2.1 The Geometry of Real-Valued Functions

Note Title 1/4/2016

2.

(a) vector-valued
(b) scalar-valued
(c) scalar-valued

4.

(a) $(x-y)^2 = c \Leftrightarrow x-y = \pm\sqrt{c}$, or $y = x \pm \sqrt{c}$

$\therefore$ $c = 0$: (iii) $y = 0$
$c = 1$: (ii) $y = x+1$, $y = x-1$
$c = 4$: (i) $y = x+2$, $y = x-2$

(b) $(x+y)^2 = C \iff x+y = \pm\sqrt{C}$, or $y = -x \pm \sqrt{C}$

$\therefore$ $C=0$: $y=-x$ (vi) or (iv)

$C=1$: $y=-x+1$, $y=-x-1$ (v) assuming each tick is $\frac{1}{2}$

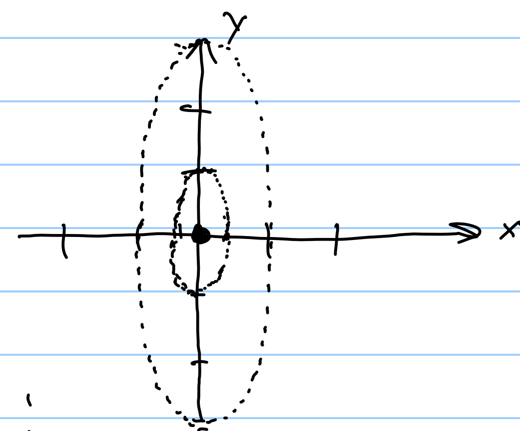
$C=4$: $y=-x+2$, $y=-x-2$ (v)

6.

(a) $0 = 9x^2+y^2 \Rightarrow x,y=0$

$9x^2+y^2=1 \iff \dfrac{x^2}{\left(\frac{1}{3}\right)^2} + y^2 = 1$

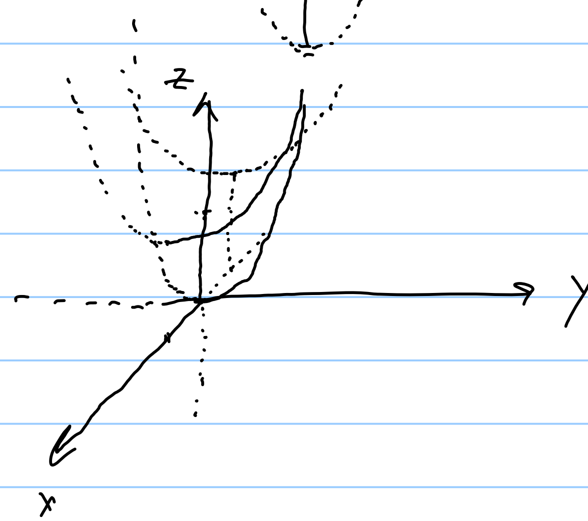
$9x^2+y^2=9 \iff x^2 + \dfrac{y^2}{3^2} = 1$



(b) $x=-1$: $z = 9+y^2$

$x=0$: $z = y^2$

$x=1$: $z = 9+y^2$



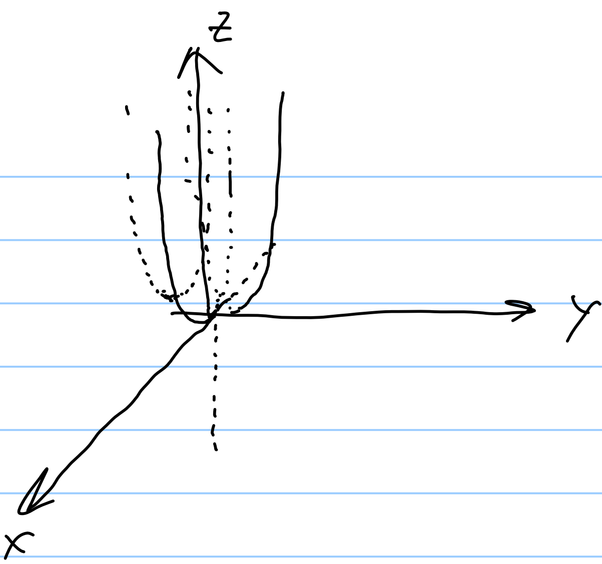
parabolas, which dip down to origin when $x=0$

(c) $y=-1$: $z = 9x^2+1$

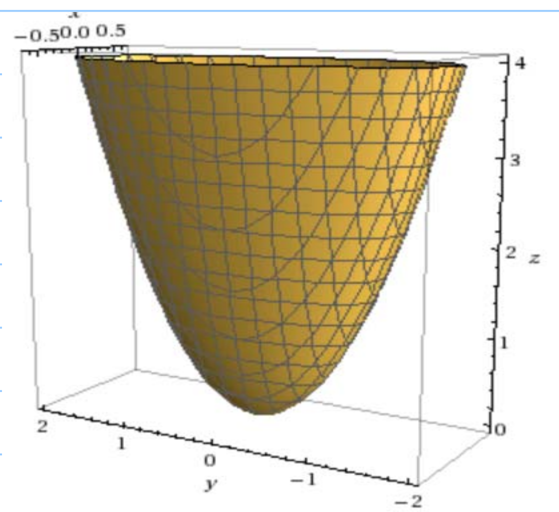
$y=0$: $z=9x^2$

$y=1$: $z=9x^2+1$

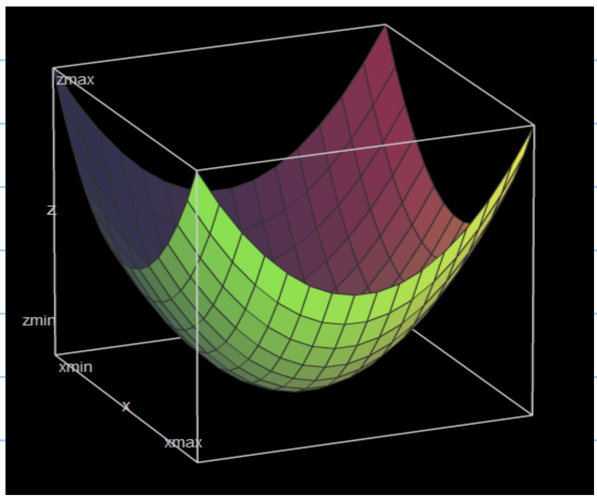
parabolas, which dip to origin when $y=0$.



(d) An elliptic paraboloid



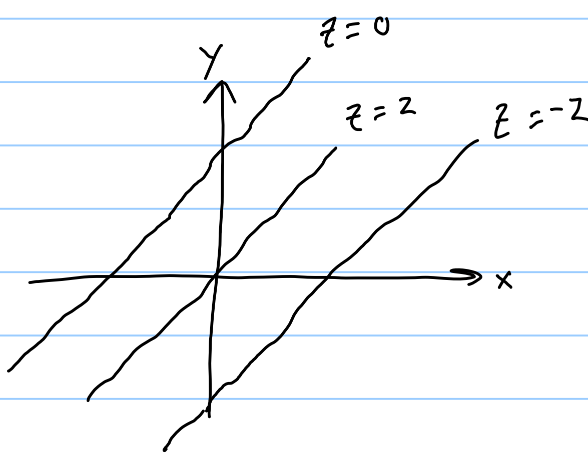
or



7.

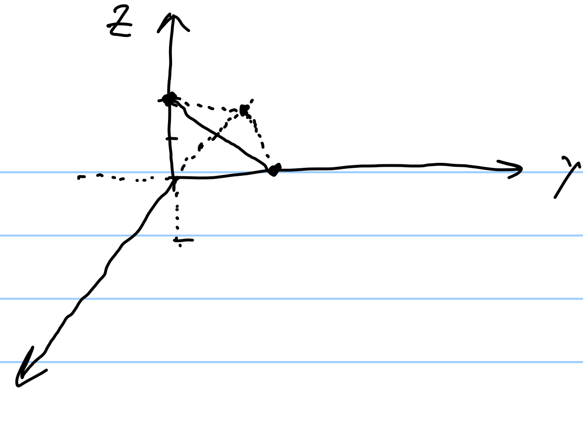
(a) Level curves:

parallel lines

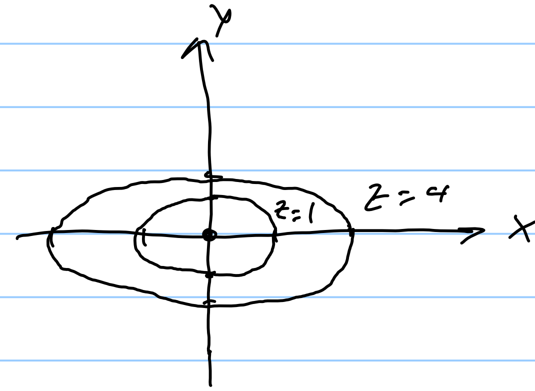


Graph:

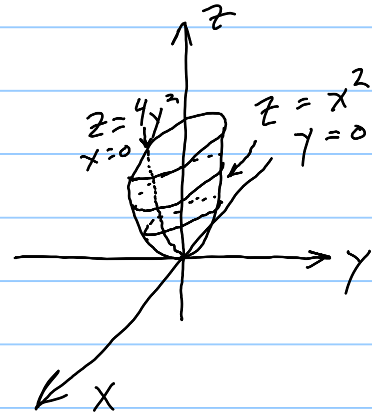
a plane seen behind yz plane.



(b) Level curves:

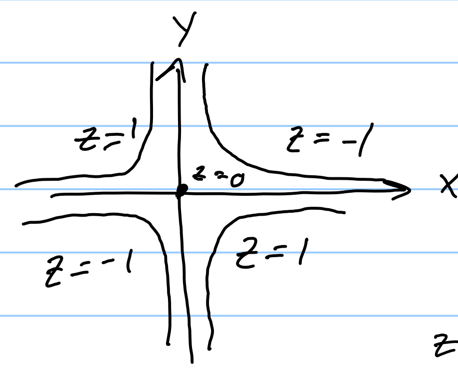


Graph:

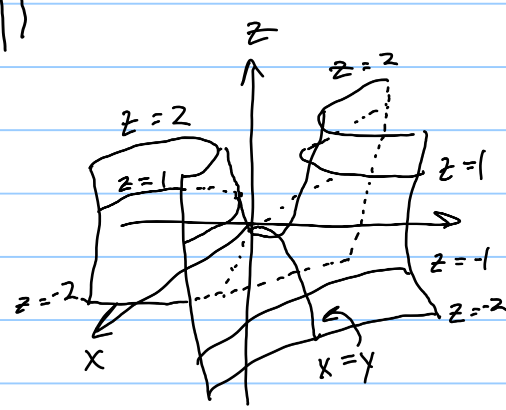


(c) Level curves:

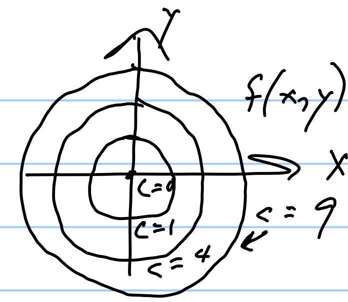
hyperbolas dipping down for $z<0$ in quadrants I, III
dipping up for quadrants II, IV



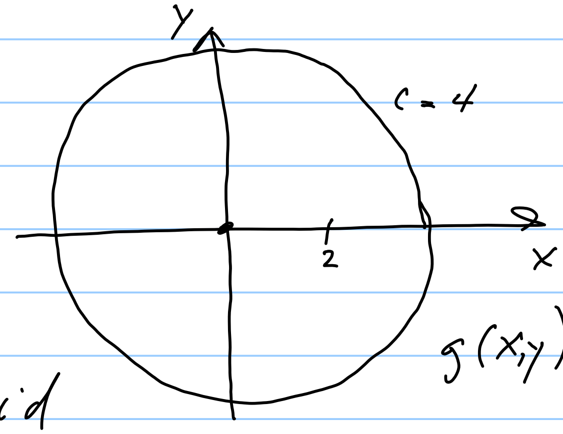
Graph: saddle or hyperbolic paraboloid



8.

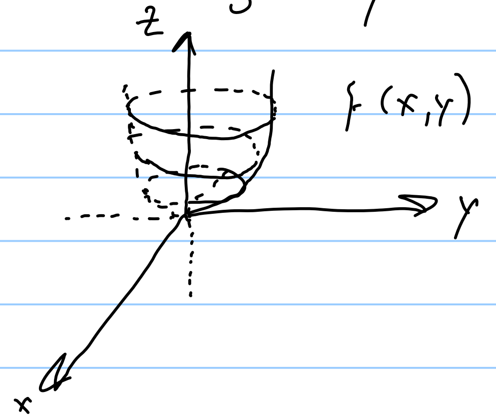


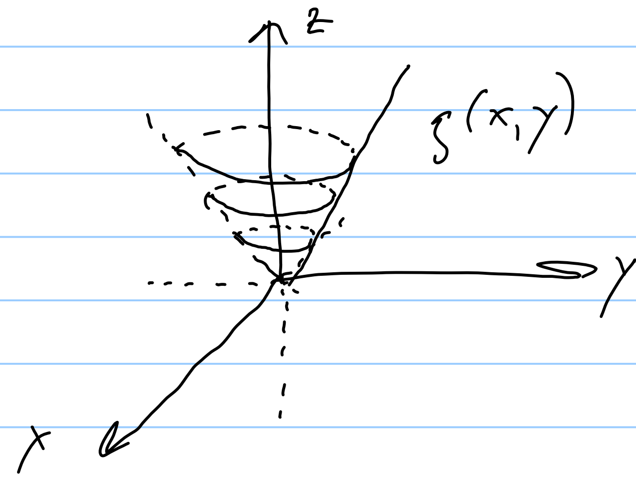
Level sets look similar (circles), but $g(x,y)$ curves are "expanded", or larger for same level values.



Graphs: $f(x,y)$: paraboloid

$g(x,y)$: cone





9.

(a) Let $f(x,y,z) = x^2y^6 - 2z$, let $c=3$

$$\therefore S = \{(x,y,z) \mid f(x,y,z) = 3\} \subset \mathbb{R}^3$$

(5) $x^2y^6 - 2z = 3 \iff x^2y^6 - 3 = 2z \iff z = \frac{x^2y^6-3}{2}$

$\therefore$ let $g(x,y) = \frac{x^2y^6-3}{2}$

$\therefore$ graph $g = \{(x,y,g(x,y)) \in \mathbb{R}^3\} = S$

i.e., graph of g = level set of f at $c=3$.

Sketch the surfaces in Exercises 27 to 39

28.

y doesn't matter. For $y=0$, this is a line with intercepts at $(4,0,0)$ and $(0,0,2)$.

$\therefore$ A plane parallel to y-axis intersecting x-axis at $(4,0,0)$ and z-axis at $(0,0,2)$

30.

Completing the square, $x^2+y^2-2x=0 \iff (x-1)^2+y^2=1$

When $z=0$, this is a circle with center at $(1,0)$.

z doesn't matter. $\therefore$ A circular cylinder perpendicular to xy-plane, centered at $(1,0,z)$.

32. $$\frac{y^2}{3^2}+\frac{z^2}{2^2}-\frac{x^2}{4^2}=1$$

When $z=0$, $\frac{y^2}{3^2}-\frac{x^2}{4^2}=1$ is an hyperbola intersecting y-axis at ± 3

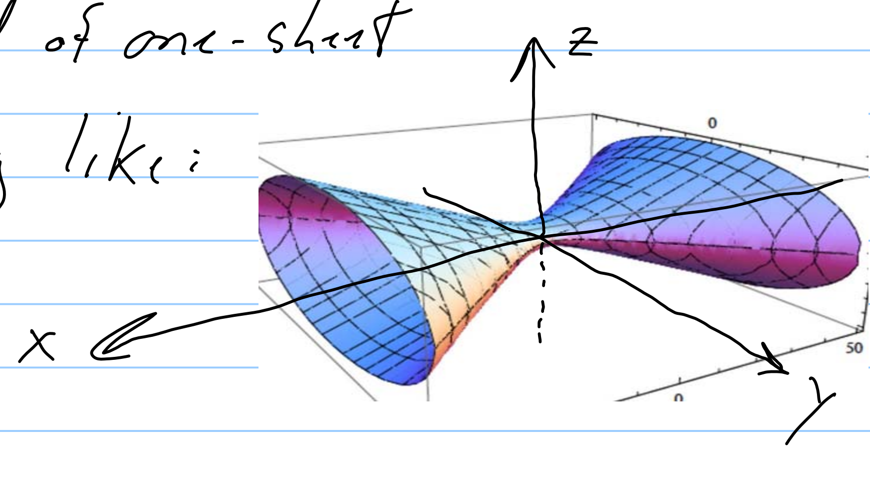
When $y=0$, $\frac{z^2}{2^2}-\frac{x^2}{4^2}=1$ is an hyperbola intersecting z-axis at ± 2

When $x=0$, $\frac{y^2}{3^2}+\frac{z^2}{2^2}=1$, an ellipse in yz-plane intercepts at $(0,\pm 3,0)$ and $(0,0,\pm 2)$

As $\pm x$ gets bigger, the ellipse proportionately expands, like a cylinder parallel to x-axis.

$\therefore$ Hyperboloid of one-sheet

Something like:



34.

x doesn't matter. For any plane $x = c$,

$y^2 + z^2 = 4$ is a circle of radius 2.

$\therefore$ A circular cylinder parallel to x-axis.

36.

For any $y = \pm c$, $x^2 + z^2 = c^2$ is a circle of radius c.

The origin $(0,0,0)$ is on the graph.

$\therefore$ A circular cone, parallel to y axis, centered at origin

38. For $x = 0$, or $y = 0$, or $z = 0$, the section is an ellipse.

$\therefore$ An ellipsoid with intercepts at

$(\pm 3, 0, 0)$, $(0, \pm\sqrt{12}, 0)$, $(0, 0, \pm 3)$

40.

$x = r\cos\theta$, $y = r\cos\theta$. $\therefore x^2 + y^2 = r^2$

$2xy = 2r^2\cos\theta\sin\theta$

$\therefore \dfrac{2xy}{x^2+y^2} \Leftrightarrow \dfrac{2r^2\cos\theta\sin\theta}{r^2} = 2\cos\theta\sin\theta$

$= \sin 2\theta$

$\therefore z = f(x,y) = g(r,\theta) = \sin 2\theta$

r does not matter.

$\therefore$ Visualize a ray $(r \geq 0)$ sweeping out a surface as θ increases from 0 to 2π.

z increases from $\theta = 0$ to $\frac{\pi}{4}$, then decreases to 0 from $\frac{\pi}{4}$ to $\frac{\pi}{2}$, then becomes negative from $\frac{\pi}{2}$ to $\frac{3}{4}\pi$, etc.

$\therefore$ An undulating surface centered at $(0,0,0)$

42.

Note that for any z, the graph is symmetric to the x-axis as $\pm y$ values leave z unchanged, and also symmetric to y-axis as $\pm x$ values leave z unchanged.

Consider $x=0$. $\therefore$ $z = 3y^2 e^{1-y^2}$

If $y=1$, $z=3$.

$$\frac{dz}{dy} = 6y e^{1-y^2} + 3y^2 e^{1-y^2}(-2y) = 6ye^{1-y^2} - 6y^3 e^{1-y^2}$$

and $\therefore$ $\left.\frac{dz}{dy}\right|_{y=1} = 6-6=0$

$$\frac{d^2z}{dy^2} = 6e^{1-y^2} + 6ye^{1-y^2}(-2y) - 18y^2 e^{1-y^2} - 6y^3 e^{1-y^2}(-2y)$$

$\therefore$ $\left.\frac{d^2z}{dy^2}\right|_{y=1} = 6 - 12 - 18 + 12 = -12 < 0$.

$\therefore$ $y=1$ a local max in $x=0$ plane

Consider $y=1$ plane. $\therefore$ $z = (x^2+3)e^{-x^2} = \frac{x^2+3}{e^{x^2}}$

At $x=0$, $z=3$. Since e^{x^2} increases faster than x^2 as x moves away from 0, $x=0$ is a local max in $y=1$ plane.

Now consider circles of constant value in xy-plane, $x^2+y^2=K$.

$\therefore$ $z = \frac{K+2y^2}{e^{K-1}}$, or $z = K' + K''y^2$

$\therefore$ z depends on y^2, like a parabola, and increases the same for $\pm y$.

Since $e^{x^2+y^2}$ increases faster than x^2+3y^2, z must get smaller on these circles.

As shown above, for $x=0$, $z=3$ was a local max, and so on the circle with $(0, \pm 1)$, z is a local max at $z=3$.

$\therefore$ Just two points at $z=3$

2.2 Limits and Continuity

Note Title 1/14/2016

1.

Since $f(x,y)$ is not defined, you don't know if $(1,3)$ is even in domain of f. $\therefore$ Can't say anything about $f(1,3)$.

2.

Since f is continuous, then the limit at (x,y) = the value at (x,y). $\therefore$ $f(1,3)$ is defined, and $\therefore$ $f(1,3) = 5$.

3.

(a) x^3y is continuous, so $\lim\limits_{(x,y)\to(0,1)} x^3y = 0^3\cdot 1 = \underline{0}$

(b) Using L'Hopital's Rule, $\lim\limits_{x\to 0} \frac{\cos x - 1}{x^2} = \lim\limits_{x\to 0} \frac{-\sin x}{2x} =$

$\lim\limits_{x\to 0} \frac{-\cos x}{2} = \underline{-\frac{1}{2}}$

(c) Using L'Hopital's Rule, $\lim\limits_{h\to 0} \frac{e^h - 1}{h} = \lim\limits_{h\to 0} \frac{e^h}{1} = e^0 = \underline{1}$

4.

(a) $\left[\lim\limits_{(x,y)\to(0,1)} e^{x}\right]\left[\lim\limits_{(x,y)\to(0,1)} y\right] = [1][1] = \underline{1}$

(b) $\left[\lim\limits_{x\to 0} \dfrac{\sin x}{x}\right]\left[\lim\limits_{x\to 0} \sin x\right] = [1][0] = \underline{0}$

(c) $\left[\lim\limits_{x\to 0} \dfrac{\sin x}{x}\right]^2 = 1^2 = \underline{1}$

5.

(a) $\lim\limits_{x\to 3}(x^2-3x+5) = 3^2-3(3)+5 = \underline{5}$

(b) $\lim\limits_{x\to 0} \sin x = \sin 0 = \underline{0}$

(c) $\lim\limits_{h\to 0} (x+h)^2-x^2 = 0$, $\lim\limits_{h\to 0} h = 0$, $\therefore$ use L'Hopital's

$$D_h\left[(x+h)^2-x^2\right] = 2(x+h) \qquad D_h h = 1.$$

$$\therefore \lim_{h\to 0} \frac{(x+h)^2-x^2}{h} = \lim_{h\to 0} \frac{2(x+h)}{1} = \underline{2x}$$

6.

(a) when $x=0$, $f(x,y) = \frac{0}{y^6} = 0$

$\therefore \lim_{(x,y)\to(0,0)} 0 = \underline{0}$

(b) When $x=y^3$, $f(x,y) = \frac{y^3 y^3}{y^6+y^6} = \frac{1}{2}$

$\therefore \lim_{(x,y)\to(0,0)} f(x,y) = \underline{\frac{1}{2}}$

(c) if $\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0$, then limit along

any path toward $(0,0)$ should be same.

(a) & (b) show this not to be so, so

$f(x,y)$ not continuous at $(0,0)$.

i.e., there is some $\epsilon > 0$ (take $\epsilon = \frac{1}{4}$)
s.t. for every $\delta > 0$ there is some
x with $0 < |x - 0| < \delta$ but
not $|f(0,0) - 0| < \epsilon$

So, from (6), for every $\delta > 0$, with $x = y^3$, $(x,y) \neq (0,0)$, $f(x,y) - 0 = \frac{1}{2}$, so

That $|f(x,y) - 0| = \frac{1}{2} < \frac{1}{4} = \epsilon$ is not true.

7.

$$f(1,2,3) = \frac{e^3}{10}$$

$$f(1,2+h,3) = \frac{e^{3+h}}{10}$$

$$\therefore \lim_{h \to 0} \frac{\frac{e^{3+h}}{10} - \frac{e^3}{10}}{h} = \frac{1}{10} \lim_{h \to 0} \frac{e^{3+h} - e^3}{h}$$

$$= \frac{1}{10} \lim_{h \to 0} \frac{D_h(e^{3+h} - e^3)}{D_h(h)} = \frac{1}{10} \lim_{h \to 0} e^{3+h} = \underline{\underline{\frac{e^3}{10}}}$$

8.

(a) $$\lim_{(x,y) \to (0,0)} \frac{(x+y)^2 - (x-y)^2}{xy} = \lim_{(x,y) \to (0,0)} \frac{(x^2 + 2xy + y^2) - (x^2 - 2xy + y^2)}{xy}$$

$$= \lim_{(x,y) \to (0,0)} \frac{4xy}{xy} = \underline{\underline{4}}$$

(b) For any x, $\lim_{y \to 0} \frac{\sin xy}{y} = \lim_{y \to 0} \frac{x \cos xy}{1} = x$

$\therefore$ For small $|x|$, it seems $\lim_{(x,y) \to (0,0)} \frac{\sin xy}{y} = 0$

Note also, $\sin(xy) = xy - \frac{(xy)^3}{3!} + \frac{(xy)^5}{5!} - \frac{(xy)^7}{7!} + \dots$

$\therefore \frac{\sin(xy)}{y} = x - \frac{x^3y^2}{3!} + \dots$, so $\lim\limits_{(x,y)\to(0,0)} = 0$

$\therefore$ Let $\epsilon > 0$

Since $\lim\limits_{x\to 0} x = 0$, choose $\delta_1 = \epsilon$

$\therefore$ if $|x| < \delta_1$ then $|x| < \epsilon$

Note $x - \sin x \geq 0$ for $0 \leq x < \frac{\pi}{2}$

since $\frac{d}{dx}(x - \sin x) = 1 - \cos x > 0$

since $\cos x < 1$ for $x < \frac{\pi}{2}$

$\therefore |\sin xy| \leq |xy|$ for $|xy| < \frac{\pi}{2}$

$\therefore$ For the above ϵ, choose $\delta = \min\{\delta_1, \frac{1}{2}\}$

$\therefore$ if $0 < \sqrt{x^2+y^2} < \delta$, and $y \neq 0$, then

$|x| < \sqrt{x^2+y^2} < \frac{1}{2}$, and $|y| < \frac{1}{2}$

$\therefore |xy| < \frac{\pi}{2}$

$\therefore |\sin xy| < |xy|$

$\therefore \left|\frac{\sin xy}{y}\right| < \frac{|xy|}{|y|} = |x| < \delta_1 = \epsilon$

$\therefore \left|\frac{\sin xy}{y}\right| < \epsilon$

$$\therefore \lim_{(x,y)\to(0,0)} \frac{\sin xy}{y} = 0$$

(c) First prove $\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} = 0$.

Let $\epsilon > 0$. Choose $\delta = \epsilon$

$\therefore$ if $0 < |(x,y)-(0,0)| = \sqrt{x^2+y^2} < \delta = \epsilon$,

Then $|x| < \sqrt{x^2+y^2}$, so $|x| < \epsilon$

$\therefore |x^3| < \epsilon x^2 < \epsilon(x^2+y^2)$

$\therefore \left|\frac{x^3}{x^2+y^2}\right| < \epsilon$, or $\left|\frac{x^3}{x^2+y^2} - 0\right| < \epsilon$

$\therefore \lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} = 0$ Similarly, $\lim_{(x,y)\to(0,0)} \frac{y^3}{x^2+y^2} = 0$

$$\therefore \lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2} = 0$$

9.

(a) Since $\lim_{y\to 0} \frac{e^{xy}-1}{y} = \lim_{y\to 0} \frac{xe^{xy}}{1} = x$, for all x

using L'Hopital's Rule.

$\therefore$ Let $\epsilon > 0$. $\exists\, \delta'$ s.t. if $0<|y|<\delta'$, then

$$\left| \frac{e^{xy}-1}{y} - x \right| < \frac{\epsilon}{2}, \text{ or } \left| \frac{e^{xy}-1}{y} \right| < \frac{\epsilon}{2} + |x|, \text{ all } x$$

since $|a|-|b| \le |a-b|$, by triangle inequality

$\therefore$ Choose $\delta = \min\left\{ \frac{\epsilon}{2}, \delta' \right\}$

$\therefore$ if $0 < |(x,y)-(0,0)| = \sqrt{x^2+y^2} < \delta$, and $y \neq 0$,

Then $0 < |y| < \delta < \delta'$ and $0 \le |x| < \delta \le \frac{\epsilon}{2}$

$$\therefore \left| \frac{e^{xy}-1}{y} - x \right| < \frac{\epsilon}{2}, \text{ and}$$

$$\therefore \left| \frac{e^{xy}-1}{y} - 0 \right| < \frac{\epsilon}{2} + |x| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

$$\therefore \lim_{(x,y)\to(0,0)} \frac{e^{xy}-1}{y} = \underline{0}$$

(b) Since $\lim_{x\to 0} \frac{\cos x - 1}{x^2} = \lim_{x\to 0} \frac{-\sin x}{2x} = \lim_{x\to 0} \frac{-\cos x}{2} = -\frac{1}{2}$

by L'Hopital's Rule, Then

Given any $\epsilon > 0$, $\exists\, \delta'$ s.t. if $0 < |x| < \delta'$

Then $\left| \frac{\cos x - 1}{x^2} + \frac{1}{2} \right| < \epsilon$

$\therefore$ Choose $\delta = \min\{\delta', 0.9\}$. $\therefore \delta < 1$.

$\therefore$ if $0 < |(x,y) - (0,0)| < \delta$, and $(x,y) \neq (0,0)$, then

$0 < \sqrt{x^2 + y^2} < \delta \Rightarrow 0 < |x| < \delta, 0 < |y| < \delta$

and $\therefore 0 < |xy| < \delta^2 < \delta \leq \delta'$

$\therefore \left| \frac{\cos(xy) - 1}{(xy)^2} - \left(-\frac{1}{2}\right) \right| < \epsilon$

$\therefore \lim\limits_{(x,y)\to(0,0)} \frac{\cos(xy) - 1}{x^2 y^2} = \underline{-\frac{1}{2}}$

(c) $\lim\limits_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2 + 2} = \frac{\lim\limits_{(x,y)\to(0,0)} xy}{\lim\limits_{(x,y)\to(0,0)} (x^2 + y^2 + 2)} = \frac{0}{2} = \underline{0}$

10.

(a) $\lim\limits_{(x,y)\to(0,0)} \frac{e^{xy}}{x+1} = \frac{\lim\limits_{(x,y)\to(0,0)} e^{xy}}{\lim\limits_{(x,y)\to(0,0)} x+1} = \frac{e^0}{0+1} = \underline{1}$

(b) With $y=0$, $\lim\limits_{x\to 0} \dfrac{\cos x - 1 - \frac{x^2}{2}}{x^4} = \lim\limits_{x\to 0} \dfrac{-\sin x - x}{4x^3} =$

$\lim\limits_{x\to 0} \dfrac{-\cos x - 1}{12x^2} = -\infty$.

$\therefore$ Limit doesn't exist

(c) With $y=0$, $\lim\limits_{(x,y)\to(0,0)} \dfrac{(x-y)^2}{x^2+y^2} = \lim\limits_{x\to 0} \dfrac{x^2}{x^2} = 1$

With $y=x$, $\lim\limits_{(x,y)\to(0,0)} \dfrac{(x-y)^2}{x^2+y^2} = \lim\limits_{x\to 0} \dfrac{(x-x)^2}{x^2+x^2} = 0$

$\therefore$ Limit doesn't exist

11.

(a) $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(xy)}{xy} = \underline{1}$

pf: Since $\lim\limits_{z\to 0} \dfrac{\sin z}{z} = 1$, and $\lim\limits_{(x,y)\to(0,0)} xy = 0$

Then use limit of composite function.

If need $\frac{\sin xy}{xy}$ to be continuous at $(0,0)$, define $\frac{\sin xy}{xy} = 1$ at $(x,y)=(0,0)$.

$\therefore f: \mathbb{R}^2 \to \mathbb{R}$, where $f(x,y) = (x,y)$

$g: \mathbb{R} \to \mathbb{R}$, $g(z) = \frac{\sin z}{z}$.

$\therefore g \circ f : \mathbb{R}^2 \to \mathbb{R}$, $(g\circ f)(x,y) = \frac{\sin (xy)}{xy}$

(3) $$\lim_{(x,y,z)\to(0,0,0)} \frac{\sin(xyz)}{xyz} = 1$$

Again, use limit of composite functions, and $\lim_{z\to 0} \frac{\sin z}{z} = 1$.

Let $f(x,y,z) = xyz$. $\therefore \lim_{(x,y,z)} xyz = 0$.

Let $g(z) = \frac{\sin z}{z}$. $\therefore (g\circ f)(x,y,z) = \frac{\sin(xyz)}{xyz}$

Using ϵ & δ, let $\epsilon > 0$. $\exists\, \epsilon' > 0$ s.t. if $0 < |r| < \epsilon'$, then $\left|\frac{\sin r}{r} - 1\right| < \epsilon$. Given $\epsilon' > 0$, $\exists\, \delta > 0$ s.t. if $0 < |xyz| < \delta$ then $|xyz| < \epsilon'$.

$\therefore$ Letting $r = xyz$, if $0 < |xyz| < \delta$, then $\left|\frac{\sin(xyz)}{xyz} - 1\right| < \epsilon$. $\therefore \lim_{(x,y,z)\to(0,0,0)} \frac{\sin(xyz)}{xyz} = 1$

(c) $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{x^2+3y^2}{x+1} = \dfrac{\lim\limits_{(x,y,z)\to(0,0,0)} x^2+3y^2}{\lim\limits_{(x,y,z)\to(0,0,0)} x+1} = \dfrac{0}{1}$

$= \underline{0}$

12.

(a) Using L'Hopital's Rule, $\lim\limits_{x\to 0} \dfrac{\sin 2x - 2x}{x^3} =$

$\lim\limits_{x\to 0} \dfrac{2\cos 2x - 2}{3x^2} = \lim\limits_{x\to 0} \dfrac{-4\sin 2x}{6x} = \lim\limits_{x\to 0} \dfrac{-8\cos 2x}{6} = \underline{-\dfrac{4}{3}}$

(b) When $y=0$, $\lim\limits_{x\to 0} \dfrac{\sin 2x - 2x}{x^3} = -\dfrac{4}{3}$ from (a)

when $x=0$, $\lim\limits_{y\to 0} \dfrac{y}{y} = 1$.

$\therefore$ Limit does not exist

(c) Note: $\lim\limits_{(y,z)\to(0,0)} 2y\cos z = 0$.

Note also if $y=z=0$, then $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{2x^2 y\cos z}{x^2+y^2} = 0$

$\therefore$ Assume $(y,z) \neq (0,0)$

$\therefore$ Given any $\epsilon > 0$, $\exists \delta > 0$ s.t. if

$0 < |(y,z) - (0,0,)| = \sqrt{y^2+z^2} < \delta$, Then

$|2y\cos z| < \epsilon$

$\therefore$ Choose above δ.

$\therefore$ If $(x,y) \neq (0,0)$, and $\sqrt{x^2+y^2+z^2} < \delta$

Then $\sqrt{y^2+z^2} \le \sqrt{x^2+y^2+z^2}$

so $\sqrt{y^2+z^2} < \delta$, $(\therefore |2y\cos z| < \epsilon)$

and $x^2+y^2 \geq x^2$ and $x^2+y^2 \neq 0$

$\therefore \quad \left|\dfrac{2x^2y\cos z}{x^2+y^2}\right| \le \left|\dfrac{2x^2y\cos z}{x^2}\right| = |2y\cos z| < \epsilon$

$\therefore \displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{2x^2y\cos z}{x^2+y^2} = \underline{0}$

13.

(a) $\lim_{x \to x_0} |x| = |x_0|$. $\therefore$ For $x_0 = 1$, $\lim_{x \to 1} |x| = \lim_{x \to 1} x = 1$

(b) $f(\vec{x}) = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$

Since $\lim_{\vec{x} \to \vec{c}} \vec{a} \cdot \vec{x} = \vec{a} \cdot \vec{c}$, $\vec{a}, \vec{x}, \vec{c} \in \mathbb{R}^n$

then $\lim_{\vec{x} \to \vec{c}} x_i = c_i$, where $x = (x_1, x_2, \ldots, x_n)$
$c = (c_1, c_2, \ldots, c_n)$

and $\therefore$ $\lim_{\vec{x} \to \vec{c}} x_i^2 = c_i^2$, so

$$\lim_{\vec{x} \to \vec{c}} x_1^2 + x_2^2 + \ldots + x_n^2 = c_1^2 + c_2^2 + \ldots + c_n^2$$

$$\therefore \lim_{\vec{x} \to \vec{c}} \sqrt{x_1^2 + \ldots + x_n^2} = \sqrt{c_1^2 + \ldots + c_n^2}$$

$$\therefore \lim_{\vec{x} \to \vec{x}_0} \|\vec{x}\| = \|\vec{x}_0\|$$

(c) $f(x) = (g(x), h(x))$, where $g(x) = x^2$, $h(x) = e^x$

Both $g(x)$ and $h(x)$ are continuous for all $x \in \mathbb{R}$.

$\therefore \lim_{x \to x_0} f(x) = (g(x_0), h(x_0))$ since $\lim_{x \to x_0} g(x) = g(x_0)$

and $\lim_{x \to x_0} h(x) = h(x_0)$

$$\therefore \lim_{x\to 1} f(x) = f(1) = (1^2, e^1) = \underline{(1, e)}$$

(d) $$f(x,y) = \frac{\left(\sin(x-y),\ e^{x(y+1)} - x - 1\right)}{\sqrt{x^2+y^2}}$$

Consider $$\lim_{(x,y)\to(0,0)} \frac{\sin(x-y)}{\sqrt{x^2+y^2}}$$

When $y=0$, $\lim_{x\to 0^+} \frac{\sin x}{x} = 1$, $\lim_{x\to 0^-} \frac{\sin x}{|x|} = -1$

$\therefore \lim_{x\to 0} \frac{\sin x}{|x|}$ doesn't exist.

$\therefore$ for $f(x,y) = (g(x,y), h(x,y))$,

$\lim_{(x,y)\to(0,0)} g(x,y)$ doesn't exist,

$\therefore \lim_{(x,y)\to(0,0)} f(x,y)$ <u>doesn't exist</u>

14.

$f(x,y,z)$ is not defined for $x^2+y^2+z^2=1$, so

$f(x,y,z)$ is not continuous for the sphere centered at the origin with radius of 1.

15.

$f(x,y)$ is not defined, and $\therefore$ not continuous, at $x^2+y^2=0$, or $(x,y)=(0,0)$.

16.

(a) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+2y \\ 3x+4y \end{bmatrix}$ $\therefore$ $\underline{\underline{f(x,y) = (x+2y,\ 3x+4y)}}$

(b) Both $g(x,y)=x+2y$ and $h(x,y)=3x+4y$ are continuous. $\therefore$ $f(x,y) = (g(x,y), h(x,y))$ is continuous, for all (x,y)

17.

Note: $\lim\limits_{r\to 0} r^2 \log r = \lim\limits_{r\to 0} \dfrac{\log r}{\frac{1}{r^2}} = \lim\limits_{r\to 0} \dfrac{\frac{1}{r}}{\frac{-2}{r^3}}$

using L'Hopital's Rule, and $\lim\limits_{r\to 0} \dfrac{\frac{1}{r}}{\frac{-2}{r^3}} = \lim\limits_{r\to 0} -\frac{1}{2}r^2 = 0$.

$\therefore \lim\limits_{r\to 0} 3r^2 \log r^2 = 6 \lim\limits_{r\to 0} r^2 \log r = 6\cdot 0 = 0$.

$\therefore$ Given any $\epsilon > 0$, $\exists\, \delta > 0$ s.t. if $0 < |r| < \delta$, Then $|3r^2 \log r^2| < \epsilon$

$\therefore$ Let $r = \sqrt{x^2+y^2}$. $\therefore r^2 = x^2+y^2$

$\therefore$ if $0 < \sqrt{x^2+y^2} < \delta$, Then

$|3(x^2+y^2)\log(x^2+y^2)| < \epsilon$

$\therefore \lim_{(x,y)\to(0,0)} (3x^2+3y^2)\log(x^2+y^2) = \underline{0}$

18.

Let $\vec{P_0} = (x_0, y_0) \in A$. $\therefore -1 < x_0 < 1$ and $-1 < y < 1$.

