4.1 Acceleration and Newton's Second Law

Note Title 8/16/2016

1.

$\vec{r}'(t) = -\sin t\,\hat{i} + 2\cos 2t\,\hat{j}$ = velocity vector

$\therefore\ \underline{\vec{r}'(0) = 2\hat{j}}$

$\vec{r}''(t) = -\cos t\,\hat{i} - 4\sin 2t\,\hat{j}$ = acceleration vector

$\therefore\ \vec{r}''(0) = \underline{-\hat{i}}$

Tangent line at $t=0$: $\vec{r}(0) + s\vec{v}(0)$, $s \in \mathbb{R}$

$\therefore\ \vec{r}(0) = [1, 0]$ $\quad \vec{v}(0) = \vec{r}'(0) = [0, 2]$

$\therefore\ \underline{\ell(s)} = [1,0] + s[0,2] = \underline{[1, 2s]} = \underline{\hat{i} + 2s\hat{j}}$

2.

$\vec{c}'(t) = [\sin t + t\cos t, \cos t - t\sin t, \sqrt{3}]$ = velocity vector

$\therefore\ \underline{\vec{c}'(0) = [0, 1, \sqrt{3}]} = \underline{\hat{j} + \sqrt{3}\,\hat{k}}$

$\vec{c}''(t) = [\cos t + \cos t - t\sin t, \sin t - \sin t - t\cos t, 0]$

$= [2\cos t - t\sin t, -t\cos t, 0] =$ acceleration vector

$\therefore \vec{c}\,''(0) = [2,0,0] = 2\hat{i}$

Tangent line at $t=0$: $\vec{c}(0) + s\vec{v}(0)$, $s \in R$

$\vec{c}(0) = (0,0,0) \quad \vec{v}(0) = \vec{c}\,'(0) = (0,1,\sqrt{3})$

$\therefore \ell(s) = s(0,1,\sqrt{3}) = s\hat{j} + s\sqrt{3}\hat{k}$

3.

$\vec{r}\,'(t) = (\sqrt{2}, e^{t}, -e^{-t}) =$ velocity vector

$\therefore \vec{r}\,'(0) = (\sqrt{2}, 1, -1) = \sqrt{2}\hat{i} + \hat{j} - \hat{k}$

$\vec{r}\,''(t) = (0, e^{t}, e^{-t}) =$ acceleration vector

$\therefore \vec{r}\,''(0) = (0,1,1) = \hat{j} + \hat{k}$

Tangent line at $t=0$: $\vec{r}(0) + s\vec{v}(0)$, $s \in R$

$\vec{r}(0) = (0,1,1) \quad \vec{v}(0) = \vec{c}\,'(0) = (\sqrt{2}, 1, -1)$

$\therefore \ell(s) = (0,1,1) + s(\sqrt{2},1,-1)$

$= s\sqrt{2}\hat{i} + (1+s)\hat{j} + (1-s)\hat{k}$

4.

$\vec{c}\,'(t) = (1, 1, t^{1/2})$ = velocity vector

$\therefore \underline{\vec{c}\,'(9)} = \underline{(1, 1, 3)} = \underline{\hat{i} + \hat{j} + 3\hat{k}}$

$\vec{c}\,''(t) = (0, 0, \frac{1}{2}t^{-\frac{1}{2}})$ = acceleration vector

$\therefore \underline{\vec{c}\,''(9)} = \underline{(0, 0, \frac{1}{6})} = \underline{\frac{1}{6}\hat{k}}$

Tangent line at $t = 9$: $\vec{c}(9) + s\,\vec{v}(9)$, $s \in \mathbb{R}$

$\vec{c}(9) = (9, 9, 18)$ $\quad \vec{v}(9) = \vec{c}\,'(9) = (1, 1, 3)$

$\therefore \underline{\ell(s)} = \underline{(9, 9, 18) + s(1, 1, 3)}$

$= \underline{(9+s)\hat{i} + (9+s)\hat{j} + (18+3s)\hat{k}}$

5.

$\vec{c}_1\,'(t) = (e^{t}, \cos t, 3t^2)$ $\quad \vec{c}_2\,'(t) = (-e^{-t}, -\sin t, -6t^2)$

$\therefore \underline{\vec{c}\,'(t) + \vec{c}_2\,'(t)} = \underline{(e^{t} - e^{-t}, \cos t - \sin t, -3t^2)}$

$$\vec{c}_1(t) + \vec{c}_2(t) = (e^{t} + e^{-t},\ \sin t + \cos t,\ -t^3)$$

$$\therefore \underline{\frac{d}{dt}\left[\vec{c}_1(t) + c_2(t)\right]} = \underline{(e^{t} - e^{-t},\ \cos t - \sin t,\ -3t^2)}$$

6.

$$\vec{c}_1(t) \cdot \vec{c}_2(t) = (e^{t} \cdot e^{-t}) + (\sin t \cdot \cos t) + (t^3 \cdot (-2t^3))$$

$$= 1 + \sin t \cos t - 2t^6$$

$$\therefore \underline{\frac{d}{dt}\left[\vec{c}_1(t) \cdot \vec{c}_2(t)\right]} = \underline{\cos^2 t - \sin^2 t - 12t^5}$$

$$\vec{c}_1{}'(t) = (e^{t},\ \cos t,\ 3t^2) \qquad \vec{c}_2{}'(t) = (-e^{-t},\ -\sin t,\ -6t^2)$$

$$\therefore \vec{c}_1{}'(t) \cdot \vec{c}_2(t) = (e^{t}, \cos t, 3t^2) \cdot (e^{-t}, \cos t, -2t^3)$$

$$= 1 + \cos^2 t - 6t^5$$

$$\vec{c}_1(t) \cdot \vec{c}_2{}'(t) = (e^{t}, \sin t, t^3) \cdot (-e^{-t}, -\sin t, -6t^2)$$

$$= -1 - \sin^2 t - 6t^5$$

$$\therefore \underline{\vec{c}_1{}'(t) \cdot \vec{c}_2(t) + \vec{c}_1(t) \cdot \vec{c}_2{}'(t)} = \underline{\cos^2 t - \sin^2 t - 12t^5}$$

7.

$$\vec{c}_1(t) \times \vec{c}_2(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ e^{t} & \sin t & t^3 \\ e^{-t} & \cos t & -2t^3 \end{vmatrix}$$

$$= \left[-2t^3 \sin t - t^3 \cos t,\ t^3 e^{-t} + 2t^3 e^{t},\ e^{t}\cos t - e^{-t}\sin t\right]$$

$$\therefore \frac{d}{dt}\left\{\vec{c}_1(t) \times c_2(t)\right\} =$$

$$\boxed{\begin{aligned} &\left(-6t^2 \sin t - 2t^3 \cos t - 3t^2 \cos t + t^3 \sin t,\right. \\ &\ 3t^2 e^{-t} - t^3 e^{-t} + 6t^2 e^{t} + 2t^3 e^{t}, \\ &\left.\ e^{t}\cos t - e^{t}\sin t + e^{-t}\sin t - e^{-t}\cos t\right) \end{aligned}} \quad [1]$$

$$\vec{c}_1\,'(t) \times \vec{c}_2(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ e^{t} & \cos t & 3t^2 \\ e^{-t} & \cos t & -2t^3 \end{vmatrix}$$

$$= \left(-2t^3 \cos t - 3t^2 \cos t,\ 3t^2 e^{-t} + 2t^3 e^{t},\ e^{t}\cos t - e^{-t}\cos t\right)$$

$$\vec{c}_1(t) \times \vec{c}_2\,'(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ e^{t} & \sin t & t^3 \\ -e^{-t} & -\sin t & -6t^2 \end{vmatrix}$$

$$= \left(-6t^2 \sin t + t^3 \sin t,\ -e^{-t} t^3 + 6t^2 e^{t},\ -e^{t}\sin t + e^{-t}\sin t\right)$$

$$\therefore \vec{c}_1\,'(t) \times \vec{c}_2(t) + \vec{c}_1(t) \times \vec{c}_2\,'(t) =$$

$$\boxed{\begin{aligned}&\big(-2t^3\cos t-3t^2\cos t-6t^2\sin t+t^3\sin t,\\&3t^2e^{-t}+2t^3e^{t}-e^{-t}t^3+6t^2e^{t},\\&e^{t}\cos t-e^{-t}\cos t-e^{t}\sin t+e^{-t}\sin t\big)\end{aligned}}$$ [2]

[1] = [2] $\therefore \frac{d}{dt}\left[\vec{c}_1\times\vec{c}_2\right]=\vec{c}_1{}'\times\vec{c}_2+\vec{c}_1\times\vec{c}_2{}'$

8.

(a) $2\vec{c}_2(t)=(2e^{-t},\ 2\cos t,\ -4t^3)$

$\vec{c}_1(t)=(e^{t},\ \sin t,\ t^3)$

$\therefore 2\vec{c}_2+\vec{c}_1=(2e^{-t}+e^{t},\ 2\cos t+\sin t,\ -3t^3)$

$\therefore \vec{c}_1\cdot(2\vec{c}_2+\vec{c}_1)=2+e^{2t}+2\sin t\cos t+\sin^2 t-3t^6$

$\therefore \frac{d}{dt}\left[\vec{c}_1\cdot(2\vec{c}_2+\vec{c}_1)\right]=$

$\underline{\underline{2e^{2t}+2\cos^2 t-2\sin^2 t+2\sin t\cos t-18t^5}}$

(b) $\frac{d}{dt}\left[2\vec{c}_2(t)\right]=(-2e^{-t},\ -2\sin t,\ -12t^2)$

$\vec{c}_1{}'(t)=(e^{t},\ \cos t,\ 3t^2)$

$$\therefore \frac{d}{dt}\left[2\vec{c_2}(t)+\vec{c_1}(t)\right] =$$

$$(-2e^{-t}+e^{t}, -2\sin t+\cos t, -9t^2)$$

$$\therefore \vec{c_1}(t)\cdot \frac{d}{dt}\left[2\vec{c_2}(t)+\vec{c_1}(t)\right] =$$

$$(e^{t}, \sin t, t^3)\cdot(-2e^{-t}+e^{t}, -2\sin t+\cos t, -9t^2)$$

$$= -2+e^{2t}-2\sin^2 t+\sin t\cos t-9t^5 \quad [1]$$

$$\vec{c_1}'(t)\cdot\left[2\vec{c_2}(t)+\vec{c_1}(t)\right] =$$

$$(e^{t}, \cos t, 3t^2)\cdot(2e^{-t}+e^{t}, 2\cos t+\sin t, -3t^3)$$

$$= 2+e^{2t}+2\cos^2 t+\cos t\sin t-9t^5 \quad [2]$$

$$[2]+[1] = \vec{c_1}'\cdot\left[2\vec{c_2}+\vec{c_1}\right]+\vec{c_1}\cdot\frac{d}{dt}\left[2\vec{c_2}+\vec{c_1}\right]$$

$$= \underline{2e^{2t}+2\cos^2 t-2\sin^2 t+2\sin t\cos t-18t^5}$$

(a) = (b)

9.

$$\vec{c}'(t) = (-a\sin t, a\cos t, b)$$

$$\vec{c}''(t) = (-a\cos t, -a\sin t, 0) = \text{acceleration vector}$$

z-component is 0. $\therefore \vec{c}''$ parallel to xy-plane.

10.

Let $\vec{b}(t) = \{b_1(t), b_2(t), b_3(t)\}$

$\vec{c}(t) = \{c_1(t), c_2(t), c_3(t)\}$

$\therefore \vec{b}(t) \cdot \vec{c}(t) = b_1(t)c_1(t) + b_2(t)c_2(t) + b_3(t)c_3(t)$

$\therefore \frac{d}{dt}\left[\vec{b}(t) \cdot \vec{c}(t)\right] = b_1'c_1 + b_1c_1' + b_2'c_2 + b_2c_2' + b_3'c_3 + b_3c_3'$ [1]

$$\frac{d}{dt}\vec{b}(t) \cdot \vec{c}(t) = (b_1', b_2', b_3') \cdot (c_1, c_2, c_3)$$
$$= b_1'c_1 + b_2'c_2 + b_3'c_3$$

$$\vec{b}(t) \cdot \frac{d}{dt}\vec{c}(t) = (b_1, b_2, b_3) \cdot (c_1', c_2', c_3')$$
$$= b_1c_1' + b_2c_2' + b_3c_3'$$

$\therefore \frac{d}{dt}\vec{b}(t) \cdot \vec{c}(t) + \vec{b}(t) \cdot \frac{d}{dt}\vec{c}(t) =$

$b_1'c_1 + b_1c_1' + b_2'c_2 + b_2c_2' + b_3'c_3 + b_3c_3'$ [2]

[1] = [2] $\quad \therefore \frac{d}{dt}\left[\vec{b} \cdot \vec{c}\right] = \frac{d}{dt}\vec{b} \cdot \vec{c} + \vec{b} \cdot \frac{d}{dt}\vec{c}$

11.

(a) $\vec{c}\,'(t) = (-\sin t, \cos t, 1)$. $\therefore \vec{c}\,'(t) \neq \vec{0}$.

$\therefore \vec{c}(t)$ is regular

(b) $\vec{c}\,'(t) = (3t^2, 5t^4, -\sin t)$. For $t=0$, $\vec{c}\,'(t) = 0$.

$\therefore$ Not regular

(c) $\vec{c}\,'(t) = (2t, e^t, 3)$. $\therefore \vec{c}\,'(t) \neq \vec{0}$

$\therefore$ Regular

12.

$\vec{v}(t) = (c_1, c_2, 6t + c_3)$, c_1, c_2, c_3 are constants.

$\vec{v}(0) = (1, 1, -2)$, $\therefore c_1 = 1$, $c_2 = 1$, $c_3 = -2$.

$\therefore \vec{v}(t) = (1, 1, 6t - 2)$.

$\therefore \vec{c}(t) = (t + k_1, t + k_2, 3t^2 - 2t + k_3)$, k_i constants.

$\vec{c}(0) = (3, 4, 0) = (k_1, k_2, k_3)$. $\therefore k_1 = 3, k_2 = 4, k_3 = 0$.

$$\therefore \vec{c}(t) = (t+3,\ t+4,\ 3t^2-2t)$$

13.

$$\vec{a}(t) = (2,-6,-4) \Rightarrow \vec{v}(t) = (2t+c_1, -6t+c_2, -4t+c_3)$$

$$\vec{v}(0) = (-5,1,3) = (c_1, c_2, c_3)$$

$$\therefore \vec{v}(t) = (2t-5,\ -6t+1,\ -4t+3)$$

$$\therefore \vec{r}(t) = (t^2-5t+k_1, -3t^2+t+k_2, -2t^2+3t+k_3),$$

k_i constants.

$$\vec{r}(0) = (6,-2,1) = (k_1, k_2, k_3)$$

$$\therefore \vec{r}(t) = (t^2-5t+6, -3t^2+t-2, -2t^2+3t+1)$$

Cross yz plane when x component $= 0$.

$$\therefore t^2-5t+6 = 0 = (t-3)(t-2) \Rightarrow t = 2,3.$$

$$\therefore t=2:\ \vec{r}(2) = \underline{(0,-12,-1)}$$

$$t=3:\ \vec{r}(3) = \underline{(0,-26,-8)}$$

14.

$\vec{a}(t) = (-6, 2, 4) \Rightarrow \vec{v}(t) = (-6t + c_1, 2t + c_2, 4t + c_3)$,

c_i constants

$\vec{v}(0) = (2, -5, 1) = (c_1, c_2, c_3)$

$\therefore \vec{v}(t) = (-6t + 2, 2t - 5, 4t + 1)$

$\therefore \vec{r}(t) = \int \vec{v}(t)\,dt = (-3t^2 + 2t + k_1, t^2 - 5t + k_2, 2t^2 + t + k_3)$,

k_i constants

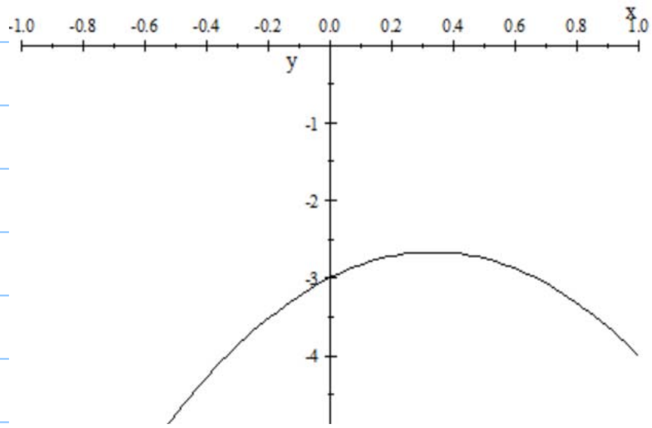
$\vec{r}(0) = (-3, 6, 2) = (k_1, k_2, k_3)$

$\therefore \vec{r}(t) = (-3t^2 + 2t - 3, t^2 - 5t + 6, 2t^2 + t + 2)$

Cross yz plane when x component $= 0$.

$\therefore -3t^2 + 2t - 3 = 0$

$t = \dfrac{-2 \pm \sqrt{4 - 36}}{-6}$. No real solutions.



Problem probably meant crossing xz plane (y component $= 0$, at $t = 2, 3$).

15.

$\vec{r}'(t) = (6, 6t, 3t^2)$, $\vec{a}(t) = r''(t) = (0, 6, 6t)$

at $t=0$, $\vec{a}(0) = (0,6,0)$.

$\therefore \vec{F} = m\vec{a}(0) = (0, 6m, 0) = 6m\hat{j}$

16.

From #1,

$\vec{r}''(t) = -\cos t\,\hat{i} - 4\sin 2t\,\hat{j}$ = acceleration vector

$\therefore \vec{r}''(0) = (-1, 0)$

$\vec{F}(0) = m\vec{r}''(0) = 1g\,(-1,0) = -\hat{i}\ \underline{\underline{\frac{gm\text{-}cm}{sec^2}}}$

17.

Circumference $= 2\pi(3\text{ meter}) = 6\pi$ meters

Speed $= \frac{6\pi}{5}$ meters/sec.

Acceleration $= \frac{v^2}{r} = \frac{(6\pi/5)^2}{(3\text{ meters})} = \frac{36\pi^2}{25}$

$$\therefore F = ma = (2\,kg)\left(\frac{36\pi^2}{75}\right) = \underline{\frac{72}{75}\pi^2 \text{ Newtons}}$$

Using text $\vec{r}(t) = r_0\left(\cos\frac{st}{r_0}, \sin\frac{st}{r_0}\right)$,

assuming $\vec{r}(0) = (r_0, 0)$, here $r_0 = 3$ meters

$$\omega = \frac{s}{r_0} = \frac{2\pi r_0 / 5\text{ sec}}{r_0} = \frac{2\pi}{5}\text{ sec}^{-1}$$

$$\therefore \vec{r}(t) = 3\left(\cos\frac{2\pi}{5}t, \sin\frac{2\pi}{5}t\right)$$

$$\therefore \vec{r}\,'(t) = \frac{6\pi}{5}\left(-\sin\frac{2\pi}{5}t, \cos\frac{2\pi}{5}t\right)$$

$$\vec{r}\,''(t) = \frac{12\pi^2}{25}\left(-\cos\frac{2\pi}{5}t, -\sin\frac{2\pi}{5}t\right)$$

$$\therefore \vec{F}(t) = (2\,kg)\,\vec{r}\,''(t)$$

$$= \underline{-\frac{24}{25}\pi^2\left(\cos\frac{2\pi}{5}t, \sin\frac{2\pi}{5}t\right)}$$

18.

Let $\vec{r}(t) = r_0(\cos\omega t, \sin\omega t)$

Here, $r_0 = 10\,m$, $\omega = \frac{s}{r_0} = \frac{2(2\pi r_0)}{r_0} = 4\pi\text{ sec}^{-1}$

$$\therefore \vec{r}(t) = 10(\cos 4\pi t, \sin 4\pi t)$$

$\therefore \vec{r}'(t) = 40\pi\langle -\sin 4\pi t, \cos 4\pi t\rangle$

$\vec{r}''(t) = -160\pi^2\langle \cos 4\pi t, \sin 4\pi t\rangle$

$\vec{F}(t) = m\,\vec{a}(t) = (4\,Kg)(-160\pi^2)\langle \cos 4\pi t, \sin 4\pi t\rangle$

$= \underline{-640\pi^2\langle \cos 4\pi t, \sin 4\pi t\rangle \text{ Newtons}}$

19.

Let $\vec{r}(t)$ = position of object at time t

$\therefore \vec{r}'(t)$ = velocity, $\vec{r}''(t)$ = acceleration.

Given: $\vec{r}'(t) \cdot \vec{r}''(t) = 0$

$\therefore \vec{r}'(t) \cdot \vec{r}''(t) + \vec{r}''(t) \cdot \vec{r}'(t) = 0$

$\therefore \frac{d}{dt}\left[\vec{r}'(t) \cdot \vec{r}'(t)\right] = 0$

$\therefore \vec{r}'(t) \cdot \vec{r}'(t) = c$, a constant

$\therefore \|\vec{r}'(t)\|^2 = c$, so $\|\vec{r}'(t)\| = \sqrt{c}$, a constant.

$\therefore$ Speed $= \|\vec{r}'(t)\|$ is a constant

20.

A local maximum or minimum for $\|\vec{r}(t)\|$ will also be a local max or min for $\|\vec{r}(t)\|^2$, since both are > 0.

At a local max or min, $\frac{d}{dt}\left(\|\vec{r}(t)\|^2\right) = 0$

$$\therefore \frac{d}{dt}\left(\vec{r}(t) \cdot \vec{r}(t)\right) = 0 \Rightarrow$$

$$2\,\vec{r}(t) \cdot \vec{r}\,'(t) = 0 \Rightarrow \vec{r}(t) \cdot \vec{r}\,'(t) = 0$$

$\therefore \vec{r}(t) \perp \vec{r}\,'(t)$ at a local max or min of $\|\vec{r}(t)\|$.

21.

$$R_{satellite} = 4500 \text{ miles} = 6.436 \times 10^6 \left(\frac{4500}{4000}\right) = 7.24 \times 10^6 \text{ m}$$

$$T^2 = R^3\left(\frac{4\pi^2}{GM}\right) = \frac{(7.24 \times 10^6)^3\, 4\pi^2}{(6.67 \times 10^{-11})(5.98 \times 10^{24})} = 375.6 \times 10^5$$

$$\therefore T = \sqrt{37.56 \times 10^6} = 6.129 \times 10^3 = \underline{6{,}129 \text{ secs}}$$

$$= \underline{102.1 \text{ min.}}$$

22.

(a) Acceleration $= \frac{V^2}{R} = \frac{(2\pi R/T)^2}{R} = \frac{4\pi^2 R}{T^2}$

$$= \frac{4\pi^2(7.24\times10^6\,m)}{(6.129\times10^3\,sec)^2} = \underline{7.61\,m/sec^2}$$

(b) $F = ma$. Need to know mass of satellite.

23.

$$\int \vec{c}\,'(t) = \int (t, e^t, t^2) = \left(\tfrac{1}{2}t^2 + c_1, e^t + c_2, \tfrac{1}{3}t^3 + c_3\right),$$

where c_i are constants.

$$\vec{c}(0) = (0,-5,1) = \left(\tfrac{1}{2}(0)^2 + c_1, e^0 + c_2, \tfrac{1}{3}(0)^3 + c_3\right)$$

$$= (c_1, 1 + c_2, c_3)$$

$\therefore c_1 = 0, c_2 = -6, c_3 = 1$

$$\therefore \underline{\vec{c}(t) = \left(\frac{t^2}{2}, e^t - 6, \frac{t^3}{3} + 1\right)}$$

24.

$$\vec{c}\,''(t) = (0,0,0) \quad \therefore \vec{c}\,'(t) = \int \vec{c}\,''(t) = [c_1, c_2, c_3],$$

where c_i are constants.

$$\therefore \vec{c}(t) = \int \vec{c}\,'(t) = [c_1 t + k_1, c_2 t + k_2, c_3 t + k_3],$$

where c_i and k_i are constants.

If $c_1 = c_2 = c_3 = 0$, then $\vec{c}(t) = [k_1, k_2, k_3]$,

so $\vec{c}(t)$ is a point (no motion over time).

If just one of $c_i \neq 0$, then $[c_1 t + k_1, c_2 t + k_2, c_3 t + k_3]$

describes a line.

