6.1 The Geometry of Maps from R^2 to R^2

Note Title 11/22/2016

1.

(a)

$$T(x,y) = T(x',y') \Rightarrow (2x,y) = (2x',y')$$

$$\Rightarrow x = x',\ y = y' \Rightarrow (x,y) = (x',y')$$

$\therefore$ One-to-one

Let $(u,v) \in \mathbb{R}^2$. $\therefore (2x,y) = (u,v) \Rightarrow x = \frac{u}{2},\ y = v$

$\therefore T(x,y) = T(\frac{1}{2}u, v) = (2(\frac{1}{2}u), v) = (u,v)$.

$\therefore$ Onto

(b)

Not one-to-one since $T(x,y) = T(-x,y)$

Not onto: for $(-1,0) \in \mathbb{R}^2$, there is no $(x,y) \in \mathbb{R}^2$ s.t. $T(x,y) = (-1,0)$.

(c)

$$T(x,y) = T(x',y') \Rightarrow (\sqrt[3]{x}, \sqrt[3]{y}) = (\sqrt[3]{x'}, \sqrt[3]{y'})$$

$$\Rightarrow x = x',\ y = y'.$$

$\therefore$ One-to-one

Let $(u,v) \in \mathbb{R}^2$. $\therefore (\sqrt[3]{x}, \sqrt[3]{y}) = (u,v) \Rightarrow$

$x = u^3, \; y = v^3$

$\therefore T(x,y) = T(u^3, v^3) = (u,v)$. $\therefore$ Onto

(d)

Since $T(x,y) = T(x,-y)$ as $\cos(y) = \cos(-y)$,

Then not one-to-one

Not onto since there is no $(x,y) \in \mathbb{R}^2$

s.t. $T(x,y) = (a,b)$ for $|a|>1, |b|>1$

2.

Actually, $T: \mathbb{R}^3 \to \mathbb{R}^3$

(a)

If $T(x,y,z) = T(x',y',z')$ then

$$2x + y + 3z = 2x' + y' + 3z' \quad [1]$$

$$3y - 4z = 3y' - 4z' \quad [2]$$

$$5x = 5x' \quad [3]$$

$[3] \Rightarrow x = x'$ $\therefore [1]$ becomes

$$y + 3z = y' + 3z' \quad [4]$$

Multiply [2] by 3, [4] by 4:

$$9y - 12z = 9y' - 12z' \quad [3^*]$$

$$4y + 12z = 4y' + 12z' \quad [4^*]$$

Add to get: $13y = 13y' \Rightarrow y = y'$

$\therefore$ [2] becomes $-4z = -4z' \Rightarrow z = z'$

$\therefore (x, y, z) = (x', y', z')$

$\therefore$ <u>One-to-one</u>

Let $(u, v, w) \in \mathbb{R}^3$

$$\therefore \begin{bmatrix} 2 & 1 & 3 \\ 0 & 3 & -4 \\ 5 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

Let $A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 3 & -4 \\ 5 & 0 & 0 \end{bmatrix}$ Since $\det A = 5(-4-9) \neq 0$, then A^{-1} exists.

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1} \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

$\therefore$ T is <u>onto</u>

(b)

Since $T(x,y,z) = T(-x,-y,z)$, not one-to-one

Consider $(u,v,0)$ where $u \neq 0$.

There is no (x,y,z) s.t. $(y\sin x, z\cos y, xy) = (u,v,0)$

for $xy=0 \Rightarrow x$ or y must be 0.

$\therefore y\sin x = 0$.

$\therefore$ Not onto

(c)

Since $T(1,1,1) = T(-1,-1,-1)$, not one-to-one

Consider $(1,1,-1) \in \mathbb{R}^3$

Is there an (x,y,z) s.t. $(xy, yz, xz) = (1,1,-1)$?

$xz = -1 \Rightarrow x<0$ and $z>0$ or $x>0$ and $z<0$

(1) Suppose $x<0$ and $z>0$

$xy = 1 \Rightarrow y<0$.