Let $r_x = \min\{1-x_0, x_0+1\}$

$r_y = \min\{1-y_0, y_0+1\}$

Since $x_0 < 1$, Then $0 < 1 - x_0$
$-1 < x_0$, Then $0 < x_0 + 1$ $\}$ $\therefore r_x > 0$

Since $y_0 < 1$, Then $0 < 1 - y_0$
$-1 < y_0$, Then $0 < y_0 + 1$ $\}$ $\therefore r_y > 0$

Let $r = \min\{r_x, r_y\}$

Consider $D_r(\vec{P_0})$. Let $\vec{X} = (x,y) \in D_r(\vec{P_0})$

$\therefore |\vec{x} - \vec{P}_0| < r$, or $\sqrt{(x-x_0)^2 + (y-y_0)^2} < r$,

$\therefore (x-x_0)^2 + (y-y_0)^2 < r^2$

$\therefore |x-x_0| < r \leq r_x$ and $|y-y_0| < r \leq r_y$

$\therefore -r_x < x - x_0 < r_x$

But $r_x \leq 1 - x_0$, and $r_x \leq x_0 + 1$

$\therefore -1 - x_0 \leq -r_x$

$\therefore -1 - x_0 \leq -r_x < x - x_0 < r_x \leq 1 - x_0$

$\therefore -1 - x_0 < x - x_0$ and $x - x_0 < 1 - x_0$

$\therefore -1 < x$ and $x < 1$ [1]

For y: $-r_y < y - y_0 < r_y$

But $r_y \leq 1 - y_0$ and $r_y \leq y_0 + 1$

$\therefore -1 - y_0 \leq -r_y$

$\therefore -1 - y_0 \leq -r_y < y - y_0 < r_y \leq 1 - y_0$

$\therefore -1 - y_0 < y - y_0$ and $y - y_0 < 1 - y_0$

$\therefore -1 < y$ and $y < 1$ [2]

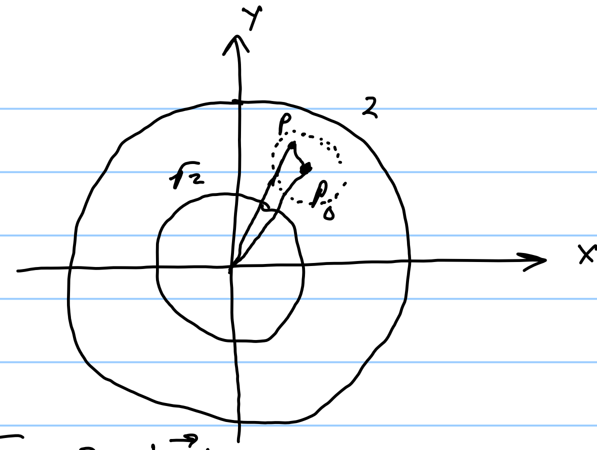
[1] + [2] : $-1 < x < 1$, $-1 < y < 1 \Rightarrow \vec{x} \in A$

$\therefore$ For any $\vec{P}_0 \in A$, $\exists\, r > 0$ s.t. $D_r(\vec{P}_0) \subset A$

$\therefore$ **A is open**

20.

Let $\vec{P_0} = (x_0, y_0) \in C$



Look at distances of $\vec{P_0}$ to edge of circles:

$|\vec{P_0}| - \sqrt{2}, \; 2 - |\vec{P_0}|$

Note: $\vec{P_0} \in C \Rightarrow 2 < |\vec{P_0}|^2 < 4$, so $\sqrt{2} < |\vec{P_0}|$ and $|\vec{P_0}| < 2$. $\therefore$ $0 < |\vec{P_0}| - \sqrt{2}$, and $0 < 2 - |\vec{P_0}|$

$\therefore$ Let $0 < r < \min\{|\vec{P_0}| - \sqrt{2}, \; 2 - |\vec{P_0}|\}$

Consider $D_r(\vec{P_0})$, let $\vec{P} = (x, y) \in D_r(\vec{P_0})$

Must show $\sqrt{2} < |\vec{P}| < 2$

Note from triangle inequality,

$$|\vec{P}| - |\vec{P_0}| \le |\vec{P} - \vec{P_0}|$$

and $|\vec{P_0}| - |\vec{P}| \le |\vec{P_0} - \vec{P}| = |\vec{P} - \vec{P_0}|$

If $\vec{P} \in D_r(\vec{P_0})$, then $|\vec{P} - \vec{P_0}| < r$

But $|\vec{P}| - |\vec{P_0}| \le |\vec{P} - \vec{P_0}| < r < 2 - |\vec{P_0}|$

$\therefore$ $|\vec{P}| - |\vec{P_0}| < 2 - |\vec{P_0}| \Rightarrow |\vec{P}| < 2$

Also, $|\vec{P_0}| - |\vec{P}| \le |\vec{P} - \vec{P_0}| < r < |\vec{P_0}| - \sqrt{2}$

$\therefore\ |\vec{P_0}| - |\vec{P}| < |\vec{P_0}| - \sqrt{2} \Rightarrow \sqrt{2} < |\vec{P}|$

$\therefore\ \sqrt{2} < |\vec{P}| < 2 \Rightarrow 2 < |\vec{P}|^2 < 4 \Rightarrow \vec{P} \in C$

$\therefore$ Given any $\vec{P_0} \in C$, $\exists\ r > 0$ s.t. $D_r(\vec{P_0}) \subset C$

$\therefore$ <u>C is open</u>

22.

Given any $\epsilon > 0$, need to find an $\delta > 0$ s.t. if
$0 < \|\vec{x} - \vec{x}_0\| < \delta$ and $\vec{x} \in A$, then $|f(\vec{x}) - 1| < \epsilon$.

But for all $\vec{x} \in A$, $f(\vec{x}) = 1 \ \therefore\ |f(\vec{x}) - 1| = |0|$, so $|f(\vec{x}) - 1| < \epsilon$ will always be true for $\vec{x} \in A$.

Here, we can choose $\delta = \epsilon$, and $\therefore$ must show that for this δ, there are values of $\vec{x} \in A$ in which $0 < \|\vec{x} - (1,0)\| < \delta$.
Then, every $\vec{x} \in A \cap D_\delta(1,0)$ will be within δ of $(1,0)$, and will also be in A, so $|f(\vec{x}) - 1| < \epsilon$ since $f(\vec{x}) = 1$. That is, $A \cap D_\epsilon(1,0) \neq \emptyset$ for all $\epsilon > 0$.

$\therefore$ Let $\epsilon > 0$ and let $\vec{x} = (x_1, x_2)$.

Choose $\delta = \min\left\{1, \frac{\epsilon}{2}\right\}$

$\therefore$ Let $x_2 = 0$ and $x_1 = 1 - \delta$. $\therefore x_2^2 = 0$.

Since $\delta < 1$, then $1 - \delta > 0$, so $x_1 > 0$.

Also, since $x_1 + \delta = 1$, $x_1 < 1$.

$\therefore 0 < x_1^2 < 1$, and so $0 < x_1^2 + x_2^2 < 1$

$\therefore (x_1, x_2) \in A$ [1]

Since $x_1 - 1 = -\delta$, $(x_1 - 1)^2 = \delta^2$

$\therefore (x_1 - 1)^2 + x_2^2 = \delta^2$

$\therefore \|\vec{x} - (1,0)\| = \delta < \epsilon$

$\therefore \vec{x} \in D_\epsilon(\vec{x}_0)$ [2]

$\therefore \vec{x} \in A \cap D_\epsilon(\vec{x}_0)$, so $A \cap D_\epsilon(\vec{x}_0) \neq \emptyset$ for any $\epsilon > 0$.

$\therefore$ Given any $\epsilon > 0$, $\exists\, \delta > 0$ s.t. if $\vec{x} \in A$ and $0 < \|\vec{x} - \vec{x}_0\| < \delta$, then $|f(\vec{x}) - 1| < \epsilon$

$\therefore \lim_{x \to x_0} f(\vec{x}) = 1$

25.

(a) $\lim_{(x,y)\to(0,0)} \frac{\sin(x+y)}{x+y} = 1$ since $\lim_{r\to 0} \frac{\sin r}{r} = 1$.

$\therefore$ Define $\frac{\sin(x+y)}{x+y} = 1$ at $(x,y) = (0,0)$

(b) $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ doesn't exist.

For $x=0$, $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = 0$

For $y=x$, $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{x^2}{2x^2} = \frac{1}{2}$.

$\therefore$ Can't give $\frac{xy}{x^2+y^2}$ a unique value at $(0,0)$ so that limit = unique value.

(c) $\lim_{(x,y)\to(a,b)} y = b$, $\lim_{(x,y)\to(a,b)} e^x = e^a$, $\lim_{(x,y)\to(a,b)} \sin x = \sin a$,

$\lim_{(x,y)\to(a,b)} (xy)^4 = (ab)^4$.

$\therefore$ Components are continuous, $\therefore$ sum of components is continuous.

26.

Let $\epsilon > 0$. Choose $\delta = \epsilon$.

If $0 < \|(x,y,z)-(0,0,0)\| = \sqrt{x^2+y^2+z^2} < \delta$, then

$|x| \le \sqrt{x^2+y^2+z^2} < \delta$

$\therefore\ |x| < \delta \Rightarrow |x^3| < \delta x^2$

Similarly $|y^3| < \delta y^2$, $|z^3| < \delta z^2$

Letting $s = \max\{|x|, |y|, |z|\}$,

then $|x| \le s, |y| \le s, |z| \le s$, so $|xyz| \le s^3$

and s^3 is one of $|x^3|, |y^3|$, or $|z^3|$

$\therefore\ s^3 \le |x^3| + |y^3| + |z^3|$

$< \delta x^2 + \delta y^2 + \delta z^2 = \delta(x^2+y^2+z^2)$

$\therefore\ |xyz| < \delta(x^2+y^2+z^2)$

$\therefore\ \left|\dfrac{xyz}{x^2+y^2+z^2} - 0\right| = \left|\dfrac{xyz}{x^2+y^2+z^2}\right| < \delta = \epsilon$

$$\therefore\ \lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2+y^2+z^2} = 0$$

Using spherical coordinates:

Let $\rho = \sqrt{x^2+y^2+z^2}$, $z = \rho\cos\phi$, $x = \rho\sin\phi\cos\theta$
$y = \rho\sin\phi\sin\theta$

$\therefore\ xyz = \rho^3\cos\phi\sin^2\phi\cos\theta\sin\theta$

$x^2+y^2+z^2 = \rho^2$

$$\therefore\ \frac{xyz}{x^2+y^2+z^2} = \rho\cos\phi\sin^2\phi\cos\theta\sin\theta$$

$$\therefore\ \lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2+y^2+z^2} = \lim_{\rho\to 0} \rho\cos\phi\sin^2\phi\cos\theta\sin\theta$$

$$= (\cos\phi\sin^2\phi\cos\theta\sin\theta)\lim_{\rho\to 0}\rho = 0$$

Note: $\|(x,y,z)-(0,0,0)\| < \delta \iff \sqrt{x^2+y^2+z^2} < \delta$

$\iff \rho < \delta \iff |\rho - 0| < \delta$

27.

(a) $\lim_{x\to 2} x^2 = 4$

Proof: Let $\epsilon > 0$. Choose $\delta = \min\left\{\sqrt{\frac{\epsilon}{2}}, \frac{\epsilon}{8}, 1\right\}$

$\therefore$ if $0 < |x-2| < \delta$, then $-\delta < x-2 < \delta$

$\therefore\ 2-\delta < x < 2+\delta$

$4-\delta < x+2 < 4+\delta$

Since $\delta \le 1$, then $4-\delta > 0$.

$\therefore\ 0 < x+2 < 4+\delta$

$\therefore\ |x+2| < 4+\delta$

$\therefore\ |x-2||x+2| < \delta(4+\delta)$

or, $|x^2-4| < \delta^2 + 4\delta$

But $\delta \le \sqrt{\frac{\epsilon}{2}}$, so $\delta^2 \le \frac{\epsilon}{2}$

and $\delta \le \frac{\epsilon}{8} \Rightarrow 4\delta \le \frac{\epsilon}{2}$

$\therefore\ |x^2-4| < \delta^2 + 4\delta \le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$

$\therefore\ 0 < |x-2| < \delta \Rightarrow |x^2-4| < \epsilon$

(b) $\lim_{x\to 2} x = 2$, $\therefore \lim_{x\to 2} x^2 = \left(\lim_{x\to 2} x\right)\left(\lim_{x\to 2} x\right) = 2\cdot 2 = 4$

28.

(a) $B_s(\vec{x}) = \{\vec{y} \in R^n : \|\vec{y}-\vec{x}\| < s\}$

Since $s < t$, then for all $\vec{y} \in B_s(\vec{x})$,

$\|\vec{y}-\vec{x}\| < s < t$. $\therefore\ y \in B_t(\vec{x})$

$\therefore\ B_s(\vec{x}) \subset B_t(\vec{x})$

(b) $U = D_r(\vec{x})$, some r, $V = D_s(\vec{x})$, some s.

If $r < s$, then by (a), $D_r(\vec{x}) \subset D_s(\vec{x})$

$\therefore U \cap V = D_r(\vec{x}) \cap D_s(\vec{x}) = D_r(\vec{x})$, a neighborhood of $\vec{x}$.

Similarly, if $s < r$, $U \cap V = D_s(\vec{x}) \cap D_r(\vec{x}) = D_s(\vec{x})$, a neighborhood of $\vec{x}$.

If $r = s$, $U \cap V = D_r(\vec{x}) \cap D_s(\vec{x}) = D_r(\vec{x}) = D_s(\vec{x})$, a neighborhood of $\vec{x}$.

(c) For each $x \in (a,b)$, there is an r s.t. $D_r(x) \subset (a,b)$.
$\therefore$ No point of (a,b) is a boundary point.

However, every $D_r(a)$ and $D_r(b)$ contains elements of (a,b), and elements not in (a,b).
$\therefore$ a and b are boundary points.

e.g.: $D_r(a) = \{x : |x-a| < r\}$, any $r > 0$.

$\therefore -r < x - a < r$, or $a - r < x < a + r$

$\therefore x = a + \frac{1}{2}r < a + r$ and $a < a + \frac{1}{2}r = x$.
$\therefore x \in (a,b)$ and $x \in D_r(a)$

And $x = a - \frac{1}{2}r < a$, and $a - r < a - \frac{1}{2}r$.
$\therefore x \notin (a,b)$ and $x \in D_r(a)$

29.

Since $\vec{x} \neq \vec{y}$, $\|\vec{x}-\vec{y}\| \neq 0$. Let $r = \|\vec{x}-\vec{y}\|$

$\therefore$ Let $f(\vec{z}) = \dfrac{\|\vec{z}-\vec{y}\|}{\|\vec{x}-\vec{y}\|}$. $\therefore f(\vec{y}) = 0$, $f(\vec{x}) = 1$.

But $f(\vec{z})$ will only be ≤ 1 when $\|\vec{z}-\vec{y}\| \leq \|\vec{x}-\vec{y}\|$

$\therefore$ Define $f(\vec{z}) = \begin{cases} \dfrac{\|\vec{z}-\vec{y}\|}{\|\vec{x}-\vec{y}\|}, & \text{for } \vec{z} \in D_r(\vec{y}) \\ 1, & \text{for } \vec{z} \notin D_r(\vec{y}) \end{cases}$

$f(\vec{z})$ is continuous, since

$f(\vec{z}) = \frac{1}{r}\sqrt{(z_1-y_1)^2 + \dots + (z_n-y_n)^2}$ is a

continuous function of $\vec{z} \in R^n$

and $f(\vec{z}) = 1$ is continuous for $\vec{z} \in R^n$

and $f(\vec{z}) = 1$ at the boundary of $D_r(\vec{y})$

30.

(a) Let $N > 0$. Look at $\frac{1}{(x-1)^2} > N$

$\therefore \frac{1}{N} > (x-1)^2$, or $(x-1)^2 < \frac{1}{N}$, $|x-1| < \frac{1}{\sqrt{N}}$

$\therefore$ Choose $\delta = \frac{1}{\sqrt{N}}$.

$\therefore$ if $0 < |x-1| < \frac{1}{\sqrt{N}}$, Then $|x-1|^2 = (x-1)^2 < \frac{1}{N}$,

and $\therefore$ $f(x) = \frac{1}{(x-1)^2} > N$ $\quad\therefore \lim\limits_{x \to 1} \frac{1}{(x-1)^2} = \infty$

(b) Let $N > 0$. Look at $\frac{1}{|x|} > N$, or $\frac{1}{N} > |x|$,

i.e. $|x| < \frac{1}{N}$. $\therefore$ Choose $\delta = \frac{1}{N}$.

$\therefore$ If $\big|\,|x| - 0\,\big| = \big|\,|x|\,\big| = |x| < \delta = \frac{1}{N}$,

Then $\frac{1}{|x|} = f(x) > N$.

$\therefore \lim\limits_{x \to 0} \frac{1}{|x|} = \infty$

$\lim\limits_{x \to 0^+} \frac{1}{x} = +\infty$, $\lim\limits_{x \to 0^-} \frac{1}{x} = -\infty$. $\therefore \lim\limits_{x \to 0} \frac{1}{x} \neq \infty$

(c) Let $N > 0$ $\quad$ Choose $\delta = \frac{1}{\sqrt{N}}$

$\therefore$ if $0 < |(x,y) - (0,0)| < \delta$, Then

$$0 < \sqrt{x^2+y^2} < \frac{1}{\sqrt{N}} \Rightarrow x^2 + y^2 < \frac{1}{N},$$

and $\therefore \frac{1}{x^2+y^2} > N$ $\quad\therefore \lim\limits_{(x,y) \to (0,0)} \frac{1}{x^2+y^2} = \infty$

31.

(a) Let $b \in \mathbb{R}$ and $f: \mathbb{R} \setminus \{b\} \to \mathbb{R}$ be a function.

$\lim_{x \to b^+} f(x) = L \iff$ for every $\epsilon > 0$ there is a

$\delta > 0$ s.t. $b < x$ and $0 < x - b < \delta$

imply $|f(x) - L| < \epsilon$.

(b) $\lim_{x \to 0^-} \frac{1}{1+e^{1/x}}$

$\lim_{x \to 0^-} \frac{1}{x} = -\infty$, and $\lim_{x \to -\infty} e^x = 0$

$\therefore \lim_{x \to 0^-} e^{\frac{1}{x}} = 0$. $\therefore \lim_{x \to 0^-} \frac{1}{1+e^{1/x}} = 1$

$\lim_{x \to 0^+} \frac{1}{1+e^{1/x}} = \lim_{x \to 0^+} \frac{e^{-\frac{1}{x}}}{e^{-1/x}+1}$

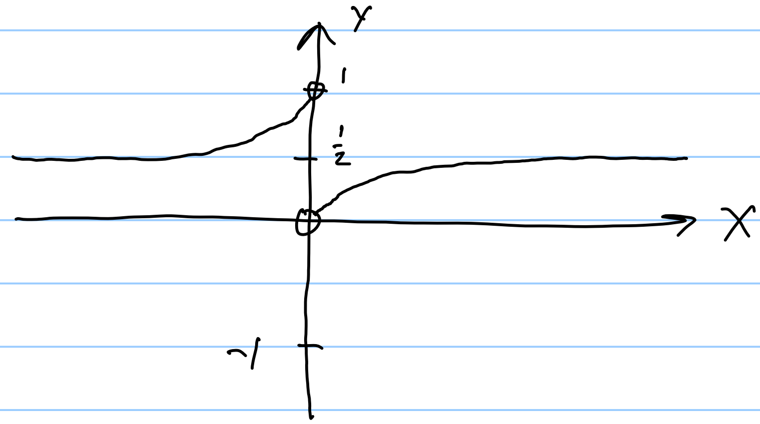
$\lim_{x \to 0^+} -\frac{1}{x} = -\infty$. $\lim_{x \to -\infty} e^x = 0$. $\therefore \lim_{x \to 0^+} e^{-\frac{1}{x}} = 0$

$\therefore \lim_{x \to 0^+} \frac{e^{-\frac{1}{x}}}{e^{-1/x}+1} = \frac{0}{0+1} = 0 = \lim_{x \to 0^+} \frac{1}{1+e^{1/x}}$

(c) As $x \to +\infty$, $\frac{1}{1+e^{1/x}} \to \frac{1}{2}$

As $x \to -\infty$, $\frac{1}{1+e^{1/x}} \to \frac{1}{2}$

$\therefore$



32.

Note that $\| f(\vec{x}) - f(\vec{x}_0) \|$ is equivalent to

$\| \| f(\vec{x}) - f(\vec{x}_0) \| - 0 \|$

$\therefore$ Given $\epsilon > 0$

(1) If $\lim_{\vec{x} \to \vec{x}_0} f(\vec{x}) = f(\vec{x}_0)$, then $\exists\, \delta > 0$ s.t. if $0 < \| \vec{x} - \vec{x}_0 \| < \delta$ then $\| f(\vec{x}) - f(\vec{x}_0) \| < \epsilon$, and

$\therefore \| \| f(\vec{x}) - f(\vec{x}_0) \| - 0 \|$. $\therefore \lim_{\vec{x} \to \vec{x}_0} \| f(\vec{x}) - f(\vec{x}_0) \| = 0$

(2) If $\lim_{\vec{x} \to \vec{x}_0} \| f(\vec{x}) - f(\vec{x}_0) \| = 0$, then $\exists\, \delta > 0$ s.t.

if $0 < \|\vec{x} - \vec{x}_0\| < \delta$ then $\big\| \|f(\vec{x}) - f(\vec{x}_0)\| - 0 \big\| < \epsilon$,

and $\therefore \|f(\vec{x}) - f(\vec{x}_0)\| < \epsilon$. $\therefore \lim_{\vec{x} \to \vec{x}_0} f(\vec{x}) = f(\vec{x}_0)$

33.

For a given function f, K and α are known and the condition is true for all $\vec{x}, \vec{y} \in A$.

$\therefore$ Let $\epsilon > 0$ and let $\vec{x}_0$ be any element in A.

Choose $\delta = \sqrt[\alpha]{\frac{\epsilon}{K}}$. Note $K, \alpha > 0$.

$\therefore$ if $\|\vec{x} - \vec{x}_0\| < \delta$, then

$\|\vec{x} - \vec{x}_0\| < \sqrt[\alpha]{\frac{\epsilon}{K}}$, so $K\|\vec{x} - \vec{x}_0\|^\alpha < \epsilon$

$\therefore \|f(\vec{x}) - f(\vec{x}_0)\| \le K\|\vec{x} - \vec{x}_0\| < \epsilon$

$\therefore \lim_{\vec{x} \to \vec{x}_0} f(\vec{x}) = f(\vec{x}_0)$ and $\therefore f$ is continuous

at x_0, so f is continuous on A.

34.

(a) Suppose f is continuous at all points of R^n.

Let B be any open set of R^m containing values of $f(\vec{x})$, where $\vec{x} \in R^n$. Since B is open, $\exists\, D_\epsilon(f(\vec{x})) \subset B$, $\epsilon > 0$. Since f is continuous, $\exists\, \delta > 0$ s.t. for $\vec{y} \in D_\delta(\vec{x})$, $f(\vec{y}) \in D_\epsilon(f(\vec{x}))$.

Note that $D_\delta(\vec{x})$ is open.

$\therefore$ For each point $f(\vec{x})$ in B, let $A =$ union of all such corresponding $D_\delta(\vec{x})$.

The union of open sets is open. $\therefore$ The union of all such $D_\delta(\vec{x})$ is open, and the union of all the $D_\delta(\vec{x})$ represents the inverse image of B.

(b) Assume inverse image of every open set of R^m is open.

Let $\vec{x}_0 \in R^n$ and let $\epsilon > 0$. $D_\epsilon(f(\vec{x}_0))$ is open. Let A be the inverse image of

$D_\epsilon(f(\vec{x}_0))$. Since $\vec{x}_0 \in A$ and A is open, $\exists\ \delta > 0$ s.t. $D_\delta(\vec{x}_0) \subset A$. But since A is the inverse image, then all of the image of $D_\delta(\vec{x}_0)$ is contained on $D_\epsilon(f(\vec{x}_0))$.

$\therefore$ Given any $\epsilon > 0$, $\exists\ \delta$ s.t. if $\vec{x} \in D_\delta(\vec{x}_0)$ then $\|\vec{x} - \vec{x}_0\| < \delta$ by definition of $D_\delta(\vec{x}_0)$, and $\|f(\vec{x}) - f(\vec{x}_0)\| < \epsilon$ since $f(\vec{x}) \in D_\epsilon(f(\vec{x}_0))$.

$\therefore \lim_{\vec{x} \to \vec{x}_0} f(\vec{x}) = f(\vec{x}_0)$, so f is continuous.

35.

(a) $|a^3 + 3a^2 + a| \leq |a^3| + 3|a^2| + |a|$

If $|a| < 1$, then $|a^2| < |a|$, $|a|^3 < a$.

$\therefore |a^3| + 3|a^2| + |a| < |a| + 3|a| + |a| = 5|a|$

$\therefore$ if $|a| < \frac{1}{500}$, then $5|a| < \frac{1}{100}$. $\therefore$ let $\underline{\delta = \frac{1}{500}}$

(6) $|x^2+y^2+3xy+180xy^5| \le |x^2+y^2| + 3|xy| + 180|xy|y^4$

Note if $x^2+y^2<\delta^2$, then $x^2<\delta^2$, $y^2<\delta^2$, so

$|x|<\delta$, $|y|<\delta$, $\therefore |xy|<\delta^2$

If $\delta<1$, then $y^2<\delta^2 \Rightarrow y^4<\delta^4<\delta^2$

$\therefore |x^2+y^2| + 3|xy| + 180|xy|y^4 < \delta^2+3\delta^2+180\delta^2\delta^2$

$< 184\delta^2 < (14\delta)^2$

$\therefore$ if $(14\delta)^2 < \frac{1}{10^4}$, or $14\delta < \frac{1}{10^2}$, or

$\underline{\delta < \frac{1}{1400}} \approx 0.0007$, then

$|x^2+y^2+3xy+180xy^5| < \frac{1}{10{,}000}$

2.3 Differentiation

Note Title 2/2/2016

1.

(a) $f_x = y$, $f_y = x$

(b) $f_x = ye^{xy}$, $f_y = xe^{xy}$

(c) $f_x = (\cos y)\frac{\partial}{\partial x}(x\cos x) = (\cos y)(\cos x - x\sin x)$

$= \cos x\cos y - x\sin x\cos y$

$f_y = -x\cos x\sin y$

(d) $f_x = \frac{\partial(x^2+y^2)}{\partial x}\left[\log(x^2+y^2)\right] + \left[x^2+y^2\right]\frac{\partial \log(x^2+y^2)}{\partial x}$

$= 2x\left[\log(x^2+y^2)\right] + \left[x^2+y^2\right]\frac{1}{x^2+y^2}(2x)$

$= 2x\left[1+\log(x^2+y^2)\right]$

$f_y = 2y\log(x^2+y^2) + (x^2+y^2)\frac{1}{x^2+y^2}(2y)$

$= 2y\left[1+\log(x^2+y^2)\right]$

2.

(a) $f_x = \frac{1}{2}(a^2-x^2-y^2)^{-\frac{1}{2}}(-2x) = -x\,(a^2-x^2-y^2)^{-\frac{1}{2}}$

$f_y = -y\,(a^2-x^2-y^2)^{-\frac{1}{2}}$

$\therefore\ \underline{f_x(0,0) = 0}\ ,\quad \underline{f_y(0,0) = 0}$

$f_x\left(\frac{a}{2},\frac{a}{2}\right) = -\frac{a}{2}\left(a^2 - \frac{a^2}{4} - \frac{a^2}{4}\right)^{-\frac{1}{2}} = -\frac{\frac{a}{2}}{\sqrt{\frac{a^2}{2}}}$

$= \underline{-\frac{a}{|a|}\frac{\sqrt{2}}{2}}$

$f_y\left(\frac{a}{2},\frac{a}{2}\right) = \underline{-\frac{a}{|a|}\frac{\sqrt{2}}{2}}$

(b) $f_x = \frac{1}{\sqrt{1+xy}} \cdot \frac{1}{2}(1+xy)^{-\frac{1}{2}}(y) = \frac{y}{2(1+xy)}$

$f_y = \frac{x}{2(1+xy)}$

$\therefore\ f_x(1,2) = \frac{2}{2(1+2)} = \underline{\frac{1}{3}} \qquad \underline{f_y(1,2) = \frac{1}{6}}$

$\underline{f_x(0,0) = 0} \qquad \underline{f_y(0,0) = 0}$

(c) $f_x = ae^{ax}\cos(bx+y) - e^{ax}b\sin(bx+y)$

$f_y = -e^{ax}\sin(bx+y)$

$\therefore f_x\left(\frac{2\pi}{b}, 0\right) = \underline{ae^{\frac{2a\pi}{b}}}$

$f_y\left(\frac{2\pi}{b}, 0\right) = \underline{0}$

3.

(a) $w_x = e^{x^2+y^2} + xe^{x^2+y^2}(2x)$
$= \underline{(2x^2+1)e^{x^2+y^2}}$

$w_y = xe^{x^2+y^2}(2y) = \underline{2xye^{x^2+y^2}}$

(b) $w_x = \dfrac{(x^2-y^2)(2x) - (x^2+y^2)(2x)}{(x^2-y^2)^2}$

$= \underline{\dfrac{-4xy^2}{(x^2+y^2)^2}}$

$w_y = \dfrac{(x^2-y^2)(2y) - (x^2+y^2)(-2y)}{(x^2-y^2)^2} = \underline{\dfrac{4x^2y}{(x^2+y^2)^2}}$

(c) $w_x = ye^{xy}\log(x^2+y^2) + e^{xy}\cdot\frac{1}{x^2+y^2}\cdot 2x$

$w_y = xe^{xy}\log(x^2+y^2) + e^{xy}\cdot\frac{1}{x^2+y^2}\cdot 2y$

(d) $w_x = \frac{1}{y}$ $\quad w_y = -\frac{x}{y^2}$

(e) $w_x = -\sin(ye^{xy})\cdot(y^2e^{xy})\sin x + \cos(ye^{xy})\cos x$

$w_y = -\sin(ye^{xy})\cdot(e^{xy} + xye^{xy})\sin x$

4.

(a) $f_x = \frac{(x^2+y^2)^2 2y - 2xy[2(x^2+y^2)\cdot 2x]}{(x^2+y^2)^4}$

$f_y = \frac{(x^2+y^2)^2 2x - 2xy[2(x^2+y^2)2y]}{(x^2+y^2)^4}$

$\therefore$ f_x and f_y are continuous for $(x,y)\neq(0,0)$.

$\therefore$ $f(x,y)$ is differentiable and C^1 in its domain: $\mathbb{R}^2-(0,0)$.

(b) $f_x = \frac{1}{y} - \frac{y}{x^2}$, $f_y = -\frac{x}{y^2} + \frac{1}{x}$

$\therefore$ f_x and f_y are continuous for all $(x,y) \neq (0,0)$.

$\therefore$ $f(x,y)$ is differentiable and C' in its domain: $R^2 - (0,0)$

(c) $f_r = \frac{1}{2} \sin 2\theta$, $f_\theta = \frac{1}{2} r (2\cos 2\theta) = r\cos 2\theta$

f_r, f_θ are continuous for $r > 0$, all θ.

$\therefore$ $f(r,\theta)$ is C' for its domain.

(d) $$f_x = \frac{(x^2+y^2)^{1/2}\, y - xy\left[\frac{1}{2}(x^2+y^2)^{-\frac{1}{2}} 2x\right]}{(x^2+y^2)}$$

$$f_y = \frac{(x^2+y^2)^{1/2}\, x - xy\left[\frac{1}{2}(x^2+y^2)^{-\frac{1}{2}} 2y\right]}{(x^2+y^2)}$$

f_x, f_y are continuous for all $(x,y) \neq (0,0)$

$\therefore$ $f(x,y)$ is C' in its domain: $R^2 - (0,0)$

(e) $f_x = \frac{(x^4+y^2)\, 2xy - x^2 y (4x^3)}{(x^4+y^2)^2}$, continuous for $(x,y) \neq (0,0)$

$f_y = \frac{(x^4+y^2)\, x^2 - x^2 y (2y)}{(x^4+y^2)^2}$, continuous for $(x,y) \neq (0,0)$

$\therefore$ $f(x,y)$ is C' it its domain; $R^2 - (0,0)$

5.

$f_x = 2x$, $\therefore f_x(3,1) = 6$

$f_y = 3y^2$, $\therefore f_y(3,1) = 3$

$\therefore z = f(3,1) + f_x(3,1)(x-3) + f_y(3,1)(y-1)$

$\therefore z = 10 + 6(x-3) + 3(y-1)$

$= 10 + 6x - 18 + 3y - 3$

$\therefore \underline{z = 6x + 3y - 11}$

6.

$f_x = f_y = e^{x+y}$ $\quad f_x(0,0) = f_y(0,0) = e^0 = 1$

$f(0,0) = e^0 = 1.$ $\quad \therefore z = 1 + 1(x-0) + 1(y-0)$

$\therefore \underline{z = x + y + 1}$

7.

$f(1,1) = e^0 = 1$, $f_x = e^{x-y}$, $f_y = -e^{x-y}$

$\therefore f_x(1,1) = e^0 = 1$, $f_y(1,1) = -e^0 = -1$

$\therefore\ z = 1 + 1(x-1) - 1(y-1)$
$= 1 + x - 1 - y + 1$

$\therefore\ \underline{z = x - y + 1}$

8.

(a) $f(x,y) = xy$. $f_x = y$, $f_y = x$ $\quad f(0,0) = 0$
$f_x(0,0) = f_y(0,0) = 0$

$\therefore\ z = 0 + 0(x-0) + 0(y-0)$
$\therefore\ \underline{z = 0}$

(b) $f(x,y) = e^{xy}$. $\therefore\ f_x = ye^{xy}$, $f_y = xe^{xy}$

$f(0,1) = e^0 = 1 \quad f_x(0,1) = 1 \cdot e^0 = 1 \quad f_y(0,1) = 0 \cdot e^0 = 0$

$\therefore\ z = 1 + 1(x-0) + 0(y-1)$

$\therefore\ \underline{z = x + 1}$

(c) $f(x,y) = x\cos x \cos y \qquad f(0,\pi) = 0$

$f_x = \cos x \cos y - x \sin x \cos y \quad f_x(0,\pi) = -1$

$f_y = -x\cos x \sin y \qquad f_y(0,\pi) = 0$

$\therefore z = 0 - 1(x-0) + 0(y-\pi)$

$\therefore \underline{z = -x}$

(d) $f(x,y) = (x^2+y^2)\log(x^2+y^2)$ $\quad f(0,1) = 0$

$f_x = 2x[1+\log(x^2+y^2)]$ $\quad f_x(0,1) = 0$

$f_y = 2y[1+\log(x^2+y^2)]$ $\quad f_y(0,1) = 2$

$\therefore z = 0 + 0(x-0) + 2(y-1)$

$\therefore \underline{z = 2y-2}$

9.

(a) $f_1(x,y) = x$ $\quad \therefore f_{1x} = 1 \quad f_{1y} = 0$
$f_2(x,y) = y$ $\quad f_{2x} = 0 \quad f_{2y} = 1$

$$\therefore \Delta f(\vec{x}) = \underline{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$$

(b) $f_1(x,y) = xe^y + \cos y$ $\quad \therefore f_{1x} = e^y, \ f_{1y} = xe^y - \sin y$
$f_2(x,y) = x$ $\quad f_{2x} = 1, \ f_{2y} = 0$
$f_3(x,y) = x + e^y$ $\quad f_{3x} = 1, \ f_{3y} = e^y$

$$\therefore Df(\vec{x}) = \begin{bmatrix} e^y & xe^y - \sin y \\ 1 & 0 \\ 1 & e^y \end{bmatrix}$$

(c) $f_1(x,y,z) = x + e^z + y \quad \therefore f_{1x} = 1 \quad f_{1y} = 1 \quad f_{1z} = e^z$

$f_2(x,y,z) = yx^2 \quad f_{2x} = 2xy \quad f_{2y} = x^2 \quad f_{2z} = 0$

$$\therefore Df(\vec{x}) = \begin{bmatrix} 1 & 1 & e^z \\ 2xy & x^2 & 0 \end{bmatrix}$$

(d) $f_1(x,y) = xye^{xy} \quad f_{1x} = ye^{xy} + xy^2e^{xy} \quad f_{1y} = xe^{xy} + x^2ye^{xy}$

$f_2(x,y) = x\sin y \quad f_{2x} = \sin y \quad f_{2y} = x\cos y$

$f_3(x,y) = 5xy^2 \quad f_{3x} = 5y^2 \quad f_{3y} = 10xy$

$$\therefore Df(\vec{x}) = \begin{bmatrix} ye^{xy} + xy^2e^{xy} & xe^{xy} + x^2ye^{xy} \\ \sin y & x\cos y \\ 5y^2 & 10xy \end{bmatrix}$$

10.

(a) $f_1(x,y) = e^x \quad f_{1x} = e^x \quad f_{1y} = 0$

$f_2(x,y) = \sin xy \quad f_{2x} = y\cos xy \quad f_{2y} = x\cos xy$

$$\therefore \Delta f(\vec{x}) = \begin{bmatrix} e^x & 0 \\ y\cos xy & x\cos xy \end{bmatrix}$$

(b) $f_1(x,y,z) = x-y \qquad f_{1x} = 1 \qquad f_{1y} = -1 \qquad f_{1z} = 0$

$f_2(x,y,z) = y+z \qquad f_{2x} = 0 \qquad f_{2y} = 1 \qquad f_{2z} = 1$

$$\therefore \Delta f(\vec{x}) = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

(c) $f_1(x,y) = x+y \qquad f_{1x} = 1 \qquad f_{1y} = 1$

$f_2(x,y) = x-y \qquad f_{2x} = 1 \qquad f_{2y} = -1$

$f_3(x,y) = xy \qquad f_{3x} = y \qquad f_{3y} = x$

$$\therefore \Delta f(\vec{x}) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ y & x \end{bmatrix}$$

(d) $f_1(x,y,z) = x+z \qquad f_{1x} = 1 \qquad f_{1y} = 0 \qquad f_{1z} = 1$

$f_2(x,y,z) = y-5z \qquad f_{2x} = 0 \qquad f_{2y} = 1 \qquad f_{2z} = -5$

$f_3(x,y,z) = x-y \qquad f_{3x} = 1 \qquad f_{3y} = -1 \qquad f_{3z} = 0$

$$\therefore \Delta f(\vec{x}) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -5 \\ 1 & -1 & 0 \end{bmatrix}$$

11.

$f_x = 2x - 2y \qquad f_y = -2x + 4y$

Let (x,y) be the point of tangency

$\therefore f_x(x,y) = 2 = 2x - 2y$
$\quad f_y(x,y) = 4 = -2x + 4y$

Solving, $6 = 2y \Rightarrow y = 3$
$\therefore 2x - 2(3) = 2 \Rightarrow x = 4$

$\therefore$ tangent point $= (4,3)$

$\therefore f_x(4,3) = 2, \quad f_y(4,3) = 4$

$f(4,3) = 4^2 - 2(4)(3) + 2(3)^2 = 16 - 24 + 18 = 10$

$\therefore z = 10 + 2(x-4) + 4(y-3)$
$= 10 + 2x - 8 + 4y - 12$

$\therefore \underline{z = 2x + 4y - 10}$

12.

(a) $f(0,0) = e^0 = 1$

$f_x = 2e^{(2x+3y)} \qquad f_y = 3e^{(2x+3y)}$

$f_x(0,0) = 2 \qquad f_y(0,0) = 3$

$\therefore\ z = 1 + 2(x-0) + 3(y-0)$
$= \underline{2x + 3y + 1}$

(b) $f(.1, 0)$: $\Delta x = 0.1$, $\Delta y = 0$

$\therefore\ f(\Delta x, \Delta y) \approx 1 + 2\Delta x + 3\Delta y$
$= 1 + 2(0.1) + 3(0)$
$\therefore\ \underline{f(.1, 0) \approx 1.2}$

$f(0, .1)$: $\Delta x = 0$, $\Delta y = 0.1$

$\therefore\ f(\Delta x, \Delta y) \approx 1 + 2\Delta x + 3\Delta y$
$= 1 + 2(0) + 3(0.1)$
$\therefore\ \underline{f(0, .1) \approx 1.3}$

(c) $f(.1, 0) = e^{(2(.1) + 3(0))} = e^{.2} = 1.2214$

$f(0, .1) = e^{(2(0) + 3(.1))} = e^{.3} = 1.34986$

13.

$f(x,y) = e^{x-y}$ $\quad f(1,1) = 1$

$f_x = e^{x-y}$ $\quad f_x(1,1) = e^0 = 1$

$f_y = -e^{x-y}$ $\quad f_y(1,1) = -e^0 = -1$

$\therefore$ Plane: $z = 1 + 1(x-1) - 1(y-1)$
$= x - y + 1$

Meet z-axis at $(0,0,z)$

$\therefore z = (0) - (0) + 1 = 1$

$\therefore$ Meet z-axis at $\underline{(0,0,1)}$

14.

$f(0,0) = 0 \quad f_x = 2x \quad f_y = 2y \quad f_x(0,0) = 0, \; f_y(0,0) = 0$

Tangent plane: $z = 0 + 0(x-0) + 0(y-0) = 0$

$\therefore z = 0$

$g(0,0) = 0 \quad g_x = -2x + y^3 \quad g_y = -2y + 3xy^2$

$g_x(0,0) = 0 \quad g_y(0,0) = 0$

Tangent plane: $z = 0 + 0(x-0) + 0(y-0) = 0$

$\therefore z = 0$.

Both $f(x,y)$ and $g(x,y)$ have a <u>tangent plane of $z=0$ at $(0,0)$</u>.

15.

$f_x = y e^{xy} \quad f_y = x e^{xy}$.

$\therefore x \dfrac{\partial f}{\partial x} = xy e^{xy} = yx e^{xy} = y \dfrac{\partial f}{\partial y}$

16.

