1.

(a) $$\int_0^1 (1-x^3+xy)\,dx = x - \frac{x^4}{4} + \frac{x^2}{2}y \Big|_0^1 = 1 - \frac{1}{4} + \frac{1}{2}y$$
$$= \frac{3}{4} + \frac{1}{2}y$$
$$\therefore \int_0^1 \left(\frac{3}{4} + \frac{1}{2}y\right) dy = \frac{3}{4}y + \frac{y^2}{4}\Big|_0^1 = \frac{3}{4} + \frac{1}{4} = \underline{1}$$

(b) $$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x \sin y \, dx = \sin x \sin y \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$
$$= (1)\sin y - (-1)\sin y = 2\sin y$$
$$\therefore \int_0^{\frac{\pi}{2}} 2\sin y \, dy = -2\cos y \Big|_0^{\frac{\pi}{2}} = 0 - (-2(1)) = \underline{2}$$

(c) $$\int_2^4 \left(\frac{x}{y} + \frac{y}{x}\right) dx = \frac{x^2}{2y} + y\ln x \Big|_2^4$$
$$= \frac{16}{2y} + y\ln 4 - \left(\frac{4}{2y} + y\ln 2\right)$$

$$= \frac{6}{y} + y(\ln 4 - \ln 2) = \frac{6}{y} + y\ln 2$$

$$\therefore \int_1^2 \left(\frac{6}{y} + y\ln 2\right) dy = 6\ln y + \frac{y^2}{2}\ln 2 \Big|_1^2$$

$$= 6\ln 2 + \frac{4}{2}\ln 2 - \left(0 + \frac{1}{2}\ln 2\right)$$

$$= 8\ln 2 - \frac{1}{2}\ln 2 = \underline{\frac{15}{2}\ln 2}$$

$$= 7\ln 2 + \frac{1}{2}\ln 2 = \underline{\ln 128 + \ln\sqrt{2}}$$

(d) $$\int_0^{\pi/4} \tan x \sec^2 y \, dx = -\log|\cos x| \sec^2 y \Big|_0^{\pi/4}$$

$$= -\log\left(\frac{\sqrt{2}}{2}\right)\sec^2 y - [0 \sec^2 y]$$

$$= \log\sqrt{2} \sec^2 y$$

$$\therefore \int_0^{\pi/4} \log\sqrt{2} \sec^2 y \, dy = \log\sqrt{2} \tan y \Big|_0^{\pi/4}$$

$$= \log\sqrt{2}(1) - 0 = \underline{\ln\sqrt{2}}$$

2.

(a) $$\int_0^1 \int_0^1 (1 - x^3 + xy) \, dy \, dx$$

$$\int_0^1 (1 - x^3 + xy) \, dy = y - x^3 y + x\frac{y^2}{2} \Big|_0^1 = 1 - x^3 + \frac{x}{2}$$

$$\therefore \int_0^1 \left(1 - x^3 + \frac{x}{2}\right) dx = x - \frac{x^4}{4} + \frac{x^2}{4}\Big|_0^1 = 1 - \frac{1}{4} + \frac{1}{4} = \underline{1}$$

(b)

$$\int_{-\pi/2}^{\pi/2} \int_0^{\pi/2} \cos x \sin y \, dy \, dx$$

$$\int_0^{\pi/2} \cos x \sin y \, dy = -\cos x \cos y \Big|_0^{\pi/2} = 0 - (-\cos x (1))$$

$$= \cos x$$

$$\therefore \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x \, dx = \sin x \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 1 - (-1) = \underline{2}$$

(c)

$$\int_2^4 \int_1^2 \left(\frac{x}{y} + \frac{y}{x}\right) dy \, dx$$

$$\int_1^2 \left(\frac{x}{y} + \frac{y}{x}\right) dy = x \ln y + \frac{y^2}{2x} \Big|_1^2$$

$$= x \ln 2 + \frac{2}{x} - \left(x \cdot 0 + \frac{1}{2x}\right)$$

$$= x \ln 2 + \frac{3}{2x}$$

$$\therefore \int_2^4 x \ln 2 + \frac{3}{2x} \, dx = \frac{x^2}{2} \ln 2 + \frac{3}{2} \ln x \Big|_2^4$$

$$= 8 \ln 2 + \frac{3}{2} \ln 4 - \left(2 \ln 2 + \frac{3}{2} \ln 2\right)$$

$$= 8 \ln 2 + \frac{3}{2} \ln 2^2 - 2 \ln 2 - \frac{3}{2} \ln 2$$

$= (7\tfrac{1}{2})\ln 2 = \underline{\tfrac{15}{2}\ln 2} = 7\ln 2 + \ln\sqrt{2}$

$= \ln 2^7 + \ln\sqrt{2} = \underline{\ln 128 + \ln\sqrt{2}}$

(d) $$\int_0^{\pi/4}\int_0^{\pi/2} \tan x \sec^2 y \, dy\, dx$$

$$\int_0^{\pi/4} \tan x \sec^2 y \, dy = \tan x \tan y \Big|_0^{\pi/4}$$

$$= \tan x (1) - \tan x (0) = \tan x$$

$$\therefore \int_0^{\pi/4} \tan x \, dx = -\log|\cos x| \Big|_0^{\pi/4}$$

$$= -\log\left(\frac{\sqrt{2}}{2}\right) - (0) = -\log \frac{\sqrt{2}}{2}$$

$$= \underline{\ln\sqrt{2}}$$

3.

(a) $$\int_0^1 (x^4 y + y^2)\, dy = x^4 \frac{y^2}{2} + \frac{y^3}{3} \Big|_0^1 = \frac{x^4}{2} + \frac{1}{3}$$

$$\therefore \int_{-1}^{1}\left(\frac{x^4}{2}+\frac{1}{3}\right)dx = \frac{x^5}{10}+\frac{x}{3}\Big|_{-1}^{1} = \frac{1}{10}+\frac{1}{3}-\left(\frac{-1}{10}-\frac{1}{3}\right)$$

$$= \frac{1}{5}+\frac{2}{3} = \underline{\frac{13}{15}}$$

(5) $$\int_0^1 (y\cos x+2)\,dy = \frac{y^2}{2}\cos x+2y\Big|_0^1 = \frac{1}{2}\cos x+2$$

$$\therefore \int_0^{\pi/2}\left(\frac{1}{2}\cos x+2\right)dx = \frac{1}{2}\sin x+2x\Big|_0^{\frac{\pi}{2}} =$$

$$\frac{1}{2}(1)+\pi-0 = \underline{\pi+\frac{1}{2}}$$

(c) $$\int_0^1 xye^{x+y}\,dy = xye^{x+y}\Big|_0^1 - \int_0^1 xe^{x+y}\,dy$$ (integration by parts)

$$= xe^{x+1} - xe^{x+y}\Big|_0^1$$

$$= xe^{x+1} - \left[xe^{x+1}-(xe^{x})\right\} = xe^{x}$$

$$\therefore \int_0^1 xe^{x}\,dx = xe^{x}\Big|_0^1 - \int_0^1 e^{x}\,dx$$

$$= e-0 - e^{x}\Big|_0^1 = e-(e-1) = \underline{1}$$

(d) $\int_1^2 (-x\log y)\,dy = -x\,(y\log y - y)\Big|_1^2$

$= -xy\log y\Big|_1^2 + xy\Big|_1^2$

$= -2x\log 2 + x(0) + 2x - x$

$= x - 2x\log 2 = x(1-2\log 2)$

$\therefore \int_{-1}^{0} (1-2\log 2)\,x\,dx = (1-2\log 2)\,\frac{x^2}{2}\Big|_{-1}^{0}$

$= -(1-2\ln 2)\,\frac{1}{2} = \underline{\ln 2 - \frac{1}{2}}$

4.

$= \int_0^1 \int_{-1}^1 (x^4 y + y^2)\,dx\,dy$

$\int_{-1}^1 (x^4 y + y^2)\,dx = \frac{x^5}{5}y + xy^2\Big|_{-1}^1 = \frac{y}{5} + y^2 - \left(-\frac{y}{5} - y^2\right)$

$= \frac{2y}{5} + 2y^2$

$\therefore \int_0^1 \left(\frac{2y}{5} + 2y^2\right)dy = \frac{y^2}{5} + \frac{2}{3}y^3\Big|_0^1 = \frac{1}{5} + \frac{2}{3} - 0$

$= \underline{\frac{13}{15}}$

$$= \int_0^1 \int_0^{\pi/2} (y\cos x + 2)\, dx\, dy$$

$$\int_0^{\pi/2} (y\cos x + 2)\, dx = y\sin x + 2x \Big|_0^{\pi/2} = y + \pi$$

$$\therefore \int_0^1 (y + \pi)\, dy = \frac{y^2}{2} + \pi y \Big|_0^1 = \underline{\frac{1}{2} + \pi}$$

$$= \int_0^1 \int_0^1 (xye^{x+y})\, dx\, dy$$

$$\int_0^1 xye^{x+y}\, dx = yxe^{x+y} \Big|_0^1 - \int_0^1 ye^{x+y}\, dx$$

$$= ye^{1+y} - ye^{x+y} \Big|_0^1$$

$$= ye^{1+y} - \left[ye^{1+y} - ye^{y}\right] = ye^{y}$$

$$\therefore \int_0^1 ye^{y}\, dy = ye^{y} \Big|_0^1 - \int_0^1 e^{y}\, dy$$

$$= e - 0 - e^{y} \Big|_0^1 = e - (e - 1) = \underline{1}$$

$$= \int_1^2 \int_{-1}^0 (-x\log y)\, dx\, dy$$

$$\int_{-1}^{0} (-x \log y)\,dx = -\frac{x^2}{2} \log y \Big|_{-1}^{0} = \frac{1}{2} \log y$$

$$\therefore \int_1^2 \frac{1}{2} \log y \, dy = \frac{1}{2} y \log y - \frac{1}{2} y \Big|_1^2$$

$$= \log 2 - 1 - \left[0 - \frac{1}{2}\right] = \underline{\log 2 - \frac{1}{2}}$$

5.

Take a cross section perpendicular to h.

Each has cross section of $\pi r^2 = A(h)$ at height h.

$\therefore$ Each has volume $\int_0^h A(h)\,dh = \int_0^h \pi r^2 \, dh$

$$= \pi r^2 h \Big|_0^h = \underline{\pi r^2 h}$$

6.

Let h = height of solid, perpendicular to rectangular base. $\therefore A(h) = 3 \times 5 = 15$

$\therefore$ Volume $= \int_0^7 A(h)\,dh = \int_0^7 15\,dh = 15h\Big|_0^7$

$= 15(7) = \underline{105}$

7.

Using the figure, the area of the triangular cross section is $A(x) = \frac{1}{2}bh$

$b = \sqrt{r^2 - x^2}$, where $x \in [-r, r]$

$h = b \tan\theta = \tan\theta \sqrt{r^2 - x^2}$

$\therefore A(x) = \frac{1}{2} \tan\theta \, (r^2 - x^2)$

$\therefore$ Volume $= \displaystyle\int_{-r}^{r} A(x)\,dx = \int_{-r}^{r} \frac{1}{2} \tan\theta \, (r^2 - x^2)\,dx$

$= \frac{1}{2} \tan\theta \, r^2 x - \frac{1}{2} \tan\theta \, \frac{x^3}{3} \Big|_{-r}^{r}$

$= \frac{1}{2} \tan\theta \, r^3 - \frac{1}{2} \tan\theta \, \frac{r^3}{3} - \left[-\frac{1}{2} \tan\theta \, r^3 + \frac{1}{2} \tan\theta \, \frac{r^3}{3} \right]$

$= \tan\theta \, r^3 - \tan\theta \, \frac{r^3}{3} = \underline{\frac{2}{3} \tan\theta \, r^3}$

8.

(a) Take a cross section perpendicular to the x-axis.

This is a circle of radius $f(x)$ (mistake in text).

$\therefore A(x) = \pi [f(x)]^2$

$\therefore$ Volume $= \displaystyle\int_a^b \pi [f(x)]^2 \, dx = \pi \int_a^b [f(x)]^2 \, dx$

(6)

Take a cross section perpendicular to x-axis.

This is a circle of radius y.

$\therefore A(x) = \pi y^2 = \pi(-x^2+2x+3)^2$

$\therefore$ Volume $= \displaystyle\int_{-1}^{3} \pi(-x^2+2x+3)^2\,dx$

$= \pi \displaystyle\int_{-1}^{3} (x^4 - 4x^3 - 2x^2 + 12x + 9)\,dx$

$= \pi\left(\dfrac{x^5}{5} - x^4 - \dfrac{2}{3}x^3 + 6x^2 + 9x\right)\Big|_{-1}^{3}$

$= \pi\left(\dfrac{243}{5} - 81 - 18 + 54 + 27\right)$

$\quad - \pi\left(-\dfrac{1}{5} - 1 + \dfrac{2}{3} + 6 - 9\right)$

$= \pi\left(\dfrac{153}{5}\right) - \pi\left(\dfrac{7}{15} - 4\right) = \pi\left(\dfrac{459}{15} - \dfrac{7}{15} + \dfrac{60}{15}\right)$

$= \underline{\dfrac{512}{15}\pi}$

9. $[0,2] \times [-1,0]$ means $dx \times dy$

$$\int_{-1}^{0} (x^2y^2 + x)\,dy = \frac{x^2y^3}{3} + xy \Big|_{-1}^{0} = 0 - \left(-\frac{x^2}{3} - x\right)$$

$$= \frac{x^2}{3} + x$$

$$\therefore \int_0^2 \frac{x^2}{3} + x\,dx = \frac{x^3}{9} + \frac{x^2}{2}\Big|_0^2 = \frac{8}{9} + \frac{4}{2} - 0$$

$$= 2\frac{8}{9} = \underline{\frac{26}{9}}$$

10.

$$\int_{-1}^{0} |y| \cos\frac{\pi x}{4}\,dy = \int_{-1}^{0} -y\cos\frac{\pi x}{4}\,dy \quad \text{since } |y| = -y \text{ for } -1 \le y \le 0$$

$$= -\frac{y^2}{2}\cos\frac{\pi x}{4}\Big|_{-1}^{0} = 0 - \left(-\frac{1}{2}\cos\frac{\pi x}{4}\right) = \frac{1}{2}\cos\frac{\pi x}{4}$$

$$\therefore \int_0^2 \frac{1}{2}\cos\frac{\pi x}{4} = \left(\frac{1}{2}\right)\left(\frac{4}{\pi}\right)\sin\frac{\pi x}{4}\Big|_0^2 = \left(\frac{2}{\pi}\right)\sin\frac{\pi}{2} - 0$$

$$= \underline{\frac{2}{\pi}}$$

11.

$$\int_{-1}^{0} -xe^x \sin\frac{\pi y}{2}\,dy = xe^x\left(\frac{2}{\pi}\right)\cos\frac{\pi y}{2}\Big|_{-1}^{0}$$

$$= xe^x\left(\frac{2}{\pi}\right)(1) - xe^x\left(\frac{2}{\pi}\right)(0)$$

$$= \left(\frac{2}{\pi}\right)xe^x$$

$$\therefore \frac{2}{\pi}\int_0^2 xe^x\,dx = \frac{2}{\pi}\left[xe^x\Big|_0^2 - \int_0^2 e^x\,dx\right]$$

$$= \frac{2}{\pi}\left\{2e^2 - 0 - e^x\Big|_0^2\right\}$$

$$= \frac{2}{\pi}\left[2e^2 - (e^2 - 1)\right)$$

$$= \underline{\frac{2}{\pi}\left(e^2 + 1\right)}$$

12.

$$\int_1^2 xy\left(x^2+y^2\right)^{-\frac{3}{2}}dx = \left(\frac{1}{2}\right)(-2)\,y\left(x^2+y^2\right)^{-\frac{1}{2}}\Big|_1^2$$

$$= -y(4+y^2)^{-\frac{1}{2}} - \left[-y(1+y^2)^{-\frac{1}{2}}\right]$$

$$= -\frac{y}{\sqrt{4+y^2}} + \frac{y}{\sqrt{1+y^2}}$$

$$\therefore \int_1^3 \left[-y(4+y^2)^{-\frac{1}{2}} + y(1+y^2)^{-\frac{1}{2}}\right] dy =$$

$$-(4+y^2)^{\frac{1}{2}} + (1+y^2)^{\frac{1}{2}} \Big|_1^3 =$$

$$\left[-\sqrt{13} + \sqrt{10}\right] - \left[-\sqrt{5} + \sqrt{2}\right] =$$

$$\underline{\sqrt{10} + \sqrt{5} - \sqrt{13} - \sqrt{2}}$$

13.

$$\int_0^1 (3x+2y)^7 dx = \left(\frac{1}{3}\right)\left(\frac{1}{8}\right)(3x+2y)^8 \Big|_0^1$$

$$= \frac{1}{24}(3+2y)^8 - \frac{1}{24}(2y)^8$$

$$\therefore \int_0^1 \frac{1}{24}(3+2y)^8 - \frac{1}{24}(2y)^8 \, dy =$$

$$\frac{1}{24}\left(\frac{1}{2}\right)\left(\frac{1}{9}\right)(3+2y)^9 - \frac{1}{24}\left(\frac{1}{2}\right)\left(\frac{1}{9}\right)(2y)^9 \Bigg|_0^1$$

$$\frac{1}{432}(5)^9 - \frac{1}{432}(2)^9 - \left[\frac{1}{432}(3)^9 - 0\right]$$

$$= \frac{5^9 - 2^9 - 3^9}{432} = \frac{1932930}{432} = \frac{2 \times 3^3 \times 5 \times 7159}{2^4 \times 3^3}$$

$$= \frac{(5)(7159)}{2^3} = \underline{\underline{\frac{35{,}795}{8}}}$$

14.

$$\text{Volume} = \int_0^1 \int_1^2 (1+2x+3y)\,dx\,dy$$

Note: $1+2x+3y \geq 0$ for $1 \leq x \leq 2,\ 0 \leq y \leq 1$

$$\therefore \int_1^2 (1+2x+3y)\,dx = x + x^2 + 3xy \Big|_1^2$$

$$= (2+4+6y) - (1+1+3y)$$

$$= 4 + 3y$$

$$\therefore \int_0^1 (4+3y)\,dy = 4y + \frac{3y^2}{2}\Bigg|_0^1 = 4 + \frac{3}{2} = \underline{\frac{11}{2}}$$

If $f(x,y) < 0$ over some portion of $[1,2] \times [0,1]$, Then we would have to break up rectangle into pieces where $f(x,y)$ is ≥ 0 and < 0, integrate over those pieces, taking the absolute value over areas where $f(x,y) < 0$.

15.

As in #14, note $f(x,y) \geq 0$ for $[-1,1] \times [-3,-2]$

$$\therefore \text{Volume} = \int_{-1}^{1} \int_{-3}^{-2} (x^4 + y^2)\, dy\, dx$$

$$\therefore \int_{-3}^{-2} (x^4 + y^2)\, dy = x^4 y + \frac{y^3}{3} \Big|_{-3}^{-2}$$

$$= -2x^4 - \frac{8}{3} - \left[-3x^4 - \frac{27}{3}\right]$$

$$= x^4 + \frac{19}{3}$$

$$\therefore \int_{-1}^{1} \left(x^4 + \frac{19}{3}\right) dx = \frac{x^5}{5} + \frac{19}{3}x \Big|_{-1}^{1} = \frac{1}{5} + \frac{19}{3} - \left[-\frac{1}{5} - \frac{19}{3}\right]$$

$$= \frac{2}{5} + \frac{38}{3} = \underline{\underline{\frac{196}{15}}}$$

1.

(a)

$$\int_R (x^3+y^2)\,dA = \int_0^1\int_0^1 (x^3+y^2)\,dx\,dy$$

$$\int_0^1 (x^3+y^2)\,dx = \frac{x^4}{4} + xy^2\Big|_0^1 = \frac{1}{4} + y^2$$

$$\int_0^1 \left(\frac{1}{4}+y^2\right)dy = \frac{y}{4} + \frac{y^3}{3}\Big|_0^1 = \frac{1}{4}+\frac{1}{3} = \underline{\frac{7}{12}}$$

(b)

$$\iint_R ye^{xy}\,dA = \int_0^1\int_0^1 ye^{xy}\,dx\,dy$$

$$\int_0^1 ye^{xy}\,dx = e^{xy}\Big|_0^1 = e^y - 1$$

$$\int_0^1 (e^y-1)\,dy = e^y - y\Big|_0^1 = e-1-(1-0) = \underline{e-2}$$

(c)

$$\iint_R (xy)^2\cos x^3\,dA = \int_0^1\int_0^1 (xy)^2\cos x^3\,dx\,dy$$

$$\int_0^1 (xy)^2 \cos x^3 \, dx = \frac{1}{3} y^2 \sin(x^3) \Big|_0^1 = \frac{y^2 \sin(1)}{3}$$

$$\int_0^1 y^2 \frac{\sin(1)}{3} \, dy = \frac{\sin(1)}{3} \frac{y^3}{3} \Big|_0^1 = \underline{\frac{\sin(1)}{9}}$$

(d)

$$\iint_R \ln[(x+1)(y+1)] \, dA = \int_0^1 \int_0^1 \ln[(x+1)(y+1)] \, dx \, dy$$

$$\int_0^1 \ln[(x+1)(y+1)] \, dx = \int_0^1 \ln(x+1) \, dx + \ln(y+1) \int_0^1 dx$$

$$= (x+1)\ln(x+1) - x \Big|_0^1 + [\ln(y+1)] \, x \Big|_0^1$$

$$= 2\ln 2 - 1 + \ln(y+1) = \ln 4 - 1 + \ln(y+1)$$

$$\therefore \int_0^1 [\ln 4 - 1 + \ln(y+1)] \, dy = (\ln 4 - 1) y + (y+1)\ln(y+1) - y \Big|_0^1$$

$$= \ln 4 - 1 + 2\ln 2 - 1$$

$$= \underline{2\ln 4 - 2} = \underline{\ln 16 - 2}$$

2.

(a)

$$\iint_R (x^m y^n)\,dx\,dy = \int_0^1\int_0^1 x^m y^n\,dx\,dy$$

$$\int_0^1 x^m y^n\,dx = \frac{x^{m+1}}{m+1} y^n \Big|_0^1 = \frac{y^n}{m+1}$$

$$\therefore \int_0^1 \frac{y^n}{m+1}\,dy = \frac{y^{n+1}}{(m+1)(n+1)}\Big|_0^1 = \underline{\underline{\frac{1}{(m+1)(n+1)}}}$$

(b)

$$\iint_R (ax+by+c)\,dx\,dy = \int_0^1\int_0^1 (ax+by+c)\,dx\,dy$$

$$\int_0^1 (ax+by+c)\,dx = \frac{ax^2}{2} + byx + cx \Big|_0^1 = \frac{a}{2} + by + c$$

$$\int_0^1 \left(\frac{a}{2} + by + c\right)dy = \frac{a}{2}y + \frac{b}{2}y^2 + cy\Big|_0^1 = \underline{\underline{\frac{a}{2} + \frac{b}{2} + c}}$$

(c)

$$\iint_R \sin(x+y)\,dx\,dy = \int_0^1\int_0^1 \sin(x+y)\,dx\,dy$$

$$\int_0^1 \sin(x+y)\,dx = -\cos(x+y)\Big|_0^1 = -\cos(1+y) - [-\cos(y)]$$

$$= \cos y - \cos(1+y)$$

$$\therefore \int_0^1 \cos y - \cos(1+y)\, dy = \sin y - \sin(1+y) \Big|_0^1$$

$$= \sin(1) - \sin(2) - (\sin(0) - \sin(1))$$

$$= \underline{2\sin(1) - \sin(2)}$$

(d)

$$= \int_0^1 \int_0^1 (x^2 + 2xy + y\sqrt{x})\, dx\, dy$$

$$\int_0^1 (x^2 + 2xy + y\sqrt{x})\, dx = \frac{x^3}{3} + x^2 y + \frac{2}{3} y x^{\frac{3}{2}} \Big|_0^1$$

$$= \frac{1}{3} + y + \frac{2}{3} y = \frac{1}{3} + \frac{5}{3} y$$

$$\therefore \int_0^1 \left(\frac{1}{3} + \frac{5}{3} y\right) dy = \frac{1}{3} y + \frac{5}{6} y^2 \Big|_0^1 = \frac{1}{3} + \frac{5}{6} = \underline{\frac{7}{6}}$$

3.

$$\int_0^2 \frac{y x^3}{y^2+2}\, dx = \frac{y}{y^2+2} \cdot \frac{1}{4} x^4 \Big|_0^2 = \frac{4y}{y^2+2}$$

$$\therefore \int_{-1}^1 \frac{4y}{y^2+2}\, dy = 2\ln(y^2+2) \Big|_{-1}^1 = 2\ln(3) - 2\ln(3)$$

$$= \underline{0}$$

4.

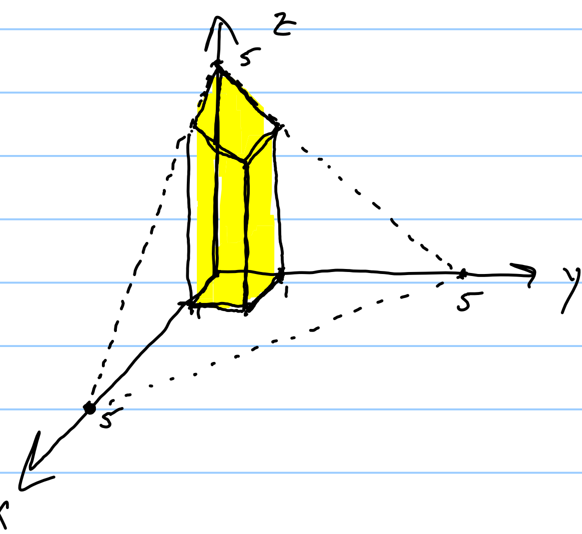
$$= \int_{-2}^{2} \int_{0}^{1} \frac{y}{1+x^2}\, dx\, dy$$

$$\int_0^1 \frac{y}{1+x^2}\, dx = y \arctan(x) \Big|_0^1 = y\arctan(1) - y\arctan(0)$$

$$= y\left(\frac{\pi}{4}\right) - 0 = \frac{\pi}{4} y$$

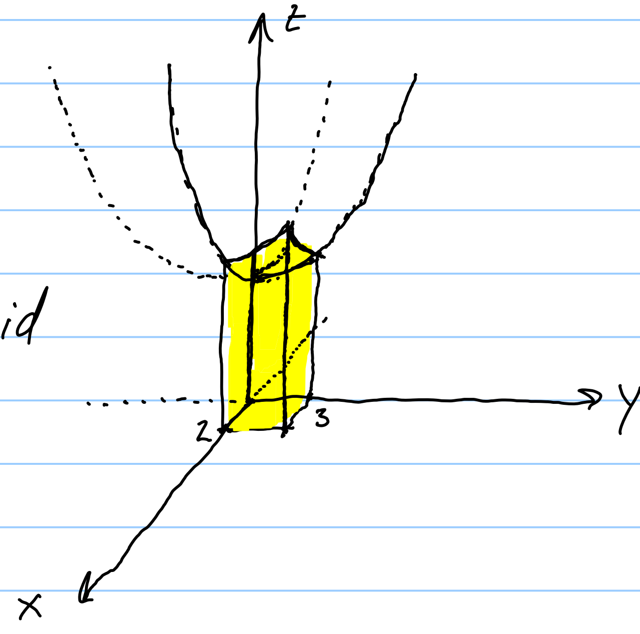
$$\therefore \int_{-2}^{2} \frac{\pi}{4} y\, dy = \frac{\pi}{8} y^2 \Big|_{-2}^{2} = \frac{\pi}{8}(4) - \frac{\pi}{8}(4) = \underline{0}$$

5.



$z = 5 - x - y$ is a plane intersecting axes at $(5,0,0)$, $(0,5,0)$ and $(0,0,5)$.

6.



$z = 9 + x^2 + y^2$ is a paraboloid whose inferior apex is 9 units above xy-plane. The solid is underneath the paraboloid.

7.

$$\iint_R xy\, dA = \int_0^1 \int_0^1 xy\, dx\, dy = \int_0^1 \left[\int_0^1 xy\, dx \right] dy$$

$$= \int_0^1 \left(\frac{x^2}{2} y \Big|_0^1 \right) dy = \int_0^1 \frac{y}{2}\, dy = \frac{y^2}{4} \Big|_0^1 = \underline{\frac{1}{4}}$$

8.

$$R = [0,1] \times [0,1]$$

$$\iint_R (x^2 + y^4)\, dA = \int_0^1 \int_0^1 (x^2 + y^4)\, dx\, dy$$

$$\int_0^1 (x^2 + y^4)\, dx = \frac{x^3}{3} + y^4 x \Big|_0^1 = \frac{1}{3} + y^4$$

$$\therefore \int_0^1 \left(\frac{1}{3} + y^4 \right) dy = \frac{y}{3} + \frac{y^5}{5} \Big|_0^1 = \frac{1}{3} + \frac{1}{5} = \underline{\frac{8}{15}}$$

9.

$$\iint_R [f(x)\, g(y)]\, dx\, dy = \int_c^d \int_a^b [f(x)\, g(y)]\, dx\, dy$$

$$= \int_c^d \left[\int_a^b [g(y)\, f(x)]\, dx \right] dy$$

$$= \int_c^d g(y) \left[\int_a^b f(x)\, dx \right] dy \qquad \text{using } \int c\, f(x)\, dx = c \int f(x)\, dx$$

$$= \left[\int_a^b f(x)\, dx \right] \int_c^d g(y)\, dy \qquad \text{using } \int g(y)\, k\, dy = k \int g(y)\, dy$$

where $k = \int_a^b f(x)\, dx$

10.

$$R = [0,1] \times [0, \tfrac{\pi}{2}]$$

$$\therefore \iint_R \sin y\, dA = \int_0^1 \int_0^{\frac{\pi}{2}} \sin y\, dy\, dx$$

$$\int_0^{\frac{\pi}{2}} \sin y\, dy = -\cos y \Big|_0^{\pi/2} = 0 - (-\cos(0)) = 1$$

$$\therefore \int_0^1 1\,dx = x\Big|_0^1 = \underline{\underline{1}}$$

11.

$$\iint_R (x^2+y)\,dA = \int_1^2 \int_0^1 (x^2+y)\,dx\,dy$$

$$\int_0^1 (x^2+y)\,dx = \frac{x^3}{3} + yx\Big|_0^1 = \frac{1}{3} + y$$

$$\int_1^2 \left(\frac{1}{3}+y\right)dy = \frac{y}{3} + \frac{y^2}{2}\Big|_1^2 = \frac{2}{3} + 2 - \left(\frac{1}{3}+\frac{1}{2}\right)$$

$$= \frac{1}{3} + \frac{3}{2} = \underline{\underline{\frac{11}{6}}}$$

12.

Fubini's Theorem says,

$$\int_a^x \left[\int_c^y f(u,v)\,dv\right] du = \int_c^y \left[\int_a^x f(u,v)\,du\right] dv \quad [1]$$

Let $G(u,y) = \int_c^y f(u,v)\,dv \quad \therefore \frac{\partial G}{\partial y}(u,y) = f(u,y)$ [2]

Let $H(x,v) = \int_a^x f(u,v)\,du \quad \therefore \frac{\partial H}{\partial x}(x,v) = f(x,v)$ [3]

[2], [3] follow from The Fundamental Theorem of Calculus.

$\therefore$ [1] can be written as,

$$F(x,y) = \int_a^x G(u,y)\,du = \int_c^y H(x,v)\,dv$$

$$\therefore \frac{\partial F}{\partial x} = G(x,y) \qquad \frac{\partial F}{\partial y} = H(x,y)$$

$$\therefore \frac{\partial^2 F}{\partial y \partial x} = \frac{\partial}{\partial y} G(x,y) = f(x,y) \text{ from [2]}$$

$$\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial}{\partial x} H(x,y) = f(x,y) \text{ from [3]}$$

$$\therefore f(x,y) = \frac{\partial^2 F}{\partial y \partial x} = \frac{\partial^2 F}{\partial x \partial y}$$

$\therefore$ Fubini's Theorem $\Rightarrow$ equality of mixed partial derivatives.

Note G, H are C^1, F is C^2.

13.

From 2(a), $\iint_R x^m y^n \, dx\, dy = \frac{1}{(m+1)(n+1)}$, $m,n > 0$

$\therefore \lim_{m,n \to \infty} f(m,n) = \lim_{m,n\to\infty} \frac{1}{(m+1)(n+1)} = \underline{0}$.

