therefore x \cdot y = x \cdot 0 = \int f \cdot 0 = \int 0 = 0$$

$$\text{and } (x \cdot x) \cdot (y \cdot y) = (x \cdot x) \cdot \int 0^2 = (x \cdot x) \cdot 0$$

$$= 0\left(\int f^2 dx\right) \quad \text{by (ii)}$$

$$= \int_0^1 0(f^2)\,dx = 0$$

$$\therefore (x \cdot y)^2 = 0^2 \leq 0 = (x \cdot x) \cdot (y \cdot y)$$

(b) Let $a = y \cdot y$, $b = -x \cdot y$, and assume $a \neq 0$

$$\therefore 0 \leq (ax + by) \cdot (ax + by) \quad \text{by (iii)}$$

$$= a^2 x \cdot x + 2ab\, x \cdot y + b^2 y \cdot y \quad \text{by (i), (ii), and 4(b), (8) above}$$

$$= (y \cdot y)^2 (x \cdot x) - 2(y \cdot y)(x \cdot y)^2 + (x \cdot y)^2 (y \cdot y)$$

$$= (y\cdot y)^2 (x\cdot x) - (y\cdot y)(x\cdot y)^2$$

since, for $r,s \in R$, $-2rs^2 + s^2 r = -rs^2$

$$\therefore (y\cdot y)(x\cdot y)^2 \leq (y\cdot y)^2 (x\cdot x)$$

Since $y\cdot y > 0$, divide both sides by $y \cdot y$.

$$\therefore (x\cdot y)^2 \leq (y\cdot y)(x\cdot x)$$

Taking square roots on real numbers,

$$|(x\cdot y)| \leq \sqrt{y\cdot y}\sqrt{x\cdot x}$$

$$\therefore \left| \int_0^1 f(x)g(x)\,dx \right| \leq \sqrt{\int_0^1 f(x)^2\,dx}\sqrt{\int_0^1 g(x)^2\,dx}$$

Note The proof above does not depend on $[0,1]$, so that it is true for any $[a,b]$.

20.

$$(A^T x)\cdot y = \sum_{i=1}^{n} (A^T x)_i \cdot y_i$$

But $(A^T x)_i = \sum_{k=1}^{n} (A^T)_{ik} x_k = \sum_{k=1}^{n} A_{ki} x_k$

$$\therefore (A^T x)\cdot y = \sum_{i=1}^{n} \left(\sum_{k=1}^{n} A_{ki} x_k \right) \cdot y_i$$

$$= \sum_{i=1}^{n} \left(\sum_{k=1}^{n} x_k A_{ki} \right) \cdot y_i$$

$$= \sum_{i=1}^{n} \left(x_1 A_{1i} y_i + x_2 A_{2i} y_i + \dots + x_n A_{ni} y_i \right)$$

$$\begin{aligned} = \; & x_1 A_{11} y_1 + x_2 A_{21} y_1 + \dots + x_n A_{n1} y_1 \\ & + x_1 A_{12} y_2 + x_2 A_{22} y_2 + \dots + x_n A_{n2} y_2 \\ & + \quad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ & + x_1 A_{1n} y_n + x_2 A_{2n} y_n + \dots + x_n A_{nn} y_n \end{aligned}$$

$$= x_1 \left(\sum_{k=1}^{n} A_{1k} y_k \right) + x_2 \left(\sum_{k=1}^{n} A_{2k} y_k \right) + \dots + x_n \left(\sum_{k=1}^{n} A_{nk} y_k \right)$$

$$= \sum_{i=1}^{n} x_i \left(\sum_{k=1}^{n} A_{ik} y_k \right)$$

$$= \sum_{i=1}^{n} x_i \, (Ay)_i = x \cdot (Ay)$$

21.

(a) $$\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ad-bc & bd-bd \\ -ca+ac & ad-bc \end{bmatrix}$$

$$= \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}$$

$$\therefore \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(b) $$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\left(\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\right) =$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} = \begin{bmatrix} \frac{ad-bc}{ad-bc} & \frac{-ab+ab}{ad-bc} \\ \frac{cd-cd}{ad-bc} & \frac{-bc+ad}{ad-bc} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

22.

$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned}$$ can be written as $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$

$\therefore$ If $\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} e \\ f \end{bmatrix}$, Then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\left(\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} e \\ f \end{bmatrix}\right) =$$

$$\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\right)\begin{bmatrix} e \\ f \end{bmatrix},$$ since $A(BC) = (AB)C$

Exercise #21 showed $\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\therefore \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\right)\begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$$

$\therefore$ This $\begin{bmatrix} x \\ y \end{bmatrix}$ is a solution.

Note, if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, from #21,

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

$\therefore A^{-1}A\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}\begin{bmatrix} e \\ f \end{bmatrix}$, or $I\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}\begin{bmatrix} e \\ f \end{bmatrix}$

Since inverses are unique, $A^{-1}\begin{bmatrix} e \\ f \end{bmatrix}$ is the only solution.

23.

If A has an inverse, then $AA^{-1} = I$

$\therefore \det(AA^{-1}) = \det(A)\det(A^{-1}) = \det(I) = 1$,

which means both $\det(A)$ and $\det(A^{-1})$ are nonzero.

24.

Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$

$\therefore AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\quad BA = \begin{bmatrix} 2 & 2 \\ -2 & -2 \end{bmatrix}$

Review Exercises for Chapter 1

Note Title 12/10/2015

4.

(a) $(0,1,0) + t(3,0,1)$, t any real value

(b) $(0,1,1) - (0,1,0) = (0,0,1)$

$\therefore (0,1,1) + t(0,0,1)$, t any real value

(c) $(-1,1,-1)\cdot[(x,y,z) - (1,1,1)] = 0$

$\therefore -x + y - z + (1 + (-1) + 1) = 0$, or

$x - y + z - 1 = 0$

5.

$(3,0,2) - (2,1,-1) = (1,-1,3)$

$(4,-3,1) - (2,1,-1) = (2,-4,2)$

$\therefore \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 3 \\ 2 & -4 & 2 \end{vmatrix} = (10,4,-2)$ = normal to plane

$\therefore (10,4,-2)\cdot[(x,y,z) - (2,1,-1)] = 0$, or

$10x + 4y - 2z - (20 + 4 + 2) = 0$, or

$10x + 4y - 2z - 26 = 0$, or $\underline{5x + 2y - z - 13 = 0}$

6.

Normal to plane: $(2, -3, 5)$

$\therefore$ Vector perpendicular to $(2, -3, 5)$: $(1, -1, -1)$

since $(1, -1, -1) \cdot (2, -3, 5) = 0$

$\therefore$ One such line: $(-1, 7, 4) + t\,(1, -1, -1)$,

t any real value.

11.

One side of the triangle is described by

$s\vec{a}$, $0 \le s \le 1$.

Another side by $t\vec{b}$, $0 \le t \le 1$

The third side is described by $\vec{a} + r(\vec{b} - \vec{a})$, $0 \le r \le 1$,

or $(1-r)\vec{a} + r\vec{b}$.