(a) Let $f(x,y) = (x e^{y})^{8} = x^{8} e^{8y}$

Let $x = 1$, $y = 0$, $\Delta x = -0.01$, $\Delta y = 0.02$

$\therefore f(1,0) = 1^{8} e^{0} = 1$ $\quad f_x = 8x^{7} e^{8y}$ $\quad f_y = 8x^{8} e^{8y}$

$\therefore f_x(1,0) = 8$ $\quad f_y(1,0) = 8$

$\therefore z = 1 + 8(x-1) + 8(y-0)$

or $f(x,y) \approx 1 + 8(x-1) + 8(y-0)$

$\therefore f(1+\Delta x, 0+\Delta y) \approx 1 + 8\Delta x + 8\Delta y$

$$\begin{aligned}\therefore f(0.99, 0.02) &= f(1+(-0.01), 0+0.02) \\ &\approx 1 + 8(-0.01) + 8(0.02) \\ &= 1 - 0.08 + 0.16 \\ &= 1.08\end{aligned}$$

$\therefore \underline{(0.99 e^{0.02})^{8} \approx 1.08}$

(b) $f(x,y) = x^{3} + y^{3} - 6xy$

$x = 1.00$, $\Delta x = -0.01 \Rightarrow 0.99$

$y = 2.00$, $\Delta y = 0.01 \Rightarrow 2.01$

$f(1,2) = 1^3 + 2^3 - 6(1)(2) = -3$

$f_x = 3x^2 - 6y \qquad f_x(1,2) = -9$

$f_y = 3y^2 - 6x \qquad f_y(1,2) = 6$

$\therefore\ z = -3 - 9(x-1) + 6(y-2)$

$\therefore\ f(1+\Delta x,\ 2+\Delta y) \approx -3 - 9\Delta x + 6\Delta y$

$$\begin{aligned}\therefore\ f(0.99,\ 2.01) &\approx -3 - 9(-0.01) + 6(0.01)\\ &= -3 + 0.09 + 0.06\\ &= 2.85\end{aligned}$$

$\therefore\ (0.99)^3 + (2.01)^3 - 6(0.99)(2.01) \approx \underline{2.85}$

(c) $f(x,y,z) = \sqrt{x^2+y^2+z^2}$

$x = 4.00 \quad \Delta x = 0.01 \implies 4.01$

$y = 4.00 \quad \Delta y = -0.02 \implies 3.98$

$z = 2.00 \quad \Delta z = 0.02 \implies 2.02$

$f(4,4,2) = \sqrt{4^2+4^2+2^2} = \sqrt{36} = 6$

$f_x = \frac{1}{2}(x^2+y^2+z^2)^{-\frac{1}{2}}(2x) = x/\sqrt{x^2+y^2+z^2}$

$\therefore\ f_x(4,4,2) = \frac{4}{6} = \frac{2}{3}$

$f_y = y/\sqrt{x^2+y^2+z^2} \qquad f_y(4,4,2) = 2/3$

$f_z = z/\sqrt{x^2+y^2+z^2} \qquad f_z(4,4,2) = \frac{2}{6} = \frac{1}{3}$

$\therefore\ z = 6 + \frac{2}{3}(x-4) + \frac{2}{3}(y-4) + \frac{1}{3}(z-2)$

$$\therefore f(4+\Delta x, 4+\Delta y, 2+\Delta z) \approx 6 + \tfrac{2}{3}\Delta x + \tfrac{2}{3}\Delta y + \tfrac{1}{3}\Delta z$$

$$\therefore f(4.01, 3.98, 2.02) \approx 6 + \tfrac{2}{3}(0.01) + \tfrac{2}{3}(-0.02) + \tfrac{1}{3}(0.02)$$

$$= 6 + \frac{2-4+2}{3(100)} = 6+0 = 6$$

$$\therefore \underline{\sqrt{(4.01)^2+(3.98)^2+(2.02)^2} \approx 6.00}$$

17.

$g(1,2) = -6 \quad g_x = -4x \quad g_x(1,2) = -4$

$\qquad\qquad\qquad\;\; g_y = -6y \quad g_y(1,2) = -12$

$\therefore z = -6 - 4(x-1) - 12(y-2)$

or $4(x-1) + 12(y-2) + z = -6$

or $(4,12,1)\cdot(x-1, y-2, z) = -6$ (dot product)

$\therefore$ normal to plane $= (4,12,1)$

$f_x = -2x \quad f_y = -2y$

$\therefore z = f(x_0,y_0) - 2x_0(x-x_0) - 2y_0(y-y_0)$

$\therefore (2x_0, 2y_0, 1)\cdot(x-x_0, y-y_0, z) = f(x_0,y_0)$ (dot product)

Make normals parallel

$\therefore (2x_0, 2y_0, 1) = (4,12,1) \therefore x_0 = 2, y_0 = 6$

$\therefore f(2,6) = -36 \quad \therefore$ At $\underline{(2,6,-36)}$

18.

(a) $f(1,2) = e^4 - 2e$

$f_x = e^{y^2} - 2xye^{x^2} \qquad f_y = 2xye^{y^2} - e^{x^2}$

$f_x(1,2) = e^4 - 4e \qquad f_y(1,2) = 4e^4 - e$

$$\therefore \underline{z = (e^4 - 2e) + (e^4 - 4e)(x-1) + (4e^4 - e)(y-2)}$$

Or, $(4e - e^4, e - 4e^4, 1)\cdot(x-1, y-2, z) = e^4 - 2e$

$\therefore$ Normal is: $(4e - e^4, e - 4e^4, 1)$

(b) Let (x_0, y_0, z_0) be the tangent point, $f(x,y) = x^2 - y^2$

$\therefore f_x = 2x \quad f_y = -2y$

$\therefore$ Tangent plane: $z = (x_0^2 - y_0^2) + 2x_0(x - x_0) - 2y_0(y - y_0)$

or $(-2x_0, 2y_0, 1)\cdot(x - x_0, y - y_0, z) = x_0^2 - y_0^2$

$\therefore$ Normal at (x_0, y_0) is $(-2x_0, 2y_0, 1)$

Make normals parallel.

$\therefore \ (-2x_0, 2y_0, 1) = (4e - e^4, e - 4e^4, 1)$

$\therefore \ x_0 = \frac{4e - e^4}{-2} = \frac{1}{2}e^4 - 2e$

$y_0 = \frac{e - 4e^4}{2} = \frac{1}{2}e - 2e^4$

$\therefore$ at $\left(\frac{1}{2}e^4 - 2e, \frac{1}{2}e - 2e^4, z_0\right)$

$z_0 = \left(\frac{1}{2}e^4 - 2e\right)^2 - \left(\frac{1}{2}e - 2e^4\right)^2$ (from $z = f(x,y)$)

$= \frac{1}{4}e^8 - 2e^5 + 4e^2 - \left[\frac{1}{4}e^2 - 2e^5 + 4e^8\right]$

$= -\frac{15}{4}e^8 + \frac{15}{4}e^2$

$\therefore$ at $\underline{\left(\frac{1}{2}e^4 - 2e, \frac{1}{2}e - 2e^4, -\frac{15}{4}e^8 + \frac{15}{4}e^2\right)}$

19.

(a) $f_x = e^{-x^2-y^2-z^2} - 2x^2e^{-x^2-y^2-z^2}$

$f_y = -2xye^{-x^2-y^2-z^2}$ $\qquad f_z = -2xze^{-x^2-y^2-z^2}$

$\therefore \ \nabla f = \underline{e^{-x^2-y^2-z^2}(1-2x^2, -2xy, -2xz)}$

(b) $f_x = \dfrac{(x^2+y^2+z^2)(yz) - xyz(2x)}{(x^2+y^2+z^2)^2}$

$f_y = \dfrac{(x^2+y^2+z^2)(xz) - xyz(2y)}{(x^2+y^2+z^2)^2}$

$f_z = \dfrac{(x^2+y^2+z^2)(xy) - xyz(2z)}{(x^2+y^2+z^2)^2}$

$\therefore \nabla f = \dfrac{1}{x^2+y^2+z^2}\left[yz(-x^2+y^2+z^2),\ xz(x^2-y^2+z^2),\ xy(x^2+y^2-z^2)\right]$

(c) $f_x = (z^2\cos y)e^x \quad f_y = -z^2e^x\sin y \quad f_z = 2z\,e^x\cos y$

$\therefore \nabla f = \left[z^2e^x\cos y,\ -z^2e^x\sin y,\ 2ze^x\cos y\right]$

20.

(a) $\nabla f(1,0,1) = e^{-2}(-1,0,-2) = (-e^{-2}, 0, -2e^{-2})$

$\therefore (e^{-2}, 0, -2e^{-2})\cdot(x-1, y-0, z-1) = 0 \quad [\vec{n}\cdot\vec{r} = 0]$

Or, $e^{-2}(x-1) - 2e^{-2}(z-1) = 0$ [now multiply by e^2]

Or, $(x-1) - 2(z-1) = 0$, or $x - 2z + 1 = 0$

(b) $\nabla f(1,0,1) = \frac{1}{2}[0, 2, 0] = (0,1,0)$

$\therefore (0,1,0)\cdot(x-1, y-0, z-1) = 0$

Or, $\underline{y = 0}$

(c) $\nabla f(1,0,1) = (e, 0, 2e)$

$\therefore (e,0,2e)\cdot(x-1, y-0, z-1) = 0$

Or, $e(x-1) + 2e(z-1) = 0$, $\therefore x - 1 + 2z - 2 = 0$

$\therefore \underline{x + 2z - 3 = 0}$

21.

Let $z = f(x,y)$ $\therefore f(1,1) = 3$ $f_x = 2x$ $f_y = 6y^2$

$f_x(1,1) = 2$ $f_y(1,1) = 6$

$\therefore z = 3 + 2(x-1) + 6(y-1)$

Or, $\underline{2x + 6y - z - 5 = 0}$

22.

(a) $\frac{\partial f}{\partial x}(0,0) = \lim_{h \to 0} \frac{f(x+h,0) - f(x,0)}{h} \Big|_{x=0}$

$$\lim_{h \to 0} \frac{\frac{(0+h)^2 0^4}{(0+h)^4 + 6 0^8} - \frac{0^2 0^4}{0^4 + 6 0^8}}{h} = \lim_{h \to 0} \frac{\frac{0}{h^4+0} - 0}{h}$$

$$= \lim_{h \to 0} 0 = 0 \qquad \therefore \frac{\partial f}{\partial x}(0,0) = 0$$

Similarly, $\frac{\partial f}{\partial y}(0,0) = \lim_{h \to 0} \frac{f(0,y+h) - f(0,y)}{h} \Big|_{y=0}$

$$\lim_{h \to 0} \frac{\frac{0^2(0+h)^4}{0^4 + 6(0+h)^8} - \frac{0^2 0^4}{0^4 + 6 0^8}}{h} = \lim_{h \to 0} 0 = 0$$

$$\therefore \frac{\partial f}{\partial y}(0,0) = 0$$

(b) Along the path $y = \sqrt{x}$,

$$\lim_{(x,y) \to (0,0)} \frac{x^2 y^4}{x^4 + 6y^8} = \lim_{x \to 0} \frac{x^2 (\sqrt{x})^4}{x^4 + 6(\sqrt{x})^8} = \lim_{x \to 0} \frac{x^4}{x^4 + 6x^4}$$

$$= \lim_{x \to 0} \frac{x^4}{7x^4} = \lim_{x \to 0} \frac{1}{7} = \frac{1}{7} \neq 0.$$

$\therefore \lim_{(x,y) \to (0,0)} f(x,y) \neq f(0,0)$. $\therefore$ <u>$f(x,y)$ is not continuous</u>

23.

The tangent plane contains the vector from $(1,2,8)$ to $(1,3,20) = (0,1,12)$

and the vector from $(1,2,8)$ to $(2,1,z) = (1,-1,z-8)$

These two vectors are perpendicular to normal vector for P.

$\therefore$ cross product is parallel to normal.

Cross product: $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 12 \\ 1 & -1 & z-8 \end{vmatrix} = (z+4, 12, -1)$

To find normal to tangent plane:

$f_x = 2xy^3 \qquad f_y = 3y^2x^2$

$f_x(1,2) = 16 \qquad f_y(1,2) = 12 \qquad f(1,2) = 8$

$\therefore$ Tangent plane: $z = 8 + 16(x-1) + 12(y-2)$

or, $16x + 12y - z = 32$

$\therefore$ Normal to plane: $(16, 12, -1)$

$\therefore (16, 12, -1) = (z+4, 12, -1)$

$\therefore$ For line l, $z = 12$, and $(2,1,z) = (2,1,12)$

$\therefore$ l contains $(1,3,20)$ and $(2,1,12)$
and is parallel to vector $(-1,2,8)$

$$\therefore \underline{l(t) = (1,3,20) + t(-1,2,8)}$$

24.

$h_x = e^{x-y} + (x+z)e^{x-y}$ $\quad h_x(1,1,1) = 1+2 = 3$

$h_y = -(x+z)e^{x-y}$ $\quad h_y(1,1,1) = -2$

$h_z = e^{x-y}$ $\quad h_z(1,1,1) = 1$

$$\therefore \nabla h(1,1,1) = \underline{(3,-2,1)}$$

25.

$f_x = 2x$ $\quad f_x(0,0,1) = 0$

$f_y = 2y$ $\quad f_y(0,0,1) = 0$

$f_z = -2z$ $\quad f_z(0,0,1) = -2$

$$\therefore \nabla f(0,0,1) = \underline{(0,0,-2)}$$

26.

$$f_x = \frac{2x}{x^2+y^2+z^2} \qquad f_x(1,0,1) = 1$$

$$f_y = \frac{2y}{x^2+y^2+z^2} \qquad f_y(1,0,1) = 0$$

$$f_z = \frac{2z}{x^2+y^2+z^2} \qquad f_z(1,0,1) = 1$$

$$\therefore \nabla f(1,0,1) = \underline{\underline{(1,0,1)}}$$

27.

From Exercise 33, Section 2.2.,

$\therefore$ Let $f: A \subset \mathbb{R}^n \to \mathbb{R}^m$. Since it's Hölder-continuous, $\exists\, K, \alpha$ s.t. $\|f(\vec{x}) - f(\vec{y})\| \leq K\|\vec{x}-\vec{y}\|^{\alpha}$, $\forall \vec{x}, \vec{y} \in A$, and $\alpha > 1$.

Note: $\dfrac{\|f(\vec{x}) - f(\vec{y})\|}{\|\vec{x}-\vec{y}\|} \leq K\|x-y\|^{\alpha-1}$, for $\vec{x} \neq \vec{y}$.

Let $\epsilon > 0$. Choose $\delta = \sqrt[\alpha-1]{\dfrac{\epsilon}{K}}$. Note $\alpha - 1 > 0$

Let any $\vec{y} \in A$, and let $\vec{y} = (y_1, \ldots, y_n)$.

$\therefore$ if $\|\vec{x}-\vec{y}\| < \delta$, and $x \in A$, Then

$\|\vec{x}-\vec{y}\| < \sqrt[\alpha-1]{\frac{\epsilon}{K}}$, so $\|\vec{x}-\vec{y}\|^{\alpha-1} < \frac{\epsilon}{K}$

(since for $0<a<b$, $r>0$, Then $a^r < b^r$)

$\therefore K\|\vec{x}-\vec{y}\|^{\alpha-1} < \epsilon \Rightarrow K\|\vec{x}-\vec{y}\|^{\alpha} < \epsilon\|\vec{x}-\vec{y}\|$

But $\|f(\vec{x})-f(\vec{y})\| < K\|\vec{x}-\vec{y}\|^{\alpha}$

So $\|f(\vec{x})-f(\vec{y})\| < \epsilon\|\vec{x}-\vec{y}\|$

Note, for $i=1,\ldots,m$, $|f_i(\vec{x})-f_i(\vec{y})| \leq \|f(\vec{x})-f(\vec{y})\|$

and for $j=1,\ldots,n$, $\|(y_1,\ldots,y_j+h,\ldots,y_n)-(y_1,\ldots,y_j,\ldots,y_n)\| = |h|$

$\therefore$ Consider $\vec{x_j} = (y_1,\ldots,y_j+h,\ldots,y_n)$

Given The $\epsilon>0$ above and $\delta = \sqrt[\alpha-1]{\frac{\epsilon}{K}}$ as above,

if $\|\vec{x_j}-\vec{y}\| = |h| < \delta$, Then

$|f_i(\vec{x_j})-f_i(\vec{y})| \leq \|f(\vec{x_j})-f(\vec{y})\| < \epsilon\|\vec{x_j}-\vec{y}\| = \epsilon|h|$

$\therefore \left|\frac{f_i(\vec{x_j})-f(\vec{y})}{h} - 0\right| < \epsilon$

or $\left|\frac{f_i(y_1,\ldots,y_j+h,\ldots,y_n)-f_i(y_1,\ldots,y_j,\ldots,y_n)}{h} - 0\right| < \epsilon$

$$\therefore \lim_{h \to 0} \frac{f_i(y_1, \dots, y_j + h, \dots, y_n) - f_i(y_1, \dots, y_j, \dots, y_n)}{h} = 0$$

$$\therefore \frac{\partial f_i}{\partial y_j}(\vec{y}) = 0 \quad , \text{ for all } y \in A, \text{ and } j = 1, 2, \dots, n$$

$\therefore f_i(\vec{y})$ is constant along the y_j component of $\vec{y}$. Call this vector $\vec{y_j}$. $f_i(\vec{y_j}) = C_{i_j}$

Similarly, $\frac{\partial f_i}{\partial y_k}(\vec{y}) = 0 \Rightarrow f_i(\vec{y_k}) = C_{i_k}$, $j \neq k$.

But $f(\vec{y})$ is continuous (Exercise 33, sec. 2.2)

$\therefore f_i(\vec{y})$ is continuous, and in particular is continuous at $(y_1, \dots, y_j, \dots, y_k, \dots, y_n)$ so that $C_{i_j} = C_{i_k}$.

That is, the value of $f_i(\vec{y})$ along the j component must equal the value along the k component

$\therefore$ for all $y \in A$, $f_i(\vec{y}) = C_i$ $(c_i = c_{i_j} = c_{i_k})$.

$\therefore f(\vec{y}) = (c_1, c_2, \dots, c_m)$, a constant for all $y \in A$.

$\therefore$ Hölder-continuous function with $\alpha > 1 \Rightarrow$ function is constant

Note: $\alpha > 1$ was needed for $0 < a < b \Rightarrow a^{\alpha - 1} < b^{\alpha - 1}$, since $\alpha - 1 > 0$.

28.

For $\vec{x} \in \mathbb{R}^n$, $\vec{x} = (x_1, \ldots, x_n) = x_1\hat{e}_1 + \ldots + x_n\hat{e}_n$,

where $\hat{e}_i = (0, \ldots, \underset{\substack{\uparrow \\ \text{ith component}}}{1}, \ldots 0)$, the natural basis.

$\therefore f(\vec{x}) = f(x_1\hat{e}_1 + \ldots + x_n\hat{e}_n) = x_1 f(\hat{e}_1) + \ldots + x_n f(\hat{e}_n)$.

Note $f(\vec{x}) = [f_1(\vec{x}), \ldots, f_m(\vec{x})]$

$\therefore f(\vec{x}) = x_1[f_1(\hat{e}_1), \ldots, f_m(\hat{e}_1)] + \ldots + x_n[f_1(\hat{e}_n), \ldots, f_m(\hat{e}_n)]$

$$\therefore f(\vec{x}) = \begin{bmatrix} f_1(\hat{e}_1) & f_1(\hat{e}_2) & \ldots & f_1(\hat{e}_n) \\ f_2(\hat{e}_1) & f_2(\hat{e}_2) & \ldots & f_2(\hat{e}_n) \\ \vdots & \vdots & & \vdots \\ f_m(\hat{e}_1) & f_m(\hat{e}_2) & & f_m(\hat{e}_n) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Let T be the matrix $\begin{bmatrix} f_1(\hat{e}_1) & \cdots & f_1(\hat{e}_n) \\ \vdots & & \vdots \\ f_m(\hat{e}_1) & \cdots & f_m(\hat{e}_n) \end{bmatrix}$. $\therefore f(\vec{x}) = T\vec{x}$

Must prove: (1) Partial derivates of f exist.
(2) Above matrix T contains the partial derivatives of f
(3) $\displaystyle\lim_{\vec{x} \to \vec{x}_0} \frac{\| f(\vec{x}) - f(\vec{x}_0) - T(\vec{x} - \vec{x}_0) \|}{\| \vec{x} - \vec{x}_0 \|} = 0$

Then the derivate of f will be the matrix T.

(3) Assuming (1) + (2) are true, letting $\vec{h} = \vec{x} - \vec{x_0}$,

Then $$\lim_{\vec{h} \to \vec{0}} \frac{\| f(\vec{x_0} + \vec{h}) - f(\vec{x_0}) - T\vec{h} \|}{\| h \|} =$$

$$\lim_{\vec{h} \to \vec{0}} \frac{\| f(\vec{x_0}) + f(\vec{h}) - f(\vec{x_0}) - T\vec{h} \|}{\| \vec{h} \|} =$$

$$\lim_{\vec{h} \to \vec{0}} \frac{\| f(\vec{h}) - T\vec{h} \|}{\| \vec{h} \|} = \lim_{\vec{h} \to \vec{0}} \frac{\| T\vec{h} - T\vec{h} \|}{\| \vec{h} \|} = 0$$

since, as defined above, $f(\vec{x}) = T\vec{x}$ since f is linear.

(1) Note: if $g : R^n \to R^m$, where

$$g(\vec{h}) = (g_1(\vec{h}), \ldots, g_m(\vec{h})), \; \vec{h} \in R^n, \; g_i : R^n \to R$$

Then $$\lim_{\vec{h} \to \vec{0}} \frac{\| g(\vec{h}) \|}{\| \vec{h} \|} = 0 \iff \lim_{\vec{h} \to \vec{0}} \frac{|g_i(\vec{h})|}{\| \vec{h} \|} = 0,$$ $i = 1, 2, \ldots, m$

$\therefore$ Letting $g(\vec{h}) = f(\vec{x_0} + \vec{h}) - f(\vec{x_0}) - T\vec{h}$,

only need to prove: $$\lim_{\vec{h} \to \vec{0}} \frac{| f_i(\vec{x_0} + \vec{h}) - f_i(\vec{x_0}) - (T\vec{h})_i |}{\| h \|} = 0$$ where $i = 1, \ldots, m$

and $(T\vec{h})_i$ = ith entry in $m \times 1$ column vector.

But $f_i(\vec{x}_0 + \vec{h}) = f_i(\vec{x}_0) + f_i(\vec{h})$ since

f is linear, and $\therefore$ so is f_i

$$\therefore \left| f_i(\vec{x}_0 + \vec{h}) - f_i(\vec{x}_0) - (T\vec{h})_i \right| = \left| f_i(\vec{h}) - (T\vec{h})_i \right|$$

Let $\vec{h} = (h_1, \dots, h_n) = h_1\hat{e}_1 + \dots h_n\hat{e}_n$

Then $f_i(\vec{h}) = h_1 f_i(\hat{e}_1) + \dots + h_n f_i(\hat{e}_n)$

But $(T\vec{h})_i = h_1 f_i(\hat{e}_1) + \dots + h_n f_i(\hat{e}_n)$

$$\therefore \quad f_i(\vec{h}) - (T\vec{h})_i = 0$$

$$\therefore \lim_{\vec{h}\to\vec{0}} \frac{\left| f_i(\vec{x}_0 + \vec{h}) - f_i(\vec{x}_0) - (T\vec{h})_i \right|}{\|h\|} =$$

$$\lim_{\vec{h}\to\vec{0}} \frac{0}{\|\vec{h}\|} = 0$$

Now consider $\vec{h} = a\hat{e}_j$, $j = 1, \dots, n$

$\therefore \|\vec{h}\| = |a|$, and $\|\vec{h}\| \to \vec{0} \iff a \to 0$.

$$\therefore \lim_{a\to 0} \frac{\left| f_i(\vec{x}_0 + a\hat{e}_j) - f_i(\vec{x}_0) - T(a\hat{e}_j)_i \right|}{|a|} = 0$$

$$\text{or } \lim_{a\to 0} \left| \frac{f_i(\vec{x}_0 + a\hat{e}_j) - f_i(\vec{x}_0) - a(T\hat{e}_j)_i}{a} \right| = 0$$

$$\text{or} \lim_{a\to 0} \left| \frac{f_i(\vec{x}_0 + a\hat{e}_j) - f_i(\vec{x}_0)}{a} - (T\hat{e}_j)_i \right| = 0$$

$$\therefore \lim_{a\to 0} \frac{f_i(\vec{x}_0 + a\hat{e}_j) - f_i(\vec{x}_0)}{a} = (T\hat{e}_j)_i$$

$$\text{But } \frac{\partial f_i}{\partial x_j} = \lim_{a\to 0} \frac{f_i(\vec{x}_0 + a\hat{e}_j) - f_i(\vec{x}_0)}{a}$$

$\therefore \dfrac{\partial f_i}{\partial x_j}$ exists and is $(T\hat{e}_j)_i = f_i(\hat{e}_j)$

i.e., the partials exist, proving (1),

and T is the matrix of partial derivatives

$$(2) \quad \begin{bmatrix} f_1(\hat{e}_1) & \cdots & f_1(\hat{e}_n) \\ \vdots & & \\ f_m(\hat{e}_1) & \cdots & f_m(\hat{e}_n) \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

$\therefore$ For linear $f: R^n \to R^m$, the derivative is the linear map itself, represented by its matrix T.

Example: $f(\vec{x}) = (\vec{a}\cdot\vec{x}, \vec{b}\cdot\vec{x}, \dots \vec{p}\cdot\vec{x})$

$$\therefore T = \begin{bmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \\ \vdots & & & \\ \vdots & & & \\ p_1 & p_2 & \dots & p_n \end{bmatrix}$$

and so $\dfrac{\partial f_2}{\partial x_7} = b_7$ (assuming $n \geq 7$)

2.4 Introduction to Paths and Curves

Note Title 2/9/2016

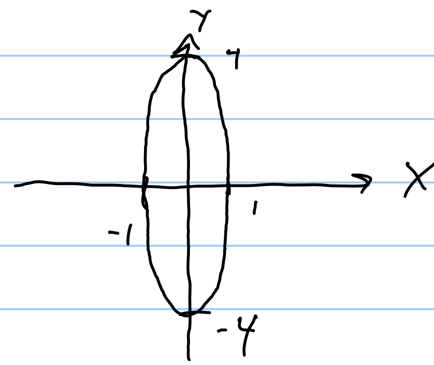
1.

At $t=0: (0,4)$ $\therefore$ Ellipse

$t=\pi/2: (1,0)$

$t=\pi: (0,-4)$ $\left(\frac{x}{1}\right)^2+\left(\frac{y}{4}\right)^2=1$

$t=3\pi/2: (-1,0)$



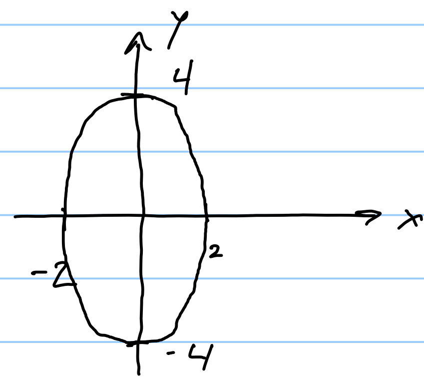
2.

$t=0: (0,4)$ $\therefore$ Ellipse

$t=\pi/2: (2,0)$

$t=\pi: (0,-4)$ $\left(\frac{x}{2}\right)^2+\left(\frac{y}{4}\right)^2=1$

$t=3\pi/2: (-2,0)$

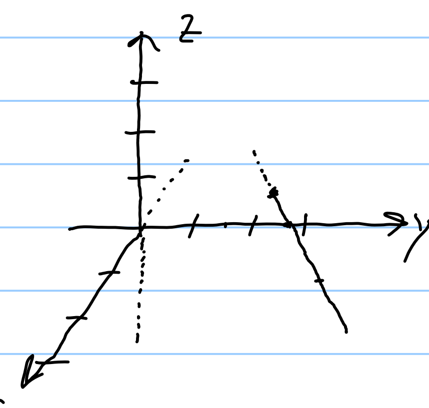


3.

$t=0: (-1,2,0)$ $(-1,2,0)+t(2,1,1)$

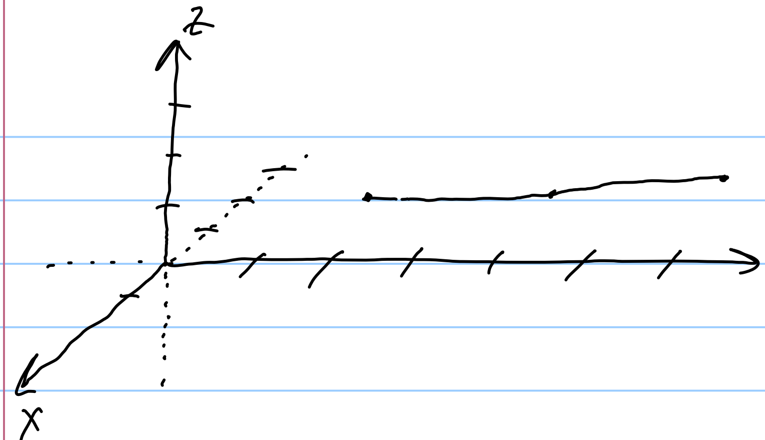
$t=1: (1,3,1)$ a line

$t=2: (3,4,2)$



4.

$t=1: (-1,2,1)$ $t=2: (-2,4,\frac{1}{2})$ $t=3: (-3,6,\frac{1}{3})$



Part of "$\frac{1}{x}$" curve
That gets closer to
xy-plane as projection
onto xy-plane goes out
toward $(-\infty, +\infty)$.

5.

$\vec{c}(t) = 2(\sim, \sim)$, where $\sim$ is some function of $\cos t$ or $\sin t$

at $t=0$, $(2,0)$, so $2(\cos \sim, \sin \sim)$

$\therefore$ $\underline{\vec{c}(t) = 2(\cos t, \sin t)}$

(b) To shift starting point $\frac{\pi}{2}$ radians from (a),

use $2\left(\cos\left(\frac{\pi}{2}+t\right), \sin\left(\frac{\pi}{2}+t\right)\right)$

To make it clockwise, use a reflection with respect to y-axis: $(x,y) \to (-x,y)$.

$\therefore$ $2\left[-\cos\left(\frac{\pi}{2}+t\right), \sin\left(\frac{\pi}{2}+t\right)\right]$

To simplify: $\cos(\frac{\pi}{2}+x) = \cos\frac{\pi}{2}\cos x - \sin\frac{\pi}{2}\sin x$
$= -\sin x$
$\sin(\frac{\pi}{2}+x) = \sin\frac{\pi}{2}\cos x + \cos\frac{\pi}{2}\sin x$
$= \cos x$

$\therefore \underline{\vec{c}(t) = 2(\sin t, \cos t)}$

(c) From (a), $\vec{c}(t) = 2(\cos t, \sin t) + (4,7)$

$= \underline{(4 + 2\cos t, 7 + 2\sin t)}$

6.

(a) $(1,2,3) - (-2,0,7) = (3,2,-4)$.

$\therefore \underline{\vec{c}(t) = (1,2,3) + t(3,2,-4)}$

(b) $\underline{\vec{c}(t) = (t, t^2)}$

(c) $(0,0)$ to $(0,1)$: $(0,1)-(0,0) = (0,1)$.
$\therefore (0,0) + s(0,1) = (0,s)$, $0 \le s \le 1$

$(0,1)$ to $(1,1)$: $(1,1)-(0,1) = (1,0)$
$\therefore (0,1) + s(1,0) = (s,1)$, $0 \le s \le 1$
or $(s-1,1)$, $1 \le s \le 2$

$(1,1)$ to $(1,0)$: $(1,0)-(1,1) = (0,-1)$

$\therefore (1,1) + s(0,-1) = (1, 1-s),\ 0 \le s \le 1$

or, $(1, 1-(s-2)) = (1, 3-s),\ 2 \le s \le 3$

$(1,0)$ to $(0,0)$: $(0,0)-(1,0) = (-1,0)$

$\therefore (1,0) + s(-1,0) = (1-s, 0),\ 0 \le s \le 1$

or $(1-(s-3), 0) = (4-s, 0),\ 3 \le s \le 4$

$$\therefore \quad \vec{c}(t) = \begin{cases} (0,s), & 0 \le s \le 1 \\ (s-1,1), & 1 \le s \le 2 \\ (1,3-s), & 2 \le s \le 3 \\ (4-s,0), & 3 \le s \le 4 \end{cases}$$

(d) If $x = 3\cos\theta,\ y = 5\sin\theta,\ 0 \le \theta \le 2\pi$, then

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{5}\right)^2 = \frac{9\cos^2\theta}{9} + \frac{25\sin^2\theta}{25} = 1.$$

$$\therefore \underline{\vec{c}(\theta) = (3\cos\theta, 5\sin\theta)},\ 0 \le \theta \le 2\pi$$

7.

$$\vec{c}\,'(t) = 6\hat{i} + 6t\hat{j} + 3t^2\hat{k}$$

8.

$$\vec{c}\,'(t) = (3\cos t)\hat{i} - (3\sin t)\hat{j} + 3t^{\frac{1}{2}}\hat{k}$$

9.

$$\vec{r}\,'(t) = (2\cos t(-\sin t), 3-3t^2, 1)$$

$$= (-\sin 2t, 3-3t^2, 1)$$ since $\sin 2\theta = 2\sin\theta\cos\theta$

10.

Assume typo, and e' means e^t

$$\vec{r}\,'(t)\ (4e^t, 24t^3, -\sin t)$$

11.

$$\vec{c}\,'(t) = (e^t, -\sin t)$$

12.

$$\vec{c}\,(t) = (6t, 3t^2)$$

13.

$$\vec{c}\,'(t) = (\sin t + t\cos t, 4)$$

14.

$$\vec{c}\,(t) = (2t, 0)$$

15.

From Example 4, p. 119, $\mathbf{c}(t) = \left(vt - r\sin\frac{vt}{R},\ R - r\cos\frac{vt}{R}\right).$

(a) For a point on a wheel, $r = R$

$$\therefore \vec{c}(t) = \left(Vt - R\sin\frac{Vt}{R},\ R - R\cos\frac{Vt}{R}\right)$$

$$\therefore \vec{c}\,'(t) = \left(V - V\cos\frac{Vt}{R},\ Vt\sin\frac{Vt}{R}\right)$$

$$= V\left(1 - \cos\frac{Vt}{R},\ t\sin\frac{Vt}{R}\right)$$

$\vec{c}\,'(t)$ is horizontal when $t\sin\frac{Vt}{R} = 0$.

Neglecting $t = 0$, $\sin\frac{Vt}{R} = 0$ when

$$\frac{Vt}{R} = n\pi,\quad n = 0, 1, 2, 3, \ldots$$

$$\therefore t = \underline{\underline{\frac{nR\pi}{V}}},\quad n = 0, 1, 2, 3, \ldots$$

(b) Velocity at these points is

$$\vec{c}\,'(t) = V(1 - \cos n\pi,\ 0)$$

$$= 0, \quad \text{for } n = 0, 2, 4, \ldots \text{ (even)}$$

$$2V, \quad \text{for } n = 1, 3, 5, \ldots \text{ (odd)}$$

$\therefore$ Speed: 0 when rim contacts road

2V when point of rim at top.

16.

$\vec{c}'(t) = (6, 6t, 3t^2)$. $\therefore$ $\underline{\vec{c}'(0) = (6, 0, 0)}$

17.

$\vec{c}'(t) = (3\cos 3t, -3\sin 3t, 5t^{3/2})$

$\therefore \vec{c}'(1) = (3\cos 3, -3\sin 3, 5)$

$\vec{c}(1) = (\sin 3, \cos 3, 2)$

$\therefore$ tangent line at $t=1$:

$\underline{\vec{l}(s) = (\sin 3, \cos 3, 2) + (s-1)(3\cos 3, -3\sin 3, 5)}$

18.

$\vec{c}(0) = (1, 0, 0)$

$\vec{c}'(t) = (2\cos t(-\sin t), 3-3t^2, 1)$

$\therefore \vec{c}'(0) = (0, 3, 1)$

$\therefore$ tangent line at point $t=0$:

$\underline{\vec{l}(s) = (1,0,0) + s(0,3,1)} = (1, 3s, s)$

19.

$\vec{c}(t_0) = (4,0,0)$ $\quad \vec{c}\,'(t) = (2t,\ 3t^2-4,\ 0)$

$\therefore \vec{c}\,'(t_0) = (4,8,0)$

$\therefore$ tangent line at $t_0 = 2$:

$$\vec{l}(s) = (4,0,0) + (s-2)(4,8,0)$$

$$\therefore \vec{l}(3) = (4,0,0) + (3-2)(4,8,0) = (8,8,0)$$

$\therefore$ at $\underline{(8,8,0)}$

20.

$\vec{c}(t_0) = (e, \frac{1}{e}, \cos 1)$ $\quad \vec{c}\,'(t) = (e^t, -e^{-t}, -\sin t)$

$\therefore \vec{c}\,'(t_0) = (e, -\frac{1}{e}, -\sin 1)$

$\therefore$ Tangent line at $t_0 = 1$:

$$\vec{l}(s) = (e, \tfrac{1}{e}, \cos 1) + (s-1)(e, -\tfrac{1}{e}, -\sin 1)$$

$$\therefore \vec{l}(2) = (e, \tfrac{1}{e}, \cos 1) + (2-1)(e, -\tfrac{1}{e}, -\sin 1)$$

$$= \underline{(2e, 0, \cos 1 - \sin 1)}$$

21.

$\vec{c}(t_0) = (4, 0, 1)$ $\quad \vec{c}\,'(t) = (4e^{t}, 24t^3, -\sin t)$

$\vec{c}\,'(t_0) = (4, 0, 0)$

$\therefore$ Tangent line at $t_0 = 0$:

$\vec{l}(s) = (4,0,1) + (s-0)(4,0,0)$

$\therefore \vec{l}(1) = (4,0,1) + (4,0,0) = \boxed{(8,0,1)}$

22.

$\vec{c}(t_0) = (\sin e, 1, 3)$ $\quad \vec{c}\,'(t) = (e^t \cos e^t, 1, -3t^2)$

$\vec{c}\,'(t_0) = (e \cos e, 1, -3e^2)$

$\therefore$ Tangent line at $t_0 = 1$:

$\vec{l}(s) = (\sin e, 1, 3) + (s-1)(e\cos e, 1, -3e^2)$

$\therefore \vec{l}(2) = (\sin e, 1, 3) + (e\cos e, 1, -3e^2)$

$= \boxed{(\sin e + e\cos e, 2, 3 - 3e^2)}$

23.

(a) $\vec{c}\,'(t) = (-\sin t, \cos t, 2t)$ $\quad \therefore \vec{c}\,'(4\pi) = (0, 1, 8\pi)$

$\therefore$ Speed $= \|\vec{c}\,'(4\pi)\| = \sqrt{0^2+1^2+(8\pi)^2} = \underline{\sqrt{1+64\pi^2}}$

(b) $\vec{c}(t)\cdot\vec{c}\,'(t) = [-\cos t\sin t,\ \sin t\cos t,\ 2t^3]$

$2t^3 = 0$ only when $t=0$.

$\therefore$ $\underline{\vec{c}(t) \perp \vec{c}\,'(t)}$ at $t=0$

(c) $\vec{c}(t_0) = (1, 0, 16\pi^2)$, From (a), $\vec{c}\,'(t_0) = (0,1,8\pi)$

$\therefore$ $\underline{\vec{l}(t) = (1,0,16\pi^2) + (t-4\pi)(0,1,8\pi)}$

(d) When z-component is 0, $\vec{l}(t)$ intersects xy-plane.

$\therefore$ $16\pi^2 + (t-4\pi)\,8\pi = 0$

$\therefore$ $\dfrac{-16\pi^2}{8\pi} = t-4\pi$, $t = -2\pi + 4\pi = 2\pi$

$\therefore$ $\vec{l}(2\pi) = (1,0,16\pi^2) + (2\pi-4\pi)(0,1,8\pi)$

$= (1,0,16\pi^2) + (0,-2\pi,-16\pi^2)$

$= \underline{(1,-2\pi,0)}$

24.

$$\vec{c}\,'(t) = (e^t\cos(t) - e^t\sin(t),\ e^t\sin(t) + e^t\cos(t))$$

$$= e^t(\cos t - \sin t,\ \sin t + \cos t)$$

$$\vec{c}(t) = e^t(\cos t,\ \sin t)$$

$$\therefore\ \vec{C}(t)\cdot\vec{c}\,'(t) = e^{2t}\left[\cos^2 t - \sin t\cos t + \sin^2 t + \sin t\cos t\right]$$

$$= e^{2t}$$

$$\|\vec{c}(t)\| = \sqrt{e^{2t}(\cos^2 t + \sin^2 t)} = e^t$$

$$\|c'(t)\| = e^t\sqrt{\cos^2 t - 2\cos t\sin t + \sin^2 t + \sin^2 t + 2\sin t\cos t + \cos^2 t}$$

$$= e^t\sqrt{2}$$

$$\therefore\ \cos\theta = \frac{\vec{c}(t)\cdot\vec{C}\,'(t)}{\|\vec{c}(t)\|\ \|\vec{c}\,'(t)\|} = \frac{e^{2t}}{(e^t)(e^t\sqrt{2})} = \frac{\sqrt{2}}{2}$$

$$\therefore\ \theta = \frac{\pi}{4} \text{ radians}$$

$\therefore$ For all t, $\theta = \frac{\pi}{4}$ = angle between $\vec{c}(t)$ and $\vec{c}\,'(t)$ is constant

25.