14.

$$\int_{-\pi}^{\pi} \cos(nx)\sin(my)\,dx = \frac{\sin(nx)}{n}\sin(my)\Big|_{-\pi}^{\pi}$$

$$= \frac{1}{n}\left[\sin(n\pi)\sin(my) - (\sin(-n\pi)\sin(my))\right]$$

$$= \frac{2}{n}\sin(n\pi)\sin(my), \quad n \neq 0.$$

If n is an integer, this is 0. $\therefore$ Assume n is not necessarily an integer, so $\sin(n\pi)$ may not be 0.

$$\therefore \int_{-\pi}^{\pi} \frac{2}{n} \sin(n\pi) \sin(my)\, dy = -\frac{2}{mn} \sin(n\pi) \cos(my) \Big|_{-\pi}^{\pi}$$

$$= -\frac{2\sin(n\pi)}{mn} \left[\cos(m\pi) - \cos(-m\pi) \right]$$

$$= -\frac{2\sin(n\pi)}{mn} [0] = 0 \quad , m \neq 0 .$$

$\therefore f(m,n) = 0.$ $\therefore \underline{\underline{\lim_{m,n\to\infty} f(m,n) = 0}}$

15.

Consider $\int_0^1 f(x,y)\, dy$

For x rational: $\int_0^1 f(x,y) = \int_0^1 1\, dy = y \Big|_0^1 = 1$

For x irrational: $\int_0^1 f(x,y) = \int_0^1 2y\, dy = y^2 \Big|_0^1 = 1$

$\therefore$ For all x, $\int_0^1 f(x,y)\, dy = 1.$

$$\therefore \int_0^1 \left[\int_0^1 f(x,y)\,dy \right] dx = \int_0^1 1\,dx = x\Big|_0^1 = \underline{1}.$$

f is not integrable: definition (p. 272 of text) states the sequence of sums, S_n, depends on the choice c_{jk} of the point in the rectangle R_{jk}, and the limit of S_n should be the same, S, for all choices of c_{jk}.

For every rectangle, R_{jk}, one can always find a rational or irrational coordinate for c_{jk}. The above showed the limit is the same if you choose all your coordinates to be one or the other. The trick is to choose some c_{jk} as rational coordinates, and others as irrational coordinates. Then the limit won't be the same.

For example: For R_{jk} with $0 \le y \le \frac{1}{2}$, choose

$c_{jk} = (\text{rational}, \text{rational})$, and for R_{jk} with $\frac{1}{2} \le y \le 1$, choose $c_{jk} = (\text{irrational}, \text{irrational})$.

Then
$$\int_0^1 f(x,y)\,dy = \int_0^{1/2} f(x,y)\,dy + \int_{\frac{1}{2}}^1 f(x,y)\,dy$$
$$= \int_0^{\frac{1}{2}} 1\,dy + \int_{\frac{1}{2}}^1 2y\,dy$$
$$= y\Big|_0^{\frac{1}{2}} + y^2\Big|_{\frac{1}{2}}^1$$
$$= \frac{1}{2} + \left(1 - \frac{1}{4}\right) = \frac{5}{4} \neq 1.$$

$\therefore \lim_{n\to\infty} S_n$ differs depending on choice of c_{jk},

so <u>f is not</u> integrable.

16.

Let $0 = x_0 < x_1 < \dots < x_n = 1$ be a regular partition of $[0,1]$
$0 = y_0 < y_1 < \dots < y_n = 1$ be a regular partition of $[0,1]$

Let c_{jk} be any point in $[x_j, x_{j+1}] \times [y_k, y_{k+1}]$,

$0 \le j \le n-1$, $0 \le k \le n-1$.

Let $S_n = \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} \cosh(c_{jk})(x_{j+1}-x_j)(y_{k+1}-y_k)$

Let $\Delta x = x_{j+1}-x_j$, $\Delta y = y_{k+1}-y_k$

Since $\cosh(xy)$ is continuous on $R=[0,1]\times[0,1]$,

Then $\cosh(xy)$ is integrable on R, so that

$\lim_{n\to\infty} S_n$ converges.

$$\iint_R \cosh(xy)\,dA = \lim_{n\to\infty} \sum_{j=0}^{n-1}\sum_{k=0}^{n-1} \cosh(c_{jk})\,\Delta x\,\Delta y$$

17.

(a) Use $1+\tan^2\theta = \sec^2\theta$. $x^2+y^2 = x^2\left(1+\left(\frac{y}{x}\right)^2\right)$

$= x^2(1+\tan^2\theta)$ using $\frac{y}{x} = \tan\theta$

$x^2-y^2 = x^2\left(1-\frac{y^2}{x^2}\right) = x^2(1-\tan^2\theta)$

$$\therefore \frac{x^2-y^2}{(x^2+y^2)^2} = \frac{x^2\left((1-\tan^2\theta)\right)}{x^4\,(1+\tan^2\theta)^2} = \frac{1}{x^2}\,\frac{(1-\tan^2\theta)}{\sec^4\theta}$$

If $y = x\tan\theta$, Then $dy = x\sec^2\theta\, d\theta$

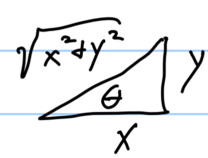
$$\therefore \int \frac{x^2-y^2}{(x^2+y^2)^2}\,dy = \int \frac{1}{x^2}\,\frac{(1-\tan^2\theta)}{\sec^4\theta}\; x\sec^2\theta\, d\theta$$

$$= \int \frac{1}{x}\,\frac{(1-\tan^2\theta)}{\sec^2\theta}\,d\theta = \int \frac{1}{x}\,\frac{\left(1-\frac{\sin^2\theta}{\cos^2\theta}\right)}{\frac{1}{\cos^2\theta}}\,d\theta$$

$$= \int \frac{1}{x}\cos^2\theta - \sin^2\theta\; d\theta = \int \frac{1}{x}\cos(2\theta)\,d\theta$$

$$= \frac{1}{2x}\sin(2\theta) + C = \frac{\sin\theta\cos\theta}{x} + C$$

For $y = x\tan\theta$, $\frac{y}{x} = \tan\theta$,



$$\therefore \cos\theta = \frac{x}{\sqrt{x^2+y^2}},\quad \sin\theta = \frac{y}{\sqrt{x^2+y^2}}$$

$$\therefore \frac{\sin\theta\cos\theta}{x} = \frac{xy}{x(x^2+y^2)} = \frac{y}{x^2+y^2}$$

$$\therefore \int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}\,dy = \left.\frac{y}{x^2+y^2}\right|_0^1 = \frac{1}{x^2+1} - 0 = \frac{1}{x^2+1}$$

Now, for $\int_0^1 \frac{1}{x^2+1}\,dx = \arctan x \Big|_0^1 = \frac{\pi}{4} - 0 = \frac{\pi}{4}$

$$\therefore \int_0^1 \int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}\,dy\,dx = \underline{\underline{\frac{\pi}{4}}}$$

(b) For $\int \frac{x^2-y^2}{(x^2+y^2)^2}\,dx$, use $x = y\tan\theta$ so that

$$dx = y\sec^2\theta\,d\theta$$

$$\therefore x^2 - y^2 = y^2\tan^2\theta - y^2 = y^2(\tan^2\theta - 1)$$

$$(x^2+y^2)^2 = (y^2\tan^2\theta + y^2)^2 = y^4\sec^4\theta$$

$$\therefore \frac{x^2-y^2}{(x^2+y^2)^2} = \frac{\tan^2\theta - 1}{y^2\sec^4\theta}$$

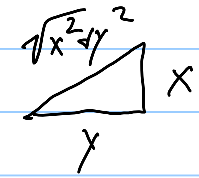
$$\therefore \int \frac{x^2-y^2}{(x^2+y^2)^2}\,dx = \int \frac{\tan^2\theta - 1}{y^2\sec^4\theta}(y\sec^2\theta)\,d\theta$$

$$= \int \frac{\tan^2\theta - 1}{y\sec^2\theta}\,d\theta = \int \frac{1}{y}\,\frac{\frac{\sin^2\theta}{\cos^2\theta} - 1}{\frac{1}{\cos^2\theta}}\,d\theta$$

$$= \int \frac{1}{y}(\sin^2\theta - \cos^2\theta)\,d\theta = -\int \frac{1}{y}\cos(2\theta)\,d\theta$$

$$= -\frac{\sin(2\theta)}{2y} + C = -\frac{\sin\theta\cos\theta}{y} + C$$

Going back, $\sin\theta = \frac{x}{\sqrt{x^2+y^2}}$, $\cos\theta = \frac{y}{\sqrt{x^2+y^2}}$



$$\therefore \int \frac{x^2-y^2}{(x^2+y^2)^2}\,dx = -\frac{xy}{y(x^2+y^2)} + C = -\frac{x}{x^2+y^2} + C$$

$$\therefore \int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}\,dx = -\frac{x}{x^2+y^2}\Big|_0^1 = -\frac{1}{1+y^2}$$

$$\therefore \int_0^1 -\frac{1}{1+y^2}\,dy = -\arctan y\Big|_0^1 = -\frac{\pi}{4}$$

$$\therefore \int_0^1\int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}\,dx\,dy = \underline{\underline{-\frac{\pi}{4}}}$$

Fubini's Theorem can't be used here since $\frac{x^2-y^2}{(x^2+y^2)^2}$ is not defined at $(0,0)$, and so is not bounded on $[0,1]\times[0,1]$.

18.

Suppose $f > 0$ for some $(x,y) \in R$.

Since f is continuous, $\exists$ a rectangle $R' \subset R$ s.t.

$f(x,y) > 0$ for all $(x,y) \in R'$. Let $R' = [a,b] \times [c,d]$

Let $S_n = \sum_j^{n-1} \sum_k^{n-1} f(c_{jk}) \Delta x \Delta y$.

Note that each term of S_n is ≥ 0, so $S_n \geq 0$.

Eventually, for some N, the size of the regular partition for S_N will be smaller than the size of $[a,b] \times [c,d]$.

For $n \geq N$, S_n will then always have a term $f(c_{jk}) \Delta x \Delta y > 0$

$\therefore$ If choose $\epsilon = \dfrac{f(x,y)(b-a)(d-c)}{2}$, for some $(x,y) \in [a,b] \times [c,d]$

Then there is an $N > 0$ s.t. for all $n > N$,

it is not true that $S_n < \epsilon$.

$\therefore \lim_{n \to \infty} S_n \neq 0$, contradicting $\iint_R f \, dA = 0$

$\therefore$ Assumption of $f > 0$ is false, so $f = 0$ on R.

1.

(a) dy is "inside" and $y=\ln x$ to $y=e^x$ with $x=1$ to $x=2$ is shown in (iii).

(b) $y=\frac{x}{8}$ to $y=x^{1/3}$ are y-simple graphs, with $x=0$ to $x=2$.

This is (iv)

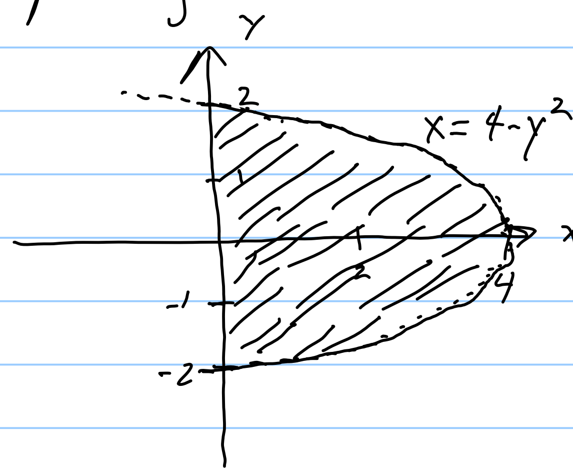
(c) $x=-\sqrt{9-y^2}$ to $x=0$ are the x-simple graphs, with $y=0$ to $y=2$ is shown in (ii)

(d) dx "inside" so look for $x=\arccos\frac{y}{3}$ to $x=0$ with $y=0$ to $y=3$, is shown in (i)

2.

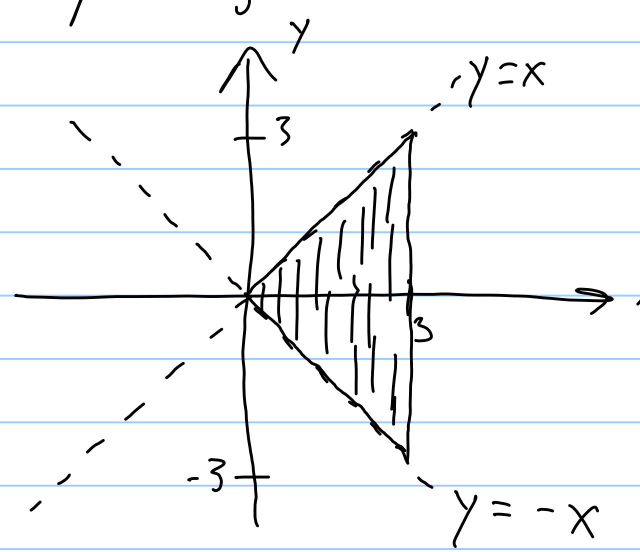
(a) dx "inside" $\Rightarrow$ look for x-simple regions, $x=0$ to $x=4-y^2$ (a parabola).

The y limits are $y=-2$ to $y=2$



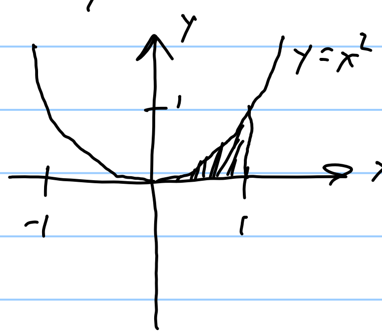
(b) dy "inside" $\Rightarrow$ look for y-simple regions, from $y=-x$ to $y=x$.

The x limits are $x=0$ to $x=3$



3.

(a) dy "inside" $\Rightarrow$ look at y-simple, $y=0$ to $y=x^2$, bounded by $x=0$ to $x=1$.

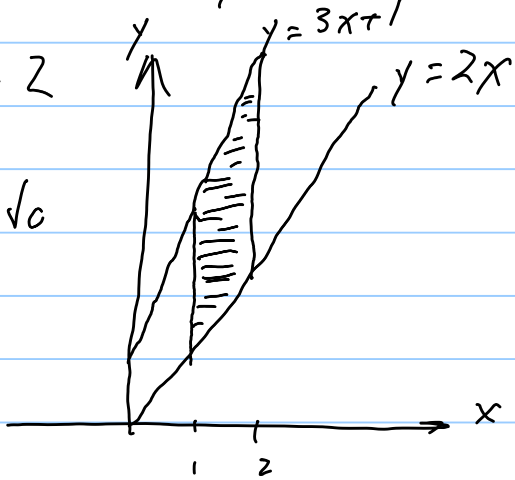


$$\int_0^{x^2} dy = y\Big|_{y=0}^{y=x^2} = x^2 - 0 = x^2$$

$$\therefore \int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_0^1 = \underline{\underline{\frac{1}{3}}}$$

A is both y-simple and x-simple.

(b) dy "inside" $\Rightarrow$ look for y-simple curves, $\therefore$ $y=2x$ to $y=3x+1$, bounded by $x=1$ to $x=2$



The figure can be broken up into either y-simple or x-simple curves.

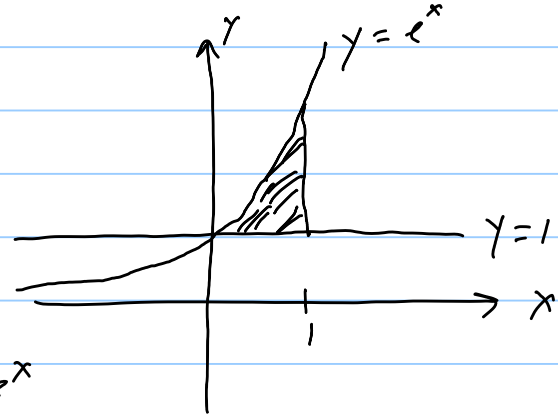
$$\int_{2x}^{3x+1} dy = y\Big|_{2x}^{3x+1} = x+1$$

$$\therefore \int_1^2 (x+1)\,dx = \frac{x^2}{2} + x\Big|_1^2 = \left(\frac{4}{2}+2\right) - \left(\frac{1}{2}+1\right)$$

$$= \underline{\underline{\frac{5}{2}}}$$

(c) dy "inside" $\Rightarrow$ look for y-simple curves.

$\therefore$ look at $y=1$ to $y=e^x$, bounded from $x=0$ to $x=1$



Region can be viewed as x-simple or y-simple.

$$\int_1^{e^x} (x+y)\,dy = xy + \frac{y^2}{2}\Bigg|_{y=1}^{y=e^x}$$

$$= xe^x + \frac{e^{2x}}{2} - \left(x+\frac{1}{2}\right)$$

$$= xe^x + \frac{e^{2x}}{2} - x - \frac{1}{2}$$

$$\therefore \int_0^1 \left(xe^x + \frac{e^{2x}}{2} - x - \frac{1}{2}\right)dx = xe^x - e^x + \frac{e^{2x}}{4} - \frac{x^2}{2} - \frac{x}{2}\Bigg|_0^1$$

$$= e - e + \frac{e^2}{4} - \frac{1}{2} - \frac{1}{2} - \left(0 - 1 + \frac{1}{4} - 0 - 0\right)$$

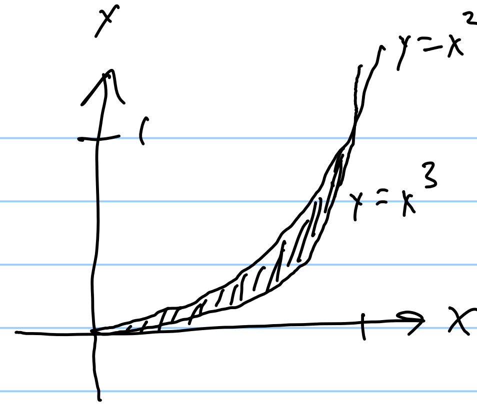
$$= \frac{e^2}{4} - 1 + 1 - \frac{1}{4} = \underline{\underline{\frac{e^2-1}{4}}}$$

(d) dy "inside" $\Rightarrow$ look for y-simple curves.

$\therefore$ $y=x^3$ to $y=x^2$, bounded by $x=0$ to $x=1$.

$$\int_{x^3}^{x^2} y\,dy = \frac{y^2}{2}\Bigg|_{x^3}^{x^2} = \frac{x^4}{2} - \frac{x^6}{2}$$

$$\therefore \int_0^1 \left(\frac{x^4}{2} - \frac{x^6}{2}\right) dx =$$



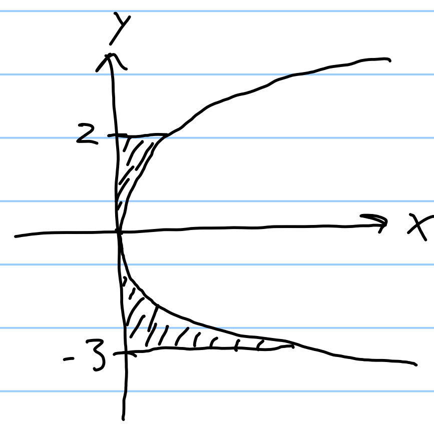
$$\left.\frac{x^5}{10} - \frac{x^7}{14}\right|_0^1 = \frac{1}{5} - \frac{1}{14} = \underline{\frac{9}{70}}$$

Domain figure can be viewed either as x-simple or y-simple.

4.

(a)

$x = 0$ to $x = y^2$
$y = -3$ to $y = 2$



$$\int_0^{y^2} (x^2 + y)\, dx = \left.\frac{x^3}{3} + xy\right|_{x=0}^{x=y^2}$$

$$= \frac{y^6}{3} + y^3$$

$$\therefore \int_{-3}^{2} \left(\frac{y^6}{3} + y^3\right) dy = \left.\frac{y^7}{21} + \frac{y^4}{4}\right|_{-3}^{2}$$

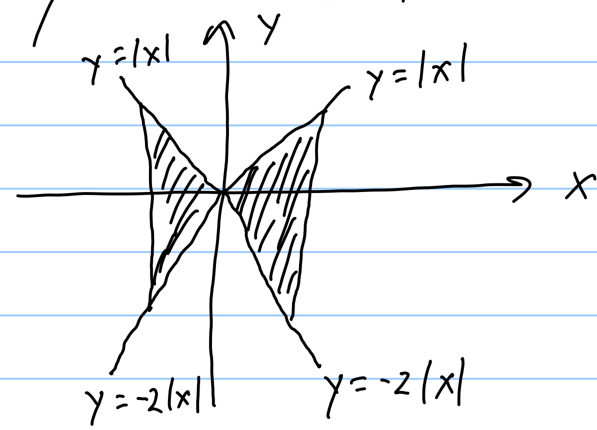
$$= \frac{2^7}{21} + \frac{2^4}{4} - \left(\frac{(-3)^7}{21} + \frac{(-3)^4}{4}\right)$$

$$= \frac{128}{21} + \frac{16}{4} + \frac{2187}{21} - \frac{81}{4} = \underline{\frac{2059}{21} - \frac{65}{4}}$$

(b)

$y=-2|x|$ to $y=|x|$
bounded by $x=-1$ to $x=1$

$$\int_{-1}^{1}\int_{-2|x|}^{|x|} e^{x+y}\,dy\,dx =$$



$$\int_{-1}^{0}\int_{2x}^{-x} e^{x+y}\,dy\,dx + \int_{0}^{1}\int_{-2x}^{x} e^{x+y}\,dy\,dx$$

$$\int_{2x}^{-x} e^{x+y}\,dy = e^{x+y}\Big|_{y=2x}^{y=-x} = e^{x+(-x)} - e^{x+2x} = 1 - e^{3x}$$

and $$\int_{-2x}^{x} e^{x+y}\,dy = e^{x+y}\Big|_{y=-2x}^{y=x} = e^{2x} - e^{-x}$$

$$\therefore \int_{-1}^{0}(1-e^{3x})\,dx + \int_{0}^{1}(e^{2x}-e^{-x})\,dx =$$

$$x - \frac{e^{3x}}{3}\Big|_{-1}^{0} + \frac{e^{2x}}{2} + e^{-x}\Big|_{0}^{1} =$$

$$\left(0-\tfrac{1}{3}\right)-\left(-1-\frac{e^{-3}}{3}\right) + \left(\frac{e^{2}}{2}+e^{-1}\right)-\left(\tfrac{1}{2}+1\right) =$$

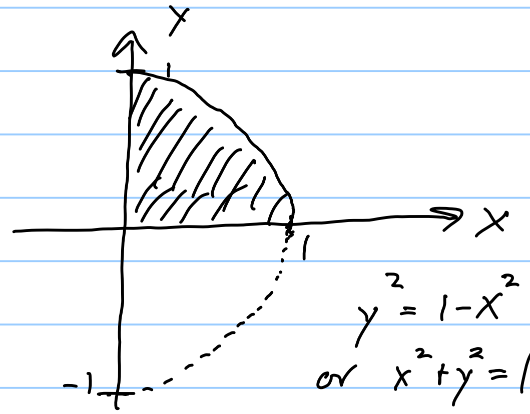
$$\frac{2}{3} + \frac{e^{-3}}{3} + \frac{e^{2}}{2} + e^{-1} - \frac{3}{2} = \underline{\frac{e^{-3}}{3} + \frac{e^{2}}{2} + e^{-1} - \frac{5}{6}}$$

(c)

$y = 0$ to $y = (1-x^2)^{1/2}$ bounded by
$x = 0$ to $x = 1$

$$\int_0^{(1-x^2)^{1/2}} dy = y \Big|_0^{(1-x^2)^{\frac{1}{2}}}$$

$$= \sqrt{1-x^2}$$



$$\therefore \int_0^1 \sqrt{1-x^2}\, dx \quad \text{Let } x = \sin\theta \;\therefore\; dx = \cos\theta\, d\theta$$

$$x = 0: \theta = 0 \qquad x = 1: \theta = \frac{\pi}{2}$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2\theta}\, \cos\theta\, d\theta = \int_0^{\frac{\pi}{2}} \cos^2\theta\, d\theta$$

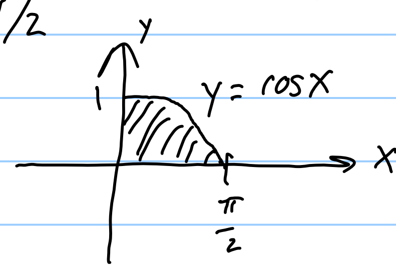
$$= \int_0^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{2}\, d\theta = \frac{1}{2}\theta + \frac{\sin 2\theta}{4} \Big|_0^{\frac{\pi}{2}} = \frac{\pi}{4} + 0 - (0+0)$$

$$= \underline{\frac{\pi}{4}}$$

(d)

$y = 0$ to $y = \cos x$ bounded by
$x = 0$ to $x = \pi/2$

$$\int_0^{\cos x} y \sin x\, dy =$$



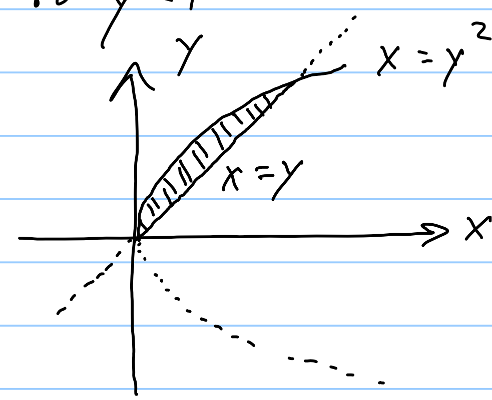
$$\frac{y^2}{2}\sin x\Big|_{y=0}^{y=\cos x} = \frac{\cos^2 x \sin x}{2}$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{\cos^2 x \sin x}{2}\,dx = -\frac{\cos^3 x}{6}\Big|_0^{\frac{\pi}{2}}$$

$$= 0 - \left(-\frac{1}{6}\right) = \underline{\frac{1}{6}}$$

(e)

$x = y^2$ to $x = y$ bounded by $y = 0$ to $y = 1$



$$\int_{y^2}^{y} (x^n + y^m)\,dx =$$

$$\frac{x^{n+1}}{n+1} + xy^m \Big|_{x=y^2}^{x=y}$$

$$= \frac{y^{n+1}}{n+1} + y^{m+1} - \left(\frac{y^{2n+2}}{n+1} + y^{2+m}\right)$$

$$\therefore \int_0^1 \left(\frac{y^{n+1}}{n+1} - \frac{y^{2n+2}}{n+1} + y^{m+1} - y^{m+2}\right) dy$$

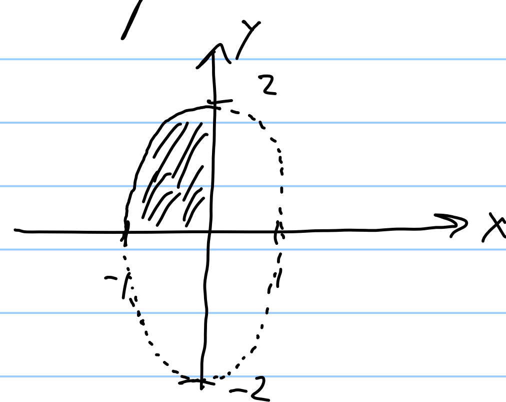
$$= \frac{y^{n+2}}{(n+2)(n+1)} - \frac{y^{2n+3}}{(2n+3)(n+1)} + \frac{y^{m+2}}{m+2} - \frac{y^{m+3}}{m+3}\Big|_0^1$$

$$= \frac{1}{(n+2)(n+1)} - \frac{1}{(2n+3)(n+1)} + \frac{1}{m+2} - \frac{1}{m+3}$$

(f) (f) $\int_{-1}^{0}\int_{0}^{2(1-x^2)^{1/2}} x\,dy\,dx$ $\quad$ $y=0$ to $y=2(1-x^2)^{1/2}$ bounded by $x=-1$ to $x=0$

$$\int_0^{2(1-x^2)^{1/2}} x\,dy =$$

Note: $y^2 = 4(1-x^2)$, $x^2 + \frac{y^2}{4} = 1$, an ellipse.



$$xy\Big|_{y=0}^{y=2(1-x^2)^{1/2}} = 2x(1-x^2)^{\frac{1}{2}}$$

$$\therefore \int_{-1}^{0} 2x(1-x^2)^{\frac{1}{2}}dx = -\frac{2}{3}(1-x^2)^{\frac{3}{2}}\Big|_{-1}^{0}$$

$$= -\frac{2}{3}(1-0)^{3/2} - \left(-\frac{2}{3}(1-1)^{\frac{3}{2}}\right) = -\frac{2}{3}$$

5.

Use $f(x,y)=1$, so that $f(x,y)\,dx\,dy = dx\,dy$, and the volume effectively becomes the area.

From $x^2+y^2=r^2$, $y = \pm\sqrt{r^2-x^2}$

$$\therefore \iint_R dA = \int_{-r}^{r}\int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} dy\,dx$$

$$\int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} dy = \sqrt{r^2-x^2} - \left(-\sqrt{r^2-x^2}\right) = 2\sqrt{r^2-x^2}$$

$$\therefore \int_{-r}^{r} 2\sqrt{r^2-x^2}\, dx$$

Use a substitution $x = r\cos\theta$

$\therefore dx = -r\sin\theta\, d\theta$

$\theta = -\pi \rightarrow x = -r$

$\theta = 0 \rightarrow x = r$

$$\therefore \int_{-r}^{r} 2\sqrt{r^2-x^2}\, dx = \int_{-\pi}^{0} 2\sqrt{r^2 - r^2\cos^2\theta}\,(-r\sin\theta)\, d\theta$$

$$= \int_{-\pi}^{0} 2\sqrt{r^2\sin^2\theta}\,(-r\sin\theta)\, d\theta = \int_{-\pi}^{0} 2r(-\sin\theta)(-r\sin\theta)\, d\theta$$

since $\sqrt{\sin^2\theta} = -\sin\theta$, $-\pi \le \theta \le 0$

$$= \int_{-\pi}^{0} 2r^2\sin^2\theta\, d\theta$$

Now use $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$\therefore \cos 2\theta = (1-\sin^2\theta) - \sin^2\theta$

$= 1 - 2\sin^2\theta$

$\therefore 2\sin^2\theta = 1 - \cos 2\theta$

$$= \int_{-\pi}^{0} r^2(1-\cos 2\theta)\, d\theta = r^2\theta - \frac{r^2}{2}\sin 2\theta \Big|_{-\pi}^{0}$$

$$= 0 - 0 - \left[-\pi r^2 - \frac{r^2}{2}\sin(-2\pi)\right] = \underline{\pi r^2}$$

6.

Let $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ be the ellipse.

Rewrite as $b^2x^2 + a^2y^2 = a^2b^2$. Let $f(x,y) = 1$ be the function on the region D defined by the ellipse.

For a given x, $a^2y^2 = a^2b^2 - b^2x^2$, $y = \pm\sqrt{b^2 - \frac{b^2}{a^2}x^2}$

$\therefore \int_{-\sqrt{b^2-\frac{b^2}{a^2}x^2}}^{\sqrt{b^2-\frac{b^2}{a^2}x^2}} dy$ is the area of a slice at a given x, for the function $f(x,y) = 1$, from the bottom of the ellipse to the top.

$\therefore \int_{-a}^{a} \int_{-\sqrt{b^2-\frac{b^2}{a^2}x^2}}^{\sqrt{b^2-\frac{b^2}{a^2}x^2}} dy\,dx$ is the volume of the solid of height $1 = f(x,y)$ using the ellipse as the base. This effectively is the formula for the area of the ellipse: Volume $=$ (height)(base) $= 1$(base).

$$\therefore \int_{-\sqrt{b^2-\frac{b^2}{a^2}x^2}}^{\sqrt{b^2-\frac{b^2}{a^2}x^2}} dy = 2\sqrt{b^2-\frac{b^2}{a^2}x^2}$$

$$\therefore \int_{-a}^{a} 2\sqrt{b^2-\frac{b^2}{a^2}x^2}\,dx$$

Use $x = a\cos\theta \therefore dx = -a\sin\theta\,d\theta$
$\theta = -\pi \to x = -a$, $\theta = 0 \to x = a$

$$= \int_{-\pi}^{0} 2\sqrt{b^2-\frac{b^2}{a^2}(a^2\cos^2\theta)}\,(-a\sin\theta)\,d\theta$$

$$= \int_{-\pi}^{0} 2b\sqrt{\sin^2\theta}\,(-a\sin\theta)\,d\theta$$

$$= \int_{-\pi}^{0} 2b(-\sin\theta)(-a\sin\theta)\,d\theta$$

as $\sqrt{\sin^2\theta} = -\sin\theta$ for $-\pi \le \theta \le 0$

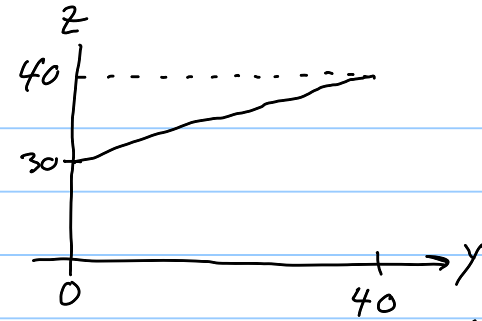
$$= \int_{-\pi}^{0} 2ab\sin^2\theta\,d\theta$$

Now use $1-\cos 2\theta = 2\sin^2\theta$

$$= \int_{-\pi}^{0} ab(1-\cos 2\theta)\,d\theta = ab\theta - \frac{ab}{2}\sin 2\theta \Big|_{-\pi}^{0}$$

$$= 0 - 0 - \left\{-ab\pi - \frac{ab}{2}\sin(-2\pi)\right\} = \underline{\pi ab}$$

7.