Using similar triangles, the triangular region can be described as The base sliding continuously from The vertex to The base?

Use The $(1-r)\vec{a}+r\vec{b}$ vector as The side That slides down in a parallel fashion from apex (The origin) to The base.

$\therefore\ s\left[(1-r)\vec{a}+r\vec{b}\right]$, $0 \le s \le 1$, $0 \le r \le 1$

If s is fixed, as r varies between 0 and 1, The segment is described by $(1-r)\vec{a}+r\vec{b}$, and is parallel to $\vec{b}-\vec{a}$

$\therefore\ \underline{s(1-r)\vec{a}+sr\vec{b}}$, $0 \le s \le 1$, $0 \le r \le 1$

12.

(1) If all 3 line in a line, Then after choosing one vector, say $\vec{a}$, The other two can be described as a scalar multiple of $\vec{a}$.
Let $\vec{b}=\frac{1}{\beta}\vec{a}$, $\vec{c}=\frac{1}{\gamma}\vec{a}$.

$\therefore\ \beta\vec{b}+\gamma\vec{c}-2\vec{a}=0$.

$\therefore$ Assume one, $\vec{a}$, is not collinear to $\vec{b}$ and $\vec{c}$.

$\therefore$ All points in plane can be described by

$\vec{a}$ and $\vec{b}$ $(\alpha \vec{a} + \beta \vec{b})$. Since $\vec{c}$ is in the plane, then there must be values for α, β, s.t.

$$\alpha \vec{a} + \beta \vec{b} = \vec{c}, \text{ or}$$

$$\alpha \vec{a} + \beta \vec{b} + \gamma \vec{c} = 0, \quad \gamma = -1.$$

(2) Suppose there exists scalars α, β, γ, not all zero, s.t.

$$\alpha \vec{a} + \beta \vec{b} + \gamma \vec{c} = 0.$$

Choose a sclar that is not zero, say α.

$$\therefore \frac{\beta}{\alpha} \vec{b} + \frac{\gamma}{\alpha} \vec{c} = \vec{a}.$$

$\therefore$ The plane through the origin perpendicular to $\vec{b} \times \vec{c}$ contains $\vec{b}$, $\vec{c}$ and $\vec{a}$,

since $\vec{a} \cdot (\vec{b} \times \vec{c}) = \frac{\beta}{\alpha} \vec{b} \cdot (\vec{b} \times \vec{c}) + \frac{\gamma}{\alpha} \vec{c} \cdot (\vec{b} \times \vec{c}) = 0$

since $\vec{b} \cdot (\vec{b} \times \vec{c}) = (\vec{b} \times \vec{b}) \cdot \vec{c} = 0$

$$\vec{c} \cdot (\vec{b} \times \vec{c}) = -\vec{c} \cdot (\vec{c} \times \vec{b}) = -(\vec{c} \times \vec{c}) \cdot \vec{b} = 0$$

14.

$$\vec{a} \cdot \vec{u} = (\alpha \vec{u} + \beta \vec{v} + \gamma \vec{w}) \cdot \vec{u}$$

$$= \alpha \vec{u} \cdot \vec{u} + \beta \vec{v} \cdot \vec{u} + \gamma \vec{w} \cdot \vec{u}$$

$$= \alpha \vec{u} \cdot \vec{u}, \text{ since } \vec{v} \cdot \vec{u} = 0, \vec{w} \cdot \vec{u} = 0$$

$$= \alpha, \text{ since } \vec{u} \cdot \vec{u} = \|\vec{u}\| \|\vec{u}\| \cos\theta = 1 \cdot 1 \cdot 1 \ (\theta = 0)$$

Similarly $\vec{a} \cdot \vec{v} = \beta$, $\vec{a} \cdot \vec{w} = \gamma$

=

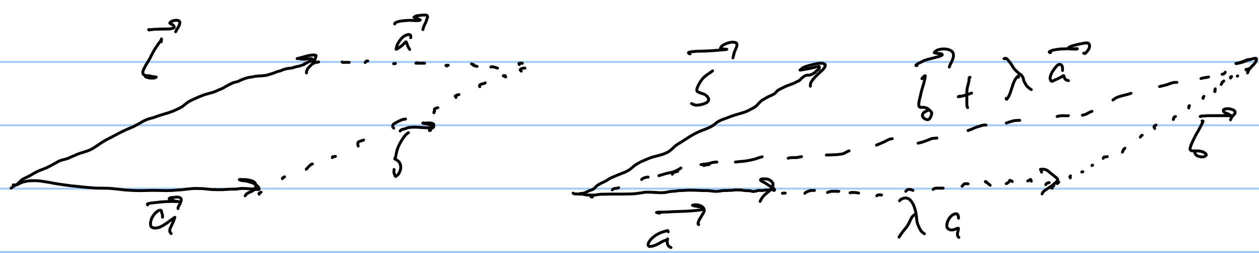
$$\alpha = \vec{a} \cdot \vec{u} = \|\vec{a}\| \|\vec{u}\| \cos\theta = \|\vec{a}\| \cos\theta$$

$\therefore$ α = scalar value of projection of $\vec{a}$ onto $\vec{u}$

similarly for β, γ.

(similar to direction numbers).

17.



Same height, same base ($\vec{a}$)

$$\text{Area}(1) = \|\vec{a} \times \vec{b}\| = (\|b\| \sin\theta)\|a\|$$

$$\text{Area}(2) = \|(\vec{b} + \lambda\vec{a}) \times \vec{a}\|$$

$$= \|(\vec{b} \times \vec{a}) + \lambda(\vec{a} \times \vec{a})\|$$

$$= \|(\vec{b} \times \vec{a}) + 0\| = (\|\vec{b}\| \sin\theta)\|\vec{a}\|$$

$$\therefore \text{Area}(1) = \text{Area}(2)$$

Property: when add scalar multiple of one row to another row, determinant stays unchanged.

19. show 2 angles are equal.

Angle between $\vec{v}, \vec{a}$: $\vec{v} \cdot \vec{a} = \|\vec{a}\| \vec{b} \cdot \vec{a} + \|\vec{b}\| \vec{a} \cdot \vec{a}$

$$= \|\vec{v}\| \|\vec{a}\| \cos\theta_{va}$$

$$\therefore \cos\theta_{va} = \frac{\vec{b} \cdot \vec{a} + \|\vec{b}\| \|\vec{a}\|}{\|\vec{v}\|}$$

Angle between $\vec{v}, \vec{b}$: $\vec{v} \cdot \vec{b} = \|\vec{a}\| \vec{b} \cdot \vec{b} + \|\vec{b}\| \vec{a} \cdot \vec{b}$

$$= \|\vec{v}\| \|\vec{b}\| \cos\theta_{vb}$$

$$\therefore \cos\theta_{vb} = \frac{\|\vec{a}\| \|\vec{b}\| + \vec{a} \cdot \vec{b}}{\|\vec{v}\|}$$

$\therefore \cos\theta_{vA} = \cos\theta_{vb}, \therefore \theta_{vA} = \pm\theta_{vb}$, or $|\theta_{vA}| = |\theta_{vb}|$

$\therefore \vec{v}$ bisects angle between $\vec{a}, \vec{b}$

21.