(a) $(f\circ\vec{c})(t) = f(t^3, t^2, 2t)$

$$= \left[(t^3)^2 - (t^2)^2,\ 2(t^3)(t^2),\ (2t)^2 \right]$$

$$= \underline{\left(t^6 - t^4,\ 2t^5,\ 4t^2 \right)}$$

(3) $(f \circ \vec{c})'(t) = (6t^5 - 4t^3,\ 10t^2,\ 8t)$

$(f \circ \vec{c})(1) = (0, 2, 4)$

$(f \circ \vec{c})'(1) = (2, 10, 8)$

$\therefore$ Tangent line of $f \circ \vec{c}$ at $t = 1$:

$$\underline{\vec{l}(s) = (0, 2, 4) + (s - 1)(2, 10, 8)}$$

2.5 Properties of the Derivative

Note Title 2/16/2016

1.

By Theorem 10 of the text, using the product rule,

$$\mathbf{D} f^2(\vec{x}) = [\mathbf{D} f(\vec{x})] f(\vec{x}) + f(\vec{x}) \mathbf{D} f(\vec{x})$$

where $\mathbf{D} f(\vec{x})$ is a $1 \times n$ matrix, $f(\vec{x}) \in \mathbb{R}$

Using the constant multiple rule,

$$\mathbf{D}(2 f(\vec{x})) = 2 \mathbf{D} f(\vec{x}),$$

and using the sum rule,

$$\mathbf{D}[f^2(\vec{x}) + 2 f(\vec{x})] = [\mathbf{D} f(\vec{x})] f(\vec{x}) + f(\vec{x}) \mathbf{D} f(\vec{x}) + 2 \mathbf{D} f(\vec{x})$$

$$= 2 f(\vec{x}) \mathbf{D} f(\vec{x}) + 2 \mathbf{D} f(\vec{x})$$

$$= \underline{(2 f(\vec{x}) + 2) \mathbf{D} f(\vec{x})}$$

2.

(a) $\partial f/\partial x = 0$, $\partial f/\partial y = 0$, both are continuous, so f is differentiable. $\mathbf{D} f(x,y) = \underline{[0 \quad 0]}$

(b) $\partial f/\partial x = 1$, $\partial f/\partial y = 1$, both are continuous,

$\therefore Df(x,y) = \underline{[1 \ \ 1]}$. Could use sum rule as well.

(c) $\partial f/\partial x = 1$, $\partial f/\partial y = 1$. Both partial are continuous,

f is differentiable. $Df(x,y) = [1 \ \ 1]$

(d) $\partial f/\partial x = 2x$, $\partial f/\partial y = 2y$. Both continuous $\Rightarrow f$ is differentiable. $Df(x,y) = \underline{[2x \ \ 2y]}$

(e) x is differentiable, y is differentiable, $\therefore$ by product rule, xy is differentiable.
By chain rule, composite function e^{xy} is differentiable.

$\partial f/\partial x = y e^{xy}$ $\quad \partial f/\partial y = x e^{xy}$

$\therefore Df(x,y) = \underline{[ye^{xy} \ \ xe^{xy}]}$

By chain rule: Let $h(x,y) = xy$, $g(x) = e^x$

$\therefore f(x,y) = (g \circ h)(x,y)$.

$\therefore Df(x,y) = Dg(h(x,y))\, Dg(x,y)$

$= [e^{xy}]\,[y \ \ x] = [ye^{xy} \ \ xe^{xy}]$

(f) $1-x^2-y^2$ is differentiable by product rule (for x^2, y^2), sum rule (for $1-x^2-y^2$), and
$\therefore f$ is differentiable by chain rule (for $\sqrt{x}$, $x>0$).

Let $h(x,y) = 1-x^2-y^2$. $\therefore Dh(x,y) = \begin{bmatrix} -2x & -2y \end{bmatrix}$

$g(z) = \sqrt{z}$. $\therefore Dg(z) = \begin{bmatrix} \frac{1}{2\sqrt{z}} \end{bmatrix}$

$$\therefore Df(x,y) = Dg(h(x,y))\,Dh(x,y)$$

$$= \begin{bmatrix} \frac{1}{2\sqrt{1-x^2-y^2}} \end{bmatrix} \begin{bmatrix} -2x & -2y \end{bmatrix}$$

$$= \underline{\begin{bmatrix} \frac{-x}{\sqrt{1-x^2-y^2}} & \frac{-y}{\sqrt{1-x^2-y^2}} \end{bmatrix}}, \quad 1-x^2-y^2>0$$

(g) By product rule (for x^4, y^4) and sum rule (for x^4-y^4), f is differentiable. $\partial f/\partial x = 4x^3$ $\partial f/\partial y = -4y^3$

$$\therefore Df(x,y) = \underline{\begin{bmatrix} 4x^3 & -4y^3 \end{bmatrix}}$$

3.

"The first special case" is according to text on page 127, which is $f: \mathbb{R}^3 \to \mathbb{R}$, $c: \mathbb{R} \to \mathbb{R}^3$

(a) (1) $f \circ c = e^t \cos t$.

$\therefore (f \circ c)'(t) = e^t \cos t - \sin t \, e^t$

(2) $Df(\vec{c}(t)) \, D\vec{c}(t)$

$$Df(x,y) = \begin{bmatrix} y & x \end{bmatrix} \quad D\vec{c}(t) = \begin{bmatrix} e^t \\ -\sin t \end{bmatrix}$$

$$\therefore Df(\vec{c}(t)) \, D\vec{c}(t) = \begin{bmatrix} \cos t & e^t \end{bmatrix} \begin{bmatrix} e^t \\ -\sin t \end{bmatrix}$$

$$= \cos t \, e^t - \sin t \, e^t$$

(1) = (2)

(b) (1) $(f \circ c)(t) = e^{(3t^2)(t^3)} = e^{3t^5}$

$\therefore (f \circ c)'(t) = 15t^4 e^{3t^5}$

(2) $Df(x,y) = \begin{bmatrix} y e^{xy} & x e^{xy} \end{bmatrix} \quad D\vec{c}(t) = \begin{bmatrix} 6t \\ 3t^2 \end{bmatrix}$

$$\therefore Df(\vec{c}(t)) \, D\vec{c}(t) = \begin{bmatrix} t^3 e^{3t^5} & 3t^2 e^{3t^5} \end{bmatrix} \begin{bmatrix} 6t \\ 3t^2 \end{bmatrix}$$

$$= 6t^4 e^{3t^5} + 9t^4 e^{3t^5} = 15t^4 e^{3t^5}$$

$\therefore$ (1) = (2)

(c)

(1) $(f \circ c)(t) = (e^{2t} + e^{-2t}) \log\sqrt{e^{2t} + e^{-2t}}$

$$\therefore (f \circ c)'(t) = 2(e^{2t} - e^{-2t}) \log\sqrt{e^{2t} + e^{-2t}}$$
$$+ (e^{2t} + e^{-2t}) \frac{1}{\sqrt{e^{2t} + e^{-2t}}} \left(\frac{1}{2} \frac{2e^{2t} - 2e^{-2t}}{\sqrt{e^{2t} + e^{-2t}}}\right)$$
$$= 2(e^{2t} - e^{-2t}) \log\sqrt{e^{2t} + e^{-2t}} + (e^{2t} - e^{-2t})$$
$$= (e^{2t} - e^{-2t})(2 \log\sqrt{e^{2t} + e^{-2t}} + 1)$$

(2) $\partial f/\partial x = 2x \log\sqrt{x^2 + y^2} + \dfrac{(\frac{1}{2}) 2x (x^2 + y^2)}{x^2 + y^2}$

$\partial f/\partial y = 2y \log\sqrt{x^2 + y^2} + \dfrac{(\frac{1}{2})(2y)(x^2 + y^2)}{x^2 + y^2}$

$D\vec{c}(t) = \begin{bmatrix} e^{t} \\ -e^{-t} \end{bmatrix}$

$\therefore Df(\vec{c}(t))\, D\vec{c}(t) =$

$$\begin{bmatrix} 2e^{t} \log\sqrt{e^{2t} + e^{-2t}} + e^{t} & 2e^{-t} \log\sqrt{e^{2t} + e^{-2t}} + e^{-t} \end{bmatrix} \begin{bmatrix} e^{t} \\ -e^{-t} \end{bmatrix}$$

$$= 2e^{2t} \log\sqrt{e^{2t} + e^{-2t}} + e^{2t} - 2e^{-2t} \log\sqrt{e^{2t} + e^{-2t}} - e^{-2t}$$
$$= 2(e^{2t} - e^{-2t}) \log\sqrt{e^{2t} + e^{-2t}} + (e^{2t} - e^{-2t})$$
$$= (e^{2t} - e^{-2t})(2 \log\sqrt{e^{2t} + e^{-2t}} + 1)$$

$\therefore (1) = (2)$ Note: $2 \log\sqrt{e^{2t} + e^{-2t}} = \log(e^{2t} + e^{-2t})$

(d)

(1) $(f \circ c)(t) = t e^{t^2+t^2} = te^{2t^2}$

$\therefore (f \circ c)'(t) = e^{2t^2} + t(4t)e^{2t^2} = (1+4t^2)e^{2t^2}$

(2) $\partial f/\partial x = e^{x^2+y^2} + 2x^2e^{x^2+y^2}$

$\partial f/\partial y = 2xye^{x^2+y^2}$

$D\vec{c}(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\therefore Df(\vec{c}(t)) = \begin{bmatrix} e^{2t^2} + 2t^2e^{2t^2} & -2t^2e^{2t^2} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$= e^{2t^2} + 2t^2e^{2t^2} + 2t^2e^{2t^2}$

$= (1+4t^2)e^{2t^2}$

$\therefore (1) = (2)$

4.

(a) $\vec{c}'(t) = (e^t, -\sin t)$

(b) $\vec{c}'(t) = (6t, 3t^2)$

(c) $\vec{c}(t) = (e^t, e^{-t})$ $\vec{c}'(t) = (e^t, -e^{-t})$

(d) $\vec{c}\,'(t) = (1,-1)$

5.

$$\nabla(fg) = \left[\frac{\partial fg}{\partial x}, \frac{\partial fg}{\partial y}, \frac{\partial fg}{\partial z}\right]$$, now use product rule

$$= \left[g\frac{\partial f}{\partial x} + f\frac{\partial g}{\partial x}, g\frac{\partial f}{\partial y} + f\frac{\partial g}{\partial y}, g\frac{\partial f}{\partial z} + f\frac{\partial g}{\partial z}\right]$$

$$= \left[f\frac{\partial g}{\partial x}, f\frac{\partial g}{\partial y}, f\frac{\partial g}{\partial z}\right] + \left[g\frac{\partial f}{\partial x}, g\frac{\partial f}{\partial y}, g\frac{\partial f}{\partial z}\right]$$

$$= f\nabla g + g\nabla f$$

6.

$$\frac{\partial f}{\partial \rho} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \rho} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial \rho} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \rho}$$

$$= \cos\theta\sin\phi\frac{\partial f}{\partial x} + \sin\theta\sin\phi\frac{\partial f}{\partial y} + \cos\phi\frac{\partial f}{\partial z}$$

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \theta}$$

$$= -\rho \sin\theta \sin\phi \frac{\partial f}{\partial x} + \rho \cos\theta \sin\phi \frac{\partial f}{\partial y}$$

$$\frac{\partial f}{\partial \phi} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \phi} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \phi} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \phi}$$

$$= \rho \cos\theta \cos\phi \frac{\partial f}{\partial x} + \rho \sin\theta \cos\phi \frac{\partial f}{\partial y} - \rho \sin\phi \frac{\partial f}{\partial \phi}$$

7.

(a) $$(f \circ g)(x,y) = \left[\tan(e^{x-y}-1) - e^{x-y}, (e^{x-y})^2 - (x-y)^2\right]$$

$$= \left[\tan(e^{x-y}-1) - e^{x-y}, e^{2x-2y} - x^2 + 2xy + y^2\right]$$

(b) $$Df(u,v) = \begin{bmatrix} \sec^2(u-1) & -e^v \\ 2u & -2v \end{bmatrix}$$

$$g(1,1) = (e^0, 0) = (1,0)$$

$$\therefore Df(g(1,1)) = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$$

$$Dg(x,y) = \begin{bmatrix} e^{x-y} & -e^{x-y} \\ 1 & -1 \end{bmatrix} \quad \therefore Dg(1,1) = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

$$\therefore Df(g(1,1))Dg(1,1) = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 0 & 0 \\ 2 & -2 \end{bmatrix}}}$$

8.

(a) $(f \circ g)(x,y) =$

$$\left[e^{e^x - e^{-y}},\ \cos(e^x + \cos(y-x)) + \sin(e^x + \cos(y-x) + e^{-y}) \right]$$

(b) $$Df(u,v,w) = \begin{bmatrix} e^{u-w} & 0 & -e^{u-w} \\ -\sin(v+u) + \cos(u+v+w) & -\sin(v+u) + \cos(u+v+w) & \cos(u+v+w) \end{bmatrix}$$

$$g(0,0) = (e^0, \cos 0, e^0) = (1,1,1)$$

$$\therefore Df(g(0,0)) = Df(1,1,1) = \begin{bmatrix} 1 & 0 & -1 \\ \cos 3 - \sin 2 & \cos 3 - \sin 2 & \cos 3 \end{bmatrix}$$

$$Dg(x,y) = \begin{bmatrix} e^x & 0 \\ \sin(y-x) & -\sin(y-x) \\ 0 & -e^{-y} \end{bmatrix} \quad Dg(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\therefore Df(g(0,0))Dg(0,0) = \begin{bmatrix} 1 & 0 & 1 \\ \cos 3 - \sin 2 & \cos 3 - \sin 2 & \cos 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= \underline{\begin{bmatrix} 1 & -1 \\ \cos 3 - \sin 2 & -\cos 3 \end{bmatrix}}$$

9.

$$f \circ T(s,t) = \cos(\cos(t^2 s))\sin(\log\sqrt{1+s^2})$$

$$\therefore \frac{\partial f\circ T}{\partial s} = \sin(\log\sqrt{1+s^2})\left[-\sin(\cos(t^2 s))(-\sin(t^2 s))\,t^2\right]$$
$$+ \cos(\cos(t^2 s))\left[\cos(\log\sqrt{1+s^2})\left(\frac{1}{\sqrt{1+s^2}}\right)\left(\frac{2s}{2\sqrt{1+s^2}}\right)\right]$$

$$\therefore \frac{\partial f\circ T}{\partial s}(1,0) = \sin(\log 2)[0] + \cos 1\left[\cos(\log\sqrt{2})\left(\tfrac{1}{2}\right)\right]$$
$$= \underline{\tfrac{1}{2}\cos 1(\cos(\log\sqrt{2}))}$$

Another way: $$DT(s,t) = \begin{bmatrix} -\sin(t^2 s)(t^2) & -\sin(t^2 s)(2st) \\ \frac{s}{1+s^2} & 0 \end{bmatrix}$$

$$Df(u,v) = \begin{bmatrix} -\sin u \sin v & \cos u \cos v \end{bmatrix}$$

$f\circ T: \mathbb{R}^2 \to \mathbb{R}$, $\therefore$ $Df\circ T = \begin{bmatrix} \frac{\partial f\circ T}{\partial s} & \frac{\partial f\circ T}{\partial t} \end{bmatrix}$

$$Df\circ T = Df(T(s,t))\cdot DT(s,t)$$
$$= Df(u,v)\cdot DT(s,t)\Big|_{\substack{u=\cos(t^2 s)\\ v=\log\sqrt{1+s^2}}}\Big|_{\substack{s=1\\ t=0}}$$

$$= Df(u,v)\cdot DT(1,0)\Big|_{\substack{u=1\\ v=\log\sqrt{2}}}$$

$$= D f(1, \log\sqrt{2}) \cdot D T(1,0)$$

$$= \begin{bmatrix} -\sin 1 \sin(\log\sqrt{2}) & \cos 1 \cos(\log\sqrt{2}) \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \frac{1}{2} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\cos 1 \cos(\log\sqrt{2})}{2} & 0 \end{bmatrix}$$

$$\therefore \frac{\partial f \circ T}{\partial s}(1,0) = \frac{\cos(1)\cos(\log\sqrt{2})}{2}$$

10.

(a) $T(t) = \cos^2 t + \sin^2 t + t^2 = 1 + t^2$

$\therefore T'(t) = 2t$

(b) $T(t) \simeq T(t_0) + T'(t_0)(t - t_0)$

$\therefore$ let $t_0 = \pi/2$, $t = \frac{\pi}{2} + 0.01$

$$\therefore T\left(\frac{\pi}{2} + 0.01\right) \simeq T\left(\frac{\pi}{2}\right) + 2\left(\frac{\pi}{2}\right)(0.01)$$

$$= 1 + \frac{\pi^2}{4} + \pi(0.01) \simeq 3.5$$

11.

(a) $\vec{p}(t) = f \circ \vec{c}(t) = (3\sin t + 2, \cos^2 t + \sin^2 t, \cos t + t^2)$

$= (3\sin t + 2, 1, \cos t + t^2)$

$\vec{p}\,'(t) = (3\cos t, 0, -\sin t + 2t)$

$\therefore \underline{p'(\pi) = (-3, 0, 2\pi)}$

(b) $\vec{c}(\pi) = (\cos\pi, \sin\pi, \pi) = \underline{(-1, 0, \pi)}$

$\vec{c}\,'(t) = (-\sin t, \cos t, 1) \quad \therefore \vec{c}\,'(\pi) = \underline{(0, -1, 1)}$

$$Df(x,y,z) = \begin{bmatrix} 0 & 3 & 0 \\ 2x & 2y & 0 \\ 1 & 0 & 2z \end{bmatrix} \quad \therefore Df(-1,0,\pi) = \underline{\begin{bmatrix} 0 & 3 & 0 \\ -2 & 0 & 0 \\ 1 & 0 & 2\pi \end{bmatrix}}$$

(c)
$$\begin{bmatrix} 0 & 3 & 0 \\ -2 & 0 & 0 \\ 1 & 0 & 2\pi \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \underline{\begin{bmatrix} -3 \\ 6 \\ 2\pi \end{bmatrix}}$$

12.

$$Dh(x,y,z) = \begin{bmatrix} yz & xz & xy \\ ze^{xz} & 0 & xe^{xz} \\ \sin(y) & x\cos(y) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad Dg(u,v) = \begin{bmatrix} 2 & 2v \\ 0 & 0 \\ \frac{1}{\sqrt{u}} & 0 \end{bmatrix}$$

$$Dg(1,1) = \begin{bmatrix} 2 & 2 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$g(1,1) = (3, \pi, 2)$$

$$\therefore Dh(3,\pi,2) = \begin{bmatrix} 2\pi & 6 & 3\pi \\ 2e^6 & 0 & 3e^6 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore D(h \circ g)(1,1) = Dh(3,\pi,2)\,Dg(1,1) =$$

$$\begin{bmatrix} 2\pi & 6 & 3\pi \\ 2e^6 & 0 & 3e^6 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 7\pi & 4\pi \\ 7e^6 & 4e^6 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

13.

(a) $DT(x,y) = \begin{bmatrix} 2xe^y - y^3 & x^2e^y - 3xy^2 \end{bmatrix}$

Let $f(t) = (\cos t, \sin t)$. $\therefore Df(t) = \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}$

$$\therefore D(T\circ f)(t) = \begin{bmatrix} 2xe^y - y^3 & x^2e^y - 3xy^2 \end{bmatrix} \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} \Bigg|_{\substack{x=\cos t \\ y=\sin t}}$$

$$= -\sin t\left(2xe^y - y^3\right) + \cos t\left(x^2e^y - 3xy^2\right) \Bigg|_{\substack{x=\cos t \\ y=\sin t}}$$

$$= -\sin t\left(2(\cos t)e^{\sin t} - \sin^3 t\right) + \cos t\left((\cos^2 t)e^{\sin t} - 3\cos t\sin^2 t\right)$$

$$\therefore T'(t) = \underline{e^{\sin t}\left(\cos^3 t - 2\sin t\cos t\right) + \sin^4 t - 3\cos^2 t\sin^2 t}$$

(b) $T(t) = (\cos^2 t)e^{\sin t} - (\cos t)\sin^3 t$

$$T'(t) = e^{\sin t}\left(2\cos t(-\sin t)\right) + e^{\sin t}(\cos t)(\cos^2 t)$$

$$+ (\sin t)(\sin^3 t) - (\cos t)(3\sin^2 t)(\cos t)$$

$$= \underline{e^{\sin t}\left(\cos^3 t - 2\sin t\cos t\right) + \sin^4 t - 3\cos^2 t\sin^2 t}$$

14.

Let $f: \mathbb{R}^n \to \mathbb{R}^m$ be $T\vec{x}$. It was shown in Ex. 28 of

Sec. 2.3 That $T = Df = \begin{bmatrix} f_1(\hat{e}_1) \cdots f_1(\hat{e}_n) \\ \vdots \\ f_m(\hat{e}_1) \cdots f_m(\hat{e}_n) \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} \cdots \frac{\partial f_1}{\partial x_n} \\ \vdots \\ \frac{\partial f_m}{\partial x_1} \cdots \frac{\partial f_m}{\partial x_n} \end{bmatrix}$

Let $g : R^m \to R^p$ be $M\vec{y}$ so that

$$M = Dg = \begin{bmatrix} g_1(\hat{e}_1) \cdots g_1(\hat{e}_m) \\ \vdots \\ g_p(\hat{e}_1) \cdots g_p(\hat{e}_m) \end{bmatrix} = \begin{bmatrix} \frac{\partial g_1}{\partial y_1} & \cdots & \frac{\partial g_1}{\partial y_m} \\ \vdots & & \vdots \\ \frac{\partial g_p}{\partial y_1} & \cdots & \frac{\partial g_p}{\partial y_m} \end{bmatrix}$$

The derivative is the linear map (the matrix)

Note: $f(\vec{x}) = T\vec{x}$, $Df(\vec{x}) = T$, <u>not</u> $T\vec{x}$

T is composed of real numbers, it has no variables to evaluate for.
So, "Df at $\vec{x}$ is T".

Let $\vec{y} = f(\vec{x}) = T\vec{x}$

$\therefore (g \circ f)(\vec{x}) = g(f(\vec{x})) = g(\vec{y})$

$$= g(T\vec{x}) = M(T\vec{x}) = (MT)\vec{x}$$

$\therefore g \circ f = MT : R^n \to R^p$

so $D(g \circ f)(\vec{x}) = MT$

By chain rule, $D(g \circ f)(\vec{x}) = Dg(f(\vec{x}))\, Df(\vec{x})$

= "Derivative of g at $f(\vec{x})$" times

"Derivative of f at $\vec{x}$"

$= MT$

Again, $Dg(f(\vec{x})) = Dg(\vec{y}) = M$, $Df(\vec{x}) = T$

since $Dg = M$ and M has only real number entries, no variables.

Just like if $f(x) = 3x$, $f' = 3$, so $f'(1) = f'(12) = 3$.

15.

$D(f \circ \vec{c})(0) = Df(\vec{c}(0)) \cdot D\vec{c}(0)$

$Df(x,y) = \begin{bmatrix} e^{x+y} & e^{x+y} \\ e^{x-y} & -e^{x-y} \end{bmatrix}$

$\therefore Df(\vec{c}(0)) = Df((0,0)) = \begin{bmatrix} e^0 & e^0 \\ e^0 & -e^0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$D\vec{c}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\therefore Df(\vec{c}(0)) \cdot D\vec{c}(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

or, $\underline{(2,0)}$

16.

$$\nabla f(x,y) = [f_x \quad f_y]$$

$$= \left[-\frac{1}{2}(x^2+y^2)^{-3/2}(2x) \quad -\frac{1}{2}(x^2+y^2)^{-3/2}(2y) \right]$$

$$= \left[-x(x^2+y^2)^{-3/2} \quad -2y(x^2+y^2)^{-\frac{3}{2}} \right]$$

17.

(a) Let $v(x,y) = x$. $\therefore h(x,y) = f(v(x,y), u(x,y))$

Note $\frac{\partial v(x,y)}{\partial x} = 1$, $\frac{\partial v}{\partial y}(x,y) = 0$

Let $g(x,y) = (v(x,y), u(x,y))$

$\therefore$ have $f(u,v)$, and $h(x,y) = (f \circ g)(x,y)$

$\therefore D h(x,y) = D f(g(x,y)) \cdot D g(x,y)$

$D h(x,y) = \left[\frac{\partial h}{\partial x} \quad \frac{\partial h}{\partial y} \right]$, $D f(v,u) = \left[\frac{\partial f}{\partial v} \quad \frac{\partial f}{\partial u} \right]$

$$Dg(x,y) = \begin{bmatrix} \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \end{bmatrix}$$

$$\therefore \begin{bmatrix} \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial v} & \frac{\partial f}{\partial u} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial f}{\partial v} + \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} & \frac{\partial f}{\partial u} \frac{\partial u}{\partial y} \end{bmatrix}$$

$$\therefore \underline{\frac{\partial h}{\partial x} = \frac{\partial f}{\partial v} + \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x}}, \text{ where } f(v,u) = f(x, u(x,y))$$

(b) Let $w(x) = x$, so $h(x) = f(w(x), u(x), v(x))$

where $f : R^3 \to R$, let $g(x) = (w(x), u(x), V(x))$

$g : R \to R^3$

$\therefore h(x) = (f \circ g)(x)$, $h : R \to R$

By chain rule, $Dh(x) = Df(g(x)) \cdot Dg(x)$

$$Dg(x) = \begin{bmatrix} w'(x) \\ u'(x) \\ V'(x) \end{bmatrix} \qquad Df = \begin{bmatrix} \frac{\partial f}{\partial w} & \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \end{bmatrix}$$

and $Dh(x) = \left[\frac{dh}{dx} \right]$

$$\therefore Dh(x) = \left[\frac{dh}{dx}\right] = \left[\frac{\partial f}{\partial w}\ \frac{\partial f}{\partial u}\ \frac{\partial f}{\partial v}\right] \begin{bmatrix} w'(x) \\ u'(x) \\ v'(x) \end{bmatrix}$$

$$= \frac{\partial f}{\partial w}\frac{dw}{dx} + \frac{\partial f}{\partial u}\cdot\frac{du}{dx} + \frac{\partial f}{\partial v}\cdot\frac{dv}{dx}$$

But $\frac{dw}{dx} = 1$ since $w(x) = x$

$$\therefore \frac{dh}{dx} = \frac{\partial f}{\partial w} + \frac{\partial f}{\partial u}\cdot\frac{du}{dx} + \frac{\partial f}{\partial v}\cdot\frac{dv}{dx},$$

where $f(w,u,v) = F(x, u(x), v(x))$

so, could write $\underline{\frac{dh}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial u}\cdot\frac{du}{dx} + \frac{\partial f}{\partial v}\cdot\frac{dv}{dx}}$

(c)

Let $g: \mathbb{R}^3 \to \mathbb{R}^3$ be $g(x,y,z) = (u(x,y,z), v(x,y), w(x))$

$f: \mathbb{R}^3 \to \mathbb{R}^1$, $f(u,v,w)$ $\therefore h(x,y,z) = (f \circ g)(x,y,z)$

By chain rule, $Dh(x,y,z) = Df(g(x,y,z)) \cdot Dg(x,y,z)$

$Dh = \left[\frac{\partial h}{\partial x}\ \frac{\partial h}{\partial y}\ \frac{\partial h}{\partial z}\right]$ $Df = \left[\frac{\partial f}{\partial u}\ \frac{\partial f}{\partial v}\ \frac{\partial f}{\partial w}\right]$

$$Dg = \begin{bmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{bmatrix} = \begin{bmatrix} u_x & u_y & u_z \\ v_x & v_y & 0 \\ w_x & 0 & 0 \end{bmatrix}$$

$$\therefore \left\{\frac{\partial h}{\partial x}\right\} = \begin{bmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} & \frac{\partial f}{\partial w} \end{bmatrix} \begin{bmatrix} u_x \\ v_x \\ w_x \end{bmatrix}$$

$$= \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x} + \frac{\partial f}{\partial w} \cdot \frac{dw}{dx}$$

18

(a) $$h(x,y) = \frac{(e^{-x-y})^2 + (e^{xy})^2}{(e^{-x-y})^2 - (e^{xy})^2} = \frac{e^{-2x-2y} + e^{2xy}}{e^{-2x-2y} - e^{2xy}}$$

$$\therefore \frac{\partial h}{\partial x} = \frac{(e^{-2x-2y} - e^{2xy})(-2e^{-2x-2y} + 2ye^{2xy}) - (e^{-2x-2y} + e^{2xy})(-2e^{-2x-2y} - 2ye^{2xy})}{(e^{-2x-2y} - e^{2xy})^2}$$

$$= \frac{-2e^{-4x-4y} + 2e^{-2x-2y+2xy} + 2ye^{-2x-2y+2xy} - 2ye^{4xy}}{e^{-4x-4y} - 2e^{-2x-2y+2xy} + e^{4xy}}$$

$$- \frac{(-2e^{-4x-4y} - 2e^{-2x-2y+2xy} - 2ye^{-2x-2y+2xy} - 2ye^{4xy})}{e^{-4x-4y} - 2e^{-2x-2y+2xy} + e^{4xy}}$$

$$= \frac{4e^{-2x-2y+2xy} + 4ye^{-2x-2y+2xy}}{e^{-4x-4y} - 2e^{-2x-2y+2xy} + e^{4xy}}$$

(b) $\frac{\partial h}{\partial x} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}$

$\frac{\partial u}{\partial x} = -e^{-x-y} \qquad \frac{\partial v}{\partial x} = ye^{xy}$

$$\left.\begin{aligned}\frac{\partial f}{\partial u} &= \frac{(u^2-v^2)(2u)-(u^2+v^2)(2u)}{(u^2-v^2)^2}\\ \frac{\partial f}{\partial v} &= \frac{(u^2-v^2)(2v)-(u^2+v^2)(-2v)}{(u^2-v^2)^2}\end{aligned}\right\} \quad \begin{aligned}u &= e^{-x-y}\\ v &= e^{xy}\end{aligned}$$

$$\therefore \frac{\partial f}{\partial u} = \frac{(e^{-2x-2y}-e^{2xy})(2e^{-x-y})-(e^{-2x-2y}+e^{2xy})(2e^{-x-y})}{(e^{-2x-2y}-e^{2xy})^2}$$

$$= \frac{2e^{-3x-3y}-2e^{-x-y+2xy}-2e^{-3x-3y}-2e^{-x-y+2xy}}{e^{-4x-4y}-2e^{-2x-2y+2xy}+e^{4xy}}$$

$$= \frac{-4e^{-x-y+2xy}}{e^{-4x-4y}-2e^{-2x-2y+2xy}+e^{4xy}}$$

$$\frac{\partial f}{\partial v} = \frac{(e^{-2x-2y}-e^{2xy})(2e^{xy})-(e^{-2x-2y}+e^{2xy})(-2e^{xy})}{e^{-4x-4y}-2e^{-2x-2y+2xy}+e^{4xy}}$$

$$= 2e^{-2x-2y+xy}-2e^{3xy}+2e^{-2x-2y+xy}+2e^{3xy}$$

$$= \frac{4e^{-2x-2y+xy}}{e^{-4x-4y}-2e^{-2x-2y+2xy}+e^{4xy}}$$

$$\therefore \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x} =$$

$$\frac{-4e^{-x-y+2xy}}{e^{-4x-4y}-2e^{-2x-2y+2xy}+e^{4xy}}\left(-e^{-x-y}\right)$$

$$+\ \frac{4e^{-2x-2y+xy}}{e^{-4x-4y}-2e^{-2x-2y+2xy}+e^{4xy}}\left(ye^{xy}\right)$$

$$= \underline{\underline{\frac{4e^{-2x-2y+2xy}+4ye^{-2x-2y+2xy}}{e^{-4x-4y}-2e^{-2x-2y+2xy}+e^{4xy}}}}$$

Amazingly, (a) = (b) = $\frac{\partial h}{\partial x}$

19.

(a) Let $h(x) = G(x, y(x)) = 0$.

By the chain rule, $h'(x) = \frac{\partial G}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial G}{\partial y}\cdot\frac{dy}{dx} = 0$

Note $\frac{dx}{dx} = 1$.

$$\therefore \frac{\partial G}{\partial x} \cdot 1 + \frac{\partial G}{\partial y} \cdot \frac{dy}{dx} = 0, \quad \frac{\partial G}{\partial y} \cdot \frac{dy}{dx} = -\frac{\partial G}{\partial x}$$

$$\therefore \frac{dy}{dx} = -\frac{\partial G/\partial x}{\partial G/\partial y}, \text{ assuming } \frac{\partial G}{\partial y}(x) \neq 0.$$

(5)

$$\frac{\partial G_1}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial G_1}{\partial y_1} \cdot \frac{dy_1}{dx} + \frac{\partial G_1}{\partial y_2} \frac{dy_2}{dx} = 0$$

$$\frac{\partial G_2}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial G_2}{\partial y_1} \frac{dy_1}{dx} + \frac{\partial G_2}{\partial y_2} \frac{dy_2}{dx} = 0$$

$$\therefore \begin{bmatrix} \frac{\partial G_1}{\partial y_1} & \frac{\partial G_1}{\partial y_2} \\ \frac{\partial G_2}{\partial y_1} & \frac{\partial G_2}{\partial y_2} \end{bmatrix} \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} -\frac{\partial G_1}{\partial x} \\ -\frac{\partial G_2}{\partial x} \end{bmatrix}$$

$$\therefore \begin{bmatrix} \frac{dy_1}{dx} \\ \frac{dy_2}{dx} \end{bmatrix} = \frac{1}{\frac{\partial G_1}{\partial y_1} \frac{\partial G_2}{\partial y_2} - \frac{\partial G_1}{\partial y_2} \frac{\partial G_2}{\partial y_1}} \begin{bmatrix} \frac{\partial G_2}{\partial y_2} & -\frac{\partial G_1}{\partial y_2} \\ -\frac{\partial G_2}{\partial y_1} & \frac{\partial G_1}{\partial y_1} \end{bmatrix} \begin{bmatrix} -\frac{\partial G_1}{\partial x} \\ \frac{\partial G_2}{\partial x} \end{bmatrix}$$

$$\text{assuming } \frac{\partial G_1}{\partial y_1} \frac{\partial G_2}{\partial y_2} - \frac{\partial G_1}{\partial y_2} \frac{\partial G_2}{\partial y_1} \neq 0$$

(c)

Let $G(x, y(x)) = x^2 + y^3 + e^y = 0$

By (a), $\dfrac{dy}{dx} = -\dfrac{\partial G/\partial x}{\partial G/\partial y} = -\dfrac{2x}{3y^2 + e^y}$

20.

Let $G(y, z) = F(x(y,z), y, z) = 0$ [1]

$H(x, z) = F(x, y(x,z), z) = 0$ [2]

$J(x, y) = F(x, y, z(x,y)) = 0$ [3]

From [1], $G_y = F_x \cdot \dfrac{\partial x}{\partial y} + F_y \dfrac{dy}{dy} + F_z \dfrac{\partial z}{\partial y} = 0$

Since for $G(y,z)$, z is not a function of y, $\dfrac{\partial z}{\partial y} = 0$

Also, $\dfrac{dy}{dy} = 1$

$\therefore G_y = F_x \dfrac{\partial x}{\partial y} + F_y = 0$

Similarly, $G_z = F_x \dfrac{\partial x}{\partial z} + F_z = 0 \quad \left(\dfrac{\partial y}{\partial z} = 0\right)$

$[1'] \therefore \dfrac{\partial x}{\partial y} = \dfrac{-F_y}{F_x}$, $\dfrac{\partial x}{\partial z} = \dfrac{-F_z}{F_x}$, assuming $F_x \neq 0$

From [2], $H_x = F_x \dfrac{dx}{dx} + F_y \dfrac{\partial y}{\partial x} + F_z \dfrac{\partial z}{\partial x}$

Or, $F_x + F_y \dfrac{\partial y}{\partial x} = 0 \quad \left(\dfrac{\partial z}{\partial x} = 0\right)$

Similarly, $H_z = F_y \dfrac{\partial y}{\partial z} + F_z = 0 \quad \left(\dfrac{\partial x}{\partial z} = 0\right)$

$[2'] \quad \therefore \dfrac{\partial y}{\partial x} = -\dfrac{F_x}{F_y} \quad \dfrac{\partial y}{\partial z} = -\dfrac{F_z}{F_y}$, $F_y \neq 0$

From [3], $J_x = F_x \dfrac{dx}{dx} + F_z \dfrac{\partial z}{\partial x} = 0 \quad \left(\dfrac{\partial y}{\partial x} = 0\right)$

Or, $F_x + F_z \dfrac{\partial z}{\partial x} = 0$

Similarly, $J_y = F_y + F_z \dfrac{\partial z}{\partial y} = 0$

$[3'] \quad \therefore \dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z}$, $\dfrac{\partial z}{\partial y} = \dfrac{-F_y}{F_z}$, $F_z \neq 0$

From [2'] $\dfrac{\partial y}{\partial x} = -\dfrac{F_x}{F_y}$ From [3'] $\dfrac{\partial z}{\partial y} = -\dfrac{F_y}{F_z}$

From [1'] $\dfrac{\partial x}{\partial z} = -\dfrac{F_z}{F_x}$

$$\therefore \left(\frac{\partial y}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right)\left(\frac{\partial x}{\partial z}\right) = \left(-\frac{F_x}{F_y}\right)\left(-\frac{F_y}{F_z}\right)\left(-\frac{F_z}{F_x}\right) = -1$$

assuming $F_x, F_y, F_z \neq 0$

21.

Rewrite equation: $\log(V-b) + a/RVT = \log\left(\frac{RT}{p}\right)$

$$\therefore \frac{\partial \log(V-b)}{\partial T} + \frac{\partial}{\partial T}\left(\frac{a}{RVT}\right) = \frac{\partial}{\partial T}\left(\log\frac{RT}{p}\right)$$

$$\frac{1}{V-b}\frac{\partial V}{\partial T} + \frac{a}{R}\left(-\frac{1}{(VT)^2}\right)\left(V + T\frac{\partial V}{\partial T}\right) = \frac{1}{\frac{RT}{p}}\cdot\frac{R}{p}$$

$$\therefore \frac{1}{V-b}\frac{\partial V}{\partial T} - \frac{a}{RVT^2} - \frac{a}{RTV^2}\frac{\partial V}{\partial T} = \frac{1}{T}$$

$$\therefore \frac{\partial V}{\partial T}\left(\frac{1}{V-b} - \frac{a}{RTV^2}\right) = \frac{1}{T} + \frac{a}{RVT^2}$$

$$\therefore \frac{\partial V}{\partial T} = \frac{\frac{1}{T} + \frac{a}{RVT^2}}{\frac{1}{V-b} - \frac{a}{RTV^2}} \cdot \frac{RT}{RT} = \frac{R + \frac{a}{VT}}{\frac{RT}{V-b} - \frac{a}{V^2}}$$

22.

(a) Computing f_x directly won't help since

$$f_x = \frac{(x^2+y^2)y^2 - xy^2(2x)}{(x^2+y^2)^2} = \frac{(-x^2+y^2)y^2}{(x^2+y^2)^2}, \quad x,y \neq 0$$

and nothing cancels to get rid of denominator. So, look at original definition:

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{(0+h)0^2}{(0+h)^2+0^2} - 0}{h} = \lim_{h\to 0} 0 = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{f(0,0+h) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{0(0+h)^2}{0+(0+h)^2} - 0}{h} = \lim_{h\to 0} 0 = 0$$

(b) $$f \circ \vec{g}(t) = \frac{(at)(bt)^2}{(at)^2+(bt)^2} = \frac{ab^2t^3}{a^2t^2+b^2t^2} = \frac{ab^2t}{a^2+b^2}$$

$\therefore \dfrac{d}{dt}(f \circ g)(0) = \dfrac{ab^2}{a^2+b^2} \neq 0$ if $a, b \neq 0$

But, by chain rule, $D(f \circ \vec{g})(0) = Df(\vec{g}(0)) \cdot g'(0)$

$= Df(0,0) \cdot g'(0) = \begin{bmatrix} f_x(0,0) & f_y(0,0) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$

$= 0$ since $f_x(0,0) = 0, f_y(0,0)$ by (a).

$\therefore D(f \circ \vec{g})(0) = \nabla f(0,0) \cdot \vec{g}'(0) = 0$

so $D(f \circ \vec{g})(0) \neq \dfrac{d}{dt}(f \circ \vec{g})(0)$

23.