Sideways view of barn gives $z = 30 + \frac{10}{40} y = 30 + \frac{1}{4} y$

$\therefore$ For a fixed y, the height is $30 + \frac{1}{4} y$.

$\therefore$ An area slice parallel to x-axis is

$$\int_0^{20} \left(30 + \tfrac{1}{4} y\right) dx \qquad \therefore \text{Volume} = \int_0^{40}\int_0^{20} \left(30 + \tfrac{y}{4}\right) dx\,dy$$

$$\int_0^{20} \left(30 + \tfrac{y}{4}\right) dx = 30x + \tfrac{y}{4} x \Big|_0^{20} = 600 + 5y$$

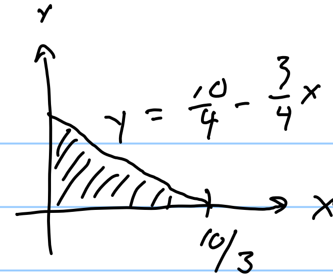
$$\therefore \int_0^{40} (600 + 5y)\, dy = 600y + \tfrac{5}{2} y^2 \Big|_0^{40}$$

$$= 600(40) + \tfrac{5}{2}(40)^2 = 28{,}000 \text{ ft.}^3$$

8.

For a fixed x, $y = \dfrac{10-3x}{4}$. The line intersects the x-axis at $3x + 4(0) = 10$, $x = \frac{10}{3}$

$$\therefore \int_0^{\frac{10}{3}} \int_0^{\frac{10-3x}{4}} (x^2+y^2)\, dy\, dx$$



$$\int_0^{\frac{10}{4}-\frac{3}{4}x} (x^2+y^2)\,dy = x^2 y + \frac{y^3}{3} \Big|_0^{\frac{10-3x}{4}}$$

$$= x^2\left(\frac{10-3x}{4}\right) + \frac{(10-3x)^3}{3\cdot 4^3}$$

$$= \frac{5x^2}{2} - \frac{3x^3}{4} + \frac{(10-3x)^3}{192}$$

$$\therefore \int_0^{\frac{10}{3}} \frac{5x^2}{2} - \frac{3x^3}{4} + \frac{(10-3x)^3}{192}\, dx$$

$$= \frac{5x^3}{6} - \frac{3}{16}x^4 + \frac{(10-3x)^4}{192(4)(-3)} \Big|_0^{\frac{10}{3}}$$

$$= \frac{5}{6}\left(\frac{10}{3}\right)^3 - \frac{3}{16}\left(\frac{10}{3}\right)^4 + 0 - \left[0 - 0 + \frac{10^4}{(192)(4)(-3)}\right]$$

$$= \frac{5(1000)}{\underset{2\cdot 3^4}{6(27)}} - \frac{3(10{,}000)}{\underset{2^4\cdot 3^4}{16(81)}} + \frac{10{,}000}{\underset{2^8\cdot 3^2}{192(4)(3)}}$$

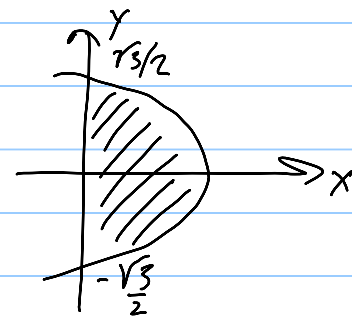
$$= \frac{5(128)(1000) - 48(10{,}000) + 9(10{,}000)}{2^8\cdot 3^4}$$

$$\frac{640{,}000 - 480{,}000 + 90{,}000}{2^8 \cdot 3^4} = \frac{250{,}000}{2^8 \cdot 3^4}$$

$$= \frac{5^2 \cdot 10{,}000}{2^8 \cdot 3^4} = \frac{5^2 \cdot (5 \cdot 2)^4}{2^8 \cdot 3^4} = \frac{5^3 \cdot 2^4}{2^8 \cdot 3^4} = \frac{5^3}{2^4 \cdot 3^4}$$

$$= \underline{\frac{125}{1296}}$$

9.



For $x=0$, $4y^2 = 3$, $y = \pm\frac{\sqrt{3}}{2}$

For a fixed y, an area slice is from $x=0$ to $x=-4y^2+3$

$$\therefore \int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \int_0^{-4y^2+3} (x^3 y)\,dx\,dy = \iint_D x^3 y\,dx\,dy$$

$$\int_0^{-4y^2+3} (x^3 y)\,dx = \frac{x^4}{4} y \Big|_{x=0}^{x=-4y^2+3} = \frac{(-4y^2+3)^4 y}{4}$$

$$\therefore \int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \frac{(-4y^2+3)^4 y}{4}\,dy = \frac{(-4y^2+3)^5}{(5)(-8)(4)} \Bigg|_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}}$$

$$= \frac{\left[-4\left(\frac{\sqrt{3}}{2}\right)^2+3\right]^5}{(5)(-8)(4)} - \frac{\left[-4\left(-\frac{\sqrt{3}}{2}\right)^2+3\right]^5}{(5)(-8)(4)} = \underline{\underline{0}}$$

10.

dy is "inside", so the area slice is from $y=0$ to $y=x^2$ for a given x. x varies from 0 to 1.

$\therefore$ Δ is area under parabola $y=x^2$, $x \in [0,1]$.

$$\therefore \int_0^{x^2} (x^2+xy-y^2)\,dy = x^2y + \frac{xy^2}{2} - \frac{y^3}{3} \Bigg|_{y=0}^{y=x^2}$$

$$= x^4 + \frac{x^5}{2} - \frac{x^6}{3}$$

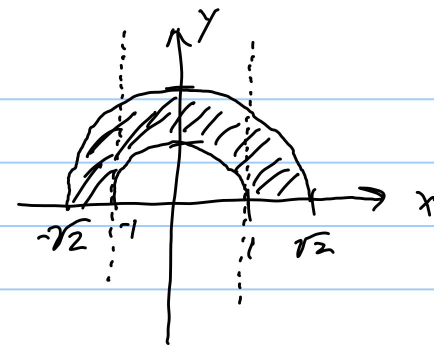
$$\therefore \int_0^1 \left(x^4+\frac{x^5}{2}-\frac{x^6}{3}\right)dx = \frac{x^5}{5} + \frac{x^6}{12} - \frac{x^7}{21}\Bigg|_0^1$$

$$= \frac{1}{5} + \frac{1}{12} - \frac{1}{21} = \frac{84+35-20}{(5)(4)(3)(7)} = \frac{99}{(5)(4)(3)7}$$

$5 \times 4 \times 3 \times 7$

$$= \underline{\underline{\frac{33}{140}}}$$

11.



(a) yes, D is 3 adjacent y-simple elementary regions. For $-\sqrt{2} \le x \le -1$ and $1 \le x \le \sqrt{2}$, use $y_2 = \sqrt{2-x^2}$ and $y_1 = 0$. For $-1 \le x \le 1$, use $y_2 = \sqrt{2-x^2}$ and $y_1 = \sqrt{x^2-1}$

(b) $$\iint_D (1+xy)\,dA = \int_{-\sqrt{2}}^{-1}\int_0^{\sqrt{2-x^2}} (1+xy)\,dy\,dx + \int_{-1}^{1}\int_{\sqrt{1-x^2}}^{\sqrt{2-x^2}} (1+xy)\,dy\,dx$$

$$+ \int_1^{\sqrt{2}}\int_0^{\sqrt{2-x^2}} (1+xy)\,dy\,dx$$

[1] $$\int_{-\sqrt{2}}^{-1}\int_0^{\sqrt{2-x^2}} (1+xy)\,dy\,dx = \int_{-\sqrt{2}}^{-1}\left[y + x\frac{y^2}{2} \Big|_0^{\sqrt{2-x^2}} \right] dx$$

$$= \int_{-\sqrt{2}}^{-1} \sqrt{2-x^2} + \frac{x(2-x^2)}{2}\,dx = \int_{-\sqrt{2}}^{-1} \sqrt{2-x^2}\,dx + \frac{x^2}{2} - \frac{x^4}{8}\Big|_{-\sqrt{2}}^{-1}$$

$$= \int_{-\sqrt{2}}^{-1} \sqrt{2-x^2}\,dx + \left(\frac{1}{2} - \frac{1}{8}\right) - \left(1 - \frac{1}{2}\right) = \int_{-\sqrt{2}}^{-1} \sqrt{2-x^2}\,dx - \frac{1}{8}$$

Now let $x = \sqrt{2}\sin\theta$. $\therefore dx = \sqrt{2}\cos\theta \, d\theta$

$\theta = -\frac{\pi}{2} \to x = -\sqrt{2}$ $\quad \theta = -\frac{\pi}{4} \to x = -1$

$$\therefore \int_{-\sqrt{2}}^{-1} \sqrt{2-x^2}\, dx = \int_{-\frac{\pi}{2}}^{-\frac{\pi}{4}} \sqrt{2-2\sin^2\theta}\,(\sqrt{2}\cos\theta)\, d\theta$$

$$\int_{-\frac{\pi}{2}}^{-\frac{\pi}{4}} 2\sqrt{\cos^2\theta}\,(\cos\theta)\, d\theta = \int_{-\frac{\pi}{2}}^{-\frac{\pi}{4}} 2\cos^2\theta\, d\theta$$

as $\cos\theta \geq 0$ for $-\frac{\pi}{2} \leq \theta \leq -\frac{\pi}{4}$

$$= \int_{-\frac{\pi}{2}}^{-\frac{\pi}{4}} (1+\cos 2\theta)\, d\theta$$

As $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$= \cos^2 - (1-\cos^2)$

$= 2\cos^2\theta - 1$

$$= \theta + \frac{\sin 2\theta}{2}\Big|_{-\frac{\pi}{2}}^{-\frac{\pi}{4}} = -\frac{\pi}{4} - \frac{1}{2} - \left(-\frac{\pi}{2} + 0\right)$$

$$= \underline{\frac{\pi}{4} - \frac{1}{2}}$$

$$\therefore \int_{-\sqrt{2}}^{-1}\int_{0}^{\sqrt{2-x^2}} (1+xy)\, dy\, dx = \frac{\pi}{4} - \frac{1}{2} - \frac{1}{8} = \underline{\frac{\pi}{4} - \frac{5}{8}} \quad [1]$$

[2] $$\int_{-1}^{1}\int_{\sqrt{1-x^2}}^{\sqrt{2-x^2}} (1+xy)\, dy\, dx = \int_{-1}^{1} \left[y + \frac{xy^2}{2} \Big|_{\sqrt{1-x^2}}^{\sqrt{2-x^2}} \right] dx$$

$$= \int_{-1}^{1} \sqrt{2-x^2} + \frac{x(2-x^2)}{2} - \sqrt{1-x^2} - \frac{x(1-x^2)}{2} \, dx$$

$$= \int_{-1}^{1} \sqrt{2-x^2}\, dx - \int_{-1}^{1} \sqrt{1-x^2}\, dx + \int_{-1}^{1} \frac{x}{2}\, dx$$

$$= \int_{-1}^{1} \sqrt{2-x^2}\, dx - \int_{-1}^{1} \sqrt{1-x^2}\, dx + x^2 \Big|_{-1}^{1}$$

$$= \int_{-1}^{1} \sqrt{2-x^2}\, dx - \int_{-1}^{1} \sqrt{1-x^2}\, dx + 0$$

use $x = \sqrt{2} \sin\theta$

$dx = \sqrt{2} \cos\theta\, d\theta$

$\theta = -\frac{\pi}{4} \rightarrow x = -1$

$\theta = \frac{\pi}{4} \rightarrow x = 1$

use $x = \sin\theta$

$dx = \cos\theta\, d\theta$

$\theta = -\frac{\pi}{2} \rightarrow x = -1$

$\theta = \frac{\pi}{2} \rightarrow x = 1$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sqrt{2}\sqrt{\cos^2\theta}\,(\sqrt{2}\cos\theta)\, d\theta - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\cos^2\theta}\,(\cos\theta)\, d\theta$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 2\cos^2\theta\, d\theta - \int_{-\pi/2}^{\pi/2} \cos^2\theta\, d\theta$$

using $1+\cos 2\theta = 2\cos^2\theta$ $\qquad \cos^2\theta = \frac{1}{2} + \frac{\cos 2\theta}{2}$

$$= \int_{-\frac{\pi}{4}}^{\pi/4} (1+\cos 2\theta)\, d\theta - \int_{-\pi/2}^{\pi/2} \left(\frac{1}{2} + \frac{\cos 2\theta}{2}\right) d\theta$$

$$= \theta + \frac{\sin 2\theta}{2}\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} - \left[\frac{1}{2}\theta + \frac{\sin 2\theta}{4}\Big|_{-\frac{\pi}{2}}^{\pi/2}\right]$$

$$= \left(\frac{\pi}{4} + \frac{1}{2}\right) - \left(-\frac{\pi}{4} - \frac{1}{2}\right) - \left[\frac{\pi}{4} + 0 - \left(-\frac{\pi}{4} - 0\right)\right]$$

$$= \frac{\pi}{2} + 1 - \left(\frac{\pi}{2}\right) = \underline{\underline{1}}$$

[2]

[3] $$\int_1^{\sqrt{2}}\int_0^{\sqrt{2-x^2}} (1+xy)\,dy\,dx = \int_1^{\sqrt{2}} \left[y + \frac{xy^2}{2}\Big|_0^{\sqrt{2-x^2}}\right] dx$$

$$= \int_1^{\sqrt{2}} \sqrt{2-x^2} + \frac{x(2-x^2)}{2}\, dx = \int_1^{\sqrt{2}} \sqrt{2-x^2}\,dx + \frac{x^2}{2} - \frac{x^4}{8}\Big|_1^{\sqrt{2}}$$

$$= \int_1^{\sqrt{2}} \sqrt{2-x^2}\,dx + \left(1 - \frac{1}{2}\right) - \left(\frac{1}{2} - \frac{1}{8}\right) = \int_1^{\sqrt{2}} \sqrt{2-x^2}\,dx + \underline{\frac{1}{8}}$$

Again use $x = \sqrt{2}\sin\theta$ $\quad dx = \sqrt{2}\cos\theta\, d\theta$

$\theta = \frac{\pi}{4} \to x = 1 \qquad \theta = \frac{\pi}{2} \to x = \sqrt{2}$

$$\therefore \int_1^{\sqrt{2}} \sqrt{2-x^2}\,dx = \int_{\frac{\pi}{4}}^{\pi/2} \sqrt{2-2\sin^2\theta}\,(\sqrt{2}\cos\theta)\,d\theta$$

$$= \int_{\pi/4}^{\pi/2} 2\sqrt{\cos^2\theta}\,(\cos\theta)\,d\theta = \int_{\pi/4}^{\pi/2} 2\cos^2\theta\,d\theta$$

as $\cos\theta \geq 0$ for $\pi/4 \leq \theta \leq \pi/2$

$$= \int_{\pi/4}^{\pi/2} (1+\cos 2\theta)\,d\theta \quad \text{using } 2\cos^2\theta = 1+\cos 2\theta$$

$$= \theta + \frac{\sin 2\theta}{2} \Bigg|_{\pi/4}^{\pi/2} = \left(\frac{\pi}{2}+0\right) - \left(\frac{\pi}{4}+\frac{1}{2}\right)$$

$$= \frac{\pi}{4} - \frac{1}{2}$$

$$\therefore \int_1^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} (1+xy)\,dy\,dx = \frac{\pi}{4} - \frac{1}{2} + \frac{1}{8} = \underline{\frac{\pi}{4} - \frac{3}{8}} \quad [3]$$

$$\therefore [1] + [2] + [3] = \left(\frac{\pi}{4} - \frac{5}{8}\right) + (1) + \left(\frac{\pi}{4} - \frac{3}{8}\right)$$

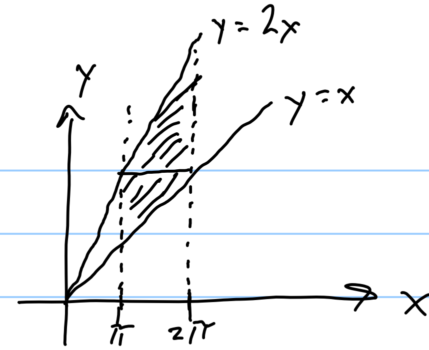
$$= \underline{\frac{\pi}{2}}$$

Check: circle radius $\sqrt{2}$: $\pi(\sqrt{2})^2 = 2\pi$

circle radius 1 : $\pi(1)^2 = \pi$

$$\frac{1}{2}(2\pi - \pi) = \frac{\pi}{2}$$

12.



D is both a y-simple region and x-simple region

Here, dx is "inside", so use x-simple regions

Region 1: $\pi \le y \le 2\pi$, use from $x = \pi$ to $x = y$

$$\therefore \int_{\pi}^{2\pi} \int_{\pi}^{y} \cos y \, dx \, dy$$

Region 2: $2\pi \le y \le 4\pi$, use $x = \frac{y}{2}$ to $x = 2\pi$

$$\therefore \int_{2\pi}^{4\pi} \int_{\frac{y}{2}}^{2\pi} \cos y \, dx \, dy$$

$\therefore$ Region 1: $\displaystyle\int_{\pi}^{2\pi} \int_{\pi}^{y} \cos y \, dx \, dy = \int_{\pi}^{2\pi} \left[x \cos y \Big|_{\pi}^{y} \right] dy$

$$= \int_{\pi}^{2\pi} \left[y \cos y - \pi \cos y \right] dy$$

$$= y\sin y \Big|_{\pi}^{2\pi} - \int_{\pi}^{2\pi} \sin y \, dy - \int_{\pi}^{2\pi} \pi \cos y \, dy$$

$$= 0 - 0 - (-\cos y)\Big|_{\pi}^{2\pi} - \pi \sin y \Big|_{\pi}^{2\pi}$$

$$= 0 - [-1 - (1)] - (0 - 0) = \underline{\underline{2}} \quad [1]$$

Region 2: $\displaystyle\int_{2\pi}^{4\pi} \int_{\frac{y}{2}}^{2\pi} \cos y \, dx \, dy = \int_{2\pi}^{4\pi} \left[x\cos y \Big|_{\frac{y}{2}}^{2\pi} \right] dy$

$$= \int_{2\pi}^{4\pi} \left[2\pi \cos y - \frac{y}{2} \cos y \right] dy$$

$$= 2\pi \sin y \Big|_{2\pi}^{4\pi} - \frac{1}{2} \int_{2\pi}^{4\pi} y \cos y \, dy$$

$$= 0 - \frac{1}{2} \left[y \sin y \Big|_{2\pi}^{4\pi} - \int_{2\pi}^{4\pi} \sin y \, dy \right]$$

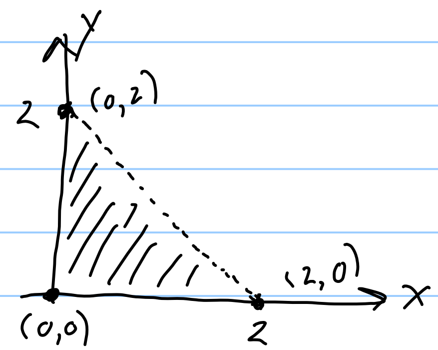
$$= -\frac{1}{2} \left[0 + \cos y \Big|_{2\pi}^{4\pi} \right]$$

$$= -\frac{1}{2} [1 - 1] = \underline{\underline{0}} \quad [2]$$

$$\therefore\ [1] + [2] = 2 + 0 = \underline{2}$$

$$\therefore \iint_D \cos y \, dx\, dy = \underline{2}$$

13.



D is a y-simple region. The hypotenuse is $y = -x + 2$

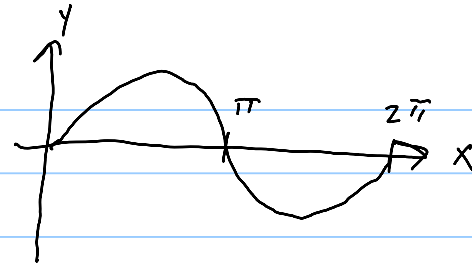
$$\therefore \int_0^2 \int_0^{-x+2} (xy)\, dy\, dx = \int_0^2 \left[\frac{xy^2}{2} \Big|_{y=0}^{y=-x+2} \right] dx$$

$$= \int_0^2 \frac{x(-x+2)^2}{2}\, dx = \int_0^2 \frac{x(x^2 - 4x + 4)}{2}\, dx$$

$$= \int_0^2 \frac{x^3}{2} - 2x^2 + 2x = \frac{x^4}{8} - \frac{2}{3}x^3 + x^2 \Big|_0^2$$

$$= 2 - \frac{16}{3} + 4 = \underline{\frac{2}{3}}$$

14.



Two symmetrical y-simple regions : $0 \le x \le \pi$, $\pi \le x \le 2\pi$

$$\therefore \int_0^{\pi}\int_0^{\sin x} dy\,dx + \int_{\pi}^{2\pi}\int_{\sin x}^{0} dy\,dx$$

$$= \int_0^{\pi}\left[y \Big|_0^{\sin x} \right] dx + \int_{\pi}^{2\pi}\left[y \Big|_{\sin x}^{0} \right] dx$$

$$= \int_0^{\pi} \sin x\,dx + \int_{\pi}^{2\pi} (-\sin x)\,dx$$

$$= -\cos x \Big|_0^{\pi} + \cos x \Big|_{\pi}^{2\pi}$$

$$= 1-(-1) + 1-(-1) = \underline{\underline{4}}$$

15.

The domain D is a disc centered at $(x,y) = (0,0)$ with radius $\sqrt{10}$, from $10 = x^2 + y^2$.

$\therefore$ a y-simple region from $y = -\sqrt{10-x^2}$ to $y = \sqrt{10-x^2}$, and x ranges $-\sqrt{10} \le x \le \sqrt{10}$

The height of any one rectangular solid is $10 - (x^2 + y^2)$ for any $(x,y) \in dx\,dy$

$$\therefore \text{Volume} = \iint_D (10 - x^2 - y^2)\, dA$$

$$= \int_{-\sqrt{10}}^{\sqrt{10}} \int_{-\sqrt{10-x^2}}^{\sqrt{10-x^2}} (10 - x^2 - y^2)\, dy\, dx$$

$$\int_{-\sqrt{10-x^2}}^{\sqrt{10-x^2}} (10 - x^2 - y^2)\, dy = 10y - x^2 y - \frac{y^3}{3} \Bigg|_{y=-\sqrt{10-x^2}}^{y=\sqrt{10-x^2}}$$

$$= 20\sqrt{10-x^2} - 2x^2\sqrt{10-x^2} - \frac{2}{3}(10-x^2)^{3/2}$$

$$\therefore \text{Volume} = \int_{-\sqrt{10}}^{\sqrt{10}} \left[20\sqrt{10-x^2} - 2x^2\sqrt{10-x^2} - \frac{2}{3}(10-x^2)^{3/2} \right] dx$$

Using a table of integrals,

[1] $$20\int_{-\sqrt{10}}^{\sqrt{10}} \sqrt{10-x^2} = 20\left[\frac{x}{2}\sqrt{10-x^2} + \frac{10}{2}\arcsin\frac{x}{\sqrt{10}}\right]_{-\sqrt{10}}^{\sqrt{10}}$$

$$= 20\left[\frac{\sqrt{10}}{2}(0) + 5\arcsin(1) - \frac{-\sqrt{10}}{2}(0) - 5\arcsin(-1)\right]$$

$$= 20\left[5\left(\frac{\pi}{2}\right) - 5\left(-\frac{\pi}{2}\right)\right] = \underline{100\pi}$$

[2] $$-2\int_{-\sqrt{10}}^{\sqrt{10}} x^2\sqrt{10-x^2}\,dx =$$

$$-2\left[-\frac{x}{4}(10-x^2)^{\frac{3}{2}} + \frac{10}{8}x\sqrt{10-x^2} + \frac{100}{8}\arcsin\frac{x}{\sqrt{10}}\right]_{-\sqrt{10}}^{\sqrt{10}}$$

$$= -2\left[0 + 0 + \frac{100}{8}\arcsin(1) - \left(0 + 0 + \frac{100}{8}\arcsin(-1)\right)\right]$$

$$= -2\left[\frac{25}{2}\left(\frac{\pi}{2}\right) - \frac{25}{2}\left(-\frac{\pi}{2}\right)\right] = \underline{-25\pi}$$

[3] $$-\frac{2}{3}\int_{-\sqrt{10}}^{\sqrt{10}} (10-x^2)^{3/2}\,dx =$$

$$-\frac{2}{3}\left[-\frac{x}{8}\left(2x^2 - 5(10)\right)\sqrt{10-x^2} + \frac{3(100)}{8}\arcsin\frac{x}{\sqrt{10}}\right]_{-\sqrt{10}}^{\sqrt{10}}$$

$$= -\frac{2}{3}\left[0 + \frac{300}{8}\arcsin(1) - \left(0 + \frac{300}{8}\arcsin(-1)\right)\right]$$

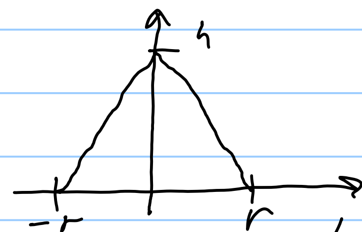
$$= -\frac{2}{3}\left[\frac{300}{8}\left(\frac{\pi}{2}\right) - \frac{300}{8}\left(-\frac{\pi}{2}\right)\right]$$

$$= -\frac{2}{3}\left(\frac{300}{8}\right)(\pi) = \underline{-25\pi}$$

$[1] + [2] + [3] = 100\pi - 25\pi - 25\pi = \underline{\underline{50\pi}}$

$$\therefore \iint_D (10 - x^2 - y^2)\, dA = \underline{50\pi}$$

16.



You can view the cone standing on a circular disc – in which case the height is a function of (x,y), using $x^2+y^2=d^2$, d = diameter of a circle, and $0 \le d \le r$. If $d=r$, height $=0$, if $d=0$, height $=h$.

Let $f(x,y)$ = height function.

$$\therefore f(x,y) = h - d\left(\frac{h}{r}\right) = h - \sqrt{x^2+y^2}\left(\frac{h}{r}\right),$$

$$-r \le x \le r, \quad -r \le y \le r$$

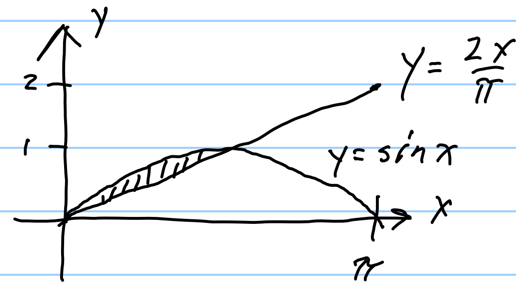
For the base, region D, can be viewed as a

y-simple region, $-\sqrt{r^2-x^2} \le y \le \sqrt{r^2-x^2}$

$$\therefore \text{Volume} = \iint_D f(x,y)\, dA = \int_{-r}^{r} \int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} f(x,y)\, dy\, dx$$

$$= \int_{-r}^{r} \int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} \left[h - \sqrt{x^2+y^2}\left(\frac{h}{r}\right) \right] dy\, dx$$

17.



First figure out what D is:

$y \le \sin x$ means values of y under $y = \sin x$ curve.

$\frac{2x}{\pi} \le y$ means those values of y above the line $y = \frac{2}{\pi} x$

Note that $0 \le x$ and $y = \frac{2x}{\pi} = \sin x$ when $x = \frac{\pi}{2}$, $y = \frac{2\left(\frac{\pi}{2}\right)}{\pi} = 1$, and $y = \sin\frac{\pi}{2} = 1$

$\therefore\ 0 \le x \le \frac{\pi}{2}$

View D as a y-simple region.

$$\therefore \int_0^{\frac{\pi}{2}} \int_{\frac{2x}{\pi}}^{\sin x} y\, dy\, dx = \int_0^{\frac{\pi}{2}} \left[\frac{y^2}{2} \Big|_{y=\frac{2x}{\pi}}^{y=\sin x} \right] dx$$

$$= \int_0^{\pi/2} \left[\frac{\sin^2 x}{2} - \frac{1}{2}\left(\frac{2x}{\pi}\right)^2 \right] dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin^2 x \, dx - \frac{2}{\pi^2}\int_0^{\frac{\pi}{2}} x^2 dx$$

Use $\cos^2\theta - \sin^2\theta = \cos 2\theta$

$$(1-\sin^2\theta) - \sin^2\theta = \cos 2\theta$$

$$\frac{1-\cos 2\theta}{2} = \sin^2\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{1-\cos 2x}{2} dx - \frac{2}{\pi^2}\left[\frac{x^3}{3}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left[\frac{x}{2} - \frac{\sin 2x}{4}\right]_0^{\frac{\pi}{2}} - \frac{2}{\pi^2}\left(\frac{\pi^3}{(3)(8)}\right)$$

$$= \frac{\pi}{8} - \frac{\sin \pi}{8} - \frac{\pi}{12} = \frac{\pi}{8} - \frac{\pi}{12} = \underline{\frac{\pi}{24}}$$

18.

For a y-simple region, $y_1 = \phi_1(x) \le y \le \phi_2(x) = y_2$ and $a \le x \le b$

$$\therefore \iint_D f(x)g(y)\,dx\,dy = \iint_D f(x)g(y)\,dy\,dx$$

$$= \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x) g(y) \, dy \, dx$$

$$= \int_a^b f(x) \left[G(\phi_2(x)) - G(\phi_1(x)) \right] dx$$

assuming $G'(y) = g(y)$.

You can't factor out $[G(\phi_2(x)) - G(\phi_1(x))]$ since it is not a constant.

$\therefore$ <u>No</u>, The assertion is false.

19.

$$\iint_D f(x,y) \, dA = \int_a^b \int_{-\phi(x)}^{\phi(x)} f(x,y) \, dy \, dx$$

$$= \int_a^b \left[\int_{-\phi(x)}^{0} f(x,y) \, dy + \int_0^{\phi(x)} f(x,y) \, dy \right] dx$$

$$= \int_a^b \left[- \int_0^{-\phi(x)} f(x,y) \, dy + \int_0^{\phi(x)} f(x,y) \, dy \right] dx$$

$$= \int_a^b \left[\int_0^{-\phi(x)} -f(x,y)\,dy + \int_0^{\phi(x)} f(x,y)\,dy \right] dx$$

$$= \int_a^b \left[\int_0^{-\phi(x)} f(x,-y)\,dy + \int_0^{\phi(x)} f(x,y)\,dy \right] dx \quad [1]$$

Let $u = -y$. $\therefore du = -dy$

when $y = -\phi(x)$, $u = -y = \phi(x)$

$y = 0 \rightarrow u = 0$

$$\therefore \int_0^{-\phi(x)} f(x,-y)\,dy = \int_0^{\phi(x)} -f(x,u)\,du = -\int_0^{\phi(x)} f(x,u)\,du$$

Now use $y = u$, $dy = du$, $u = 0 \rightarrow y = 0$

$u = \phi(x) \rightarrow y = \phi(x)$

$$= -\int_0^{\phi(x)} f(x,y)\,dy$$

$\therefore$ [1] becomes

$$\int_a^b \left[\int_0^{-\phi(x)} f(x,-y)\,dy + \int_0^{\phi(x)} f(x,y)\,dy \right] dx =$$

$$\int_a^b \left[-\int_0^{\phi(x)} f(x,y)\,dy + \int_0^{\phi(x)} f(x,y)\,dy \right] dx =$$

$$\int_a^b 0 \, dx = \underline{0}$$

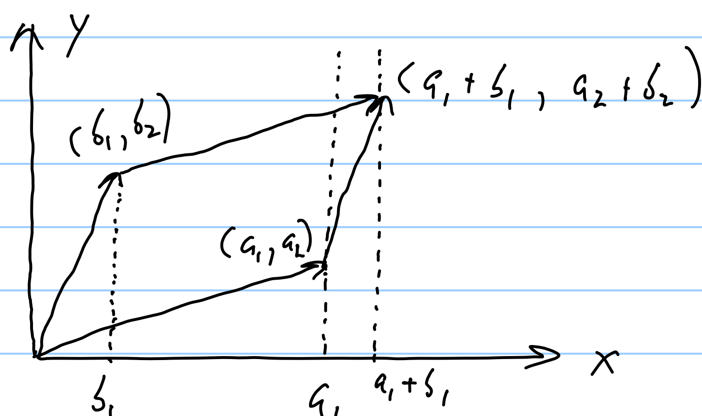
20.

Assume $\vec{a}$ and $\vec{b}$ not parallel, and not $\vec{0}$.