$\therefore yz<0$ so $yz \neq 1$

(2) Suppose $x > 0$ and $z < 0$

$\therefore xy = 1 \Rightarrow y > 0$

$\therefore yz < 0$ so $yz \neq 1$

$\therefore$ There is no (x,y,z) s.t. $T(x,y,z) = (1,1,-1)$.

$\therefore$ <u>Not onto</u>

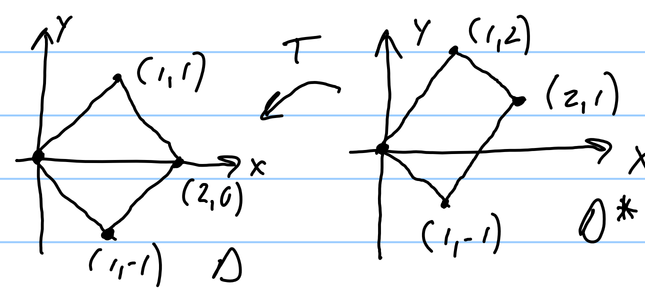
(d)

Since $e^u = e^v \Rightarrow u = v$, Then T <u>is one-to-one</u>

Since $e^x > 0$ for all x, there is no (x,y,z)

s.t. $T(x,y,z) = (-1,-1,-1)$. $\therefore$ <u>Not onto</u>

3.



Let $T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ Assign vertices in D^* to D and try to solve, using Theorem 1, p. 310 of text.

$$\therefore \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \begin{aligned} a + 2b &= 1 \quad [1] \\ c + 2d &= 1 \quad [2] \end{aligned}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{ll} 2a + b = 2 & [3] \\ 2c + d = 0 & [4] \end{array}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow \begin{array}{ll} a - b = 1 & [5] \\ c - d = -1 & [6] \end{array}$$

[1], [3], [5]: $$\begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad \begin{array}{l} [1] \\ [3] \\ [5] \end{array}$$

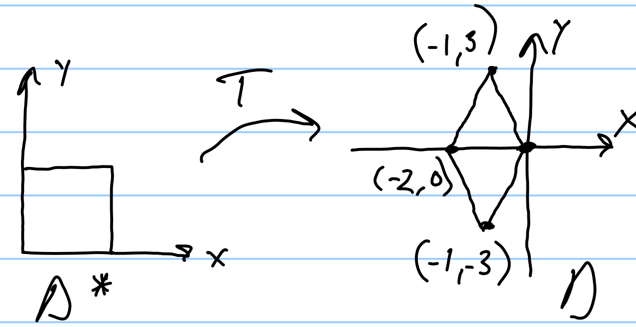
$$\Rightarrow \begin{bmatrix} 1 & 2 \\ 0 & -3 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow a = 1,\ b = 0$$

[2], [4], [6] $$\begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad \begin{array}{l} [2] \\ [4] \\ [6] \end{array}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 \\ 0 & -3 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix} \Rightarrow d = \frac{2}{3},\ c = -\frac{1}{3}$$

$$\therefore\ T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

4.



Use, Theorem 1, p. 310 of text.

You always want $(0,0) \to (0,0)$ for linear maps.

Here, imagine a counterclockwise rotation so that $(0,1) \to (-1,-3)$, $(1,1) \to (-2,0)$, $(1,0) \to (-1,3)$

$\therefore$ Let $T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\therefore \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -2 & -1 \\ -3 & 0 & 3 \end{bmatrix}$$

Looking at the transpose of each side,

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} -1 & -3 \\ -2 & 0 \\ -1 & 3 \end{bmatrix}$$ add $(-1)(\text{row } 3)$ to row 2

$$\therefore \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} -1 & -3 \\ -1 & -3 \\ -1 & 3 \end{bmatrix} \Rightarrow b = -1,\ d = -3,\ a = -1,\ c = 3$$

$\therefore$ $\underline{T = \begin{bmatrix} -1 & -1 \\ -3 & 3 \end{bmatrix}}$, or $\underline{T(x,y) = (-x-y, -3x+3y)}$

5.