25.

(a) Let $\vec{c}(t) = (t, e^t)$ for $t \in \mathbb{R}$ (or $-\infty < t < \infty$)

(b) $y^2 = 1 - 4x^2$, so $y = \pm\sqrt{1-4x^2}$. Can't neatly

describe all the points with one $\vec{c}(t)$.

$\therefore$ Try polar coordinates.

Let $x = \frac{\cos\theta}{2}$, $y = \sin\theta$, $0 \le \theta < 2\pi$

$\therefore 4x^2 + y^2 = 4\frac{\cos^2\theta}{4} + \sin^2\theta = 1$.

$\therefore \vec{c}(\theta) = \underline{\left(\frac{\cos\theta}{2}, \sin\theta\right)}$, $0 \le \theta < 2\pi$

(c) $(0,0,0) + t[(a,b,c) - (0,0,0)] = t(a,b,c)$.

$\therefore \vec{c}(t) = \underline{t(a,b,c)}$, $t \in \mathbb{R}$

(d) $9x^2 + 16y^2 = 4 \Leftrightarrow \frac{9}{4}x^2 + 4y^2 = 1$

$\therefore x = \frac{2}{3}\cos\theta \quad y = \frac{1}{2}\sin\theta$

$\therefore \frac{9}{4}\left(\frac{2}{3}\cos\theta\right)^2 + 4\left(\frac{1}{2}\sin\theta\right)^2 = \cos^2\theta + \sin^2\theta = 1$

$\therefore \vec{c}(\theta) = \underline{\left(\frac{2}{3}\cos\theta, \frac{1}{2}\sin\theta\right)}$, $0 \le \theta < 2\pi$

26.

(a) $\frac{d}{dt}\left[m\vec{c}(t)\times\vec{v}(t)\right] = m\frac{d}{dt}\vec{c}(t)\times\vec{v}(t) + m\vec{c}(t)\times\frac{d}{dt}\vec{v}(t)$

$= m\,\vec{v}(t)\times\vec{v}(t) + m\vec{c}(t)\times\vec{a}(t)$

$= 0 + \vec{c}(t)\times m\,\vec{a}(t)$ since $\vec{v}\times\vec{v} = 0$

$= \vec{c}(t)\times F(\vec{c}(t))$

(b) if $\vec{F}(\vec{c}(t))$ is parallel to $\vec{c}(t)$, then

$\vec{c}(t)\times F(\vec{c}(t)) = 0$, so

$\frac{d}{dt}\left[m\vec{c}(t)\times\vec{v}(t)\right] = 0$, $\therefore$ $\underline{m\vec{c}(t)\times\vec{v}(t) = \text{constant}}$

(c) $\vec{F}(\vec{c}(t))$ is parallel to $\vec{c}(t)$ in planetary motion as gravity is a radial force.

$\therefore$ angular momentum of a planet is constant over time.

27.

From #26, $m\vec{c}(t)\times\vec{v}(t)$ = a constant vector, one that doesn't change size or direction over time.

Let $\vec{L} = m\vec{c}(t) \times \vec{v}(t)$.

Note that $\vec{L} \perp \vec{c}(t)$ for all t, as

$(\vec{a} \times \vec{b}) \perp \vec{a}$ and $(\vec{a} \times \vec{b}) \perp b$ for any $\vec{a}, \vec{b} \neq 0$.

$\therefore$ $\vec{L}$ is the normal to a plane, and since

$\vec{c}(t) \perp \vec{L}$, $\vec{c}(t)$ stays in the plane for all t.

4.2 Arc Length

Note Title 8/23/2016

1.

$$\int_0^{2\pi} \sqrt{(-2\sin t)^2 + (2\cos t)^2 + 1}\, dt$$

$$= \int_0^{2\pi} \sqrt{4(\sin^2 t + \cos^2 t) + 1}\, dt = \int_0^{2\pi} \sqrt{5}\, dt = \sqrt{5}\, t \Big|_0^{2\pi}$$

$$= \underline{2\sqrt{5}\,\pi}$$

2.

$$\int_0^1 \sqrt{0 + (6t)^2 + (3t^2)^2}\, dt = \int_0^1 \sqrt{36t^2 + 9t^4}\, dt$$

$$= \int_0^1 3t\sqrt{t^2 + 4}\, dt \quad \text{since } \sqrt{t^2} = t \text{ for } 0 \le t \le 1$$

$$= \frac{3}{2}\int_0^1 2t\sqrt{t^2 + 4}\, dt = \frac{3}{2}\left[\frac{2}{3}(t^2 + 4)^{3/2}\right]\Big|_0^1$$

$$= \frac{3}{2}\left[\frac{2}{3}(t^2+4)^{3/2}\right]\Big|_0^1 \quad \text{since } \frac{d}{dt}(t^2+4) = 2t$$

$$= (5)^{3/2} - (4)^{3/2} = \underline{5^{3/2} - 8} \approx 3.18$$

3.

$$\int_0^1 \sqrt{(3\cos 3t)^2 + (-3\sin 3t)^2 + (3t^{\frac{1}{2}})^2}\, dt$$

$$= \int_0^1 \sqrt{9(\cos^2 3t + \sin^2 3t) + 9t}\, dt$$

$$= 3\int_0^1 \sqrt{t+1}\, dt = 3\left[\frac{2}{3}(t+1)^{3/2}\right]\Big|_0^1$$

$$= 2\left[2^{3/2} - 1^{3/2}\right] = 2(2\sqrt{2} - 1) = \underline{4\sqrt{2} - 2}$$

4.

$$\int_1^2 \sqrt{1 + (\sqrt{2}\,t^{\frac{1}{2}})^2 + t^2}\, dt = \int_1^2 \sqrt{t^2 + 2t + 1}\, dt$$

$$\int_1^2 \sqrt{(t+1)^2}\, dt = \int_1^2 (t+1)\, dt \quad \text{since } \sqrt{(t+1)^2} = t+1 \text{ for } 1 \le t \le 2$$

$$= \frac{1}{2}(t+1)^2 \Big|_1^2 = \frac{9}{2} - \frac{4}{2} = \underline{\frac{5}{2}}$$

5.

$$\int_1^2 \sqrt{1 + 1 + (2t)^2}\, dt = \int_1^2 \sqrt{2 + 4t^2}\, dt$$

$$= 2\int_1^2 \sqrt{t^2 + \frac{1}{2}}\, dt \quad \left(\text{using } \frac{1}{2}\left[x\sqrt{x^2+a^2} + a^2 \log\left(x + \sqrt{x^2+a^2}\right)\right] + C\right)$$

$$= \left[t\sqrt{t^2 + \frac{1}{2}} + \frac{1}{2}\log\left(t + \sqrt{t^2 + \frac{1}{2}}\right)\right]_1^2$$

$$= 2\sqrt{4 + \frac{1}{2}} + \frac{1}{2}\log\left(2 + \sqrt{4 + \frac{1}{2}}\right) - \left[\sqrt{1\tfrac{1}{2}} + \frac{1}{2}\log\left(1 + \sqrt{1\tfrac{1}{2}}\right)\right]$$

$$= 2\frac{3}{\sqrt{2}} + \frac{1}{2}\log\left(2 + \frac{3}{\sqrt{2}}\right) - \sqrt{\frac{3}{2}} - \frac{1}{2}\log\left(1 + \sqrt{\frac{3}{2}}\right)$$

$$= \frac{6}{\sqrt{2}} - \frac{\sqrt{3}}{\sqrt{2}} + \frac{1}{2}\log\left[\frac{2 + \frac{3}{\sqrt{2}}}{1 + \frac{\sqrt{3}}{\sqrt{2}}}\right]$$

$$= \frac{6-\sqrt{3}}{\sqrt{2}} + \frac{1}{2}\log\left(\frac{2\sqrt{2}+3}{\sqrt{2}+\sqrt{3}}\right)$$

6.

$$\vec{c}\,'(t) = (1, \sin t + t\cos t, \cos t - t\sin t)$$

$$\therefore \int_0^{\pi} \sqrt{1 + (\sin t + t\cos t)^2 + (\cos t - t\sin t)^2}\, dt$$

$$= \int_0^{\pi} \sqrt{1 + (\sin^2 t + 2t\sin t\cos t + t^2\cos^2 t) + (\cos^2 t - 2t\sin t\cos t + t^2\sin^2 t)}\, dt$$

$$= \int_0^{\pi} \sqrt{1 + (\sin^2 t + \cos^2 t) + t^2(\cos^2 t + \sin^2 t)}\, dt$$

$$= \int_0^{\pi} \sqrt{t^2 + 2}\, dt$$

using: $\int \sqrt{x^2+a^2}\,dx = \frac{1}{2}\left[x\sqrt{x^2+a^2} + a^2\log\left(x+\sqrt{x^2+a^2}\right)\right] + C$

$$= \frac{1}{2}\left[t\sqrt{t^2+2} + 2\log\left(t+\sqrt{t^2+2}\right)\right]\Big|_0^{\pi}$$

$$= \frac{1}{2}\left[\pi\sqrt{\pi^2+2} + 2\log\left(\pi+\sqrt{\pi^2+2}\right) - 2\log\sqrt{2}\right]$$

7.

Break up into $-1 \le t \le 0$ and $0 \le t \le 1$

$-1 \le t \le 0$: $\vec{c}(t) = (t, -t)$ $\therefore$ $\vec{c}\,'(t) = (1, -1)$

$0 \le t \le 1$: $\vec{c}(t) = (t, t)$ $\therefore$ $\vec{c}\,'(t) = (1, 1)$

$$\therefore \int_{-1}^{0} \sqrt{1^2 + (-1)^2}\, dt + \int_{0}^{1} \sqrt{1^2 + 1^2}\, dt$$

$$= \sqrt{2}\, t \Big|_{-1}^{0} + \sqrt{2}\, t \Big|_{0}^{1} = \sqrt{2} + \sqrt{2} = \underline{2\sqrt{2}}$$

8.

$\vec{c}\,'(t) = (R - R\cos t, R\sin t)$

$$\therefore \int_{0}^{2\pi} \sqrt{(R - R\cos t)^2 + (R\sin t)^2}\, dt$$

$$= \int_{0}^{2\pi} R\sqrt{(1 - 2\cos t + \cos^2 t + \sin^2 t)}\, dt$$

$$= R\int_0^{2\pi}\sqrt{2-2\cos t}\,dt = \sqrt{2}R\int_0^{2\pi}\sqrt{1-\cos t}\,dt$$

$$= \sqrt{2}R\int_0^{2\pi}\sqrt{2}\sqrt{\frac{1-\cos t}{2}}\,dt = 2R\int_0^{2\pi}\sqrt{\frac{1-\cos t}{2}}\,dt$$

$$= 2R\int_0^{2\pi}\sin\frac{t}{2}\,dt \qquad \text{since } \sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}}$$

$$\text{for } 0 \le \theta \le 2\pi$$

$$= 2R\left[-2\cos\frac{t}{2}\right]\Bigg|_0^{2\pi} = 2R\left[2-(-2)\right] = 8R$$

$\therefore$ Length of one arch $= 8R$

Diameter of rolling circle $= 2R$

$\therefore$ Length of one arch $= 4 \times$ Diameter of rolling circle

9.

(a) $\vec{c}(t) = \vec{p} + t(\vec{q}-\vec{p})$, $0 \le t \le 1$

$$= (1,2,0) + t\left[(0,1,-1)-(1,2,0)\right]$$

$$= (1,2,0) + t(-1,-1,-1) = \underline{(1-t,\ 2-t,\ -t),\ 0 \le t \le 1}$$

(b) $\vec{c}\,'(t) = (-1, -1, -1)$

$$\therefore \int_0^1 \sqrt{(-1)^2 + (-1)^2 + (-1)^2}\, dt = \sqrt{3}\int_0^1 dt = \underline{\sqrt{3}}$$

(c) $\vec{p} - \vec{q} = (1, 2, 0) - (0, 1, -1) = (1, 1, 1)$

$$\therefore \|\vec{p} - \vec{q}\| = \sqrt{1^2 + 1^2 + 1^2} = \underline{\sqrt{3}}$$

10.

$$\vec{c}\,'(t) = \left(\frac{1}{\sqrt{t}} \cdot \frac{1}{2}\frac{1}{\sqrt{t}}, \sqrt{3}, 3t\right) = \left(\frac{1}{2t}, \sqrt{3}, 3t\right)$$

$$\therefore \int_1^2 \sqrt{\left(\frac{1}{2t}\right)^2 + (\sqrt{3})^2 + (3t)^2}\, dt$$

$$= \int_1^2 \sqrt{\frac{1}{4t^2} + 3 + 9t^2}\, dt = \int_1^2 \sqrt{\frac{1 + 12t^2 + 36t^4}{4t^2}}\, dt$$

$$= \int_1^2 \sqrt{\left(\frac{6t^2+1}{2t}\right)^2}\, dt = \int_1^2 \frac{6t^2+1}{2t}\, dt$$

since $\frac{6t^2+1}{2t} > 0$ for $1 \le t \le 2$

$$= \int_1^2 3t\, dt + \int_1^2 \frac{1}{2t}\, dt$$

$$= \frac{3}{2}t^2 \Big|_1^2 + \frac{1}{2}\log t \Big|_1^2$$

$$= \left(6 - \frac{3}{2}\right) + \frac{1}{2}(\log 2 - 0)$$

$$= \underline{\underline{\frac{9}{2} + \frac{1}{2}\log 2}}$$

11.

Note: The curves are connected at $t = 2\pi$

$$\vec{c}_1{}'(t) = (-2\sin t, 2\cos t, 1)$$

$$\vec{c}_2{}'(t) = (0, 1, 1)$$

$$\therefore \int_0^{2\pi} \sqrt{(-2\sin t)^2 + (2\cos t)^2 + 1}\, dt + \int_{2\pi}^{4\pi} \sqrt{(0^2 + 1^2 + 1^2)}\, dt$$

$$= \int_0^{2\pi} \sqrt{4\sin^2 + 4\cos^2 + 1}\, dt + \int_{2\pi}^{4\pi} \sqrt{2}\, dt$$

$$= \int_0^{2\pi} \sqrt{5}\, dt + \sqrt{2}\, t \Big|_{2\pi}^{4\pi}$$

$$= \sqrt{5}\, t \Big|_0^{2\pi} + (4\sqrt{2}\pi - 2\sqrt{2}\pi)$$

$= 2\sqrt{5}\pi + 2\sqrt{2}\pi = \underline{2\pi(\sqrt{5} + \sqrt{2})}$

12.

$(0,0,0)$ corresponds to $t=0$, $(\pi, 0, -\pi)$ to $t=\pi$

$\vec{c}\,'(t) = (1, \sin t + t\cos t, \cos t - t\sin t)$

$$\therefore \int_0^{\pi} \sqrt{1^2 + (\sin t + t\cos t)^2 + (\cos t - t\sin t)^2}\, dt$$

$$= \int_0^{\pi} \sqrt{1 + (\sin^2 t + 2t\sin t\cos t + t^2\cos^2 t) + (\cos^2 t - 2t\sin t\cos t + t^2\sin^2 t}\, dt$$

$$= \int_0^{\pi} \sqrt{1 + \sin^2 t + \cos^2 t + t^2(\cos^2 t + \sin^2 t)}\, dt$$

$$= \int_0^{\pi} \sqrt{t^2 + 2}\, dt$$

using: $\int \sqrt{x^2 + a^2}\, dx = \frac{1}{2}\left[x\sqrt{x^2 + a^2} + a^2 \log\left(x + \sqrt{x^2 + a^2}\right)\right] + C$

$$= \frac{1}{2}\left[t\sqrt{t^2+2} + 2\log\left(t + \sqrt{t^2+2}\right)\right]\Bigg|_0^{\pi}$$

$$= \frac{1}{2}\left[\pi\sqrt{\pi^2+2} + 2\log\left(\pi+\sqrt{\pi^2+2}\right) - 2\log\sqrt{2}\right]$$

Note: same as problem #6

13.

(2, 1, 0) corresponds to $t = 1$
(4, 4, log 2) corresponds to $t = 2$

$$\vec{c}\,'(t) = \left(2, 2t, \frac{1}{t}\right).$$

$$\therefore \int_1^2 \sqrt{2^2 + (2t)^2 + \left(\frac{1}{t}\right)^2}\, dt$$

$$= \int_1^2 \sqrt{\frac{4t^2 + 4t^4 + 1}{t^2}}\, dt = \int_1^2 \frac{\sqrt{(2t^2+1)^2}}{t}\, dt$$

$$= \int_1^2 \frac{2t^2+1}{t}\, dt = \int_1^2 2t\, dt + \int_1^2 \frac{1}{t}\, dt$$

$$= t^2\Big|_1^2 + \log t\Big|_1^2 = (4-1) + (\log 2 - \log 1)$$

$$= \underline{3 + \log 2}$$

14.

(a) $\vec{\alpha}'(x) = (\sinh(x), \cosh(x), 1)$

$$\therefore \quad s(t) = \int_0^t \sqrt{\sinh^2(x) + \cosh^2(x) + 1}\, dx$$

$$= \int_0^t \sqrt{\sinh^2(x) + \cosh^2(x) + [\cosh^2 x - \sinh^2 x]}\, dx$$

$$= \int_0^t \sqrt{2\cosh^2 x}\, dx = \int_0^t \sqrt{2} \cosh x \, dx$$ using $\cosh x > 0$ for all x.

$$= \sqrt{2} \sinh x \Big|_0^t = \sqrt{2} \sinh t$$

$$\therefore \quad \underline{\underline{s(t) = \sqrt{2} \sinh t}}$$

(b) $\vec{\beta}'(x) = (-\sin x, \cos x, 1)$

$$\therefore \quad s(t) = \int_0^t \sqrt{(-\sin x)^2 + \cos^2 x + 1}\, dx$$

$$= \int_0^t \sqrt{2}\, dx = \sqrt{2}\, t$$

$\therefore$ $\underline{s(t) = \sqrt{2}\, t}$

15.

(a) For all $t \in [a,b]$, $\vec{d}(s) = \vec{d}(\alpha(t)) = \vec{c}(t)$

$\therefore$ On $[a,b]$, for each $t \in [a,b]$, $\vec{d} = \vec{c}$

(b)

Let l = the arc length

$$\therefore\ l = \int_{\alpha(a)}^{\alpha(b)} \|\vec{d}'(s)\|\, ds = \int_a^b \|\vec{d}'(\alpha(t))\alpha'(t)\|\, dt$$

$$= \int_a^b \|\vec{c}'(t)\|\, dt$$

since $\vec{c}'(t) = \frac{d}{dt}\vec{d}(\alpha(t))$

$= \vec{d}'(\alpha(t)) \cdot \alpha'(t)$

(c)

$$\frac{d}{dt}\vec{d}(s) = \frac{d}{ds}\vec{d}(s) \cdot \frac{d}{dt}s(t) = \frac{d}{ds}\vec{d}(s) \cdot \alpha'(t)$$

$$\therefore \frac{\frac{d}{dt}\vec{d}(s)}{\alpha'(t)} = \frac{d}{ds}\vec{d}(s)$$

But $\vec{d}(s) = \vec{c}(t)$. $\therefore \frac{d}{dt}\vec{d}(s) = \vec{c}\,'(t)$

$$\therefore \frac{\vec{c}\,'(t)}{\alpha'(t)} = \frac{d}{ds}\vec{d}(s) \qquad [1]$$

But $\frac{d}{dt}\alpha(t) = \frac{d}{dt}\int_a^t \|\vec{c}\,'(\tau)\|\,d\tau = \|\vec{c}\,'(t)\|$

$\therefore$ [1] becomes $\dfrac{\vec{c}\,'(t)}{\|\vec{c}\,'(t)\|} = \dfrac{d}{ds}\vec{d}(s) \qquad [2]$

$\dfrac{\vec{c}\,'(t)}{\|\vec{c}\,'(t)\|}$ is a unit vector, so $\left\|\dfrac{\vec{c}\,'(t)}{\|\vec{c}\,'(t)\|}\right\| = 1$

$\therefore$ From [2], $\underline{\left\|\dfrac{d}{ds}\vec{d}(s)\right\| = 1}$

16.

(a) Since $\|\vec{T}(t)\| = 1$, then $\|\vec{T}(t)\|^2 = 1$,

so $\vec{T}(t) \cdot \vec{T}(t) = 1$.

$$\therefore \frac{d}{dt}\left\{\vec{T}(t) \cdot \vec{T}(t)\right\} = 0$$

$$\therefore \vec{T}'(t) \cdot \vec{T}(t) + \vec{T}(t) \cdot \vec{T}'(t) = 2\vec{T}'(t) \cdot \vec{T}(t) = 0$$

$$\therefore \vec{T}'(t) \cdot \vec{T}(t) = 0$$

(b)
$$\vec{T}'(t) = \frac{d}{dt}\left[\frac{\vec{c}'(t)}{\|\vec{c}'(t)\|}\right]$$

$$= \frac{\|\vec{c}'(t)\|\,\vec{c}''(t) - \vec{c}'(t) \cdot \frac{d}{dt}\left[\sqrt{\vec{c}'(t) \cdot \vec{c}'(t)}\right]}{\|\vec{c}'(t)\|^2}$$

$$= \frac{\|\vec{c}'(t)\|\,\vec{c}''(t) - \vec{c}'(t)\left[\frac{1}{2}\,\frac{2\left[\vec{c}'(t) \cdot \vec{c}''(t)\right]}{\|\vec{c}'(t)\|}\right]}{\|\vec{c}'(t)\|^2}$$

$$= \underline{\underline{\frac{\|\vec{c}'(t)\|^2\,\vec{c}''(t) - \left[\vec{c}'(t) \cdot \vec{c}''(t)\right]\vec{c}'(t)}{\|\vec{c}'(t)\|^3}}}$$

$$\therefore \vec{T}'(t) = \frac{1}{\|\vec{c}'(t)\|}\left[\vec{c}''(t) - \frac{\vec{c}'(t)\cdot\vec{c}''(t)}{\|\vec{c}'(t)\|^2}\,\vec{c}'(t)\right] \quad [1]$$

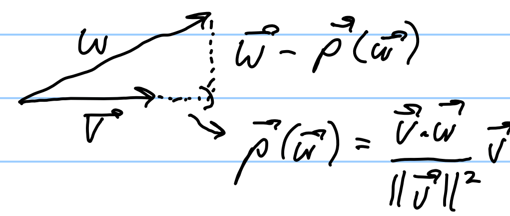
Note: $\dfrac{\vec{c}'(t)\cdot\vec{c}''(t)}{\|\vec{c}'(t)\|^2}\,\vec{c}'(t) = \dfrac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|^2}\,\vec{v}$, the projection of $\vec{w}$ onto $\vec{v}$.

$$\therefore \vec{T}'(t) = \frac{1}{\|\vec{c}'(t)\|}\left[\vec{c}''(t) - \text{projection of } \vec{c}''(t) \text{ onto } \vec{c}'(t)\right]$$

Note: $\vec{w} - \dfrac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|^2}\,\vec{v}$ is $\perp$ to $\vec{v}$, since

$$\vec{v}\cdot\left(\vec{w} - \frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|^2}\,\vec{v}\right) = \vec{v}\cdot\vec{w} - \frac{(\vec{v}\cdot\vec{w})(\vec{v}\cdot\vec{v})}{\|\vec{v}\|^2}$$

$$= \vec{v}\cdot\vec{w} - \vec{v}\cdot\vec{w} = 0$$



$$\therefore \|\vec{v}\times\vec{w}\| = \|\vec{v}\|\,\|\vec{w}\|\sin\theta$$

$$\therefore \|\vec{v}\times\vec{w}\| = \|\vec{v}\|\,\|\vec{w} - \vec{p}(\vec{w})\|$$

$$\therefore \frac{\|\vec{v}\times\vec{w}\|}{\|\vec{v}\|^2} = \frac{1}{\|\vec{v}\|}\,\|\vec{w} - \vec{p}(\vec{w})\| \quad [2]$$

Letting $\vec{v} = \vec{c}'(t)$, $\vec{w} = \vec{c}''(t)$, [1], [2] become

$$\|\vec{T}'(t)\| = \frac{\|\vec{c}'(t)\times\vec{c}''(t)\|}{\|\vec{c}'(t)\|^2} \quad [3]$$

17.