Since $\|a+b\| \leq \|a\| + \|b\|$

(1) Let $\vec{a} = \vec{w}, \vec{b} = \vec{v} - \vec{w}$

$\therefore \|\vec{v}\| = \|\vec{w} + (\vec{v} - \vec{w})\| \leq \|w\| + \|\vec{v} - \vec{w}\|$

$\therefore \|\vec{v}\| - \|\vec{w}\| \leq \|\vec{v} - \vec{w}\|$

(2) Now let $\vec{a} = \vec{v}, \vec{b} = \vec{w} - \vec{v}$

$\therefore \|\vec{w}\| = \|\vec{v} + (\vec{w} - \vec{v})\| \leq \|\vec{v}\| + \|\vec{w} - \vec{v}\|$

$\therefore \|\vec{w}\| - \|\vec{v}\| \leq \|\vec{w} - \vec{v}\| = \|\vec{v} - \vec{w}\|$

$\therefore$ (1) & (2) $\Rightarrow \big| \|\vec{w}\| - \|\vec{v}\| \big| \leq \|\vec{v} - \vec{w}\|$

20.

$(\|\vec{b}\|\vec{a} + \|\vec{a}\|\vec{b}) \cdot (\|\vec{b}\|\vec{a} - \|\vec{a}\|\vec{b}) =$

$\|\vec{b}\|^2 \vec{a}\cdot\vec{a} - \|\vec{b}\|\|\vec{a}\| \vec{a}\cdot\vec{b} + \|\vec{a}\|\|\vec{b}\| \vec{b}\cdot\vec{a} - \|\vec{a}\|^2 \vec{b}\cdot\vec{b}$

$= \|\vec{b}\|^2 \vec{a}\cdot\vec{a} - \|\vec{a}\|^2 \vec{b}\cdot\vec{b} = \|\vec{b}\|^2\|\vec{a}\|^2 - \|\vec{a}\|^2\|\vec{b}\|^2 = \underline{0}$

22.

Let (x_0, y_0) be a point on the line $\therefore ax_0 + by_0 = c$

$\therefore a(x - x_0) + b(y - y_0) = 0$, or

$(a, b) \cdot (x - x_0, y - y_0) = 0$

$\therefore$ The line consists of all points (x, y)

s.t. $(x - x_0, y - y_0)$ is $\perp$ to (a, b).

Let $\vec{n} = (a, b)$, the normal to the line.

$\therefore$ Distance from (x_1, y_1) to line is

the magnitude of the projection of

$(x_1, y_1) - (x_0, y_0)$ onto $\vec{n}$, or

$\|(x_1, y_1) - (x_0, y_0)\| \cos\theta$, θ = angle between $\vec{n}$ and $(x_1, y_1) - (x_0, y_0)$

But $\vec{n} \cdot [(x_1, y_1) - (x_0, y_0)] = \vec{n} \cdot (x_1, y_1) - \vec{n} \cdot (x_0, y_0)$

$= (a, b) \cdot (x_1, y_1) - (a, b) \cdot (x_0, y_0) =$

$ax_1 + by_1 - c$

And $\vec{n} \cdot [(x_1, y_1) - (x_0, y_0)] =$

$$\|\vec{n}\| \cdot \|(x_1, y_1) - (x_0, y_0)\| \cos\theta$$

$\therefore \; ax_1 + by_1 - c = \|\vec{n}\| \; \|(x_1, y_1) - (x_0, y_0)\| \cos\theta$

$\therefore \; \dfrac{ax_1 + by_1 - c}{\|\vec{n}\|} = \|(x_1, y_1) - (x_0, y_0)\| \cos\theta$

and $\|\vec{n}\| = \sqrt{a^2 + b^2}$

$\therefore \; \dfrac{|ax_1 + by_1 - c|}{\sqrt{a^2 + b^2}} = \Big| \|(x_1, y_1) - (x_0, y_0)\| \cos\theta \Big|$

= magnitude of projection of $(x_1, y_1) - (x_0, y_0)$ onto $\vec{n}$

= distance of (x_1, y_1) to line.

23.

Choose $\vec{b}$ & $\vec{c}$ to be in xy-plane, with $\vec{b}$ along x-axis. $\therefore \vec{b} = b\hat{i}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j}$, b, c_1, c_2 some real numbers.

$\therefore \vec{b} \times \vec{c} = (b\hat{i}) \times (c_1\hat{i} + c_2\hat{j})$

$= (bc_1)\,\hat{i} \times \hat{i} + (bc_2)\,\hat{i} \times \hat{j}$

$= 0 + bc_2\hat{k} = bc_2\hat{k}$

$\therefore$ Only component of $\vec{b} \times \vec{c}$ is along $\hat{k}$, same as right-hand rule.

24.

(a) Let $\vec{a} = (a_1, a_2, a_3)$, $\vec{a}' = (a_1', a_2', a_3')$

$\vec{a} \cdot (1,0,0) = a_1 \quad \vec{a}' \cdot (1,0,0) = a_1' \quad \therefore a_1 = a_1'$

$\vec{a} \cdot (0,1,0) = a_2 \quad \vec{a}' \cdot (0,1,0) = a_2' \quad \therefore a_2 = a_2'$

$\vec{a} \cdot (0,0,1) = a_3 \quad \vec{a}' \cdot (0,0,1) = a_3' \quad \therefore a_3 = a_3'$

$\therefore \vec{a} = \vec{a}'$

(b) Since $\vec{a} \times \vec{a}' = \vec{a}' \times \vec{a}' = 0$, and assuming $\vec{a}, \vec{a}' \neq 0$,

Then $\|\vec{a} \times \vec{a}'\| = \|\vec{a}\|\|\vec{a}'\| \sin\theta \Rightarrow \sin\theta = 0$.

$\therefore \vec{a}' = \lambda\vec{a}$, some scalar $\lambda \neq 0$.

But $\vec{a} \times \hat{i} = \vec{a}' \times \hat{i}$

$\therefore \|\vec{a} \times \hat{i}\| = \|\vec{a}\| \sin\theta = \|\vec{a}' \times \hat{i}\|$

$= \|\lambda \vec{a} \times \hat{i}\| = |\lambda| \, \|\vec{a}\| \sin\theta$

$\therefore |\lambda| = 1$

If $\lambda = -1$, then $\vec{a} \times \vec{b} = -\vec{a} \times \vec{b}$, for all $\vec{b}$,

$\therefore 2\,\vec{a} \times \vec{b} = 0$, or $\vec{a} \times \vec{b} = 0$ for all $\vec{b}$.

Since $\vec{a} \neq 0$, then $\lambda \neq -1$.