Since f is differentiable at $\vec{x}_0 \in U$, then given any $\epsilon > 0$, $\exists\, \delta > 0$ s.t. if $0 < \|\vec{x} - \vec{x}_0\| < \delta$, then

[1] $\dfrac{\| f(\vec{x}) - f(\vec{x}_0) - [Df(\vec{x}_0)](\vec{x} - \vec{x}_0) \|}{\| \vec{x} - \vec{x}_0 \|} < \epsilon$, by definition

$\therefore$ Define $V = D_\delta(\vec{0})$, and define $\vec{h} = \vec{x} - \vec{x}_0$, where

$\vec{x} \in D_\delta(\vec{x}_0)$. $\therefore$ $\vec{h} \in V \iff \vec{x}_0 + \vec{h} \in D_\delta(\vec{x}_0)$, as $\vec{h} \in V$ means $\|\vec{h}\| < \delta$

Define $R_1(\vec{h}) = f(\vec{x}_0 + \vec{h}) - f(\vec{x}_0) - [Df(\vec{x}_0)]\vec{h}$

$\therefore$ Given the above $\epsilon > 0$, choose the above δ, and if $0 < \|\vec{h}\| < \delta$, then $0 < \|\vec{x} - \vec{x}_0\| < \delta$, since $\vec{h} = \vec{x} - \vec{x}_0$

and $\therefore$ [1] above $\Rightarrow$

$$\frac{\|f(\vec{x}_0 + \vec{h}) - f(\vec{x}_0) - [Df(\vec{x}_0)]\vec{h}\|}{\|\vec{h}\|} < \epsilon, \text{ or}$$

$$\frac{|R_1(\vec{h})|}{\|\vec{h}\|} < \epsilon, \quad \text{since } \|R_1(\vec{h})\| = |R_1(\vec{h})| \text{ as } R_1(\vec{h}) : \mathbb{R}^n \to \mathbb{R}$$

$$\therefore \lim_{\vec{h} \to \vec{0}} \frac{|R_1(\vec{h})|}{\|\vec{h}\|} = 0$$

Since $\left|\frac{|R_1(\vec{h})| - 0}{\|\vec{h}\|}\right| = \left|\frac{R_1(\vec{h}) - 0}{\|\vec{h}\|}\right|$, $\lim_{\vec{h} \to \vec{0}} \frac{R_1(\vec{h})}{\|\vec{h}\|} = 0$

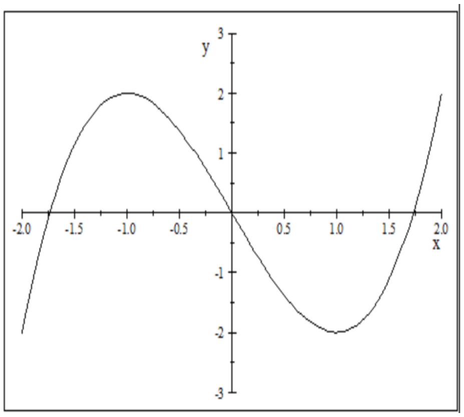
$\therefore \forall \vec{h} \in V$, $\vec{x}_0 + \vec{h} \in U$, $f(\vec{x}_0 + \vec{h}) = f(\vec{x}_0) + [Df(\vec{x}_0)]\vec{h} + R_1(\vec{h})$

and $\lim_{\vec{h} \to \vec{0}} \frac{R_1(\vec{h})}{\|h\|} = 0$

24.

$f(x) = ax^3 + bx^2$ are C^1 functions, $a, b \neq 0$.

A plot of, e.g., $y = x^3 - 3x$ looks like:



The idea is to use the part of the function between the max & min, where slope $= 0$, and define the function to be 0 to the right, and 1 to the left, using r_1 and r_2.

We want r_1 to be the local max and r_2 to be the local min (since $r_1 < r_2$).

$\therefore$ set $y' = (x - r_1)(x - r_2) = 0$

$$\text{or } y' = x^2 - (r_1 + r_2)x + r_1 r_2 = 0$$

$y'' = 2x - (r_1 + r_2)$, $y''(r_2) = r_2 - r_1 > 0$, $y''(r_1) = r_1 - r_2 < 0$

$\therefore$ r_2 will be local min, r_1 a local max.

Integrate y': $y = \frac{1}{3}x^3 - \frac{(r_1 + r_2)}{2}x^2 + r_1 r_2 x + C$

$$\text{or } f(x) = x^3 - 3(r_1 + r_2)x^2 + 6 r_1 r_2 x + K$$

We want $f(r_2)=0$, so

$$f(r_2) = r_2^3 - 3(r_1+r_2)r_2^2 + 6r_1r_2^2 + k = 0$$

$$= r_2^3 - 3r_2^3 - 3r_1r_2^2 + 6r_1r_2^2 + k = 0$$

$$= -2r_2^3 + 3r_1r_2^2 + k = 0, \quad k = 2r_2^3 - 3r_1r_2^2$$

$$\therefore f(x) = x^3 - 3(r_1+r_2)x^2 + 6r_1r_2x + (2r_2^3 - 3r_1r_2^2)$$

We want $f(r_1)=1$, so

$$f(r_1) = r_1^3 - 3r_1^3 - 3r_1^2r_2 + 6r_1^2r_2 + (2r_2^3 - 3r_1r_2^2)$$

$$= -2r_1^3 + 3r_1^2r_2 + 2r_2^3 - 3r_1r_2^2$$

$$= 2(r_2^3 - r_1^3) - 3r_1r_2(r_2 - r_1)$$

$$= 2(r_2-r_1)(r_2^2 + r_2r_1 + r_1^2) - (r_2-r_1)(3r_1r_2)$$

$$= (r_2-r_1)(2r_2^2 - r_1r_2 + 2r_1^2)$$

$$= (r_2-r_1)[r_2(2r_2-r_1) + 2r_1^2] > 0$$

since $r_2 > r_1 \geq 0$

The above $f(x)$ is centered around 0. Now define it by shifting according to x_0.

$\therefore$ Let $a = (r_2-r_1)[r_2(2r_2-r_1) + 2r_1^2]$

$$f(\vec{x}) = \begin{cases} 1, & \text{for } \|\vec{x}-\vec{x}_0\| \le r_1 \\ \frac{1}{a}\left[\|\vec{x}-\vec{x}_0\|^3 - 3(r_1+r_2)\|\vec{x}-\vec{x}_0\|^2 + 6r_1r_2\|\vec{x}-\vec{x}_0\| + (2r_2^2 - 3r_1r_2^2)\right] & \\ \qquad \text{for } r_1 < \|\vec{x}-\vec{x}_0\| < r_2 & \\ 0, & \text{for } r_2 \le \|\vec{x}\| \end{cases}$$

The above $f(\vec{x})$ meets the requirement of $0 < f(\vec{x}) < 1$ for $r_1 < \|\vec{x}-\vec{x}_0\| < r_2$, from the construction above.

$f(\vec{x})$ is also C^1 since, along any x_i, $f(\vec{x})$ is the cubic polnomial from local max to local min, and $\therefore$ the partial derivative at the endpoints (r_1, r_2) exists (and is 0).

25.

From the answer in the back of the book:

From #24 above, let $g_1(\vec{x}): R^3 \to R$ and $g_2(\vec{x}): R^3 \to R$ be C^1 functions such that:

$$g_1(\vec{x}) = \begin{cases} 1, & \|\vec{x}\| \le \sqrt{2}/3 \\ \text{cubic polynomial described in Ex. \#24}, & \frac{\sqrt{2}}{3} < \|x\| < \frac{2\sqrt{2}}{3} \\ 0, & \frac{2\sqrt{2}}{3} \le \|\vec{x}\| \end{cases}$$

and

$$g_2(\vec{x}) = \begin{cases} 1, & \|\vec{x}-(1,1,0)\| \le \frac{\sqrt{2}}{3} \\ \text{cubic polynomial described in Ex. \#24, for} & \frac{\sqrt{2}}{3} < \|\vec{x}-(1,1,0)\| < \frac{2\sqrt{2}}{3} \\ 0, & \frac{2\sqrt{2}}{3} \le \|\vec{x}-(1,1,0)\| \end{cases}$$

In the above, $r_1 = \frac{\sqrt{2}}{3}$, $r_2 = \frac{2\sqrt{2}}{3}$, using Ex. #24 above.

$g_1(\vec{x})$ is for vectors arising from the origin

$g_2(\vec{x})$ is for vectors arising from $(1,1,0)$

Let $h_1(\vec{x}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, for vectors arising from $(0,0,0)$

$h_2(\vec{x}) = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, for vectors arising from $(1,1,0)$

Note $h_1(\vec{x})$ and $h_2(\vec{x})$ are C^1, since, if $j(\vec{x}) = T\vec{x}$, the $Dj(\vec{x}) = T$, and so the matrix T of partial derivatives (just real numbers in this case) are all continuous (since they are real constants).

$\therefore$ Let $f(\vec{x}) = g_1(\vec{x})h_1(\vec{x}) + g_2(\vec{x})h_2(\vec{x})$

This is the sum and product of C^1 functions, and so is a C^1 function itself.

26.

Write $w(x,y) = f(x, y, g(x,y))$

The $\frac{\partial w}{\partial x}$ on the left side means the derivative from holding the 2nd independent variable constant.

The $\frac{\partial w}{\partial x}$ on the right is really $\frac{\partial f}{\partial x}$, and means the derivative of the 1st independent variable while holding the 2nd & 3rd independent variables constant.

Thus, it's an abuse of terminology, and the two are different derivatives.

27.

Note: $Dh(\vec{x}_0)$, $Df(\vec{x}_0)$, $Dg(\vec{x}_0)$ are $1 \times n$ matrices

$$h(\vec{x}) - h(\vec{x}_0) - [g(\vec{x}_0)Df(\vec{x}_0) + f(\vec{x}_0)Dg(\vec{x}_0)](\vec{x}-\vec{x}_0)$$

$$= g(\vec{x})f(\vec{x}) - g(\vec{x}_0)f(\vec{x}_0) - [g(\vec{x}_0)Df(\vec{x}_0) + f(\vec{x}_0)Dg(\vec{x}_0)](\vec{x}-\vec{x}_0) \quad [0]$$

$$= g(\vec{x})f(\vec{x}) - g(\vec{x}_0)f(\vec{x}) + g(\vec{x}_0)f(\vec{x}) \quad \{\text{adding/subtracting } g(\vec{x}_0)f(\vec{x})\}$$

$$- g(\vec{x}_0)f(\vec{x}_0) - [g(\vec{x}_0)Df(\vec{x}_0) + f(\vec{x}_0)Dg(\vec{x}_0)](\vec{x}-\vec{x}_0)$$

$$= [g(\vec{x}) - g(\vec{x}_0)]f(\vec{x}) + g(\vec{x}_0)[f(\vec{x}) - f(\vec{x}_0)]$$

$$- [g(\vec{x}_0)Df(\vec{x}_0) + f(\vec{x}_0)Dg(\vec{x}_0)](\vec{x}-\vec{x}_0)$$

$$\{\text{now subtract/add } [Dg(\vec{x}_0)(\vec{x}-\vec{x}_0)]f(\vec{x})\}$$

$$= [g(\vec{x}) - g(\vec{x}_0) - Dg(\vec{x}_0)(\vec{x}-\vec{x}_0)]f(\vec{x}) + [Dg(\vec{x}_0)(\vec{x}-\vec{x}_0)]f(\vec{x})$$

$$\{\text{and subtract/add } g(\vec{x}_0)[Df(\vec{x}_0)(\vec{x}-\vec{x}_0)]\}$$

$$+ g(\vec{x}_0)[f(\vec{x}) - f(\vec{x}_0) - Df(\vec{x}_0)(\vec{x}-\vec{x}_0)] + g(\vec{x}_0)[Df(\vec{x}_0)(\vec{x}-\vec{x}_0)]$$

$$- [g(\vec{x}_0)Df(\vec{x}_0) + f(\vec{x}_0)Dg(\vec{x}_0)](\vec{x}-\vec{x}_0)$$

$$= [g(\vec{x}) - g(\vec{x}_0) - Dg(\vec{x}_0)(\vec{x}-\vec{x}_0)]f(\vec{x}) \qquad [1]$$

$$+ g(\vec{x}_0)[f(\vec{x}) - f(\vec{x}_0) - Df(\vec{x}_0)(\vec{x}-\vec{x}_0)] \qquad [2]$$

$$+ [f(\vec{x}) - f(\vec{x}_0)][Dg(\vec{x}_0)(\vec{x}-\vec{x}_0)] \qquad [3]$$

For [1], $\lim\limits_{\vec{x}\to\vec{x}_0} \dfrac{[g(\vec{x})-g(\vec{x}_0)-Dg(\vec{x}_0)(\vec{x}-\vec{x}_0)]f(\vec{x})}{\|\vec{x}-\vec{x}_0\|} =$

$$\lim_{\vec{x}\to\vec{x}_0} \frac{[g(\vec{x})-g(\vec{x}_0)-Dg(x_0)(\vec{x}-\vec{x}_0)}{\|\vec{x}-\vec{x}_0\|} \cdot \lim_{\vec{x}\to\vec{x}_0} f(\vec{x}) =$$

$$0 \cdot f(\vec{x}_0) = 0$$

as $f(\vec{x})$ is continuous at $\vec{x}_0$ since it is differentiable at x_0. Also, the first limit is from the definition of the derivative of $g(\vec{x})$.

Note: as shown in file "Chapter 2 Notes",

$$\lim_{\vec{x}\to\vec{x}_0} \frac{\|f(\vec{x})-f(\vec{x}_0)-T(\vec{x}-\vec{x}_0)\|}{\|\vec{x}-\vec{x}_0\|} = 0 \Leftrightarrow \lim_{\vec{x}\to\vec{x}_0} \frac{f(\vec{x})-f(\vec{x}_0)-T(\vec{x}-\vec{x}_0)}{\|\vec{x}-\vec{x}_0\|} = \vec{0}$$

For [2] above, since f is differentiable at $\vec{x}_0$

$$\lim_{\vec{x}\to\vec{x}_0} g(\vec{x}_0)\frac{[f(\vec{x})-f(\vec{x}_0)-Df(\vec{x}_0)(\vec{x}-\vec{x}_0)]}{\|\vec{x}-\vec{x}_0\|} =$$

$$g(\vec{x}_0)\cdot 0 = 0$$

For [3], consider $\lim\limits_{\vec{x}\to\vec{x}_0} [f(\vec{x})-f(\vec{x}_0)]\dfrac{[Dg(\vec{x}_0)(\vec{x}-\vec{x}_0)]}{\|\vec{x}-\vec{x}_0\|}$

$\lim\limits_{\vec{x}\to\vec{x}_0}[f(\vec{x})-f(\vec{x}_0)] = 0$, as f is continuous at $\vec{x}_0$ since f is differentiable at $\vec{x}_0$.

In "Chapter 2 Notes", it is shown that for any matrix T, $\|T\vec{x}\| \le k\|\vec{x}\|$, some real value k specific to T. Actually, $k = \|T\hat{e}_1\| + \dots + \|T\hat{e}_n\|$, where $\hat{e}_i$ is the natural basis, and T is $m \times n$ and $T_{ij} \in \mathbb{R}$.

$\therefore$ Let k be s.t. $\|Dg(\vec{x}_0)(\vec{x}-\vec{x}_0)\| \le k\|\vec{x}-\vec{x}_0\|$

$\therefore$ Let $\epsilon > 0$. $\exists\, \delta > 0$ s.t. if $0 < \|\vec{x}-\vec{x}_0\| < \delta$, then

$$|f(\vec{x}) - f(\vec{x}_0)| < \frac{\epsilon}{k} \qquad \{f(\vec{x}) \text{ continuous at } \vec{x}_0\}$$

$$\therefore |f(\vec{x}) - f(\vec{x}_0)|\,\|Dg(\vec{x}_0)(\vec{x}-\vec{x}_0)\| < \frac{\epsilon}{k}\cdot k\|\vec{x}-\vec{x}_0\| = \epsilon\|\vec{x}-\vec{x}_0\|$$

$$\therefore \frac{\left|\, |f(\vec{x}) - f(\vec{x}_0)|\,\|Dg(\vec{x}_0)(\vec{x}-\vec{x}_0)\| - 0 \,\right|}{\|\vec{x}-\vec{x}_0\|} < \epsilon$$

$$\therefore \lim_{\vec{x}\to\vec{x}_0} [f(\vec{x}) - f(\vec{x}_0)]\frac{[Dg(\vec{x}_0)(\vec{x}-\vec{x}_0)]}{\|\vec{x}-\vec{x}_0\|} = 0$$

$\therefore$ [1], [2], and [3] $\Rightarrow$ for [0],

$$\lim_{\vec{x}\to\vec{x}_0} \frac{f(\vec{x})g(\vec{x}) - f(\vec{x}_0)g(\vec{x}_0) - [g(\vec{x}_0)Df(\vec{x}_0) + f(\vec{x}_0)Dg(\vec{x}_0)](\vec{x}-\vec{x}_0)}{\|\vec{x}-\vec{x}_0\|} = 0$$

$$\therefore \underline{Dh(\vec{x}_0) = g(\vec{x}_0)Df(\vec{x}_0) + f(\vec{x}_0)Dg(\vec{x}_0)}$$

It will probably be easier to prove just for $\frac{1}{g(\vec{x})}$, and then apply (iii) for $\frac{f(\vec{x})}{g(\vec{x})}$, so prove:

$$D\left(\frac{1}{g}\right)(\vec{x}_0) = -\frac{1}{g(\vec{x}_0)^2} Dg(\vec{x}_0)$$

$\therefore$ Look at $\frac{1}{g(\vec{x})} - \frac{1}{g(\vec{x}_0)} - \left(-\frac{1}{g(\vec{x}_0)^2} Dg(\vec{x}_0)\right)(\vec{x}-\vec{x}_0)$ [0]

$$= \frac{g(\vec{x}_0) - g(\vec{x})}{g(\vec{x})\,g(\vec{x}_0)} + \frac{Dg(\vec{x}_0)(\vec{x}-\vec{x}_0)}{g(\vec{x}_0)^2}$$

$\left\{\text{adding/subtracting } \frac{g(\vec{x}_0) - g(\vec{x})}{g(\vec{x}_0)^2}\right\}$

$$= \frac{g(\vec{x}_0) - g(\vec{x}) + Dg(\vec{x}_0)(\vec{x}-\vec{x}_0)}{g(\vec{x}_0)^2} - \frac{g(\vec{x}_0) - g(\vec{x})}{g(\vec{x}_0)^2} + \frac{g(\vec{x}_0) - g(\vec{x})}{g(\vec{x})\,g(\vec{x}_0)}$$

$$= -\frac{1}{g(\vec{x}_0)^2}\left[g(\vec{x}) - g(\vec{x}_0) - Dg(\vec{x}_0)(\vec{x}-\vec{x}_0)\right] \quad [1]$$

$$+ \left[\frac{g(\vec{x}) - g(\vec{x}_0)}{g(\vec{x}_0)}\right]\left[\frac{1}{g(\vec{x}_0)} - \frac{1}{g(\vec{x})}\right] \quad [2]$$

For [1], since $g(\vec{x})$ is differentiable at $\vec{x}_0$,

$$\lim_{\vec{x}\to\vec{x}_0}\left[-\frac{1}{g(\vec{x}_0)^2}\right]\left[\frac{g(\vec{x})-g(\vec{x}_0)-Dg(\vec{x}_0)(\vec{x}-\vec{x}_0)}{\|\vec{x}-\vec{x}_0\|}\right]$$

$$=\left[-\frac{1}{g(\vec{x}_0)^2}\right]\lim_{\vec{x}\to\vec{x}_0}\frac{g(\vec{x})-g(\vec{x}_0)-Dg(\vec{x}_0)(\vec{x}-\vec{x}_0)}{\|\vec{x}-\vec{x}_0\|}$$

$$=-\frac{1}{g(\vec{x}_0)^2}\cdot 0=0$$

For [2], $\left[\dfrac{g(\vec{x})-g(\vec{x}_0)}{g(\vec{x}_0)}\right]\left[\dfrac{1}{g(\vec{x}_0)}-\dfrac{1}{g(\vec{x})}\right]$

{adding/subtracting $Dg(\vec{x}_0)(\vec{x}-\vec{x}_0)$}

$$=\left[\frac{g(\vec{x})-g(\vec{x}_0)-Dg(\vec{x}_0)(\vec{x}-\vec{x}_0)+Dg(\vec{x}_0)(\vec{x}-\vec{x}_0)}{g(\vec{x}_0)}\right]\left[\frac{1}{g(\vec{x}_0)}-\frac{1}{g(\vec{x})}\right]$$

$$=\frac{1}{g(\vec{x}_0)}\left[g(\vec{x})-g(\vec{x}_0)-Dg(\vec{x}_0)(\vec{x}-\vec{x}_0)\right]\left[\frac{1}{g(\vec{x}_0)}-\frac{1}{g(\vec{x})}\right]\qquad [3]$$

$$+\frac{1}{g(\vec{x}_0)}\left[Dg(\vec{x}_0)(\vec{x}-\vec{x}_0)\right]\left[\frac{1}{g(x_0)}-\frac{1}{g(\vec{x})}\right]\qquad [4]$$

For [3], $\displaystyle\lim_{\vec{x}\to\vec{x}_0}\frac{1}{g(\vec{x}_0)}\frac{[g(\vec{x})-g(\vec{x}_0)-Dg(\vec{x}_0)(\vec{x}-\vec{x}_0)]}{\|\vec{x}-\vec{x}_0\|}\left[\frac{1}{g(\vec{x}_0)}-\frac{1}{g(\vec{x})}\right]$

$g(\vec{x})$ continuous at $\vec{x}_0$

$$=\frac{1}{g(\vec{x}_0)}\cdot 0\cdot 0=0$$

$g(\vec{x})$ differentiable at $\vec{x}_0$

For [4], using $\|T\vec{x}\| \leq k\|\vec{x}\|$, where T is a matrix and k is a real number specific to T,

Let K be s.t. $\|Dg(\vec{x}_0)(\vec{x}-\vec{x}_0)\| \leq K\|x-x_0\|$

Let $\epsilon > 0$. Since $\lim_{\vec{x}\to\vec{x}_0}\left[\frac{1}{g(\vec{x})} - \frac{1}{g(\vec{x}_0)}\right] = 0$ as

g is continuous at x_0, $\exists\, \delta > 0$ s.t.

if $0 < \|\vec{x}-\vec{x}_0\| < \delta$, then $\left|\frac{1}{g(\vec{x})} - \frac{1}{g(\vec{x}_0)}\right| < \frac{\epsilon}{K}\cdot|g(\vec{x}_0)|$

$$\therefore \left|\frac{1}{g(\vec{x}_0)}\right| \|Dg(x_0)(x-x_0)\| \left|\frac{1}{g(\vec{x})} - \frac{1}{g(\vec{x}_0)}\right|$$

$$\leq \left|\frac{1}{g(\vec{x}_0)}\right| K\|\vec{x}-\vec{x}_0\| \frac{\epsilon}{K}|g(\vec{x}_0)| = \epsilon\|\vec{x}-\vec{x}_0\|$$

$\therefore$ Given the above $\epsilon > 0$, $\exists\, \delta > 0$ s.t. if $0 < \|\vec{x}-\vec{x}_0\| < \delta$, then

$$\frac{\left|\frac{1}{g(\vec{x}_0)}\right| \|Dg(\vec{x}_0)(\vec{x}-\vec{x}_0)\| \left|\frac{1}{g(\vec{x})} - \frac{1}{g(\vec{x}_0)}\right|}{\|\vec{x}-\vec{x}_0\|} < \epsilon$$

$$\therefore \lim_{\vec{x}\to\vec{x}_0} \frac{1}{g(\vec{x}_0)} \frac{Dg(\vec{x}_0)(\vec{x}-\vec{x}_0)}{\|\vec{x}-\vec{x}_0\|}\left[\frac{1}{g(\vec{x})} - \frac{1}{g(\vec{x}_0)}\right] = 0$$

$\therefore$ [3] & [4] $\Rightarrow$

$$\lim_{\vec{x}\to\vec{x}_0} \frac{\left(\frac{g(\vec{x}) - g(\vec{x}_0)}{g(\vec{x}_0)}\right)\left[\frac{1}{g(\vec{x}_0)} - \frac{1}{g(\vec{x})}\right]}{\|\vec{x}-\vec{x}_0\|} = 0$$

which is [2].

$\therefore$ [1] & {2} $\Rightarrow$

$$\lim_{\vec{x} \to \vec{x}_0} \frac{\frac{1}{g(\vec{x})} - \frac{1}{g(\vec{x}_0)} - \left[\left(-\frac{1}{g(\vec{x}_0)^2}\right) Dg(\vec{x}_0)\right](\vec{x} - \vec{x}_0)}{\|\vec{x} - \vec{x}_0\|} = 0$$

which is [0]

$$\therefore D\left(\frac{1}{g}\right)(x_0) = -\frac{1}{g(\vec{x}_0)^2} Dg(\vec{x}_0)$$

28.

should be,

$\in [h_i(\vec{x}) - h_i(\vec{x}_0) - Dh_i(\vec{x}_0)(\vec{x}-\vec{x}_0)]^2$

Note: $|a_i| \le \sqrt{a_1^2 + \ldots + a_m^2} \le |a_1| + \ldots + |a_m|$, $i = 1, 2, \ldots, m$

This is easily proved by squaring all sides.

(1) Assume $h: \mathbb{R}^n \to \mathbb{R}^m$ is differentiable, and let $\vec{x}_0 \in \mathbb{R}^n$

$\therefore$ Given $\epsilon > 0$, $\exists \delta$ s.t. if $0 < \|\vec{x} - \vec{x}_0\| < \delta$, then

$$\frac{\|h(\vec{x}) - h(\vec{x}_0) - Dh(\vec{x}_0)(\vec{x} - \vec{x}_0)\|}{\|\vec{x} - \vec{x}_0\|} < \epsilon$$

Note: $h(\vec{x})-h(\vec{x}_0)-Dh(\vec{x}_0)(\vec{x}-\vec{x}_0) =$

$$\begin{bmatrix} h_1(\vec{x}) \\ \vdots \\ h_m(\vec{x}) \end{bmatrix} - \begin{bmatrix} h_1(\vec{x}_0) \\ \vdots \\ h_m(\vec{x}_0) \end{bmatrix} - \begin{bmatrix} \frac{\partial h_1}{\partial x_1}(\vec{x}_0) & \cdots & \frac{\partial h_1}{\partial x_n}(\vec{x}_0) \\ \vdots & & \vdots \\ \frac{\partial h_m(\vec{x}_0)}{\partial x_1} & & \frac{\partial h_m(\vec{x}_0)}{\partial x_n} \end{bmatrix} \begin{bmatrix} (\vec{x}-\vec{x}_0)_1 \\ \vdots \\ (\vec{x}-\vec{x}_0)_n \end{bmatrix}$$

Let a_i = value of the i^{th} row:

dot product

$$a_i = h_i(\vec{x}) - h_i(\vec{x}_0) - \left[\frac{\partial h_i}{\partial x_1}(\vec{x}_0), \ldots, \frac{\partial h_i}{\partial x_n}(\vec{x}_0)\right] \cdot \left[(\vec{x}-\vec{x}_0)_1, \ldots, (\vec{x}-\vec{x}_0)_n\right]$$

$\therefore \sqrt{a_1^2 + \ldots + a_m^2} = \|h(\vec{x})-h(\vec{x}_0)-Dh(\vec{x}_0)(\vec{x}-\vec{x}_0)\|$

and $|a_i| = |h_i(\vec{x})-h_i(\vec{x}_0)-Dh_i(x_0)(\vec{x}-\vec{x}_0)|$

Since $|a_i| \le \sqrt{a_1^2+\ldots+a_m^2}$, then if $0<\|\vec{x}-\vec{x}_0\|<\delta$,

$$\frac{|h_i(\vec{x})-h_i(\vec{x}_0)-Dh_i(\vec{x}_0)(\vec{x}-\vec{x}_0)|}{\|\vec{x}-\vec{x}_0\|} \le \frac{\|h(\vec{x})-h(\vec{x}_0)-Dh(\vec{x}_0)(\vec{x}-\vec{x}_0)\|}{\|\vec{x}-\vec{x}_0\|} < \epsilon$$

$$\therefore \lim_{\vec{x}\to\vec{x}_0} \frac{|h_i(\vec{x})-h_i(\vec{x}_0)-Dh_i(x_0)(\vec{x}-\vec{x}_0)|}{\|\vec{x}-\vec{x}_0\|} = 0$$

so $h_i: R^n \to R^m$ is differentiable wherever h is.

(2) Assume for $i=1,\ldots,m$, $h_i: R^n \to R^m$ are differentiable. Let $\vec{x}_0 \in R^n$.

Let $\epsilon>0$. Then $\exists\, \delta>0$ s.t. if $0<\|\vec{x}-\vec{x}_0\|<\delta_i$, then

$$\frac{|h_i(\vec{x}) - h_i(\vec{x}_0) - Dh_i(\vec{x}_0)(\vec{x}-\vec{x}_0)|}{\|\vec{x}-\vec{x}_0\|} < \frac{\epsilon}{m}$$

Let $a_i = h_i(\vec{x}) - h_i(\vec{x}_0) - Dh_i(\vec{x}_0)(\vec{x}-\vec{x}_0)$, $i = 1, \ldots, m$

The above says $\dfrac{|a_i|}{\|\vec{x}-\vec{x}_0\|} < \epsilon$

But $\sqrt{a_1^2 + \ldots + a_m^2} = \|h(\vec{x}) - h(\vec{x}_0) - Dh(\vec{x}_0)(\vec{x}-\vec{x}_0)\|$

and $\sqrt{a_1^2 + \ldots + a_m^2} \le |a_1| + \ldots + |a_m|$

$\therefore$ Let $\delta = \min\{\delta_1, \ldots, \delta_m\}$. $\therefore$ if $0 < \|\vec{x}-\vec{x}_0\| < \delta$, then

$$\therefore \frac{\|h(\vec{x}) - h(\vec{x}_0) - Dh(\vec{x}_0)(\vec{x}-\vec{x}_0)\|}{\|\vec{x}-\vec{x}_0\|} \le \frac{|a_1|}{\|\vec{x}-\vec{x}_0\|} + \ldots + \frac{|a_m|}{\|\vec{x}-\vec{x}_0\|} < \frac{\epsilon}{m} + \ldots + \frac{\epsilon}{m} = \epsilon$$

$$\therefore \lim_{\vec{x}\to\vec{x}_0} \frac{\|h(\vec{x}) - h(\vec{x}_0) - Dh(\vec{x}_0)(\vec{x}-\vec{x}_0)\|}{\|\vec{x}-\vec{x}_0\|} = 0$$

$\therefore h: \mathbb{R}^n \to \mathbb{R}^m$ is differentiable wherever $h_i: \mathbb{R}^n \to \mathbb{R}^m$ are.

29.

To understand $\frac{d}{dx}\int_a^b f(x,y)\,dy = \int_a^b \frac{\partial f}{\partial x}(x,y)\,dy$,

look at Riemann sum:

$$\frac{d}{dx}\sum_{i=1}^{n} f(x,\xi_i)\,\Delta x_i = \sum_{i=1}^{n}\frac{\partial}{\partial x}\left[f(x,\xi_i)\,\Delta x_i\right]$$

where Δx_i goes from a to b. Since Δx_i only depend on a and b, not x, then $\frac{\partial}{\partial x}\left[f(x,\xi_i)\,\Delta x_i\right]$
$= \left[\frac{\partial f(x,\xi_i)}{\partial x}\right]\Delta x_i$

The limit, as the norm of the partition goes to 0, exists if $f(x,y)$ is sufficiently "nice".

Apparently, in this problem, $f(x,y)$ is sufficiently "nice."

For this problem, consider splitting up the two references to x. i.e.,

Let

$$F(a,x) = \int_0^a f(x,y)\,dy \qquad [1]$$

$$\therefore \text{ If } a=x,\quad F(x,x) = \int_0^x f(x,y)\,dy$$

By the Fundamental Theorem of Calculus,

$$\text{if } F(x) = \int_0^x f(t)\,dt, \text{ then } F'(x) = f(x).$$

$\therefore$ Holding x constant in [1], by the Fundamental Theorem,

$$\frac{\partial F}{\partial a}(a, x) = \frac{d}{da}\int_0^a f(x,y)\,dy = f(x,a) \quad [2]$$

Differentiating under the integral,

$$\frac{\partial F}{\partial x}(a,x) = \frac{d}{dx}\int_0^a f(x,y)\,dy = \int_0^a \frac{\partial}{\partial x} f(x,y)\,dy \quad [3]$$

Now let $a(x) = x$. By chain rule,

$$\therefore \frac{d}{dx} F(a(x), x) = F_a \cdot \frac{da}{dx} + F_x \quad . \text{ But } \frac{da}{dx} = 1$$

$$\therefore \frac{d}{dx}\int_0^x f(x,y)\,dy = \frac{d}{dx} F(a(x),x) = F_a + F_x$$

using [2], [3]

$$= f(x, a(x)) + \int_0^{a(x)} \frac{\partial}{\partial x} f(x,y)\,dy$$

$$= f(x, x) + \int_0^x \frac{\partial}{\partial x} f(x,y)\,dy$$

30.

(a) For $x \neq 0$, and all integers $p > 0$, $f(x)$ is differentiable.

For $x = 0$, look at $\lim_{x \to 0} \frac{x^p \sin(\frac{1}{x}) - 0}{x - 0} = \lim_{x \to 0} \frac{x^p \sin(\frac{1}{x})}{x}$

Let $\epsilon > 0$. Since $|\sin \frac{1}{x}| \leq 1$ for all $x \neq 0$,

Then $|x^p \sin \frac{1}{x}| \leq |x^p|$ for all $x \neq 0$.

$\therefore \left| \frac{x^p \sin \frac{1}{x}}{x} \right| \leq |x^{p-1}| = |x|^{p-1}$, all $x \neq 0$.

$|x|^{p-1} < \epsilon \iff |x| < \sqrt[p-1]{\epsilon}$, $x \neq 0$, $p - 1 > 0$.

$\therefore$ For all $p \geq 2$, choose $\delta = \sqrt[p-1]{\epsilon}$.

Then if $0 < |x| < \delta$, $\left| \frac{x^p \sin \frac{1}{x}}{x} \right| < \epsilon$

$\therefore$ $f(x)$ differentiable $\forall x$ for all integers $\underline{p \geq 2}$

(b) Look at $f'(x) = p x^{p-1} \sin(\frac{1}{x}) + x^p \cos(\frac{1}{x})(-\frac{1}{x^2})$

$= p x^{p-1} \sin(\frac{1}{x}) - x^{p-2} \cos(\frac{1}{x})$

From (a) $x^{p-1}\sin(\frac{1}{x})$ is differentiable if $p-1 \geq 2$, or $p \geq 3$

Similar reasoning to (a) shows $x^{p-2}\cos(\frac{1}{x})$ is differentiable if $p-2 \geq 2$, or $p \geq 4$

$\therefore$ Derivative continuous for $\underline{\underline{p \geq 4}}$.

31.

Consider $g: R^n \to R^m$. If $\vec{x} \in R^n$, then

$g(\vec{x}) = (g_1(\vec{x}), \dots, g_m(\vec{x}))$, where $g_i : R^n \to R$, $i=1,\dots,m$

$\therefore$ For $f: R^n \to R$,

$h(\vec{x}) = f(\vec{x})g(\vec{x}) = (f(\vec{x})g_1(\vec{x}), \dots, f(\vec{x})g_m(\vec{x}))$

As shown in problem #28, $h(\vec{x})$ is differentiable $\Leftrightarrow$ each $f(\vec{x})g_i(\vec{x})$ is differentiable.

But by Theorem 10 (iii), and shown in problem #27, each $f(\vec{x})g_i(\vec{x})$ is differentiable since $f(\vec{x})$ is differentiable and $g(\vec{x})$ is differentiable (and $\therefore$ so is each $g_i(\vec{x})$).

$\therefore h(\vec{x})$ is differentiable

If $h_i(\vec{x}) = f(\vec{x})g_i(\vec{x})$, by the product rule,

$$\underset{[1\times n]}{Dh_i(\vec{x})} = \underset{[1\times 1]}{f(\vec{x})}\,\underset{[1\times n]}{Dg_i(\vec{x})} + \underset{[1\times 1]}{g_i(\vec{x})}\,\underset{[1\times n]}{Df(\vec{x})}$$

Where the matrix sizes are shown.

$\therefore Dh(\vec{x}_0)$

$$f(\vec{x}_0)\overset{m\times n}{\begin{bmatrix} g_{1_{x_1}}(\vec{x}_0) & \cdots & g_{1_{x_n}}(\vec{x}_0) \\ \vdots & & \\ g_{m_{x_1}}(\vec{x}_0) & \cdots & g_{m_{x_n}}(\vec{x}_0) \end{bmatrix}} + \overset{m\times n}{\begin{bmatrix} g_1(\vec{x}_0)f_{x_1}(\vec{x}_0) & \cdots & g_1(\vec{x}_0)f_{x_n}(\vec{x}_0) \\ \vdots & & \\ g_m(\vec{x}_0)f_{x_1}(\vec{x}_0) & \cdots & g_m(\vec{x}_0)f_{x_n}(\vec{x}_0) \end{bmatrix}}$$

$\therefore$ for $\vec{y} \in R^n$, $Dh(\vec{x}_0)$, an $m\times n$ matrix, maps $\vec{y}$, an $n\times 1$ matrix, as follows:

$$f(\vec{x}_0)\begin{bmatrix} g_{1_{x_1}}(\vec{x}_0) & \cdots & g_{1_{x_n}}(\vec{x}_0) \\ \vdots & & \\ g_{m_{x_1}}(\vec{x}_0) & \cdots & g_{m_{x_n}}(\vec{x}_0) \end{bmatrix}\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} + \begin{bmatrix} g_1(\vec{x}_0)f_{x_1}(\vec{x}_0) & \cdots & g_1(\vec{x}_0)f_{x_n}(\vec{x}_0) \\ \vdots & & \\ g_m(\vec{x}_0)f_{x_1}(\vec{x}_0) & \cdots & g_m(\vec{x}_0)f_{x_n}(\vec{x}_0) \end{bmatrix}\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

$$= \overset{[1\times1]}{f(\vec{x}_0)}\,\overbrace{[Dg(\vec{x}_0)]\,\vec{y}}^{m\times1} + \overbrace{\begin{bmatrix} g_1(\vec{x}_0)f_{x_1}(\vec{x}_0)y_1+\dots+g_1(\vec{x}_0)f_{x_n}(\vec{x}_0)y_n \\ \vdots \\ g_m(\vec{x}_0)f_{x_1}(\vec{x}_0)y_1+\dots+g_m(\vec{x}_0)f_{x_n}(\vec{x}_0)y_n \end{bmatrix}}^{m\times1}$$

$$= f(\vec{x}_0)\left[Dg(\vec{x}_0)\,\vec{y}\right] + \overbrace{\left(f_{x_1}(\vec{x}_0)\,y_1+\dots+f_{x_n}(\vec{x}_0)\,y_n\right)}^{[1\times1],\text{ a dot product}} \overset{m\times1}{\begin{bmatrix} g_1(\vec{x}_0) \\ \vdots \\ g_m(\vec{x}_0) \end{bmatrix}}$$

$$= \overset{[1\times1]}{f(\vec{x}_0)}\,\overset{[m\times n]\,[n\times1]}{\left[Dg(\vec{x}_0)\,\vec{y}\right]} + \overset{[1\times n]\cdot[n\times1]}{\left[Df(\vec{x}_0)\cdot\vec{y}\right]}\,\overset{[m\times1]}{\vec{g}(\vec{x}_0)}$$

32.

First, $f(0,1,0) = (0,0)$

$\therefore D(g\circ f)(0,1,0) = Dg(0,0)\cdot Df(0,1,0)$

$$Dg = \begin{bmatrix} g_{1u} & g_{1v} \\ g_{2u} & g_{2v} \end{bmatrix} = \begin{bmatrix} e^u & 0 \\ 1 & \cos v \end{bmatrix} \quad \therefore Dg(0,0) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$Df = \begin{bmatrix} f_{1x} & f_{1y} & f_{1z} \\ f_{2x} & f_{2y} & f_{2z} \end{bmatrix} = \begin{bmatrix} y & x & 0 \\ 0 & z & y \end{bmatrix} \quad \therefore Df(0,1,0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore D(g\circ f)(0,1,0) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

33.

$$\frac{d}{dt}(f \circ c)(\pi) = Df(c(\pi)) \cdot c'(\pi)$$

$$= Df(1, 1, \pi, e^6) \cdot (19, 11, 0, 1)$$

$$= (0, 1, 3, -7) \cdot (19, 11, 0, 1)$$

$$= 0 + 11 + 0 - 7 = \underline{4}$$

34.

(a) $g(\vec{x}) \in R^q$ must be in domain of $f \subset R^n$

$\therefore \underline{q = n}$ p and m can be any positive integer.

(b) $f(\vec{x}) \in R^m$ must be in domain of $g \subset R^p$.

$\therefore \underline{m = p}$ n and q can be any positive integer.

(c) When range is a subset of domain.

$\therefore \underline{m = n}$

35.

Let $z(x,y) = f(x-y)$, so $z : \mathbb{R}^2 \to \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$

Let $g : \mathbb{R}^2 \to \mathbb{R}$ be $g(x,y) = x - y$

$\therefore z = (f \circ g)(x,y)$

$$\therefore Dz = [z_x \quad z_y] = f'(g(x,y))\,[g_x \quad g_y]$$
$$= f'(g(x,y))\,[1 \quad -1]$$
$$= [f'(g(x,y)) \quad -f'(g(x,y))]$$

$$\therefore \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = f'(g(x,y)) - f'(g(x,y)) = 0$$

36.