(a) $\displaystyle\int_a^b \|\vec{c}\,'(s)\|\,ds = \int_a^b 1\,ds = \underline{b-a}$

(b) Not mentioned, $\vec{T}$ is defined as the unit tangent vector. $\therefore$ $\vec{T}(t) = \dfrac{\vec{c}\,'(t)}{\|\vec{c}\,'(t)\|}$, $\vec{T}(s) = \dfrac{\vec{c}\,'(s)}{\|\vec{c}\,'(s)\|}$

But $\|\vec{c}\,'(s)\| = 1$, from (a)

$\therefore$ $\vec{T}(s) = \vec{c}\,'(s)$ $\quad\therefore$ $\dfrac{d}{ds}\vec{T}(s) = \dfrac{d}{ds}\vec{c}\,'(s) = \vec{c}\,''(s)$

$\therefore$ $K = \|T'(s)\| = \|\vec{c}\,''(s)\|$

(c) From 16(b) above, using [3],

$$\frac{\|\vec{T}'(t)\|}{\|\vec{c}\,'(t)\|} = \frac{\|\vec{c}\,'(t)\times\vec{c}\,''(t)\|}{\|\vec{c}\,'(t)\|^3}$$

But $\|\vec{T}'(t)\| = \left\|\dfrac{d}{ds}\vec{T}(s)\cdot\dfrac{ds}{dt}\right\|$

$$= \|\vec{T}'(s)\| \, \|\vec{c}'(t)\| = K \|\vec{c}'(t)\|$$

$$\therefore K = \frac{\|\vec{T}'(t)\|}{\|\vec{c}'(t)\|} = \frac{\|\vec{c}'(t) \times \vec{c}''(t)\|}{\|\vec{c}'(t)\|^3}$$

(d)

Using formula in (c),

$$\vec{c}'(t) = \frac{1}{\sqrt{2}}(-\sin t, \cos t, 1) \quad \therefore \|\vec{c}'(t)\| = \frac{1}{\sqrt{2}}\sqrt{1+1} = 1$$

$$\vec{c}''(t) = \frac{1}{\sqrt{2}}(-\cos t, -\sin t, 0) \quad \therefore \|\vec{c}''(t)\| = \frac{1}{\sqrt{2}}$$

$$\vec{c}'(t) \times \vec{c}''(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\frac{\sin t}{\sqrt{2}} & \frac{\cos t}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{\cos t}{\sqrt{2}} & -\frac{\sin t}{\sqrt{2}} & 0 \end{vmatrix}$$

$$= \left(\frac{\sin t}{2}, -\frac{\cos t}{2}, \frac{\sin^2 t + \cos^2 t}{2} \right) = \frac{1}{2}(\sin t, -\cos t, 1)$$

$$\therefore \|\vec{c}'(t) \times \vec{c}''(t)\| = \frac{1}{2}\sqrt{1+1} = \frac{\sqrt{2}}{2}$$

$$\therefore K = \frac{\frac{\sqrt{2}}{2}}{1} = \underline{\frac{\sqrt{2}}{2}}$$

18.

$$\vec{l}'(t) = \vec{v}, \quad \vec{l}''(t) = \vec{0} \quad \therefore \|\vec{l}'(t)\| = \|\vec{v}\| = 1.$$

From 17(c), $K = \dfrac{\|\vec{l}'(t) \times \vec{l}''(t)\|}{\|\vec{l}'(t)\|^3} = \dfrac{\|\vec{l}'(t) \times \vec{l}''(t)\|}{1}$

$= \|\vec{v} \times \vec{0}\| = \underline{0}$

19.

(a) $\vec{c}'(t) = [-\sin(t), \cos(t)]$

$\therefore \|\vec{c}'(t)\| = \sqrt{\sin^2 t + \cos^2 t} = 1$

$\therefore$ By definition given in 17(a), $\vec{c}(t)$ is parametrized by arc length.

(b) Write $\vec{c}(t) = (\cos t, \sin t, 0)$, so $\vec{c}(t) \in R^3$

By 17(c), $K = \dfrac{\|\vec{c}'(t) \times \vec{c}''(t)\|}{\|\vec{c}'(t)\|} = \|\vec{c}'(t) \times \vec{c}''(t)\|$

From (a), $\|\vec{c}(t)\| = 1$.

$\vec{c}''(t) = [-\cos(t), -\sin(t), 0]$

$\therefore \vec{c}'(t) \times \vec{c}''(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin t & \cos t & 0 \\ -\cos t & -\sin t & 0 \end{vmatrix}$

$$= (0, 0, \sin^2 t + \cos^2 t) = (0,0,1)$$

$$\therefore \underline{K} = \frac{\|(0,0,1)\|}{1} = \underline{\underline{1}}$$

20.

(a) $\|\vec{B}\| = \|\vec{T} \times \vec{N}\| = \|\vec{T}\| \|\vec{N}\| \sin\theta$

But $\|\vec{T}\| = 1$, $\|\vec{N}\| = 1$, and $\vec{N} \perp \vec{T} \Rightarrow \sin\theta = 1$.

$$\therefore \|\vec{B}\| = 1 \Rightarrow \|\vec{B}\|^2 = 1 \Rightarrow \vec{B} \cdot \vec{B} = 1$$

$$\Rightarrow \frac{d}{dt}(\vec{B} \cdot \vec{B}) = 0 \Rightarrow 2 \frac{d\vec{B}}{dt} \cdot B = 0$$

$$\Rightarrow \frac{d\vec{B}}{dt} \cdot \vec{B} = 0.$$

(b) Since $\vec{B} = \vec{T} \times \vec{N}$, $\vec{B} \perp \vec{T}$. $\therefore \vec{B} \cdot \vec{T} = 0$.

$$\therefore \frac{d}{dt}(\vec{B} \cdot \vec{T}) = 0 \Rightarrow \frac{d\vec{B}}{dt} \cdot \vec{T} + \vec{B} \cdot \frac{d\vec{T}}{dt} = 0 \quad [1]$$

But $\frac{d\vec{T}}{dt} = \|\vec{T}'\|\vec{N}$ by definition of $\vec{N}$,

and $\vec{B} = \vec{T} \times \vec{N} \Rightarrow \vec{B} \perp \vec{N} \Rightarrow \vec{B} \perp \|\vec{T}'\|\vec{N}$

$\therefore \vec{B} \cdot \frac{d\vec{T}}{dt} = \vec{B} \cdot (\|\vec{T}'\|\vec{N}) = 0.$

$\therefore$ [1] becomes $\frac{d}{dt}(\vec{B} \cdot \vec{T}) = \frac{d\vec{B}}{dt} \cdot \vec{T} + 0 = 0$

$\therefore \underline{\frac{d\vec{B}}{dt} \cdot \vec{T} = 0}$

(c) (a) + (b) $\Rightarrow \frac{d\vec{B}}{dt} \perp \vec{B}$ and $\frac{d\vec{B}}{dt} \perp \vec{T}$

But $\vec{N} \perp \vec{B}$ and $\vec{N} \perp \vec{T}$.

$\therefore \frac{d\vec{B}}{dt}$ and $\vec{N}$ are parallel.

$\therefore \underline{\frac{d\vec{B}}{dt} = s\vec{N}, \quad s \in R}$

21.

(a) $\vec{c}(s)$ parametrized by arc length $\Rightarrow \|\vec{c}\,'(s)\| = 1$ by definition given in 17(a)

Note that the definitions of $\vec{T}, \vec{N}, \vec{B}$ are irrelevant to the letter of the parameterization.

i.e., $T(s) = \dfrac{\vec{c}\,'(s)}{\|\vec{c}\,'(s)\|}$ or $\vec{T}(x) = \dfrac{\vec{c}\,'(x)}{\|\vec{c}\,'(x)\|}$

So, all the properties of problems #16, 17, 20 hold irrespective to the letter of the parameter.

Here, since $\|\vec{c}\,(s)\| = 1$, $\vec{T}(s) = \vec{c}\,'(s)$ and

$\vec{T}\,'(s) = \vec{c}\,''(s)$. $\therefore \vec{N} = \dfrac{\vec{T}\,'}{\|\vec{T}\,'\|} = \dfrac{\vec{c}\,''(s)}{\|\vec{c}\,''(s)\|}$

$\therefore \vec{B} = \vec{T} \times \vec{N} \Rightarrow \vec{B}\,' = \vec{T}\,' \times \vec{N} + \vec{T} \times \vec{N}\,'$

$\therefore \vec{B}\,'(s) = \vec{T}\,'(s) \times \vec{N}(s) + \vec{T}(s) \times \vec{N}\,'(s)$

$$= \frac{\vec{c}''(s) \times \vec{c}''(s)}{\|\vec{c}''(s)\|} + \vec{c}'(s) \times \frac{d}{ds}\left[\frac{\vec{c}''(s)}{\|\vec{c}''(s)\|}\right]$$

$$= 0 + \vec{c}'(s) \times \frac{d}{ds}\left[\frac{\vec{c}''(s)}{\|\vec{c}''(s)\|}\right]$$

as $\vec{c}'' \times \vec{c}'' = 0$

$$\therefore \frac{d\vec{B}(s)}{ds} = \vec{c}'(s) \times \frac{d}{ds}\left[\frac{\vec{c}''(s)}{\|\vec{c}''(s)\|}\right]$$

But $\displaystyle \frac{d}{ds}\left(\frac{\vec{c}''(s)}{\|\vec{c}''(s)\|}\right) = \frac{\|\vec{c}''(s)\|\vec{c}'''(s) - \vec{c}''(s)\frac{d}{ds}\|\vec{c}''(s)\|}{\|\vec{c}''(s)\|^2}$

and $\displaystyle \frac{d}{ds}\|\vec{c}''(s)\| = \frac{d}{ds}\sqrt{\vec{c}''(s)\cdot\vec{c}''(s)}$

$$= \frac{1}{2}\frac{2\left[\vec{c}'''(s)\cdot\vec{c}''(s)\right]}{\sqrt{\vec{c}''(s)\cdot\vec{c}''(s)}} = \frac{\vec{c}'''(s)\cdot\vec{c}''(s)}{\|\vec{c}''(s)\|}$$

$$\therefore \frac{d}{ds}\left(\frac{\vec{c}''(s)}{\|\vec{c}''(s)\|}\right) = \frac{\vec{c}'''(s)}{\|\vec{c}''(s)\|} - \left[\frac{\vec{c}'''(s)\cdot\vec{c}''(s)}{\|\vec{c}''(s)\|^3}\right]\vec{c}''(s)$$

$$\therefore \frac{d\vec{B}(s)}{ds} = \vec{c}'(s) \times \left[\frac{\vec{c}'''(s)}{\|\vec{c}''(s)\|} - \left(\frac{\vec{c}'''(s)\cdot\vec{c}''(s)}{\|\vec{c}''(s)\|^3}\right)\vec{c}''(s)\right]$$

$$= \vec{T} \times \left[\frac{\vec{c}'''(s)}{\|\vec{c}''(s)\|} - \left(\frac{\vec{c}'''(s)}{\|\vec{c}''(s)\|}\cdot\frac{\vec{c}''(s)}{\|\vec{c}''(s)\|}\right)\frac{\vec{c}''(s)}{\|\vec{c}''(s)\|}\right]$$

$$= \vec{T} \times \left[\frac{\vec{C}'''(s)}{\|\vec{C}''(s)\|} - \left(\frac{\vec{C}'''(s)}{\|\vec{C}''(s)\|} \cdot \vec{N} \right) \vec{N} \right]$$

$$\therefore \frac{d\vec{B}(s)}{ds} = \vec{T} \times \frac{\vec{C}'''(s)}{\|\vec{C}''(s)\|} - \left(\frac{\vec{C}'''(s)}{\|\vec{C}''(s)\|} \cdot \vec{N} \right) \vec{T} \times N \quad [1]$$

Since $\frac{d\vec{B}}{ds} = -\tau \vec{N}$, $\frac{d\vec{B}}{ds} \cdot \vec{N} = -\tau \vec{N} \cdot \vec{N} = -\tau$

since $\vec{N} \cdot \vec{N} = 1$.

$$\therefore -\frac{d\vec{B}}{ds} \cdot \vec{N} = \tau$$

$\therefore$ Using [1],

$$-\frac{d\vec{B}}{ds} \cdot \vec{N} = -\left[\vec{T} \times \frac{\vec{C}'''(s)}{\|\vec{C}''(s)\|} \right] \cdot \vec{N} + \left(\frac{\vec{C}'''(s)}{\|\vec{C}''(s)\|} \cdot \vec{N} \right) (\vec{T} \times \vec{N}) \cdot \vec{N}$$

But $(\vec{T} \times \vec{N}) \cdot \vec{N} = \vec{N} \cdot (\vec{T} \times \vec{N}) = \begin{vmatrix} n_1 & n_2 & n_3 \\ T_1 & T_2 & T_3 \\ n_1 & n_2 & n_3 \end{vmatrix} = 0$

as 2 rows of the determinant are equal.

$$\therefore \tau = -\frac{d\vec{B}}{ds} \cdot \vec{N} = -\left[\vec{T} \times \frac{\vec{C}'''(s)}{\|\vec{C}''(s)\|} \right] \cdot \vec{N}$$

$$= -\left[\vec{C}'(s) \times \frac{\vec{C}'''(s)}{\|\vec{C}''(s)\|} \right] \cdot \frac{\vec{C}''(s)}{\|\vec{C}''(s)\|}$$

$$= -\frac{1}{\|\vec{C}''(s)\|^2} \left[\vec{C}'(s) \times \vec{C}'''(s) \right] \cdot \vec{C}''(s)$$

$$\therefore \quad \tau = \frac{1}{\|\vec{c}''(s)\|^2} \left[\vec{c}'(s) \times \vec{c}''(s) \right] \cdot \vec{c}'''(s)$$

Using $-(\vec{a} \times \vec{c}) \cdot \vec{b} = (\vec{a} \times \vec{b}) \cdot \vec{c}$

as $$-\begin{vmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$ row exchange

(6)

From $\dfrac{d\vec{B}(s)}{ds} = -\tau \vec{N}(s)$ and $\therefore$ since $\vec{N}(s) \cdot \vec{N}(s) = 1$,

$$\tau = -\frac{d\vec{B}(s)}{ds} \cdot \vec{N}(s)$$

Let $\vec{c}(t)$ be another parametrization s.t.

$$s = \int_a^s \|\vec{c}'(t)\|\, dt, \text{ assuming } \vec{c}(t) \text{ is } C^1.$$

$$\therefore \quad \frac{ds}{dt} = \|\vec{c}'(t)\| \quad \text{and} \quad \frac{d\vec{B}}{dt} = \frac{d\vec{B}}{ds} \cdot \frac{ds}{dt}$$ by chain rule

$$\therefore \quad \tau = -\frac{d\vec{B}/dt}{\|\vec{c}'(t)\|} \cdot \vec{N}(s(t))$$ [0]

$\therefore$ Need to find $\vec{B}$ as a function of t.

Note: $\vec{N}$ is defined as $\dfrac{\vec{T}'(x)}{\|\vec{T}'(x)\|}$, irrespective

of the parametrization of $\vec{T}$ (and $\therefore$ of $\vec{c}$).

$\vec{B} = \vec{T} \times \vec{N}$ From 16(6),

$$\vec{T}'(t) = \frac{\|\vec{c}'(t)\|^2 \vec{c}''(t) - [\vec{c}'(t) \cdot \vec{c}''(t)]\vec{c}'(t)}{\|\vec{c}'(t)\|^3}$$

and from [3] of 16(6),

$$\|\vec{T}'(t)\| = \frac{\|\vec{c}'(t) \times \vec{c}''(t)\|}{\|\vec{c}'(t)\|^2}$$

$$\therefore \vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|} = \frac{\|\vec{c}'(t)\|^2 \vec{c}''(t) - [\vec{c}'(t) \cdot \vec{c}''(t)]\vec{c}'(t)}{\|\vec{c}'(t)\|^3} \cdot \frac{\|\vec{c}'(t)\|^2}{\|\vec{c}'(t) \times \vec{c}''(t)\|}$$

$$\frac{\|\vec{c}'(t)\|}{\|\vec{c}'(t) \times \vec{c}''(t)\|}\vec{c}''(t) - \left(\frac{\vec{c}'(t) \cdot \vec{c}''(t)}{\|\vec{c}'(t) \times \vec{c}''(t)\|}\right)\frac{\vec{c}'(t)}{\|\vec{c}'(t)\|} \quad [1]$$

$$\therefore \vec{B}(t) = \vec{T}(t) \times \vec{N}(t) = \frac{\vec{c}'(t)}{\|\vec{c}'(t)\|} \times \vec{N}(t)$$

$$= \frac{\vec{c}'(t)}{\|\vec{c}'(t)\|} \times \left[\frac{\|\vec{c}'(t)\|\,\vec{c}''(t)}{\|\vec{c}'(t) \times \vec{c}''(t)\|} - \left(\frac{\vec{c}'(t) \cdot \vec{c}''(t)}{\|\vec{c}'(t) \times \vec{c}''(t)\|}\right)\frac{\vec{c}'(t)}{\|\vec{c}'(t)\|}\right]$$

$$\therefore \vec{B}(t) = \frac{\vec{c}'(t) \times \vec{c}''(t)}{\|\vec{c}'(t) \times \vec{c}''(t)\|}, \quad \text{as } \vec{c}'(t) \times \vec{c}'(t) = 0.$$

$$= \vec{c}' \times \frac{\vec{c}''}{\|\vec{c}' \times \vec{c}''\|}$$

$$\therefore \frac{d\vec{B}}{dt} = \frac{\vec{c}'' \times \vec{c}''}{\|\vec{c}' \times \vec{c}''\|} + \vec{c}' \times \frac{d}{dt}\left(\frac{\vec{c}''}{\|\vec{c}' \times \vec{c}''\|}\right)$$

$$= \vec{c}' \times \frac{d}{dt}\left(\frac{\vec{c}''}{\|\vec{c}' \times \vec{c}''\|}\right) \qquad \text{as } \vec{c}'' \times \vec{c}'' = 0$$

$$= \vec{c}' \times \left[\frac{\|\vec{c}' \times \vec{c}''\| \, \vec{c}''' - \vec{c}'' \cdot \frac{d}{dt}(\|\vec{c}' \times \vec{c}''\|)}{\|\vec{c}' \times \vec{c}''\|^2}\right]$$

$$= \frac{\vec{c}' \times \vec{c}'''}{\|\vec{c}' \times \vec{c}''\|} - \frac{\vec{c}' \times \vec{c}''}{\|\vec{c}' \times \vec{c}''\|^2} \cdot \frac{d}{dt}\left[(\vec{c}' \times \vec{c}'') \cdot (\vec{c}' \times \vec{c}'')\right]^{\frac{1}{2}}$$

$$= \frac{\vec{c}' \times \vec{c}'''}{\|\vec{c}' \times \vec{c}''\|} - \frac{\vec{c}' \times \vec{c}''}{\|\vec{c}' \times \vec{c}''\|^2}\left[\frac{2(\vec{c}' \times \vec{c}'') \cdot \frac{d}{dt}(\vec{c}' \times \vec{c}'')}{2\|\vec{c}' \times \vec{c}''\|}\right]$$

$$\therefore \frac{d\vec{B}}{dt} = \frac{\vec{c}' \times \vec{c}'''}{\|\vec{c}' \times \vec{c}''\|} - \frac{\vec{c}' \times \vec{c}''}{\|\vec{c}' \times \vec{c}''\|^3}\left[(\vec{c}' \times \vec{c}'') \cdot (\vec{c}' \times \vec{c}''')\right] \quad [2]$$

$$\text{as } \vec{c}'' \times \vec{c}'' = 0$$

From [1],

$$\vec{N}(t) = \frac{\|\vec{c}'(t)\|}{\|\vec{c}'(t) \times \vec{c}''(t)\|}\vec{c}''(t) - \left(\frac{\vec{c}'(t) \cdot \vec{c}''(t)}{\|\vec{c}'(t) \times \vec{c}''(t)\|}\right)\frac{\vec{c}'(t)}{\|\vec{c}'(t)\|}$$

Now for the horrible task of computing $\frac{d\vec{B}(t)}{dt} \cdot \vec{N}(t)$

Using [1], [2], computing the 2 terms of $\vec{N}$ separately,

$$\frac{d\vec{B}}{dt} \cdot \frac{\|\vec{c}\,'(t)\|}{\|\vec{c}\,'(t) \times \vec{c}\,''(t)\|} \vec{c}\,''(t) =$$

$$\left[\frac{\vec{c}\,' \times \vec{c}\,'''}{\|\vec{c}\,' \times \vec{c}\,''\|} - \frac{\vec{c}\,' \times \vec{c}\,''}{\|\vec{c}\,' \times \vec{c}\,''\|^3} \left[(\vec{c}\,' \times \vec{c}\,'') \cdot (\vec{c}\,' \times \vec{c}\,''') \right] \right] \cdot \frac{\|\vec{c}\,'\| \, \vec{c}\,''}{\|\vec{c}\,' \times \vec{c}\,''\|}$$

$$= \frac{\|\vec{c}\,'\|}{\|\vec{c}\,' \times \vec{c}\,''\|^2} (\vec{c}\,' \times \vec{c}\,''') \cdot \vec{c}\,'' - 0 \, , \text{ as } [\vec{c}\,' \times \vec{c}\,''] \cdot \vec{c}\,'' = 0$$

$$= - \frac{\|\vec{c}\,'\|}{\|\vec{c}\,' \times \vec{c}\,''\|^2} (\vec{c}\,' \times \vec{c}\,'') \cdot \vec{c}\,''' \, , \qquad [3]$$

as $(\vec{a} \times \vec{c}) \cdot \vec{b} = -(\vec{a} \times \vec{b}) \cdot \vec{c}$

$$\frac{d\vec{B}}{dt} \cdot -\left(\frac{\vec{c}\,' \cdot \vec{c}\,''}{\|\vec{c}\,' \times \vec{c}\,''\|} \right) \frac{\vec{c}\,'}{\|\vec{c}\,'\|} =$$

$$-\left[\frac{\vec{c}\,' \times \vec{c}\,'''}{\|\vec{c}\,' \times \vec{c}\,''\|} - \frac{\vec{c}\,' \times \vec{c}\,''}{\|\vec{c}\,' \times \vec{c}\,''\|^3} \left[(\vec{c}\,' \times \vec{c}\,'') \cdot (\vec{c}\,' \times \vec{c}\,''') \right] \right] \cdot \left(\frac{\vec{c}\,' \cdot \vec{c}\,''}{\|\vec{c}\,' \times \vec{c}\,''\|} \right) \frac{\vec{c}\,'}{\|\vec{c}\,'\|}$$

$= -0 + 0 = 0$, as $(\vec{c}\,' \times \vec{c}\,''') \cdot \vec{c}\,' = 0$ (first term)
and $(\vec{c}\,' \times \vec{c}\,'') \cdot \vec{c}\,' = 0$ (second term).

$= 0 \qquad [4]$

$\therefore$ From [3], [4]: $\displaystyle \frac{d\vec{B}}{dt} \cdot \vec{N} = \frac{-\|\vec{c}\,'\|}{\|\vec{c}\,' \times \vec{c}\,''\|^2} (\vec{c}\,' \times \vec{c}\,'') \cdot \vec{c}\,''' \quad [5]$

$\therefore$ From [0], [5], $\tau = -\dfrac{d\vec{B}(t)/dt}{\|\vec{c}\,'(t)\|} \cdot \vec{N}(t)$

$$= -\frac{1}{\|\vec{c}\,'\|}\left[-\frac{\|\vec{c}\,'\|}{\|\vec{c}\,'\times\vec{c}\,''\|^2}(\vec{c}\,'\times\vec{c}\,'')\cdot\vec{c}\,'''\right]$$

$$= \frac{\left[\vec{c}\,'(t)\times\vec{c}\,''(t)\right]\cdot\vec{c}\,'''(t)}{\|\vec{c}\,'(t)\times\vec{c}\,''(t)\|^2}$$

(c)

$$\vec{c}\,' = \frac{1}{\sqrt{2}}(-\sin(t), \cos(t), 1)$$

$$\vec{c}\,'' = \frac{1}{\sqrt{2}}(-\cos(t), -\sin(t), 0)$$

$$\vec{c}\,''' = \frac{1}{\sqrt{2}}(\sin(t), -\cos(t), 0)$$

$$\vec{c}\,'\times\vec{c}\,'' = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\frac{\sin}{\sqrt{2}} & \frac{\cos}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{\cos}{\sqrt{2}} & -\frac{\sin}{\sqrt{2}} & 0 \end{vmatrix} = \left(0 + \frac{\sin}{2},\ -\frac{\cos}{2} - 0,\ \frac{\sin^2}{2} + \frac{\cos^2}{2}\right)$$

$$= \frac{1}{2}(\sin(t), -\cos(t), 1)$$

$$\therefore \|\vec{c}\,'\times\vec{c}\,''\|^2 = \frac{\sin^2}{4} + \frac{\cos^2}{4} + \frac{1}{4} = \frac{1}{2}$$

$$(\vec{c}\,'\times\vec{c}\,'')\cdot\vec{c}\,''' = \frac{1}{2\sqrt{2}}(\sin^2 + \cos^2 + 0) = \frac{\sqrt{2}}{4}$$

From (b) above,

$$\tau = \frac{(\vec{c}\,' \times \vec{c}\,'') \cdot \vec{c}\,'''}{\|\vec{c}\,' \times \vec{c}\,''\|^2} = \frac{\frac{\sqrt{2}}{4}}{\frac{1}{2}} = \underline{\underline{\frac{\sqrt{2}}{2}}}$$

22.