$T(r,\theta) = (x,y)$. $x^2+y^2 = r^2\cos^2\theta + r^2\sin^2\theta = 1$, a circle

$\therefore$ T appears to map to a disc of radius 1, minus $(0,0)$

as there is no (r,θ) s.t. $T(r,\theta)=(0,0)$.

$T(r,\theta) = T(r',\theta') \Rightarrow (r\cos\theta, r\sin\theta) = (r'\cos\theta', r'\sin\theta')$

(a) For $0 \leq \theta < \frac{\pi}{2}$: $\sin\theta$ and $\cos\theta$ are one-to-one.

$\therefore \sin\theta = \sin\theta' \Rightarrow \theta = \theta'$

$\cos\theta = \cos\theta' \Rightarrow \theta = \theta'$

$\therefore (r\cos\theta, r\sin\theta) = (r'\cos\theta', r'\sin\theta')$

$= (r'\cos\theta, r'\sin\theta)$

$\Rightarrow r' = r$ since $\cos\theta \neq 0$ on $0 \leq \theta < \frac{\pi}{2}$

$\therefore T(r,\theta) = T(r',\theta') \Rightarrow (r,\theta) = (r',\theta')$

(b) For $\theta = \frac{\pi}{2}$: $T(r, \frac{\pi}{2}) = (0, r)$

$\therefore T(r, \frac{\pi}{2}) = T(r', \frac{\pi}{2}) \Rightarrow (0,r) = (0,r') \Rightarrow r = r'$

(c) For $\frac{\pi}{2} < \theta \leq \pi$: as in (a), $\sin\theta$ and $\cos\theta$ are one-to-one on this interval.

$\therefore (r\cos\theta, r\sin\theta) = (r'\cos\theta', r'\sin\theta')$

$= (r'\cos\theta, r'\sin\theta)$

$\Rightarrow r' = r$ since $\cos\theta \neq 0$ on $\pi/2 < \theta \leq \pi$

(d) For $\pi < \theta \leq \frac{3}{2}\pi$: as in (a), $\sin\theta$ and $\cos\theta$ are one-to-one on this interval.

$\therefore (r\cos\theta, r\sin\theta) = (r'\cos\theta', r'\sin\theta')$

$= (r'\cos\theta, r'\sin\theta)$

$\Rightarrow r' = r$ since $\sin\theta \neq 0$ on this interval.

(e) $\frac{3}{2}\pi < \theta < 2\pi$: as in (a), $\sin\theta$ and $\cos\theta$ are one-to-one on this interval.

$$\therefore (r\cos\theta, r\sin\theta) = (r'\cos\theta', r'\sin\theta')$$
$$= (r'\cos\theta, r'\sin\theta)$$
$\Rightarrow r' = r$ since $\cos\theta \neq 0$ on this interval.

$\therefore$ (a) $\to$ (e) $\Rightarrow$ when $T(r,\theta) = T(r',\theta')$ for $0 < r \leq 1$, $0 \leq \theta < 2\pi$, then $(r,\theta) = (r',\theta')$

$\therefore T$ is one-to-one on S^*

6.

To show rotation, must show every point (x^*, y^*) of the original unit square maps to the same angle θ from the original point, and is at the same distance from the origin as the original point.

(1) The origin of this rotation is $(0,0)$ as

$$T(0,0) = \left(\frac{0-0}{\sqrt{2}}, \frac{0+0}{\sqrt{2}}\right) = (0,0).$$

(2) The distance from the origin after the mapping is:

$$\sqrt{\left(\frac{x^*-y^*}{\sqrt{2}}\right)^2+\left(\frac{x^*+y^*}{\sqrt{2}}\right)^2}=\sqrt{\frac{(x^*)^2-2x^*y^*+(y^*)^2}{2}+\frac{(x^*)^2+2x^*y^*+(y^*)^2}{2}}$$

$$=\sqrt{\frac{2(x^*)^2+2(y^*)^2}{2}}=\sqrt{(x^*)^2+(y^*)^2}$$

$\therefore$ Distance from the origin is preserved.