$w(x,y,z) = x^2 + y^2 + z^2$ Let $f(u,v) = (uv, u\cos v, u\sin v)$

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial u}$$

$$= 2x(v) + 2y(\cos v) + 2z(\sin v)$$

At $(u,v)=(1,0)$, $x=0$, $y=1$, $z=0$

$$\therefore \frac{\partial w}{\partial u} = 2x(0) + 2y(1) + 2z(0) = 2y = \underline{2}$$

Or, $w = (uv)^2 + (u\cos v)^2 + (u\sin v)^2$

$$= u^2v^2 + u^2\cos^2 v + u^2\sin^2 v$$

$$\therefore w_u = 2uv^2 + 2u\cos^2 v + 2u\sin^2 v$$

$$\therefore w_u(1,0) = 0 + 2(1)(1) + 0 = \underline{2}$$

2.6 Gradients and Directional Derivatives

Note Title 2/29/2016

1.

$\nabla f = (z^2, 3y^2, x)$. $\therefore \nabla f(1,1,2) = (4,3,1)$

$(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0)$ is a unit vector.

$\therefore \nabla f(1,1,2) \cdot (\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0) = (4,3,1) \cdot (\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0) =$

$\frac{4}{\sqrt{5}} + \frac{6}{\sqrt{5}} + 0 = \frac{10}{\sqrt{5}} = \underline{2\sqrt{5}}$

2.

(a) $\nabla f = (1+2y, 2x-6y)$. $\nabla f(1,2) = (5, -10)$

$\therefore \nabla f(1,2) \cdot (\frac{3}{5}, \frac{4}{5}) = \frac{15}{5} - \frac{40}{5} = \frac{-25}{5} = \underline{-5}$

(b) $f_x = \frac{1}{\sqrt{x^2+y^2}} \cdot \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}} \cdot 2x = \frac{x}{x^2+y^2}$. $f_y = \frac{y}{x^2+y^2}$

$\therefore \nabla f(1,0) = (1, 0)$. $\therefore \nabla f(1,0) \cdot \vec{v} = \underline{\frac{2}{\sqrt{5}}}$

(c) $\nabla f = (e^x\cos(\pi y), -\pi e^x\sin(\pi y))$

$\therefore \nabla f(0,-1) = (-1, 0)$. $\therefore \nabla f(0,-1)\cdot\left(-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right) =$

$\frac{1}{\sqrt{5}} + 0 = \underline{\frac{1}{\sqrt{5}}}$

(d) $\nabla f = (y^2 + 3x^2y, 2xy + x^3)$ $\therefore \nabla f(4,-2) = (-92, 48)$

$\therefore \nabla f(4,-2)\cdot\left(\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right) = -\frac{92}{\sqrt{10}} + \frac{144}{\sqrt{10}} = \underline{\frac{52}{\sqrt{10}}}$

3.

(a) $f_x = yx^{y-1}$ $f_y = x^y\ln x$, assuming $x > 0$.

$\therefore \nabla f(e,e) = (ee^{e-1}, e^e\ln e) = (e^e, e^e)$

$\frac{\vec{d}}{\|\vec{d}\|} = \left(\frac{5}{13}, \frac{12}{13}\right)$. $\therefore \nabla f(e,e)\cdot\left(\frac{5}{13}, \frac{12}{13}\right) = \underline{\frac{17}{13}e^e}$

(b) $\nabla f = (e^x, z, y)$. $\therefore \nabla f(1,1,1) = (e, 1, 1)$

$\frac{\vec{d}}{\|\vec{d}\|} = \frac{1}{\sqrt{3}}(1,-1,1)$. $\therefore \nabla f(1,1,1)\cdot\frac{1}{\sqrt{3}}(1,-1,1) = \underline{\frac{e}{\sqrt{3}}}$

(c) $\nabla f = (yz, xz, xy)$. $\therefore \nabla f(1,0,1) = (0,1,0)$

$\frac{\vec{d}}{\|\vec{d}\|} = \frac{1}{\sqrt{2}}(1,0,-1)$ $\therefore \nabla f(1,0,1)\cdot\frac{1}{\sqrt{2}}(1,0,-1) = \underline{0}$

4.

$f_x = -\pi y \sin(\pi x) - \cos(\pi y)$ $f_x(2,1) = 1$
$f_y = \cos(\pi x) + \pi x \sin(\pi y)$ $f_y(2,1) = 1$

$\therefore \nabla f(2,1) = (1,1).$

To stay at same level, go perpendicular to ∇f (i.e., in between steepest ascent & descent).

$\therefore \pm(1,-1)$, since $(1,-1)\cdot(1,1) = 0$

or $\underline{\hat{i} - \hat{j}}$ or $\underline{-\hat{i} + \hat{j}}$ $= \pm(1,-1)$

5.

(a) $\nabla f(\vec{x}_0)\cdot\vec{v} = \|\nabla f(\vec{x}_0)\|\,\|\vec{v}\|\cos\theta$

But $\|\vec{v}\| = 1$, and $\cos\theta$ is max at $\cos\theta = 1$.

$\therefore$ Max value is $\|\nabla f(\vec{x}_0)\|$

(b) $f_x = 3x^2$, $f_y = -3y^2$, $f_z = 3z^2$.

$\nabla f(1,2,3) = (3,-12,27)$

$\therefore \|\nabla f(1,2,3)\| = \sqrt{3^2+12^2+27^2} = \sqrt{882} = \underline{\underline{21\sqrt{2}}}$

6.

$f_x = 3x^2+y, \quad f_y = x+3y^2. \quad \therefore \nabla f(1,2) = \underline{(5,13)}$

7.

Let $g(x,y,z) = yx^2 + xy^2 + yz^2$

$g_x = 2yx + y^2 \quad g_y = x^2 + 2xy + z^2 \quad g_z = 2yz$

$\therefore \nabla g(1,1,1) = (3,4,2) =$ normal to surface at $(1,1,1)$.

$\therefore$ unit normal $= \dfrac{(3,4,2)}{\sqrt{3^2+4^2+2^2}} = \dfrac{1}{\sqrt{29}}(3,3,2) = \vec{n}$

For $f(x,y,z)$, $f_x = yz$, $f_y = xz$, $f_z = xy$.

$\therefore \nabla f(1,1,1)\cdot\vec{n} = (1,1,1)\cdot\frac{1}{\sqrt{29}}(3,4,2) = \underline{\underline{\frac{9}{\sqrt{29}}}}$

8.

(a) $\nabla f = (2x+3z,\ 4y,\ 3x)$. $\therefore \nabla f(1,2,\frac{1}{3}) = (2+1, 8, 3) = (3,8,3)$

$\therefore (3,8,3)\cdot(x-1,\ y-2,\ z-\frac{1}{3}) = 0$, or

$3(x-1) + 8(y-2) + 3z - 1$, or

$\underline{3x + 8y + 3z - 20 = 0}$

(b) $\nabla f(x,y,z) = (-2x,\ 2y,\ 0)$. $\therefore \nabla f(1,2,8) = (-2, 4, 0)$

$\therefore (-2,4,0)\cdot(x-1,\ y-2,\ z-8) = 0$, or

$-2x + 2 + 4y - 8 = 0$, or $\underline{2x - 4y + 6 = 0}$

(c) $\nabla f = (yz,\ xz,\ xy)$, $\therefore \nabla f(1,1,1) = (1,1,1)$

$\therefore (1,1,1)\cdot(x-1,\ y-1,\ z-1) = 0$, $\therefore \underline{x + y + z - 3 = 0}$

9.

(a) Let $g(x,y,z) = x^3 + y^3 - 6xy - z = 0$ $\quad \nabla g = (3x^2 - 6y,\ 3y^2 - 6x,\ -1)$

$\therefore \nabla g(1,2,-3) = (3-12, 12-6, -1) = (-9, 6, -1).$

$\therefore (-9, 6, -1)\cdot(x-1, y-2, z+3) = 0$, or

$-9x + 9 + 6y - 12 - z - 3 = 0$, or

$\underline{9x - 6y + z + 6 = 0}$

(b) $g(x,y,z) = (\cos x)(\cos y) - z = 0$, $\nabla g = (-\sin x \cos y, -\cos x \sin y, -1)$

$\therefore \nabla g(0, \frac{\pi}{2}, 0) = (0, -1, -1)$

$\therefore (0,-1,-1)\cdot(x-0, y-\frac{\pi}{2}, z-0) = 0$, or

$-y + \frac{\pi}{2} - z = 0$, or $\underline{y + z - \frac{\pi}{2} = 0}$

(c) $g(x,y,z) = \cos x \sin y - z$, $\nabla g = (-\sin x \sin y, \cos x \cos y, -1)$

$\therefore \nabla g(0, \frac{\pi}{2}, 1) = (0, 0, -1).$

$\therefore (0,0,-1)\cdot(x-0, y-\frac{\pi}{2}, z-1) = 0$, or $\underline{z - 1 = 0}$

10.

(a) $\nabla f(x,y,z) = -\dfrac{1}{x^2+y^2+z^2}\cdot\dfrac{1}{\sqrt{x^2+y^2+z^2}}(x,y,z)$

(b) $\nabla f(x,y,z) = (y+z,\ x+z,\ y+x)$

(c) $\nabla f(x,y,z) = -\dfrac{2}{(x^2+y^2+z^2)^2}(x,y,z)$

11.

(a) $\nabla f(1,1,1) = -\dfrac{1}{3\sqrt{3}}(1,1,1)$

(b) $\nabla f(1,1,1) = (2,2,2)$

(c) $\nabla f(1,1,1) = -\frac{2}{9}(1,1,1)$

$\therefore$ For (a) & (c), direction is same as $(-1,-1,-1)$

For (b), direction is $(1,1,1)$

12.

$f_x = 3x^2y^3 \quad f_y = 3y^2x^3+1 \quad f_z = -1$

$\therefore \nabla f(0,0,2) = (0,1,-1)$. $\sqrt{1^2+(-1)^2} = \sqrt{2}$

$\therefore$ normal : $\underline{\frac{1}{\sqrt{2}}(0,1,-1)} = \underline{\frac{1}{\sqrt{2}}(\hat{j}-\hat{k})}$

13. **13.** Find a unit normal to the surface $\cos(xy) = e^z - 2$ at $(1, \pi, 0)$.

$f(x,y,z) = \cos(xy) - e^z + 2 = 0$

$f_x = -y\sin(xy) \quad f_y = -x\sin(xy) \quad f_z = -e^z$

$\therefore \nabla f(1,\pi,0) = (-\pi\sin(\pi), -\sin(\pi), -e^0) = (0,0,-1)$

$\therefore \underline{(0,0,-1)}$ or $\underline{(0,0,1)}$

14.

$f(x,y,z)$ is a sphere, so unit normal pointing at fastest rate of increase should be pointing directly away from origin (e.g., an expanding bubble). Also, normal to tangent plane should point away from origin.

$\nabla f(x,y,z) = (f_x, f_y, f_z) = (2x, 2y, 2z) = 2(x,y,z)$.

Unit normal: $\frac{1}{\sqrt{3}}(x,y,z)$, and

This points away from origin.

Alternatively, let $\vec{x_0} = (x_0, y_0, z_0)$ be a point on

the sphere, let $\vec{n}$ be a unit vector in some direction at $\vec{x}_0$. So the line through $\vec{x}_0$ in the $\vec{v}$ direction is: $\vec{x}_0 + t\vec{n}$, $\vec{v} = (n_1, n_2, n_3)$

$$\lim_{t \to 0} \frac{f(\vec{x}_0 + t\vec{n}) - f(\vec{x}_0)}{t} =$$

$$\lim_{t \to 0} \frac{(x_0 + tn_1)^2 + (y_0 + tn_2)^2 + (z_0 + tn_3)^2 - (x_0^2 + y_0^2 + z_0)^2}{t} =$$

$$\lim_{t \to 0} \frac{2x_0n_1t + 2y_0n_2t + 2z_0n_3t + t^2(n_1^2 + n_2^2 + n_3^2)}{t} =$$

$$\lim_{t \to 0} \left[2x_0n_1 + 2y_0n_2 + 2z_0n_3 + t(n_1^2 + n_2^2 + n_3^2)\right] =$$

$$2x_0n_1 + 2y_0n_2 + 2z_0n_3 = 2\vec{x}_0 \cdot \vec{n}$$

$$= \nabla f(\vec{x}_0) \cdot \vec{n}$$

$\vec{x}_0 = (x_0, y_0, z_0)$ which points away from origin.

$\nabla f(\vec{x}_0) \cdot \vec{n}$ is max when $\vec{n}$ is parallel to $\vec{x}_0$, i.e., away from origin.

15.

If $F(x,y,z) = f(x,y) - z$, then $\nabla F = (f_x, f_y, -1)$

$\therefore$ Tangent plane at $F(x_0, y_0, z_0) = f(x_0, y_0) - z_0$

is $\nabla F(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0$

or $[f_x(x_0, y_0), f_y(x_0, y_0), -1] \cdot (x - x_0, y - y_0, z - z_0) = 0$

or $f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0$,

or $\underline{z = z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)}$

which is the tangent plane definition on p. 110.

16.

Let $F(x, y, z) = f(x, y) - z$.

$$\therefore F_x = -\frac{1}{2}(1 - x^2 - y^2)^{-\frac{1}{2}} \cdot 2x = -\frac{x}{\sqrt{1 - x^2 - y^2}}$$

$$F_y = -\frac{y}{\sqrt{1 - x^2 - y^2}}$$

$$F_z = -1$$

$\therefore$ Normal to tangent plane at $[x_0, y_0, f(x_0, y_0)]$ is:

$$\left(\frac{-x_0}{\sqrt{1 - x_0^2 - y_0^2}}, \frac{-y_0}{\sqrt{1 - x_0^2 - y_0^2}}, -1 \right) = \vec{N}.$$

Multiply $\vec{N}$ by the constant $-\sqrt{1 - x_0^2 - y_0^2} = k$

and get $k\vec{N} = (x_0, y_0, \sqrt{1-x_0^2-y_0^2}) = (x_0, y_0, f(x_0, y_0))$

$\therefore \vec{N}$ is parallel to $(x_0, y_0, f(x_0, y_0))$.

$\therefore$ The tangent plane, perpendicular to $\vec{N}$, is perpendicular to $(x_0, y_0, f(x_0, y_0))$.

Geometrically, $z = -(1-x^2-y^2)^{\frac{1}{2}}$ is the bottom of a sphere $(x^2+y^2+z^2=1)$. The tangent plane to a point on a sphere is perpendicular to the radius vector from origin to the point.

17.

(a) $\nabla f = (z+y, z+x, x+y)$, $g'(t) = (e^t, -\sin t, \cos t)$

$$(f \circ \vec{g})'(1) = \nabla f(\vec{g}(1)) \cdot \vec{g}'(1)$$

$$= \nabla f(e, \cos 1, \sin 1) \cdot (e, -\sin 1, \cos 1)$$

$$= (\cos 1 + \sin 1, e + \sin 1, e + \cos 1) \cdot (e, -\sin 1, \cos 1)$$

$$= (e\cos 1 + e\sin 1) - e\sin 1 - \sin^2 1 + (e\cos 1 + \cos^2 1)$$

$$= \underline{\underline{2e\cos 1 + \cos^2 1 - \sin^2 1}}$$

(6) $\nabla f = (yze^{xyz}, xze^{xyz}, xye^{xyz})$, $\vec{g}'(t) = (6, 6t, 3t^2)$

$$(f\circ\vec{g})'(1) = \nabla f(\vec{g}(1)) \cdot \vec{g}'(1)$$

$$= \nabla f(6,3,1) \cdot (6,6,3)$$

$$= (3e^{18}, 6e^{18}, 18e^{18}) \cdot (6,6,3)$$

$$= e^{18}(18 + 36 + 54) = \underline{\underline{108e^{18}}}$$

(c) $f_x = 2x \log\sqrt{x^2+y^2+z^2} + (x^2+y^2+z^2) \cdot \frac{1}{\sqrt{x^2+y^2+z^2}} \cdot \frac{1}{2}(x^2+y^2+z^2)^{-\frac{1}{2}} \cdot 2x$

$$= x\log(x^2+y^2+z^2) + x$$

$$f_y = y\log(x^2+y^2+z^2) + y$$

$$f_z = z\log(x^2+y^2+z^2) + z$$

$g'(t) = (e^t, -e^{-t}, 1)$ $\qquad$ $g(1) = (e, e^{-1}, 1)$

$$\therefore (f\circ\vec{g})'(1) = \nabla f(\vec{g}(1)) \cdot \vec{g}'(1)$$

$$= [e\log(e^2+e^{-2}+1)+e, e^{-1}\log(e^2+e^{-2}+1)+e^{-1}, \log(e^2+e^{-2}+1)+1) \cdot (e, e^{-1}, 1)$$

$$= e^2\log(e^2+e^{-2}+1)+e^2 + e^{-2}\log(e^2+e^{-2}+1)+e^{-2} + \log(e^2+e^{-2}+1)+1$$

$$= (e^2+e^{-2}+1)\log(e^2+e^{-2}+1) + (e^2+e^{-2}+1)$$

$$= \left(e^{2}+e^{-2}+1\right)\left[\log\left(e^{2}+e^{-2}+1\right)+1\right]$$

18.

Note $\vec{v}$ is a unit vector for (a) + (b).

(a) $\nabla f = \left(y^2+z^3,\ 2xy+2yz^3,\ 3y^2z^2+3xz^2\right)$

$\therefore \nabla f(4,-2,-1) = (3,\ -16+4,\ 6+12) = (3,-12,18)$

$\therefore \nabla f(4,-2,-1)\cdot\vec{v} = (3,-12,18)\cdot(1,3,2)\frac{1}{\sqrt{14}}$

$$= \frac{3-36+36}{\sqrt{14}} = \frac{3}{\sqrt{14}}$$

(b) $\nabla f = \left(yz\,x^{yz-1},\ (z\log x)x^{yz},\ (y\log x)\,x^{yz}\right)$

$\therefore \nabla f(e,e,0) = (0,\ 0,\ e\cdot e^{0}) = (0,0,e)$

$\therefore \nabla f(e,e,0)\cdot\vec{v} = (0,0,e)\cdot(12,3,4)\frac{1}{13}$

$$= \frac{4}{13}e$$

19.

(a) x and y can only subtract from 100. $\therefore$ max value is when $x = y = 0$.
$\therefore (0, 0, 100)$.

(b) $\nabla f(x,y) = (-4x, -6y)$. $\therefore \nabla f(0,0) = (0,0)$

20.

The plane can be written as $(2,2,1)\cdot(x-0, y-0, z-5) = 0$.
$\therefore$ Normal to plane is $(2,2,1)$.

Tangent planes parallel to above plane will have same normal.

$\therefore \nabla f(x,y,z) = (2x, 8y, -2z) = \pm(2,2,1)$

$\therefore (1, \frac{1}{4}, -\frac{1}{2})$ gives $\nabla f = (2,2,1)$

$(-1, -\frac{1}{4}, \frac{1}{2})$ gives $\nabla f = -(2,2,1)$

$\therefore \underline{(1, \frac{1}{4}, -\frac{1}{2}) \text{ and } (-1, -\frac{1}{4}, \frac{1}{2})}$

21.

$$r = \|\vec{r}\| = \sqrt{x^2+y^2+z^2} \quad \therefore \frac{1}{r(x,y,z)} = (x^2+y^2+z^2)^{-\frac{1}{2}}$$

$$\frac{\partial\left(\frac{1}{r}\right)}{\partial x} = -\frac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}(2x) = -\frac{x}{(\sqrt{x^2+y^2+z})^3} = \frac{-x}{r^3}$$

$$\therefore \frac{\partial\left(\frac{1}{r}\right)}{\partial y} = \frac{-y}{r^3}, \quad \frac{\partial\left(\frac{1}{r}\right)}{\partial z} = \frac{-z}{r^3}$$

$$\therefore \nabla\left(\frac{1}{r}\right) = -\left(\frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3}\right) = -\frac{\vec{r}}{r^3}$$

22.

(a) $T_x = -2xe^{-x^2-2y^2-3z^2}$ $\quad T_y = -4ye^{-x^2-2y^2-3z^2}$

$T_z = -6ze^{-x^2-2y^2-3z^2}$ $\quad T(1,1,1) = e^{-1-2-3} = e^{-6}$

$$\therefore \nabla T(1,1,1) = e^{-6}(-2,-4,-6).$$

$\therefore -\nabla T(1,1,1) = e^{-6}(2,4,6) \equiv K(1,2,3) \equiv \frac{(1,2,3)}{\sqrt{14}}$

$\therefore$ proceed toward $\frac{1}{\sqrt{14}}(1,2,3)$

(b) The rate of change in the direction of (a) is

$-\|\nabla T(1,1,1)\| = -\|e^{-6}(2,4,6)\| = -e^{-6}\sqrt{2^2+4^2+6^2}$

$= -2e^{-6}\sqrt{14}$ degrees/m.

$\therefore$ If travelling at e^8 m/sec, then rate of change is $(e^8 \text{ m/sec})(-2e^{-6}\sqrt{14} \text{ degrees/m})$

$= -2\sqrt{14}\,e^2$ degrees/sec

(c) Look at $\nabla T(1,1,1)\cdot\vec{v}$, where $\|\vec{v}\| = e^8$

$\therefore \nabla T(1,1,1)\cdot\vec{v} = \|\nabla T(1,1,1)\|\,\|\vec{v}\|\cos\theta$

$= \left[e^{-6}\sqrt{(-2)^2+(-4)^2+(-6)^2}\right] e^8 \cos\theta$

$= e^2\sqrt{56}\cos\theta = 2e^2\sqrt{14}\cos\theta$

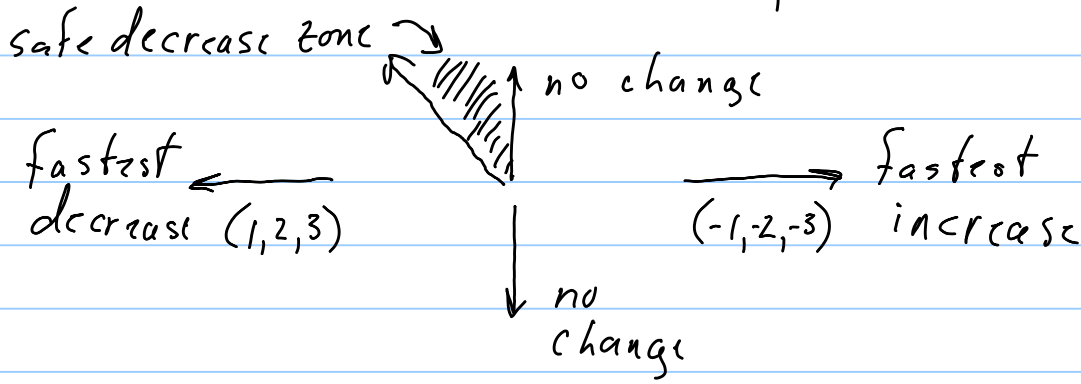
To produce a temp decrease, want $\cos\theta < 0$.

And want $|\text{rate}| \leq \sqrt{14}\,e^2$

$\therefore$ Want θ s.t. $\left|2e^2\sqrt{14}\cos\theta\right| \leq 14e^2$,

or $|\cos\theta| \leq \frac{1}{2}$, $\therefore -\frac{1}{2} \leq \cos\theta \leq \frac{1}{2}$,

so $-\frac{1}{2} \leq \cos\theta \leq 0$, $\therefore 120^\circ \leq \theta \leq 90^\circ$



i.e., between <u>90° to 120° from direction of gradient</u>

23.

$\nabla f(x,y) = (f_x, f_y)$. $\dfrac{\partial f(x,y)}{\partial x} = \dfrac{d}{dx} g(x) = g'(x)$

$\dfrac{\partial f(x,y)}{\partial y} = \dfrac{\partial}{\partial y} g(x) = 0$. $\therefore \underline{\nabla f(x,y) = (g'(x), 0)}$

24.

Let $\vec{c}(t)$ be a curve on a sphere centered at $\vec{0}$.

To show $f(\vec{c}(t)): \mathbb{R} \to \mathbb{R}$ is constant,

it suffices to show $\frac{d}{dt} f(\vec{c}(t))\Big|_{t=0} = 0$, where

$\vec{c}(0)$ is some point on a sphere, call it $\vec{c}(0) = \vec{x}_0$.

$$\frac{d}{dt} f(\vec{c}(t))\Big|_{t=0} = \nabla f(\vec{c}(0)) \cdot \vec{c}\,'(0). \text{ Let } \vec{v} = \vec{c}\,'(0)$$

Let $h(x,y,z) = x^2 + y^2 + z^2 = r^2$ be any sphere

centered at the origin. $\therefore \nabla h(x,y,z) = (2x, 2y, 2z)$.

$\therefore \nabla h(\vec{x}_0) = 2\vec{x}_0$. But $h(\vec{c}(t)) = r^2$,

so $\frac{d}{dt} h(\vec{c}(t)) = 0 = \nabla h(\vec{c}(t)) \cdot \vec{c}\,'(t)$. At $t=0$,

$\nabla h(\vec{x}_0) \cdot \vec{v} = 0$, so $(2\vec{x}_0) \cdot \vec{v} = 0$, so $\vec{x} \cdot \vec{v} = 0$.

$\therefore \nabla f(\vec{x}_0) \cdot \vec{v} = g(\vec{x}_0)\, \vec{x}_0 \cdot \vec{v} = 0$

$\therefore \frac{d}{dt} f(\vec{c}(t))\Big|_{t=0} = \nabla f(\vec{c}(0)) \cdot \vec{v} = 0$,

Since $\vec{c}(0)$ was an arbitrary point on a sphere,

f is constant on the arbitrary sphere

centered at the origin.

25.

$Df(x) = Df(-x)$

Let $g(\vec{x}) = -\vec{x}$. $\therefore D(f \circ g)(\vec{x}) = Df(g(\vec{x})) \cdot Dg(\vec{x})$

But $Dg(\vec{x}) = -I$, I the identity matrix.

$\therefore Df(g(\vec{x})) \cdot Dg(\vec{x}) = Df(-\vec{x}) \cdot (-I) = -Df(-\vec{x})$

$\therefore Df(\vec{x}) = Df(-\vec{x}) = -Df(-\vec{x})$

$\therefore$ For $Df(\vec{x}) = -Df(-\vec{x})$, let $\vec{x} = \text{origin} = \vec{0}$.

$\therefore Df(\vec{0}) = -Df(\vec{0})$, so $2Df(\vec{0}) = \vec{0}$,

$\therefore \underline{Df(\vec{0}) = \vec{0} = [0 \ldots 0]}$, a $1 \times n$ matrix.

26.

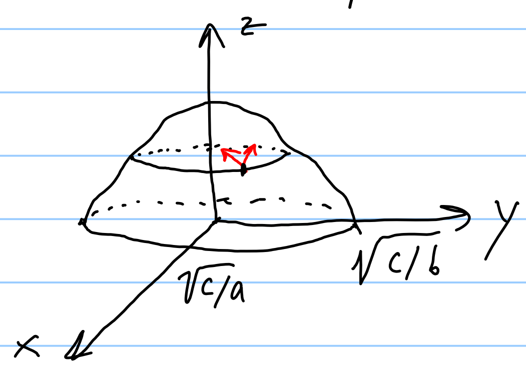
(a) $z = f(x,y) = c - ax^2 - by^2$. $\therefore \nabla f = (-2ax, -2by)$

$\therefore$ At $(x,y) = (1,1)$, $f(x,y)$ increases most in the direction of $(-2a,-2b)$ [i.e., toward origin].

(b) In the opposite direction of (a), so in the direction of $(2a,2b)$, or (a,b). This is the direction of fastest decrease.

Note also: consider tangent plane at $(1,1)$, as it should have no lateral forces at the point. For $F(x,y,z) = c - ax^2 - by^2 - z$, normal to tangent plane is $\pm(-2ax, -2by, -1)$. So at $(x,y) = (1,1)$, normal is $\pm(-2a,-2b,-1)$.
$\therefore$ It will move along (x,y) vector of $\pm(a,b)$.
From (a), $(-a,b)$ is uphill. $\therefore$ (a,b) is downhill with the only force of gravity on it.

27.



$\nabla f(x,y)\cdot\vec{v}$ = rate of change of $f(x,y)$ in direction of unit vector $\vec{v}$. Let r = grade = 0.03

Let $\vec{v} = (x,y)$ be the unit vector in xy-plane. $x^2+y^2=1$

$\nabla f(1,1) = (-2a,-2b)$. Assume $a>0$, $b>0$.

$\therefore \nabla f(1,1)\cdot(x,y) = -2ax - 2by = r$

Or, $ax + by = \frac{-r}{2}$ [1]

$\therefore ax + \frac{r}{2} = by$

$a^2x^2 + arx + \frac{r^2}{4} = b^2y^2 = b^2(1-x^2)$

$\therefore (a^2+b^2)x^2 + arx + \frac{r^2}{4} - b^2 = 0$ [2]

Checking the discriminant,

$a^2r^2 - 4(a^2+b^2)\left(\frac{r^2}{4} - b^2\right) > 0 \iff$

$a^2r^2 > (a^2+b^2)(r^2 - 4b^2) \iff$

$a^2r^2 > a^2r^2 - 4a^2b^2 + b^2r^2 - 4b^4 \iff$

$4a^2 + 4b^2 > r^2$, or $a^2+b^2 > \frac{r^2}{4}$

For $r = 0.03$ and any reasonable mountain (i.e., values for a, b), this will be true

$\therefore$ For [2] with $r = 0.03$,

$(a^2+b^2)x^2 + (0.03)ax + \frac{(0.03)^2}{4} - b^2 = 0$ [3]

has two real solutions, call them x_1, x_2.

From [1], $ax + by = -\frac{r}{2}$, or $y = \frac{-\frac{r}{2} - ax}{b}$, or

$$y = \frac{-(r + 2ax)}{2b}.$$

$$\therefore \underline{\left[x_1, \frac{-(0.03 + 2ax_1)}{2b}\right]} \text{ and } \underline{\left[x_2, \frac{-(0.03 + 2ax_2)}{2b}\right]} \quad [4]$$

where x_1, x_2 are solutions to [3]:

$$(a^2 + b^2)x^2 + (0.03)ax + \frac{(0.03)^2}{4} - b^2 = 0$$

As a concrete example, let $a = 40$, $b = 50$, $c = 100$, so

$z = 100 - 40x^2 - 50y^2$. $\therefore \nabla f(1,1) = (-80, -100)$

[3] becomes: $4100x^2 + 1.2x - 2500 = 0$,
$x = 0.781, -0.781$

From [4], $(x,y) = [0.781, -0.625]$ and $[-0.781, 0.625]$

28.

$$f(x,y,z) = \frac{-k}{\sqrt{x^2+y^2+z^2}} = -k(x^2+y^2+z^2)^{-\frac{1}{2}}$$

$$\therefore f_x = -\frac{1}{2}k(x^2+y^2+z^2)^{-\frac{3}{2}}(2x) = -kx(x^2+y^2+z^2)^{-3/2}$$

$$= -\frac{Kx}{(\sqrt{x^2+y^2+z^2})^3} = -\frac{Kx}{\|\vec{r}\|^3}$$

Similarly, $f_y = -\frac{Ky}{\|\vec{r}\|^3}$, $f_z = \frac{-Kz}{\|\vec{r}\|^3}$

$$\therefore \nabla f(x,y,z) = -\frac{k}{\|\vec{r}\|^3}(x,y,z) = -\frac{K\vec{r}}{\|\vec{r}\|^3} = \vec{F}$$

29.

$$V(x,y) = \frac{\lambda}{2\pi\epsilon_0}\cdot\frac{1}{2}\ln\left(\frac{r_2}{r_1}\right)^2 = \frac{\lambda}{4\pi\epsilon_0}\ln\frac{r_2^2}{r_1^2}$$

$$= \frac{\lambda}{4\pi\epsilon_0}\ln\left[\frac{(x+x_0)^2+y^2}{(x-x_0)^2+y^2}\right]$$

$$\therefore \frac{\partial V}{\partial x} = \frac{\lambda}{4\pi\epsilon_0}\cdot\left[\frac{(x-x_0)^2+y^2}{(x+x_0)^2+y^2}\right]\left[\frac{[(x-x_0)^2+y^2]\cdot[2(x+x_0)] - [(x+x_0)^2+y^2][2(x-x_0)]}{[(x-x_0)^2+y^2]^2}\right]$$

$$= \frac{\lambda}{2\pi\epsilon_0}\left[\frac{r_1^2}{r_2^2}\right]\left[\frac{r_1^2(x+x_0) - r_2^2(x-x_0)}{r_1^4}\right]$$

$$= \frac{\lambda}{2\pi\epsilon_0}\left[\frac{r_1^2(x+x_0) - r_2^2(x-x_0)}{r_1^2 r_2^2}\right] = \frac{\lambda}{2\pi\epsilon_0}\left[\frac{(x+x_0)}{r_2^2} - \frac{(x-x_0)}{r_1^2}\right]$$

$$\frac{\partial V}{\partial y} = \frac{\lambda}{4\pi\epsilon_0}\left[\frac{(x-x_0)^2+y^2}{(x+x_0)^2+y^2}\right]\left[\frac{[(x-x_0)^2+y^2][2y]-[(x+x_0)^2+y^2][2y]}{[(x-x_0)^2+y^2]^2}\right]$$

$$= \frac{\lambda}{2\pi\epsilon_0} y\left[\frac{r_1^2}{r_2^2}\right]\left[\frac{r_1^2-r_2^2}{r_1^4}\right] = \frac{\lambda y}{2\pi\epsilon_0}\left[\frac{r_1^2-r_2^2}{r_1^2 r_2^2}\right]$$

$$= \frac{\lambda}{2\pi\epsilon_0} y\left[\frac{1}{r_2^2}-\frac{1}{r_1^2}\right]$$

$$\therefore \nabla V(x,y) = \frac{\lambda}{2\pi\epsilon_0}\left[\frac{(x+x_0)}{r_2^2}-\frac{(x-x_0)}{r_1^2},\ y\left(\frac{1}{r_2^2}-\frac{1}{r_1^2}\right)\right]$$

30.

(a) $(f\circ c)(t) = (\cos t)(\sin t) = \frac{1}{2}\sin 2t$

$$\therefore (f\circ c)'(t) = \cos 2t,\quad 0 \leq t \leq 2\pi$$

$$(f\circ c)''(t) = -2\sin 2t$$

$$(f\circ c)'(t) = 0 \iff \cos 2t = 0,\quad 0 \leq t \leq 2\pi$$

$$\iff 2t = \frac{\pi}{2}, \frac{3}{2}\pi, \frac{5}{2}\pi, \ldots$$

$$\iff t = \frac{\pi}{4}, \frac{3}{4}\pi, \frac{5}{4}\pi, \frac{7}{4}\pi \quad (0 \leq t \leq 2\pi)$$

$$(f \circ c)''\left(\frac{\pi}{4}\right) = -2\sin\left(2\cdot\frac{\pi}{4}\right) = -2 < 0 \qquad \therefore \text{ a max}$$
$$(f \circ c)''\left(\tfrac{3}{4}\pi\right) = -2\sin(2\cdot 3\pi/4) = 2 > 0 \qquad \therefore \text{ a min}$$
$$(f \circ c)''(5\pi/4) = -2\sin(2\cdot 5\pi/4) = -2 < 0 \qquad \therefore \text{ a max}$$
$$(f \circ c)''(7\pi/4) = -2\sin(2\cdot 7\pi/4) = 2 > 0 \qquad \therefore \text{ a min}$$

$$\therefore (f \circ c)\left(\frac{\pi}{4}\right) = \cos\frac{\pi}{4}\sin\frac{\pi}{4} = \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{1}{2}$$

$$(f \circ c)\left(\frac{5\pi}{4}\right) = \cos\frac{5\pi}{4}\sin\frac{5\pi}{4} = \left(-\frac{\sqrt{2}}{2}\right)\left(-\frac{\sqrt{2}}{2}\right) = \frac{1}{2}$$

$$(f \circ c)\left(\frac{3\pi}{4}\right) = \cos\frac{3\pi}{4}\sin\frac{3\pi}{4} = \left(-\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = -\frac{1}{2}$$

$$(f \circ c)(7\pi/4) = \cos\frac{7\pi}{4}\sin\frac{7\pi}{4} = \left(\frac{\sqrt{2}}{2}\right)\left(-\frac{\sqrt{2}}{2}\right) = -\frac{1}{2}$$

$$\therefore (f \circ c)(t) = \frac{1}{2}, \text{ a max, at } t = \frac{\pi}{4}, \frac{5\pi}{4}$$

$$(f \circ c)(t) = -\frac{1}{2}, \text{ a min, at } t = \frac{3\pi}{4}, \frac{7\pi}{4}$$

(3)

$$(f \circ c)(t) = \cos^2 t + 4\sin^2 t, \quad 0 \le t \le 2\pi$$

Let $h(t) = (f \circ c)t$.

$$\therefore h'(t) = -2\cos t\sin t + 8\sin t\cos t$$
$$= 6\sin t\cos t$$
$$= 3\sin 2t$$

$$\therefore h''(t) = (3\cos 2t)(2) = 6\cos 2t$$

$$\therefore h'(t) = 3\sin 2t = 0 \iff 2t = 0, \pi, 2\pi, 3\pi, \ldots$$

$\Leftrightarrow t = 0, \frac{\pi}{2}, \pi, \frac{3}{2}\pi, 2\pi$

$$h''(0) = 6\cos(0) = 6 > 0 \quad \therefore \text{min}$$
$$h''(\tfrac{\pi}{2}) = 6\cos(2 \cdot \pi/2) = -6 < 0 \quad \therefore \text{max}$$
$$h''(\pi) = 6\cos(2\pi) = 6 > 0 \quad \therefore \text{min}$$
$$h''(3\pi/2) = 6\cos(2 \cdot 3\pi/2) = -6 < 0 \quad \therefore \text{max}$$
$$h''(2\pi) = 6\cos(2 \cdot 2\pi) = 6 > 0 \quad \therefore \text{min}$$

$\therefore (f \circ c)(t) = 1$, a min, at $t = 0, \pi, 2\pi$

$(f \circ c)(t) = 4$, a max, at $t = \frac{\pi}{2}, \frac{3\pi}{2}$

31. should read, "... normal to the surface directed toward the xy plane at..."

Find line from $(1,1,\sqrt{3})$ to xy plane, along path of normal to surface.

$\therefore$ Normal to surface: $\nabla f(x,y,z) = (2x, 2y, -2z)$

$\therefore \nabla f(1,1,\sqrt{3}) = (2, 2, -2\sqrt{3})$.

$\therefore$ line is $(1,1,\sqrt{3}) + s(2,2,-2\sqrt{3})$.

This intersects xy plane when z coord. $= 0$.

$\therefore \sqrt{3} - s\,2\sqrt{3} = 0, \quad s = \frac{1}{2}$

$\therefore$ intersects xy plane at $(1,1,\sqrt{3}) + \frac{1}{2}(2,2,-2\sqrt{3}) = (2,2,0)$

$\therefore$ Distance from $(1,1,\sqrt{3})$ to $(2,2,0)$ =

$$\sqrt{(2-1)^2+(2-1)^2+(0-\sqrt{3})^2} = \sqrt{5}$$

$\therefore$ Takes $\dfrac{\sqrt{5} \text{ units}}{10 \text{ units/sec}} = \dfrac{\sqrt{5}}{10}$ secs.

$\therefore$ $\dfrac{\sqrt{5}}{10}$ secs to reach $(2,2,0)$.

32.

$$Df(x,y,z) = [f_x \;\; f_y \;\; f_z]$$

Let $\vec{x} \in \mathbb{R}^3$, so $\vec{x}=(x,y,z)$ or $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$\therefore [Df(x,y,z)]\,\vec{x} = [f_x \;\; f_y \;\; f_z]\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= f_x(x,y,z)\cdot x + f_y(x,y,z)\cdot y + f_z(x,y,z)\cdot z$$

$$= \nabla f(x,y,z)\cdot \vec{x} \quad \text{(a dot product).}$$

$\therefore$ Kernal of Df is all $\vec{x} \in \mathbb{R}^3$ s.t. $\nabla f \cdot \vec{x} = 0$.

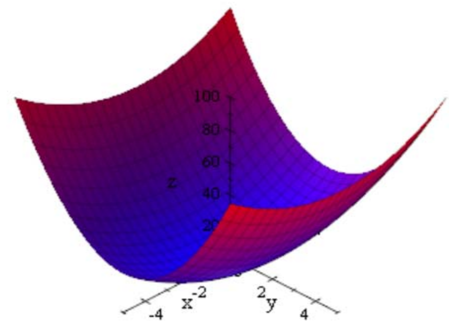
i.e., Kernal of Df is all $\vec{x} \in \mathbb{R}^3$ perpendicular to $\nabla f(x,y,z)$.

But this is just the plane in $\mathbb{R}^3$
orthogonal to $\nabla f(x,y,z)$.

Review Exercises for Chapter 2

Note Title 3/9/2016

1.