Let $\vec{c}(t) = [f(t), g(t), h(t)]$.

If $\vec{c}(t)$ lies in a plane, then one of the components is zero.

$\therefore$ The same component will be zero for $\vec{c}\,'(t)$, $\vec{c}\,''(t)$, and $\vec{c}\,'''(t)$.

From #21(6), $\tau = \frac{[\vec{c}\,'(t) \times \vec{c}\,''(t)] \cdot \vec{c}\,'''(t)}{\|\vec{c}\,'(t) \times \vec{c}\,''(t)\|^2}$

$$= \frac{1}{\|\vec{c}\,' \times \vec{c}\,''\|^2} \begin{vmatrix} f'(t) & g'(t) & h'(t) \\ f''(t) & g''(t) & h''(t) \\ f'''(t) & g'''(t) & h'''(t) \end{vmatrix}$$

The same component being $0 \Rightarrow$ one column of $(\vec{c}\,' \times \vec{c}\,'') \cdot \vec{c}\,'''$ will be 0, so the entire determinant evaluates to 0. $\therefore \underline{\tau = 0}$

Alternatively, $\vec{B} = \vec{T} \times \vec{N}$. $\therefore \|\vec{B}\| = \|\vec{T}\| \, \|\vec{N}\| \sin\theta$

By definition, $\|\vec{T}\| = 1$, $\|\vec{N}\| = 1$. And if $\vec{c}$ is in a plane, $\vec{T} = \dfrac{\vec{c}\,'}{\|\vec{c}\,'\|}$ is in that plane,

and $\therefore$ so is $\vec{N} = \dfrac{\vec{T}\,'}{\|\vec{T}\,'\|}$. Since $\vec{T} \perp \vec{N}$, $\sin\theta = 1$.

$\therefore \|\vec{B}\| = 1$, so magnitude is constant.

Direction of $\vec{B}$ is always $\perp$ $\vec{T}$ and $\vec{N}$, so $\vec{B}$ is always $\perp$ to plane of $\vec{c}$.

$\therefore$ $\vec{B}$ never changes, so $\dfrac{d\vec{B}}{ds} = 0 = -\tau \vec{N}$.

Assuming $\vec{N} \neq 0$, then $\tau = 0$.

Note: $\vec{N}$ is only defined when $\vec{T}\,' \neq 0$, and $\vec{N} = 0 \Rightarrow \vec{T}\,' = 0$, so torsion not defined when $\vec{N} = 0$

23.

(a) Unit speed $\Rightarrow \|\vec{c}\,'(s)\| = 1$. $\therefore \vec{T}(s) = \dfrac{\vec{c}\,'(s)}{\|\vec{c}\,'(s)\|} = \vec{c}\,'(s)$

$\kappa = \| \vec{T}'(s) \|$ by definition

$$\therefore \vec{N}(s) = \frac{\vec{T}'(s)}{\| \vec{T}'(s) \|} = \frac{\vec{T}'(s)}{\kappa}$$

$$\therefore \underline{\vec{T}'(s) = \kappa \vec{N}(s)} \qquad [1]$$

Since $\vec{T}, \vec{N}, \vec{B}$ are mutually orthogonal and $\vec{B} = \vec{T} \times \vec{N}$ by definition,

Then $\vec{N} \times \vec{B} = \vec{N} \times (\vec{T} \times \vec{N}) = (\vec{N} \cdot \vec{N})\vec{T} - (\vec{N} \cdot \vec{T})\vec{N}$
$= (1)\vec{T} - (0)\vec{N} = \vec{T}$

$\therefore \vec{T} = \vec{N} \times \vec{B}$

Also, $\vec{B} \times \vec{T} = (\vec{T} \times \vec{N}) \times \vec{T} = (\vec{T} \cdot \vec{T})\vec{N} - (\vec{N} \cdot \vec{T})\vec{T}$
$= (1)\vec{N} - (0)\vec{T} = \vec{N}$

$\therefore \vec{N} = \vec{B} \times \vec{T}$

$$\therefore \frac{d\vec{N}}{ds} = \frac{d\vec{B}}{ds} \times \vec{T} + \vec{B} \times \frac{d\vec{T}}{ds}$$

$$= (-\tau \vec{N}) \times \vec{T} + \vec{B} \times (\kappa \vec{N})$$

using definition of τ and using [1]

$$= -\tau(-\vec{B}) + \kappa(\vec{B} \times \vec{N})$$

$$= \tau \vec{B} - \kappa \vec{T}$$

$$\therefore \frac{d\vec{N}}{ds} = -\kappa\vec{T} + \tau\vec{B} \qquad [2]$$

By definition, $\frac{d\vec{B}}{ds} = -\tau\vec{N}$ [3]

(5)

Since $\vec{T}, \vec{N}, \vec{B}$ form a basis, There are scalars a, b, c s.t. $\vec{\omega} = a\vec{T} + b\vec{N} + c\vec{B}$

$\vec{\omega} \times \begin{pmatrix} \vec{T} \\ \vec{N} \\ \vec{B} \end{pmatrix}$ is interpreted as $\begin{pmatrix} \vec{\omega} \times \vec{T} \\ \vec{\omega} \times \vec{N} \\ \vec{\omega} \times \vec{B} \end{pmatrix}$

Also,

$$\frac{d}{ds}\begin{pmatrix} \vec{T} \\ \vec{N} \\ \vec{B} \end{pmatrix} = \begin{pmatrix} \frac{d\vec{T}}{ds} \\ \frac{d\vec{N}}{ds} \\ \frac{d\vec{B}}{ds} \end{pmatrix} = \begin{pmatrix} \kappa\vec{N} \\ -\kappa\vec{T} + \tau\vec{B} \\ -\tau\vec{N} \end{pmatrix}$$

$$\therefore \vec{\omega} \times \vec{T} = (a\vec{T} + b\vec{N} + c\vec{B}) \times \vec{T}$$

$$= a\vec{T} \times \vec{T} + b\vec{N} \times \vec{T} + c\vec{B} \times \vec{T}$$

$$= \vec{0} + (-b\vec{B}) + (c\vec{N})$$

$\therefore$ For $\vec{w} \times \vec{T} = \kappa \vec{N}$, let $b=0$, $c=\kappa$

$$\therefore \vec{w} = a\vec{T} + \kappa \vec{B}$$

$$\vec{w} \times \vec{N} = (a\vec{T} + \kappa\vec{B}) \times \vec{N} = a\vec{T} \times \vec{N} + \kappa \vec{B} \times \vec{N}$$

$$= a\vec{B} + \kappa(-\vec{T})$$

For $\vec{w} \times \vec{N} = -\kappa\vec{T} + \tau\vec{B}$, let $a = \tau$

$$\therefore \vec{w} = \tau\vec{T} + \kappa\vec{B}$$

$$\vec{w} \times \vec{B} = (\tau\vec{T} + \kappa\vec{B}) \times \vec{B} = \tau\vec{T} \times \vec{B} + \kappa\vec{B} \times \vec{B}$$

$$= \tau(-\vec{N}) + \vec{0}$$

$$= -\tau N$$

This is consistent with $\frac{d\vec{B}}{ds} = -\tau\vec{N}$

$$\therefore \vec{w} = \underline{\tau\vec{T} + \kappa\vec{B}}$$

24.

The "speed" along path AC is zero. i.e., $x' = 0$, $y' = 0$, $z' = 0$.

$\therefore$ path will just be

$$\sqrt{-0^2 - 0^2 - 0^2 + c^2 t^2} = \sqrt{c^2 t^2} = ct$$

For person going from $A \to B$, path is:

$$\sqrt{-(x_1')^2 - (y_1')^2 - (z_1')^2 + c^2 t_1^2} < \sqrt{c^2 t_1^2} = ct_1$$

and path from $B \to C$ is:

$$\sqrt{-(x_2')^2 - (y_2')^2 - (z_2')^2 + c^2 t_2^2} < \sqrt{c^2 t_2^2} = ct_2$$

$\therefore$ proper time (AB) + proper time (BC) $< ct_1 + ct_2$
$= c(t_1 + t_2)$
$= ct$

as $t = t_1 + t_2$ from perspective of person on AC path.

$\therefore$ Proper time (AB) + Proper time (BC) < Proper time (AC)

25.

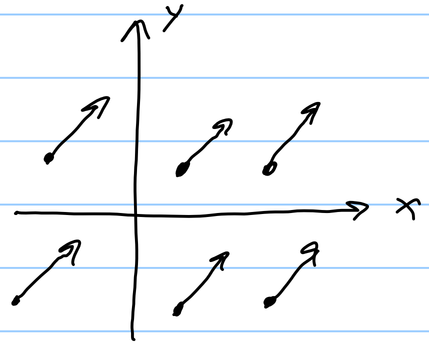
For the first three endpoints along the path, by triangle inequality, $C_0 \to C_2 < C_0 \to C_1 + C_1 \to C_2$.

For the next point, C_3, $C_0 \to C_3 < C_0 \to C_2 + C_2 \to C_3$

$$< C_0 \to C_1 + C_1 \to C_2 + C_2 \to C_3.$$

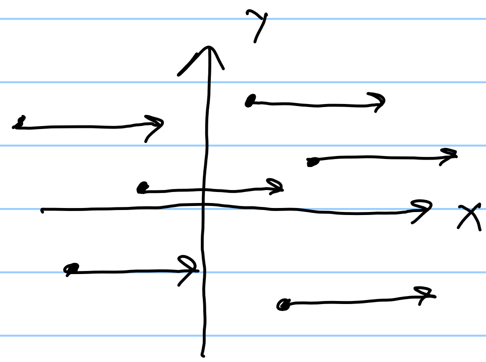
Finally, $C_0 \to C_n < C_0 \to C_1 + C_1 \to C_2 + \dots + C_{n-1} \to C_n$

1.



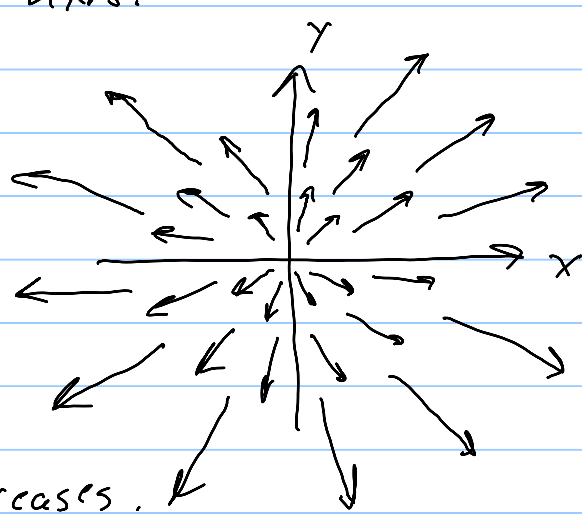
Every point in xy-plane has same vector, length of $\sqrt{2^2+2^2} = 2\sqrt{2}$, angle $45°$ w/ x-axis

2.



Every point in xy-plane has same vector, length of $\sqrt{4^2+0^2} = 4$, $0°$ angle with x-axis.

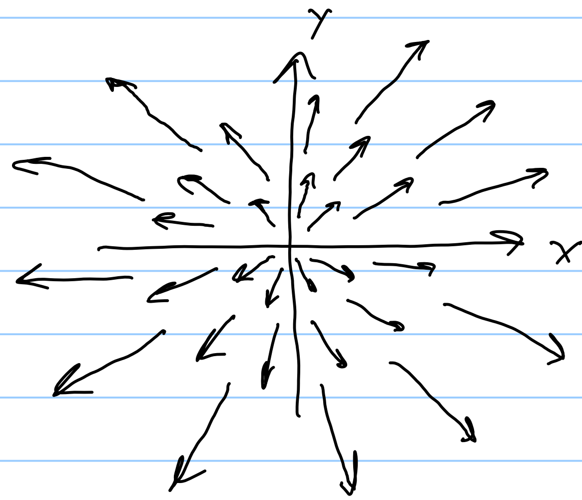
3.



Each point radially pointed outward, magnitude increases as distance from origin increases.

Each vector is parallel to the line between the origin and the base of the vector.

4.



Each point is a reflection, via the y-axis. Magnitude increases with distance from origin. Vectors parallel to line through origin + base of vector. Field looks identical to #3.

5.

Consider axis points:

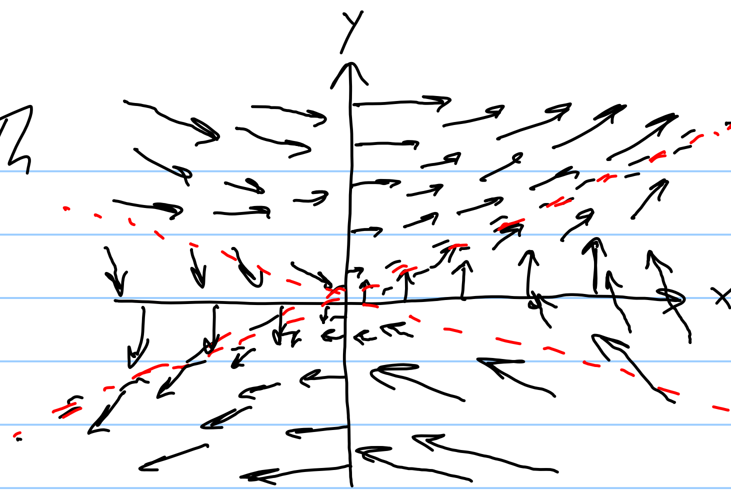
x-axis: $(2,0) \to (0,2)$, $(4,0) \to (0,4)$ i.e., vertical
$(-2,0) \to (0,-2)$, $(-4,0) \to (0,-4)$ vectors

y-axis: $(0,2) \to (4,0)$, $(0,4) \to (8,0)$ i.e., horizontal
$(0,-2) \to (-4,0)$, $(0,-4) \to (-8,0)$ vectors

Points on $y=x$ line become $(2y, x)$, so angle of less than $45°$ with x-axis.

Points on $y = \frac{1}{2}x$ line go to $y = x$ orientation.
i.e., $(x, \frac{1}{2}x) \to (x, x)$.

Magnitude increases with distance from origin



4th quadrant points:

$(1,-1) \to (-2,1)$, i.e., point toward 3rd quadrant

$(10,-1) \to (-2, 10)$ $\quad$ $(1,-10) \to (-20,1)$

3rd quadrant points: $(-5,5) \to (10,-5)$

$(-1,5) \to (10,-1)$

$(-5,1) \to (2,-10$

$(-1,1) \to (2,-1)$

The field flows around the lines $y = \pm \frac{1}{2} x$

6.

Look at axis points

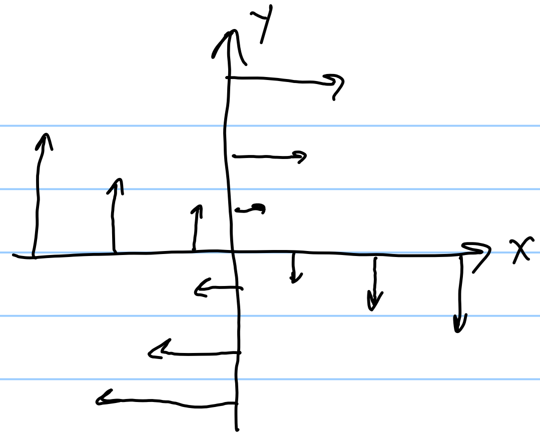
$(1,0) \to (0,-2)$ $\quad$ $(2,0) \to (0,-4)$ $\quad$ $(3,0) \to (0,-6)$

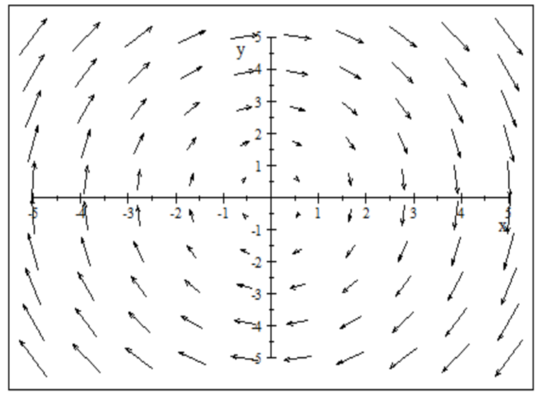
$(0,1) \to (1,0)$ $\quad$ $(0,2) \to (2,0)$ $\quad$ $(0,3) \to (3,0$

$(-1,0) \to (0,2$ $\quad$ $(-2,0) \to (0,4)$ $\quad$ $(-3,0) \to (0,6)$

$(0,-1) \to (-1,0)$ $\quad$ $(0,-2) \to (-2,0)$ $\quad$ $(0,-3) \to (-3,0)$

$\therefore$ Clockwise rotation, magnitude increases with distance from origin



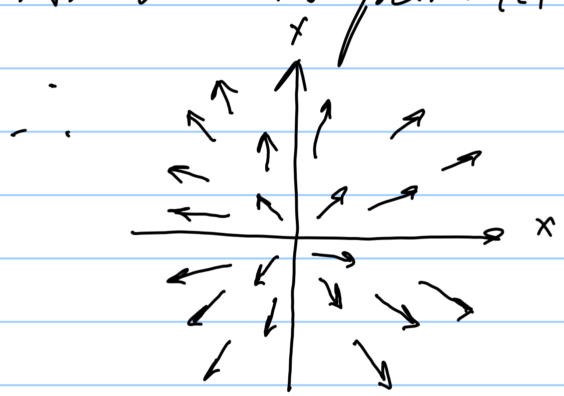


7.

Note $\|\vec{F}\| = 1$, so always a unit vector.

This is much like $f(x,y) = (x,y)$, a radial vector field pointing away from origin.

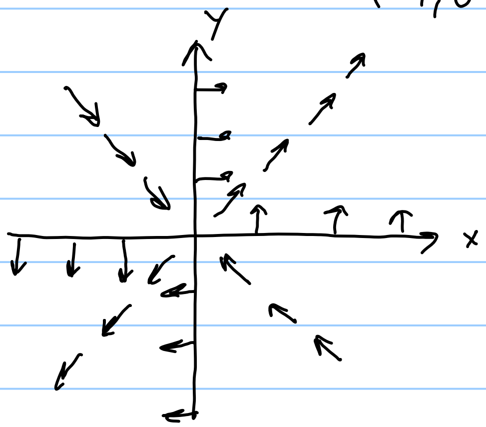
All vectors parallel to line from origin.

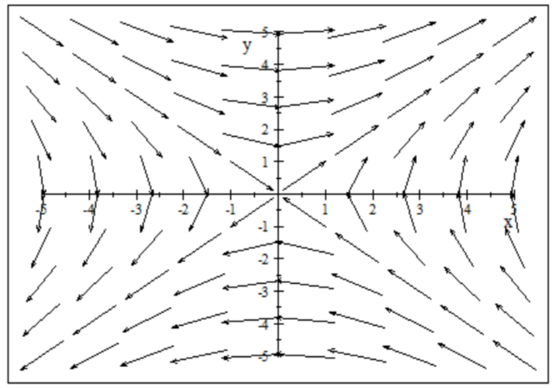


8.

Like #7, $\|\vec{F}\| = 1$, so always a unit vector.

Look at axes: $(1,0)\to(0,1)$ $\quad (0,1)\to(1,0)$
$(-1,0)\to(0,-1)$ $\quad (0,-1)\to(-1,0)$





9.

(a) a radial outward field, magnitude increasing with distance from origin.
$\therefore$ (ii)

(b) At axes: $(1,0)\to(0,-1)$ $\quad (0,1)\to(1,0)$
$(-1,0)\to(0,1)$ $\quad (0,-1)\to(-1,0)$

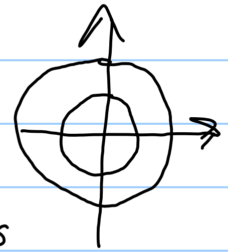
$\therefore$ A clockwise rotation. $\therefore$ (i)

10.

(a) $\|\vec{v}\| = 1$. Like 9(b), a clockwise rotation
Not defined at origin (division by 0).
$\therefore$ (i)

(b) $\|\vec{v}\| = 1$. Like 9(a), a radial outward field.
Not defined at origin (division by zero).
$\therefore$ (ii)

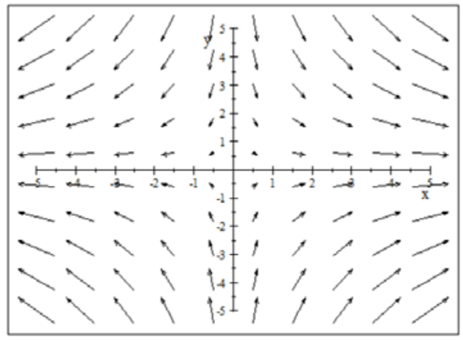
11.



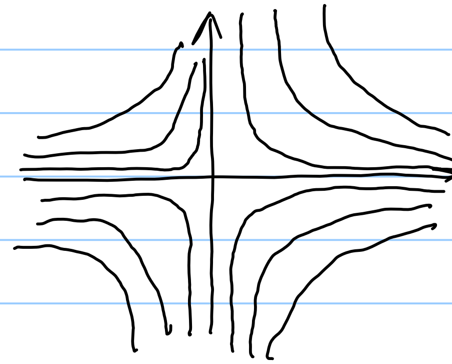
The field looks like 9(b),
a clockwise rotation. $\therefore$ Concentric circles

12.

The field looks like:



$\therefore$ The flow lines look like hyperbolas:

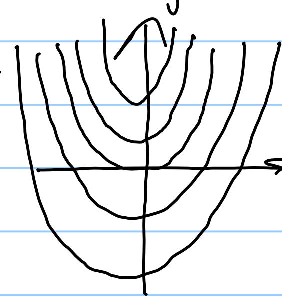


13.

A series of parabolas as $y=x^2$ for (x, x^2)

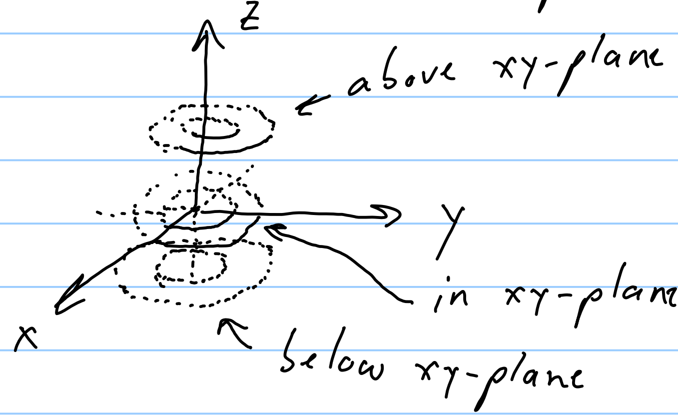
Note for each y of $\vec{F}(x,y)$, the field vector is identical. So for $x=c$, vectors along the vertical line are parallel and equal in magnitude.

$\therefore$ Flow lines are "parallel" parabolas:



14.

Field vectors are parallel to xy-plane. In each plane, the $(y,-x)$ looks like 9(b) above, a clockwise rotation. $\therefore$ Flow lines are concentric circles in a plane, stacked on top of one another.



15.