$\therefore \ \vec{a} \times \vec{b} = \|\vec{a}\| \, \|\vec{b}\| \sin\theta \neq 0$

so $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & 0 \\ b_1 & b_2 & 0 \end{vmatrix} = \vec{a} \times \vec{b} = a_1 b_2 - a_2 b_1 \neq 0.$

Consider



We have 3 y-simple regions

$0 \leq x \leq b_1 \quad y_1 = \frac{a_2}{a_1} x$ to $y_2 = \frac{b_2}{b_1} x$

$b_1 \leq x \leq a_1 \quad y_1 = \frac{a_2}{a_1} x$ to $y_2 = \frac{a_2}{a_1} x + \left(b_2 - \frac{a_2}{a_1} b_1\right)$

$a_1 \leq x \leq a_1 + b_1 \quad y_1 = \frac{b_2}{b_1} x + \left(a_2 - \frac{b_2}{b_1} a_1\right)$ to

$y_2 = \frac{a_2}{a_1} x + \left(b_2 - \frac{a_2}{a_1} b_1\right)$

Area of parallelogram $= \iint_D dy\,dx$

$$= \iint_{A_1} dy\,dx + \iint_{A_2} dy\,dx + \iint_{A_3} dy\,dx$$

A_1 : $$\int_0^{b_1} \int_{\frac{a_2}{a_1}x}^{\frac{b_2}{b_1}x} dy\,dx = \int_0^{b_1} \left(\frac{b_2}{b_1}x - \frac{a_2}{a_1}x \right) dx$$

$$= \frac{b_2}{b_1}\frac{x^2}{2} - \frac{a_2}{a_1}\frac{x^2}{2} \Big|_0^{b_1} = \frac{b_2}{b_1}\cdot\frac{b_1^2}{2} - \frac{a_2}{a_1}\cdot\frac{b_1^2}{2}$$

$$= \underline{\frac{b_1 b_2}{2} - \frac{a_2 b_1^2}{2a_1}}$$

A_2 : $$\int_{b_1}^{a_1} \int_{\frac{a_2}{a_1}x}^{\frac{a_2}{a_1}x + \left(b_2 - \frac{a_2}{a_1}b_1\right)} dy\,dx =$$

$$\int_{b_1}^{a_1} \left(b_2 - \frac{a_2}{a_1}b_1 \right) dx = \left(b_2 x - \frac{a_2}{a_1}b_1 x \right) \Big|_{b_1}^{a_1}$$

$$= a_1 b_2 - a_2 b_1 - \left(b_1 b_2 - \frac{a_2}{a_1} b_1^2 \right)$$

$$= \underline{a_1 b_2 - a_2 b_1 - b_1 b_2 + \frac{a_2 b_1^2}{a_1}}$$

$$A_3 : \int_{a_1}^{a_1+b_1} \int_{\frac{b_2}{b_1}x + \left(a_2 - \frac{b_2}{b_1}a_1\right)}^{\frac{a_2}{a_1}x + \left(b_2 - \frac{a_2}{a_1}b_1\right)} dy\, dx =$$

$$\int_{a_1}^{a_1+b_1} \frac{a_2}{a_1}x - \frac{b_2}{b_1}x + b_2 - a_2 - \frac{a_2}{a_1}b_1 + \frac{b_2}{b_1}a_1 \quad dx$$

$$= \frac{a_2}{a_1}\frac{x^2}{2} - \frac{b_2}{b_1}\frac{x^2}{2} + b_2x - a_2x - \frac{a_2}{a_1}b_1x + \frac{b_2}{b_1}a_1x \Big|_{a_1}^{a_1+b_1}$$

$$= \frac{a_2}{a_1}\frac{(a_1+b_1)^2}{2} - \frac{b_2}{b_1}\frac{(a_1+b_1)^2}{2} + a_1b_2 + b_1b_2 - a_1a_2 - a_2b_1$$

$$- a_2b_1 - \frac{a_2b_1^2}{a_1} + \frac{b_2a_1^2}{b_1} + a_1b_2$$

$$- \frac{a_1a_2}{2} + \frac{b_2}{b_1}\frac{a_1^2}{2} - a_1b_2 + a_1a_2 + a_2b_1 - \frac{b_2}{b_1}a_1^2$$

$$= \frac{a_2}{a_1}\frac{(a_1+b_1)^2}{2} - \frac{b_2}{b_1}\frac{(a_1+b_1)^2}{2} + b_1b_2 - a_2b_1 + a_1b_2 - \frac{a_2b_1^2}{a_1} + \frac{b_2a_1^2}{b_1} - \frac{a_1a_2}{2} - \frac{a_1^2b_2}{2b_1}$$

$$= \frac{a_2}{2a_1}\left(a_1^2 + 2a_1b_1 + b_1^2\right) - \frac{b_2}{2b_1}\left(a_1^2 + 2a_1b_1 + b_1^2\right)$$

$$+ b_1b_2 - a_2b_1 + a_1b_2 - \frac{a_2b_1^2}{a_1} + \frac{a_1^2b_2}{b_1} - \frac{a_1a_2}{2} - \frac{a_1^2b_2}{2b_1}$$

$$= \frac{a_1 a_2}{2} + a_2 b_1 + \frac{a_2 b_1^2}{2a_1} - \frac{a_1^2 b_2}{2b_1} - a_1 b_2 - \frac{b_1 b_2}{2}$$

$$+ b_1 b_2 - a_2 b_1 + a_1 b_2 - \frac{a_2 b_1^2}{a_1} + \frac{a_1^2 b_2}{b_1} - \frac{a_1 a_2}{2} - \frac{a_1^2 b_2}{2b_1}$$

$$= -\frac{a_2 b_1^2}{2a_1} + \frac{b_1 b_2}{2}$$

$$\therefore A_1 + A_2 + A_3 =$$

$$\left(\frac{b_1 b_2}{2} - \frac{a_2 b_1^2}{2a_1}\right) + \left(a_1 b_2 - a_2 b_1 - b_1 b_2 + \frac{a_2 b_1^2}{a_1}\right) + \left(-\frac{a_2 b_1^2}{2a_1} + \frac{b_1 b_2}{2}\right)$$

$$= a_1 b_2 - a_2 b_1$$

$$\therefore \text{Area} = |A_1 + A_2 + A_3| = \underline{|a_1 b_2 - a_2 b_1|}$$

Note: any parallelogram can be rotated so that the acute angle is in Quadrant I.

If a rectangle, then $(a_1, a_2) = (a_1, 0)$, $(b_1, b_2) = (0, b_2)$. $\therefore |a_1 b_2 - a_2 b_1| = |a_1 b_2|$ so formula still works.

21.

This question seems to be identical to Example 3.

$$A(D) = \lim_{n\to\infty} \sum_{j,k=0}^{n-1} f(c_{jk})\,\Delta x\,\Delta y, \qquad \begin{array}{l}\Delta x = x_{j+1} - x_j \\ \Delta y = y_{j+1} - y_j\end{array}$$

where $f(c_{jk}) = \begin{cases} 1, & \text{if } c_{jk} \in D \\ 0, & \text{if } c_{jk} \notin D \end{cases}$, c_{jk} any point in R_{jk}

where R_{jk} is a subrectangle of the partition R, a rectangle that contains D.

5.4 Changing the Order of Integration

Note Title 10/27/2016

1.

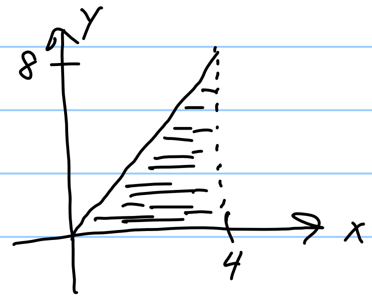
(a) The original integral must be $\int_0^8 \int_{\frac{y}{2}}^4 dx\,dy$

as $y \neq 0$ for $\frac{1}{2y}$

$\therefore$ For $\frac{y}{2} \le x \le 4$ and $0 \le y \le 8$

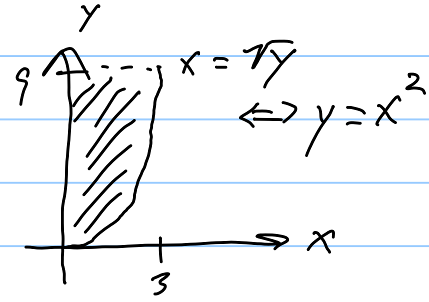
Can be also described as

$0 \le x \le 4$ and $0 \le y \le 2x$



$\therefore \int_0^4 \int_0^{2x} dy\,dx$

(b) $0 \le y \le 9$ and $0 \le x \le \sqrt{y}$

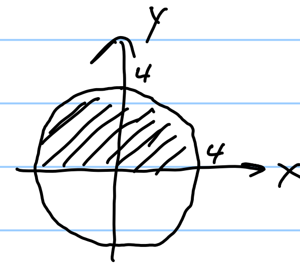


so x ranges from 0 to $\sqrt{9} = 3$

$\therefore$ $0 \le x \le 3$ and $x^2 \le y \le 9$

$$\therefore \int_0^3 \int_{x^2}^9 dy\,dx$$

(c)



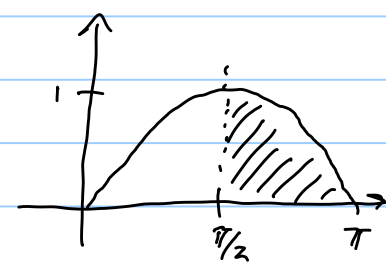
$-\sqrt{16-y^2} \le x \le \sqrt{16-y^2} \quad 0 \le y \le 4$

is the same region as $-4 \le x \le 4$

and $0 \le y \le \sqrt{16-x^2}$

$$\therefore \int_{-4}^{4} \int_0^{\sqrt{16-x^2}} dy\,dx$$

(d)



$0 \le y \le \sin x \ , \ \frac{\pi}{2} \le x \le \pi$

The max & min values for $\sin x$ are $\sin(\frac{\pi}{2})$, $\sin(\pi)$, or, between 0 and 1.

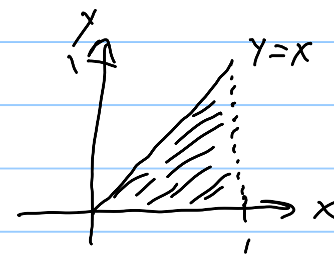
$\therefore$ after switching, y will range $0 \le y \le 1$

The relationship for x will be the inverse of $\sin x$.

$\therefore \arcsin y$, so $\frac{\pi}{2} \le x \le \arcsin y$

$$\therefore \int_0^1 \int_{\frac{\pi}{2}}^{\arcsin y} dx\,dy$$

2.



Initially, $0 \le y \le 1$, $y \le x \le 1$

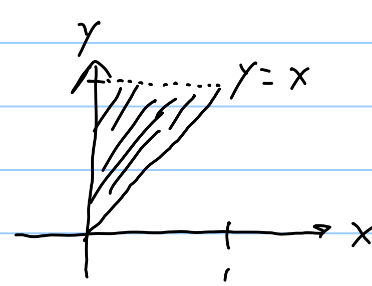
$\therefore$ Same region D described as: $0 \le x \le 1$, $0 \le y \le x$

$$\therefore \int_0^1 \int_0^x \sin(x^2)\,dy\,dx = \int_0^1 \left[y \sin(x^2) \Big|_{y=0}^{y=x} \right] dx$$

$$= \int_0^1 x \sin(x^2)\,dx = -\frac{1}{2}\cos(x^2) \Big|_0^1 =$$

$$-\frac{1}{2}\cos(1) - \left(-\frac{1}{2}\cos(0)\right) = \frac{1}{2} - \frac{1}{2}\cos(1)$$

3.



(a)

$0 \le x \le 1$, $x \le y \le 1$

$$\int_0^1 \int_x^1 (xy)\,dy\,dx = \int_0^1 \left[\frac{xy^2}{2} \Bigg|_{y=x}^{y=1} \right] dx$$

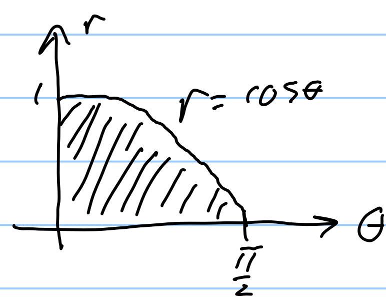
$$= \int_0^1 \left(\frac{x}{2} - \frac{x^3}{2} \right) dx = \frac{x^2}{4} - \frac{x^4}{8} \Bigg|_0^1 = \frac{1}{4} - \frac{1}{8} = \underline{\frac{1}{8}}$$

Change order: $0 \le x \le y$, $0 \le y \le 1$

$$\therefore \int_0^1 \int_0^y (xy)\,dx\,dy = \int_0^1 \left[\frac{x^2}{2} y \Bigg|_{x=0}^{x=y} \right] dx$$

$$= \int_0^1 \frac{y^3}{2}\,dy = \frac{y^4}{8} \Bigg|_0^1 = \underline{\frac{1}{8}}$$

(3)



Initially, $0 \le r \le \cos\theta$
$0 \le \theta \le \pi/2$

$$\int_0^{\frac{\pi}{2}} \int_0^{\cos\theta} \cos\theta\,dr\,d\theta = \int_0^{\frac{\pi}{2}} \left[r\cos\theta \Bigg|_{r=0}^{r=\cos\theta} \right] d\theta$$

$$= \int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta = \int_0^{\frac{\pi}{2}} \frac{\cos 2\theta + 1}{2}\,d\theta$$

using $\cos 2\theta =$
$\cos^2\theta - \sin^2\theta$
$= \cos^2\theta - (1 - \cos^2\theta)$
$= 2\cos^2\theta - 1$

$$= \frac{\sin 2\theta}{4} + \frac{\theta}{2} \Big|_0^{\frac{\pi}{2}} = 0 + \frac{\pi}{4} - (0+0) = \underline{\frac{\pi}{4}}$$

The inverse of $r = \cos\theta$ is $\theta = \arccos r$

Initially, r varied between 0 and $\max(\cos\theta)$

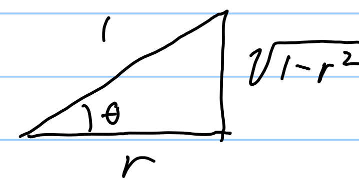
Here, $\max(\cos\theta) = \cos(0) = 1$.

For the inverse, θ varies between 0 and $\arccos(r)$ and $0 \le r \le 1$.

$$\therefore \int_0^1 \int_0^{\arccos(r)} \cos\theta \, d\theta \, dr = \int_0^1 \left[\sin\theta \Big|_{\theta=0}^{\theta=\arccos(r)} \right] dr$$

$$= \int_0^1 \sin(\arccos(r)) \, dr$$

For $\theta = \arccos(r)$, $r = \cos\theta$. $\therefore$



$$\therefore \sin\theta = \sqrt{1-r^2} = \sin(\arccos(r))$$

$$\therefore \int_0^1 \sin(\arccos(r)) \, dr = \int_0^1 \sqrt{1-r^2} \, dr$$

$$= \frac{r}{2}\sqrt{1-r^2} + \frac{1}{2}\arcsin(r) \Big|_0^1$$ using table of integrals

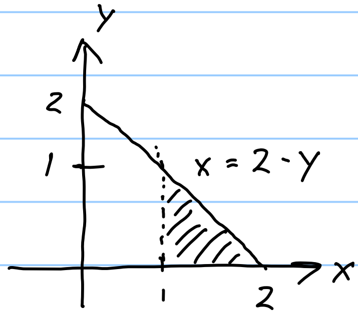
$$= \frac{1}{2}\arcsin(1) = \frac{1}{2}\left(\frac{\pi}{2}\right) = \underline{\frac{\pi}{4}}$$

(c) (c) $\int_0^1 \int_1^{2-y} (x+y)^2 \, dx \, dy$

$$\int_0^1 \left[\frac{(x+y)^3}{3} \Big|_{x=1}^{x=2-y} \right] dy = \int_0^1 \left[\frac{8}{3} - \frac{(1+y)^3}{3} \right] dy$$

$$= \frac{8}{3}y - \frac{(1+y)^4}{12} \Big|_0^1 = \frac{8}{3} - \frac{16}{12} - \left(0 - \frac{1}{12}\right) = \underline{\frac{17}{12}}$$

To change,



$x = f(y) = 2 - y$

$y = f^{-1}(x) = 2 - x$

$\left.\begin{array}{l} f(0) = 2 - 0 = 2 \\ f(1) = 2 - 1 = 1 \end{array}\right\} \therefore 1$ is min.

$\therefore 0 \leq y \leq f^{-1}(x) = 2 - x$

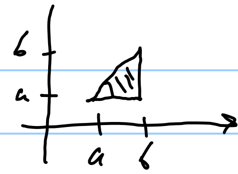
$1 \leq x \leq 2$ or $1 \leq x \leq f(y_{min})$

$$\therefore \int_1^2 \int_0^{2-x} (x+y)^2 \, dy \, dx = \int_1^2 \left[\frac{(x+y)^3}{3} \Big|_{y=0}^{y=2-x} \right] dx$$

$$= \int_1^2 \left[\frac{8}{3} - \frac{x^3}{3} \right] dx = \frac{8}{3}x - \frac{x^4}{12} \Big|_1^2 = \frac{16}{3} - \frac{16}{12} - \left(\frac{8}{3} - \frac{1}{12}\right)$$

$$= \underline{\frac{17}{12}}$$

(d)



Let $\frac{\partial G}{\partial x}(x,y) = f(x,y)$. $\therefore \int_a^y f(x,y)\,dx = G(y,y) - G(a,y)$

Now let $F(y)$ be s.t. $F'(y) = G(y,y) - G(a,y)$

$$\therefore \int_a^b \left[G(y,y) - G(a,y)\right] dy = \underline{F(b) - F(a)}$$

Switching, $\displaystyle\int_a^b \int_a^x f(x,y)\,dy\,dx$

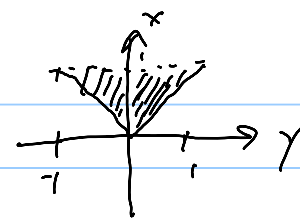
Let $H(x,y)$ be s.t. $\frac{\partial H}{\partial y}(x,y) = f(x,y)$

$$\therefore \int_a^x f(x,y)\,dy = H(x,x) - H(x,a)$$

Now let $J(x)$ be s.t. $J'(x) = H(x,x) - H(x,a)$

$$\therefore \int_a^b \left[H(x,x) - H(x,a)\right] dx = J(b) - J(a)$$

4.



(a) The "inner" variable limit is $x = |y|$ for $-1 \le y \le 1$,
or $x = -y$, $-1 \le y \le 0$, $x = y$, $0 \le y \le 1$

$$\therefore \int_{-1}^{1} \int_{|y|}^{1} (x+y)^2 \, dx \, dy = \int_{-1}^{0} \int_{-y}^{1} (x+y)^2 \, dx \, dy + \int_{0}^{1} \int_{y}^{1} (x+y)^2 \, dx \, dy$$

$$\int_{-1}^{0} \int_{-y}^{1} (x+y)^2 \, dx \, dy = \int_{-1}^{0} \left[\frac{1}{3}(x+y)^3 \Big|_{x=-y}^{x=1} \right] dy$$

$$= \int_{-1}^{0} \frac{1}{3}(1+y)^3 \, dy = \frac{1}{12}(1+y)^4 \Big|_{-1}^{0} = \frac{1}{12} - 0 = \underline{\frac{1}{12}}$$

$$\int_{0}^{1} \int_{y}^{1} (x+y)^2 \, dx \, dy = \int_{0}^{1} \left[\frac{1}{3}(x+y)^3 \Big|_{x=y}^{x=1} \right] dy$$

$$= \int_{0}^{1} \left[\frac{1}{3}(1+y)^3 - \frac{1}{3}(2y)^3 \right] dy = \frac{1}{12}(1+y)^4 - \frac{1}{24}(2y)^4 \Big|_{0}^{1}$$

$$= \frac{1}{12}(2)^4 - \frac{1}{24}(2)^4 - \left[\frac{1}{12} - 0 \right] = \frac{16}{12} - \frac{16}{24} - \frac{1}{12}$$

$$= \frac{16}{12} - \frac{8}{12} - \frac{1}{12} = \frac{7}{12}$$

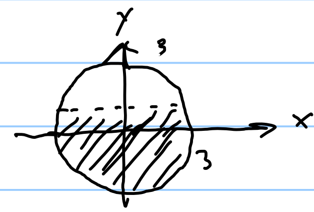
$$\therefore \frac{1}{12} + \frac{7}{12} = \frac{8}{12} = \underline{\frac{2}{3}}$$

Note: The "change of order" integration would be

$$\int_0^1 \int_{-x}^{x} (x+y)^2\,dy\,dx \quad \int_0^1 \left[\frac{1}{3}(x+y)^3 \Big|_{y=-x}^{y=x}\right] dx$$

$$= \int_0^1 \frac{1}{3}(2x)^3\,dx = \int_0^1 \frac{8}{3}x^3\,dx = \frac{8}{12}x^4 \Big|_0^1 = \frac{8}{12} = \underline{\frac{2}{3}}$$

(b)



$$\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} x^2\,dx = \frac{x^3}{3}\Big|_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} = \frac{2(9-y^2)^{3/2}}{3}$$

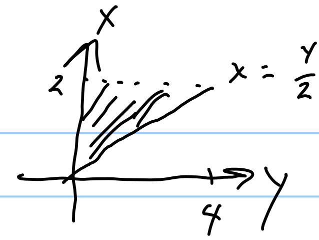
$$\therefore \frac{2}{3}\int_{-3}^{1} (9-y^2)^{3/2}\,dy =$$ (using #39 table of integrals from text)

$$\frac{2}{3}\left[\frac{y}{8}(5\cdot 9 - 2y^2)\sqrt{9-y^2} + \frac{3(9)}{8}\arcsin\left(\frac{y}{3}\right)\right]_{-3}^{1}$$

$$= \frac{2}{3}\left[\frac{43}{8}\sqrt{8} + \frac{27}{8}\arcsin\left(\frac{1}{3}\right) - \left(0 + \frac{27}{8}\left(-\frac{\pi}{2}\right)\right)\right]$$

$$= \underline{\frac{43}{6}\sqrt{2} + \frac{27}{8}\arcsin\left(\frac{1}{3}\right) + \frac{27}{16}\pi}$$

(c)



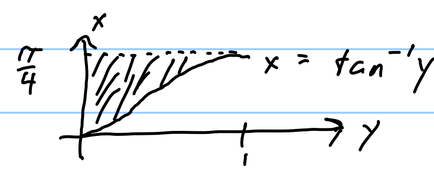
Change order to get an "x" in front of e^{x^2}.

$$\therefore \int_0^4 \int_{y/2}^{2} e^{x^2}\,dx\,dy = \int_0^2 \int_0^{2x} e^{x^2}\,dy\,dx$$

$$= \int_0^2 \left[y e^{x^2} \Big|_{y=0}^{y=2x} \right] dx = \int_0^2 2x e^{x^2}\,dx$$

$$= e^{x^2} \Big|_0^2 = \underline{e^4 - 1}$$

(d)



Change order to get a "tan" in front of $\sec^5 x$.

Over the interval $0 \le y \le 1$, $x = \tan^{-1} y$ is increasing

$\therefore \tan^{-1}(0) = 0,\ \tan^{-1}(1) = \frac{\pi}{4} \Rightarrow$ new outer limits

$$\therefore \int_0^1 \int_{\tan^{-1} y}^{\frac{\pi}{4}} (\sec^5 x)\,dx\,dy = \int_0^{\frac{\pi}{4}} \int_0^{\tan x} (\sec^5 x)\,dy\,dx$$

$$= \int_0^{\frac{\pi}{4}} \tan x \sec^5 x \, dx = \int_0^{\frac{\pi}{4}} (\sec x \tan x) \sec^4(x) \, dx$$

$$= \int_0^{\frac{\pi}{4}} \sec^4(x) \, d(\sec x) = \frac{\sec^5 x}{5} \Big|_0^{\frac{\pi}{4}} = \frac{1}{5\cos^5(x)} \Big|_0^{\frac{\pi}{4}}$$

$$= \frac{1}{5\left(\frac{1}{\sqrt{2}}\right)^5} - \frac{1}{5} = \frac{(\sqrt{2})^5}{5} - \frac{1}{5} = \underline{\underline{\frac{4\sqrt{2}-1}{5}}}$$

5.

$x = \sqrt{y}$ is increasing on $0 \le y \le 1$ (The "outer" limits)

$\therefore$ The new limits become $\int_{\sqrt{0}}^{\sqrt{1}} \int_0^{x^2} dy\,dx = \int_0^1 \int_0^{x^2} dy\,dx$

$$\therefore \int_0^1 \int_0^{x^2} e^{x^3} \, dy\,dx = \int_0^1 \left[y e^{x^3} \Big|_0^{y=x^2} \right] dx = \int_0^1 x^2 e^{x^3} \, dx$$

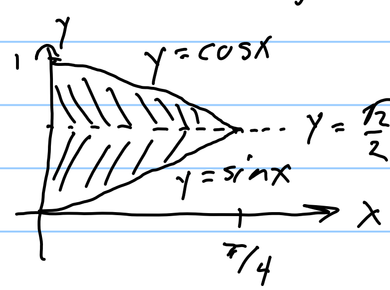
$$= \frac{1}{3} e^{x^3} \Big|_0^1 = \underline{\frac{1}{3} e - \frac{1}{3}}$$

6.

(a)

(a) $\displaystyle\int_0^{\pi/4}\int_{\sin x}^{\cos x} dy\,dx = \int_0^{\sqrt{2}/2}\int_0^{\arcsin y} dx\,dy + \int_{\sqrt{2}/2}^{2}\int_0^{\arccos y} dx\,dy$

Over the interval $0 \le x \le \frac{\pi}{4}$, $y = \cos x$ is decreasing, $y = \sin x$ is increasing.



$\cos(0) = 1 \quad \cos(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$

$\sin(0) = 0 \quad \sin(\pi/4) = \sqrt{2}/2$

∴ To change order, split the integral as:

$$\int_0^{\frac{\pi}{4}}\int_{\sin x}^{\cos x} dy\,dx = \int_0^{\frac{\pi}{4}}\int_{\sin x}^{\frac{\sqrt{2}}{2}} dy\,dx + \int_0^{\frac{\pi}{4}}\int_{\frac{\sqrt{2}}{2}}^{\cos x} dy\,dx$$

$$= \int_{\sin(0)}^{\sin(\frac{\pi}{4})}\int_0^{\arcsin y} dx\,dy + \int_{\cos(\pi/4)}^{\cos(0)}\int_0^{\arccos(y)} dx\,dy$$

$$= \int_0^{\frac{\sqrt{2}}{2}}\int_0^{\arcsin(y)} dx\,dy + \int_{\sqrt{2}/2}^{1}\int_0^{\arccos(y)} dx\,dy$$

∴ **False** the second out limit is wrong

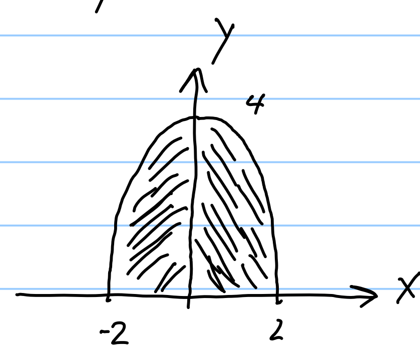
(b)

Over interval $-2 \le x \le 2$, $y = 4 - x^2$ is both increasing and decreasing. $\therefore$ Break it up.

$$\int_{-2}^{2}\int_{0}^{4-x^2} dy\,dx = \underbrace{\int_{-2}^{0}\int_{0}^{4-x^2} dy\,dx}_{\text{increasing}} + \underbrace{\int_{0}^{2}\int_{0}^{4-x^2} dy\,dx}_{\text{decreasing}}$$

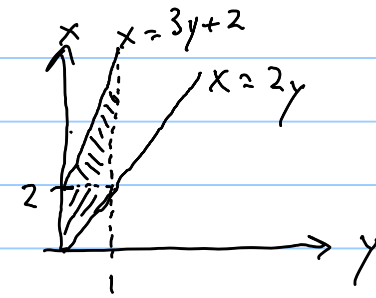
$$= \int_{0}^{4}\int_{-\sqrt{4-y}}^{0} dx\,dy + \int_{0}^{4}\int_{0}^{\sqrt{4-y}} dx\,dy$$

$$= \int_{0}^{4}\int_{-\sqrt{4-y}}^{\sqrt{4-y}} dx\,dy$$



$\therefore$ **True**

(c)



Over the interval $0 \le y \le 1$, $x = 2y$ and $x = 3y + 2$ are both increasing. Note $2y$ max is at $y = 1$ and $3y + 2$ minimum is at $y = 0$.

$$\left.\begin{array}{l} x = 2y \text{ at } y = 1 \text{ is } 2 \\ x = 3y + 2 \text{ at } y = 0 \text{ is } 2 \end{array}\right\} \therefore \text{ Break up at } x = 2$$

For $x = 2y$, $y = \frac{x}{2}$ For $x = 3y+2$, $y = \frac{x-2}{3}$

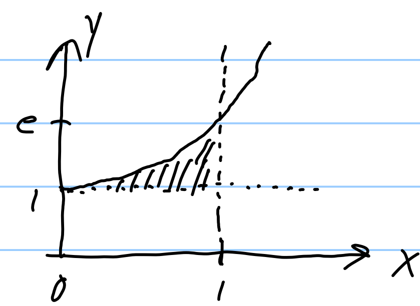
$2(0) = 0$, $2(1) = 2$ $3(0)+2 = 2$, $3(1)+2 = 5$

$$\therefore \int_0^1 \int_{2y}^{3y+2} dx\,dy = \int_0^1 \int_{2y}^{2} dx\,dy + \int_0^1 \int_2^{3y+2} dx\,dy$$

$$= \int_0^2 \int_0^{\frac{x}{2}} dy\,dx + \int_2^5 \int_{\frac{x-2}{3}}^{1} dy\,dx$$

$\therefore$ True

(d)



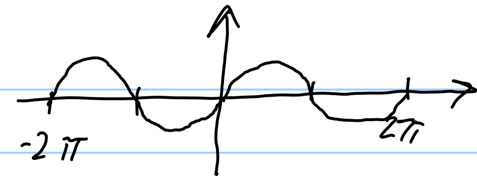
Over the interval $0 \le x \le 1$, $y = e^x$ is increasing

$e^{(0)} = 1$, $e^{(1)} = e$. $y = e^x \Leftrightarrow x = \ln y$

$$\therefore \int_0^1 \int_1^{e^x} dy\,dx = \int_{e^{(0)}}^{e^{(1)}} \int_{\ln y}^{1} dx\,dy = \int_1^e \int_{\ln y}^{1} dx\,dy$$

$\therefore$ True

7.



Over the interval $-2\pi \le u \le 2\pi$, $\sin(u)$ takes on

a max at $u = \frac{\pi}{2}$ and $u = -\frac{3}{2}\pi$: $\sin(u) = 1$

a min at $u = 3/2\,\pi$, $u = -\pi/2$: $\sin(u) = -1$

$\therefore$ $\sin(x+y)$ over $[-\pi,\pi] \times [-\pi,\pi]$ has

a max of 1 and a min of -1.

$\therefore$ e^z is an increasing function.

$\therefore$ $e^{\sin(x+y)}$ is a max at $e^1 = e$

a min at $e^{-1} = 1/e$.

Area of $D = (2\pi)(2\pi) = 4\pi^2$.

$\therefore$ $\iint_D e^{\sin(x+y)}\,dA$ has a min of $\frac{1}{e}(4\pi^2)$

and a max of $e(4\pi^2)$

$$\therefore \quad \frac{1}{e}(4\pi^2) \le \iint_D f(x,y)\,dA \le e(4\pi^2)$$

$$\therefore \quad \frac{1}{e} \le \frac{1}{4\pi^2}\iint_D f(x,y)\,dA \le e$$

8.

The min for $1+(xy)^4$ is 1.
The max for $1+(xy)^4$ is 2 $(x=1, y=1$ on $[0,1]\times[0,1])$.

Max for $\sin(x)$ on $[0,1]$ is $\sin(1)$ since $\sin(x)$ is increasing on $[0,1]$, and $1 \leq \frac{\pi}{2}$

$\therefore$ Max for $\frac{\sin(x)}{1+(xy)^4} = \frac{\sin(1)}{1} = \sin(1) \leq 1 = \sin(\frac{\pi}{2})$.

Min for $\sin(x)$ on $[0,1]$ is $0 = \sin(0)$.

$\therefore$ Min for $\frac{\sin(x)}{1+(xy)^4} = \frac{\sin(0)}{2} = 0$.