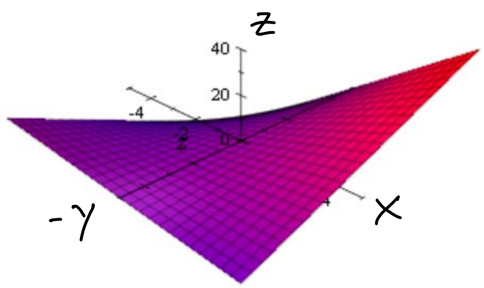


(a) for $y=0$, $z=3x^2$, a parabola
for $x=0$, $z=y^2$, a parabola

for level set $K=3x^2+y^2 \iff \frac{K}{3}=x^2+\frac{y^2}{3}$, an ellipse.

$\therefore$ Elliptical paraboloid.

(b) for $y=0$, $z=3x$
for $y=5$, $z=8x$
for $y=-5$, $z=-2x$



This is a "twisted" sheet,
That rises (from neg. x to pos. x) for large pos. y, and decreases for large negative y. The sheet is smooth.

3.

(a) $f: \mathbb{R}^2 \to \mathbb{R}^2$, so $Df(\vec{x})$ is 2×2 matrix

$$\frac{\partial\, x^2y}{\partial x}=2xy\,,\quad \frac{\partial\, x^2y}{\partial y}=x^2$$

$$\frac{\partial e^{-xy}}{\partial x} = -y e^{-xy} \qquad \frac{\partial e^{-xy}}{\partial y} = -x e^{-xy}$$

$$\therefore Df(\vec{x}) = \begin{bmatrix} 2xy & x^2 \\ -ye^{-xy} & -xe^{-xy} \end{bmatrix}$$

(b) $f: \mathbb{R} \to \mathbb{R}^2$, $\therefore$ a 2×1 matrix $\quad \frac{\partial x}{\partial x} = 1$

$$\therefore \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(c) $f: \mathbb{R}^3 \to \mathbb{R}^1$, $\therefore$ a 1×3 matrix

$$\frac{\partial f}{\partial x} = e^x \qquad \frac{\partial f}{\partial y} = e^y \qquad \frac{\partial f}{\partial z} = e^z$$

$$\therefore \begin{bmatrix} e^x & e^y & e^z \end{bmatrix}$$

(d) $f: \mathbb{R}^3 \to \mathbb{R}^3$, $\therefore$ a 3×3 matrix

$$\left.\begin{aligned} &\frac{\partial x}{\partial x} = 1 \quad \frac{\partial x}{\partial y} = 0 \quad \frac{\partial x}{\partial z} = 0 \\ &\frac{\partial y}{\partial x} = 0 \quad \frac{\partial y}{\partial y} = 1 \quad \frac{\partial y}{\partial z} = 0 \\ &\frac{\partial z}{\partial x} = 0 \quad \frac{\partial z}{\partial y} = 0 \quad \frac{\partial z}{\partial z} = 1 \end{aligned}\right\} \therefore \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

4.

$$\frac{\partial f}{\partial x}(a,b) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

$$= \lim_{h\to 0} \frac{f(b,a+h) - f(b,a)}{h} = \frac{\partial f}{\partial y}(b,a)$$

5.

$$D(f\circ g)(0,1,1) = Df(g(0,1,1))\cdot Dg(0,1,1)$$

Note $g(0,1,1) = (\pi, 0)$

$$Df(u,v) = \begin{bmatrix} -\sin u & 0 \\ \cos u & 1 \end{bmatrix} \therefore Df(\pi,0) = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}$$

$$Dg(x,y,z) = \begin{bmatrix} 2x & 2\pi y & 0 \\ z & 0 & x \end{bmatrix} \therefore Dg(0,1,1) = \begin{bmatrix} 0 & 2\pi & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\therefore Df(g(0,1,1))\cdot Dg(0,1,1) = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 2\pi & 0 \\ 1 & 0 & 0 \end{bmatrix} = \underline{\begin{bmatrix} 0 & 0 & 0 \\ 1 & -2\pi & 0 \end{bmatrix}}$$

6.

$$D(f\circ g)(-2,1) = Df(g(-2,1))\cdot Dg(-2,1)$$

$$g(-2,1) = (-2,3,-1)$$

$$Df(u,v,w) = \begin{bmatrix} w & 2v & u \\ 2u & 0 & 2w \\ 2uv & u^2 & -3w^2 \end{bmatrix} \quad \therefore Df(-2,3,-1) = \begin{bmatrix} -1 & 6 & -2 \\ -4 & 0 & -2 \\ -12 & 4 & -3 \end{bmatrix}$$

$$Dg(x,y) = \begin{bmatrix} y^3 & 3xy^2 \\ 2x & -2y \\ 3 & 5 \end{bmatrix} \quad \therefore Dg(-2,1) = \begin{bmatrix} 1 & -6 \\ -4 & -2 \\ 3 & 5 \end{bmatrix}$$

$$\therefore Df(g(-2,1)) \cdot Dg(-2,1) = \begin{bmatrix} -1 & 6 & -2 \\ -4 & 0 & -2 \\ -12 & 4 & -3 \end{bmatrix} \begin{bmatrix} 1 & -6 \\ -4 & -2 \\ 3 & 5 \end{bmatrix} = \underline{\underline{\begin{bmatrix} -31 & -16 \\ -10 & 14 \\ -37 & 49 \end{bmatrix}}}$$

7.

$$D(f \circ g)(-1,2) = Df(g(-1,2)) \cdot Dg(-1,2)$$

$$g(-1,2) = (1,-2,3)$$

$$Df(u,v,w) = \begin{bmatrix} 0 & 2v & 2w \\ 3u^2 & -w & -v \\ 2uv & u^2 & 1 \end{bmatrix} \quad \therefore Df(1,-2,3) = \begin{bmatrix} 0 & -4 & 6 \\ 3 & -3 & 2 \\ -4 & 1 & 1 \end{bmatrix}$$

$$Dg(x,y) = \begin{bmatrix} 3 & 2 \\ 3x^2y & x^3 \\ -2x & 2y \end{bmatrix} \quad \therefore Dg(-1,2) = \begin{bmatrix} 3 & 2 \\ 6 & -1 \\ 2 & 4 \end{bmatrix}$$

$$\therefore Df(g(-1,2)) \cdot Dg(-1,2) = \begin{bmatrix} 0 & -4 & 6 \\ 3 & -3 & 2 \\ -4 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 6 & -1 \\ 2 & 4 \end{bmatrix} = \underline{\begin{bmatrix} -12 & 28 \\ -5 & 17 \\ -4 & -5 \end{bmatrix}}$$

8.

$$D(f\circ g)(3,1,0) = Df(g(3,1,0))\cdot Dg(3,1,0)$$

Note: $g(3,1,0) = (3e,1)$

$$Df(x,y) = \begin{bmatrix} y & x \\ \frac{1}{y} & -\frac{x}{y^2} \\ 1 & 1 \end{bmatrix} \therefore Df(3e,1) = \begin{bmatrix} 1 & 3e \\ 1 & -3e \\ 1 & 1 \end{bmatrix}$$

$$Dg(w,s,t) = \begin{bmatrix} e^s & we^s & 0 \\ tse^{wt} & e^{wt} & wse^{wt} \end{bmatrix}$$

$$\therefore Dg(3,1,0) = \begin{bmatrix} e & 3e & 0 \\ 0 & 1 & 3 \end{bmatrix}$$

$$\therefore D(f\circ g)(3,1,0) = \begin{bmatrix} 1 & 3e \\ 1 & -3e \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e & 3e & 0 \\ 0 & 1 & 3 \end{bmatrix} = \underline{\underline{\begin{bmatrix} e & 6e & 9e \\ e & 0 & -9e \\ e & 3e+1 & 3 \end{bmatrix}}}$$

9.

$$\vec{r}'(t) = [\cos(\pi t) - \pi t\sin(\pi t),\ \sin(\pi t) + \pi t\cos(\pi t),\ 1]$$

$$\therefore \vec{r}'(5) = [-1-0,\ 0-5\pi,\ 1] = [-1,-5\pi,1]$$

$$\vec{r}(5) = [-5,0,5].$$

$\therefore$ Tangent line : $(-5,0,5) + a(-1,-5\pi,1)$

At xy plane intersection, z-component $= 0$.

$5 + a(1) = 0, \quad a = -5$

$\therefore$ At $(-5,0,5) - 5(-1,-5\pi,1) = \underline{(0, 25\pi, 0)}$

10.

(a) $z = x^2 e^{-xy}$. Let $F(x,y,z) = x^2 e^{-xy} - z = 0$. $f(1,2) = e^{-2}$

$\nabla F(x,y,z) = (F_x, F_y, F_z) = (2xe^{-xy} - yx^2e^{-xy}, -x^3e^{-xy}, -1)$

$\therefore \nabla F(1,2,e^{-2}) = (2e^{-2} - 2e^{-2}, -e^{-2}, -1) = \underline{(0, e^{-2}, -1)}$

(b) Equation of plane: $\vec{n} \cdot (\vec{x} - \vec{x_0}) = 0$, $\vec{n}$ = normal.

$\therefore (0, e^{-2}, -1) \cdot [(x,y,z) - (1,2,e^{-2})]$

$= (0, e^{-2}, -1) \cdot (x-1, y-2, z-e^{-2})$

$= e^{-2}(y-2) - (z-e^{-2}) = 0$, or

$\underline{e^{-2}y - z - e^{-2} = 0}$

(c) The planes will have the same normal vectors.

Let $F(x,y,z) = x^2 - y^2 - z = 0$.

normal $= \nabla F = (2x, -2y, -1)$

$\therefore (2x, -2y - 1) = (0, e^{-2}, -1)$, from normal in (a).

$\therefore x = 0,\ -2y = e^{-2},\ y = -\frac{1}{2}e^{-2}$.

$\therefore z = x^2 - y^2 = 0 - (-\frac{1}{2}e^{-2}) = \frac{1}{4}e^{-4}$

$\therefore$ At the point: $\underline{(0, -\frac{1}{2}e^{-2}, \frac{1}{4}e^{-4})}$

11.

$z = (1 - x^2 - y^2)^{1/2}$, $\therefore$ Let $F(x,y,z) = (1 - x^2 - y^2)^{\frac{1}{2}} - z = 0$.

$\therefore$ Normal for tangent plane $= \nabla F(x,y,z)$

$F_x = \frac{1}{2}(1 - x^2 - y^2)^{-\frac{1}{2}}(-2x) = -\frac{x}{\sqrt{1 - x^2 - y^2}}$

$F_y = -\frac{y}{\sqrt{1 - x^2 - y^2}}$ $\qquad F_z = -1$

$\therefore \nabla F(x_0, y_0, f(x_0, y_0)) = \left[-\frac{x_0}{\sqrt{1 - x_0^2 - y_0^2}}, -\frac{y_0}{\sqrt{1 - x_0^2 - y_0^2}}, -1\right]$

$$= \left[\frac{-x_0}{f(x_0,y_0)}, \frac{-y_0}{f(x_0,y_0)}, -1 \right]$$

Let $k = -f(x_0, y_0)$.

$$\therefore\ k \nabla F(x_0, y_0, f(x_0,y_0)) = -f(x_0,y_0) \left[\frac{-x_0}{f(x_0,y_0)}, \frac{-y_0}{f(x_0,y_0)}, -1 \right]$$

$$= (x_0, y_0, f(x_0,y_0))$$

$\therefore$ normal to tangent plane is parallel to $(x_0, y_0, f(x_0,y_0))$.

$\therefore$ <u>Tangent plane is perpendicular to $(x_0, y_0, f(x_0,y_0))$.</u>

Geometrically, $z = (1-x^2-y^2)^{\frac{1}{2}}$ is a hemisphere, as $z^2 = 1-x^2-y^2$, or $x^2+y^2+z^2=1$. $\therefore$ Plane tangent to a sphere is $\perp$ vector from the origin: $(x_0, y_0, f(x_0,y_0))$.

12.

$$\nabla f(x,y,z) = [f_x, f_y, f_z]$$

$$\frac{\partial f}{\partial x} = \frac{\partial F}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial F}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$= \frac{\partial F}{\partial h} \cdot \frac{\partial h}{\partial x} + \frac{\partial F}{\partial k} \cdot \frac{\partial k}{\partial x}$$

Similarly, $\frac{\partial f}{\partial y} = \frac{\partial F}{\partial h} \cdot \frac{\partial h}{\partial y} + \frac{\partial F}{\partial K} \cdot \frac{\partial K}{\partial y}$

$$\frac{\partial f}{\partial z} = \frac{\partial F}{\partial h} \cdot \frac{\partial h}{\partial z} + \frac{\partial F}{\partial K} \cdot \frac{\partial K}{\partial z}$$

13.

$F(x,y,z) = \log(x+y) + x\cos y + \arctan(x+y) - z = 0.$

$f(1,0) = 0 + 1(1) + \arctan 1 = 1 + \frac{\pi}{4}$

$$F_x = \frac{1}{x+y} + \cos y + \frac{1}{1+(x+y)^2}$$

$$F_y = \frac{1}{x+y} - x\sin y + \frac{1}{1+(x+y)^2}$$

$F_z = -1.$

$\therefore \nabla F\left(1,0,1+\frac{\pi}{4}\right) = \left(1+1+\frac{1}{2}, 1-0+\frac{1}{2}, -1\right) = \left(\frac{5}{2}, \frac{3}{2}, -1\right)$

$\therefore \vec{n} \cdot (\vec{x} - \vec{x_0}) = 0: \left(\frac{5}{2}, \frac{3}{2}, -1\right) \cdot \left(x-1, y-0, z-\left(1+\frac{\pi}{4}\right)\right) = 0$

or $\frac{5}{2}(x-1) + \frac{3}{2}y - \left(z - 1 - \frac{\pi}{4}\right) = 0$

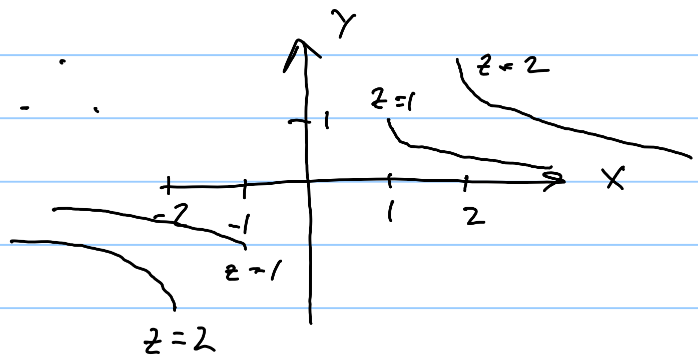
or $5(x-1) + 3y - 2z + 2 + \frac{\pi}{2} = 0$

or $\underline{10x + 6y - 4z = 6 - \pi}$

16.

For any x, $z = x\sin y$ is just the sine function with amplitude x.

For level curves, $\frac{z}{x} = \sin y$, or $y = \arcsin\left(\frac{z}{x}\right)$.
Note arcsin is defined only for $-1 \le \frac{z}{x} \le 1$, where z is fixed.



$\nabla T = (\sin y, x\cos y)$. This gives the direction for which the temperature changes the greatest.

17.

(a) Using L'Hopital's just for h, $\lim_{h\to 0} \frac{\cos h - 1}{h} = \lim_{h\to 0}(-\sin h) = 0$

$\therefore$ Given any $\epsilon > 0$, $\exists\, \delta > 0$ s.t. if $0 < |h| < \delta$,

then $\left|\frac{\cos h - 1}{h}\right| < \epsilon$.

$\therefore$ If $0 < |xy| < \delta$, then $\left| \frac{\cos xy - 1}{xy} \right| < \epsilon$.

$\therefore |\cos xy - 1| < \epsilon |xy|$ if $0 < |xy| < \delta$. [1]

If $0 < \sqrt{(x-0)^2 + (y-0)^2} < \delta$, then $|x| < \sqrt{x^2+y^2} < \delta$ and $|y| < \sqrt{x^2+y^2} < \delta$.

$\therefore$ if $\delta < 1$, then $|x| < 1, |y| < 1$, so $|xy| < |x| < \delta$.

$\therefore$ Given any $\epsilon > 0$, choose the $\delta > 0$ from

$\lim\limits_{h \to 0} \frac{\cos h - 1}{h} = 0$. $\therefore$ Let $\delta' = \min\{1, \delta\}$.

$\therefore$ if $0 < \sqrt{x^2+y^2} < \delta'$, then $0 < |xy| < \delta'$ as shown above. And from [1],

$|\cos xy - 1| < \epsilon |xy| < \epsilon |x|$, since $|y| < 1$.

$\therefore \left| \frac{\cos xy - 1}{x} - 0 \right| < \epsilon$ if $0 < \sqrt{x^2+y^2} < \delta'$

$$\therefore \lim_{(x,y) \to (0,0)} \frac{\cos xy - 1}{x} = 0$$

(3)

Along the line $y = 2x$, $\sqrt{\left| \frac{x+y}{x-y} \right|} = \sqrt{\left| \frac{3x}{-x} \right|} = \sqrt{3}$

Along the line $y=3x$, $\sqrt{\left|\frac{x+y}{x-y}\right|} = \sqrt{\left|\frac{4x}{-2x}\right|} = \sqrt{2}$

$\therefore$ limit doesn't exist

18.

(a) $f_x = e^z - y\sin x$
$f_y = \cos x$
$f_z = xe^z$ $\quad\therefore \nabla f(x,y,z) = (e^z - y\sin x, \cos x, xe^z)$

(b) $f_x = f_y = f_z = 10(x+y+z)^9$

$\therefore \nabla f(x,y,z) = 10\left[(x+y+z)^9, (x+y+z)^9, (x+y+z)^9\right]$

(c) $f_x = \frac{2x}{z}$ $\quad f_y = \frac{1}{z}$ $\quad f_z = -\frac{(x^2+y)}{z^2}$

$\therefore \nabla f(x,y,z) = \left(\frac{2x}{z}, \frac{1}{z}, -\frac{(x^2+y)}{z^2}\right).$

19.

$$\frac{\partial}{\partial x} = e^{1+x^2+y^2} + xe^{1+x^2+y^2}(2x)$$
$$= (1+2x^2)e^{1+x^2+y^2}$$

20.

$$D(F\circ g)(1,2) = Df(g(1,2)) \cdot Dg(1,2)$$

$$g(1,2) = (2e, e^4)$$

$$Df(x,y) = \begin{bmatrix} 2x & -2y \\ 0 & 0 \\ y\cos(xy) & x\cos(xy) \\ 0 & 0 \end{bmatrix}$$

$$\therefore Df(g(1,2)) = Df(2e, e^4) = \begin{bmatrix} 4e & -2e^4 \\ 0 & 0 \\ e^4\cos(2e^5) & 2e\cos(2e^5) \\ 0 & 0 \end{bmatrix}$$

$$Dg(x,y) = \begin{bmatrix} 2xye^{x^2} & e^{x^2} \\ e^{y^2} & 2xye^{y^2} \end{bmatrix} \quad \therefore Dg(1,2) = \begin{bmatrix} 4e & e \\ e^4 & 4e^4 \end{bmatrix}$$

$$\therefore Df(g(1,2)) \cdot Dg(1,2) = \begin{bmatrix} 4e & -2e^4 \\ 0 & 0 \\ e^4\cos(2e^5) & 2e\cos(2e^5) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 4e & e \\ e^4 & 4e^4 \end{bmatrix}$$

$$= \begin{bmatrix} 16e^2 - 2e^8 & 4e^2 - 8e^8 \\ 0 & 0 \\ 6e^5\cos(2e^5) & 9e^5\cos(2e^5) \\ 0 & 0 \end{bmatrix}$$

21.

"Toward the origin" from $(2,1)$ is $(0,0)-(2,1)=(-2,-1)$.
Unit vector is $-\frac{1}{\sqrt{5}}(2,1)$.

Rate of change in direction toward origin: $\nabla f \cdot \vec{v}$

$$\nabla f(x,y) = (f_x, f_y)$$

$$f_x = 2x e^{-(x^2+y^2+10)} + (x^2+y^2) e^{-(x^2+y^2+10)}(-2x)$$

$$f_y = 2y e^{-(x^2+y^2+10)} + (x^2+y^2) e^{-(x^2+y^2+10)}(-2y)$$

$$\therefore f_x(2,1) = 4e^{-15} + (-4)(5)e^{-15} = -16e^{-15}$$

$$f_y(2,1) = 2e^{-15} + (-2)(5)e^{-15} = -8e^{-15}$$

$$\therefore \nabla f(2,1)\cdot \vec{v} = (-16e^{-15}, -8e^{-15})\cdot\left(-\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\right)$$

$$= \frac{32}{\sqrt{5}}e^{-15} + \frac{8}{\sqrt{5}}e^{-15} = \frac{40}{\sqrt{5}}e^{-15}$$

$$= \underline{\underline{8\sqrt{5}e^{-15}}}$$

22.

Since $\frac{\partial F}{\partial x} > 0$, $\frac{\partial F}{\partial y} > 0$, then $\frac{\partial F/\partial x}{\partial F/\partial y} > 0$.

$\therefore -\frac{\partial F/\partial x}{\partial F/\partial y} < 0$, so $\frac{dy}{dx} < 0$.

$\therefore$ y decreases as x increases, for any fixed z.

Note also, $\frac{dx}{dy} = -\frac{\partial F/\partial y}{\partial F/\partial x}$, and since

$\frac{\partial F/\partial y}{\partial F/\partial x} > 0$, so $-\frac{\partial F/\partial y}{\partial F/\partial x} < 0$, so $\frac{dx}{dy} < 0$.

$\therefore$ x decreases as y increases, for any fixed z.

This obviously agrees w/ the minus sign in the formula for dy/dx.

Considering $z \simeq F(x,y) + \frac{\partial F}{\partial x}\Delta x + \frac{\partial F}{\partial y}\Delta y$, for $z = 0$, to keep balance, Δx and Δy must work in opposite ways since $F_x > 0$ and $F_y > 0$.

23.

(a)(i) Tangent plane at $(x_0, y_0, f(x_0, y_0))$ is

$$z = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0) \quad [1]$$

$$\nabla f(x_0, y_0) = \left(\frac{\partial f}{\partial x}(x_0, y_0), \frac{\partial f}{\partial y}(x_0, y_0)\right).$$

A vector, in the xy-plane, perpendicular to $\nabla f(x_0, y_0)$ is of the form $\left(-\frac{\partial f}{\partial y}(x_0, y_0), \frac{\partial f}{\partial x}(x_0, y_0)\right)$, since its dot product with $\nabla f(x_0, y_0)$ is 0.

$\therefore$ a line parallel to this vector, through $(x_0, y_0, f(x_0, y_0))$ is:

$$\ell(t) = (x_0, y_0, f(x_0, y_0)) + t\left[-f_y(x_0, y_0), f_x(x_0, y_0), 0\right]$$

$\therefore$ if, for every t, if $l(t)$ satisfies [1], the tangent plane, then $l(t)$ lies in the plane.

Every point of $l(t)$ is of the form:

$$x = x_0 - t f_y(x_0, y_0)$$

$$y = y_0 + t f_x(x_0, y_0)$$

$$z = f(x_0, y_0)$$

Substituting into [1],

$$f(x_0, y_0) = f(x_0, y_0) + f_x(x_0, y_0)\left[x_0 - t f_y(x_0, y_0) - x_0\right]$$
$$+ f_y(x_0, y_0)\left[y_0 + t f_x(x_0, y_0) - y_0\right]$$

$$\therefore f(x_0, y_0) = f(x_0, y_0) + f_x(x_0, y_0)(-t f_y(x_0, y_0))$$
$$+ f_y(x_0, y_0)(t f_x(x_0, y_0)),$$

or $f(x_0, y_0) = f(x_0, y_0) - t f_x f_y + t f_y f_x$

so $f(x_0, y_0) = f(x_0, y_0)$

$\therefore l(t)$ satisfies the equation of the plane for all t, so the line lies in the plane.

(ii) The normal to the tangent plane at $(x_0, y_0, f(x_0,y_0))$ is: $(f_x(x_0,y_0), f_y(x_0,y_0), -1)$, so the "upward" normal is: $(-f_x(x_0,y_0), -f_y(x_0,y_0), 1) = \vec{n}$

$\therefore \vec{n} \cdot \hat{k} = \|\vec{n}\| \, \|\hat{k}\| \cos\theta$

Since $\vec{n} \cdot \hat{k} = 1$, $\|\hat{k}\| = 1$, then

$1 = \sqrt{f_x^2 + f_y^2 + 1} \cos\theta$, or

$\sec\theta = \sqrt{f_x^2 + f_y^2 + 1}$, $\sec^2\theta = f_x^2 + f_y^2 + 1$

But $1 + \tan^2\theta = \sec^2\theta$, so

$\sec^2\theta - 1 = \tan^2\theta = f_x^2 + f_y^2$

$\therefore \underline{\tan\theta} = \sqrt{f_x^2 + f_y^2} = \underline{\|\nabla f(x_0,y_0)\|}$

(b)

$z = f(x,y) = x^3 + x^2\cos y$

$\nabla f(x,y) = (3x^2 + 2x\cos y, -x^2\sin y)$

$\therefore \nabla f(1,0) = (5, 0)$. A vector perpendicular to

This is $(0,-5)$. $\therefore$ $\ell(t) = (1,0) + t(0,-5)$ in the xy-plane, and through $(1,0,2)$,

$$\ell(t) = (1,0,2) + t(0,-5,0).$$

Tangent plane at $(1,0,2)$ is: $z = 2 + 5(x-1) + 0(y-0)$
or: $z = 5x - 3$

Does $z = 5x-3$ contain $\ell(t)$?

For $\ell(t)$: $x = 1 + t \cdot 0 = 1$
$y = 0 - t \cdot 5 = -5t$
$z = 2 + t \cdot 0 = 2$

$\therefore$ $2 \stackrel{?}{=} 5(1) - 3 = 2$. yes

24.

$f_x = 2x$, $f_y = 2y$. $\therefore$ $f_x(1,-2) = 2$, $f_y(1,-2) = -4$

$\therefore$ Tangent plane: $z = 5 + 2(x-1) - 4(y+2)$

or, $\underline{\underline{z = 2x - 4y - 5}}$

$\nabla f(x,y)$ = gradient, gives projection into the xy-plane of the normal to the plane.

25.

$$f_x = \frac{(x^2+y^2)(2x) - (x^2-y^2)(2x)}{(x^2+y^2)^2} = \frac{4xy^2}{(x^2+y^2)^2}$$

$$f_y = \frac{(x^2+y^2)(-2y) - (x^2-y^2)(2y)}{(x^2+y^2)^2} = \frac{-4yx^2}{(x^2+y^2)^2}$$

$\therefore\ f_x(1,1) = 1 \qquad f_y(1,1) = -1$

Directional derivative $= \nabla f(1,1) \cdot \vec{v}$, $\vec{v}$ a unit vector.

$\therefore\ (1,-1)\cdot(a,b) = 0$, $a-b = 0 \Rightarrow a = b$. (e.g., $a=b=1$).

Unit vector for $(1,1)$ is: $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. $\therefore$ along $\underline{\pm\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)}$

26.

(a) $f_x = e^x\cos(yz) \qquad f_y = -ze^x\sin(yz) \qquad f_z = -ye^x\sin(yz)$

$f_x(0,0,0) = 1 \qquad f_y(0,0,0) = 0 \qquad f_z(0,0,0) = 0$

$\|\vec{v}\| = 3$. $\therefore\ \vec{u} = \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)$.

$\therefore\ \nabla f(0,0,0)\cdot\vec{u} = (1,0,0)\cdot\left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right) = \underline{\frac{2}{3}}$

(5) $f_x = y + z \quad f_y = x + z \quad f_z = y + x$

$f_x(1,1,2) = 3 \quad f_y(1,1,2) = 3 \quad f_z(1,1,2) = 2$

$\|\vec{v}\| = \sqrt{105} \quad \therefore \vec{u} = \frac{1}{\sqrt{105}}(10,-1,2)$

$\nabla f(1,1,2) \cdot \vec{u} = (3,3,2) \cdot \frac{1}{\sqrt{105}}(10,-1,2) = \frac{30-3+4}{\sqrt{105}}$

$= \frac{31}{\sqrt{105}}$

27.

$f_x = 2x \quad f_y = 2y \quad f_z = -2z \quad \nabla f(3,5,-4) = (6,10,8)$.

$\therefore \nabla f(3,5,-4) \cdot (\vec{x} - (3,5,-4)) = 0 \implies$

$6(x-3) + 10(y-5) + 8(z+4) = 0$, or

$\underline{3x + 5y + 4z = 18}$ Normal: $\underline{(3,5,4)}$

28.

Let $h(t) = f(x(t), y(t))$, $0 \le t \le 1$.

$\therefore h'(t) = (f_x)(dx/dt) + (f_y)(dy/dt)$ by chain rule.

$\therefore h'(t) \le 0$, since $(f_x)\left(\frac{dx}{dt}\right) + (f_y)\left(\frac{dy}{dt}\right) \le 0$.

$\therefore h(t)$ is a decreasing function, and $\therefore h(0) \ge h(1)$.

$\therefore h(0) = f(x(0), y(0)) \ge f(x(1), y(1)) = h(1)$.

29.

The bug should move in the $-\nabla T$ direction, which is opposite ∇T, the direction of fastest increase.

$\nabla T(x,y) = (T_x, T_y) = (4x, -8y)$.

$\therefore -\nabla T(x,y) = (-4x, 8y)$. $\therefore -\nabla T(-1,2) = \underline{(4, 16)}$.

30.

$\nabla w(x,y) = (2x+y, x)$ $\therefore \underline{\nabla w(-1,1) = (-1,-1)}$.

$\|\nabla w(-1,1)\| = \sqrt{(-1)^2+1^2} = \sqrt{2}$. Thus, $w(x,y)$ changes most rapidly by $\sqrt{2}$ /unit.

31.

Note: should state $f: S \to R^1$

(a) Let $\vec{x}$ be any "fixed" point in R^n, and define

$$g(\lambda) = f(\lambda \vec{x}) = \lambda^p f(\vec{x}) \quad , \quad h(\lambda) = \lambda \vec{x}$$

$\therefore g'(\lambda) = p\lambda^{p-1} f(\vec{x})$ (here, $f(\vec{x})$ acts as a constant).

and as $g(\lambda) = f(h(\lambda)) = f(\lambda \vec{x})$, Then $g'(\lambda) = D(f \circ h)(\lambda)$

$$= Df(h(\lambda)) \cdot Dh(\lambda) = \underset{1\times n}{Df(\lambda\vec{x})} \cdot \underset{n\times 1}{D(\lambda\vec{x})}$$ ← matrix multiplication

$$= \nabla f(\lambda \vec{x}) \cdot \vec{x}$$ (vector dot product)

$$\therefore p\lambda^{p-1} f(\vec{x}) = \nabla f(\lambda\vec{x}) \cdot \vec{x}$$

Letting $\lambda = 1$, $pf(\vec{x}) = \nabla f(\vec{x}) \cdot \vec{x}$.

Note: if $h(\lambda) = (\lambda x_1, \lambda x_2, \dots, \lambda x_n)$, $\vec{x} \in R^n$,

where $h: R^1 \to R^n$, and $h(\lambda) = (h_1(\lambda), \dots, h_n(\lambda))$,

Then $Dh(\lambda) = \begin{bmatrix} \partial h_1/\partial\lambda \\ \vdots \\ \partial h_n/\partial\lambda \end{bmatrix} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, the x_i act as constants

(b) Note $f(\lambda\vec{x}) = f(\lambda x, \lambda y, \lambda z) = (\lambda x) - 2(\lambda y) - \sqrt{(\lambda x)(\lambda z)}$

$$= \lambda x - 2\lambda y - \lambda\sqrt{xz}$$

$$= \lambda(x - 2y - \sqrt{xz})$$

$$= \lambda f(x,y,z).$$

$\therefore$ f is homogeneous of degree 1.

$\therefore$ Check for $\nabla f \cdot \vec{x} = f(\vec{x})$

$$f_x = 1 - \tfrac{1}{2}(xz)^{-\frac{1}{2}}(z) = 1 - \frac{z}{2\sqrt{xz}}$$

$$f_y = -2$$

$$f_z = -\tfrac{1}{2}(xz)^{-\frac{1}{2}}(x) = -\frac{x}{2\sqrt{xz}}$$

$$\therefore \nabla f \cdot \vec{x} = (f_x, f_y, f_z) \cdot (x, y, z)$$

$$= x f_x + y f_y + z f_z$$

$$= \left(x - \frac{xz}{2\sqrt{xz}}\right) + (-2y) + \left(-\frac{xz}{2\sqrt{xz}}\right)$$

$$= x - 2y - \frac{xz}{\sqrt{xz}} = x - 2y - \sqrt{xz} = f(x,y,z)$$

$\therefore$ $\underline{\nabla f \cdot \vec{x} = f(\vec{x})}$

32.

$$\frac{\partial z}{\partial x} = Df(x-y)\cdot\frac{1}{y} \qquad \frac{\partial z}{\partial y} = \frac{y[Df(x-y)](-1) - f(x-y)(1)}{y^2}$$

$$= -\frac{Df(x-y)}{y} - \frac{f(x-y)}{y^2}$$

$$\therefore z + y\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} =$$

$$\frac{f(x-y)}{y} + y\left(\frac{Df(x-y)}{y}\right) + y\left(-\frac{Df(x-y)}{y} - \frac{f(x-y)}{y^2}\right) =$$

$$\frac{f(x-y)}{y} + Df(x-y) - Df(x-y) - \frac{f(x-y)}{y} = \underline{\underline{0}}$$

33.

$f: R^1 \to R^1$. Let $h: R^2 \to R^1$, $h(x,y) = \frac{x+y}{x-y}$.

$\therefore z = g(x,y) = (f \circ h)(x,y) \quad \therefore g: R^2 \to R^1$, Dg a 1×2 matrix.

$$Dg = [\partial g/\partial x \quad \partial g/\partial y] = \left[\frac{\partial z}{\partial x} \quad \frac{\partial z}{\partial y}\right].$$

$$\frac{\partial z}{\partial x} = f'(h(x,y))\cdot\frac{\partial h}{\partial x} \qquad \frac{\partial h}{\partial x} = \frac{(x-y)(1) - (x+y)(1)}{(x-y)^2} = \frac{-2y}{(x-y)^2}$$

$$\frac{\partial z}{\partial y} = f'(h(x,y))\cdot\frac{\partial h}{\partial y} \qquad \frac{\partial h}{\partial y} = \frac{(x-y)(1) - (x+y)(-1)}{(x-y)^2} = \frac{2x}{(x-y)^2}$$

$$\therefore\ x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = x\left[f'(h(x,y))\left(\frac{-2y}{(x-y)^2}\right)\right] + y\left[f'(h(x,y))\left(\frac{2x}{(x-y)^2}\right)\right]$$

$$= \frac{f'(h(x,y))}{(x-y)^2}\left[-2xy + 2xy\right] = 0$$

34.

Since f has a local max. or local min. at $\vec{x_0}$, $\exists\,\delta > 0$ s.t. $f(\vec{x_0}) \geq f(\vec{x})\ \forall\, \vec{x} \in D_\delta(\vec{x_0})$, or $f(\vec{x_0}) \leq f(\vec{x})\ \forall\, \vec{x} \in D_\delta(\vec{x_0})$.

Let $g_{\vec{x}}(x_i) = f(\vec{x})$, for $\vec{x} \in D_\delta(x_0)$

(i.e., holding all components of $\vec{x}$ constant except the "i"th)

$$\therefore\ \frac{d}{dx_i} g_{\vec{x}}(x_i) = \frac{\partial f}{\partial x_i}(\vec{x}), \quad \text{for all } x \in D_\delta(\vec{x_0})$$

and $g_{\vec{x}}((\vec{x_0})_i)$ is a local max or local min on $((\vec{x_0})_i - \delta, (\vec{x_0})_i + \delta)$, since $g_{\vec{x}}((\vec{x_0})_i) = f(\vec{x_0})$.

$\therefore\ g'_{\vec{x}}((\vec{x_0})_i) = 0$ from calculus of one-variable.

$$\therefore\ \frac{\partial f}{\partial x_i}(\vec{x_0}) = g'_{\vec{x}}((\vec{x_0})_i) = 0$$

35.

(a) for (i) and (ii) f_x and f_y clearly exist for $(x,y) \neq (0,0)$ since they are the product, sum, and quotient of differentiable scalar-valued functions.

For $(x,y) = (0,0)$, must refer to limit definitions.

(i) $$f_x(0,0) = \lim_{h \to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{f(0+h,0) - 0}{h}$$

$$= \lim_{h \to 0} \frac{\frac{h \cdot 0}{h^2 + 0^2}}{h} = \lim_{h \to 0} \frac{0}{h^3} = 0$$

$$f_y(0,0) = \lim_{h \to 0} \frac{f(0,0+h) - f(0,0)}{h} = \lim_{h \to 0} \frac{f(0,0+h) - 0}{h}$$

$$= \lim_{h \to 0} \frac{\frac{0 \cdot h}{0^2 + h^2}}{h} = \lim_{h \to 0} \frac{0}{h^3} = 0$$

(ii) $$f_x(0,0) = \lim_{h \to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{f(h,0) - 0}{h}$$

$$= \lim_{h \to 0} \frac{\frac{h^2 \cdot 0^2}{h^2 + 0^4}}{h} = \lim_{h \to 0} \frac{0}{h^3} = 0$$

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,0+h)-f(0,0)}{h} = \lim_{h\to 0} \frac{f(0,h)-0}{h}$$

$$= \lim_{h\to 0} \frac{\frac{0^2 \cdot h^2}{0^2+h^4}}{h} = \lim_{h\to 0} \frac{0}{h^5} = 0.$$

(b) (i) is not differentiable at $(0,0)$. If it were, it would be continuous at $(0,0)$. But along $y=x$ path, (i) becomes $\lim\limits_{(x,y)\to(0,0)} \frac{x^2}{x^2+x^2} = \frac{1}{2} \neq 0 = f(0,0)$.

For (ii), use definition: f is differentiable at $\vec{x}_0$ if

$$\lim_{\vec{x}\to\vec{x}_0} \frac{\| f(\vec{x}) - f(\vec{x}_0) - T(\vec{x}-\vec{x}_0) \|}{\|\vec{x}-\vec{x}_0\|} = 0, \qquad [1]$$

Where $\vec{x}_0 = (0,0)$, $f(\vec{x}_0) = 0$, and

$$Df(\vec{x}_0) = T = \left[\frac{\partial f}{\partial x}(0,0) \quad \frac{\partial f}{\partial y}(0,0) \right] = \left[0 \quad 0 \right] \text{ from (a)}$$

$$\therefore T(\vec{x}-\vec{x}_0) = 0.$$

$\therefore$ [1] reduces to $\lim\limits_{\vec{x}\to(0,0)} \frac{\| f(\vec{x}) \|}{\| x \|} = 0,$

$$\text{or } \lim_{(x,y)\to(0,0)} \frac{\frac{x^2y^2}{x^2+y^4}}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{x^2y^2}{(x^2+y^4)\sqrt{x^2+y^2}} = 0$$

If switch to polar coordinates,

$x = r\cos\theta$, $y = r\sin\theta$, Then $0 < \sqrt{x^2+y^2} < \delta$ becomes $0 < r < \delta$, so $(x,y) \to (0,0)$ becomes $r \to 0$.

$$\therefore \lim_{r\to 0} \frac{r^4\cos^2\theta\sin^2\theta}{(r^2\cos^2\theta + r^4\sin^2\theta)(r)} = \lim_{r\to 0} \frac{r\cos^2\theta\sin^2\theta}{\cos^2\theta + r^2\sin^2\theta} \quad [2]$$

if $\cos\theta = 0$, Then [2] becomes $(\sin^2\theta = 1)$:

$$\lim_{r\to 0} \frac{0}{0+r^2} = 0, \text{ so } f \text{ is differentiable.}$$

if $\cos\theta \neq 0$, Then $\lim_{r\to 0} \frac{r\cos^2\theta\sin^2\theta}{\cos^2 + r^2\sin^2\theta} = \frac{0}{\cos^2\theta + 0} = 0$

so f is differentiable.

$\therefore$ For (ii), f is diffentiable at $(0,0)$

36.