$\therefore \lambda = 1$, and $\therefore \vec{a} = \vec{a}'$

$\therefore$ True, $\vec{a} = \vec{a}'$

Perhaps an easier proof:

$\vec{a} \times \hat{i} = \vec{a}' \times \hat{i}$ means

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ 1 & 0 & 0 \end{vmatrix} = (0, a_3, -a_2) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1' & a_2' & a_3' \\ 1 & 0 & 0 \end{vmatrix} = (0, a_3', -a_2')$$

$\therefore a_3 = a_3', \; a_2 = a_2'$

Similarly, $$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1' & a_2' & a_3' \\ 0 & 1 & 0 \end{vmatrix}$$

from $\vec{a} \times \hat{j} = \vec{a}' \times \hat{j}$, and $\therefore$

$$(-a_3, 0, a_1) = (-a_3', 0, a_1'), \therefore a_1 = a_1'$$

$$\therefore (a_1, a_2, a_3) = (a_1', a_2', a_3'), \text{ or } \underline{\underline{\vec{a} = \vec{a}'}}$$

25.

$$l_1 : \vec{v}_1 + r\vec{a}_1$$

$$l_2 : \vec{v}_2 + s\vec{a}_2$$

$$r, s \in R$$

(a) A vector perpendicular to l_1 and l_2 is $\vec{a}_1 \times \vec{a}_2$

There is a plane, P_1, containing l_1 $\perp$ $\vec{a}_1 \times \vec{a}_2$,

and There is a plane, P_2, containing l_2 $\perp$ $\vec{a}_1 \times \vec{a}_2$

$\therefore$ Consider The distance between l_1 and l_2 as The distance between The planes P_1, P_2.

$\vec{v}_1 \in P_1$ since l_1 is in P_1, $\vec{v}_2 \in P_2$ since l_2 is in P_2.

$\therefore$ The magnitude of The projection of $\vec{v}_2 - \vec{v}_1$

onto $\vec{a}_1 \times \vec{a}_2$, a vector perpendicular to so P_1 and P_2, is the distance between P_1 & P_2, and $\therefore$ the distance between l_1 & l_2.

$\left| \|\vec{v}_2 - \vec{v}_1\| \cos\theta \right|$ is the magnitude of the projection of $\vec{v}_2 - \vec{v}_1$ onto $\vec{a}_1 \times \vec{a}_2$, θ = angle between $(\vec{v}_2 - \vec{v}_1)$ and $\vec{a}_1 \times \vec{a}_2$

But $(\vec{v}_2 - \vec{v}_1) \cdot (\vec{a}_1 \times \vec{a}_2) = \|\vec{v}_2 - \vec{v}_1\| \, \|\vec{a}_1 \times \vec{a}_2\| \cos\theta$

$$\therefore \|\vec{v}_2 - \vec{v}_1\| \cos\theta = \frac{(\vec{v}_2 - \vec{v}_1) \cdot (\vec{a}_1 \times \vec{a}_2)}{\|\vec{a}_1 \times \vec{a}_2\|}$$

$$\therefore \left| \|v_2 - v_1\| \cos\theta \right| = \frac{\left| (\vec{v}_2 - \vec{v}_1) \cdot (\vec{a}_1 \times \vec{a}_2) \right|}{\|\vec{a}_1 \times \vec{a}_2\|}$$

(6) $l_1 : (0,0,0) + r(-1,-1,-1)$

$l_2 : (2,0,5) + s\left[(2,0,5) - (0,-2,0)\right] = (2,0,5) + s(2,2,5)$

$\therefore \vec{v}_2 - \vec{v}_1 = (2,0,5) - (0,0,0) = (2,0,5)$

$$\vec{a}_1 \times \vec{a}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & -1 & -1 \\ 2 & 2 & 5 \end{vmatrix} = (-3, 3, 0)$$

$\|\vec{a}_1 \times \vec{a}_2\| = \sqrt{9+9+0} = 3\sqrt{2}$

$$(\vec{v}_2 - \vec{v}_1)\cdot(\vec{a}_1 \times \vec{a}_2) = (2,0,5)\cdot(-3,3,0) = -6$$

$$\therefore d = \frac{|-6|}{3\sqrt{2}} = \underline{\underline{\sqrt{2}}}$$

26.

Normal for each plane is: (A, B, C)

$\therefore$ same normal $\Rightarrow$ planes parallel.

Let (x_0, y_0, z_0) be a point on $Ax + By + Cz + D_1 = 0$

$$\therefore Ax_0 + By_0 + Cz_0 + D_1 = 0,$$

$$\therefore -D_1 = Ax_0 + By_0 + Cz_0.$$

Distance between (x_0, y_0, z_0) and second plane is:

$$d = \frac{|Ax_0 + By_0 + Cz_0 + D_2|}{\sqrt{A^2 + B^2 + C^2}} = \frac{|-D_1 + D_2|}{\sqrt{A^2 + B^2 + C^2}}$$

But $|-D_1 + D_2| = |D_2 - D_1| = |D_1 - D_2|$

$$\therefore d = \frac{|D_1 - D_2|}{\sqrt{A^2 + B^2 + C^2}}$$

27.

Let $P_1 = (x_1, y_1, 0)$
$P_2 = (x_2, y_2, 0)$
$P_3 = (x_3, y_3, 0)$

(a) $\therefore$ Area $= \frac{1}{2} \left| (\vec{P_2} - \vec{P_1}) \times (\vec{P_3} - \vec{P_1}) \right|$

$$= \frac{1}{2} \left\| \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_2 - x_1 & y_2 - y_1 & 0 \\ x_3 - x_1 & y_3 - y_1 & 0 \end{vmatrix} \right\|$$

$$= \frac{1}{2} \left\| \left(0, 0, (x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)\cdot(y_2 - y_1)\right) \right\|$$

$$= \frac{1}{2} \sqrt{0^2 + 0^2 + \left[(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)\right]^2}$$

$$= \frac{1}{2} \left| (x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1) \right|$$

$$= \frac{1}{2} \left| \begin{vmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{vmatrix} \right|$$

$$= \frac{1}{2} \left| \begin{vmatrix} 0 & 0 & 1 \\ x_2 - x_1 & x_3 - x_1 & x_1 \\ y_2 - y_1 & y_3 - y_1 & y_1 \end{vmatrix} \right|$$

= after adding Colum3 to Column1

$$= \frac{1}{2}\left|\begin{vmatrix} 1 & 0 & 1 \\ x_2 & x_3 - x_1 & x_1 \\ y_2 & y_3 - y_1 & y_1 \end{vmatrix}\right|$$

= after adding Column 3 to Column 2

$$= \frac{1}{2}\left|\begin{vmatrix} 1 & 1 & 1 \\ x_2 & x_3 & x_1 \\ y_2 & y_3 & y_1 \end{vmatrix}\right|$$

= after swapping Column 3 for Column 2,

Then swapping Column 2 for Column 1

$$= \frac{1}{2}\left|\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}\right|$$

(b) Area $= \frac{1}{2}\left|\begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 2 & 1 & 1 \end{vmatrix}\right| = \frac{1}{2}\left|1 - 3 + 1\right| = \underline{\underline{\frac{1}{2}}}$

28.