Area of $[0,1]\times[0,1] = 1$.

$$\therefore\ 0(1) \leq \iint_D f(x,y)\,dA \leq \sin(1) \leq 1$$

$$\therefore\ 0 \leq \iint_{[0,1]\times[0,1]} \frac{\sin(x)}{1+(xy)^4}\,dx\,dy \leq 1$$

Note also $\frac{\sin(x)}{2} \leq \frac{\sin(x)}{1+(xy)^4}$ on $[0,1]\times[0,1]$.

$$\therefore\ \int_0^1\int_0^1 \frac{\sin(x)}{2}\,dx\,dy \leq \int_0^1\int_0^1 \frac{\sin(x)}{1+(xy)^4}\,dx\,dy$$

$$\text{But } \int_0^1\int_0^1 \frac{\sin(x)}{2}\,dx\,dy = \int_0^1\int_0^1 \frac{\sin(x)}{2}\,dy\,dx$$

$$= \int_0^1 \left[y\frac{\sin(x)}{2} \Big|_{y=0}^{y=1} \right] dx = \int_0^1 \frac{\sin(x)}{2}\,dx$$

$$= -\frac{\cos(x)}{2}\Big|_0^1 = -\frac{\cos(1)}{2} + \frac{1}{2} = \frac{1-\cos(1)}{2}$$

And $\frac{1-\cos(1)}{2} > 0$.

$$\therefore \frac{1-\cos(1)}{2} \leq \iint_{[0,1]\times[0,1]} \frac{\sin x}{1+(xy)^4}\,dx\,dy \leq 1$$

9.

For all x, y, $1 \leq x^2+y^2+1$ $\therefore \frac{1}{x^2+y^2+1} \leq 1$

On $[-1,1]\times[-1,2]$, $x^2+y^2+1 \leq (1)^2+(2)^2+1 = 6$

$$\therefore \frac{1}{6} \leq \frac{1}{x^2+y^2+1}$$

$$\therefore \frac{1}{6} \leq \frac{1}{x^2+y^2+1} \leq 6$$

$$\therefore \frac{1}{6}(D_{Area}) \leq \iint_D \frac{1}{x^2+y^2+1}\,dA \leq 1\,(D_{Area})$$

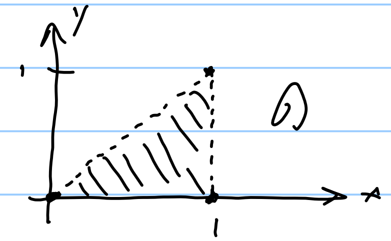
For $[-1,1]\times[-1,2]$, Area $= (2)(3) = 6$

$$\therefore \frac{1}{6}(6) \le \iint_D \frac{1}{x^2+y^2+1}\,dA \le 1(6)$$

Or,

$$1 \le \iint_D \frac{1}{x^2+y^2+1}\,dx\,dy \le 6$$

10.



Area of $D = \frac{1}{2}$

Min of $y-x+3 = 2$ $(y=0,\ x=1)$

Max of $y-x+3 = 3$ $(y=x$, for any $x\in D)$

Note $y-x+3 \ge 0$ on D.

$$\therefore 2 \le y-x+3 \le 3 \Rightarrow \frac{1}{3} \le \frac{1}{y-x+3} \le \frac{1}{2}$$

$$\frac{1}{3}(\text{Area}) = \frac{1}{3}\left(\frac{1}{2}\right) \le \iint_D \frac{dA}{y-x+3} \le \frac{1}{2}(\text{Area}) = \frac{1}{2}\left(\frac{1}{2}\right)$$

$$\therefore \frac{1}{6} \le \iint_D \frac{dA}{y-x+3} \le \frac{1}{4}$$

11.

A general formula is $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

$\therefore$ Upper half is: $\frac{z}{c} = \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$, $a, b, c > 0$.

$\therefore$ Upper half volume $= \iint_D f(x,y)\, dA$

where $f(x,y) = c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$, where D is

$-a \le x \le a$, $-b\sqrt{1 - \frac{x^2}{a^2}} \le y \le b\sqrt{1 - \frac{x^2}{a^2}}$

$$\therefore \frac{1}{2} V = \int_{-a}^{a} \int_{-b\sqrt{1 - \frac{x^2}{a^2}}}^{b\sqrt{1 - \frac{x^2}{a^2}}} c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\, dy\, dx$$

Using symmetry again, just find the volume in the positive quadrant.

$$\therefore \frac{1}{8} V = \int_{0}^{a} \int_{0}^{b\sqrt{1 - \frac{x^2}{a^2}}} c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\, dy\, dx$$

(1) First consider $\int_{0}^{b\sqrt{1 - x^2/a^2}} c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\, dy$

Let $u^2 = 1 - \frac{x^2}{a^2}$, valid since $\frac{x^2}{a^2} \le 1$.

Let $v = \frac{y}{b}$. $\therefore b\,dv = dy$

and $y = 0 \Rightarrow v = 0$, $y = bu \Rightarrow v = u$

$$\therefore \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\,dy = \int_0^u cb\sqrt{u^2 - v^2}\,dv$$

$$= cb\left[\frac{v}{2}\sqrt{u^2 - v^2} + \frac{u^2}{2}\arcsin\frac{v}{u}\right]_{v=0}^{v=u}$$

$$= cb\left[0 + \frac{u^2}{2}\arcsin(1) - (0 + 0)\right]$$

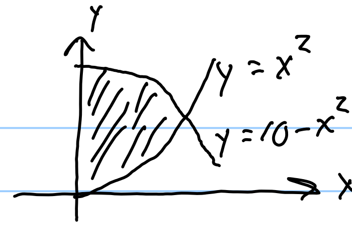
$$= cb\left[\frac{u^2}{2}\left(\frac{\pi}{2}\right)\right] = \frac{\pi}{4}cbu^2 = \frac{\pi bc}{4}\left(1 - \frac{x^2}{a^2}\right)$$

(2) $\therefore \frac{\pi bc}{4}\int_0^a \left(1 - \frac{x^2}{a^2}\right)dx = \frac{\pi bc}{4}\left[x - \frac{x^3}{3a^2}\right]_0^a$

$$= \frac{\pi bc}{4}\left[a - \frac{a^3}{3a^2}\right] = \frac{\pi bc}{4}\left(\frac{2}{3}a\right) = \frac{\pi abc}{6}$$

$\therefore \frac{1}{8}V = \frac{\pi abc}{6}$, $\underline{\underline{V = \frac{4}{3}\pi abc}}$

12.



$$x^2 = 10 - x^2 \Rightarrow x^2 = 5, \quad x = \sqrt{5} \quad \therefore \; 0 \le x \le \sqrt{5}$$

$$\therefore \int_0^{\sqrt{5}} \int_{x^2}^{10-x^2} y^2 \sqrt{x}\, dy\, dx = \int_0^{\sqrt{5}} \left[\frac{y^3}{3} \sqrt{x} \right]_{y=x^2}^{y=10-x^2} dx$$

$$= \int_0^{\sqrt{5}} \frac{(10-x^2)^3 x^{\frac{1}{2}}}{3} - \frac{x^6 x^{\frac{1}{2}}}{3} \, dx$$

$$= \int_0^{\sqrt{5}} \frac{(10-x^2)^3 x^{\frac{1}{2}}}{3} - \int_0^{\sqrt{5}} \frac{1}{3} x^{\frac{13}{2}} dx$$

(a) $$-\int_0^{\sqrt{5}} \frac{1}{3} x^{\frac{13}{2}} dx = -\frac{1}{3}\left(\frac{2}{15}\right) x^{\frac{15}{2}} \Big|_0^{\sqrt{5}} = \underline{-\frac{2}{45} 5^{\frac{15}{4}}}$$

(b) $$\int_0^{\sqrt{5}} \frac{(10-x^2)^3 x^{\frac{1}{2}}}{3} dx = \int_0^{\sqrt{5}} (1000 - 300x^2 + 30x^4 - x^6) x^{\frac{1}{2}} dx$$

$$= \int_0^{\sqrt{5}} \frac{1000x^{\frac{1}{2}} - 300x^{\frac{5}{2}} + 30x^{\frac{9}{2}} - x^{\frac{13}{2}}}{3} dx$$

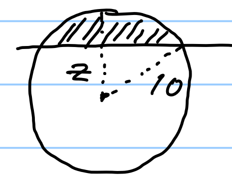
$$= \frac{1000}{3}\left(\frac{2}{3}\right) x^{\frac{3}{2}} - 100\left(\frac{2}{7}\right) x^{\frac{7}{2}} + 10\left(\frac{2}{11}\right) x^{\frac{11}{2}} - \frac{1}{3}\left(\frac{2}{15}\right) x^{\frac{15}{2}} \Big|_0^{\sqrt{5}}$$

$$= \frac{2000}{9} 5^{\frac{3}{4}} - \frac{200}{7} 5^{\frac{7}{4}} + \frac{20}{11} 5^{\frac{11}{4}} - \frac{2}{45} 5^{\frac{15}{4}}$$

$\therefore$ (a) + (b) =

$$\frac{2000}{9} 5^{3/4} - \frac{200}{7} 5^{7/4} + \frac{20}{11} 5^{\frac{11}{4}} - \frac{4}{45} 5^{\frac{15}{4}}$$

13.



This is a sphere of radius $\sqrt{10}$. $z \geq 2$ means the volume of the section above the plane $z=2$.

Using Cavalieri's principle, take a slice, perpendicular to z-axis. The disc slice has a volume of

$\pi r^2 \, dz$, since the slice is a flat cylinder.

Here, $r^2 = x^2 + y^2 = 10 - z^2$

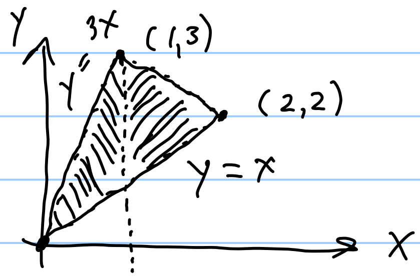
$$\therefore V = \int_2^{\sqrt{10}} \pi (10 - z^2) \, dz = \pi 10 z - \pi \frac{z^3}{3} \Big|_2^{\sqrt{10}}$$

$$= \pi 10 \sqrt{10} - \pi \frac{10^{3/2}}{3} - \left(20\pi - \frac{8}{3}\pi\right)$$

$$= \pi\left(\frac{30\sqrt{10} - 10\sqrt{10}}{3}\right) - \pi\left(\frac{60-8}{3}\right)$$

$$= \frac{\pi\left(20\sqrt{10} - 52\right)}{3}$$

14.



Break the triangle into 2 parts,

from $0 \le x \le 1$ and $1 \le x \le 2$

The left section is bounded by $y=x$ and $y=3x$

$$\therefore \int_0^1 \int_x^{3x} dy\,dx$$

The right section is bounded by $y=x$ and

$$y = \frac{3-2}{1-2}x + b,\ \text{or}\ y = -x+b,\ (3) = -(1)+b,\ b = 4$$

$$\therefore y = -x+4 \quad \therefore \int_1^2 \int_x^{-x+4} dy\,dx$$

$$\therefore \int_0^1 \int_x^{3x} e^{x-y}\,dy\,dx + \int_1^2 \int_x^{-x+4} e^{x-y}\,dy\,dx$$

(a) $$\int_0^1\int_x^{3x} e^{x-y}\,dy\,dx = \int_0^1 \left[-e^{x-y}\Big|_{y=x}^{y=3x}\right]dx$$

$$= \int_0^1 \left(-e^{-2x} + e^{0}\right)dx = \frac{1}{2}e^{-2x} + x\Big|_0^1$$

$$= \frac{1}{2}e^{-2} + 1 - \left(\frac{1}{2} + 0\right) = \underline{\frac{1}{2}e^{-2} + \frac{1}{2}}$$

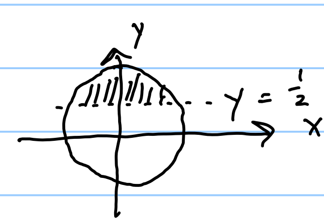
(b) $$\int_1^2\int_x^{-x+4} e^{x-y}\,dy\,dx = \int_1^2 \left[-e^{x-y}\Big|_{y=x}^{y=-x+4}\right]dx$$

$$= \int_1^2 \left(-e^{2x-4} + 1\right)dx = -\frac{1}{2}e^{2x-4} + x\Big|_1^2$$

$$= -\frac{1}{2} + 2 - \left(-\frac{1}{2}e^{-2} + 1\right) = \underline{\frac{1}{2}e^{-2} + \frac{1}{2}}$$

$$\therefore (a) + (b) = \underline{e^{-2} + 1}$$

15.



The x-simple region is $-\sqrt{1-y^2} \le x \le \sqrt{1-y^2}$
for $\frac{1}{2} \le y \le 1$

$$\therefore \iint_D f(x,y)\,dA = \int_{\frac{1}{2}}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} y^3 (x^2+y^2)^{-3/2}\,dx\,dy$$

There looks to be some cancelling of y terms when the limits $x = \pm\sqrt{1-y^2}$ are inserted for x^2+y^2, so proceed without changing order of integration limits.

From a table of integrals: $\displaystyle\int \frac{du}{(a^2+u^2)^{3/2}} = \frac{u}{a^2\sqrt{a^2+u^2}} + C$

$$\therefore \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{y^3}{(y^2+x^2)^{3/2}}\,dx = \left.\frac{y^3 x}{y^2\sqrt{y^2+x^2}}\right|_{x=-\sqrt{1-y^2}}^{x=\sqrt{1-y^2}}$$

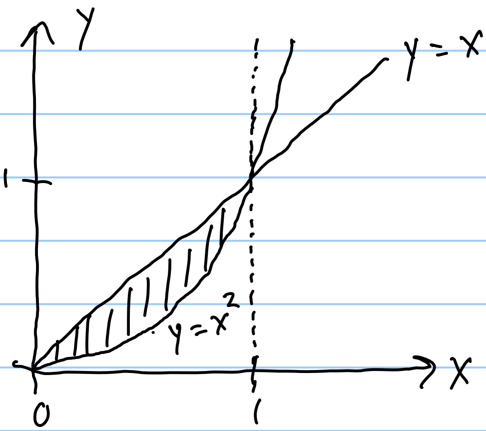
$$= \frac{y^3\sqrt{1-y^2}}{y^2\sqrt{1}} - \left(\frac{-y^3\sqrt{1-y^2}}{y^2\sqrt{1}}\right) = 2y\sqrt{1-y^2}$$

$$\therefore \int_{\frac{1}{2}}^{1} 2y\sqrt{1-y^2}\,dy = \left.-\frac{2}{3}(1-y^2)^{3/2}\right|_{\frac{1}{2}}^{1}$$

$$= -\frac{2}{3}(0) - \left(-\frac{2}{3}\left(1-\left(\frac{1}{2}\right)^2\right)^{3/2}\right)$$

$$= \frac{2}{3}\left(\frac{3}{4}\right)^{3/2} = \frac{2}{3}\left(\frac{\sqrt{3}}{2}\right)^3 = \frac{2}{3}\left(\frac{3\sqrt{3}}{8}\right) = \underline{\frac{\sqrt{3}}{4}}$$

16.



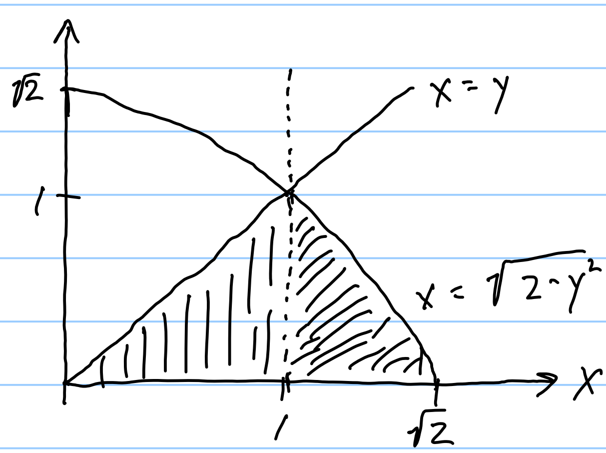
Change order: $y = x \Rightarrow x = y$

$y = x^2 \Rightarrow x = \pm\sqrt{y}$

use $x = \sqrt{y}$ since $0 \le x \le 1$.

$$\therefore \int_0^1 \int_y^{\sqrt{y}} f(x,y)\, dy\, dx$$

17.



Change order: $x = y \Rightarrow y = x$

$x = \sqrt{2-y^2} \Rightarrow$

$y = \pm\sqrt{2-x^2}$

use $y = \sqrt{2-x^2}$

since $0 \le y \le 1$

Over the interval, $0 \le x \le \sqrt{2}$,

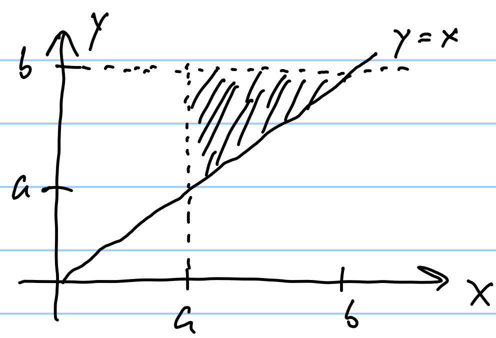
There is no simple $y = f(x)$

$\therefore$ Break up into two adjacent y-simple regions,

using $y = x$ as one region, $y = \sqrt{2-x^2}$ the other.

$$\therefore \int_0^1 \int_0^x f(x,y)\,dy\,dx + \int_1^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} f(x,y)\,dy\,dx$$

18.



Using $c\int_a^b f = \int_a^b cf$,

$$\iint_R f(x)f(y)\,dy\,dx = \int_a^b \int_a^b f(x)\,f(y)\,dy\,dx =$$

$$\int_a^b \left[\int_a^b f(x)f(y)\,dy\right] dx = \int_a^b f(x)\left[\int_a^b f(y)\,dy\right] dx$$

But $\int_a^b f(y)\,dy$ is a number (a constant)

$$= \left[\int_a^b f(y)\,dy\right] \int_a^b f(x)\,dx = \left[\int_a^b f\right]^2$$

$$\therefore \left[\int_a^b f(x)\,dx\right]^2 = \int_a^b f(x)\left[\int_a^b f(y)\,dy\right] dx$$

$$= \int_a^b f(x) \left[\int_a^x f(y)\,dy + \int_x^b f(y)\,dy \right] dx$$

as $a \leq x \leq b$

$$= \int_a^b f(x) \left[\int_a^x f(y)\,dy \right] dx + \int_a^b f(x) \left[\int_x^b f(y)\,dy \right] dx$$

$$= \int_a^b \int_a^x f(x) f(y)\,dy\,dx + \int_a^b \int_x^b f(x) f(y)\,dy\,dx$$

$$= \int_a^b \int_y^b f(x) f(y)\,dy\,dx + \int_a^b \int_x^b f(x) f(y)\,dy\,dx \quad [2]$$

from changing order of integration

But viewed as "dummy variables",

$$\int_a^b \int_y^b f(m) f(n)\,dn\,dm = \int_a^b \int_z^b f(m) f(n)\,dn\,dm$$

$\therefore$ in [2], $$\int_a^b \int_y^b f(x) f(y)\,dy\,dx = \int_a^b \int_x^b f(x) f(y)\,dy\,dx$$

and so, $$\int_a^b \int_y^b f(x) f(y)\,dy\,dx + \int_a^b \int_x^b f(x) f(y)\,dy\,dx = 2 \int_a \int_x f(x) f(y)\,dy\,dx$$

$$\therefore \left[\int_a^b f(x)\,dx \right]^2 = \underline{2 \int_a^b \int_x^b f(x) f(y)\,dy\,dx}$$

19.

On the left, the "x" variable should be viewed as two separate "dummy variables".

$$\therefore \int_a^x \int_c^d f(x,y,z)\,dz\,dy = \int_a^u \int_c^d f(v,y,z)\,dz\,dy$$

$$\therefore \text{ Let } F(u,v) = \int_a^u \int_c^d f(v,y,z)\,dz\,dy$$

By the chain rule, $\dfrac{\partial F(u,v)}{\partial u} = \dfrac{\partial F}{\partial u} + \dfrac{\partial F}{\partial v} \cdot \dfrac{\partial v}{\partial u}$

When $v(u) = u$, $\dfrac{\partial v}{\partial u} = 1$.

Letting $u = x$, $v(x) = x$, $\dfrac{\partial F(x,x)}{\partial x} = \dfrac{\partial F}{\partial x} + \dfrac{\partial F}{\partial v} \cdot \dfrac{\partial v}{\partial x}$

$$= \frac{\partial F}{\partial x} + \frac{\partial F}{\partial v} \cdot (1) = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial v}$$

and $F(x,x) = \int_a^x \int_c^d f(x,y,z)\,dz\,dy$

$$\therefore \frac{d}{dx} F(x,x) = \frac{d}{dx} \int_a^x \int_c^d f(x,y,z)\,dz\,dy$$

$$= \frac{\partial F}{\partial(\text{1st variable})} + \frac{\partial F}{\partial(\text{2nd variable})} \qquad [0]$$

By Fundamental Theorem of Calculus,

$$\frac{d}{dx} \int_a^x g(t)\,dt = g(x)$$

Now view $\displaystyle\int_a^x \left[\int_c^d f(v,y,z)\,dz \right] dy$ as

$\displaystyle\int_a^x g(y)\,dy$, where $\displaystyle g(y) = \int_c^d f(v,y,z)\,dz$

$$\therefore \frac{d}{dx} \int_a^x g(y)\,dy = g(x) = \int_c^d f(v,x,z)\,dz$$

Letting $v(x) = x$, $\displaystyle\frac{d}{dx} \int_a^x g(y)\,dy = \int_c^d f(x,x,z)\,dz$

$$\therefore \frac{\partial F}{\partial(\text{1st variable})} = \frac{\partial F}{\partial x} = \int_c^d f(x,x,z)\,dz \qquad [1]$$

$$\frac{\partial F}{\partial(\text{2nd variable})} = \frac{\partial F}{\partial v} = \frac{\partial}{\partial v}\int_a^x \int_c^d f(v,y,z)\,dz\,dy$$

$$= \int_a^x \int_c^d \frac{\partial}{\partial v} f(v,y,z)\,dz\,dy$$

$$= \int_a^x \int_c^d f_x(x,y,z)\,dz\,dy \qquad [2]$$

switching back to $v=x$, and using "differentiation under the integral".

[0], [1], [2] mean,

$$\frac{d}{dx}\int_a^x \int_c^d f(x,y,z)\,dz\,dy = \int_c^d f(x,x,z)\,dz + \int_a^x \int_c^d f_x(x,y,z)\,dz\,dy$$

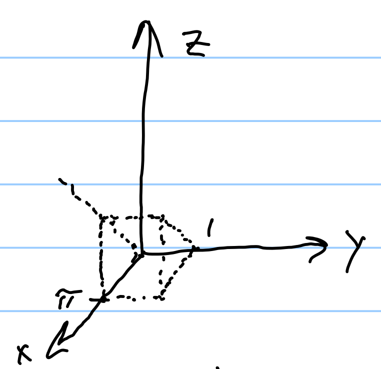
5.5 The Triple Integral

Note Title 11/6/2016

1.

(b) - (i) (a) - (ii) (c) - (iii) (d) - (iv)

2.



z has "nonconstant" bounds. ∴ Make it "inside".

$$\therefore \int_0^1 \int_0^{\pi} \int_0^x \sin x \, dz \, dx \, dy$$

$$\int_0^x \sin x \, dz = z \sin x \Big|_0^x = x \sin x$$

$$\int_0^{\pi} x\sin x\,dx = -x\cos x\Big|_0^{\pi} + \int_0^{\pi}\cos x\,dx$$

$$= \pi + \sin x\Big|_0^{\pi} = \pi$$

$$\int_0^1 \pi\,dy = \pi \qquad \therefore \iiint_W \sin x\,dx\,dy\,dz = \underline{\pi}$$

3.

$$\iiint_B x^2\,dx\,dy\,dz = \int_0^1\int_0^1\int_0^1 x^2\,dy\,dz\,dx$$

$$= \int_0^1\int_0^1 x^2\,dz\,dx = \int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_0^1 = \underline{\frac{1}{3}}$$

4.

$$= \int_0^1\int_0^1\int_0^1 ye^{-xy}\,dz\,dx\,dy = \int_0^1\int_0^1 ye^{-xy}\,dx\,dy$$

$$= \int_0^1 \left[-e^{-xy}\right]_{x=0}^{x=1} dy = \int_0^1 (-e^{-y}+1)\,dy =$$

$$e^{-y} + y \Big|_0^1 = e^{-1} + 1 - (1+0) = \underline{\frac{1}{e}}$$

5.

$$= \int_{-1}^{1}\int_{0}^{1}\int_{0}^{2} (2x+3y+z)\,dx\,dz\,dy$$

$$= \int_{-1}^{1}\int_{0}^{1} (x^2+3xy+xz)\Big|_0^2\,dz\,dy$$

$$= \int_{-1}^{1}\int_{0}^{1} (4+6y+2z)\,dz\,dy = \int_{-1}^{1} (4z+6yz+z^2)\Big|_0^1\,dy$$

$$= \int_{-1}^{1} (5+6y)\,dy = 5y+3y^2\Big|_{-1}^{1} = (5+3)-(-5+3)$$

$$= \underline{10}$$

6.

$$= \int_0^1\int_0^1\int_0^1 z e^{x+y}\,dz\,dx\,dy = \int_0^1\int_0^1 \frac{1}{2}e^{x+y}\,dx\,dy$$

$$= \int_0^1 \left[\frac{1}{2} e^{x+y} \Big|_{x=0}^{x=1} \right] dy = \int_0^1 \frac{1}{2} e^{1+y} - \frac{1}{2} e^{y} \, dy$$

$$= \frac{1}{2} e^{1+y} - \frac{1}{2} e^{y} \Big|_0^1 = \frac{1}{2} e^2 - \frac{1}{2} e - \left(\frac{1}{2} e - \frac{1}{2}\right)$$

$$= \frac{1}{2} e^2 - e + \frac{1}{2}$$

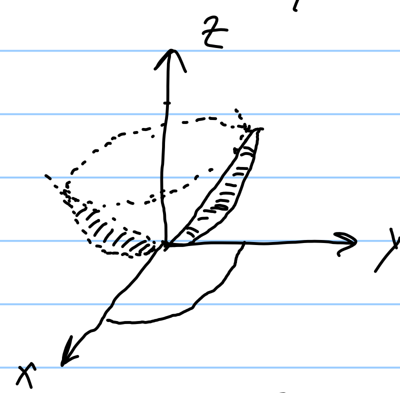
7.

Assuming the region is "underneath" the cone and "above" the paraboloid, they intersect at $x^2+y^2=1$

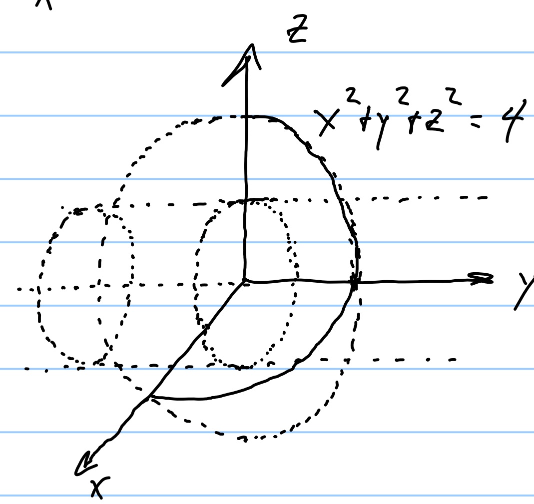
$$x^2+y^2 \le z \le \sqrt{x^2+y^2},$$

$$-\sqrt{1-x^2} \le y \le \sqrt{1-x^2},$$

$$-1 \le x \le 1$$



8.



$$\frac{x^2}{\left(\frac{1}{\sqrt{2}}\right)^2} + z^2 = 1$$

The bounds for y are only limited by the sphere.
The sphere has radius $= \sqrt{4} = 2$

$\therefore -\sqrt{4-x^2-z^2} \le y \le \sqrt{4-x^2-z^2}$

The x and z bounds are limited by the cylinder

$\therefore -\sqrt{1-2x^2} \le z \le \sqrt{1-2x^2}$, $-\frac{\sqrt{2}}{2} \le x \le \frac{\sqrt{2}}{2}$

or, $-\sqrt{(1-z^2)/2} \le x \le \sqrt{(1-z^2)/2}$, $-1 \le z \le 1$

9.

$0 \le z \le \sqrt{1-x^2-y^2}$, and

$-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$, $-1 \le x \le 1$

or

$-\sqrt{1-y^2} \le x \le \sqrt{1-y^2}$, $-1 \le y \le 1$

10.

$x=0, y=0, z=0 \Rightarrow$ one of the octants

$x+y=4 \Rightarrow$ intersect x-axis at $(4,0)$, y-axis at $(0,4)$
and so "volume" is in 1st octant

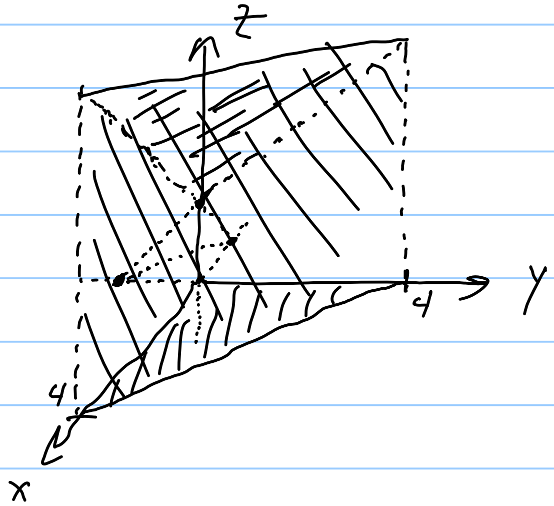
$x = z-y-1 \Leftrightarrow x+y-z=-1$ intersects axes at
$(-1,0,0)$, $(0,-1,0)$, $(0,0,1)$
and can be expressed as $z = x+y+1$

$\therefore 0 \le z \le x+y+1$

From $x+y=4$: $0 \le y \le 4-x$

The extremes of x are from $x+y=4$, so $0 \le x \le 4$

$\therefore\ 0 \le z \le x+y+1,\ 0 \le y \le 4-x,\ 0 \le x \le 4$



11.

$z = x^2+y^2$ is an "upward" paraboloid,
$z = 10-x^2-2y^2$ is a "downward paraboloid.

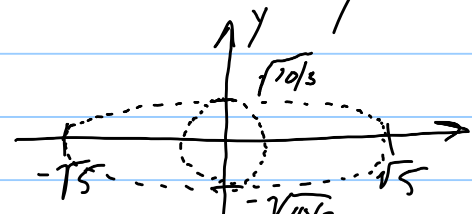
They intersect at $x^2+y^2 = 10-x^2-2y^2$, or

$2x^2+3y^2 = 10$, or $\dfrac{x^2}{(\sqrt{5})^2} + \dfrac{y^2}{\left(\sqrt{\frac{10}{3}}\right)^2} = 1$, an

ellipse with borders $(\pm\sqrt{5}, 0), (0, \pm\sqrt{\frac{10}{3}})$

At $(\pm\sqrt{5}, 0)$, $z=5$, at $(0, \pm\sqrt{\frac{10}{3}})$, $z = \frac{10}{3}$

$\therefore$ Intersection is not in a plane, but the intersection marks the extreme boundary values for x and y.



Notice the $x^2+y^2=z$ circle is limited first by the $y=\pm\sqrt{10/3}$ coordinate before the $x=\pm\sqrt{5}$ coordinate.