(3) Angle between (x^*, y^*) and $T(x^*, y^*)$:

Use dot product:

$$(x^*, y^*)\cdot\left(\frac{x^*-y^*}{\sqrt{2}}, \frac{x^*+y^*}{\sqrt{2}}\right)=\|(x^*,y^*)\|\,\|T(x^*,y^*)\|\cos\theta$$

$$\frac{(x^*)^2-x^*y^*}{\sqrt{2}}+\frac{x^*y^*+(y^*)^2}{\sqrt{2}}=\|(x^*,y^*)\|^2\cos\theta \quad [1]$$

Since, by (2) above, $\|(x^*,y^*)\|=\|T(x^*,y^*)\|=\sqrt{(x^*)^2+(y^*)^2}$

$\therefore$ [1] becomes:

$$\frac{(x^*)^2+(y^*)^2}{\sqrt{2}}=\left(\sqrt{(x^*)^2+(y^*)^2}\right)^2\cos\theta, \text{ or}$$

$$\frac{1}{\sqrt{2}}=\cos\theta, \quad \theta=\frac{\pi}{4}$$

$\therefore$ Every point (x^*, y^*) rotates $\frac{\pi}{4}$ radians counterclockwise under T.

7.

$-u^2+4u$ is a parabola, concave downward.

But on the interval $[0,1]$, it is an increasing function: $f(u) = -u^2+4u$, $f'(u) = -2u+4$

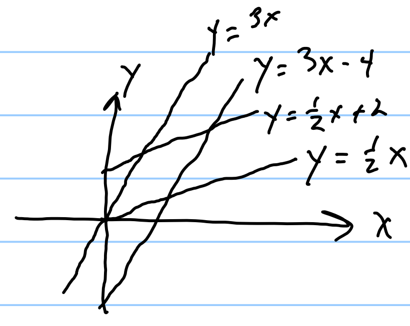
$\therefore f'(u) > 0$ for $0 \le u \le 1$. $\therefore f(1) = 3$ is the max, and $0 \le -u^2+4u \le 3$.

$\therefore D = [0,3] \times [0,1]$

$\therefore$ The image is a single-sheet parabolic cylinder

The image is <u>one-to-one</u>.

8.



Use Theorem 1, p. 310 of text

Find the corners of D^*

$y = \frac{1}{2}x$ and $y = 3x-4$: $3x - 4 = \frac{1}{2}x$, $5x = 8$, $x = \frac{8}{5}$

$\therefore y = \frac{1}{2}\left(\frac{8}{5}\right) = \frac{4}{5}$.

$\therefore \left(\frac{8}{5}, \frac{4}{5}\right)$

$y = 3x - 4$ and $y = \frac{1}{2}x + 2$: $3x - 4 = \frac{1}{2}x + 2$, $5x = 12$,

$\therefore x = \frac{12}{5}$ $\quad y = \frac{1}{2}\left(\frac{12}{5}\right) + 2 = \frac{16}{5}$

$\therefore \underline{\left(\frac{12}{5}, \frac{16}{5}\right)}$

$y = \frac{1}{2}x + 2$ and $y = 3x$: $3x = \frac{1}{2}x + 2$, $5x = 4$, $x = \frac{4}{5}$

$\therefore y = \frac{1}{2}\left(\frac{4}{5}\right) + 2 = \frac{12}{5}$

$\therefore \underline{\left(\frac{4}{5}, \frac{12}{5}\right)}$

$\therefore (0,0), \left(\frac{8}{5}, \frac{4}{5}\right), \left(\frac{12}{5}, \frac{16}{5}\right), \left(\frac{4}{5}, \frac{12}{5}\right)$

$$\therefore \begin{array}{cccc} (0,0) & \left(\frac{8}{5}, \frac{4}{5}\right) & \left(\frac{12}{5}, \frac{16}{5}\right) & \left(\frac{4}{5}, \frac{12}{5}\right) \\ \downarrow & \downarrow & \downarrow & \downarrow \\ (0,0) & (1,0) & (1,1) & (0,1) \end{array}$$