(a) Cylindrical: $r = \sqrt{0^2 + 3^2} = 3$, $\theta = \frac{\pi}{2}$. $\therefore (3, \frac{\pi}{2}, 4)$

Spherical: $\rho = \sqrt{0^2 + 3^2 + 4^2} = 5$, $\theta = \frac{\pi}{2}$, $\phi = \arcsin \frac{r}{\rho}$

$\therefore (5, \frac{\pi}{2}, \arcsin \frac{3}{5})$

(b) Cylindrical: $r = \sqrt{\sqrt{2}^2 + 1^2} = \sqrt{3}$, $\theta = \arctan \frac{1}{-\sqrt{2}} = \pi - \arctan \frac{\sqrt{2}}{2}$

$\therefore (\sqrt{3}, \pi - \arctan \frac{\sqrt{2}}{2}, 0)$

Spherical: $\rho = \sqrt{3}$, $\theta = \pi - \arctan \frac{\sqrt{2}}{2}$, $\phi = \frac{\pi}{2}$

$\therefore (\sqrt{3}, \pi - \arctan \frac{\sqrt{2}}{2}, \frac{\pi}{2})$

(c) Cylindrical: $r = 0$, θ = indeterminate, choose $\theta = 0$, $z = 0$

$\therefore (0, 0, 0)$

Spherical: $\rho = 0$, θ, ϕ indeterminate, choose 0.

$\therefore (0, 0, 0)$

(d) Cylindrical: $r = 1$, $\theta = \pi$, $z = 1$. $\therefore (1, \pi, 1)$

Spherical: $\rho = \sqrt{2}$, $\theta = \pi$, $\phi = \arcsin \frac{1}{\sqrt{2}} = \frac{\pi}{4}$

$\therefore (\sqrt{2}, \pi, \frac{\pi}{4})$

(e) Cylindrical: $r = \sqrt{12 + 4} = 4$, $\theta = \pi + \arctan \frac{2}{2\sqrt{3}}$

$\therefore (4, \pi + \arctan \sqrt{3}/3, 3)$

Spherical: $\rho = 5$, $\theta = \pi + \arctan \sqrt{3}/3$, $\phi = \arcsin \frac{4}{5}$

$\therefore \left(5, \arctan \frac{\sqrt{3}}{3}, \arcsin \frac{4}{5}\right)$

Summary:

Cartesian	Cylindrical	Spherical
$(0,3,4)$	$(3, \frac{\pi}{2}, 4)$	$(5, \frac{\pi}{2}, \arcsin \frac{3}{5})$
$(-\sqrt{2}, 1, 0)$	$(\sqrt{3}, \pi - \arctan \frac{\sqrt{2}}{2}, 0)$	$(\sqrt{3}, \pi - \arctan \frac{\sqrt{2}}{2}, \frac{\pi}{2})$
$(0,0,0)$	$(0,0,0)$	$(0,0,0)$
$(-1,0,1)$	$(1, \pi, 1)$	$(\sqrt{2}, \pi, \frac{\pi}{4})$
$(-2\sqrt{3}, -2, 3)$	$(4, \pi + \arctan \sqrt{3}/3, 3)$	$(5, \pi + \arctan \frac{\sqrt{3}}{3}, \arcsin \frac{4}{5})$

29.

(a) Cartesian: $x = \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$, $y = \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$, $z = 1$

$\therefore \left(\sqrt{2}/2, \sqrt{2}/2, 1\right)$

Spherical: $\rho^2 = r^2 + z^2 = 2$, $\theta = \frac{\pi}{4}$, $\phi = \arcsin \frac{r}{\rho} = \arcsin \frac{1}{\sqrt{2}}$

$\therefore \left(\sqrt{2}, \frac{\pi}{4}, \frac{\pi}{4}\right)$

(b) Cartesian: $x = 3 \cos \frac{\pi}{6} = \frac{3\sqrt{3}}{2}$, $y = 3 \sin \frac{\pi}{6} = \frac{3}{2}$, $z = -4$

$\therefore \left(\frac{3\sqrt{3}}{2}, \frac{3}{2}, -4\right)$

Spherical: $\rho^2 = 3^2 + (-4)^2 = 5^2$, $\theta = \frac{\pi}{6}$, $\phi = \arccos \frac{-4}{5}$

$\therefore \left(5, \frac{\pi}{6}, \pi - \arccos\frac{4}{5}\right)$

(c) Cartesian: $x = 0\cos\frac{\pi}{4} = 0$, $y = 0\sin\frac{\pi}{4} = 0$, $z = 1$

$\therefore (0, 0, 1)$

Spherical: $\rho^2 = 0 + 1^2 = 1$, $\theta = \frac{\pi}{4}$, $\phi = \arcsin\frac{0}{1} = 0$

$\therefore \left(1, \frac{\pi}{4}, 0\right)$

(d) Cartesian: $x = 2\cos -\frac{\pi}{2} = 0$, $y = 2\sin -\frac{\pi}{2} = -2$, $z = 1$

$\therefore (0, -2, 1)$

Spherical: $\rho^2 = 2^2 + 1^2 = 5$, $\theta = \frac{3}{2}\pi$, $\phi = \arccos\frac{1}{\sqrt{5}}$

$\therefore \left(\sqrt{5}, \frac{3}{2}\pi, \arccos\frac{1}{\sqrt{5}}\right)$

(e) Cartesian: $x = -2\cos -\frac{\pi}{2} = 0$, $y = -2\sin -\frac{\pi}{2} = 2$, $z = 1$

$\therefore (0, 2, 1)$

Spherical: $\rho^2 = (-2)^2 + 1^2 = 5$, $\theta = \frac{\pi}{2}$, $\phi = \arccos\frac{1}{\sqrt{5}}$

$\therefore \left(\sqrt{5}, \frac{\pi}{2}, \arccos\frac{1}{\sqrt{5}}\right)$

Summary:

Cylindrical	Cartesian	Spherical
$(1, \pi/4, 1)$	$(\sqrt{2}/2, \sqrt{2}/2, 1)$	$(\sqrt{2}, \pi/4, \pi/4)$
$(3, \pi/6, -4)$	$(\frac{3\sqrt{3}}{2}, \frac{3}{2}, -4)$	$(5, \pi/6, \pi - \arccos \frac{4}{5})$
$(0, \pi/4, 1)$	$(0, 0, 1)$	$(1, \pi/4, 0)$
$(2, -\pi/2, 1)$	$(0, -2, 1)$	$(\sqrt{5}, 3\pi/2, \arccos \frac{1}{\sqrt{5}})$
$(-2, -\pi/2, 1)$	$(0, 2, 1)$	$(\sqrt{5}, \pi/2, \arccos 1/\sqrt{5})$

30.