(a) If $(x,y) \neq (0,0)$: $f_x = 2xy^2\log(x^2+y^2) + x^2y^2 \cdot \frac{1}{x^2+y^2} \cdot 2x$

$$f_y = 2yx^2\log(x^2+y^2) + x^2y^2 \cdot \frac{1}{x^2+y^2} \cdot 2y$$

$$\therefore \nabla f(x,y) = \left(2xy^2\log(x^2+y^2) + \frac{2x^3y^2}{x^2+y^2},\ 2yx^2\log(x^2+y^2) + \frac{2x^2y^3}{x^2+y^2}\right)$$

If $(x,y)=(0,0)$, use definition.

$$f_x = \lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{(0+h)^2 0^2 \log((0+h)^2+0) - 0}{h}$$

$$= \lim_{h\to 0} \frac{0}{h} = 0$$

$$f_y = \lim_{h\to 0} \frac{f(0,0+h)-f(0,0)}{h} = \lim_{h\to 0} \frac{0^2 (0+h)^2 \log(0^2+(0+h)^2) - 0}{h}$$

$$= \lim_{h\to 0} \frac{0}{h} = 0.$$

$\therefore \nabla f(0,0) = (0,0)$

(b) For $(x,y)\neq(0,0)$ $\quad f_x = y\sin\left(\frac{1}{x^2+y^2}\right) + xy\cos\left(\frac{1}{x^2+y^2}\right)\left(-\frac{2x}{(x^2+y^2)^2}\right)$

$$f_y = x\sin\left(\frac{1}{x^2+y^2}\right) + xy\cos\left(\frac{1}{x^2+y^2}\right)\left(-\frac{2y}{(x^2+y^2)^2}\right)$$

$$\therefore \nabla f(x,y) = \left[y\sin\left(\frac{1}{x^2+y^2}\right) - \frac{2x^2y}{(x^2+y^2)^2}\cos\left(\frac{1}{x^2+y^2}\right),\right.$$
$$\left.\left(x\sin\left(\frac{1}{x^2+y^2}\right) - \frac{2xy^2}{(x^2+y^2)^2}\cos\left(\frac{1}{x^2+y^2}\right)\right]$$

For $(x,y)=(0,0)$, use definition

$$f_x = \lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{(0+h)\,0\,\sin\left(\frac{1}{(0+h)^2+0}\right) - 0}{h}$$

$$= \lim_{h\to 0} \frac{0}{h} = 0$$

$$f_y = \lim_{h\to 0} \frac{f(0,0+h)-f(0,0)}{h} = \lim_{h\to 0} \frac{0(0+h)\sin\left(\frac{1}{0+(0+h)^2}\right)-0}{h}$$

$$= \lim_{h\to 0} \frac{0}{h} = 0$$

$$\therefore \nabla f(0,0) = (0,0)$$

37.

(a) $f_x = \tan^{-1}\left(\frac{x}{y}\right) + x\left(\frac{1}{1+\left(\frac{x}{y}\right)^2}\right)\left(\frac{1}{y}\right)$

$$\therefore f_x(1,1) = \tan^{-1}(1) + \frac{1}{2} = \frac{\pi}{4} + \frac{1}{2}$$

$f_y = x\left(\frac{1}{1+\left(\frac{x}{y}\right)^2}\right)\cdot\left(-\frac{x}{y^2}\right) \quad \therefore f_y(1,1) = -\frac{1}{2}$

$$\therefore \nabla f(1,1)\cdot\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right) = \left(\frac{\pi}{4}+\frac{1}{2}\right)\frac{1}{\sqrt{2}} + \left(-\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right)$$

$$= \frac{\pi}{4\sqrt{2}} = \underline{\frac{\pi\sqrt{2}}{8}}$$

(b) $f_x = -\sin\left(\sqrt{x^2+y^2}\right)\left(\frac{1}{2}(x^2+y^2)^{-\frac{1}{2}}(2x)\right) \quad \therefore f_x(1,1) = -\frac{1}{\sqrt{2}}\sin(\sqrt{2})$

$f_y = -\sin\left(\sqrt{x^2+y^2}\right)\left(\frac{1}{2}(x^2+y^2)^{-\frac{1}{2}}(2y)\right) \quad \therefore f_y(1,1) = -\frac{1}{\sqrt{2}}\sin(\sqrt{2})$

$$\therefore \nabla f(1,1)\cdot\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right) = \left(-\frac{1}{\sqrt{2}}\sin\sqrt{2}\right)\frac{1}{\sqrt{2}} + \left(-\frac{1}{\sqrt{2}}\sin\sqrt{2}\right)\left(\frac{1}{\sqrt{2}}\right)$$

$$= \underline{-\sin\sqrt{2}}$$

(c) $f_x = e^{-x^2-y^2}(-2x) \therefore f_x(1,1) = -2e^{-2}$

$f_y = e^{-x^2-y^2}(-2y) \therefore f_y(1,1) = -2e^{-2}$

$\therefore \nabla f(1,1)\cdot\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = (-2e^{-2})\frac{1}{\sqrt{2}} + (-2e^{-2})\frac{1}{\sqrt{2}}$

$= \underline{-2\sqrt{2}e^{-2}}$

38.

(a) $\|\vec{u}\| = \sqrt{1^2+2^2+2^2} = \sqrt{9} = \underline{3}$

$\vec{u}\cdot\vec{v} = (1,-2,2)\cdot(2,1,-3) = 2-2-6 = \underline{-6}$

$\vec{u}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 2 \\ 2 & 1 & -3 \end{vmatrix} = (6-2, -(-3-4), 1+4) = \underline{(4,7,5)}$

(b) $f_x = ye^{xy}\sin(xyz) + e^{xy}\cos(xyz)(yz)$

$f_y = xe^{xy}\sin(xyz) + e^{xy}\cos(xyz)(xz)$

$f_z = e^{xy}\cos(xyz)(xy)$

$$\therefore \left.\begin{aligned} f_x(0,1,1) &= 0+1 = 1 \\ f_y(0,1,1) &= 0+0 = 0 \\ f_z(0,1,1) &= 0 \end{aligned}\right\} \therefore \nabla f(0,1,1) = (1,0,0)$$

$$\therefore \nabla f(0,1,1) \cdot \frac{\vec{u}}{\|\vec{u}\|} = (1,0,0) \cdot \left(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right) = \underline{\frac{1}{3}}$$

39.

$$h_x = 2e^{-x^2}(-2x) \quad \therefore f_x(1,0) = -4e^{-1}$$

$$h_y = e^{-3y^2}(-6y) \quad \therefore f_y(1,0) = 0$$

$$\therefore \nabla h(1,0) = \underline{\left(-\frac{4}{e}, 0\right)}$$

40.

$$z = f(x,y) = \frac{e^x}{x^2+y^2}. \quad \text{At } f(1,2),\ z = \frac{e}{1+2^2} = \frac{e}{5}$$

$\therefore$ tangent plane at $(1, 2, \frac{e}{5})$.

$$f_x = \frac{(x^2+y^2)e^x - e^x(2x)}{(x^2+y^2)^2} \quad f_x(1,2) = \frac{5e-2e}{5^2} = \frac{3e}{25}$$

$$f_y = -\frac{e^x}{(x^2+y^2)^2}(2y) = \frac{-2ye^x}{(x^2+y^2)^2} \quad f_y(1,2) = \frac{-4e}{25}$$

$\therefore$ Tangent plane: $z = f(1,2) + f_x(1,2)(x-1) + f_y(1,2)(y-2)$

$\therefore z = \frac{e}{5} + \frac{3e}{25}(x-1) - \frac{4e}{25}(y-2)$, or

$25z = 5e + 3ex - 3e - 4ey + 8e$, or

$\underline{3ex - 4ey - 25z + 10e = 0}$

41.

(a) The derivative of a composite function, $f \circ g$, at a point $\vec{x}_0$, is the Jacobian matrix of partial derivatives of f at $g(\vec{x}_0)$ times the Jacobian matrix of partial derivatives of g at $\vec{x}_0$.

(b) Directly: $f(\vec{h}(u)) = f(\sin 3u, \cos 8u)$

$$= \sin^2 3u + \cos 8u$$

$$\therefore \frac{dg}{du}(0) = 2(\sin 3u)(\cos 3u)(3) - 8\sin 8u \Big|_{u=0}$$

$$= 6\sin 3u \cos 3u - 8\sin 8u \Big|_{u=0}$$

$$= 0 - 0 = \underline{0}$$

Chain Rule: $D\vec{h}(u) = \begin{bmatrix} 3\cos 3u \\ -8\sin 3u \end{bmatrix}$ $\therefore D\vec{h}(0) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$

$$Df(x,y) = \begin{bmatrix} 2x & 1 \end{bmatrix}.$$

$$\therefore Df(\vec{h}(0)) = Df(0,1) = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

$$\therefore Df(\vec{h}(0)) \cdot D\vec{h}(0) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \underline{0}$$

42.

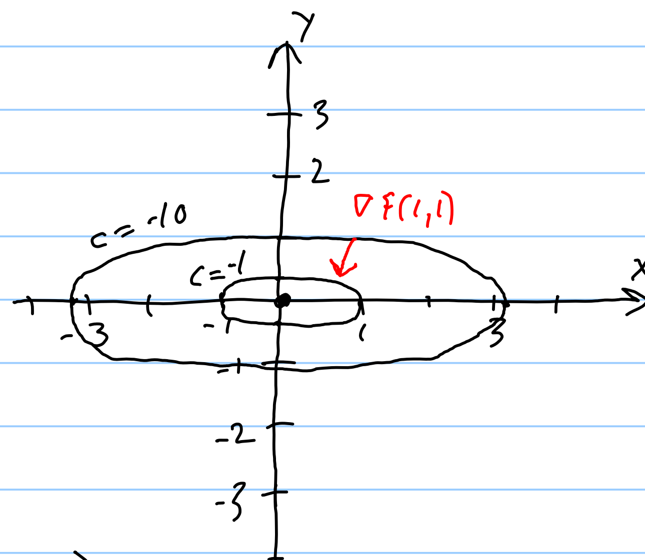
(a) Note $f(x,y) \leq 0, \forall x, y$

For $c = 0$, $x = 0, y = 0$ is only solution.

For $c = -1$, $x^2 + 9y^2 = 1$,

or $\frac{x^2}{1} + \frac{y^2}{1/9} = 1$.

$\therefore$ an ellipse with vertices at $(\pm 1, 0), (0, \pm \frac{1}{3})$.



For $c = -10$, $x^2 + 9y^2 = 10$, or

$\frac{x^2}{10} + \frac{y^2}{10/9} = 1$. $\therefore$ An ellipse with vertices at $(\pm\sqrt{10}, 0), (0, \pm \sqrt{10}/3)$

(b) $f_x = -2x \qquad f_x(1,1) = -2 \qquad \therefore \nabla f(1,1) = (-2, -18)$

$f_y = -18y \qquad f_y(1,1) = -18$

$\nabla f(1,1)$ indicates the direction (in the xy-plane) of steepest ascent.

The 3D shape is an elliptical paraboloid, with peak at $(x,y,z) = (0,0,0)$.

At $(x,y) = (1,1)$, $(x,y,z) = (1,1,-10)$, and at $(1,1,-10)$ on paraboloid, heading toward $(-2,-18)$ gives the direction of maximum increase in elevation (or Δz).

43.

Let $f(x,y,z) = x^2 + 2y^2 + 3z^2 = C$. Normal to surface is: $\nabla f(x,y,z) = (2x, 4y, 6z)$. $\therefore \nabla f(1,1,1) = (2,4,6)$.

$\therefore \ell(t) = (1,1,1) + t(s\vec{v})$, t = time, $s\vec{v} = s(2,4,6)$, where $s\vec{v}$ is a vector in the direction of $\vec{v}$.

Note: $\left|\frac{d\ell(t)}{dt}\right| = 10 = \left| s(2,4,6)\right| = s\sqrt{2^2+4^2+6^2} = s\sqrt{56}$

$\therefore s = \frac{10}{\sqrt{56}} = \frac{5}{\sqrt{14}}$

$\therefore \ell(t) = (1,1,1) + t\,\frac{5}{\sqrt{14}}(2,4,6)$

or, $\ell(t) = \left[1 + \frac{10}{\sqrt{14}}t,\ 1 + \frac{20}{\sqrt{14}}t,\ 1 + \frac{30}{\sqrt{14}}t\right]$

$\therefore$ If $g(x,y,z) = x^2+y^2+z^2 = 103$, then solve for $(g \circ l)(t) = 103$.

$$\therefore \left(1+\frac{10}{\sqrt{14}}t\right)^2 + \left(1+\frac{20}{\sqrt{14}}t\right)^2 + \left(1+\frac{30}{\sqrt{14}}t\right)^2 = 103$$

$$\left(1+\frac{20}{\sqrt{14}}t + \frac{100}{14}t^2\right) + \left(1+\frac{40}{\sqrt{14}}t+\frac{400}{14}t^2\right) + \left(1+\frac{60}{\sqrt{14}}t+\frac{900}{14}t^2\right) = 103$$

$$\therefore \frac{120}{\sqrt{14}}t + \frac{1400}{14}t^2 = 100, \text{ or } \frac{120}{\sqrt{14}}t + 100t^2 = 100$$

$$\text{or } 10t^2 + \frac{12}{\sqrt{14}}t - 10 = 0$$

$$\therefore t = \frac{-\frac{12}{\sqrt{14}} \pm \sqrt{\frac{144}{14}+400}}{20}.$$ Reject negative root.

$$\therefore t = \frac{-12}{20\sqrt{14}} + \frac{1}{20}\sqrt{\frac{5744}{14}} = -\frac{3}{5\sqrt{14}} + \frac{1}{20}\sqrt{\frac{16\times 359}{14}}$$

$$= -\frac{3}{70}\sqrt{14} + \frac{1}{5}\sqrt{\frac{359}{14}} = -\frac{3}{70}\sqrt{14} + \frac{\sqrt{14}}{70}\sqrt{359}$$

$$= \underline{\underline{\frac{\sqrt{14}}{70}\left(-3+\sqrt{359}\right)}}$$

44.

From #43, $\nabla f(x,y,z) = (2x, 4y, 6z) =$ normal to surface.

A vector parallel to line $\Rightarrow 2x = 4y = 6z$.

$\therefore x = 3z,\ y = \frac{3}{2}z.$

$\therefore (3z)^2 + 2\left(\frac{3}{2}z\right)^2 + 3z^2 = 6$, or

$9z^2 + \frac{9}{2}z^2 + 3z^2 = 6$, or $33z^2 = 12$, or

$z^2 = \frac{4}{11}$, $\therefore z = \pm\frac{2}{\sqrt{11}}$

$\therefore x = \pm\frac{6}{\sqrt{11}}$, $y = \pm\frac{3}{\sqrt{11}}$

$$\therefore \pm\left(\frac{6}{\sqrt{11}}, \frac{3}{\sqrt{11}}, \frac{2}{\sqrt{11}}\right)$$

45.

(a) $$z = \frac{(e^{-x-y})^2 + (e^{xy})^2}{(e^{-x-y})^2 - (e^{xy})^2} = \frac{e^{-2x-2y} + e^{2xy}}{e^{-2x-2y} - e^{2xy}}$$

$$z_x = \frac{(e^{-2x-2y} - e^{2xy})(-2e^{-2x-2y} + 2ye^{2xy}) - (e^{-2x-2y} + e^{2xy})(-2e^{-2x-2y} - 2ye^{2xy})}{(e^{-2x-2y} - e^{2xy})^2}$$

$$= \frac{4e^{2xy-2x-2y} + 4ye^{2xy-2x-2y}}{(e^{-2x-2y} - e^{2xy})^2}$$

$$z_y = \frac{(e^{-2x-2y} - e^{2xy})(-2e^{-2x-2y} + 2xe^{2xy}) - (e^{-2x-2y} + e^{2xy})(-2e^{-2x-2y} - 2xe^{2xy})}{(e^{-2x-2y} - e^{2xy})^2}$$

$$= \frac{4e^{2xy-2x-2y} + 4xe^{2xy-2x-2y}}{(e^{-2x-2y} - e^{2xy})^2}$$

(5) $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}$

$$\frac{\partial z}{\partial u} = \left[\frac{(u^2-v^2)2u - (u^2+v^2)(2u)}{(u^2-v^2)^2}\right] = \left[\frac{-4uv^2}{(u^2-v^2)^2}\right]$$

$$\frac{\partial z}{\partial v} = \left[\frac{(u^2-v^2)(2v) - (u^2+v^2)(-2v)}{(u^2-v^2)^2}\right] = \left[\frac{4vu^2}{(u^2-v^2)^2}\right]$$

$$\therefore \frac{\partial z}{\partial x} = \left[\frac{-4uv^2}{(u^2-v^2)^2}\right] \cdot \left(-e^{-x-y}\right) + \left[\frac{4vu^2}{(u^2-v^2)^2}\right]\left(ye^{xy}\right)$$

$$= \frac{4e^{-x-y}uv^2 + 4ye^{xy}vu^2}{(u^2-v^2)^2}$$

$$= \frac{4e^{-x-y}(e^{-x-y})(e^{xy})^2 + 4ye^{xy}(e^{xy})(e^{-x-y})^2}{(e^{-2x-2y} - e^{2xy})^2}$$

$$= \frac{4e^{2xy-2x-2y} + 4ye^{2xy-2x-2y}}{(e^{-2x-2y} - e^{2xy})^2}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y}$$

$$= \left[\frac{-4uv^2}{(u^2-v^2)^2}\right]\left(-e^{-x-y}\right) + \left[\frac{4vu^2}{(u^2-v^2)^2}\right]\left(xe^{xy}\right)$$

$$= \frac{4e^{-x-y}(e^{-x-y})(e^{xy})^2 + 4xe^{xy}(e^{xy})(e^{-x-y})^2}{(e^{-2x-2y} - e^{2xy})^2}$$

$$= \frac{4e^{2xy-2x-2y} + 4xe^{2xy-2x-2y}}{(e^{-2x-2y} - e^{2xy})^2}$$

46.

(a) $z = (x+y)(x-y) = x^2 - y^2$

$$\therefore \frac{\partial z}{\partial x} = \underline{2x} \qquad \frac{\partial z}{\partial y} = \underline{-2y}$$

(b) $$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = (v)(1) + (u)(1) = v+u$$

$$= (x-y) + (x+y) = \underline{2x}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y} = (v)(1) + (u)(-1) = v-u$$

$$= (x-y) - (x+y) = \underline{-2y}$$

47.

$\frac{\partial w}{\partial x}$ on the left side of the equation is ambiguous.

It is unclear if this means derivative with respect to first independent variable of f, or with respect to "x" of the second, composite function.

Assuming the latter, let $w = f(u,v)$, $u(x) = x$, $v(x) = x^2$.

Then $\dfrac{\partial w}{\partial x} = \dfrac{\partial w}{\partial u}\cdot\dfrac{\partial u}{\partial x} + \dfrac{\partial w}{\partial v}\cdot\dfrac{\partial v}{\partial x}$

$$= \frac{\partial w}{\partial u} + \frac{\partial w}{\partial v}(2x)$$

Example: Let $w = f(u,v) = uv$. Let $u(x) = x$, $v(x) = x^2$.

$\therefore \dfrac{\partial w}{\partial x} = \dfrac{\partial f}{\partial u}\cdot\dfrac{\partial u}{\partial x} + \dfrac{\partial f}{\partial v}\cdot\dfrac{\partial v}{\partial x}$

$$= (v)(1) + (u)(2x) = x^2 + 2x^2 = 3x^2$$

if allowed the confusion, and allowed

$\dfrac{\partial w}{\partial x} = \dfrac{\partial w}{\partial x} + 2x\dfrac{\partial w}{\partial y}$, Then $\dfrac{\partial w}{\partial y} = \dfrac{\partial w}{\partial v} = u = x.$

so $\dfrac{\partial w}{\partial y} \neq 0.$

48.

Let temp vector $\vec{T} = (-0.2^\circ c/km)\,\hat{j} + (-0.3^\circ c/km)\,\hat{i}$

velocity vector: Northeast is 45°,
and $\cos 45^\circ = \sin 45^\circ = \sqrt{2}/2$.

$\therefore (20\ km/h)\left(\frac{\sqrt{2}}{2}\right) = 10\sqrt{2}\ km/hr.$

$\therefore \vec{V} = (10\sqrt{2}\ km/hr)\,\hat{i} + (10\sqrt{2}\ km/hr)\,\hat{j}.$

$$\therefore \vec{T}\cdot\vec{v} \equiv \nabla T\cdot\vec{v}, \text{ directional derivative}$$

$$= (-0.2, -0.3)\cdot(10\sqrt{2}, 10\sqrt{2})$$

$$= -2\sqrt{2} - 3\sqrt{2} = \underline{-5\sqrt{2}\ ^\circ C/hr}$$

49.

$$\frac{d}{dt} e^{f(t)g(t)} = e^{f(t)g(t)}\left[f'(t)g(t) + f(t)g'(t)\right]$$

50.

Let $h(t) = f(t)^{g(t)}$ $\therefore \log h(t) = g(t)\log f(t)$

$$\therefore h(t) = e^{\log h(t)} = e^{g(t)\log f(t)}$$

$$\therefore \frac{d}{dt} f(t)^{g(t)} = \frac{d}{dt} e^{g(t)\log f(t)}$$

$$= e^{g(t)\log f(t)}\left[g'(t)\log f(t) + g(t)\cdot\frac{1}{f(t)}\cdot f'(t)\right]$$

$$= f(t)^{g(t)}\left[g'(t)\log f(t) + \frac{g(t)}{f(t)}\cdot f'(t)\right]$$

$$= f(t)^{g(t)}\left[g'(t)\log f(t)\right] + f(t)^{g(t)-1}\left[g(t)\cdot f'(t)\right]$$

51.

(a) $(f\circ\vec{c})(t) = \dfrac{\ln(1+t^2+2\cos^2 t)}{1+(1-t^2)^2}$

$\therefore (f\circ\vec{c})'(t) =$

$$\frac{\left[1+(1-t^2)^2\right]\cdot\dfrac{2t-4\cos t\sin t}{1+t^2+2\cos^2 t} - \left[\ln(1+t^2+2\cos^2 t)\right]2(1-t^2)(-2t)}{\left[1+(1-t^2)^2\right]^2}$$

$$= \frac{\left[1+(1-t^2)^2\right](2t-4\cos t\sin t) + 4t(1-t^2)\left[\ln(1+t^2+2\cos^2 t)\right](1+t^2+2\cos^2 t)}{\left[1+(1-t^2)^2\right]^2(1+t^2+2\cos^2 t)}$$

(b) Using chain rule, $D(f\circ\vec{c})(t) = \nabla f(\vec{c}(t))\cdot D\vec{c}(t)$

$D\vec{c}(t) = \begin{bmatrix} 1 \\ -2t \\ -\sin t \end{bmatrix}$ $\qquad f(x,y,z) = \dfrac{\ln(1+x^2+2z^2)}{1+y^2}$

$f_x = \dfrac{1}{1+y^2}\cdot\dfrac{1}{1+x^2+2z^2}\cdot 2x = \dfrac{2x}{(1+y^2)(1+x^2+2z^2)}$

$f_y = -\dfrac{\ln(1+x^2+2z^2)}{(1+y^2)^2}(2y)$

$f_z = \dfrac{1}{1+y^2}\cdot\dfrac{1}{1+x^2+2z^2}\cdot 4z = \dfrac{4z}{(1+y^2)(1+x^2+2z^2)}$

$\therefore \nabla f(\vec{c}(t))\cdot D\vec{c}(t) = f_x - 2t f_y - \sin t\, f_z$

$$= \frac{2x}{(1+y^2)(1+x^2+2z^2)} + \frac{4ty\ln(1+x^2+2z^2)}{(1+y^2)^2} - \frac{4z\sin t}{(1+y^2)(1+x^2+2z^2)}$$

$$= \frac{2x(1+y^2) + 4ty\ln(1+x^2+2z^2)[1+x^2+2z^2] - 4z\sin t\,(1+y^2)}{(1+y^2)^2(1+x^2+2z^2)}$$

$$= \frac{[1+y^2](2x-4z\sin t) + 4ty\,[\ln(1+x^2+2z^2)](1+x^2+2z^2)}{(1+y^2)^2(1+x^2+2z^2)}$$

$$= \frac{[1+(1-t^2)^2](2t-4\cos t\sin t) + 4t(1-t^2)[\ln(1+t^2+2\cos^2 t)](1+t^2+2\cos^2 t)}{[1+(1-t^2)^2]^2(1+t^2+2\cos^2 t)}$$

Note (a) = (b)

52.

$$f(x,y) = \frac{x^2}{2+\cos y} \qquad \vec{c}(t) = (e^t, e^{-t})$$

(a) $$(f \circ \vec{c})(t) = \frac{e^{2t}}{2+\cos(e^{-t})}$$

$$\therefore (f \circ \vec{c})'(t) = \frac{[2+\cos(e^{-t})]\,2e^{2t} - e^{2t}[-\sin(e^{-t})(-e^{-t})]}{[2+\cos(e^{-t})]^2}$$

$$= \frac{2e^{2t}[2+\cos(e^{-t})] - e^{t}\sin(e^{-t})}{[2+\cos(e^{-t})]^2}$$

(b) Chain rule: $D(f \circ \vec{c})(t) = Df(\vec{c}(t)) \cdot D\vec{c}(t)$

$$D\vec{c}(t) = \begin{bmatrix} e^t \\ -e^{-t} \end{bmatrix} \qquad Df(x,y) = [f_x \ \ f_y]$$

$$f_x = \frac{2x}{2+\cos y} \qquad f_y = -\frac{x^2}{(2+\cos y)^2}\cdot(-\sin y) = \frac{x^2 \sin y}{(2+\cos y)^2}$$

$$\therefore \nabla f \cdot D\vec{c}(t) = \begin{bmatrix} \frac{2x}{2+\cos y} & \frac{x^2\sin y}{(2+\cos y)^2} \end{bmatrix} \begin{bmatrix} e^t \\ -e^{-t} \end{bmatrix}$$

$$= \frac{2xe^t}{2+\cos y} - \frac{x^2e^{-t}\sin y}{(2+\cos y)^2} = \frac{2xe^t(2+\cos y) - x^2e^{-t}\sin y}{(2+\cos y)^2}$$

$$= \frac{2e^te^t[2+\cos(e^{-t})] - e^{2t}e^{-t}\sin(e^{-t})}{(2+\cos(e^{-t}))^2}$$

$$= \frac{2e^{2t}[2+\cos(e^{-t})] - e^t\sin(e^{-t})}{[2+\cos(e^{-t})]^2}$$

(a) = (b)

53.

Let $y(t) = t$, $\quad h(t) = u(f(t), t) = u(x(t), y(t))$

By chain rule, $h'(t) = u_x \cdot x'(t) + u_y\, y'(t)$

$= u_x u + u_y \cdot 1 = u_x u + u_y.$

But $u_y = u_t$ since this is the derivative of u with respect to the second variable.

$\therefore h'(t) = u_x u + u_t = 0$, so $h(t) = C$, a constant.

$\therefore h(t) = u(f(t), t) = c$, a constant.

54.

For $x = 1$, $u(t) = \sin(1-6t) + \sin(1+6t)$

$\therefore u'(t) = -6\cos(1-6t) + 6\cos(1+6t)$

$\therefore u'(\frac{1}{3}) = -6\cos(1-6\cdot\frac{1}{3}) + 6\cos(1+6\cdot\frac{1}{3})$

$= -6\cos(-1) + 6\cos(3)$

$= \underline{6(\cos 3 - \cos 1)}$

55.

(a) $P = \frac{nRT}{V}$, $V = \frac{nRT}{P}$, $n = \frac{PV}{RT}$, $T = \frac{PV}{nR}$

(b) $\frac{\partial V}{\partial T} = \frac{nR}{P}$, $\frac{\partial T}{\partial P} = \frac{V}{nR}$, $\frac{\partial P}{\partial V} = -\frac{nRT}{V^2}$

$$\therefore \frac{\partial V}{\partial T} \cdot \frac{\partial T}{\partial P} \cdot \frac{\partial P}{\partial V} = \left(\frac{nR}{P}\right)\left(\frac{V}{nR}\right)\left(-\frac{nRT}{V^2}\right)$$

$$= -\frac{nRT}{PV} = -\frac{PV}{PV} = -1$$

56.

$\theta(T, P) =$

$\theta\left(T(x,y,z,t), P(x,y,z,t)\right)$

$\theta(T, P) = (1000)^{0.286} T P^{-0.286}$

(a) $\frac{\partial \theta}{\partial x} = \frac{\partial \theta}{\partial T} \cdot \frac{\partial T}{\partial x} + \frac{\partial \theta}{\partial P} \cdot \frac{\partial P}{\partial x}$

$$= \left(\frac{1000}{P}\right)^{0.286} \cdot T_x - 0.286\,(1000)^{0.286}\, T P^{-1.286} \cdot P_x$$

Similarly, $\dfrac{\partial \theta}{\partial y} = \left(\dfrac{1000}{p}\right)^{0.286} \cdot T_y - 0.286(1000)^{0.286}\, T p^{-1.286} \cdot p_y$

$$\frac{\partial \theta}{\partial z} = \left(\frac{1000}{p}\right)^{0.286} T_z - 0.286(1000)^{0.286}\, T p^{-1.286} p_z$$

$$\frac{\partial \theta}{\partial t} = \left(\frac{1000}{p}\right)^{0.286} T_t - 0.286(1000)^{0.286}\, T p^{-1.286} p_t$$

This follows from chain rule:

Let $H(x,y,z,t) = [T(x,y,z,t), p(x,y,z,t)]$ $\quad H: R^4 \to R^2$

$\theta(T,p): R^2 \to R^1$

$\therefore$ Pot Temp $= \theta \circ H : R^4 \to R^1$

$\therefore D(\theta \circ H) = D\theta \cdot DH$

$$= \begin{bmatrix} \theta_T & \theta_p \end{bmatrix} \begin{bmatrix} T_x & T_y & T_z & T_t \\ p_x & p_y & p_z & p_t \end{bmatrix}$$

$$= \begin{bmatrix} \theta_T T_x + \theta_p p_x & \theta_T T_y + \theta_p p_y & \theta_T T_z + \theta_p p_z & \theta_T T_t + \theta_p p_t \end{bmatrix}$$

The above formulae can be simplified using:

$\dfrac{\theta}{T} = \left(\dfrac{1000}{p}\right)^{0.286}$ and $(1000)^{0.286}\, T p^{-1.286} = \left(\dfrac{1000}{p}\right)^{0.286} T p^{-1}$

$$= \frac{\theta}{T} \cdot T p^{-1} = \frac{\theta}{p}$$

$$\therefore \frac{\partial \theta}{\partial x} = \frac{\theta}{T} T_x - 0.286 \frac{\theta}{p} p_x$$

$$\frac{\partial \theta}{\partial y} = \frac{\theta}{T} T_y - 0.286 \frac{\theta}{p} p_y$$

$$\frac{\partial \theta}{\partial z} = \frac{\theta}{T} T_z - 0.286 \frac{\theta}{p} p_z$$

$$\frac{\partial \theta}{\partial t} = \frac{\theta}{T} T_t - 0.286 \frac{\theta}{p} p_t$$

(b) Given $\frac{\partial \theta}{\partial z} = \frac{\theta}{T}\left(\frac{\partial T}{\partial z} + \frac{g}{c_p}\right)$, then

$$\frac{\partial T}{\partial z} = \frac{T}{\theta}\frac{\partial \theta}{\partial z} - \frac{g}{c_p}, \quad \text{and} \quad \frac{\theta}{T} = \left(\frac{1000}{p}\right)^{0.286}$$

$$\text{so} \quad \frac{T}{\theta} = \left(\frac{p}{1000}\right)^{0.286} > 0$$

$$\therefore \frac{T}{\theta}\left(\frac{\partial \theta}{\partial z}\right) < 0, \text{ and since } c_p > 0, \ \frac{-g}{c_p} < 0$$

$\therefore \frac{\partial T}{\partial z} < 0$, i.e., temperature decreases in the upward direction.

57.

(a) From the formula, given the value of any two of V, P, T, the third can be found.

(b) From $P = \frac{RT}{V-\beta} - \frac{\alpha}{V^2}$, $\left(P + \frac{\alpha}{V^2}\right)\frac{V-\beta}{R} = T$

$\therefore T_P = \frac{V-\beta}{R}$ β, R constants

$P_V = -\frac{RT}{(V-\beta)^2} + \frac{2\alpha}{V^3}$ α, β, R constants

From $PV - \beta P + \alpha\left(\frac{V-\beta}{V^2}\right) = RT$

$$\frac{d}{dT}\left(PV - \beta P + \alpha\left(\frac{V-\beta}{V^2}\right)\right) = \frac{d}{dT}(RT)$$

$$\therefore PV_T + \alpha\left[\frac{V^2(V_T) - (V-\beta)2VV_T}{V^4}\right] = R$$

$$\therefore PV^4 V_T + \alpha V^2 V_T - \alpha(V-\beta)2VV_T = RV^4$$

$$\therefore V_T = \frac{RV^4}{PV^4 + \alpha V^2 - 2\alpha(V-\beta)V} = \frac{RV^3}{PV^3 + \alpha V - 2\alpha(V-\beta)}$$

$$= \frac{R}{P + \frac{\alpha}{V^2} - \frac{2\alpha(V-\beta)}{V^3}}$$ But $P = \frac{RT}{V-\beta} - \frac{\alpha}{V^2}$

$$= \frac{R}{\frac{RT}{(V-\beta)} - \frac{2\alpha(V-\beta)}{V^3}} = \frac{R(V-\beta)V^3}{RTV^3 - 2\alpha(V-\beta)^2}$$

(c) $$T_P \cdot P_V \cdot V_T = \left(\frac{V-\beta}{R}\right)\left(\frac{-RT}{(V-\beta)^2} + \frac{2\alpha}{V^3}\right)\left(\frac{R(V-\beta)V^3}{RTV^3 - 2\alpha(V-\beta)^2}\right)$$

$$= \left(\frac{V-\beta}{R}\right)\left(\frac{-RTV^3 + 2\alpha(V-\beta)^2}{(V-\beta)^2 V^3}\right)\left(\frac{R(V-\beta)V^3}{RTV^3 - 2\alpha(V-\beta)^2}\right)$$

$$= \frac{-RTV^3 + 2\alpha(V-\beta)^2}{RTV^3 - 2\alpha(V-\beta)^2}$$

$$= \underline{\underline{-1}}$$

58.

(a) $\vec{v} = \frac{1}{\sqrt{2}}(1,1)$, a unit vector.

Need to find $\nabla h(-2,-4) \cdot \vec{v}$, a directional derivative

$\nabla h = [-0.0013x, -0.00048y]$

$\therefore \nabla h(-2,-4) = (0.0026, 0.00192)$

$\therefore \nabla h(-2,-4) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{.0026}{\sqrt{2}} + \frac{.00192}{\sqrt{2}} = \frac{.00452}{\sqrt{2}}$

$\therefore$ Increasing at .0032 miles/mile

(b) $\nabla h(-2,-4) = (0.0026, 0.00192)$, in xy-plane.

59.

(a) $f_x = \dfrac{(x^2+y^2)(2x) - (x^2-y^2)(2x)}{(x^2+y^2)^2} = \dfrac{4xy^2}{(x^2+y^2)^2}$

$f_y = \dfrac{(x^2+y^2)(-2y) - (x^2-y^2)(2y)}{(x^2+y^2)^2} = \dfrac{-4yx^2}{(x^2+y^2)^2}$

$\therefore \nabla f(1,1) = (1,-1)$. Let $\vec{v}$ = direction.

$\therefore (1,-1) \cdot (v_1, v_2) = 0$, or $v_1 - v_2 = 0$, $v_1 = v_2$.

$\therefore$ parallel to vector $(1,1)$, or $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ as a unit vector.

(b) In first quadrant, $x_0 > 0$, $y_0 > 0$.

$$\therefore \left(\frac{4x_0 y_0^2}{(x_0^2+y_0^2)^2}, \frac{-4y_0 x_0^2}{(x_0^2+y_0^2)^2} \right) \cdot (v_1, v_2) = 0 \Rightarrow$$

$$4x_0 y_0^2 v_1 - 4y_0 x_0^2 v_2 = 0, \quad x_0 y_0^2 v_1 = y_0 x_0^2 v_2$$

$$y_0 v_1 = x_0 v_2, \text{ so } \left(v_1, v_1 \frac{y_0}{x_0}\right)$$

or parallel to $v_1\left(1, \frac{y_0}{x_0}\right)$, or (x_0, y_0).

Making it a unit vector, $\underline{\frac{1}{\sqrt{x_0^2+y_0^2}}(x_0, y_0)}$

(c) The directional derivative tangent to a level curve is 0, since the change in value is 0.

From (b), tangents to any level curve at (x,y) are parallel to a line from $(0,0)$ to (x,y).

Looking at $\frac{x^2-y^2}{x^2+y^2} = c$, if $c \neq 0$, then $(0,0)$ is not part of level curve.

$\therefore$ Level curves are half lines from origin

Example: if $\frac{x^2-y^2}{x^2+y^2} = c$, then if (x_0, y_0) is a point on the curve, so is (ax_0, ay_0), $a \neq 0$.

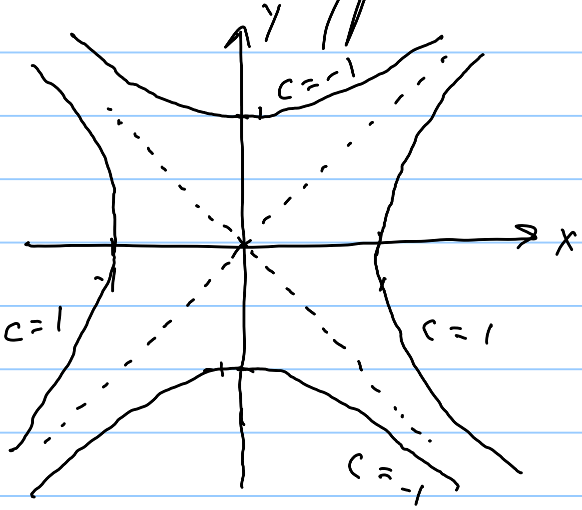
$\therefore$ The level curve for $f(x,y) = c$ is the line $a(x_0, y_0)$, a line through the origin but not containing the origin, so, 2 half-lines.

60.

(a) $\frac{d}{dx}\left(x^2 - y(x)^2\right) = 2x - 2y(x) \cdot y' = \frac{d}{dx}(c) = 0$

$\therefore 2x = 2y(x) \cdot y'$, $y' = \frac{x}{y(x)}$, or $\frac{dy}{dx} = \frac{x}{y}$

(b) These are hyperbolas.



When $y=0$, slope is vertical line.

When $c=0$, level curves are $y=x$, $y=-x$, asymptotes to the hyperbolas.

61.

Let $h(x,y) : R^2 \to R^1$ be $h(x,y) = \frac{x+y}{xy}$

$\therefore$ h is differentiable for $x \neq 0, y \neq 0$.

Let $j(x,y) = xy$, so j is differentiable.

$\therefore u = g(x,y) = j(x,y) \cdot (f \circ h)(x,y)$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial j}{\partial x}(x,y) \cdot (f \circ h)(x,y) + j(x,y) \cdot \frac{\partial}{\partial x}(f \circ h)(x,y)$$

$$\frac{\partial j}{\partial x} = y \qquad \frac{\partial}{\partial x}(f \circ h)(x,y) = f'(h(x,y)) \cdot \frac{\partial h}{\partial x}(x,y)$$

$$= f'(h(x,y)) \cdot \frac{(xy) - y}{(xy)^2}$$

$$\therefore \frac{\partial u}{\partial x} = y f\left(\frac{x+y}{xy}\right) + xy f'\left(\frac{x+y}{xy}\right)\left[\frac{x-1}{x^2 y}\right]$$

$$= y f\left(\frac{x+y}{xy}\right) + \left(\frac{x-1}{x}\right) f'\left(\frac{x+y}{xy}\right)$$

Similarly, $\frac{\partial u}{\partial y} = \frac{\partial j}{\partial y} f\left(\frac{x+y}{xy}\right) + xy f'\left(\frac{x+y}{xy}\right) \cdot \frac{xy - x}{(xy)^2}$

$$= x f\left(\frac{x+y}{xy}\right) + \left(\frac{y-1}{y}\right) f'\left(\frac{x+y}{xy}\right)$$

$$\therefore x^2 \frac{\partial u}{\partial x} = x^2 y f\left(\frac{x+y}{xy}\right) + x(x-1) f'\left(\frac{x+y}{xy}\right)$$

$$y^2 \frac{\partial u}{\partial y} = xy^2 f\left(\frac{x+y}{xy}\right) + y(y-1) f'\left(\frac{x+y}{xy}\right)$$

$$\therefore x^2 \frac{\partial u}{\partial x} - y^2 \frac{\partial u}{\partial y} = (x^2 y - xy^2) f\left(\frac{x+y}{xy}\right) + \left[x^2 - x - y^2 + y\right] f'\left(\frac{x+y}{xy}\right)$$

$$= (x-y)\, xy\, f\left(\frac{x+y}{xy}\right) + \left[x^2 - x - y^2 + y\right] f'\left(\frac{x+y}{xy}\right)$$

$$= (x-y)\, u + \left[x^2 - x - y^2 + y\right] f'\left(\frac{x+y}{xy}\right)$$

$$= u \left[(x-y) + \frac{\left[x^2 - x - y^2 + y\right] f'\left(\frac{x+y}{xy}\right)}{xy f\left(\frac{x+y}{xy}\right)} \right]$$

$$= u\, G(x,y)$$

where $G(x,y) = \left[(x-y) + \frac{(x^2 - x - y^2 + y)}{xy} \frac{f'\left(\frac{x+y}{xy}\right)}{f\left(\frac{x+y}{xy}\right)} \right]$

62.

(a) $\frac{\partial g}{\partial x} = F'(f(x,y)) \cdot \frac{\partial f}{\partial x}$

$\frac{\partial g}{\partial y} = F'(f(x,y)) \cdot \frac{\partial f}{\partial y}$

$\therefore \nabla g = F'(f(x,y)) \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial x} \right) = F'(f(x,y)) \nabla f$

$\therefore \nabla g$ is a multiple of ∇f, and so is parallel.

(b) The level curves have a similar shape, in that the tangents at any point (x,y), are parallel.

(tangents of a level curve are where directional derivative is 0).

From (a), if $g(x,y) = F(f(x,y))$, then gradients are parallel, so directional derivatives will be parallel.

$\therefore$ Look for $\lambda(x,y) = \frac{1}{F'(f(x,y))}$