Need to show $\vec{c}\,'(t) = \vec{F}(\vec{c}(t))$

$$\vec{c}\,'(t) = \left(2e^{2t}, \frac{1}{|t|}, -\frac{1}{t^2}\right)$$

$$\vec{F}(\vec{c}(t)) = \vec{F}\left(e^{2t}, \log|t|, \frac{1}{t}\right) = \left(2e^{2t}, \frac{1}{t}, -\frac{1}{t^2}\right)$$

$\therefore$ For $t > 0$, $\frac{1}{|t|} = \frac{1}{t}$, and $\underline{\vec{F}(\vec{c}(t)) = \vec{c}\,'(t)}$

16.

$$\vec{c}\,'(t) = \left(2t, 2, \frac{1}{2\sqrt{t}}\right)$$

$$\vec{F}(\vec{c}(t)) = \vec{F}\left(t^2, 2t-1, \sqrt{t}\right) = \left(2t-1+1, 2, \frac{1}{2\sqrt{t}}\right)$$

$$= \left(2t, 2, 2\tfrac{1}{\sqrt{t}}\right)$$

$$\therefore \underline{\vec{c}\,'(t) = \vec{F}(\vec{c}(t))}$$

17.

$$\vec{c}\,'(t) = (\cos t, -\sin t, e^t)$$

$$\vec{F}(\vec{c}(t)) = \vec{F}(\sin t, \cos t, e^t) = (\cos t, -\sin t, e^t)$$

$$\therefore \underline{\vec{c}\,'(t) = \vec{F}(\vec{c}(t))}$$

18.

$$\vec{c}\,'(t) = \left(-3t^{-4}, e^t, -\frac{1}{t^2}\right)$$

$$\vec{F}(\vec{c}(t)) = \vec{F}\left(\frac{1}{t^3}, e^t, \frac{1}{t}\right) = \left(-3\left(\frac{1}{t}\right)^4, e^t, -\left(\frac{1}{t}\right)^2\right)$$

$$= \left(-3t^{-4}, e^t, -\frac{1}{t^2}\right)$$

$$\therefore \vec{c}\,'(t) = \vec{F}(\vec{c}(t))$$

19.

$$\vec{c}\,'(t) = \left[-\frac{1}{(1-t)^2}(-1), 0, \frac{(1-t)e^t - e^t(-1)}{(1-t)^2}\right]$$

$$= \left[\frac{1}{(1-t)^2},\ 0,\ \frac{(1-t)e^t + e^t}{(1-t)^2} \right]$$

$$\vec{F}(\vec{c}(t)) = \vec{F}\left(\frac{1}{1-t},\ 0,\ \frac{e^t}{1-t} \right)$$

$$= \left[\frac{1}{(1-t)^2},\ 0,\ \frac{e^t}{1-t} + \frac{e^t}{1-t}\left(\frac{1}{1-t}\right) \right]$$

$$= \left[\frac{1}{(1-t)^2},\ 0,\ \frac{e^t(1-t) + e^t}{(1-t)^2} \right]$$

$$\therefore\ \underline{\underline{\vec{c}\,'(t) = \vec{F}(\vec{c}(t))}}$$

20.

$$\vec{c}\,'(t) = (-a\sin t - b\cos t,\ a\cos t - b\sin t)$$

$$\vec{F}(\vec{c}(t)) = \vec{F}(a\cos t - b\sin t,\ a\sin t + b\cos t)$$

$$= (-a\sin t - b\cos t,\ a\cos t - b\sin t)$$

$$\therefore\ \underline{\underline{\vec{c}\,'(t) = \vec{F}(\vec{c}(t))}}$$

21.

(a) $f_x = yz$, $\therefore f = xyz + g(y,z)$

$f_y = xz \Rightarrow g_y(y,z) = 0 \Rightarrow g(y,z) = h(z)$.

$\therefore f(x,y,z) = xyz + h(z)$

$f_z = xy \Rightarrow h_z = 0$, $\therefore h(z) = c$, a constant.

$\therefore \underline{f(x,y,z) = xyz + c}$, c a constant

(b) $f_x = x \Rightarrow f(x,y,z) = \frac{x^2}{2} + g(y,z)$

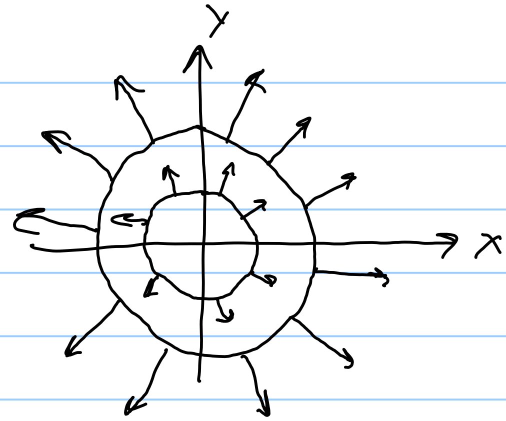
$f_y = y \Rightarrow g_y(y,z) = y \Rightarrow g(y,z) = \frac{y^2}{2} + h(z)$

$\therefore f(x,y,z) = \frac{x^2}{2} + \frac{y^2}{2} + h(z)$.

$f_z = z \Rightarrow h'(z) = z \Rightarrow h(z) = \frac{z^2}{2} + c$, c = constant

$\therefore \underline{f(x,y,z) = \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} + c}$, c a constant

22.



$f: \mathbb{R}^2 \to \mathbb{R}$ $\quad \nabla f = (2x, 2y)$

$\|\nabla f\|$ increases with distance.

Level sets : $x^2+y^2=k$,
$\therefore$ circles

The gradient field vectors are $\perp$ to the level sets.

23.

Let R_e = radius of earth, and so orbit radius of satellite also $= R_e$.

Let m = mass of satellite, M = mass earth.

Energy of satellite at any distance R from earth is $E = \frac{1}{2}mv^2 - \frac{GMm}{R}$

$E = 0$ for escape velocity v_e, so

$$E = 0 = \frac{1}{2}mv_e^2 - \frac{GMm}{R_e}, \quad \frac{1}{2}mv_e^2 = \frac{GMm}{R_e} \quad [1]$$

For orbit, at speed v_o, energy is

$$E = \frac{1}{2} m v_0^2 - \frac{GMm}{R_e}$$

In orbit, $F = ma = \frac{GMm}{R_e^2}$, so $a = \frac{GM}{R_e^2}$

In circular orbit, from p. 221 of text,

$$s^2 = \frac{GM}{R_0}, \text{ or } v_0^2 = \frac{GM}{R_0}.$$

$$\therefore \underline{\frac{1}{2} m v_0^2 = \frac{1}{2} \frac{GMm}{R_0}} \qquad [2]$$

$\therefore$ From [1], [2] $\quad \frac{1}{2} m v_0^2 = \frac{1}{2}\left(\frac{1}{2} m v_e^2\right)$

or, energy for orbit = $\frac{1}{2}$ energy for escape

24.

A decreasing function of t means for $t_1 < t_2$, $V(t_1) > V(t_2)$.

If $\frac{dV}{dt} \le 0$, then V is a decreasing function.

But $\frac{dV}{dt} = \nabla V \cdot \vec{c}\,'(t)$ by chain rule.

And since $\vec{c}(t)$ is a flow line, $\vec{c}\,'(t) = \vec{F}(\vec{c}(t))$

$$\therefore \frac{dV}{dt} = \nabla V \cdot \vec{c}\,'(t) = \nabla V \cdot \vec{F}(\vec{c}(t)) = \nabla V \cdot (-\nabla V)$$

But $\|\nabla V\|^2 = \nabla V \cdot \nabla V$

$\therefore \frac{dV}{dt} = -\|\nabla V\|^2$, and $\|\vec{a}\| \geq 0$ for any $\vec{a}$.

$$\therefore \frac{dV}{dt} = -\|\nabla V\|^2 \leq 0$$

$\therefore$ <u>$V(t)$ is</u> a <u>decreasing function of t</u>.

25.

From Example 4, p. 238 of text, $\vec{J} = -k\nabla T$,
where $\vec{J}$ = energy/heat flux vector field, k a constant, T = scalar field of temperature.

Gradient vectors, like ∇T, are perpendicular to level sets.

$\therefore$ In this case, ∇T is perpendicular to the concentric spheres, which are level sets.

$\therefore \nabla T$ points towards or away from origin.

26.

(a) Compute $\nabla V = (V_x, V_y)$.

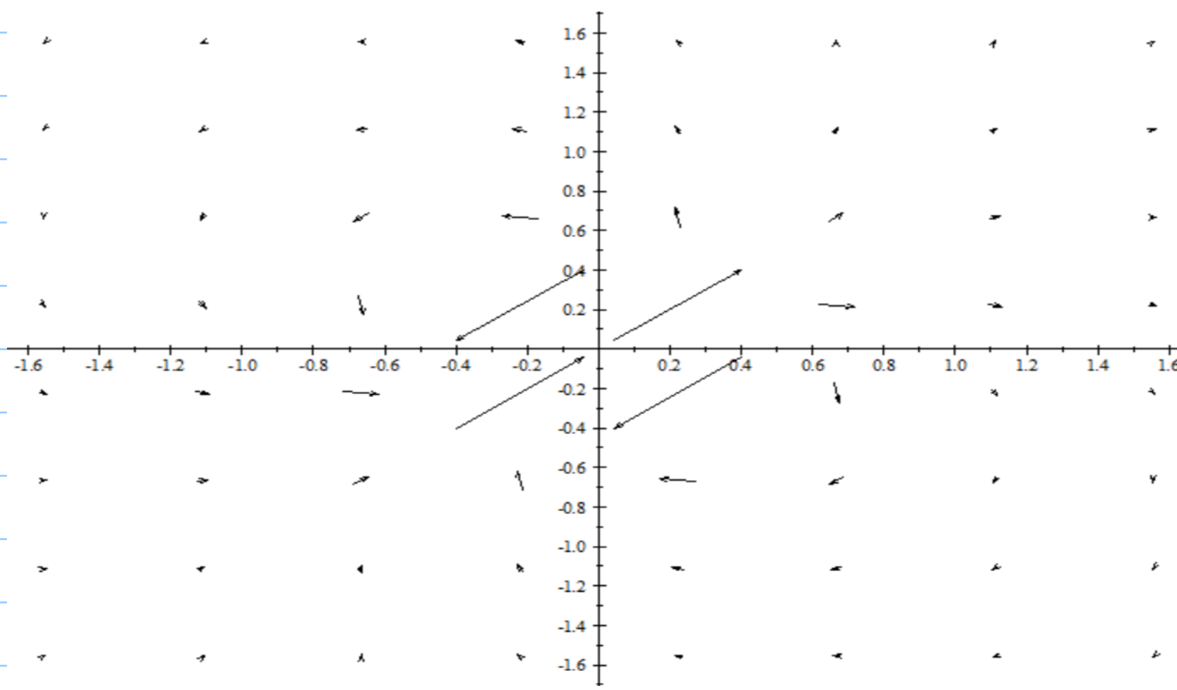
$$V_x = \frac{(x^2+y^2)(1) - (x+y)(2x)}{(x^2+y^2)^2} = \frac{-x^2-2xy+y^2}{(x^2+y^2)^2}$$

$$V_y = \frac{(x^2+y^2)(1) - (x+y)(2y)}{(x^2+y^2)^2} = \frac{x^2-2xy-y^2}{(x^2+y^2)^2}$$

$$\therefore -\nabla V = \left[\frac{x^2+2xy-y^2}{(x^2+y^2)^2}, \frac{y^2+2xy-x^2}{(x^2+y^2)^2}\right]$$

So, as $(x,y) \to \infty$, magnitude of $-\nabla V$ is very small.

as $(x,y) \to (0,0)$, magnitude of $-\nabla V$ is very large.

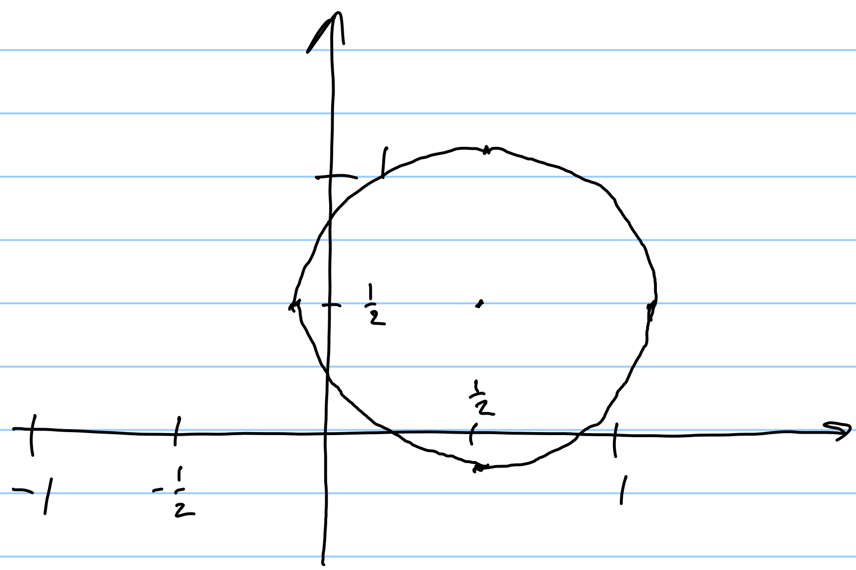


(5) For $V = 1 = \dfrac{x+y}{x^2+y^2}$, or $x^2+y^2 = x+y$,

$\therefore\ x^2 - x + y^2 - y = 0$, $(x-\frac{1}{2})^2 + (y-\frac{1}{2})^2 = \frac{1}{2}$

$\therefore$ a circle of radius $\frac{1}{\sqrt{2}}$, centered at $(\frac{1}{2}, \frac{1}{2})$

$\frac{1}{\sqrt{2}} = 0.7$



27.

Using $\vec{c}\,'(t) = F(\vec{c}(t))$, $(x', y', z') = (xe^y, y^2z^2, xyz)$.

$\therefore\ \dfrac{dx}{dt} = xe^y \quad \dfrac{dy}{dt} = y^2z^2 \quad \dfrac{dz}{dt} = xyz$

4.4 Divergence and Curl

Note Title 9/12/2016

1.

$$\frac{\partial}{\partial x} e^{xy} = ye^{xy} \quad \frac{\partial}{\partial y}(-e^{xy}) = -xe^{xy} \quad \frac{\partial}{\partial z} e^{yz} = ye^{yz}$$

$$\therefore \nabla \cdot \vec{V} = \underline{\underline{ye^{xy} - xe^{xy} + ye^{yz}}}$$

2.

$$\frac{\partial}{\partial x}(yz) = 0 \qquad \frac{\partial}{\partial y}(xz) = 0 \qquad \frac{\partial}{\partial z}(xy) = 0$$

$$\therefore \nabla \cdot \vec{V} = \underline{\underline{0}}$$

3.

$$\frac{\partial}{\partial x}(x) = 1 \qquad \frac{\partial}{\partial y}(y + \cos x) = 1 \qquad \frac{\partial}{\partial z}(z + e^{xy}) = 1$$

$$\therefore \nabla \cdot \vec{V} = 3$$

4.

$$\frac{\partial}{\partial x}(x^2) = 2x \quad \frac{\partial}{\partial y}(x+y)^2 = 2(x+y) \quad \frac{\partial}{\partial z}(x+y+z)^2 = 2(x+y+z)$$

$$\therefore \nabla \cdot \vec{V} = 2x + 2(x+y) + 2(x+y+z)$$

$$= \underline{\underline{6x + 4y + 2z}}$$

5.

div $\vec{V} > 0$ where there is expansion
so Quadrants I, III

div $\vec{V} < 0$ where there is compression
$\therefore$ Quadrants II, IV

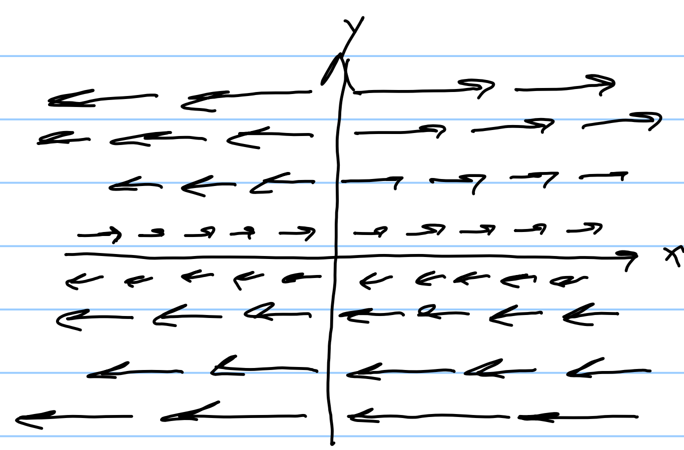
6.

Let $V(x,y,z) = (V_1(x,y,z),\ V_2(x,y,z),\ V_3(x,y,z)) = (x, 0, 0)$

$$\nabla \cdot \vec{V} = V_{1_x} + V_{2_y} + V_{3_z} = 1 + 0 + 0 = 1 > 0$$

$\therefore$ Rate of change of fluid volume = $\underline{\underline{\text{expansion}}}$

7.



$$\vec{F}(x,y) = (y, 0)$$

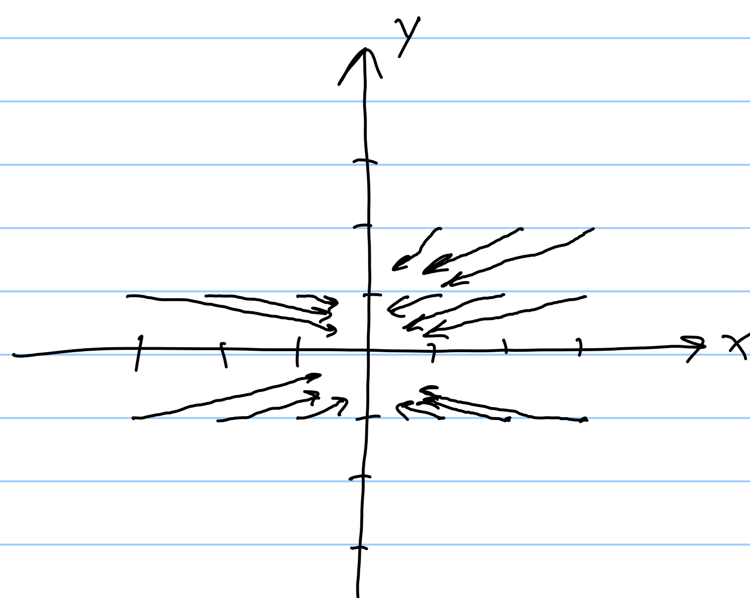
$$\therefore \nabla \cdot \vec{F} = \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial x}(0) = 0$$

A tiny rectangle around a given point does not change in area as the flow is constant for any given y : $(y, 0)$

A tiny rectangle centered at a given (x, y) moves to the right $(y > 0)$ or left $(y < 0)$.

The "shear" effect (flow at $y + \Delta y$ slightly greater than at $y - \Delta y$) becomes negligible as $\Delta y \to 0$.

8.



$\vec{F}(x, y) = (-3x, -y)$

$\therefore \nabla \cdot \vec{F} = -3 + (-1) =$

$= -4 < 0$

$\therefore$ Fluid shows compression, graph shows compression toward origin.

9.

Let $\vec{F} = (F_1, F_2) = (x^3, -x\sin(xy))$

$$\therefore \nabla \cdot \vec{F} = 3x^2 + (-x^2\cos(xy)) = \frac{\partial x^3}{\partial x} + \frac{\partial}{\partial y}(-x\sin(xy))$$

$$= \underline{3x^2 - x^2\cos(xy)}$$

10.

$\vec{F} = (y, -x) \quad \therefore \nabla \cdot \vec{F} = \frac{\partial y}{\partial x} + \frac{\partial}{\partial y}(-x) = 0 + 0 = \underline{0}$

11.

$\vec{F} = [\sin(xy), -\cos(x^2 y)]$

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(\sin(xy)) + \frac{\partial}{\partial y}(-\cos(x^2 y))$$

$$= \underline{y\cos(xy) + x^2\sin(x^2 y)}$$

12.

$\vec{F} = \left(xe^y, -\frac{y}{x+y}\right)$

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(xe^{y}) + \frac{\partial}{\partial y}\left(\frac{-y}{x+y}\right)$$

$$= e^{y} + \left[\frac{(x+y)(-1) - (-y)(1)}{(x+y)^2}\right]$$

$$= e^{y} - \frac{x}{(x+y)^2}$$

13.

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & z \end{vmatrix} = (0-0, 0-0, 0-0) = \vec{0}$$

14.

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix} = (x-x, y-y, z-z) = \vec{0}$$

15.

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3(x^2+y^2+z^2) & 4(x^2+y^2+z^2) & 5(x^2+y^2+z^2) \end{vmatrix}$$

$$= \underline{(10y - 8z,\ 6z - 10x,\ 8x - 6y)}$$

16.

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \dfrac{yz}{(x^2+y^2+z^2)} & \dfrac{-xz}{(x^2+y^2+z^2)} & \dfrac{xy}{(x^2+y^2+z^2)} \end{vmatrix} =$$

$$\frac{1}{(x^2+y^2+z^2)^2}\Big[(x^2+y^2+z^2)(x) - (xy)(2y) - [(x^2+y^2+z^2)(-x) - (-xz)(2z)],$$

$$(x^2+y^2+z^2)(y) - (yz)(2z) - [(x^2+y^2+z^2)(y) - (xy)(2x)],$$

$$(x^2+y^2+z^2)(-z) - (-xz)(2x) - [(x^2+y^2+z^2)(z) - (yz)(2y)]\Big]$$

$$= \frac{1}{(x^2+y^2+z^2)^2}\Big[2x(x^2+y^2+z^2) - 2xz^2 - 2xy^2,$$

$$-2yz^2 + 2yx^2,$$

$$-2z(x^2+y^2+z^2) + 2zx^2 + 2zy^2\Big]$$

$$= \underline{[2x^3,\ -2yz^2 + 2yx^2,\ -2z^3] \,/\, (x^2+y^2+z^2)^2}$$

17.

$$\nabla \times \vec{F} = \left[\frac{\partial}{\partial x}(\cos x) - \frac{\partial}{\partial y}(\sin x)\right]\hat{k} = -\sin x\,\hat{k}$$

$\therefore$ Scalar curl $= \underline{-\sin x}$

18.

$$\nabla \times \vec{F} = \left[\frac{\partial}{\partial x}(-x) - \frac{\partial}{\partial y}(y)\right]\hat{k} = -2\hat{k}$$

$\therefore$ Scalar curl $= \underline{-2}$

19.

$$\nabla \times \vec{F} = \left[\frac{\partial}{\partial x}(x^2-y^2) - \frac{\partial}{\partial y}(xy)\right]\hat{k} = (2x-x)\hat{k} = x\hat{k}$$

$\therefore$ Scalar curl $= \underline{x}$

20.

$$\nabla \times \vec{F} = \left[\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x)\right]\hat{k} = [0-0]\hat{k} = \vec{0}$$

$\therefore$ scalar curl $= \underline{0}$

21.

(a) $\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 & x^2 y & z + zx \end{vmatrix}$

$= (0-0,\ 0-z,\ 2yx-0) = (0, -z, 2xy)$

$\nabla \cdot (0, -z, 2xy) = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(-z) + \frac{\partial}{\partial z}(2xy)$

$\therefore \nabla \cdot (\nabla \times \vec{F}) = 0 + 0 + 0 = \underline{0}$

(b) If $\vec{F}$ is the gradient of f, then

Curl $\vec{F} = \nabla \times \vec{F} = 0 \Rightarrow (0, -z, 2xy)$

But $\nabla \times \vec{F} \neq \vec{0}$ for all x, y, z.

$\therefore$ There is __no__ such function $f: \mathbb{R}^3 \to \mathbb{R}$.

22.

(a) If the fields are gradients, then the curl of the gradient should be zero.

$\nabla \times \vec{F} = \vec{0}$ as shown, so
this could be a gradient.

$\nabla \times \vec{F} = \vec{0}$ as shown, so this
could be a gradient

$\nabla \times \vec{F} \neq \vec{0}$, so could not
be a gradient.

$\nabla \times \vec{F} \neq \vec{0}$, so could not
be a gradient.