(a) Cartesian: $r = 1 \sin \pi = 0 \therefore x = y = 0,\ z = 1 \cos \pi = -1$
$\therefore (0, 0, -1)$

Cylindrical: $r = 0,\ \theta = \frac{\pi}{2},\ z = 1 \cos \pi = -1$
$\therefore (0, \pi/2, -1)$

(b) Cartesian: $r = 2 \sin \frac{\pi}{6} = 1 \quad x = r\cos(-\frac{\pi}{2}) = 0,$
$y = r \sin(-\frac{\pi}{2}) = -1$
$z = 2 \cos \frac{\pi}{6} = \sqrt{3}$
$\therefore (0, -1, \sqrt{3})$

Cylindrical: $r = 2 \sin \frac{\pi}{6} = 1,\ \theta = \frac{-\pi}{2},\ z = 2 \cos \frac{\pi}{6} = \sqrt{3}$

$\therefore (1, -\frac{\pi}{2}, \sqrt{3})$

(c) Cartesian: $r = 0\sin\phi = 0 \Rightarrow x = y = 0$, $z = 0\cos\frac{\pi}{35} = 0$

$\therefore (0, 0, 0)$

Cylindrical: $r = 0\sin\phi = 0$, $\theta = \frac{\pi}{8}$, $z = 0\cos\phi = 0$

$\therefore \left(0, \frac{\pi}{8}, 0\right)$

(d) Cartesian: $r = 2\sin(-\pi) = 0 \Rightarrow x = y = 0$, $z = 2\cos(-\pi) = -2$

$\therefore (0, 0, -2)$

Cylindrical: $r = 2\sin(-\pi) = 0$, $\theta = -\frac{\pi}{2}$, $z = 2\cos(-\pi) - 2$

$\therefore \left(0, -\frac{\pi}{2}, -2\right)$

(e) Cartesian: $r = (-1)\sin\frac{\pi}{6} = -\frac{1}{2}$, $x = -\frac{1}{2}\cos\pi = \frac{1}{2}$

$y = -\frac{1}{2}\sin\pi = 0$

$z = (-1)\cos\frac{\pi}{6} = -\sqrt{3}/2$

$\therefore \left(\frac{1}{2}, 0, -\frac{\sqrt{3}}{2}\right)$

Cylindrical: $r = (-1)\sin\frac{\pi}{6} = -\frac{1}{2}$, $\theta = \pi$, $z = (-1)\cos\frac{\pi}{6} = -\frac{\sqrt{3}}{2}$

$\therefore \left(-\frac{1}{2}, \pi, -\sqrt{3}/2\right)$

Summary:

Spherical	Cartesian	Cylindrical
$(1, \pi/2, \pi)$	$(0, 0, -1)$	$(0, \pi/2, -1)$
$(2, -\pi/2, \pi/6)$	$(0, -1, \sqrt{3})$	$(1, -\pi/2, \sqrt{3})$
$(0, \pi/8, \pi/35)$	$(0, 0, 0)$	$(0, \pi/8, 0)$
$(2, -\pi/2, -\pi)$	$(0, 0, -2)$	$(0, -\pi/2, -2)$
$(-1, \pi, \pi/6)$	$(\frac{1}{2}, 0, -\sqrt{3}/2)$	$(-\frac{1}{2}, \pi, -\frac{\sqrt{3}}{2})$

31.

Cylindrical: $x = r\cos\theta$, so $x^2 = r^2\cos^2\theta$

$y = r\sin\theta$, $\therefore y^2 = r^2\sin^2\theta$

$\therefore z = r^2(\cos^2\theta - \sin^2\theta)$

But $\cos 2\theta = \cos\theta\cos\theta - \sin\theta\sin\theta$

$\therefore \underline{z = r^2\cos 2\theta}$

Spherical: $z = \rho\cos\phi$

$x = \rho\sin\phi\cos\theta$, $\therefore x^2 = \rho^2\sin^2\phi\cos^2\theta$

$y = \rho\sin\phi\sin\theta$, $\therefore y^2 = \rho^2\sin^2\phi\sin^2\theta$

$\therefore \rho\cos\phi = \rho^2\sin^2\phi\cos^2\theta - \rho^2\sin^2\phi\sin^2\theta$

or, $\cos\phi = \rho\sin^2\phi(\cos^2\theta - \sin^2\theta)$

$\therefore \underline{\cos\phi = \rho\sin^2\phi\cos 2\theta}$

32.

$\vec{u}\cdot\hat{k} = \|\vec{u}\|\,\|\hat{k}\|\cos\phi$, ϕ = angle between $\vec{u}$, $\hat{k}$

and $\|\hat{k}\| = 1$. $\therefore$ $\vec{u} \cdot \hat{k} = \|\vec{u}\| \cos\phi$

$$\therefore \cos\phi = \frac{\vec{u} \cdot \hat{k}}{\|\vec{u}\|}, \text{ or } \phi = \cos^{-1}\left(\frac{\vec{u} \cdot \hat{k}}{\|\vec{u}\|}\right)$$

ϕ is angle between $\vec{u}$ and z-axis.

35.

(a) x is an $n \times 1$ column vector

$$\therefore [(AB)x]_{i1} = \sum_{k=1}^{n} (AB)_{ik} x_{k1} \qquad \text{by def. of } (AB)x$$

$$= \sum_{k=1}^{n} \left(\sum_{r=1}^{n} A_{ir} B_{rk} \right) x_{k1} \qquad \text{by def. of } AB$$

$$= \sum_{k=1}^{n} \left[\sum_{r=1}^{n} (A_{ir} B_{rk} x_{k1}) \right] \qquad \text{by distrib. law}$$

$$= \sum_{r=1}^{n} \left[\sum_{k=1}^{n} A_{ir} (B_{rk} x_{k1}) \right] \qquad \text{by assoc. law}$$

$$= \sum_{r=1}^{n} A_{ir} \left[\sum_{k=1}^{n} B_{rk} x_{k1} \right] \qquad \text{by distrib. law}$$

$$= \sum_{r=1}^{n} A_{ir}(Bx)_{r1} \qquad \text{by def. of } Bx$$

$$= A(Bx)_{i1} \qquad \text{by def of } A(Bx)$$

i.e., for each i, $1 \le i \le n$, $[(AB)x]_{i1} = A(Bx)_{i1}$

$\therefore (AB)x = A(Bx)$

(6) Matrix of a composition of mappings equals matrix multiplication of the matrices of each mapping.

37.

Let $\vec{e}_i$, $i=1 \ldots n$, be a basis vector for $\mathbb{R}^n$.

Consider $T(\vec{e}_i) \in \mathbb{R}^n$. Let $T(\vec{e}_i) = \vec{c}_i$, an $n \times 1$ vector

$\therefore$ Consider an $n \times n$ matrix M in which each column of M is $T(\vec{e}_i)$.

$\therefore M\vec{e}_i = \vec{c}_i$. $\quad \therefore M_{ji} = T(\vec{e}_i)_j$

Now consider $\vec{v} \in \mathbb{R}^n$, an $n \times 1$ vector.

$\therefore \vec{v} = v_1 \vec{e}_1 + \ldots v_n \vec{e}_n$

$\therefore T\vec{v} = T(v_1 \vec{e}_1 + \ldots + v_n \vec{e}_n)$

$= v_1 T(\vec{e}_1) + \ldots + v_n T(\vec{e}_n)$, because T is linear

$= v_1 \vec{c}_1 + \ldots + v_n \vec{c}_n = v_1 (M\vec{e}_1) + \ldots v_n (M\vec{e}_n)$

$= M(v_1 \vec{e}_1) + \ldots + M(v_n \vec{e}_n) = M(v_1 \vec{e}_1 + \ldots v_n \vec{e}_n)$

$= M\vec{v}$

$\therefore T\vec{v} = M\vec{v}$, for all $\vec{v} \in \mathbb{R}^n$, where $M_{ji} = T(\vec{e}_i)_j$

38.