$\therefore$ Use $-\sqrt{\frac{10}{3}} \le y \le \sqrt{10/3}$

$$-\sqrt{(10-3y^2)/2} \le x \le \sqrt{(10-3y^2)/2}$$

$$x^2+y^2 \le z \le 10-x^2-2y^2$$

$$\therefore \int_{-\sqrt{\frac{10}{3}}}^{\sqrt{10/3}} \int_{-\sqrt{(10-3y^2)/2}}^{\sqrt{(10-3y^2)/2}} \int_{x^2+y^2}^{10-x^2-2y^2} dz\,dx\,dy$$

$$= \int_{-\sqrt{\frac{10}{3}}}^{\sqrt{\frac{10}{3}}} \int_{-\sqrt{(10-3y^2)/2}}^{\sqrt{(10-3y^2)/2}} (10-2x^2-3y^2)\,dx\,dy$$

$$\int_{-\sqrt{(10-3y^2)/2}}^{\sqrt{(10-3y^2)/2}} (10-2x^2-3y^2)\,dx = 10x-\tfrac{2}{3}x^3-3y^2x \Big|_{x=-\sqrt{(10-3y^2)/2}}^{x=\sqrt{(10-3y^2)/2}}$$

$$= 20\left(\frac{10-3y^2}{2}\right)^{\frac{1}{2}} - \frac{4}{3}\left(\frac{10-3y^2}{2}\right)^{3/2} - 6y^2\left(\frac{10-3y^2}{2}\right)^{\frac{1}{2}}$$

$$= 10\sqrt{2}\sqrt{10-3y^2} - \frac{\sqrt{2}}{3}(10-3y^2)^{3/2} - 3\sqrt{2}\,y^2\sqrt{10-3y^2}$$

Taking each term one at a time, and using

$\sqrt{10-3y^2} = \sqrt{3}\sqrt{\frac{10}{3}-y^2}$, using a table of integrals:

(1)
$$10\sqrt{6}\int_{-\sqrt{\frac{10}{3}}}^{\sqrt{\frac{10}{3}}} \sqrt{\frac{10}{3}-y^2}\,dy =$$

$$10\sqrt{6}\left[\frac{y}{2}\sqrt{\frac{10}{3}-y^2} + \frac{10}{6}\operatorname{Arcsin}\frac{y}{\sqrt{\frac{10}{3}}}\right]_{y=-\sqrt{\frac{10}{3}}}^{y=\sqrt{\frac{10}{3}}} =$$

$$10\sqrt{6}\left[0 + \frac{5}{3}\operatorname{Arcsin}(1) - \left(0 - \frac{5}{3}\operatorname{Arcsin}(1)\right)\right]$$

$$= 10\sqrt{6}\left(\frac{5}{3}\right)(\pi) = \frac{50\sqrt{6}}{3}\pi$$ [1]

(2)
$$-\frac{\sqrt{2}}{3}\int_{-\sqrt{\frac{10}{3}}}^{\sqrt{\frac{10}{3}}} (10-3y^2)^{3/2}\,dy = -\sqrt{6}\int_{-\sqrt{\frac{10}{3}}}^{\sqrt{\frac{10}{3}}} \left(\frac{10}{3}-y^2\right)^{3/2} dy$$

$$= -\sqrt{6}\left[\frac{y}{8}\left(2y^2 - \frac{50}{3}\right)\sqrt{\frac{10}{3} - y^2} + \frac{3}{8}\left(\frac{100}{9}\right)\mathrm{Arcsin}\left(\frac{y}{\sqrt{\frac{10}{3}}}\right)\right]_{-\sqrt{\frac{10}{3}}}^{\sqrt{\frac{10}{3}}}$$

$$= -\sqrt{6}\left[0 + \frac{25}{2(3)}\mathrm{Arcsin}(1) - \left(0 - \frac{25}{2(3)}\mathrm{Arcsin}(1)\right)\right]$$

$$= -\sqrt{6}\left(\frac{25}{3}\cdot\frac{\pi}{2}\right) = -\frac{25\sqrt{6}}{6}\pi \qquad [2]$$

(3)

$$-3\sqrt{2}\int_{-\sqrt{\frac{10}{3}}}^{\sqrt{\frac{10}{3}}} y^2\sqrt{10 - 3y^2}\,dy = -3\sqrt{6}\int_{-\sqrt{10/3}}^{\sqrt{10/3}} y^2\sqrt{\frac{10}{3} - y^2}\,dy$$

$$-3\sqrt{6}\left[-\frac{y}{4}\left(\frac{10}{3} - y^2\right)^{3/2} + \frac{y^2}{8}y\sqrt{\frac{10}{3} - y^2} + \left(\frac{10}{3}\right)^2\frac{1}{8}\mathrm{Arcsin}\left(\frac{y}{\sqrt{\frac{10}{3}}}\right)\right]_{-\sqrt{\frac{10}{3}}}^{\sqrt{\frac{10}{3}}}$$

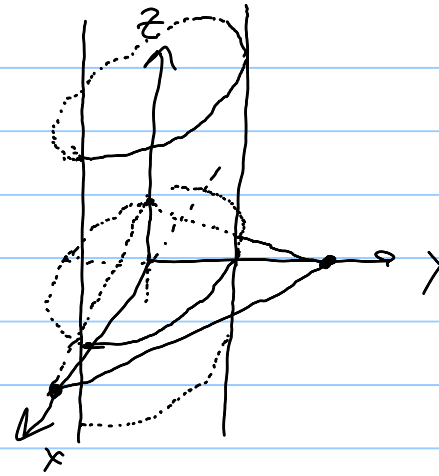
$$= -3\sqrt{6}\left[0 + 0 + \frac{100}{9(8)}\mathrm{Arcsin}(1) - \left(0 + 0 - \frac{100}{9(8)}\mathrm{Arcsin}(1)\right)\right]$$

$$= -3\sqrt{6}\left(\frac{100}{9(8)}\right)(\pi) = -\frac{25\sqrt{6}}{6}\pi \qquad [3]$$

$$[1] + [2] + [3] = \frac{50\sqrt{6}}{3}\pi + \left(-\frac{25\sqrt{6}}{6}\pi\right) + \left(-\frac{25\sqrt{6}}{6}\right)\pi$$

$$= \pi\left(\frac{100\sqrt{6}}{6} - \frac{50\sqrt{6}}{6}\right) = \underline{\frac{50\sqrt{6}}{6}\pi}$$

12.



$x^2+2y^2=2 \Leftrightarrow \frac{x^2}{(\sqrt{2})^2}+\frac{y^2}{1}=1$,

a cylinder parallel to the z-axis, in the shape of an ellipse with margins $(\pm\sqrt{2},0)$, $(0,\pm1)$. $x=\pm\sqrt{2-2y^2}$

The "top" of the volume is a plane $z=1-\frac{x}{2}-\frac{y}{2}$, with axis intercepts at $(2,0,0)$, $(0,2,0)$, $(0,0,1)$.

$\therefore$ The plane serves as the "top" to the solid, and $z=0$ serves as the "bottom." Lateral sides are bounded by the ellipse.

$\therefore 0 \le z \le 1-\frac{x}{2}-\frac{y}{2}$, $-\sqrt{2-2y^2} \le x \le \sqrt{2-2y^2}$, $-1 \le y \le 1$

$$\therefore V=\int_{-1}^{1}\int_{-\sqrt{2-2y^2}}^{\sqrt{2-2y^2}}\int_{0}^{1-\frac{x}{2}-\frac{y}{2}} dz\,dx\,dy$$

$$=\int_{-1}^{1}\int_{-\sqrt{2-2y^2}}^{\sqrt{2-2y^2}}\left(1-\frac{x}{2}-\frac{y}{2}\right)dx\,dy$$

$$\int_{-1}^{1}\left[\left(x-\frac{x^2}{4}-\frac{xy}{2}\right)\Big|_{x=-\sqrt{2-2y^2}}^{x=\sqrt{2-2y^2}}\right]dx$$

$$= \int_{-1}^{1} \left[\sqrt{2-2y^2} - \frac{(2-2y^2)}{4} - \frac{y}{2}\sqrt{2-2y^2} - \left(-\sqrt{2-2y^2} - \frac{(2-2y^2)}{4} + \frac{y}{2}\sqrt{2-2y^2} \right) \right] dx$$

$$= \int_{-1}^{1} \left(2\sqrt{2-2y^2} - y\sqrt{2-2y^2} \right) dy$$

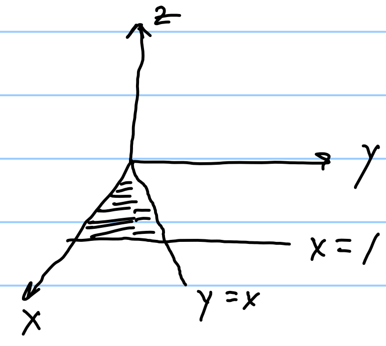
$$= 2\sqrt{2} \int_{-1}^{1} \sqrt{1-y^2}\, dy - \sqrt{2} \int_{-1}^{1} y\sqrt{1-y^2}\, dy$$

$$= 2\sqrt{2} \left[\frac{y}{2}\sqrt{1-y^2} + \frac{1}{2}\text{Arcsin}(y) \right]_{-1}^{1} + \frac{\sqrt{2}}{2}\left(\frac{2}{3}\right)\left(1-y^2\right)^{3/2} \Big|_{-1}^{1}$$

$$= 2\sqrt{2} \left\{ 0 + \frac{1}{2}\left(\frac{\pi}{2}\right) - \left(0 - \frac{1}{2}\left(\frac{\pi}{2}\right)\right) \right\} + 0$$

$$= 2\sqrt{2}\left(\frac{\pi}{2}\right) = \underline{\sqrt{2}\,\pi}$$

13.



$z = -x-y$ is a plane through $(0,0,0)$, and acts as a "floor" to the solid, $z=0$ as a "roof", as $z<0$ for $x, y > 0$.

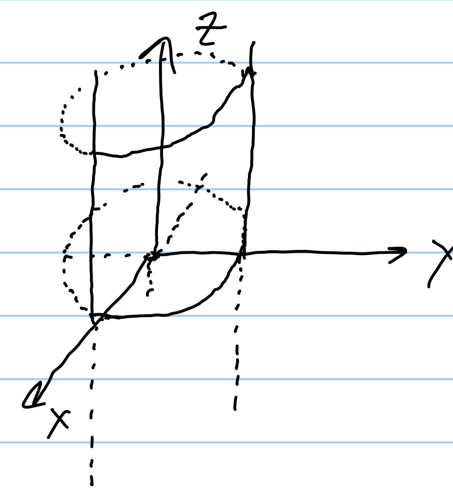
$\therefore\ 0 \le x \le 1,\ 0 \le y \le x,\ -x-y \le z \le 0$

$$\therefore V = \int_0^1 \int_0^x \int_{-x-y}^{0} dz\, dy\, dx$$

$$= \int_0^1 \int_0^x (x+y)\, dy\, dx = \int_0^1 \left[xy + \frac{y^2}{2} \Big|_{y=0}^{y=x} \right] dx$$

$$= \int_0^1 \frac{3}{2} x^2\, dx = \frac{x^3}{2} \Big|_0^1 = \underline{\underline{\frac{1}{2}}}$$

14.



$$-a \le x \le a$$

$$-\sqrt{a^2-x^2} \le y \le \sqrt{a^2-x^2}$$

$$-\sqrt{a^2-x^2} \le z \le \sqrt{a^2-x^2}$$

$$\therefore V = \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dz\, dy\, dx$$

$$= \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} 2\sqrt{a^2-x^2}\, dy\, dx$$

$$= \int_{-a}^{a} \left[2y\sqrt{a^2-x^2} \Big|_{y=-\sqrt{a^2-x^2}}^{y=\sqrt{a^2-x^2}} \right] dx$$

$$= \int_{-a}^{a} 4(a^2-x^2)\,dx = 4a^2x - \frac{4}{3}x^3 \Big|_{x=-a}^{x=a}$$

$$= 4a^3 - \frac{4}{3}a^3 - \left(-4a^3 + \frac{4}{3}a^3\right) = 8a^3 - \frac{8}{3}a^3$$

$$= \underline{\frac{16}{3}a^3}$$

15.

$$\int_2^3 \cos[\pi(x+y+z)]\,dx = \frac{1}{\pi}\sin[\pi(x+y+z)] \Big|_{x=2}^{x=3}$$

$$= \frac{1}{\pi}\left[\sin(3\pi + \pi y + \pi z) - \sin(2\pi + \pi y + \pi z)\right]$$

$$= \frac{1}{\pi}\left[-\sin(\pi y + \pi z) - \sin(\pi y + \pi z)\right]$$

$$= -\frac{2}{\pi}\sin(\pi y + \pi z)$$

$$\int_1^2 -\frac{2}{\pi}\sin(\pi y + \pi z)\,dy = \frac{2}{\pi^2}\cos(\pi y + \pi z)\Big|_{y=1}^{y=2}$$

$$= \frac{2}{\pi^2}\left[\cos(2\pi + \pi z) - \cos(\pi + \pi z)\right]$$

$$= \frac{2}{\pi^2}\left[\cos(\pi z) + \cos(\pi z)\right]$$

$$= \frac{4}{\pi^2}\cos(\pi z)$$

$$\int_0^1 \frac{4}{\pi^2}\cos(\pi z)\,dz = \frac{4}{\pi^3}\sin(\pi z)\Big|_{z=0}^{z=1}$$

$$= \frac{4}{\pi^3}\left[\sin(\pi) - \sin(0)\right] = \underline{\underline{0}}$$

16.

$$\int_0^y (y + xz)\,dz = yz + x\frac{z^2}{2}\Big|_{z=0}^{z=y} = y^2 + x\frac{y^2}{2}$$

$$\int_0^x \left(y^2 + x\frac{y^2}{2}\right)dy = \frac{y^3}{3} + \frac{xy^3}{6}\Big|_{y=0}^{y=x} = \frac{x^3}{3} + \frac{x^4}{6}$$

$$\int_0^1 \frac{x^3}{3} + \frac{x^4}{6}\,dx = \frac{x^4}{12} + \frac{x^5}{30}\Big|_0^1 = \frac{1}{12} + \frac{1}{30}$$

$$= \frac{30}{360} + \frac{12}{360} = \frac{42}{360} = \frac{7\cdot 3\cdot 2}{3^2\cdot 2^3\cdot 5} = \frac{7}{3\cdot 2^2\cdot 5} = \underline{\underline{\frac{7}{60}}}$$

17.

The plane $x+y+z=a$ intersects axes at $(a,0,0)$, $(0,a,0)$, $(0,0,a)$. At $z=0$, $x+y=a$, so $y=a-x$.

Top of W is $z=a-x-y$, bottom is $z=0$.

$\therefore$ W: $0 \le x \le a$
$0 \le y \le a-x$
$0 \le z \le a-x-y$

$$\therefore \int_0^a \int_0^{a-x} \int_0^{a-x-y} (x^2+y^2+z^2)\, dz\, dy\, dx$$

$$= \int_0^a \int_0^{a-x} \left(x^2 z + y^2 z + \frac{z^3}{3}\right)\Bigg|_{z=0}^{z=a-x-y} dy\, dx$$

$$= \int_0^a \int_0^{a-x} x^2(a-x-y) + y^2(a-x-y) + \frac{(a-x-y)^3}{3}\, dy\, dz$$

$$= \int_0^a \int_0^{a-x} ax^2 - x^3 - x^2 y + (a-x)y^2 - y^3 + \frac{(a-x-y)^3}{3}\, dy\, dz$$

$$= \int_0^a \left[ax^2 y - x^3 y - \frac{x^2 y^2}{2} + (a-x)\frac{y^3}{3} - \frac{y^4}{4} - \frac{(a-x-y)^4}{12}\right]\Bigg|_{y=0}^{y=a-x}$$

$$= \int_0^a ax^2(a-x) - x^3(a-x) - x\frac{^2(a-x)^2}{2} + \underbrace{\frac{(a-x)^4}{3} - \frac{(a-x)^4}{4} + \frac{(a-x)^4}{12}}\, dx$$

$$= \int_0^a a^2x^2 - 2ax^3 + x^4 - \frac{x^4 - 2ax^3 + a^2x^2}{2} + \frac{(a-x)^4}{6}\, dx$$

$$= \int_0^a \frac{1}{2}a^2x^2 - ax^3 + \frac{1}{2}x^4 + \frac{(a-x)^4}{6}\, dx$$

$$= \frac{a^2x^3}{6} - \frac{ax^4}{4} + \frac{x^5}{10} - \frac{(a-x)^5}{30} \Bigg|_{x=0}^{x=a}$$

$$= \frac{a^5}{6} - \frac{a^5}{4} + \frac{a^5}{10} + \frac{a^5}{30} = \frac{10a^5 - 15a^5 + 6a^5 + 2a^5}{60}$$

$$= \underline{\underline{\frac{a^5}{20}}}$$

18.

W is described by $0 \le x \le 1$, $0 \le y \le \sqrt{1-x^2}$, $0 \le z \le 1$

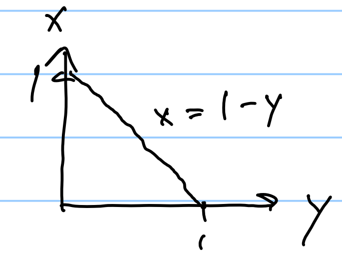
$$\therefore \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^1 z\, dz\, dy\, dx = \int_0^1 \int_0^{\sqrt{1-x^2}} \frac{1}{2}\, dy\, dx$$

$$= \int_0^1 \frac{y}{2}\Big|_{y=0}^{y=\sqrt{1-x^2}} dx = \int_0^1 \frac{\sqrt{1-x^2}}{2}\, dx$$

$$= \frac{1}{2}\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\arcsin(x)\right]_{x=0}^{x=1}$$

$$= \frac{1}{4}\arcsin(1) = \underline{\frac{\pi}{8}}$$

19.



$W: \ 0 \le z \le \pi, \ 0 \le y \le 1, \ 0 \le x \le 1-y$

$$\therefore \int_0^1 \int_0^{1-y} \int_0^{\pi} x^2 \cos z \, dz\, dx\, dy$$

$$= \int_0^1 \int_0^{1-y} \left(x^2 \sin z \Big|_{z=0}^{z=\pi} \right) dx\, dy = \int_0^1 \int_0^{1-y} 0 \, dx\, dy$$

$$= \underline{0}$$

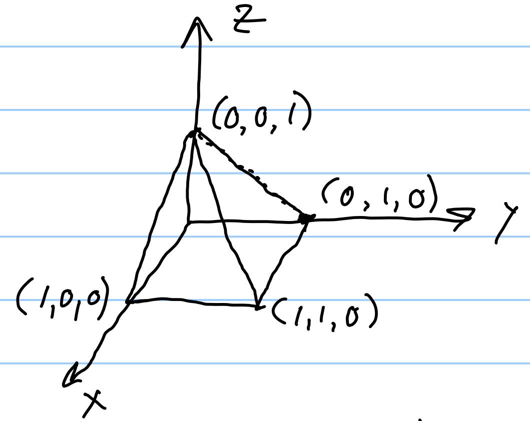
20.

$$\int_0^2 \int_0^x \left(z \Big|_{z=0}^{z=x+y} \right) dy\, dx = \int_0^2 \int_0^x (x+y)\, dy\, dx$$

$$\int_0^2 \left(xy + \frac{x^2}{2} \Big|_{y=0}^{y=x} \right) dx = \int_0^2 x^2 + \frac{x^2}{2} \, dx = \int_0^2 \frac{3}{2} x^2 \, dx$$

$$= \frac{3x^3}{6} \Big|_0^2 = \frac{1}{2}(8) = \underline{4}$$

21.



You could use the xy-plane as the base, break up the pyramid into 4 sections, using the 4-faces as tops to the $z=0$ plane, but that's a lot of work.

Better to use opposite faces as "top" and "bottom", and integrate the z component from $0 \le z \le 1$.

y-planes: $y=0$ and plane for $(0,0,1), (0,1,0), (1,1,0)$.

$\therefore Ax + By + Cz = D \Rightarrow$

$Ax + Dy + Dz = D$

$\therefore A(1) + D(1) + D(0) = D \Rightarrow A = 0$

$\therefore y + z = 1$

$\therefore 0 \le y \le 1 - z$

x-planes: $x = 0$ and $x + z = 1$

$\therefore 0 \le x \le 1 - z$

$$\therefore \int_0^1 \int_0^{1-z} \int_0^{1-z} (1-z^2)\, dx\, dy\, dz$$

$$= \int_0^1 \int_0^{1-z} \left(x - xz^2 \Big|_{x=0}^{x=1-z} \right) dy\, dz = \int_0^1 \int_0^{1-z} (1-z-(1-z)z^2\, dy\, dz$$

$$= \int_0^1 \int_0^{1-z} (1-z-z^2+z^3)\, dy\, dz = \int_0 \left(y - yz - yz^2 + yz^3 \Big|_{y=0}^{y=1-z} \right) dz$$

$$= \int_0^1 (1-z)(1-z-z^2+z^3)\, dz = \int_0^1 1 - 2z + 2z^3 - z^4\ dz$$

$$= z - z^2 + \frac{1}{2} z^4 - \frac{z^5}{5} \Big|_0^1 = \frac{1}{2} - \frac{1}{5} = \underline{\underline{\frac{3}{10}}}$$

22.

$$\int_0^1 \int_0^{1-z} \int_0^{1-z} (x^2+y^2)\, dx\, dy\, dz = \int_0^1 \int_0^{1-z} \frac{x^3}{3} + xy^2 \Big|_{x=0}^{x=1-z} dy\, dz$$

$$= \int_0^1 \int_0^{1-z} \frac{(1-z)^3}{3} + (1-z)y^2\, dy\, dz$$

$$= \int_0^1 \frac{(1-z)^3}{3} y + (1-z)\frac{y^3}{3} \Bigg|_{y=0}^{y=1-z} dz$$

$$= \int_0^1 \frac{(1-z)^4}{3} + \frac{(1-z)^4}{3} \, dz = \int_0^1 \frac{2}{3}(1-z)^4 \, dz$$

$$= \frac{2}{3}\left(-\frac{1}{5}\right)(1-z)^5 \Bigg|_0^1 = 0 - \left(-\frac{2}{15}(1)^5\right) = \underline{\frac{2}{15}}$$

23.

$$\int_{x^2+y^2}^{x+y} dz = x+y-x^2-y^2$$

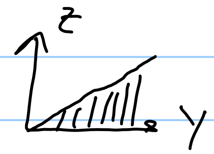
$$\int_0^{2x} x+y-x^2-y^2 \, dy = xy + \frac{y^2}{2} - x^2 y - \frac{y^3}{3} \Bigg|_{y=0}^{y=2x}$$

$$= 2x^2 + \frac{4x^2}{2} - 2x^3 - \frac{8x^3}{3} = 4x^2 - \frac{14}{3}x^3$$

$$\int_0^1 4x^2 - \frac{14}{3}x^3 \, dx = \frac{4}{3}x^3 - \frac{14}{12}x^4 \Bigg|_0^1 = \frac{4}{3} - \frac{7}{6} = \underline{\frac{1}{6}}$$

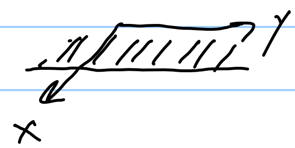
24.

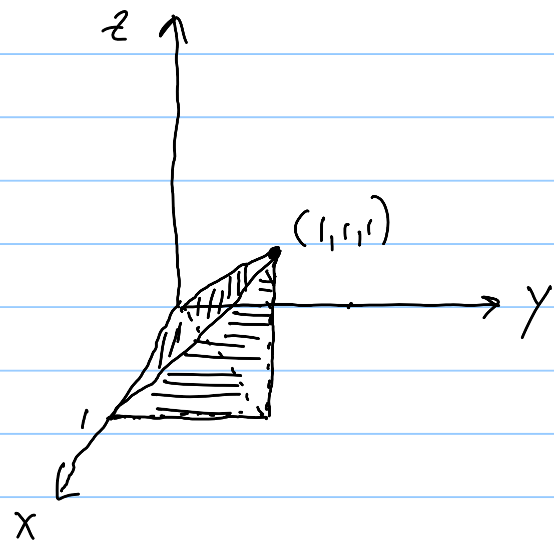
(a) $0 \le z \le y \Rightarrow z$ between $z = 0$ and $z = y$.

$\therefore$ 

$0 \le y \le x \Rightarrow y$ between $y = 0$ and $y = x$

$\therefore$ 

$0 \le x \le 1 \Rightarrow$ 

$\therefore$ 

(b) want z independent, so $\int_0^1 dz$

y a function of z: for each z', y goes from z' to 1. $\therefore z \le y \le 1$

$\therefore \int_z^1 dy$ $\quad\therefore$ So far, $\int_0^1 \int_z^1 dy\,dz$

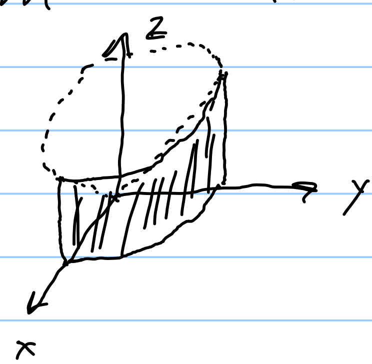
For each y', x goes from y' to 1. $\therefore \int_y^1 dx$

$$\therefore \int_0^1 \int_z^1 \int_y^1 f(x,y,z)\,dx\,dy\,dz$$

25.

From $z^2=x^2$ and $z^2=y^2$, $z^2=x^2+y^2$ is a cone, and since $0 \le \sqrt{x^2+y^2} \le z$, $z=\sqrt{x^2+y^2}$ is the "upper" portion of the cone.

$\therefore$ The solid $\sqrt{x^2+y^2} \le z \le 1$ is the volume "under" the upper portion of the cone, from $0 \le z \le 1$.



$\therefore$ $-1 \le x \le 1$
$-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$
$\sqrt{x^2+y^2} \le z \le 1$

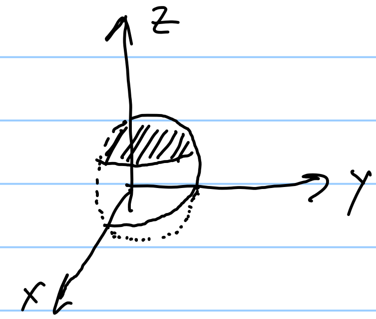
$$\therefore \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{1} f(x,y,z)\,dz\,dy\,dx$$

26.

$x^2+y^2+z^2=1$ is a sphere, ≤ 1 means inside of the sphere, and $\frac{1}{2} \le z \le 1$ means the portion of the solid sphere above $z = \frac{1}{2}$.

When $z = \frac{1}{2}$, $x^2 + y^2 + \left(\frac{1}{2}\right)^2 = 1$

becomes $x^2 + y^2 = \frac{3}{4}$. For $y = 0$,



The extremes of x come from $x^2 = \frac{3}{4}$, or $-\frac{\sqrt{3}}{2} \le x \le \frac{\sqrt{3}}{2}$

$\therefore$ $y^2 = \frac{3}{4} - x^2$ becomes $-\sqrt{\frac{3}{4} - x^2} \le y \le \sqrt{\frac{3}{4} - x^2}$

For z, $x^2 + y^2 + z^2 = 1$ becomes $\frac{1}{2} \le z \le \sqrt{1 - x^2 - y^2}$

$$\therefore \int_{-\frac{\sqrt{3}}{2}}^{\sqrt{3}/2} \int_{-\sqrt{\frac{3}{4} - x^2}}^{\sqrt{\frac{3}{4} - x^2}} \int_{\frac{1}{2}}^{\sqrt{1 - x^2 - y^2}} f(x, y, z)\, dz\, dy\, dx$$

27.

$x^2 + y^2 + z^2 \le 4 \Rightarrow$ inside sphere of radius 2.

$z \ge 0 \Rightarrow$ upper half of solid sphere.

$x^2 + y^2 \le 1 \Rightarrow$ cylinder of radius 1 inside upper half of solid sphere.

To get extremes of x, set $y = 0$ for $x^2 + y^2 = 1$ to get $-1 \le x \le 1$.

$\therefore$ Based on x, from $x^2+y^2=1$, $-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$

For z, from $x^2+y^2+z^2=4$, $0 \le z \le \sqrt{4-x^2-y^2}$

$$\therefore \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{0}^{\sqrt{4-x^2-y^2}} f(x,y,z)\, dz\,dy\,dx$$

28.

$x^2+y^2+z^2 \le 1 \Rightarrow$ inside sphere of radius 1.

$z \ge 0 \Rightarrow$ upper half of sphere.

$|x| \le 1$, $|y| \le 1 \Rightarrow$ square of side 1, which "encompasses" the sphere, and so places no additional restrictions on W.

$\therefore$ For $z=0, y=0$, $x^2+y^2+z^2 \le 1 \Rightarrow -1 \le x \le 1$, and compatible with $|x| \le 1$.

For $z=0$, $x^2+y^2+z^2=1 \Rightarrow -\sqrt{1-x^2} \le y \le \sqrt{1-y^2}$

$$\therefore \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{0}^{\sqrt{1-x^2-y^2}} f(x,y,z)\, dz\,dy\,dx$$

29.

$$\iiint_W dV \text{ becomes } \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} \int_0^{f(x,y)} dz\,dy\,dx$$

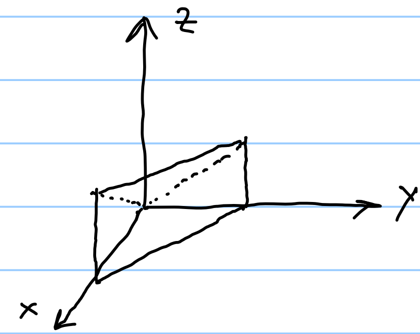
$$= \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} \left(z \Big|_0^{f(x,y)} \right) dy\,dx$$

$$= \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy\,dx = \iint_D f(x,y)\,dA$$

assuming D is described by $a \le x \le b$, $\phi_1(x) \le y \le \phi_2(x)$.

The same reasoning works for D defined by $a \le y \le b$, $\alpha_1(y) \le x \le \alpha_2(y)$, where the order of integration would be $dz\,dx\,dy$.

30.



(a) $0 \le x \le 1$, from $x + y = 1$, $0 \le y \le 1 - x$

and from $z = x + y$, $0 \le z \le x + y$.

$$\therefore \quad V = \int_0^1 \int_0^{1-x} \int_0^{x+y} dz\, dy\, dx = \int_0^1 \int_0^{1-x} (x+y)\, dy\, dx$$

$$= \int_0^1 \left(xy + \frac{y^2}{2} \Big|_{y=0}^{y=1-x} \right) dx = \int_0^1 x(1-x) + \frac{(1-x)^2}{2}\, dx$$

$$= \int_0^1 x - x^2 + \frac{x^2}{2} - x + \frac{1}{2}\, dx = \int_0^1 -\frac{1}{2}x^2 + \frac{1}{2}\, dx$$

$$= -\frac{x^3}{6} + \frac{1}{2}x \Big|_0^1 = -\frac{1}{6} + \frac{1}{2} = \underline{\frac{1}{3}}$$

(5) $$\int_0^1 \int_0^{1-x} \int_0^{x+y} x\, dz\, dy\, dx = \int_0^1 \int_0^{1-x} x(x+y)\, dy\, dz$$

$$= \int_0^1 \left(x^2 y + \frac{xy^2}{2} \Big|_{y=0}^{y=1-x} \right) dz = \int_0^1 x^2(1-x) + \frac{x(1-x)^2}{2}\, dx$$

$$= \int_0^1 x^2 - x^3 + \frac{x^3 - 2x^2 + x}{2}\, dx = \int_0^1 -\frac{1}{2}x^3 + \frac{x}{2}\, dx$$

$$= -\frac{1}{8}x^4 + \frac{x^2}{4} \Big|_0^1 = -\frac{1}{8} + \frac{1}{4} = \underline{\frac{1}{8}}$$

(c) $\displaystyle\int_0^1\int_0^{1-x}\int_0^{x+y} y\,dz\,dy\,dx = \int_0^1\int_0^{1-x} y(x+y)\,dy\,dz$

$$= \int_0^1 \left(\frac{xy^2}{2} + \frac{y^3}{3} \Bigg|_{y=0}^{y=1-x} \right) dx = \int_0^1 \frac{x(1-x)^2}{2} + \frac{(1-x)^3}{3}\,dx$$

$$= \int_0^1 \frac{x^3 - 2x^2 + x}{2} + \frac{-x^3 + 3x^2 - 3x + 1}{3}\,dx$$

$$= \int_0^1 \frac{x^3}{6} - \frac{x}{2} + \frac{1}{3}\,dx = \frac{x^4}{24} - x^2 + \frac{x}{3}\Bigg|_0^1$$

$$= \frac{1}{24} - 1 + \frac{1}{3} = \underline{-\frac{15}{24}}$$

31.

Use α instead of ϵ to avoid confusion with ϵ-δ in definition of limit.

$\therefore$ Trying to prove $\displaystyle\lim_{\alpha \to 0} \frac{1}{vol(B_\alpha)} \iiint_{B_\alpha} f(x,y,z)\,dV = f(x_0,y_0,z_0)$

Since f is continuous on closed B_α, it takes on maximum and minimum values in B_α. Let those values be Max_α and Min_α.