$\therefore$ #13, #14 could be gradients

#15, #16 could not

(b)

div (curl $\vec{F}$) $= 0$. $\therefore$ Look at $\nabla \cdot \vec{F}$

$\nabla \cdot \vec{F} \neq 0$ as shown, so can not
be the curl of some $\vec{v}$

$\nabla \cdot \vec{F} = 0$ as shown, so could be
the curl of some $\vec{v}$

$\nabla \cdot \vec{F} \neq 0$, so can not
be the curl of some $\vec{v}$

$\nabla \cdot \vec{F} \neq 0$, so can not be
the curl of some $\vec{V}$.

$\therefore$ #10 could be

#9, 11, 12 can not

23.

(a) $\frac{\partial}{\partial x}(e^{xz}) + \frac{\partial}{\partial y}(\sin(xy)) + \frac{\partial}{\partial z}(x^5y^3z^2)$

$$= \underline{ze^{xz} + x\cos(xy) + 2x^5y^3z}$$

(b) $\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^{xz} & \sin(xy) & x^5y^3z^2 \end{vmatrix}$

$$= (3x^5y^2z^2 - 0,\ xe^{xz} - 5x^4y^3z^2,\ y\cos(xy) - 0)$$

$$= \underline{(3x^5y^2z^2,\ xe^{xz} - 5x^4y^3z^2,\ y\cos(xy))}$$

24.

(a) grad f creates a vector in R^3

curl operates on a vector in R^3

so curl(grad f) makes sense, and curl generates a vector in R^3

(b) Does not make sense

f is a scalar in R^1, and curl operates on a vector on R^3, and generates a vector in R^3. Grad does not operate on a vector in R^3.

(c) Makes sense.

grad f generates a vector in R^3 (in this case) while div operates on the resultant vector.

(d) Does not make sense

div operates on a vector, not a scalar.

$\therefore$ div f makes no sense.

If $f: \mathbb{R}^1 \to \mathbb{R}^1$, Then div f could make sense.

$\text{div } f = \frac{df}{dx}$, and $\text{grad}(\text{div } f) = \frac{d^2 f}{dx^2}$,

a vector in $\mathbb{R}^1$. But here, $f: \mathbb{R}^3 \to \mathbb{R}^1$, not

$\mathbb{R}^1 \to \mathbb{R}^1$.

(e) Does not make sense.

div operates on a vector, not a scalar. See (d)

(f) Does not make sense

curl operates on a vector in $\mathbb{R}^3$, not a

scalar in $\mathbb{R}^1$.

25.

(a) Nonsense. grad operates on a scalar in $\mathbb{R}^1$

(b) Nonsense. curl $\vec{F}$ is a vector in $\mathbb{R}^3$,
grad requires a scalar in $\mathbb{R}^1$.

(c) Nonsense. grad operates on a scalar in $\mathbb{R}^1$

(d) Makes sense. div $\vec{F}$ creates a scalar in $\mathbb{R}^1$ and grad operates on a scalar in $\mathbb{R}^1$ and generates a vector in $\mathbb{R}^3$ (in this case).

(e) Nonsense. div $\vec{F}$ is a scalar in $\mathbb{R}^1$; curl requires a vector in $\mathbb{R}^3$.

(f) Makes sense. curl $\vec{F}$ generates a vector in $\mathbb{R}^3$. div operates on a vector and generates a scalar.

26.

Need to show $\nabla \times \vec{F} = \vec{0}$.

By the notation, f, g, h are only dependent on one variable.

$$\therefore \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f(x) & g(y) & h(z) \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y} h(z) - \frac{\partial}{\partial z} g(y), \frac{\partial}{\partial z} f(x) - \frac{\partial}{\partial x} h(z), \frac{\partial}{\partial x} h(z) - \frac{\partial}{\partial y} f(x) \right)$$

$$= (0-0, 0-0, 0-0) = \vec{0}$$

27.

$$\operatorname{div} \vec{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

$$= 0 + 0 + 0 = 0$$

since f is not dependent on x
g is not dependent on y
h is not dependent on z.

28.

13. div $(\nabla f \times \nabla g) = 0$

Let $\nabla f = \vec{F}$, $\nabla g = \vec{G}$.

$\therefore \operatorname{div}(\nabla f \times \nabla g) = \operatorname{div}(\vec{F} \times \vec{G})$

By identity #8, $\operatorname{div}(\vec{F} \times \vec{G}) = \vec{G} \cdot (\nabla \times \vec{F}) - \vec{F} \cdot (\nabla \times \vec{G})$

But $\operatorname{curl} \vec{F} = \nabla \times \vec{F} = \nabla \times (\nabla f) = \vec{0}$

since the curl of a gradient is $\vec{0}$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = \vec{0}$$ since mixed partials are equal.

Also, 2 rows of determinant are equal.

Similarly, $\nabla \times \vec{G} = \nabla \times (\nabla g) = \vec{0}$.

$\therefore \operatorname{div}(\vec{F} \times \vec{G}) = \vec{G} \cdot (\vec{0}) - \vec{F} \cdot (\vec{0}) = 0 + 0 = 0$

$\therefore \operatorname{div}(\nabla f \times \nabla g) = 0$

29.

$$\nabla f = \left[\frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}} \right]$$

$$\nabla \times (\nabla f) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{x}{\sqrt{x^2+y^2+z^2}} & \frac{y}{\sqrt{x^2+y^2+z^2}} & \frac{z}{\sqrt{x^2+y^2+z^2}} \end{vmatrix}$$

$$= \Big[-z(x^2+y^2+z^2)^{-\frac{3}{2}} y - \left(-y(x^2+y^2+z^2)^{-\frac{3}{2}} z\right),$$

$$-z(x^2+y^2+z^2)^{-\frac{3}{2}} x - \left(-x(x^2+y^2+z^2)^{-\frac{3}{2}} z\right),$$

$$-y(x^2+y^2+z^2)^{-3/2} x - \left(x(x^2+y^2+z^2)^{-3/2} y\right) \Big]$$

$$= [0, 0, 0] = \vec{0}$$

30.

$$\nabla f = [y+z,\ x+z,\ y+x]$$

$$\therefore \nabla \times (\nabla f) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y+z & x+z & y+x \end{vmatrix}$$

$$= [1-1,\ 1-1,\ 1-1] = [0, 0, 0] = \vec{0}$$

31.

$$\nabla f = [-(x^2+y^2+z^2)^{-2} 2x,\ -(x^2+y^2+z^2)^{-2} 2y,\ -(x^2+y^2+z^2)^{-2} 2z]$$

$$\therefore \nabla \times (\nabla f) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{-2x}{(x^2+y^2+z^2)^2} & \frac{-2y}{(x^2+y^2+z^2)^2} & \frac{-2z}{(x^2+y^2+z^2)^2} \end{vmatrix}$$

$$= [4z(x^2+y^2+z^2)^{-3} 2y - (4y(x^2+y^2+z^2)^{-3} 2z),$$

$$4x(x^2+y^2+z^2)^{-3} 2z - (4z(x^2+y^2+z^2)^{-3} 2x),$$

$$4y(x^2+y^2+z^2)^{-3} 2x - (4x(x^2+y^2+z^2)^{-3} 2y)]$$

$$= [0, 0, 0] = \vec{0}$$

32.

$$\nabla f = (2xy^2,\ 2yx^2 + 2yz^2,\ 2zy^2)$$

$$\therefore \nabla \times (\nabla f) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xy^2 & 2yx^2 + 2yz^2 & 2zy^2 \end{vmatrix}$$

$$= [4zy - 4yz,\ 0-0,\ 4yx - 4xy]$$

$$= \{0, 0, 0\} = \vec{0}$$

33.

If $\vec{F}$ is a gradient, then $\nabla \times \vec{F} = \vec{0}$.

Check: $\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y\cos x & x\sin y & 0 \end{vmatrix}$

$$= [0-0,\ 0-0,\ \sin y - \cos x]$$

$$= [0, 0, \sin y - \cos x] \neq \vec{0}$$

$\therefore \vec{F}$ not a gradient of some $f: R^3 \to R^1$

34.

If $\vec{F}$ is a gradient, then $\nabla \times \vec{F} = \vec{0}$.

Check: $\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2+y^2 & -2xy & 0 \end{vmatrix}$

$= [0-0,\ 0-0,\ -2y-2y] = [0,0,-4y] \neq \vec{0}$

$\therefore \vec{F}$ not a gradient of some $f: R^3 \to R^1$

35.

10. curl $(f\mathbf{F}) = f\,\mathrm{curl}\,\mathbf{F} + \nabla f \times \mathbf{F}$

Let $\vec{F} = (F_1, F_2, F_3)$, where $F_i : R^3 \to R^1$

$\therefore \nabla \times (f\vec{F}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ fF_1 & fF_2 & fF_3 \end{vmatrix}$

$= [\, f_y F_3 + f F_{3y} - (f_z F_2 + f F_{2z}),$

$f_z F_1 + f F_{1z} - (f_x F_3 + f F_{3x}),$

$f_x F_2 + f F_{2x} - (f_y F_1 + f F_{1y}) \,]$

$= [\, f F_{3y} - f F_{2z} \quad + \quad f_y F_3 - f_z F_2,$

$f F_{1z} - f F_{3x} \quad + \quad f_z F_1 - f_x F_3,$

$f F_{2x} - f F_{1y} \quad + \quad f_x F_2 - f_y F_1 \,]$

$$= f\begin{bmatrix} F_{3y} - F_{2z}, \\ F_{1z} - F_{3x}, \\ F_{2x} - F_{1y} \end{bmatrix} + \begin{bmatrix} f_y F_3 - f_z F_2, \\ f_z F_1 - f_x F_3, \\ f_x F_2 - f_y F_1 \end{bmatrix}$$

$$= f\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ f_x & f_y & f_z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= f(\nabla \times \vec{F}) + \nabla f \times \vec{F}$$

$$= f(\text{curl}\,\vec{F}) + \nabla f \times \vec{F}$$

36.

(a) $\nabla \cdot (\vec{F} + \vec{G}) = \nabla \cdot \vec{F} + \nabla \cdot \vec{G} = 0 + 0 = 0$

(b) $\nabla \cdot (\vec{F} \times \vec{G}) = \vec{G} \cdot (\nabla \times \vec{F}) - \vec{F} \cdot (\nabla \times \vec{G})$

No guarantee this is zero.

Example: $\vec{F} = (x, -y, 0)$ $\therefore \nabla \cdot \vec{F} = 1 - 1 + 0 = 0$

$\vec{G} = (z, 0, x)$ $\therefore \nabla \cdot \vec{G} = 0 + 0 + 0 = 0$

$$\vec{F} \times \vec{G} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & -y & 0 \\ z & 0 & x \end{vmatrix} = (-xy, -x^2, yz)$$

$$\therefore \nabla \cdot (\vec{F} \times \vec{G}) = (-y, 0, y) \neq 0$$

37.

(a) $\nabla f = (f_x, f_y, f_z) = \underline{(2xy, x^2, 0)}$, assuming $f: \mathbb{R}^3 \to \mathbb{R}^1$

(b) $$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xz^2 & 1 & y^3zx \end{vmatrix}$$

$$= (3y^2zx - 0, 4xz - y^3z, 0 - 0)$$

$$= \underline{(3y^2zx, 4xz - y^3z, 0)}$$

(c) $$\vec{F} \times \nabla f = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2xz^2 & 1 & y^3zx \\ 2xy & x^2 & 0 \end{vmatrix}$$

$$= (0 - y^3zx^3, 2y^4zx^2 - 0, 2x^3z^2 - 2xy)$$

$$= \underline{(-y^3zx^3, 2y^4zx^2, 2x^3z^2 - 2xy)}$$

(d) $\vec{F} \cdot (\nabla f) = (2xz^2, 1, y^3zx) \cdot (2xy, x^2, 0)$

$= 4x^2yz^2 + x^2 + 0 = \underline{4x^2yz^2 + x^2}$

38.

(a) (1) $\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = \frac{\partial}{\partial x}(x^2+y^2+z^2)^{-\frac{1}{2}} = -\frac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}(2x)$

$= -\frac{x}{(x^2+y^2+z^2)^{3/2}} = \frac{-x}{(\sqrt{x^2+y^2+z^2})^3} = -\frac{x}{\|\vec{r}\|^3}$

Similarly, $\frac{\partial}{\partial y}\left(\frac{1}{r}\right) = \frac{-y}{\|\vec{r}\|^3}$, $\frac{\partial}{\partial z}\left(\frac{1}{r}\right) = \frac{-z}{\|\vec{r}\|^3}$

$\therefore \nabla\left(\frac{1}{r}\right) = \left(-\frac{x}{\|\vec{r}\|^3}, \frac{-y}{\|\vec{r}\|^3}, \frac{-z}{\|\vec{r}\|^3}\right)$

$= -\frac{1}{\|\vec{r}\|^3}(x, y, z) = -\frac{\vec{r}}{\|\vec{r}\|^3} = \underline{-\frac{\vec{r}}{r^3}}$

(2) $\frac{\partial}{\partial x}(r^n) = \frac{\partial}{\partial x}\left[(x^2+y^2+z^2)^{\frac{1}{2}}\right]^n = \frac{\partial}{\partial x}(x^2+y^2+z^2)^{\frac{n}{2}}$

$= \frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1}(2x)$

$$= n(x)(x^2+y^2+z^2)^{\frac{n-2}{2}} = nx\left[(x^2+y^2+z^2)^{\frac{1}{2}}\right]^{n-2}$$

$$= nx(r)^{n-2}$$

Similarly, $\frac{\partial}{\partial y}(r^n) = ny(r)^{n-2}$, $\frac{\partial}{\partial z}(r^n) = nz(r)^{n-2}$

$$\therefore \nabla(r^n) = \left[nxr^{n-2}, nyr^{n-2}, nzr^{n-2}\right\}$$

$$= nr^{n-2}\{x, y, z\} = nr^{n-2}\vec{r}$$

(3) $\frac{\partial}{\partial x}(\log r) = \frac{\partial}{\partial x}\left(\log(x^2+y^2+z^2)^{\frac{1}{2}}\right)$

$$= \frac{\partial}{\partial x}\left(\frac{1}{2}\log(x^2+y^2+z^2)\right)$$

$$= \frac{1}{2}\,\frac{2x}{x^2+y^2+z^2} = \frac{x}{r^2}$$

Similarly, $\frac{\partial}{\partial y}(\log r) = \frac{y}{r^2}$, $\frac{\partial}{\partial z}(\log r) = \frac{z}{r^2}$

$$\therefore \nabla(\log r) = \left(\frac{x}{r^2}, \frac{y}{r^2}, \frac{z}{r^2}\right) = \frac{1}{r^2}(x, y, z)$$

$$= \underline{\frac{1}{r^2}\vec{r}}$$

(5)

(1) $\nabla^2\left(\frac{1}{r}\right) = \nabla\cdot\left(\nabla\frac{1}{r}\right) = \nabla\cdot\left(-\frac{\vec{r}}{r^3}\right) = \nabla\cdot\left(-\frac{x}{r^3}, -\frac{y}{r^3}, -\frac{z}{r^3}\right)$

$$\frac{\partial}{\partial x}\left(\frac{-x}{r^3}\right) = \frac{\partial}{\partial x}\left[\frac{-x}{(x^2+y^2+z^2)^{3/2}}\right]$$

$$= \frac{(x^2+y^2+z^2)^{3/2}(-1) + (x)\frac{3}{2}(x^2+y^2+z^2)^{\frac{1}{2}}(2x)}{(x^2+y^2+z^2)^3}$$

$$= \frac{3x^2(x^2+y^2+z^2)^{\frac{1}{2}} - (x^2+y^2+z^2)^{3/2}}{(x^2+y^2+z^2)^3}$$

$$= \frac{3x^2 r - r^3}{r^6}$$

Similarly, $\frac{\partial}{\partial y}\left(-\frac{y}{r^3}\right) = \frac{3y^2 r - r^3}{r^6}$

$$\frac{\partial}{\partial z}\left(-\frac{z}{r^3}\right) = \frac{3z^2 r - r^3}{r^6}$$

$$\therefore \nabla\cdot\left(-\frac{\vec{r}}{r^3}\right) = \frac{\partial}{\partial x}\left(-\frac{x}{r^3}\right) + \frac{\partial}{\partial y}\left(-\frac{y}{r^3}\right) + \frac{\partial}{\partial z}\left(-\frac{z}{r^3}\right)$$

$$= \frac{(3x^2r-r^3) + (3y^2r-r^3) + (3z^2r-r^3)}{r^6}$$

$$= \frac{3r(x^2+y^2+z^2) - 3r^3}{r^6} = \frac{3r(r^2) - 3r^3}{r^6}$$

$$= 0$$

$$\therefore \nabla^2\left(\frac{1}{r}\right) = 0$$

(2) From (a), $\nabla(r^n) = n r^{n-2}\vec{r} = n r^{n-2}(x, y, z)$

$$\therefore \frac{\partial}{\partial x}\left(n r^{n-2} x\right) = \frac{\partial}{\partial x}\left(n x \left[x^2+y^2+z^2\right]^{\frac{n-2}{2}}\right)$$

$$= n\left(x^2+y^2+z^2\right)^{\frac{n-2}{2}} + n x \frac{(n-2)}{2}\left(x^2+y^2+z^2\right)^{\frac{n-4}{2}}(2x)$$

$$= n(r)^{n-2} + x^2 n(n-2)(r)^{n-4}$$

Similarly, $\frac{\partial}{\partial y}\left(n r^{n-2} y\right) = n(r)^{n-2} + y^2 n(n-2)(r)^{n-4}$

$$\frac{\partial}{\partial z}\left(n r^{n-2} z\right) = n(r)^{n-2} + z^2 n(n-2)(r)^{n-4}$$

$$\therefore \nabla^2(r^n) = \frac{\partial}{\partial x}\left(n r^{n-2} x\right) + \frac{\partial}{\partial y}\left(n r^{n-2} y\right) + \frac{\partial}{\partial z}\left(n r^{n-2} z\right)$$

$$= 3n(r)^{n-2} + \left(x^2+y^2+z^2\right) n(n-2)(r)^{n-4}$$

$$= 3n(r)^{n-2} + r^2 n(n-2)(r)^{n-4}$$

$$= 3n(r)^{n-2} + n(n-2)(r)^{n-2}$$

$$= \left(3n + n^2 - 2n\right)(r)^{n-2}$$

$$= \underline{\underline{n(n+1) r^{n-2}}}$$

(c)

(1) $\frac{\vec{r}}{r^3} = -\nabla\left(\frac{1}{r}\right)$ from (a).

$\therefore \nabla\cdot\left(\frac{\vec{r}}{r^3}\right) = \nabla\cdot\left(-\nabla\frac{1}{r}\right) = -\nabla\cdot\left(\nabla\frac{1}{r}\right)$

$= -\nabla^2\left(\frac{1}{r}\right) = -(0) = 0$ from (b)

(2) From Identity #7, $\operatorname{div}(f\vec{F}) = f\operatorname{div}\vec{F} + \vec{F}\cdot\nabla f$

$\therefore \nabla\cdot(r^n\vec{r}) = r^n\nabla\cdot\vec{r} + \vec{r}\cdot(\nabla r^n)$

$\nabla\cdot\vec{r} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1+1+1 = 3$

$\therefore r^n\nabla\cdot\vec{r} = 3r^n$ [1]

From (a), $\nabla r^n = n r^{n-2}\vec{r}$

$\therefore \vec{r}\cdot(\nabla r^n) = \vec{r}\cdot(n r^{n-2}\vec{r})$

$= n r^{n-2}\,\vec{r}\cdot\vec{r} = n r^{n-2}(r^2)$

$= n r^n$ [2]

$\therefore$ From [1], [2],

$\underline{\nabla\cdot(r^n\vec{r})} = 3r^n + nr^n = \underline{(n+3)r^n}$

(d)

(1) $\nabla \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & z \end{vmatrix}$

$= (0-0,\ 0-0,\ 0-0) = \vec{0}$

(2) From Identity #10 (p. 255 of text)

$\nabla \times (f\vec{F}) = f\,\nabla \times \vec{F} + \nabla f \times \vec{F}$

$\therefore \nabla \times (r^n \vec{r}) = r^n\, \nabla \times \vec{r} + \nabla r^n \times \vec{r}$

But $\nabla \times \vec{r} = \vec{0}$ from (a)

And $\nabla r^n = n r^{n-2}\vec{r}$ from (a)

$\therefore \nabla \times (r^n \vec{r}) = \vec{0} + (n r^{n-2}\vec{r}) \times \vec{r}$

$= n r^{n-2}(\vec{r} \times \vec{r}) = \vec{0}$

$\therefore \nabla \times (r^n \vec{r}) = \underline{\underline{\vec{0}}}$

39.

No. For example, let $\vec{F} = (z, x, z)$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z & x & z \end{vmatrix} = (0, 1, 1)$$

$$\therefore \vec{F} \cdot (\nabla \times \vec{F}) = x + z$$

40.

(a) $$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3x^2y & x^3+y^3 & 0 \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(x^3+y^3), \frac{\partial}{\partial z}(3x^2y) - \frac{\partial}{\partial x}(0), \frac{\partial}{\partial x}(x^3+y^3) - \frac{\partial}{\partial y}(3x^2y) \right)$$

$$= (0-0, 0-0, 3x^2 - 3x^2) = (0,0,0) = \underline{\vec{0}}$$

(b) $$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) = (3x^2y, x^3+y^3, 0)$$

$$\therefore \frac{\partial f}{\partial x} = 3x^2y \Rightarrow f(x,y,z) = x^3y + g(y,z)$$

as $\frac{\partial}{\partial x} g(y,z) = 0$

$$\therefore \frac{\partial f}{\partial y} = x^3 + \frac{\partial g}{\partial y} = x^3 + y^3 \Rightarrow g(y,z) = \frac{y^4}{4} + h(z)$$

$$\text{as } \frac{\partial}{\partial y} h(z) = 0.$$

$$\therefore f(x,y,z) = x^3 y + \frac{y^4}{4} + h(z)$$

$$\frac{\partial f}{\partial z} = h'(z) = 0 \Rightarrow h(z) = C, \text{ a constant.}$$

$$\therefore f(x,y,z) = x^3 y + \frac{y^4}{4} + C, \quad C \text{ a constant}$$

41.

(a) $(x-iy)^2 = x^2 - y^2 - 2xy\,i$

$$\therefore \vec{F} = (x^2 - y^2, -2xy, 0)$$

$$\therefore \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 - y^2 & -2xy & 0 \end{vmatrix}$$

$$= (0-0, 0-0, -2y-(-2y)) = \underline{\vec{0}}$$

$\therefore$ irrotational

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x^2-y^2) + \frac{\partial}{\partial y}(-2xy) + \frac{\partial}{\partial z}(0)$$

$$= 2x + (-2x) = \underline{0}$$

$\therefore$ incompressible

(b) $(x-iy)^3 = x^3 + 3x^2(-iy) + 3x(-iy)^2 + (-iy)^3$

$$= x^3 - 3xy^2 + (-3x^2y + y^3)i$$

$\therefore \vec{F} = (x^3 - 3xy^2, -3x^2y + y^3, 0)$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^3-3xy^2 & -3x^2y+y^3 & 0 \end{vmatrix}$$

$$= (0-0, 0-0, -6xy - (-6xy)) = \underline{\vec{0}}$$

$\therefore$ irrotational

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x^3 - 3xy^2) + \frac{\partial}{\partial y}(-3x^2y + y^3) + \frac{\partial}{\partial z}(0)$$

$$= 3x^2 - 3y^2 + (-3x^2 + 3y^2) = \underline{0}$$

$\therefore$ incompressible

(c) $\vec{F} = (e^x \cos y, -e^x \sin y, 0)$

$$\therefore \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^x \cos y & -e^x \sin y & 0 \end{vmatrix}$$

$$= \left(0-0,\ 0-0,\ -e^x \sin y - (-e^x \sin y)\right)$$

$$= (0, 0, 0) = \underline{\underline{\vec{0}}}$$

$\therefore$ irrotational

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(e^x \cos y) + \frac{\partial}{\partial y}(-e^x \sin y) + \frac{\partial}{\partial z}(0)$$

$$= e^x \cos y + (-e^x \cos y) = \underline{0}$$

$\therefore$ incompressible.