Just need two points of line in plane: choose $t = 0, 1$.

$\therefore$ Points of plane: $(3,-1,2)$, $(2,-1,0)$, $(4,2,0)$

$\therefore$ Let $\vec{a} = (3,-1,2) - (2,-1,0) = (1,0,2)$
$\vec{b} = (3,-1,2) - (4,2,0) = (-1,-3,2)$

$\therefore \vec{a} \times \vec{b}$ = normal to plane

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2 \\ -1 & -3 & 2 \end{vmatrix} = (6, -4, -3) = \vec{n}$$

Let (x, y, z) be any point in plane.

$\therefore \vec{n} \cdot [(x, y, z) - (3, -1, 2)] = 0$, could pick any point in plane. Use $(3, -1, 2)$

$(6, -4, -3) \cdot [(x, y, z) - (3, -1, 2)] = 0$, or

$6x - 4y - 3z = (18 + 4 + (-6)) = 16$

$\therefore \underline{6x - 4y - 3z - 16 = 0}$

39.

(a) $\vec{F} \cdot \vec{r} = (10\cos\theta, 10\sin\theta) \cdot (7, 2)$

$= \underline{70\cos\theta + 20\sin\theta}$

(b) $\vec{F} = 6\cos\frac{\pi}{6}\,\hat{i} + 6\sin\frac{\pi}{6}\,\hat{j} = (3\sqrt{3}, 3)$

$\therefore \vec{F} \cdot \vec{r} = (3\sqrt{3}, 3) \cdot (7, 2) = \underline{21\sqrt{3} + 6 \text{ ft.-lbs.}}$

40.

Momentum of original particle: $(2g)(2\,m/sec)\,\hat{i}$

$= 4\ g\text{-}m/sec\ \hat{i}$

It transfers all its momentum to the two marbles (since it comes to a halt).

One marble has momentum:

$$(1g)(3\,m/sec)\cos 45°\,\hat{i} + (1g)(3\,m/sec)\sin 45°\,\hat{j}$$

$$= \left(3\sqrt{2}/2\ g\text{-}m/sec,\ 3\sqrt{2}/2\ g\text{-}m/sec\right)$$

$\therefore$ Other marble has momentum:

$$(4\ g\text{-}m/sec,\ 0) - \left(3\sqrt{2}/2\ g\text{-}m/sec,\ 3\sqrt{2}/2\ g\text{-}m/sec\right) =$$

$$\left(4 - 3\sqrt{2}/2,\ -3\sqrt{2}/2\right)\ g\text{-}m/sec = \vec{p}$$

$\therefore$ Speed $\dfrac{\|\vec{p}\|}{m} = \sqrt{\left(4-3\sqrt{2}/2\right)^2 + \left(3\sqrt{2}/2\right)^2}\ m/sec$

$= 2.83 \text{ m/sec}$

Angle: $\arctan\left(\dfrac{-3\sqrt{2}/2}{4-\frac{3\sqrt{2}}{2}}\right) = -48.5^\circ$

$\therefore$ Other marble flies off at $\underline{2.83 \text{ m/sec}}$

at an angle of -48.5° to the incident direction.

41.

$$\begin{vmatrix} x+2 & y & z \\ z & y+1 & 10 \\ 5 & 5 & 2 \end{vmatrix} = - \begin{vmatrix} y & x+2 & z \\ y+1 & z & 10 \\ 5 & 5 & 2 \end{vmatrix}$$ swap columns 1 & 2 ($\therefore$ change sign)

$$= - \begin{vmatrix} y & x+2 & z \\ 1 & z-x-2 & 10-z \\ 5 & 5 & 2 \end{vmatrix}$$ subtract row 1 from row 2 (no sign change)

42.

$$\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = \begin{vmatrix} 1 & x & x^2 \\ 0 & y-x & y^2-x^2 \\ 0 & z-x & z-x^2 \end{vmatrix}$$ subtracting row 1 from row 2 and row 3

$$= (y-x)(z^2-x^2) - (z-x)(y^2-x^2)$$

If determinant $= 0$, then

$(z-x)(y^2-x^2) = (y-x)(z^2-x^2)$, or, since $x \neq y$, $x \neq z$

$$\frac{y^2-x^2}{y-x} = \frac{z^2-x^2}{z-x}, \text{ or}$$

$$\frac{(y+x)(y-x)}{y-x} = \frac{(z+x)(z-x)}{z-x}, \text{ or } y+x = z+x$$

or $y = z$, contrary to assumption.

$\therefore$ Determinant $\neq 0$.

44.

$$\begin{vmatrix} n & n+1 & n+2 \\ n+3 & n+4 & n+5 \\ n+6 & n+7 & n+8 \end{vmatrix} =$$ subtract row 1 from row 2 and subtract row 1 from row 3

$$\begin{vmatrix} n & n+1 & n+2 \\ 3 & 3 & 3 \\ 6 & 6 & 6 \end{vmatrix} =$$ subtracting $2x$ row 2 from row 3

$$\begin{vmatrix} n & n+1 & n+2 \\ 3 & 3 & 3 \\ 0 & 0 & 0 \end{vmatrix} = \underline{0}$$ as all of row 3 is zero.

46.

Since $IA = A$, and $3IA = 3A$,

then choose $\underline{C = 3I}$.

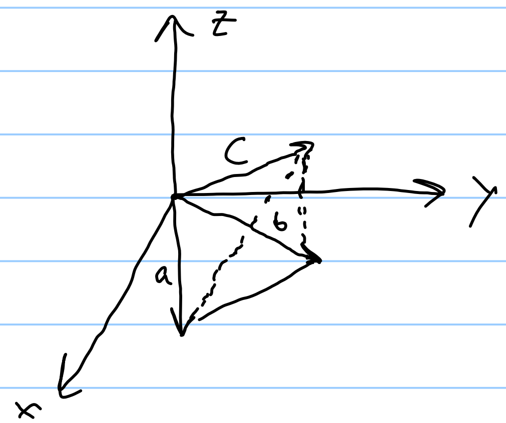
48.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The condition is that the determinant, $ad-bc$ must divide evenly each entry

50.

(a)



(b) For the base $\vec{a}$ & $\vec{b}$: $\dfrac{\vec{0}+\vec{a}+\vec{b}}{3} = \dfrac{\vec{a}+\vec{b}}{3}$

Similarly, $\dfrac{\vec{a}+\vec{c}}{3}$, $\dfrac{\vec{b}+\vec{c}}{3}$

For the last face, $\dfrac{\vec{a}+\vec{b}+\vec{c}}{3}$

51.