$\therefore$ Let $T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. $\therefore \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 8/5 & \frac{12}{5} & \frac{4}{5} \\ 0 & 4/5 & 16/5 & 12/5 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

Taking transposes,

$$\begin{bmatrix} 0 & 0 \\ 8/5 & 4/5 \\ 12/5 & 16/5 \\ 4/5 & 12/5 \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\Longleftrightarrow \begin{bmatrix} 0 & 0 \\ 8 & 4 \\ 12 & 16 \\ 4 & 12 \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 5 & 0 \\ 5 & 5 \\ 0 & 5 \end{bmatrix}$$

$$\Longleftrightarrow \begin{bmatrix} 0 & 0 \\ 0 & -20 \\ 0 & -20 \\ 4 & 12 \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 5 & -10 \\ 5 & -10 \\ 0 & 5 \end{bmatrix}$$

$$\therefore\ 0(a) - 20(b) = 5 \Rightarrow \underline{b = -\tfrac{1}{4}}$$

$$4(a) + 12(b) = 0 \Rightarrow 4a + 12(-\tfrac{1}{4}) = 0 \Rightarrow \underline{a = \tfrac{3}{4}}$$

$$0(c) - 20(d) = -10 \Rightarrow \underline{d = \tfrac{1}{2}}$$

$$4(c) + 12(d) = 5 \Rightarrow 4c + 12(\tfrac{1}{2}) = 5,\ \underline{c = -\tfrac{1}{4}}$$

$$\therefore\ T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \underline{\begin{bmatrix} 3/4 & -1/4 \\ -1/4 & \tfrac{1}{2} \end{bmatrix}}$$

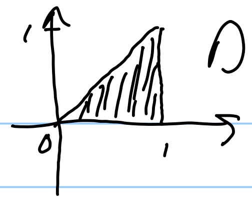
or $T(x^*, y^*) = \left(\underline{\tfrac{3}{4}x^* - \tfrac{1}{4}y^*,\ -\tfrac{1}{4}x^* + \tfrac{1}{2}y^*}\right)$

9.

(a) x^*y^* varies from 0 to 1, so $x^*y^* \in [0,1]$ and clearly $x^* \in [0,1]$

$\therefore\ D = [0,1] \times [0,1]$

Note $x^*y^* \leq x^*$ $\therefore$ The first coordinate is always "underneath" the second coordinate.

$\therefore$ 

D is bounded by the triangle whose corners are $(0,0)$, $(1,0)$, $(1,1)$, and contains all points inside the triangle.

(b) Not one-to-one as $T(0,1) = T(0,0) = (0,0)$ and $(0,1) \neq (0,0)$

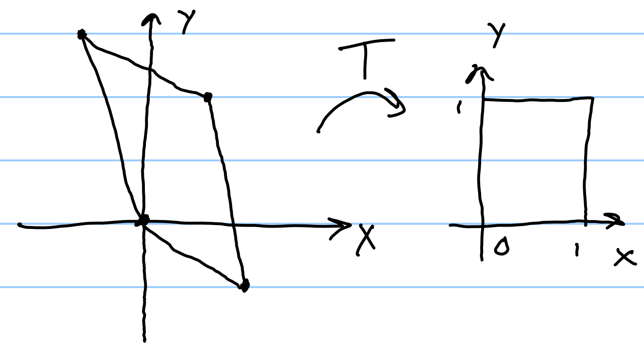
(c) If $\underline{D^* = (0,1] \times [0,1]}$ Then $T(x,y) = T(a,b)$

$\Rightarrow (xy, x) = (ab, a) \Rightarrow x = a \Rightarrow xy = xb$

$\Rightarrow y = b$ as $x \neq 0$, so $\Rightarrow (x,y) = (a,b)$

and $\therefore$ one-to-one.

10.