Review Exercises for Chapter 4

Note Title 9/19/2016

1.

$$\vec{c}\,'(t) = \left[3t^2, -e^{-t}, -\frac{\pi}{2}\sin\left(\frac{\pi t}{2}\right)\right]$$

$$\therefore \vec{c}\,'(1) = \underline{\left[3, -\frac{1}{e}, -\frac{\pi}{2}\right]} = \text{velocity vector}$$

$$\|\vec{c}\,'(1)\| = \underline{\sqrt{3^2 + \frac{1}{e^2} + \frac{\pi^2}{4}}} = \text{speed}$$

$$\vec{c}\,''(t) = \left[6t, e^{-t}, -\frac{\pi^2}{4}\cos\left(\frac{\pi t}{2}\right)\right]$$

$$\therefore \vec{c}\,''(1) = \underline{\left[6, \frac{1}{e}, 0\right]} = \text{acceleration vector}$$

$$\text{Tangent line} = \vec{c}(1) + s\,\vec{c}\,'(1), \quad s \in \mathbb{R}$$

$$= \underline{\left(2, \frac{1}{e}, 0\right) + s\left(3, -\frac{1}{e}, -\frac{\pi}{2}\right)}$$

2.

$$\text{Velocity vector: } \vec{c}\,'(t) = \left[2t, -2t\sin(t^2), 4t^3\right]$$

$$\therefore \vec{c}\,'(\sqrt{\pi}) = \underline{\left[2\sqrt{\pi}, 0, 4\pi^{3/2}\right]}$$

$$\text{speed} = \|\vec{c}\,'(\sqrt{\pi})\| = \sqrt{4\pi + 16\pi^3} = \underline{2\sqrt{\pi+\pi^3}}$$

Acceleration vector: $\vec{c}\,''(t) = [2, -2\sin(t^2) - 4t^2\cos(t^2), 12t^2]$

$$\therefore \underline{\vec{c}\,''(\sqrt{\pi}) = [2, 4\pi, 12\pi]}$$

Tangent line: $\vec{c}(\sqrt{\pi}) + s\,\vec{c}\,'(\sqrt{\pi})$, $s \in R$

$$= \underline{(\pi-1, -1, \pi^2) + s(2\sqrt{\pi}, 0, 4\pi^{3/2})}$$

3.

Velocity: $\vec{c}\,'(t) = [e^t, \cos t, -\sin t]$

$$\therefore \vec{c}\,'(0) = \underline{[1, 1, 0]}$$

Speed: $\|\vec{c}\,'(0)\| = \sqrt{1^2+1^2+0^2} = \underline{\sqrt{2}}$

Acceleration: $\vec{c}\,''(t) = [e^t, -\sin t, -\cos t]$

$$\therefore \vec{c}\,''(0) = \underline{[1, 0, -1]}$$

Tangent line: $\vec{c}(0) + s\,\vec{c}\,'(0)$, $s \in R$

$$\therefore \underline{(1, 0, 1) + s(1, 1, 0)}$$

4.

Velocity: $\vec{c}\,'(t) = \left[\dfrac{(1+t^2)2t - t^2(2t)}{(1+t^2)^2}, 1, 0 \right]$

$= \left[\dfrac{2t}{(1+t^2)^2}, 1, 0 \right]$

$\therefore \vec{c}\,'(2) = \underline{\underline{\left[\dfrac{4}{25}, 1, 0 \right]}}$

Speed: $\|\vec{c}\,'(2)\| = \sqrt{\dfrac{16}{625} + 1 + 0} = \sqrt{\dfrac{641}{625}} \approx \underline{\underline{1.01}}$

Acceleration: $\vec{c}\,''(t) = \left[\dfrac{(1+t^2)^2 2 - 2t(2)(1+t^2)(2t)}{(1+t^2)^4}, 0, 0 \right]$

$= \left[\dfrac{2t^4 + 4t^2 + 2 - 8t^2 - 8t^4}{(1+t^2)^4}, 0, 0 \right]$

$= \left[\dfrac{-6t^4 - 4t^2 + 2}{(1+t^2)^4}, 0, 0 \right]$

$\therefore \vec{c}\,''(2) = \left[\dfrac{-6(16) - 4(4) + 2}{625}, 0, 0 \right]$

$= \left[-\dfrac{110}{625}, 0, 0 \right] = \underline{\underline{\left[-\dfrac{22}{125}, 0, 0 \right]}}$

Tangent line: $\vec{c}(2) + s\,\vec{c}\,'(2), \quad s \in \mathbb{R}$

$= \underline{\underline{\left(\dfrac{4}{5}, 2, 1 \right) + s\left(\dfrac{4}{25}, 1, 0 \right)}}$

5.

Tangent: $\vec{c}\,'(t) = (-\sin t, \cos t, 1)$

$\therefore \vec{c}\,'(\frac{\pi}{4}) = \underline{(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 1)}$

Acceleration: $\vec{c}\,''(t) = (-\cos t, -\sin t, 0)$

$\therefore \vec{c}\,''(\frac{\pi}{4}) = \underline{(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 0)}$

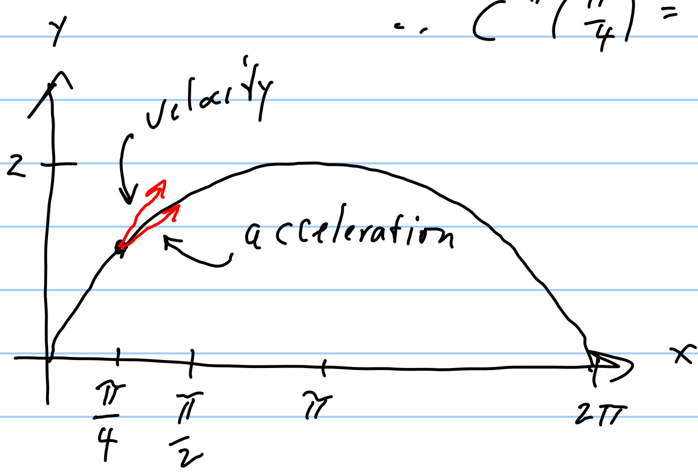
6.

Tangent: $\vec{c}\,'(t) = (1-\cos t, \sin t)$

$\therefore \vec{c}\,'(\frac{\pi}{4}) = \underline{(1-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})}$

Acceleration: $\vec{c}\,''(t) = (\sin t, \cos t)$

$\therefore \vec{c}\,''(\frac{\pi}{4}) = \underline{(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})}$



7.

$\vec{c}\,'(t) = (2t, \cos t, -\sin t) \quad \vec{c}\,''(t) = (2, -\sin t, -\cos t)$

$\therefore \vec{c}\,''(0) = (2, 0, -1)$

$\therefore \vec{F} = m\vec{a} = m\vec{c}\,''(0) = m(2, 0, -1) = \underline{(2m, 0, -m)}$

8.

(a) $\|\vec{c}(t)\| = K$, a constant $\Rightarrow$

$\|\vec{c}(t)\|^2 = K^2 \Rightarrow \vec{c}(t) \cdot \vec{c}(t) = K^2$

$\therefore \frac{d}{dt}\left(\vec{c}(t) \cdot \vec{c}(t)\right) = \frac{d}{dt}(K^2) = 0$

$\therefore \vec{c}\,'(t) \cdot \vec{c}(t) + \vec{c}(t) \cdot \vec{c}\,'(t) = 2\vec{c} \cdot \vec{c}\,' = 0$

$\therefore \vec{c}(t) \cdot \vec{c}\,'(t) = 0 \Rightarrow \underline{\vec{c}(t) \perp \vec{c}\,'(t)}$

(b) (i) Suppose $\vec{c}\,'(t) = (a, b, c)$, a, b, c are constants, not all zero.

$\therefore \frac{d}{dt}\vec{c}\,'(t) = \vec{c}\,''(t) = (0, 0, 0)$

$\therefore\ \vec{c}\,'(t) \cdot (0,0,0) = 0 \implies \vec{c}\,' \perp \vec{c}\,''$

(2) Suppose $\vec{c}\,'(t) \perp \vec{c}\,''(t)$ for all t.

$\therefore\ \vec{c}\,'(t) \cdot \vec{c}\,''(t) = 0 \implies 2\vec{c}\,'(t) \cdot \vec{c}\,''(t) = 0$

$\implies \frac{d}{dt}\left(\vec{c}\,'(t) \cdot \vec{c}\,'(t)\right) = 0$

$\implies \vec{c}\,'(t) \cdot \vec{c}\,'(t) = K$, a constant

$\implies \|\vec{c}\,'(t)\|^2 = K$

$\implies \|\vec{c}\,'(t)\| =$ speed of $\vec{c}(t) = \sqrt{K}$, a constant

9.

(a) $\vec{c}\,'(t) = (-\sin t, \cos t, \sqrt{3}) =$ velocity

$\vec{c}\,''(t) = (-\cos t, -\sin t, 0) =$ acceleration

(b) $\vec{l}(s) = \vec{c}(t) + s\,\vec{c}\,'(t)$, $s \in \mathbb{R}$

$\therefore\ \vec{l}(s) = \vec{c}(0) + s\,\vec{c}\,'(0)$

$= \underline{(1,0,0) + s(0,1,\sqrt{3})}$

(c) Arc length $= \int_0^{2\pi} \|\vec{c}\,'(t)\|\,dt$

$$= \int_0^{2\pi} \sqrt{\sin^2 t + \cos^2 t + 3}\,dt = \int_0^{2\pi} \sqrt{4}\,dt = 2t\Big|_0^{2\pi}$$

$$= \underline{\underline{4\pi}}$$

10.

(a) $\dfrac{\partial F_1}{\partial x} + \dfrac{\partial F_2}{\partial y} + \dfrac{\partial F_3}{\partial z} = \underline{\underline{z\cos(xz) + ye^{xy} + 5x^2y^3z^4}}$

(b) $\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \sin(xz) & e^{xy} & x^2y^3z^5 \end{vmatrix}$

$$= [3x^2y^2z^5 - 0,\ x\cos(xz) - 2xy^3z^5,\ ye^{xy} - 0]$$

$$= \underline{\underline{(3x^2y^2z^5,\ x\cos(xz) - 2xy^3z^5,\ ye^{xy})}}$$

11.

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \dfrac{-Ax}{(x^2+y^2+z^2)^{3/2}} & \dfrac{-Ay}{(x^2+y^2+z^2)^{3/2}} & \dfrac{-Az}{(x^2+y^2+y^2)^{3/2}} \end{vmatrix}$$

$$= \Big[\frac{3}{2} \frac{(-Az)(2y)}{(x^2+y^2+z^2)^{5/2}} - \frac{3}{2} \frac{(-Ay)(2z)}{(x^2+y^2+z^2)^{5/2}}, \quad -Azy + Ayz$$

$$\frac{3}{2} \frac{(-Ax)(2z)}{(x^2+y^2+z^2)^{5/2}} - \frac{3}{2} \frac{(-Az)(2x)}{(x^2+y^2+z^2)^{5/2}}, \quad -Axz + Azx$$

$$\frac{3}{2} \frac{(-Ay)(2x)}{(x^2+y^2+z^2)^{5/2}} - \frac{3}{2} \frac{(-Ax)(2y)}{(x^2+y^2+z^2)^{5/2}} \Big] \quad -Ayx + Axy$$

$$= [0, 0, 0]$$

12.

If $\vec{V} = \nabla \times \vec{F}$, some $\vec{F}$, then $\nabla \cdot (\nabla \times \vec{F}) = \nabla \cdot \vec{V} = 0$

But $\nabla \cdot \vec{V} = \dfrac{\partial}{\partial x}(2x) + \dfrac{\partial}{\partial y}(-3y) + \dfrac{\partial}{\partial z}(4z)$

$$= 2 + (-3) + 4 = 3 \neq 0$$

$\therefore$ $\vec{V}$ is not the curl of some $\vec{F}$.

13.

Let $x = t$. $\therefore y = t^{2/3}$ as $y^3 = t^2$

$z = t^{2/5}$ as $z^5 = t^2$

$\therefore \vec{c}(t) = (t, t^{2/3}, t^{2/5})$

$\therefore \vec{c}\,'(t) = (1, \frac{2}{3}t^{-\frac{1}{3}}, \frac{2}{5}t^{-\frac{3}{5}})$

Between $x=1$ and $x=4$, $t=1$, $t=4$

$\therefore$ Arc length $= \displaystyle\int_1^4 \sqrt{1 + \frac{4}{9}t^{-\frac{2}{3}} + \frac{4}{25}t^{-\frac{6}{5}}}\, dt$

14.

$\vec{c}\,'(t) = (1, \frac{1}{t}, 2(2t)^{-\frac{1}{2}}) = (1, \frac{1}{t}, \frac{2}{\sqrt{2t}})$

$\therefore \displaystyle\int_1^2 \sqrt{1 + \frac{1}{t^2} + \frac{4}{2t}}\, dt = \int_1^2 \sqrt{\frac{2t^2 + 2 + 4t}{2t^2}}\, dt$

$= \displaystyle\int_1^2 \sqrt{\frac{(t+1)^2}{t^2}}\, dt = \int_1^2 \frac{t+1}{t}\, dt$ since $1 \le t \le 2$

$$= \int_1^2 1 + \frac{1}{t}\, dt = \int_1^2 dt + \int_1^2 \frac{dt}{t} = t\Big|_1^2 + \ln t\Big|_1^2$$

$$= 1 + \ln 2 - \ln 1 = \underline{1 + \ln 2}$$

15.

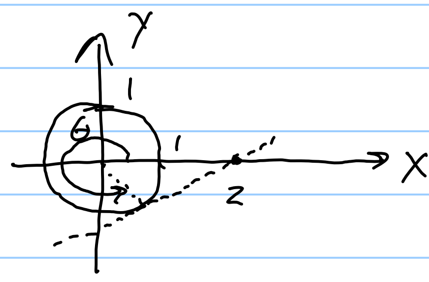
Let $\vec{c}(t) = [\cos(t^2), \sin(t^2), 0]$

(a) Velocity $= \vec{c}\,'(t) = \underline{[-2t\sin(t^2), 2t\cos(t^2), 0]}$

$$\text{Speed} = \|\vec{c}\,'(t)\| = \sqrt{4t^2\sin^2(t^2) + 4t^2\cos^2(t^2)}$$

$$= \sqrt{4t^2} = \underline{2t} \quad \text{as } t \geq 0$$

(b) Tangent line is the path the particle will follow when released.



$\therefore$ Find tangent line from unit circle to $(2, 0)$

This will give θ, then set $\theta = t^2$

$x^2 + y^2 = 1$, $y = -\sqrt{1-x^2}$, $\frac{dy}{dx} = -\frac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x)$

$\therefore$ slope of line $= \frac{x}{\sqrt{1-x^2}}$ at point $(x, -\sqrt{1-x^2})$

$\therefore$ Two points: $(2,0)$ and $(x, -\sqrt{1-x^2})$

$\therefore \frac{0-(-\sqrt{1-x^2})}{2-x} = \frac{x}{\sqrt{1-x^2}} = \frac{y_1 - y_2}{x_1 - x_2} = m$

$\therefore 1-x^2 = x(2-x) = 2x - x^2$, $x = \frac{1}{2}$

$\therefore$ release at point $\underline{\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0\right)}$

(c) $\vec{c}(t) = [\cos(t^2), \sin(t^2), 0] = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0\right)$

$\therefore t^2 = 2\pi - \frac{\pi}{3}$ since $\cos(-\frac{\pi}{3}) = \frac{1}{2}$

$\sin(-\frac{\pi}{3}) = -\frac{\sqrt{3}}{2}$

$= \frac{5\pi}{3}$

$\therefore$ Release at $\underline{t = \sqrt{\frac{5\pi}{3}}}$

(d) $\vec{c}'(\sqrt{5\pi/3}) = \left(-2\sqrt{\frac{5\pi}{3}}\right)\sin\left(\frac{5\pi}{3}\right), 2\sqrt{\frac{5\pi}{3}}\cos\left(\frac{5\pi}{3}\right), 0)$

$= \left\{-2\sqrt{\frac{5\pi}{3}}\left(-\frac{\sqrt{3}}{2}\right), 2\sqrt{\frac{5\pi}{3}}\left(\frac{1}{2}\right), 0\right\}$

$$\text{Velocity} = \left(\sqrt{5\pi}, \sqrt{5\pi/3}, 0\right)$$

$$\text{speed} = \sqrt{5\pi + \frac{5\pi}{3} + 0} = \sqrt{\frac{20\pi}{3}} = 2\sqrt{\frac{5\pi}{3}}$$

(e)

Released a time $t = \sqrt{\frac{5\pi}{3}}$.

$$\frac{\text{Distance to travel}}{\text{speed}} = \frac{\left\|\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0\right) - (2,0,0)\right\|}{2\sqrt{5\pi/3}}$$

$$= \frac{\sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 + 0}}{2\sqrt{\frac{5\pi}{3}}} = \frac{\sqrt{\frac{9}{4} + \frac{3}{4}}}{2\sqrt{5\pi/3}}$$

$$= \frac{\sqrt{3}}{2\sqrt{\frac{5\pi}{3}}} = \frac{3}{2\sqrt{5\pi}}$$

$$\therefore t = \sqrt{\frac{5\pi}{3}} + \frac{3}{2\sqrt{5\pi}} = \frac{5\pi}{3\sqrt{5\pi}} + \frac{3}{2\sqrt{5\pi}} = \left(\frac{5\pi}{3} + \frac{3}{2}\right)\frac{1}{\sqrt{5\pi}}$$

16.

Assume $\vec{F}, \vec{r} \in R^3$

(a) $\vec{F} = m\vec{a} = m\vec{r}''(t) = -k\vec{r}(t)$

Let $\vec{r}(t) = (r_x(t), r_y(t), r_z(t))$

$\therefore m\frac{d^2}{dt^2}r_x(t) = -k r_x(t)$, or $r_x'' = \frac{-k}{m}r$

$m\frac{d^2}{dt^2}r_y(t) = -k r_y(t)$, or $r_y'' = \frac{-k}{m}r$

$m\frac{d^2}{dt^2}r_z(t) = -k r_z(t)$, or $r_z'' = -\frac{k}{m}r$

(b) Let $r_i(t) = a_i\cos\left(\sqrt{\frac{k}{m}}t\right) + b_i\sin\left(\sqrt{\frac{k}{m}}t\right)$

$\therefore r_i''(t) = -a_i\frac{k}{m}\cos\left(\sqrt{\frac{k}{m}}t\right) - b_i\frac{k}{m}\sin\left(\sqrt{\frac{k}{m}}t\right)$

$= -\frac{k}{m}r_i(t)$

$\therefore$ if $\vec{r}(0) = \vec{0}$, then $r_i(0) = a_i\cos(0) + b_i\sin(0)$

$\therefore 0 = a_i$

$\therefore r_i(t) = b_i\sin\left(\sqrt{\frac{k}{m}}t\right)$

$\vec{r}'(0) = 2\hat{j} + \hat{k} \Rightarrow r_x'(0) = 0 = b_x\sqrt{\frac{k}{m}}\cos(0)$

$\Rightarrow b_x = 0$

$r_y'(0) = 2 = b_y\sqrt{\frac{k}{m}}\cos(0)$

$\Rightarrow b_y = 2\sqrt{\frac{m}{k}}$

$$r_z'(0) = 1 = b_z\sqrt{\frac{k}{m}}\cos(0)$$

$$\Rightarrow b_z = \sqrt{\frac{m}{k}}$$

$$\therefore \vec{r}(t) = \left[0,\ 2\sqrt{\frac{m}{k}}\sin\left(\sqrt{\frac{k}{m}}\,t\right),\ \sqrt{\frac{m}{k}}\sin\left(\sqrt{\frac{k}{m}}\,t\right)\right]$$

17.

Let $t = x-1 = 2y+1 = 3z+2$

$$\therefore x = t+1$$
$$y = (t-1)/2$$
$$z = (t-2)/3$$

$$\therefore \vec{c}(t) = \left[t+1,\ \frac{t-1}{2},\ \frac{t-2}{3}\right]$$

18.

Let $t = x = y^3 = z^2+1$

$$\therefore x = t$$
$$y = t^{1/3}$$
$$z = (t-1)^{\frac{1}{2}},\ t \geq 1$$

$$\therefore \vec{c}(t) = \left[t,\ t^{1/3},\ \sqrt{t-1}\right],\ t \geq 1$$

19.

To show a flow line, must show $\vec{c}\,'(t) = \vec{F}(\vec{c}(t))$, i.e., the force vector is the same as the velocity vector, at the given point.

$$\therefore \vec{F}\left(\frac{1}{1-t}, 0, \frac{e^t}{1-t}\right) = \left[\left(\frac{1}{1-t}\right)^2, 0, \frac{e^t}{1-t}\left(1+\frac{1}{1-t}\right)\right]$$

$$= \left[\frac{1}{(1-t)^2}, 0, \frac{e^t(2-t)}{(1-t)^2}\right]$$

$$\vec{c}\,'(t) = \left[-\frac{1}{(1-t)^2}(-1), 0, \frac{(1-t)e^t - e^t(-1)}{(1-t)^2}\right]$$

$$= \left[\frac{1}{(1-t)^2}, 0, \frac{e^t(2-t)}{(1-t)^2}\right]$$

$$\therefore \vec{F}(\vec{c}(t)) = \vec{c}\,'(t)$$

20.

If $\vec{c}(t)$ is a path, the velocity vector $\vec{c}\,'(t)$ must equal the force vector at point $\vec{c}(t)$.

$$\therefore \vec{F}(\vec{c}(t)) = \vec{c}\,'(t).$$

$$\vec{c}\,'(t) = g'(t)\left[-\sin(g(t)), \cos(g(t))\right].$$

$$\vec{F}(\vec{c}(t)) = f\left(\cos^2(g(t)) + \sin^2(g(t))\right)\left[-\sin(g(t)), \cos(g(t))\right]$$

$$= f(1)\left[-\sin(g(t)), \cos(g(t))\right]$$

$$\therefore \vec{F}(\vec{c}(t)) = \vec{c}\,'(t) \iff g'(t) = f(1)$$

$$\therefore \underline{g'(t) = f(1)}$$

21.

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(3y) + \frac{\partial}{\partial z}(4z)$$

$$= 2 + 3 + 4 = \underline{9}$$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x & 3y & 4z \end{vmatrix}$$

$$= [0-0, 0-0, 0-0] = \underline{\vec{0}}$$

22.

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z^2)$$

$$= \underline{2x + 2y + 2z}$$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/z \\ x^2 & y^2 & z^2 \end{vmatrix}.$$

$$= [0-0, 0-0, 0-0] = \underline{\vec{0}}$$

23.

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x+y) + \frac{\partial}{\partial y}(y+z) + \frac{\partial}{\partial z}(z+x)$$

$$= 1 + 1 + 1 = \underline{3}$$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+y & y+z & z+x \end{vmatrix}$$

$$= [0-1, 0-1, 0-1] = \underline{[-1, -1, -1]}$$

24.

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(3xy) + \frac{\partial}{\partial z}(z)$$

$$= 1 + 3x + 1 = \underline{2 + 3x}$$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & 3xy & z \end{vmatrix}$$

$$= [0-0, 0-0, 3y-0] = \underline{[0, 0, 3y]}$$

25.

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(z) + \frac{\partial}{\partial z}(x)$$

$$= 0 + 0 + 0 = \underline{0}$$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & z & x \end{vmatrix} = [0-1, 0-1, 0-1]$$

$$= \underline{[-1, -1, -1]}$$

"at the point" $(x,y,z) = (1,1,1)$ is irrelevant in this case.

26.