From the definition of $\iiint_{B_\alpha} f(x,y,z)\,dV$,

$$(Min_\alpha)\,Vol(B_\alpha) \le \iiint_{B_\alpha} f(x,y,z)\,dV \le Max_\alpha\,Vol(B_\alpha)$$

$$\therefore\ Min_\alpha \le \frac{\iiint_{B_\alpha} f(x,y,z)\,dV}{Vol(B_\alpha)} \le Max_\alpha$$

Consider $\lim\limits_{\alpha\to 0} Min_\alpha$ and $\lim\limits_{\alpha\to 0} Max_\alpha$

Given $\epsilon > 0$, since f is continuous at (x_0, y_0, z_0), $\exists\, \delta > 0$ s.t. if $\|(x,y,z) - (x_0,y_0,z_0)\| < \delta$ then $|f(x,y,z) - f(x_0,y_0,z_0)| < \epsilon$.

f continuous $\Rightarrow$ f achieves a minimum in B_δ, say at (x',y',z'). $\therefore\ \|(x',y',z') - (x_0,y_0,z_0)\| < \delta$ so that $|f(x',y',z') - f(x_0,y_0,z_0)| < \epsilon$

and f achieves a maximum in B_δ, say at (x'', y'', z''). $\therefore \|(x'', y'', z'') - (x_0, y_0, z_0)\| < \delta$

so that $|f(x'', y'', z'') - f(x_0, y_0, z_0)| < \epsilon$

Define this minimum, $f(x', y', z') = Min_\delta$

and this maximum $f(x'', y'', z'') = Max_\delta$

$\therefore |Min_\delta - f(x_0, y_0, z_0)| < \epsilon$ and $|Max_\delta - f(x_0, y_0, z_0)| < \epsilon$

$\therefore$ if $\alpha < \delta$ (same as $0 < |\alpha - 0| < \delta$), then

$B_\alpha \subset B_\delta$, both centered at (x_0, y_0, z_0),

and means that all $(x, y, z) \in B_\alpha \Rightarrow$

$(x, y, z) \in B_\delta$, so that $(x, y, z) \in B_\alpha \Rightarrow$

$|f(x, y, z) - f(x_0, y_0, z_0)| < \epsilon$, including

$|Min_\alpha - f(x_0, y_0, z_0)| < \epsilon$ and $|Max_\alpha - f(x_0, y_0, z_0)| < \epsilon$

$\therefore$ Given $\epsilon > 0$, $\exists\, \delta > 0$ s.t. if $0 < \alpha < \delta$, then

$|Min_\alpha - f(x_0, y_0, z_0)| < \epsilon$ and $|Max_\alpha - f(x_0, y_0, z_0)| < \epsilon$

$\therefore \lim_{\alpha \to 0} Min_\alpha = f(x_0, y_0, z_0)$ and $\lim_{\alpha \to 0} Max_\alpha = f(x_0, y_0, z_0)$

$$\lim_{\alpha \to 0} Min_\alpha \le \lim_{\alpha \to 0} \frac{\iiint_{B_\alpha} f(x,y,z)\,dV}{vol(B_\alpha)} \le \lim_{\alpha \to 0} Max_\alpha$$

and $\therefore$ $$f(x_0,y_0,z_0) \le \lim_{\alpha \to 0} \frac{\iiint_{B_\alpha} f(x,y,z)\,dV}{vol(B_\alpha)} \le f(x_0,y_0,z_0)$$

$$\therefore \lim_{\alpha \to 0} \frac{\iiint_{B_\alpha} f(x,y,z)\,dV}{vol(B_\alpha)} = f(x_0,y_0,z_0)$$

Review Exercises for Chapter 5

Note Title 11/11/2016

1.

$$\int_0^3 \left(\frac{xy^2}{2} \Big|_{y=-x^2+1}^{y=x^2+1} \right) dx = \int_0^3 \frac{x(x^2+1)^2}{2} - \frac{x(-x^2+1)^2}{2} \, dx$$

$$= \int_0^3 \frac{2x(x^2+1)^2}{4} dx + \int_0^3 \frac{-2x(-x^2+1)^2}{4} dx$$

$$= \frac{(x^2+1)^3}{12} \Big|_0^3 + \frac{(-x^2+1)^3}{12} \Big|_0^3$$

$$= \frac{1000}{12} - \frac{1}{12} + \frac{-512}{12} - \frac{1}{12} = \frac{486}{12} = 40\tfrac{1}{2} = \underline{\frac{81}{2}}$$

2.

Since $\frac{(x+y)^3}{3} \Big|_{y=\sqrt{x}}^{y=1}$ is messy, use change of integration limits.

$y=\sqrt{x}$ is increasing, $x=y^2$ is the inverse,

$\sqrt{0}=0$, $\sqrt{1}=1$

$$\therefore \int_0^1 \int_0^{y^2} (x+y)^2 \, dx \, dy = \int_0^1 \frac{(x+y)^3}{3} \Big|_{x=0}^{x=y^2} dy$$

$$= \int_0^1 \left[\frac{(y^2+y)^3}{3} - \frac{y^3}{3} \right] dy = \int_0^1 \frac{y^6 + 3y^5 + 3y^4 + y^3}{3} - \frac{y^3}{3} \, dy$$

$$= \int_0^1 \frac{y^6}{3} + y^5 + y^4 \, dy = \frac{y^7}{21} + \frac{y^6}{6} + \frac{y^5}{5} \Big|_0^1$$

$$= \frac{1}{21} + \frac{1}{6} + \frac{1}{5} = \frac{10 + 35 + 42}{210} = \frac{87}{210} = \underline{\underline{\frac{29}{70}}}$$

3.

$$\int_{e^x}^{e^{2x}} x \ln y \, dy = x \left[y \ln y - y \right]_{e^x}^{e^{2x}}$$

$$= x \left\{ e^{2x}(2x) - e^{2x} - (xe^x - e^x) \right\}$$

$$= 2x^2 e^{2x} - xe^{2x} - x^2 e^x + xe^x$$

$\therefore$ Evaluate: $\int_0^1 2x^2 e^{2x} - xe^{2x} - x^2 e^x + xe^x \, dx \quad [1]$

$$\int_0^1 x^2 2e^{2x} \, dx = x^2 e^{2x} \Big|_0^1 - \int_0^1 2x e^{2x} \, dx$$

$$= e^2 - 0 - \int_0^1 2x e^{2x}\,dx$$

$\therefore$ [1] becomes $e^2 + \int_0^1 -3xe^{2x} - x^2 e^x + xe^x\,dx$ [2]

$$-\frac{3}{2}\int_0^1 2xe^{2x}\,dx = -\frac{3}{2}\left[xe^{2x}\Big|_0^1 - \int_0^1 e^{2x}\,dx \right]$$

$$= -\frac{3}{2}\left[e^2 - \frac{1}{2}e^{2x}\Big|_0^1 \right] = -\frac{3}{2}\left[e^2 - \frac{1}{2}e^2 + \frac{1}{2} \right]$$

$$= -\frac{3}{4}e^2 - \frac{3}{4}$$

$\therefore$ [2] becomes $\frac{1}{4}e^2 - \frac{3}{4} + \int_0^1 -x^2e^x + xe^x\,dx$ [3]

$$-\int_0^1 x^2 e^x\,dx = -\left[x^2 e^x\Big|_0^1 - 2\int_0^1 xe^x\,dx \right]$$

$$= -e + 2\int_0^1 xe^x\,dx$$

$\therefore$ [3] becomes $\frac{1}{4}e^2 - e - \frac{3}{4} + 3\int_0^1 xe^x\,dx$ [4]

$$3\int_0^1 xe^x\,dx = 3\left[xe^x\Big|_0^1 - \int_0^1 e^x\,dx \right]$$

$$= 3\left[e - 0 - e + 1\right] = 3$$

$\therefore$ [4] becomes $\frac{1}{4}e^2 - e - \frac{3}{4} + 3 = \underline{\frac{1}{4}e^2 - e + \frac{9}{4}}$

4.

$$\int_2^3 \cos[\pi(x+y+z)]\,dx = \frac{1}{\pi}\sin[\pi(x+y+z)]\Big|_{x=2}^{x=3}$$

$$= \frac{1}{\pi}\left[\sin(3\pi + \pi y + \pi z) - \sin(2\pi + \pi y + \pi z)\right]$$

now using $\sin(n\pi + b) = \cos(n\pi)\sin(b)$

$$= \frac{1}{\pi}\left[-\sin(\pi y + \pi z) - \sin(\pi y + \pi z)\right]$$

$$= -\frac{2}{\pi}\sin(\pi y + \pi z)$$

$$\therefore \int_1^2 -\frac{2}{\pi}\sin(\pi y + \pi z)\,dy = \frac{2}{\pi^2}\cos(\pi y + \pi z)\Big|_{y=1}^{y=2}$$

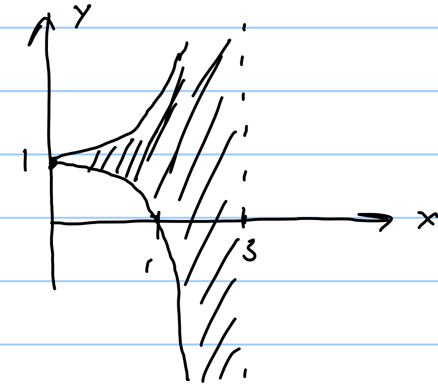
$$= \frac{2}{\pi^2}\left[\cos(2\pi + \pi z) - \cos(\pi + \pi z)\right]$$

now using $\cos(n\pi + b) = \cos(n\pi)\cos(b)$

$$= \frac{2}{\pi^2}\left[\cos(\pi z) + \cos(\pi z)\right] = \frac{4}{\pi^2}\cos(\pi z)$$

$$\therefore \int_0^1 \frac{4}{\pi^2}\cos(\pi z)\,dz = \frac{4}{\pi^3}\sin(\pi z)\Big|_0^1 = \underline{\underline{0}}$$

5.



$$y = -x^2+1 \Rightarrow x = \sqrt{1-y}$$
$$x=3 \Rightarrow y = -8$$

$$y = x^2+1 \Rightarrow x = \sqrt{y-1}$$
$$x=3 \Rightarrow y = 10$$

$$\therefore \quad 1 \le y \le 10, \quad \sqrt{y-1} \le x \le 3$$
$$-8 \le y \le 1, \quad \sqrt{1-y} \le x \le 3$$

$$\therefore \int_{-8}^{1}\int_{\sqrt{1-y}}^{3} xy\,dx\,dy + \int_{1}^{10}\int_{\sqrt{y-1}}^{3} xy\,dx\,dy$$

$$\int_{-8}^{1}\int_{\sqrt{1-y}}^{3} xy\,dx\,dy = \int_{-8}^{1} \frac{x^2y}{2}\Big|_{x=\sqrt{1-y}}^{x=3}\,dy$$

$$= \int_{-8}^{1} \frac{9y}{2} - \frac{(1-y)y}{2}\,dy = \int_{-8}^{1} 4y + \frac{y^2}{2}\,dy$$

$$= 2y^2 + \frac{y^3}{6}\Big|_{-8}^{1} = 2 + \frac{1}{6} - \left(128 - \frac{512}{6}\right)$$

$$= -126 + \frac{513}{6}$$

$$\int_1^{10}\int_{\sqrt{y-1}}^{3} xy\,dx\,dy = \int_1^{10} \frac{x^2 y}{2}\Big|_{x=\sqrt{y-1}}^{x=3} dy$$

$$= \int_1^{10} \frac{9y}{2} - \frac{(y-1)y}{2}\,dy = \int_1^{10} 5y - \frac{y^2}{2}\,dy$$

$$= \frac{5}{2}y^2 - \frac{y^3}{6}\Big|_1^{10} = 250 - \frac{1000}{6} - \frac{5}{2} + \frac{1}{6}$$

$$= 250 - \frac{5}{2} - \frac{999}{6}$$

$$\therefore\ -126 + \frac{513}{6} + 250 - \frac{5}{2} - \frac{999}{6} = 124 - \frac{5}{2} - \frac{486}{6}$$

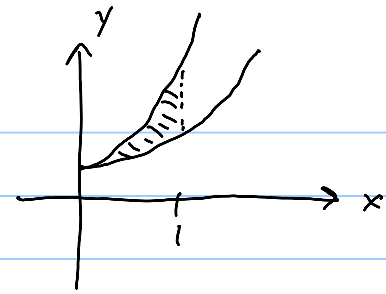
$$= 124 - \frac{5}{2} - \frac{162}{2} = \frac{248}{2} - \frac{167}{2} = \underline{\frac{81}{2}}$$

6.

This was done in #2 using $\int_0^1 \int_0^{y^2} (x+y)^2\,dx\,dy$

$$= \underline{\frac{29}{70}}$$

7.



$y = e^x \Rightarrow x = \ln y$

$e^0 \Rightarrow 1, \; e^1 \Rightarrow e$

$y = e^{2x} \Rightarrow x = \frac{1}{2}\ln y, \quad e^{2(0)} = 1, \; e^{2(1)} = e^2$

$\therefore$ Break into 2 adjacent x-simple regions.

$1 \le y \le e : \; \frac{1}{2}\ln y \le x \le \ln y$

$e \le y \le e^2 : \; \frac{1}{2}\ln y \le x \le 1$

$$\therefore \int_1^e \int_{\frac{1}{2}\ln y}^{\ln y} x \ln y \, dx\, dy + \int_e^{e^2} \int_{\frac{1}{2}\ln y}^{1} x \ln y \, dx\, dy$$

[1]: $$\int_1^e \int_{\frac{1}{2}\ln y}^{\ln y} x \ln y \, dx\, dy = \int_1^e \frac{x^2}{2} \ln y \Big|_{x=\frac{1}{2}\ln y}^{x=\ln y} dy$$

$$= \int_1^e \frac{(\ln y)^3}{2} - \frac{1}{8}(\ln y)^3 \, dy = \frac{3}{8}\int_1^e (\ln y)^3 \, dy$$

Using $\int (\log x)^m dx = x(\log x)^m - m\int (\log x)^{m-1} dx$

$$= \frac{3}{8}\left[y(\ln y)^3 \Big|_1^e - 3\int_1^e (\ln y)^2 \, dy \right]$$

$$= \frac{3}{8}\left[e - 0 - 3\left(y(\ln y)^2 \Big|_1^e - 2\int_1^e (\ln y)\, dy \right) \right]$$

$$= \frac{3}{8}\left[e - 3(e) + 6\left(y\ln y - y \right)\Big|_1^e \right]$$

$$= \frac{3}{8}\left[-2e + 6\left(e - e - (0-1) \right) \right]$$

$$= \frac{3}{8}\left[-2e + 6 \right] = \underline{-\frac{3}{4}e + \frac{9}{4}}$$

[2]: $$\int_e^{e^2}\int_{\frac{1}{2}\ln y}^{1} x \ln y \, dx\, dy = \int_e^{e^2} \frac{x^2}{2}\ln y \Big|_{x=\frac{1}{2}\ln y}^{x=1} dy$$

$$= \int_e^{e^2} \frac{1}{2}\ln y - \frac{1}{8}(\ln y)^3 \, dy$$

$$= \frac{1}{2}\left[y\ln y - y \right]_e^{e^2} - \frac{1}{8}\int_e^{e^2} (\ln y)^3 dy$$

$$= \frac{1}{2}\left[2e^2 - e^2 - (e - e) \right] - \frac{1}{8}\int_e^{e^2} (\ln y)^3 dy$$

$$= \frac{e^2}{2} - \frac{1}{8}\int_e^{e^2} (\ln y)^3 dy$$ [2]

Using $\int (\log x)^m dx = x(\log x)^m - m\int (\log x)^{m-1} dx$

$$\int_e^{e^2} (\ln y)^3 dy = y(\ln y)^3 \Big|_e^{e^2} - 3\int_e^{e^2} (\ln y)^2 dy$$

$$= (8e^2 - e) - 3\left[y(\ln y)^2 \Big|_e^{e^2} - 2\int_e^{e^2} \ln y \, dy \right]$$

$$= 8e^2 - e - 3\left[4e^2 - e - 2(y\ln y - y)\Big|_e^{e^2} \right]$$

$$= 8e^2 - e - 12e^2 + 3e + 6\left[2e^2 - e^2 - (e - e) \right]$$

$$= -4e^2 + 2e + 6e^2 = 2e^2 + 2e$$

$\therefore$ [2] becomes $\frac{e^2}{2} - \frac{1}{8}(2e^2 + 2e) = \frac{1}{4}e^2 - \frac{e}{4}$

$\therefore$ $[1] + [2] = -\frac{3}{4}e + \frac{9}{4} + \frac{1}{4}e^4 - \frac{e}{4} = \underline{\frac{1}{4}e^4 - e + \frac{9}{4}}$

8.

$$\int_2^3 \int_1^2 \int_0^1 \cos[\pi x + \pi y + \pi z] \, dz\,dy\,dx$$

$$= \int_2^3 \int_1^2 \frac{1}{\pi} \sin[\pi x + \pi y + \pi z] \Big|_{z=0}^{z=1} dy\,dx$$

$$= \int_2^3 \int_1^2 \frac{1}{\pi}\left[\sin(\pi x + \pi y + \pi) - \sin(\pi x + \pi y)\right] dy\,dx$$

Now use $\sin(\alpha + \pi) = -\sin\alpha$

$$= \int_2^3 \int_1^2 -\frac{2}{\pi} \sin(\pi x + \pi y)\, dy\,dx$$

$$= \int_2^3 \frac{2}{\pi^2} \cos(\pi x + \pi y)\Big|_{y=1}^{y=2} dx$$

$$= \int_2^3 \frac{2}{\pi^2}\left[\cos(\pi x + 2\pi) - \cos(\pi x + \pi)\right] dx$$

Now use $\cos(\alpha + n\pi) = (-1)^n \cos\alpha$

$$= \int_2^3 \frac{4}{\pi^2} \cos(\pi x)\, dx = \frac{4}{\pi^3} \sin(\pi x)\Big|_{x=2}^{x=3} = \underline{0}$$

9.

$$\int_0^y (y + xz)\, dz = yz + \frac{xz^2}{2}\Big|_{z=0}^{z=y} = y^2 + \frac{xy^2}{2}$$

$$\int_0^x \left(y^2 + \frac{xy^2}{2}\right) dy = \frac{y^3}{3} + \frac{xy^3}{6}\Big|_{y=0}^{y=x} = \frac{x^3}{3} + \frac{x^4}{6}$$

$$\int_0^1 \frac{x^3}{3} + \frac{x^4}{6}\,dx = \frac{x^4}{12} + \frac{x^5}{30}\Big|_0^1 = \frac{1}{12} + \frac{1}{30} = \frac{5}{60} + \frac{2}{60}$$

$$= \underline{\frac{7}{60}}$$

10.

$$\int_y^{y^2} e^{x/y}\,dx = y e^{\frac{x}{y}}\Big|_{x=y}^{x=y^2} = ye^y - ye$$

$$\therefore \int_0^1 (ye^y - ey)\,dy = ye^y\Big|_0^1 - \int_0^1 e^y - e\frac{y^2}{2}\Big|_0^1$$

$$= (e-0) - (e-1) - \left(\frac{e}{2} - 0\right)$$

$$= \underline{1 - \frac{e}{2}}$$

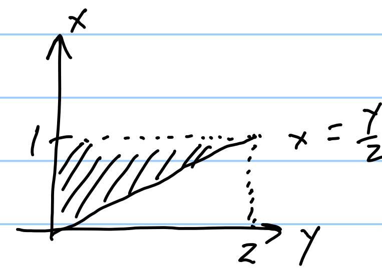
11.

$$\int_0^{\frac{\arcsin y}{y}} y\cos(xy)\,dx = \sin(xy)\Big|_{x=0}^{x=\frac{\arcsin y}{y}}$$

$$= \sin\left[y \cdot \frac{\arcsin y}{y}\right] - 0 = \sin(\arcsin y) = y$$

$$\therefore \int_0^1 y\,dy = \frac{y^2}{2}\Big|_0^1 = \underline{\underline{\frac{1}{2}}}$$

12.



$x = \frac{y}{2}$ is an increasing function of y.

Inverse function is $y = 2x$

$$\frac{y}{2}\Big|_{y=0} = 0 \quad \frac{y}{2}\Big|_{y=2} = 1$$

$$\therefore \int_0^1 \int_0^{2x} (x+y)^2\,dy\,dx$$

$$= \int_0^1 \frac{(x+y)^3}{3}\Big|_{y=0}^{y=2x} dx = \int_0^1 \frac{27}{3}x^3 - \frac{x^3}{3}\,dx$$

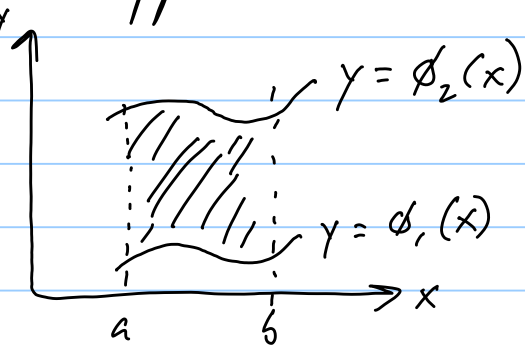
$$= \frac{26}{12}x^4\Big|_0^1 = \frac{26}{12} = \underline{\frac{13}{6}}$$

13.

Since D is a y-simple region, let the lower bound be $y = \phi_1(x)$, upper bound $y = \phi_2(x)$.

$\therefore\ \phi_1(x) \le y \le \phi_2(x)$, and suppose $a \le x \le b$.

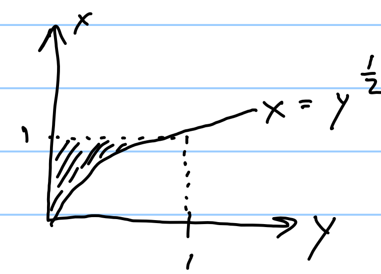
$$\therefore \iint_D dx\,dy = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} dy\,dx$$



$$= \int_a^b y \Big|_{y=\phi_1(x)}^{y=\phi_2(x)} dx$$

$$= \int_a^b [\phi_2(x) - \phi_1(x)]\, dx$$ = area between between 2 curves: $\phi_1(x)$ and $\phi_2(x)$.

14.



$x=\sqrt{y}$ is an increasing function
inverse is: $y=x^2$

$\sqrt{y}\big|_{y=0} = 0 \quad \sqrt{y}\big|_{y=1} = 1$

$$\therefore \int_0^1 \int_0^{x^2} (x^2+y^3x)\,dy\,dx = \int_0^1 x^2y + \frac{y^4x}{4}\Big|_{y=0}^{y=x^2} dx$$

$$= \int_0^1 \left(x^4 + \frac{x^9}{4} \right) dx = \frac{x^5}{5} + \frac{x^{10}}{40} \Bigg|_0^1 = \frac{1}{5} + \frac{1}{40} = \underline{\underline{\frac{9}{40}}}$$

15.

$$-1 \le x \le 1$$
$$-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$$

(a)

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} xy \, dy \, dx = \int_{-1}^{1} x \frac{y^2}{2} \Bigg|_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} dx$$

$$= \int_{-1}^{1} \frac{x(1-x^2)}{2} - \frac{x(1-x^2)}{2} \, dx = \int_{-1}^{1} 0 \, dx = \underline{\underline{0}}$$

(b)

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x^2 y^2 \, dy \, dx = \int_{-1}^{1} x^2 \frac{y^3}{3} \Bigg|_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} dx$$

$$= \int_{-1}^{1} x^2 \frac{(1-x^2)^{\frac{3}{2}}}{3} + \frac{x^2(1-x^2)^{3/2}}{3} \, dx = \frac{2}{3} \int_{-1}^{1} x^2 (1-x^2)^{3/2} \, dx$$

Let $x = \sin\theta$ $\quad x = -1 \Rightarrow \theta = -\frac{\pi}{2}, \quad x = 1 \Rightarrow \theta = \frac{\pi}{2}$

$(1-x^2)^{3/2} = \cos^3\theta \quad dx = \cos\theta \, d\theta$

$$\therefore \frac{2}{3}\int_{-1}^{1} x^2(1-x^2)^{3/2}\,dx = \frac{2}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2\theta\cos^4\theta\,d\theta$$

Now use $\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$, $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$

$$\therefore \frac{2}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1-\cos 2\theta}{2}\right)\frac{(1+\cos 2\theta)^2}{4}\,d\theta$$

$$= \frac{1}{12}\int_{-\pi/2}^{\pi/2} (1+\cos 2\theta - \cos^2 2\theta - \cos^3 2\theta)\,d\theta$$

$$= \frac{1}{12}\int_{-\frac{\pi}{2}}^{\pi/2} \left[1+\cos 2\theta - \frac{(1+\cos 2\theta)}{2} - (1-\sin^2 2\theta)\cos 2\theta\right] d\theta$$

$$= \frac{1}{12}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{1}{2} + \underbrace{\frac{\cos 2\theta}{2} - \cos 2\theta}_{} + \cos 2\theta\sin^2 2\theta\right] d\theta$$

$$= \frac{1}{12}\left[\frac{\theta}{2}\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \frac{\sin 2\theta}{4}\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{1}{6}\sin^3 2\theta\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\right]$$

$$= \frac{1}{12}\left[\frac{\pi}{2} - 0 + \frac{1}{6}(0)\right] = \underline{\frac{\pi}{24}}$$

(c)

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x^3y^3\,dy\,dx = \int_{-1}^{1} \frac{x^3y^4}{4}\Bigg|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx$$

$$= \int_{-1}^{1} \frac{x^3(1-x^2)^2}{4} - \frac{x^3(1-x^2)^2}{4}\,dx$$

$$= \int_{-1}^{1} 0\,dx = \underline{0}$$

16.

For $0 \le y \le 2$,
$0 \le x \le 4$

$$\int_0^4\int_{\sqrt{x}}^{2} y\left[1-\cos\left(\frac{\pi x}{4}\right)\right] dy\,dx$$

$$= \int_0^4 \frac{y^2}{2}\left[1-\cos\left(\frac{\pi x}{4}\right)\right]_{y=\sqrt{x}}^{y=2} dx$$

$$= \int_0^4 \left[2 - 2\cos\left(\frac{\pi x}{4}\right) - \frac{x}{2} + \frac{x}{2}\cos\left(\frac{\pi x}{4}\right)\right] dx$$

$$= 2x\Big|_0^4 - \frac{8}{\pi}\sin\left(\frac{\pi x}{4}\right)\Big|_0^4 - x^2\Big|_0^4 + \int_0^4 \frac{x}{2}\cos\left(\frac{\pi x}{4}\right)dx$$

$$= (8 - 0 - 16) + \int_0^4 \frac{x}{2}\cos\left(\frac{\pi x}{4}\right)dx$$

$$= -8 + \frac{1}{2}\int_0^4 x\cos\left(\frac{\pi x}{4}\right)dx$$

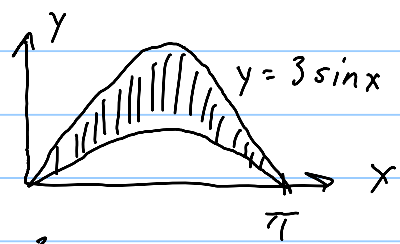
$$= -8 + \frac{1}{2}\left[\frac{4}{\pi}x\sin\left(\frac{\pi x}{4}\right)\Big|_0^4 - \frac{4}{\pi}\int_0^4 \sin\left(\frac{\pi x}{4}\right)dx\right]$$

$$= -8 + 0 - \frac{2}{\pi}\left[-\frac{4}{\pi}\cos\left(\frac{\pi x}{4}\right)\right]_0^4$$

$$= -8 + \frac{8}{\pi^2}\left[\cos(\pi) - \cos(0)\right]$$

$$= -8 - \frac{16}{\pi^2}$$

17. <u>y-simple</u> :



$$\int_{\sin x}^{3\sin x} x(1+y)\,dy = xy + \frac{xy^2}{2}\Bigg|_{y=\sin x}^{y=3\sin x}$$

$$= 3x\sin x + \frac{9}{2}x\sin^2 x - x\sin x - \frac{x}{2}\sin^2 x$$

$$= 2x\sin x + 4x\sin^2 x$$

$$\therefore \int_0^{\pi} (2x\sin x + 4x\sin^2 x)\,dx =$$

$$-2x\cos x\Big|_0^{\pi} + 2\int_0^{\pi}\cos x\,dx + \int_0^{\pi} 4x\sin^2 x\,dx$$

$$= 2\pi + 2\sin x\Big|_0^{\pi} + \int_0^{\pi} 4x\left(\frac{1-\cos 2x}{2}\right)dx$$

using $\sin^2 x = \dfrac{1-\cos 2x}{2}$

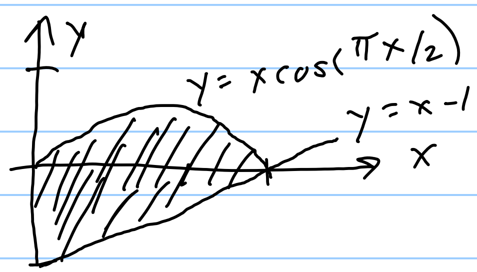
$$= 2\pi + 0 + \int_0^{\pi} 2x\,dx - \int_0^{\pi} 2x\cos 2x\,dx$$

$$= 2\pi + \pi^2 - \left[x\sin 2x\Big|_0^{\pi} - \int_0^{\pi}\sin 2x\,dx\right]$$

$$= 2\pi + \pi^2 - \left[0 + \tfrac{1}{2}\cos 2x\Big|_0^{\pi}\right]$$

$$= 2\pi + \pi^2 - \left[\tfrac{1}{2} - \tfrac{1}{2}\right] = \underline{2\pi + \pi^2}$$

18.



y-simple

$$\int_{x-1}^{x\cos\left(\frac{\pi x}{2}\right)} (x^2+xy+1)\,dy = x^2y + \frac{xy^2}{2} + y\Bigg|_{y=x-1}^{y=x\cos\left(\frac{\pi x}{2}\right)}$$

$$= x^2\left[x\cos\left(\frac{\pi x}{2}\right)\right] + \frac{x}{2}\left[x^2\cos^2\left(\frac{\pi x}{2}\right)\right] + x\cos\left(\frac{\pi x}{2}\right)$$
$$- \left[x^2(x-1) + x\frac{(x-1)^2}{2} + (x-1)\right]$$

$$= x^3\cos\left(\frac{\pi x}{2}\right) + \frac{x^3}{2}\cos^2\left(\frac{\pi x}{2}\right) + x\cos\left(\frac{\pi x}{2}\right)$$

$$- x^3 + x^2 - \frac{x^3}{2} + x^2 - \frac{x}{2} - x + 1$$

$$= x^3\cos\left(\frac{\pi x}{2}\right) + \frac{x^3}{2}\cos^2\left(\frac{\pi x}{2}\right) + x\cos\left(\frac{\pi x}{2}\right)$$

$$-\frac{3}{2}x^3 + 2x^2 - \frac{3}{2}x + 1$$

[1] $$\int_0^1 x^3\cos\left(\frac{\pi x}{2}\right)dx = \frac{2}{\pi}x^3\sin\left(\frac{\pi x}{2}\right)\Bigg|_0^1 - \frac{3\cdot 2}{\pi}\int_0^1 x^2\sin\left(\frac{\pi x}{2}\right)dx$$

$$= \frac{2}{\pi} - 0 - \frac{6}{\pi}\left[-\frac{2}{\pi}x^2\cos\left(\frac{\pi}{2}x\right)\Bigg|_0^1 + \frac{2\cdot 2}{\pi}\int_0^1 x\cos\left(\frac{\pi}{2}x\right)dx\right]$$

$$= \frac{2}{\pi} - \frac{6}{\pi}\left[0 + \frac{4}{\pi}\int_0^1 x\cos\left(\frac{\pi}{2}x\right)dx\right]$$

$$= \frac{2}{\pi} - \frac{24}{\pi^2}\left[\frac{2}{\pi}x\sin\left(\frac{\pi}{2}x\right)\Big|_0^1 - \frac{2}{\pi}\int_0^1 \sin\left(\frac{\pi x}{2}\right)dx\right]$$

$$= \frac{2}{\pi} - \frac{24}{\pi^2}\left[\frac{2}{\pi} + \frac{4}{\pi^2}\cos\left(\frac{\pi x}{2}\right)\Big|_0^1\right]$$

$$= \frac{2}{\pi} - \frac{24}{\pi^2}\left[\frac{2}{\pi} + \left(0 - \frac{4}{\pi^2}\right)\right]$$

$$= \frac{2}{\pi} - \underline{\frac{48}{\pi^3}} + \frac{96}{\pi^4}$$

[2] $$\int_0^1 \frac{x^3}{2}\cos^2\left(\frac{\pi x}{2}\right)dx = \frac{1}{2}\int_0^1 x^3\left(\frac{1+\cos(\pi x)}{2}\right)dx$$

using $\cos^2\theta = \frac{1+\cos 2\theta}{2}$

$$= \frac{1}{4}\int_0^1 x^3\,dx + \frac{1}{4}\int_0^1 x^3\cos(\pi x)\,dx$$

$$\frac{1}{4}\int_0^1 x^3\,dx = \frac{1}{16}x^4\Big|_0^1 = \underline{\frac{1}{16}}$$

$$\frac{1}{4}\int_0^1 x^3\cos(\pi x)\,dx = \frac{1}{4\pi}x^3\sin(\pi x)\Big|_0^1 - \frac{3}{4\pi}\int_0^1 x^2\sin(\pi x)\,dx$$

$$= 0 - \frac{3}{4\pi}\left[-\frac{1}{\pi}x^2\cos(\pi x)\Big|_0^1 + \frac{2}{\pi}\int_0^1 x\cos(\pi x)\,dx\right]$$

$$= -\frac{3}{4\pi}\left[\frac{1}{\pi} + \frac{1}{\pi} x \sin(\pi x)\Big|_0^1 - \frac{1}{\pi}\int_0^1 \sin(\pi x)\,dx\right]$$

$$= -\frac{3}{4\pi}\left[\frac{1}{\pi} + 0 + \frac{1}{\pi^2}\cos(\pi x)\Big|_0^1\right]$$

$$= -\frac{3}{4\pi}\left[\frac{1}{\pi} + \frac{1}{\pi^2}(-1-1)\right] = -\frac{3}{4\pi}\left[\frac{1}{\pi} - \frac{2}{\pi^2}\right]$$

$$= -\frac{3}{4\pi^2} + \frac{3}{2\pi^3}$$

$$\therefore \frac{1}{16} \underline{-\frac{3}{4\pi^2} + \frac{3}{2\pi^3}}$$

[3] $$\int_0^1 x\cos\left(\frac{\pi x}{2}\right)dx = \frac{2}{\pi} x \sin\left(\frac{\pi x}{2}\right)\Big|_0^1 - \frac{2}{\pi}\int_0^1 \sin\left(\frac{\pi x}{2}\right)dx$$

$$= \frac{2}{\pi} + \frac{2}{\pi}\left[\frac{2}{\pi}\cos\left(\frac{\pi x}{2}\right)\Big|_0^1\right]$$

$$= \frac{2}{\pi} + \frac{2}{\pi}\left[0 - \frac{2}{\pi}\right] = \underline{\frac{2}{\pi} - \frac{4}{\pi^2}}$$

$\therefore$ [1] + [2] + [3] =

$$\left(\frac{2}{\pi} - \frac{48}{\pi^3} + \frac{96}{\pi^4}\right) + \left(\frac{1}{16} - \frac{3}{4\pi^2} + \frac{3}{2\pi^3}\right) + \left(\frac{2}{\pi} - \frac{4}{\pi^2}\right)$$

$$= \frac{1}{16} + \frac{4}{\pi} - \frac{19}{4\pi^2} - \frac{93}{2\pi^3} + \frac{96}{\pi^4}$$

$$[4] \quad \int_0^1 -\frac{3}{2}x^3 + 2x^2 - \frac{3}{2}x + 1 \, dx$$

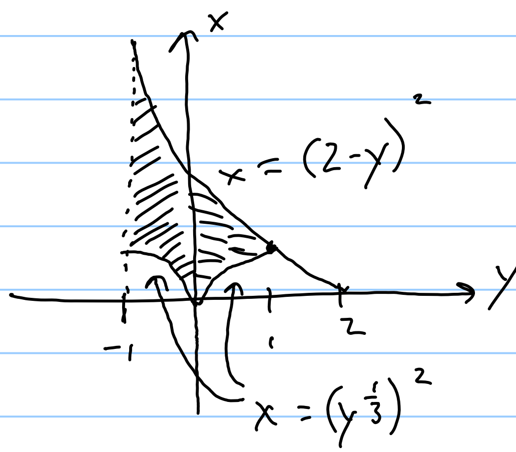
$$= -\frac{3}{8}x^4 + \frac{2}{3}x^3 - \frac{3}{4}x^2 + x \Big|_0^1$$

$$= -\frac{3}{8} + \frac{2}{3} - \frac{3}{4} + 1 = \frac{-9 + 16 - 18 + 24}{24} = \frac{13}{24}$$

$$\therefore [1] + [2] + [3] + [4] =$$

$$\frac{29}{48} + \frac{4}{\pi} - \frac{19}{4\pi^2} - \frac{93}{2\pi^3} + \frac{96}{\pi^4}$$

19.