Note:

$$\vec{C} = \frac{\sum_{i=1}^{n} m_i \vec{r}_i}{\sum_{i=1}^{n} m_i} = \frac{\sum_{i=1}^{n} m_i \vec{r}_i}{m}$$

$$\therefore m\vec{C} = \sum_{i=1}^{n} m_i \vec{r}_i$$

$$\therefore \sum_{i=1}^{n} m_i \|\vec{r}_i - \vec{c}\|^2 + m\|\vec{r} - \vec{c}\|^2 = \sum_{i=1}^{n} m_i (\vec{r}_i - \vec{c})\cdot(\vec{r}_i - \vec{c}) + m(\vec{r} - \vec{c})\cdot(\vec{r} - \vec{c})$$

$$= \sum_{i=1}^{n} m_i (\vec{r}_i \cdot \vec{r}_i - 2\vec{r}_i \cdot \vec{c} + \vec{c}\cdot\vec{c}) + m(\vec{r} - \vec{c})\cdot(\vec{r} - \vec{c})$$

$$= \sum_{i=1}^{n} m_i \vec{r}_i \cdot \vec{r}_i - 2\sum_{i=1}^{n} m_i \vec{r}_i \cdot \vec{c} + \sum_{i=1}^{n} m_i \vec{c}\cdot\vec{c} + m(\vec{r}\cdot\vec{r} - 2\vec{r}\cdot\vec{c} + \vec{c}\cdot\vec{c})$$

$$= \sum_{i=1}^{n} m_i \vec{r}_i \cdot \vec{r}_i - 2c \cdot \sum_{i=1}^{n} m_i \vec{r}_i + \vec{c}\cdot\vec{c} \sum_{i=1}^{n} m_i + m\vec{r}\cdot\vec{r} - 2m\vec{r}\cdot\vec{c} + m\vec{c}\cdot\vec{c}$$

$$= \sum_{i=1}^{n} m_i \vec{r}_i \cdot \vec{r}_i - 2\vec{c}\cdot(m\vec{c}) + m\vec{c}\cdot\vec{c} + m\vec{r}\cdot\vec{r} - 2m\vec{r}\cdot\vec{c} + m\vec{c}\cdot\vec{c}$$

$$= \sum_{i=1}^{n} m_i \vec{r}_i \cdot \vec{r}_i + m\vec{r}\cdot\vec{r} - 2m\vec{r}\cdot c$$

$$= \sum_{i=1}^{n} m_i \vec{r}_i \cdot \vec{r}_i + m\vec{r}\cdot\vec{r} - 2\vec{r}\cdot\left(\sum_{i=1}^{n} m_i \vec{r}_i\right)$$

$$= \sum_{i=1}^{n} m_i \vec{r}_i \cdot \vec{r}_i + \vec{r}\cdot\vec{r} \sum_{i=1}^{n} m_i - 2\vec{r}\cdot\left(\sum_{i=1}^{n} m_i \vec{r}_i\right)$$

$$= \sum_{i=1}^{n} m_i \vec{r}\cdot\vec{r} - \sum_{i=1}^{n} 2 m_i \vec{r}\cdot\vec{r}_i + \sum_{i=1}^{n} m_i \vec{r}_i \cdot \vec{r}_i$$

$$= \sum_{i=1}^{n} (m_i \vec{r}\cdot\vec{r} - 2m_i \vec{r}\cdot\vec{r}_i + m_i \vec{r}_i \cdot \vec{r}_i)$$

$$= \sum_{i=1}^{n} m_i (\vec{r}\cdot\vec{r} - 2\vec{r}\cdot\vec{r}_i + \vec{r}_i \cdot \vec{r}_i)$$

$$= \sum_{i=1}^{n} m_i (\vec{r}-\vec{r}_i)\cdot(\vec{r}-\vec{r}_i) = \sum_{i=1}^{n} m_i \|\vec{r}-\vec{r}_i\|^2$$

52.

$$(x,y,z) = (1,-2,-2) + t(3,16,-1)$$

$\therefore$ parallel to $(3,16,-1)$. $\sqrt{3^2+16^2+(-1)^2} = \sqrt{266}$

$266 = 2\times 7\times 19$

$$\therefore \underline{\frac{1}{\sqrt{266}}(3,16,-1)}$$

53.

$$(1,-6,1)\cdot(x,y,z) = 12$$

Normal to plane: $(1,-6,1)$ $\sqrt{1^2+(-6)^2+1^2} = \sqrt{38}$

$$\therefore \underline{\frac{1}{\sqrt{38}}(1,-6,1)}$$

54.

The normals to each plane: $(8,1,1)$ and $(1,-1,-1)$.

$\therefore$ a vector perpendicular to each normal will be parallel to each plane.

$$\therefore (8,1,1)\times(1,-1,-1) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 8 & 1 & 1 \\ 1 & -1 & -1 \end{vmatrix} = (0,9,-9)$$

or, $(0,1,-1)$. $\therefore$ Unit vector: $\underline{\frac{1}{\sqrt{2}}(0,1,-1)}$

55.

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 1 \\ 1 & 2 & -1 \end{vmatrix} = (-2,1,0) \quad \sqrt{(-2)^2+1^2+0^2} = \sqrt{5}$$

$\therefore \underline{\frac{1}{\sqrt{5}}(-2,1,0)}$

56.

line: $(x,y,z) = (-1,-1,2) + t(2,-1,1)$

$\therefore$ orthogonal to $(2,-1,1)$ and $(1,-1,0)$

$$\therefore \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 1 & -1 & 0 \end{vmatrix} = (1,1,-1). \quad \sqrt{1^2+1^2+(-1)^2} = \sqrt{3}$$

$\therefore \underline{\frac{1}{\sqrt{3}}(1,1,-1)}$

57.

Let $\vec{v}$ be the vector $= (v_1, v_2, v_3)$

$\therefore \vec{v} \cdot \hat{i} = \|\vec{v}\| \|\hat{i}\| \cos 30^\circ$, or $v_1 = \sqrt{v_1^2 + v_2^2 + v_3^2}\left(\frac{\sqrt{3}}{2}\right)$

Or, $\frac{4}{3} v_1^2 = v_1^2 + v_2^2 + v_3^2$,

$\therefore v_1^2 = 3v_2^2 + 3v_3^2$

Equal angles w/ $\hat{j}$ and $\hat{k}$: $\frac{\vec{v} \cdot \hat{j}}{\|v\| \|\hat{j}\|} = \frac{\vec{v} \cdot \vec{k}}{\|\vec{v}\| \|\hat{k}\|} = \cos\theta$

$\therefore \frac{v_2}{\|v\|} = \frac{v_3}{\|v\|}$, or $v_2 = v_3$

$\therefore v_1^2 = 3v_2^2 + 3v_3^2 = 6v_2^2$

$\therefore v_1 = \pm\sqrt{6}\, v_2$. Choose $+\sqrt{6}\, v_2$ since

$\vec{v_1} \cdot \hat{i} = \|v\| \frac{\sqrt{3}}{2}$

Let $v_2 = v_3 = 1$. $\therefore (\sqrt{6}, 1, 1)$

Unit vector: $\frac{1}{\sqrt{8}}(\sqrt{6}, 1, 1) = \underline{\left(\frac{\sqrt{3}}{2}, \frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}\right)}$