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x+y)^3 + \frac{\partial}{\partial y}(\sin(xy)) + \frac{\partial}{\partial z}(\cos(xyz))$$

$$= 3(x+y)^2 + x\cos(xy) - xy\sin(xyz)$$

$$\therefore \nabla \cdot \vec{F}(2,0,1) = 3(2)^2 + 2\cos(0) - 0$$

$$= 12 + 2 = \underline{14}$$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (x+y)^3 & \sin(xy) & \cos(xyz) \end{vmatrix}$$

$$= [-xz\sin(xyz) - 0,\ 0 - (-yz\sin(xyz)),\ y\cos(xy) - 3(x+y)^2]$$

$$= [-xz\sin(xyz),\ yz\sin(xyz),\ y\cos(xy) - 3(x+y)^2]$$

$$\therefore \nabla \times \vec{F}(2,0,1) = [0,\ 0,\ 0 - 3(2)^2] = \underline{[0, 0, -12]}$$

27.

$$\nabla f = [f_x, f_y, f_z] = \underline{[ye^{xy} - y\sin(xy),\ xe^{xy} - x\sin(xy), 0]}$$

$$\nabla \times (\nabla f) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ ye^{xy} - y\sin(xy) & xe^{xy} - x\sin(xy) & 0 \end{vmatrix}$$

$$= [0-0,\ 0-0,\ e^{xy} + xye^{xy} - \sin(xy) - xy\cos(xy) - (e^{xy} + xye^{xy} - \sin(xy) - xy\cos(xy))]$$

$$= \underline{[0,0,0]}$$

28.

$$\nabla f = \left[\frac{(x^2+y^2)(2x) - (x^2-y^2)(2x)}{(x^2+y^2)^2},\ \frac{(x^2+y^2)(-2y) - (x^2-y^2)(2y)}{(x^2+y^2)^2}\right]$$

$$= \left[\frac{4xy^2}{(x^2+y^2)^2}, \frac{-4yx^2}{(x^2+y^2)^2}, 0 \right]$$

$$\nabla \times (\nabla f) = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{4xy^2}{(x^2+y^2)^2} & \frac{-4yx^2}{(x^2+y^2)^2} & 0 \end{bmatrix}$$

$$= \left[0-0, 0-0, \frac{(x^2+y^2)^2(-8yx) - (-4yx^2)(2)(x^2+y^2)(2x)}{(x^2+y^2)^4} - \frac{\left((x^2+y^2)^2(8xy) - (4xy^2)(2)(x^2+y^2)(2y)\right)}{(x^2+y^2)^4} \right]$$

$$= \left[0, 0, \frac{(-16xy)(x^2+y^2)^2 + 16yx^3(x^2+y^2) + 16xy^3(x^2+y^2)}{(x^2+y^2)^4} \right]$$

$$= \left[0, 0, \frac{(-16xy)(x^2+y^2)^2 + 16xy(x^2+y^2)(x^2+y^2)}{(x^2+y^2)^4} \right]$$

$$= \left[0, 0, \frac{(-16xy)(x^2+y^2)^2 + 16xy(x^2+y^2)^2}{(x^2+y^2)^4} \right] = \underline{[0, 0, 0]}$$

29.

$$\nabla f = [2xe^{x^2} + y^2\sin(xy^2), 2yx\sin(xy^2), 0]$$

$$\nabla \times (\nabla f) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xe^{x^2} + y^2\sin(xy^2) & 2yx\sin(xy^2) & 0 \end{vmatrix}$$

$$= [0, 0, 2y\sin(xy^2) + 2xy^3\cos(xy^2) - (2y\sin(xy^2) + 2xy^3\cos(xy^2))]$$

$$= \underline{[0, 0, 0]}$$

30.

$$\nabla f = \left[\frac{1}{1+(x^2+y^2)^2}\cdot 2x, \frac{1}{1+(x^2+y^2)^2}\cdot 2y, 0\right]$$

$$= \underline{\left[\frac{2x}{1+(x^2+y^2)^2}, \frac{2y}{1+(x^2+y^2)^2}, 0\right]}$$

$$\nabla \times (\nabla f) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{2x}{1+(x^2+y^2)^2} & \frac{2y}{1+(x^2+y^2)^2} & 0 \end{vmatrix}$$

$$= \left[0, 0, \frac{(1+(x^2+y^2)^2)(0) - (2y)(2)(x^2+y^2)(2x)}{[1+(x^2+y^2)^2]^2} - \frac{(1+(x^2+y^2)^2)(0) - (2x)(2)(x^2+y^2)(2y)}{[1+(x^2+y^2)^2]^2}\right]$$

$$= \left[0, 0, -\frac{8xy(x^2+y^2) + 8xy(x^2+y^2)}{[1+(x^2+y^2)^2]^2}\right] = \underline{[0,0,0]}$$

31.

(a) $\nabla f = [f_x, f_y, f_z] = \underline{[yz^2, xz^2, 2xyz]}$

(b) $$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & yz & zy \end{vmatrix} = [z-y, 0-0, 0-x]$$

$$= \underline{[z-y, 0, -x]}$$

(c) $\nabla \times (f\vec{F}) = f(\nabla \times \vec{F}) + \nabla f \times \vec{F}$

$$= xyz^2 [z-y, 0, -x] + [yz^2, xz^2, 2xyz] \times [z-y, 0, -x]$$

$$= [xyz^3 - xy^2z^2, 0, -x^2yz^2] + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ yz^2 & xz^2 & 2xyz \\ xy & yz & zy \end{vmatrix}$$

$$= [xyz^3 - xy^2z^2, 0, -x^2yz^2] + [xyz^3 - 2xy^2z^2, 2x^2y^2z - y^2z^3, y^2z^3 - x^2yz^2]$$

$$= \underline{[2xyz^3 - 3xy^2z^2, 2x^2y^2z - y^2z^3, y^2z^3 - 2x^2yz^2]}$$

$$f\vec{F} = [x^2y^2z^2, xy^2z^3, xy^2z^3]$$

$$\nabla \times (f\vec{F}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2y^2z^2 & xy^2z^3 & xy^2z^3 \end{vmatrix}$$

$$= [2xyz^3 - 3xy^2z^2,\ \underline{2x^2yz - y^2z^3,\ y^2z^3} - 2x^2yz^2]$$

Same as above

32.

(a) $\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(2xye^z) + \frac{\partial}{\partial y}(e^z x^2) + \frac{\partial}{\partial z}(x^2ye^z + z^2)$

$$= 2ye^z + 0 + x^2ye^z + 2z$$

$$= \underline{2z + (2+x^2)}ye^z$$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xye^z & e^z x^2 & x^2ye^z + z^2 \end{vmatrix}$$

$$= [x^2e^z - e^zx^2,\ 2xye^z - 2xye^z,\ 2xe^z - 2xe^z]$$

$$= \underline{[0, 0, 0]}$$

(b) $\nabla f = [f_x, f_y, f_z] = [2xye^z, e^z x^2, x^2ye^z + z^2] = \vec{F}$

$f_x = 2xye^z \Rightarrow f = x^2ye^z + g(y, z)$, [1]

$g(y,z)$ some function of y and z but not x. $\therefore \frac{\partial}{\partial x} g = 0$.

Similarly, $f_y = e^z x^2 \Rightarrow f = x^2 y e^z + h(x,z)$, [2]

h some function of x, z, not y

Comparing [1], [2], $\frac{\partial g(y,z)}{\partial y} = \frac{\partial h(x,z)}{\partial x}$

which will be satisfied if $g(y,z) = h(x,z) = j(z)$,

j some function of z, not x, not y.

$\therefore$ So far, $f(x,y,z) = x^2 y e^z + j(z)$.

But $f_z = x^2 y e^z + z^2 = x^2 y e^z + j'(z)$.

$\therefore j(z) = \frac{z^3}{3} + c$, c some constant.

$\therefore \underline{f(x,y,z)} = \underline{x^2 y e^z + \frac{z^3}{3}} + c$, c some constant.

33.

$$\operatorname{div} \vec{F} = \frac{\partial}{\partial x}\left(-y f(x^2+y^2)\right) + \frac{\partial}{\partial y}\left(x f(x^2+y^2)\right)$$

$$= -y(f')(2x) + x(f')(2y)$$

$$= 2xy(f' - f') = 0 \quad (\text{note } f: \mathbb{R}^1 \to \mathbb{R}^1).$$

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y\, f(x^2+y^2) & x f(x^2+y^2) & 0 \end{vmatrix}$$

$$= [0, 0, f(x^2+y^2) + x f'(x^2+y^2)(2x) - (-f(x^2+y^2) - y f'(x^2+y^2)(2y))]$$

$$= [0, 0, 2f(x^2+y^2) + (2x^2+2y^2) f'(x^2+y^2)]$$

$\operatorname{div} \vec{F} = 0$ means the fluid is incompressible as it flows.

$\operatorname{curl} \vec{F} \neq 0$ means points of the fluid rotate as the fluid flows.

34.

(a) $\vec{l}(t) = \vec{c}(t) + s\vec{c}\,'(t)$

$\therefore \vec{c}(\frac{\pi}{4}) = (2\sqrt{2}, \frac{\sqrt{2}}{2}, \frac{\pi}{4})$

$\vec{c}\,'(t) = (-4\sin t, \cos t, 1) \quad \therefore \vec{c}\,'(\frac{\pi}{4}) = (-2\sqrt{2}, \frac{\sqrt{2}}{2}, 1)$

$\therefore \vec{l}\,(\frac{\pi}{4}) = \left[2\sqrt{2}, \frac{\sqrt{2}}{2}, \frac{\pi}{4}\right] + s\left[-2\sqrt{2}, \frac{\sqrt{2}}{2}, 1\right], \; s \in R$

(b) $\vec{F} = m\vec{a} = m\vec{c}\,''(\frac{\pi}{4})$

$\vec{c}\,''(t) = (-4\cos t, -\sin t, 0)$

$\vec{c}\,''(\frac{\pi}{4}) = (-2\sqrt{2}, -\frac{\sqrt{2}}{2}, 0)$

$\therefore \vec{F} = m(-2\sqrt{2}, -\frac{\sqrt{2}}{2}, 0)$

(c) $\displaystyle\int_0^{\frac{\pi}{4}} \|\vec{c}\,'(t)\|\,dt = \int_0^{\frac{\pi}{4}} \sqrt{16\sin^2 t + \cos^2 t + 1}\,dt$

35.

(a) Note $g(1,0,0) = 1$. $g: R^3 \to R^1$, $f: R^3 \to R^1$

$\therefore \nabla f = h'(g)(\nabla g)$ by chain rule.

$\nabla g = (3x^2, 5z, 5y + 2z) \quad \therefore \nabla g(1,0,0) = (3,0,0)$

$\therefore \nabla f(1,0,0) = h'(g(1,0,0)) \cdot \nabla g(1,0,0)$

$= h'(1)\,(3,0,0) = \frac{1}{2}(3,0,0)$

∇f is the direction of greatest change of f.

Starting at $(1,0,0)$, this direction is $(\frac{3}{2},0,0)$

If $\vec{s}$, a unit vector, is any other direction,

then we want $\nabla f \cdot \vec{s} = \|\nabla f\| \, \|\vec{s}\| \cos\theta$

$= \|\nabla f\| \cos\theta = 50\% \, \|\nabla f\|$, or $\cos\theta = \frac{1}{2}$

$\therefore \theta = 60°$, or in a $60°$ direction from $(1,0,0)$

This is a "cone" of possible directions

making an angle of $60° = \frac{\pi}{3}$ radians with $\hat{i}$

(5) As above, $\nabla g = (3x^2, 5z, 5y + 2z)$

$$\nabla \times (\nabla g) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3x^2 & 5z & 5y+2z \end{vmatrix}$$

$= [5-5,\ 0-0,\ 0-0] = [0,0,0]$

36.

(a) $x^2+y^2+z^2=3$ (a sphere) and $y=1$ (a plane)

$\Rightarrow x^2+1+z^2=3$, or $x^2+z^2=2$, a circle in

the $y=1$ plane.

$\therefore \frac{x^2}{2}+\frac{z^2}{2}=1$, or $\left(\frac{x}{\sqrt{2}}\right)^2+\left(\frac{z}{\sqrt{2}}\right)^2=1$

$\therefore$ use $\frac{x}{\sqrt{2}}=\cos t$, $\frac{z}{\sqrt{2}}=\sin t$

$\therefore \underline{\vec{c}(t)=(\sqrt{2}\cos t, 1, \sqrt{2}\sin t)}$, $0\le t\le 2\pi$

Another method:

The gradient is $\perp$ to a surface, so the curve of intersection must be $\perp$ to each gradient. Let $f(x,y,z)=x^2+y^2+z^2=3$

$g(x,y,z)=y=1$.

$\therefore \nabla f=[2x,2y,2z]$ $\nabla g=[0,1,0]$

$\nabla f \times \nabla g$ is perpendicular to each gradient.

$$\nabla f \times \nabla g = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2x & 2y & 2z \\ 0 & 1 & 0 \end{vmatrix} = [-2z, 0, 2x]$$

$\therefore$ Velocity of curve of intersection is in direction of $[-z, 0, x]$. If $\vec{c}(x(t), y(t), z(t))$ is the curve, then $x'(t) = -z(t)$, $z'(t) = x(t)$

or $x''(t) = -x(t)$, $z''(t) = -z(t)$

$\therefore$ Let $x(t) = \cos(t)$, $y(t) = \sin(t)$.

$\therefore \vec{c}\,'(t) = [-\sin(t), 0, \cos(t)]$

Since $y = 1$ is a criterion, let

$\vec{c}(t) = [\cos(t), 1, \sin(t)]$.

Since $x^2 + y^2 + z^2 = 3$ is a criterion, let

$\vec{c}(t) = [\sqrt{2}\cos(t), 1, \sqrt{2}\sin(t)]$.

$\therefore \vec{c}\,'(t)$ has direction of $[-z, 0, x]$.

$$\therefore \vec{c}(t) = [\sqrt{2}\cos(t), 1, \sqrt{2}\sin(t)], \quad 0 \le t \le 2\pi$$

(b) $\vec{l}(t) = \vec{c}(t) + s\,\vec{c}\,'(t)$

For $(1,1,1) = [\sqrt{2}\cos t, 1, \sqrt{2}\sin t]$, $t = \frac{\pi}{4}$

$\vec{c}\,'(t) = [-\sqrt{2}\sin t, 0, \sqrt{2}\cos t]$

$\therefore \vec{c}\,'(\frac{\pi}{4}) = [-1, 0, 1]$.

$\therefore \vec{l}(\frac{\pi}{4}) = \underline{(1,1,1) + s(-1,0,1)}$, $s \in R$.

(c)

$$\int_0^{2\pi} \|\vec{c}\,'(t)\|\,dt = \int_0^{2\pi} \sqrt{2\sin^2 t + 0 + 2\cos^2 t}\,dt$$

$$= \underline{\int_0^{2\pi} \sqrt{2}\,dt} = \underline{2\sqrt{2}\,\pi}$$

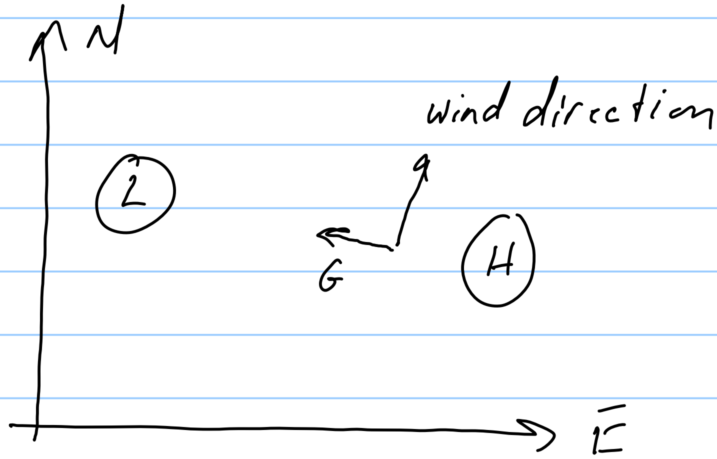
37.

(a) $\|\vec{G}\| = \sqrt{(-P_x)^2 + (-P_y)^2} = \sqrt{P_x^2 + P_y^2}$

(b) Let m = mass of a given pocket of air,

Then $\vec{F} = m\vec{a}$ becomes $\vec{G} = m\vec{a}$, or $\vec{a} = \dfrac{\vec{G}}{m}$

(c)

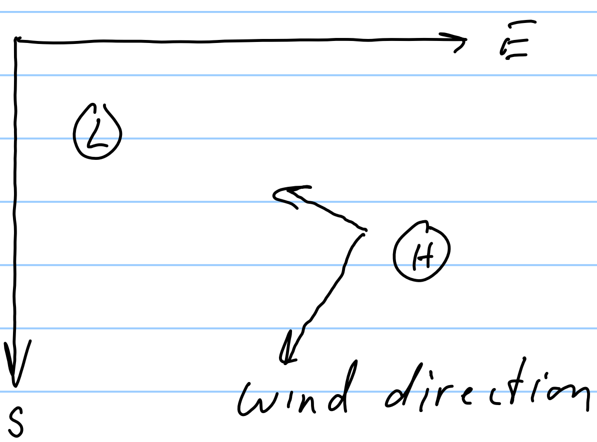


"wind at back, H on right, L on left"

H = High pressure

L = Low pressure

(d)



"wind at back, H on left, L on right"

H = High pressure

L = Low pressure

38.

(a) Outside: $\nabla u = [u_x, u_y, u_z]$, $u = \dfrac{m}{\sqrt{x^2+y^2+z^2}}$

$$u_x = -\frac{1}{2} m (x^2+y^2+z^2)^{-\frac{3}{2}}(2x) = -mx(x^2+y^2+z^2)^{3/2}$$

$$= -\frac{mx}{r^3}, \text{ as } r = \|\vec{r}\| = \sqrt{x^2+y^2+z^2}$$

Similarly, $u_y = -\dfrac{my}{r^3}$, $u_z = -\dfrac{mz}{r^3}$

$$\therefore \nabla u = \left[-\frac{m}{r^3}x, -\frac{m}{r^3}y, -\frac{m}{r^3}z\right]$$

$$= -\frac{m}{r^3}[x,y,z] = -\underline{\underline{\frac{m}{r^3}\vec{r} = \vec{F}}} \quad (r>a)$$

Inside: $u = \dfrac{3m}{2a} - \dfrac{m(x^2+y^2+z^2)}{2a^3}$

$$\therefore u_x = \frac{-m}{2a^3}(2x) = -\frac{mx}{a^3}$$

$$\therefore u_y = -\frac{my}{a^3} \quad u_z = -\frac{mz}{a^3}$$

$$\therefore \nabla u = -\frac{m}{a^3}[x,y,z] = -\underline{\underline{\frac{m}{a^3}\vec{r} = \vec{F}}} \quad (r \le a)$$

(b)

From (a), $\nabla u = -\frac{m}{a^3}[x, y, z]$

$\therefore u_{xx} = -\frac{m}{a^3}, \quad u_{yy} = -\frac{m}{a^3}, \quad u_{zz} = -\frac{m}{a^3}$

$\therefore u_{xx} + u_{yy} + u_{zz} = \underline{-\frac{3m}{a^3}}$ = a constant

(c)

From (a), $\nabla u = -\frac{m}{r^3}\vec{r} = -\frac{m}{(x^2+y^2+z^2)^{3/2}}[x, y, z]$

$$\therefore u_{xx} = \frac{(x^2+y^2+z^2)^{3/2} - x\left(\frac{3}{2}\right)(x^2+y^2+z^2)^{\frac{1}{2}}(2x)}{(x^2+y^2+y^2)^3}$$

$$= \frac{(x^2+y^2+z^2)^{3/2} - 3x^2(x^2+y^2+z^2)^{\frac{1}{2}}}{(x^2+y^2+z^2)^3}$$

$$\therefore u_{yy} = \frac{(x^2+y^2+z^2)^{3/2} - 3y^2(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3}$$

$$u_{zz} = \frac{(x^2+y^2+z^2)^{3/2} - 3z^2(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3}$$

$$\therefore \underline{u_{xx} + u_{yy} + u_{zz}} = \frac{3(x^2+y^2+z^2)^{3/2} - 3(x^2+y^2+z^2)(x^2+y^2+z^2)^{\frac{1}{2}}}{(x^2+y^2+z^2)^3}$$

$$= \frac{3(x^2+y^2+z^2)^{3/2} - 3(x^2+y^2+z^2)^{3/2}}{(x^2+y^2+z^2)^3}$$

$$\underline{= 0}$$

39.

(a) $\vec{c}(\theta) = [R\cos\theta, R\sin\theta, \rho\theta]$

$\therefore \vec{c}\,'(\theta) = [-R\sin\theta, R\cos\theta, \rho]$

Height $z_0 \Rightarrow \theta_0$ from $z = \rho\theta$, and $z_1 = \rho\theta_1$

$$\text{Arc length} = \int_{\theta_1}^{\theta_0} \|\vec{c}\,'(\theta)\| \, d\theta$$

$$= \int_{\theta_1}^{\theta_0} \sqrt{R^2\sin^2\theta + R^2\cos^2\theta + \rho^2} \, d\theta$$

$$= \int_{\theta_1}^{\theta_0} \sqrt{R^2+\rho^2}\, d\theta = \sqrt{R^2+\rho^2}\, \theta \Big|_{\theta_1}^{\theta_0} =$$

$$\sqrt{R^2+\rho^2}\,(\theta_0 - \theta_1) = \sqrt{R^2+\rho^2}\left(\frac{z_0}{\rho} - \frac{z_1}{\rho}\right)$$

$$= \underline{\frac{\sqrt{R^2+\rho^2}}{\rho}(z_0 - z_1)}$$

(b) From (a), $s(z) = \dfrac{\sqrt{R^2+\rho^2}}{\rho}(z_0 - z)$

$$\therefore \frac{ds}{dz} = -\frac{\sqrt{R^2+\rho^2}}{\rho}$$

Distance/speed = time.

$\therefore \left(\dfrac{ds}{dz}\right) \Big/ \left(\dfrac{ds}{dt}\right)$ = an increment of time as a function of $z = T(z)$ $= \dfrac{dt}{dz}$

$\therefore \displaystyle\int_{z_0}^{0} T(z)\, dz$ should give time from z_0 to $z = 0$.

$$\frac{ds}{dz} \Big/ \frac{ds}{dt} = -\frac{\sqrt{R^2+\rho^2}}{\rho} \cdot \frac{1}{\sqrt{2g(z_0 - z)}}$$

$$\therefore \int_{z_0}^{0} -\frac{\sqrt{R^2+\rho^2}}{\rho\sqrt{2g}} \cdot \frac{1}{\sqrt{z_0 - z}}\, dz = \left(-\frac{\sqrt{R^2+\rho^2}}{\rho\sqrt{2g}}\right)\left[-2(z_0 - z)^{\frac{1}{2}}\right]_{z_0}^{0}$$

$$= \left(-\sqrt{\frac{R^2+\rho^2}{2g\rho^2}}\right)\left[-2(z_0)^{\frac{1}{2}}+0\right]$$

$$= 2\sqrt{\frac{R^2+\rho^2}{2g\rho^2}}\sqrt{z_0} = \sqrt{\frac{4z_0(R^2+\rho^2)}{2g\rho^2}}$$

$$= \underline{\sqrt{\frac{2z_0(R^2+\rho^2)}{g\rho^2}}}$$

40.

(a) angular velocity 4 $\Rightarrow \|\vec{\omega}\| = 4$
counterclockwise from positive z axis $\Rightarrow \underline{\vec{\omega} = 4\hat{k}}$

(b) $\vec{\omega} \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 4 \\ 5\sqrt{2} & -5\sqrt{2} & 0 \end{vmatrix} = (0-(-20\sqrt{2}), 20\sqrt{2}-0, 0-0)$

$$= \underline{(20\sqrt{2}, 20\sqrt{2}, 0)}$$

(c) $\vec{v} = \vec{\omega} \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 4 \\ 0 & 5\sqrt{3} & 5 \end{vmatrix} = \underline{(-20\sqrt{3}, 0, 0)}$

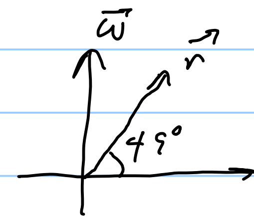
41.

$\|\vec{v}\| = \|\vec{\omega} \times \vec{r}\|$ Here, $\vec{\omega}$ = speed / (distance from axis)

If $R = 3960$, then $2\pi R$ = circumference

$\therefore\ \omega = \dfrac{2\pi(3960 \text{ miles})}{24 \text{ hrs}} \Big/ (3960 \text{ miles}) = \dfrac{2\pi}{24 \text{ hrs}}$

At 49°N, angle between $\vec{\omega}$ and $\vec{r}$ is 41° $(= 90 - 49)$



$\therefore\ \|\vec{v}\| = \|\vec{\omega}\|\,\|\vec{r}\| \sin(41^\circ)$

$= \left(\dfrac{2\pi}{24}\right)(3960) \sin(41^\circ)$

$= \left(\dfrac{2\pi}{24}\right)(3960)(0.656) = \underline{680 \text{ miles/hr}}$