2 adjacent x-simple regions

$$\int_{y^{2/3}}^{(2-y)^2} \left(\frac{3}{2}\sqrt{x} - 2y\right) dx =$$

$$\frac{3}{2}\left(\frac{2}{3}\right)x^{\frac{3}{2}} - 2xy \Big|_{x=y^{2/3}}^{x=(2-y)^2}$$

$$= (2-y)^3 - 2(2-y)^2 y - \left[y - 2y^{\frac{5}{3}}\right]$$

$$= (2-y)^3 - 2y^3 + 8y^2 - 9y + 2y^{5/3}$$

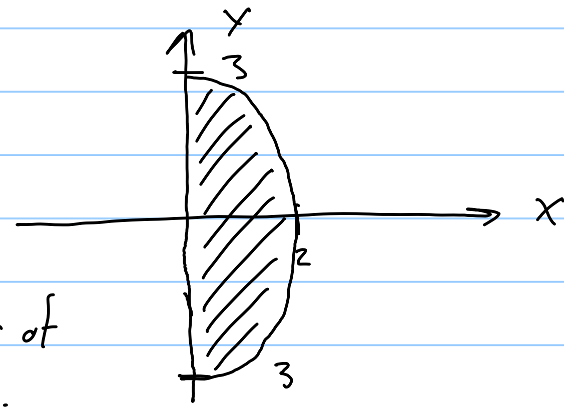
$$\therefore \int_{-1}^{1} (2-y)^3 - 2y^3 + 8y^2 - 9y + 2y^{5/3}\, dy$$

$$= -\frac{(2-y)^4}{4} - \frac{y^4}{2} + \frac{8}{3}y^3 - \frac{9}{2}y^2 + \frac{3}{4}y^{8/3} \Big|_{-1}^{1}$$

$$= -\frac{1}{4} - \frac{1}{2} + \frac{8}{3} - \frac{9}{2} + \frac{3}{4} - \left[-\frac{81}{4} - \frac{1}{2} - \frac{8}{3} - \frac{9}{2} + \frac{3}{4}\right]$$

$$= 20 + \frac{16}{3} = \underline{\frac{76}{3}}$$

20.



$y = \frac{3}{2}\sqrt{4-x^2} \Rightarrow \frac{x^2}{4} + \frac{y^2}{9} = 1$

$\therefore$ y-simple (opposite sides of ellipse).

$$\int_{-\frac{3}{2}\sqrt{4-x^2}}^{\frac{3}{2}\sqrt{4-x^2}} \left(\frac{5}{\sqrt{2+x}} + y^3\right) dy\, dx$$

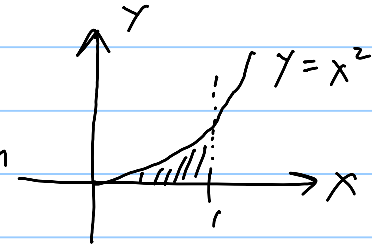
$$= \frac{5y}{\sqrt{x+2}} + \frac{y^4}{4} \Bigg|_{y=-\frac{3}{2}\sqrt{4-x^2}}^{y=\frac{3}{2}\sqrt{4-x^2}}$$

$$= \frac{15\sqrt{4-x^2}}{\sqrt{x+2}} = 15\sqrt{2-x}$$ since, for $0 \le x \le 2$, $\sqrt{x+2} \neq 0$

$$\therefore \int_0^2 15\sqrt{2-x}\,dx = -15\left(\tfrac{2}{3}\right)(2-x)^{3/2}\Big|_0^2$$

$$= 0 - \left(-10(2)^{3/2}\right) = 10\left(2\sqrt{2}\right) = \underline{\underline{20\sqrt{2}}}$$

21.

y-simple region 

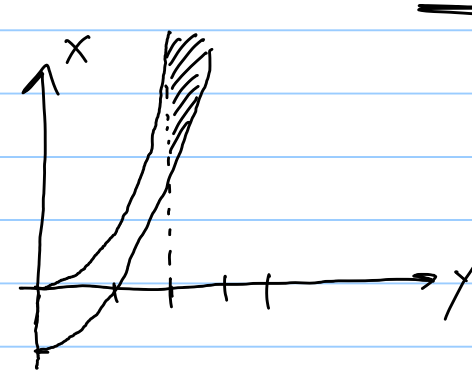
$$\int_0^{x^2} (x^2 + xy - y^2)\,dy = x^2y + \frac{xy^2}{2} - \frac{y^3}{3}\Big|_{y=0}^{y=x^2}$$

$$= x^4 + \frac{x^5}{2} - \frac{x^6}{3}$$

$$\therefore \int_0^1 x^4 + \frac{x^5}{2} - \frac{x^6}{3}\,dx = \frac{x^5}{5} + \frac{x^6}{12} - \frac{x^7}{21}\Big|_0^1$$

$$= \frac{1}{5} + \frac{1}{12} - \frac{1}{21} = \frac{84 + 35 - 20}{420} = \frac{99}{420} = \underline{\underline{\frac{33}{140}}}$$

22.

x-simple regions 

$$\int_2^4 \int_{y^2-1}^{y^3} 3\,dx\,dy = \int_2^4 3x\Big|_{x=y^2-1}^{x=y^3}\,dy$$

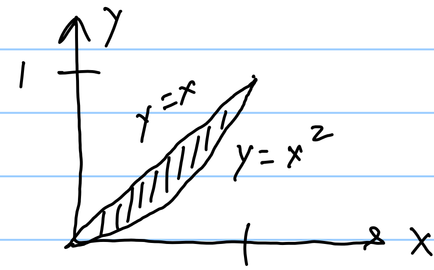
$$= \int_2^4 3y^3 - 3y^2 + 3\,dy = \frac{3y^4}{4} - y^3 + 3y \Big|_2^4$$

$$= \frac{3}{4}(256) - 64 + 12 - \left(\frac{3}{4}(16) - 8 + 6\right)$$

$$= \frac{3}{4}(240) - 56 + 6 = 180 - 50 = \underline{130}$$

23.

y-simple region



$$\int_0^1 \int_{x^2}^{x} (x+y)^2\,dy\,dx =$$

$$\int_0^1 \frac{(x+y)^3}{3} \Big|_{y=x^2}^{y=x} dx =$$

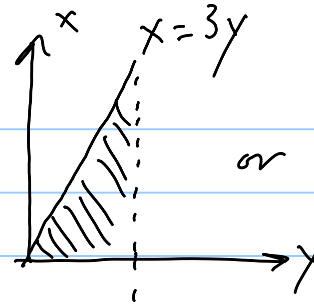
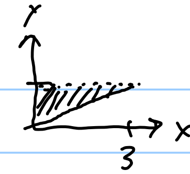
$$\int_0^1 \frac{8x^3}{3} - \frac{(x+x^2)^3}{3}\,dx = \int_0^1 \frac{8x^3 - (x^6 + 3x^5 + 3x^4 + x^3)}{3}\,dx$$

$$= \frac{1}{3}\int_0^1 -x^6 - 3x^5 - 3x^4 + 7x^3\,dx = \frac{1}{3}\left[-\frac{x^7}{7} - \frac{x^6}{2} - \frac{3}{5}x^5 + \frac{7x^4}{4}\right]_0^1$$

$$= \frac{1}{3}\left(-\frac{1}{7} - \frac{1}{2} - \frac{3}{5} + \frac{7}{4}\right) = \frac{1}{3}\left(\frac{-20 - 70 - 84 + 245}{140}\right)$$

$$= \frac{1}{3}\left(\frac{71}{140}\right) = \underline{\frac{71}{420}}$$

24.

x - simple region  or 

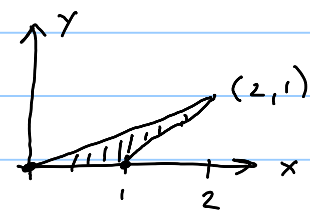
$$\int_0^1 \int_0^{3y} e^{x+y}\,dx\,dy = \int_0^1 e^{x+y}\Big|_{x=0}^{x=3y}\,dy$$

$$= \int_0^1 e^{4y} - e^{y}\,dy = \frac{1}{4}e^{4y} - e^{y}\Big|_0^1 =$$

$$\frac{1}{4}e^4 - e - \left(\frac{1}{4}e^0 - e^0\right) = \frac{1}{4}e^4 - e - \left(\frac{1}{4} - 1\right)$$

$$= \underline{\frac{e^4}{4} - e + \frac{3}{4}}$$

25.



It's easiest to make D an x-simple region: $x = y+1$ as the "top", and $x = 2y$ as the "bottom", for $0 \le y \le 1$.

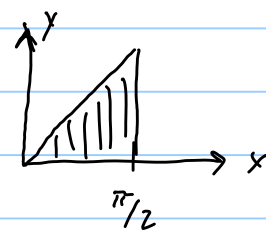
$$\therefore \int_0^1 \int_{y+1}^{2y} (x-y)\,dx\,dy = \int_0^1 \frac{x^2}{2} - xy\Big|_{x=y+1}^{x=2y}\,dy$$

$$= \int_0^1 \left(\frac{2y}{2}\right)^2 - 2y^2 - \left[\left(\frac{y+1}{2}\right)^2 - (y+1)y\right] dy$$

$$= \int_0^1 0 - \left[\frac{y^2+2y+1}{2} - y^2 - y\right] dy$$

$$= \int_0^1 \frac{-y^2+1}{2}\, dy = -\frac{y^3}{6} + \frac{y}{2}\Big|_0^1 = -\frac{1}{6} + \frac{1}{2} = \underline{\frac{1}{3}}$$

26.



$$\int_0^{\frac{\pi}{2}} \int_0^x (x^3 y + \cos x)\, dy\, dx$$

$$= \int_0^{\frac{\pi}{2}} \int_y^{\frac{\pi}{2}} (x^3 y + \cos x)\, dx\, dy$$

$$= \int_0^{\frac{\pi}{2}} \frac{x^4}{4} y + \sin x \Big|_{x=y}^{x=\frac{\pi}{2}} dy$$

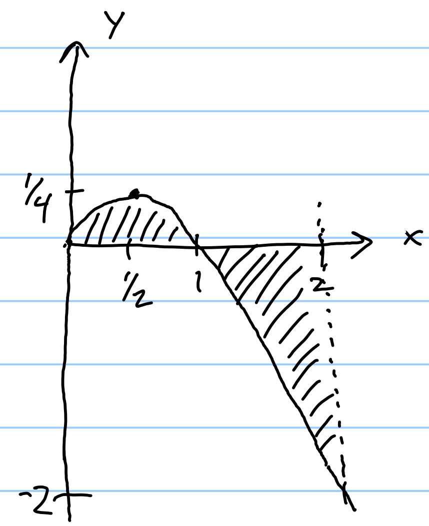
$$= \int_0^{\frac{\pi}{2}} \frac{\pi^4}{64} y + 1 - \frac{y^5}{4} - \sin y \quad dy$$

$$= \frac{\pi^4}{128} y^2 + y - \frac{y^6}{24} + \cos y \Big|_0^{\frac{\pi}{2}}$$

$$= \frac{\pi^6}{2^9} + \frac{\pi}{2} - \frac{\pi^6}{2^6 \cdot 2^3 \cdot 3} + 0 - (0+0-0+1)$$

$$= \underline{\frac{\pi^6}{256} + \frac{\pi}{2} - 1}$$

27.



$$-x^2+x = -(x^2-x) = -\left(x-\tfrac{1}{2}\right)^2 + \tfrac{1}{4}$$

Assume D is two regions: $0 \le x \le 1$ and $1 \le x \le 2$, as description in problem is imprecise.

$$\therefore \int_0^1 \int_0^{-x^2+x} (x^2+2xy^2+2)\,dy\,dx + \int_1^2 \int_{-x^2+x}^0 (x^2+2xy^2+2)\,dy\,dx$$

(1) $$\int_0^1 \int_0^{-x^2+x} (x^2+2xy^2+2)\,dy\,dx = \int_0^1 x^2y + \tfrac{2}{3}xy^3 + 2y \Big|_{y=0}^{y=-x^2+x} dx$$

$$= \int_0^1 x^2(-x^2+x) + \underbrace{\tfrac{2}{3}x(-x^2+x)^3}_{\frac{2}{3}(-x^7+3x^6-3x^5+x^4)} + 2(-x^2+x)\,dx$$

$$= \int_0^1 -x^4 + x^3 - 2x^2 + 2x - \frac{2}{3}x^7 + 2x^6 - 2x^5 + \frac{2}{3}x^4 \, dx$$

$$= -\frac{x^5}{3 \cdot 5} + \frac{x^4}{4} - \frac{2}{3}x^3 + x^2 - \frac{2}{24}x^8 + \frac{2}{7}x^7 - \frac{2}{6}x^6 \Big|_0^1$$

$$= -\frac{1}{15} + \frac{1}{4} - \frac{2}{3} + 1 - \frac{1}{12} + \frac{2}{7} - \frac{1}{3} \qquad 2^2 \cdot 3 \cdot 5 \cdot 7 = 420$$

$$= \frac{-28 + 105 - 280 + 420 - 35 + 120 - 140}{420}$$

$$= \frac{162}{420} = \frac{81}{210} = \underline{\frac{27}{70}}$$

(2) $$\int_1^2 \int_{-x^2+x}^0 (x^2 + 2xy^2 + 2)\, dy\, dx = \int_1^2 x^2 y + \frac{2}{3}xy^3 + 2y \Big|_{y=-x^2+x}^{y=0} dx$$

Now using same calculations as in (1), but using (-1)

$$= \int_1^2 x^4 - x^3 + 2x^2 - 2x + \frac{2}{3}x^7 - 2x^6 + 2x^5 - \frac{2}{3}x^4 \, dx$$

$$= \frac{x^5}{15} - \frac{x^4}{4} + \frac{2}{3}x^3 - x^2 + \frac{1}{12}x^8 - \frac{2}{7}x^7 + \frac{1}{3}x^6 \Big|_1^2$$

$$= \frac{32}{15} - \frac{16}{4} + \frac{2}{3}(8) - 4 + \frac{1}{12}(256) - \frac{2}{7}(128) + \frac{1}{3}(64) + \frac{27}{70}$$

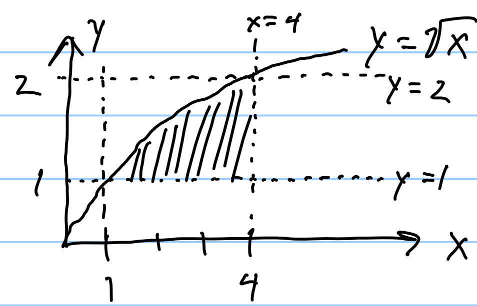
$$= \frac{32(28) - 4(420) + 16(140) - 4(420) + 256(35) - 256(60) + 64(140)}{420} + \frac{27}{70}$$

$$= \frac{2336}{420} + \frac{27}{70}$$

$$\therefore (1) + (2) = \left(\frac{2336}{420} + \frac{27}{70}\right) + \left(\frac{27}{70}\right) = \frac{2336 + 162 + 162}{420}$$

$$= \frac{2660}{420} = \frac{2^2 \cdot 5 \cdot 7 \cdot 19}{2^2 \cdot 3 \cdot 5 \cdot 7} = \frac{19}{3}$$

28.



$y=\sqrt{x}$ is increasing on $1 \le x \le 4$

$\sqrt{1} = 1$, $\sqrt{4} = 2$ inverse of $y = \sqrt{x}$ is $x = y^2$

$$\therefore \int_1^2 \int_{y^2}^4 (x^2 + y^2)\, dx\, dy = \int_1^2 \frac{x^3}{3} + xy^2 \Big|_{x=y^2}^{x=4} dy$$

$$= \int_1^2 \frac{64}{3} + 4y^2 - \left(\frac{y^6}{3} + y^4\right) dy = \frac{64}{3}y + \frac{4}{3}y^3 - \frac{y^7}{21} - \frac{y^5}{5}\Big|_1^2$$

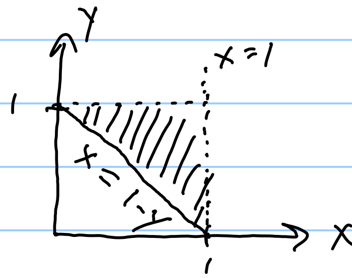
$$= \frac{128}{3} + \frac{32}{3} - \frac{128}{21} - \frac{32}{5} - \left(\frac{64}{3} + \frac{4}{3} - \frac{1}{21} - \frac{1}{5}\right)$$

$7 \cdot 3 \cdot 5 = 105$

$$= \frac{128(35) + 32(35) - 128(5) - 32(21) - 68(35) + 5 + 21}{105}$$

$$= \frac{1934}{105} = \frac{2 \cdot 967}{3 \cdot 5 \cdot 7} = \underline{\frac{1934}{105}}$$

29.



$x = f(y) = 1-y$ is a decreasing function, inverse is $y = 1-x$. $f(0)=1$, $f(1)=0$

$$\therefore \int_{f(1)}^{f(0)} \int_{1-x}^{1} \Rightarrow \int_0^1 \int_{1-x}^1 (x+y^2)\,dy\,dx$$

$$= \int_0^1 xy + \frac{y^3}{3} \Big|_{y=1-x}^{y=1} dx$$

$$= \int_0^1 x + \frac{1}{3} - \left[x(1-x) + \frac{(1-x)^3}{3} \right] dx$$

$-x^3 + 3x^2 - 3x + 1$

$$= \int_0^1 \frac{1}{3} + x^2 + \left(\frac{x^3}{3} - x^2 + x - \frac{1}{3} \right) dx$$

$$= \int_0^1 \frac{x^3}{3} + x\,dx = \frac{x^4}{12} + \frac{x^2}{2} \Big|_0^1 = \frac{1}{12} + \frac{1}{2} = \underline{\frac{7}{12}}$$

30.

The box $[1,3]\times[2,4]$ has area $(3-1)(4-2)=4$

$e^{x^2+y^2}$ has a minimum when x is smallest $(x=1)$ and y is smallest $(y=2)$ over the box.

$\therefore$ Minimum $= e^{1^2+2^2} = e^5$

Similarly, a max is obtained when x and y have maximum values: $x=3$, $y=4$.

$\therefore$ Max $= e^{3^2+4^2} = e^{25}$

$\therefore\ e^5 \le e^{x^2+y^2} \le e^{25}$ for all $(x,y)\in D = [1,3]\times[2,4]$

$$\therefore\quad 4e^5 \le \iint_{[1,3]\times[2,4]} e^{x^2+y^2}\, dA \le 4e^{25}$$

31.

A disk of radius 2 has area $\pi(2)^2 = 4\pi$

D is the region $[-2,2]\times[-2,2]$ s.t. $x^2+y^2\le 4$

$f(x,y) = x^2+y^2+1$ has a minimum at $(x,y)=(0,0)$

so that $f(0,0)=1$.

Over D, $f(x,y)$ has a maximum when $x^2+y^2=4$.

$\therefore f(x,y)_{max} = (x^2+y^2)_{max} + 1 = 4+1=5$

$\therefore \quad 1 \le f(x,y) \le 5$

$\therefore$

$$1\cdot(4\pi) = 4\pi \le \iint_D (x^2+y^2+1)\,dx\,dy \le 20\pi = 5\cdot(4\pi)$$

32.

Assume W is closed or compact, and $\therefore$ has a finite volume.

f continuous on compact $W \Rightarrow f$ achieves a minimum and maximum on W.

Let $\vec{x} \in W$ s.t. $f(\vec{x}) = m$ is a minimum.
$\vec{y} \in W$ s.t. $f(y) = M$ is a maximum.

Let $\vec{p}(t)$, $t \in R$, $\vec{p}(t) \in W$ be the continuous path connecting $\vec{x}$ and $\vec{y}$. Let $\vec{p}(a) = \vec{x}$, $\vec{p}(b) = \vec{y}$

By the mean value inequality,

$$m \cdot Vol(W) \le \iiint_W f\, dV \le M \cdot Vol(W)$$

$\therefore$

$$f(\vec{p}(a)) = f(\vec{x}) = m \le \frac{\iiint_W f\, dV}{Vol(W)} \le M = f(\vec{y}) = f(\vec{p}(b))$$

Since f and p are continuous, $f \circ p$ is continuous on $[a, b]$

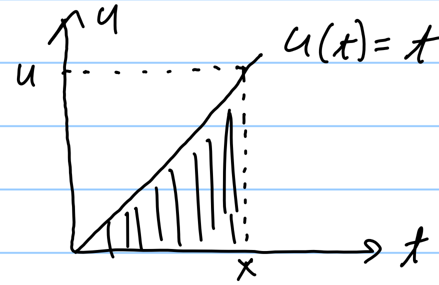
$\therefore$ By Intermediate Value Theorem, there is a $t_0 \in [a, b]$ s.t. $f(\vec{p}(t_0)) = \dfrac{\iiint_W f\, dV}{Vol(W)}$

Let $\vec{z} = \vec{p}(t_0)$. $\vec{p}$ a continuous path in W means $\vec{z} \in W$. $\therefore$ $f(\vec{z}) = \dfrac{\iiint_W f\, dV}{Vol(W)}$

If W is not connected, then can't make the assertion.

33.

Region D is $0 \le u \le t$ and $0 \le t \le x$



$u(t) = t$ is an increasing function, inverse is $t = u$ $u(0) = 0$, $u(x) = x$.

$$\therefore \int_0^x \int_0^t du\,dt \Rightarrow \int_{u(0)=0}^{u(x)=x} \int_u^x dt\,du = \int_0^x \int_u^x dt\,du$$

$\therefore$ Interchanging order of integration,

$$\int_0^x \int_0^t F(u)\,du\,dt = \int_0^x \int_u^x F(u)\,dt\,du$$

But $\displaystyle\int_u^x F(u)\,dt = F(u) \int_u^x dt$ as $F(u)$ acts like a constant relative to variable t.

and $\displaystyle\int_u^x dt = t \Big|_u^x = x - u$

$$\therefore \int_u^x F(u)\,dt = (x-u)\,F(u)$$

$$\therefore \int_0^x \int_0^t F(u)\,du\,dt = \int_0^x \int_u^x F(u)\,dt\,du = \int_0^x (x-u)\,F(u)\,du$$

34.

$$\int_0^1 \int_0^z \left[\int_0^y xy^2z^3\, dx \right] dy\, dz = \int_0^1 \int_0^z \frac{x^2}{2} y^2 z^3 \Big|_{x=0}^{x=y} dy\, dz$$

$$= \int_0^1 \int_0^z \frac{y^4 z^3}{2}\, dy\, dz = \int_0^1 \frac{y^5 z^3}{10} \Big|_{y=0}^{y=z} dz$$

$$= \int_0^1 \frac{z^8}{10}\, dz = \frac{z^9}{90} \Big|_0^1 = \underline{\frac{1}{90}}$$

35.

$$\int_0^{x/\sqrt{3}} \frac{x}{x^2+z^2}\, dz = x\left[\frac{1}{x} \arctan \frac{z}{x}\right]_{z=0}^{z=\frac{x}{\sqrt{3}}}, \quad x \neq 0$$

$$= \arctan\left(\frac{\sqrt{3}}{3}\right) = \frac{\pi}{6}$$

$$\int_0^y \frac{\pi}{6}\, dx = \frac{\pi}{6} x \Big|_0^y = \frac{\pi}{6} y$$

$$\int_0^1 \frac{\pi}{6} y\, dy = \frac{\pi}{6} \frac{y^2}{2} \Big|_0^1 = \underline{\frac{\pi}{12}}$$

Note: The first integral assumes $x \neq 0$,

but the second states $0 \leq |x| \leq y$

To justify, let $0 < |x| \leq y$

and take limit as $x \to 0$

$$\therefore \lim_{x \to 0} x \left[\frac{1}{x} \arctan \frac{z}{x} \right]_{z=0}^{z=\frac{x}{\sqrt{3}}}$$

$$= \lim_{x \to 0} \left(\arctan \frac{\sqrt{3}}{3} - \arctan(0) \right)$$

$$= \lim_{x \to 0} \frac{\pi}{6} = \frac{\pi}{6}$$

$\therefore$ Define integral as its limit

$$\int_0^{\frac{\sqrt{x}}{3}} \frac{x}{x^2 + z^2} \, dz = \frac{\pi}{6}$$

Note, if $x < 0$, $\displaystyle\int \frac{dz}{x^2 + z^2} = \frac{1}{-x} \arctan\left(\frac{z}{-x}\right) + C$

$$= \frac{1}{x} \arctan\left(\frac{z}{x}\right) + C$$

36.

$$\int_{\frac{1}{y}}^{2} yz^2\,dx = xyz^2\Big|_{x=\frac{1}{y}}^{x=2} = 2yz^2 - z^2$$

$$\int_{1}^{z} (2yz - z^2)\,dy = zy^2 - yz^2\Big|_{y=1}^{y=z} = z^3 - z^3 - (z - z^2)$$

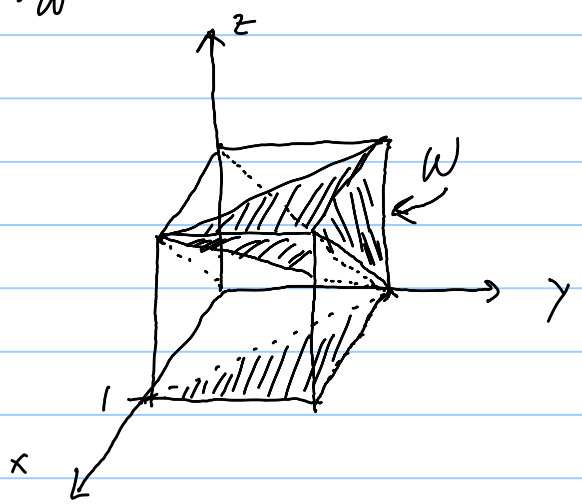
$$= z^2 - z$$

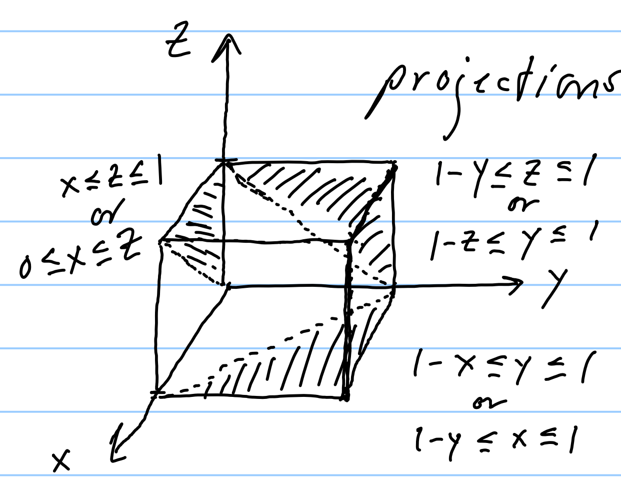
$$\therefore \int_{1}^{2} z^2 - z\,dz = \frac{z^3}{3} - \frac{z^2}{2}\Big|_{1}^{2} = \frac{8}{3} - 2 - \left(\frac{1}{3} - \frac{1}{2}\right)$$

$$= \frac{7}{3} - \frac{3}{2} = \frac{14-9}{6} = \underline{\frac{5}{6}}$$

37.

$$\iiint_W f(x,y,z)\,dV, \quad W \text{ is } 0 \le x \le 1,\ 1-x \le y \le 1,\ x \le z \le 1$$





(1) Consider the projection of the region onto the xy-plane. This is described as:

$0 \le x \le 1$ and $1-x \le y \le 1$

or

$0 \le y \le 1$ and $1-y \le x \le 1$

The z extent in both cases is $x \le z \le 1$

$$\therefore \int_0^1 \int_{1-x}^1 \int_x^1 f \, dz \, dy \, dx = \int_0^1 \int_{1-y}^1 \int_x^1 f \, dz \, dx \, dy$$

(2) Consider the projection of the region onto the yz-plane. This is described as:

$0 \le y \le 1$ and $1-y \le z \le 1$

or

$0 \le z \le 1$ and $1-z \le y \le 1$

Now consider $x = \phi_1(y,z)$ and $x = \phi_2(y,z)$.

The "bottom" of this region is a plane containing $(1,0,1)$, $(0,1,0)$, $(0,1,1)$

Using $Ax + By + Cz = D$, $(0,1,0) \Rightarrow B = D$

$\therefore Ax + By + Cz = B$ $\quad (0,1,1) \Rightarrow B + C = B \Rightarrow C = 0$

$\therefore Ax + By = B$ $\quad (1,0,1) \Rightarrow A = B \Rightarrow Bx + By = B$,

or $x + y = 1$, or $x = 1 - y = \phi_1(y,z)$

The "top" of This region is a plane containing $(1,0,1)$, $(1,1,1)$, $(0,1,0)$.

Using $Ax+By+Cz=D$, $(0,1,0) \Rightarrow B=D$

$\therefore Ax+By+Cz=B$ $\quad (1,0,1) \Rightarrow A+C=B$, $(1,1,1) \Rightarrow A+B+C=B$ $\; \therefore B=0$

$\therefore A+C=0$, $C=-A$. $\therefore Ax-Az=0$, or $x=z$

$\therefore x=z=\phi_2(y,z)$

$$\therefore \int_0^1 \int_{1-y}^1 \int_{1-y}^z f\,dx\,dz\,dy = \int_0^1 \int_{1-z}^1 \int_{1-y}^z f\,dx\,dy\,dz$$

(3) Consider The projection of the region onto the xz-plane. This is described as:

$0 \le x \le 1$ and $x \le z \le 1$

or

$0 \le z \le 1$ and $0 \le x \le z$

Now consider $y=\phi_1(x,z)$ and $y=\phi_2(x,z)$

The "top" is $y=1=\phi_2(x,z)$

The "bottom" plane contains $(0,1,1)$, $(0,1,0)$, $(1,0,1)$

This plane was described in (2) above as $x+y=1$, or $y=1-x=\phi_1(x,z)$

$$\therefore \int_0^1 \int_x^1 \int_{1-x}^1 f\,dy\,dz\,dx = \int_0^1 \int_0^z \int_{1-x}^1 f\,dy\,dx\,dz$$

Note: The answer in the back of the text

mistakenly has $\int_0^1 \int_z^1 \int_{1-x}^1 f \, dy \, dx \, dz$

The range of x is $0 \le x \le z$, not $z \le x \le 1$