Let $T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and make points

$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\begin{bmatrix} -1 \\ 3 \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 2 \\ -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\therefore \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & -1 & 1 & 2 \\ 0 & 3 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

taking transposes,

$$\begin{bmatrix} 0 & 0 \\ -1 & 3 \\ 1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}$$

add row (3) to row (2)
subtract 2 times row (3) from row (4)

$$\Rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 5 \\ 1 & 2 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 2 \\ 1 & 1 \\ -1 & -2 \end{bmatrix}$$

$\therefore\ 5b = 1,\ b = 1/5$

$5d = 2,\ d = 2/5$

$a + 2b = 1,$

$a + 2(\frac{1}{5}) = 1,\ a = \frac{3}{5}$

$c + 2d = 1,$

$c + 2(\frac{2}{5}) = 1,\ c = \frac{1}{5}$

$$\therefore\ T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 3/5 & 1/5 \\ 1/5 & 2/5 \end{bmatrix}$$

11.

(a) D is the unit sphere centered at $(0,0,0)$.

(b) T is <u>not</u> one-to-one as:

(1) For $\theta \in [0, 2\pi]$, $\cos(0) = \cos(2\pi)$

$\therefore\ (\rho, \phi, 0)$ is the same point as $(\rho, \phi, 2\pi)$

so $T(\rho,\phi,0)=T(\rho,\phi,2\pi)$

(2) For $\rho=0$, $T(0,\phi,\theta)=T(0,\phi^*,\theta^*)$ for $\phi\neq\phi^*$, $\theta\neq\theta^*$.

(3) For $\phi=0$ or $\phi=\pi$, $\sin\phi=0$, so

$T(\rho,0,\theta)=T(\rho,0,\theta^*)$ for $\theta\neq\theta^*$

and $T(\rho,\pi,\theta)=T(\rho,\pi,\theta^*)$ for $\theta\neq\theta^*$

(c) From (b), restrict D^* to

$\rho\in(0,1]$, $\theta\in[0,2\pi)$, $\phi\in(0,\pi)$

12.

(1) Suppose $T(\vec{x})=T(\vec{y})$ $\therefore$ $A\vec{x}=A\vec{y}$

Since $\det A\neq 0$, A^{-1} exists. $\therefore$ $A^{-1}A\vec{x}=A^{-1}A\vec{y}$

$\Rightarrow I\vec{x}=I\vec{y} \Rightarrow \vec{x}=\vec{y}$, where I = identity matrix

$\therefore$ $\det A\neq 0 \Rightarrow T$ is one-to-one

(2) Suppose $T(\vec{x})=T(\vec{y}) \Rightarrow \vec{x}=\vec{y}$ (i.e., T one-to-one)

$\therefore A\vec{x} = A\vec{y} \Rightarrow \vec{x} = \vec{y}$

$\therefore A\vec{x} - A\vec{y} = \vec{0} \Rightarrow \vec{x} - \vec{y} = \vec{0}$

or $A(\vec{x} - \vec{y}) = \vec{0} \Rightarrow \vec{x} - \vec{y} = \vec{0}$

Let $\vec{z} = \vec{x} - \vec{y}$. $\therefore A\vec{z} = \vec{0} \Rightarrow \vec{z} = \vec{0}$.

i.e., columns of A are independent

$\Rightarrow$ columns form a basis for R^2 since A is square

$\Rightarrow A\vec{x}_1 = \vec{e}_1$ and $A\vec{x}_2 = \vec{e}_2$ is solvable

$\Rightarrow A[\vec{x}_1\ \vec{x}_2] = I$, so $[\vec{x}_1\ \vec{x}_2] = A^{-1}$

$\Rightarrow \det A \neq 0$

13.

(1) if $\det A \neq 0$, then A^{-1} exists.

$\therefore$ For any $\vec{y} \in R^2$, let $\vec{x} = A^{-1}\vec{y}$

$\therefore A\vec{x} = AA^{-1}\vec{y} = I\vec{y} = \vec{y}$

$\therefore$ for any $\vec{y} \in R^2$, $\exists$ an $\vec{x} \in R^2$ s.t. $T\vec{x} = \vec{y}$. $\therefore$ <u>onto</u>

(2) Suppose T is onto.

$\therefore$ for every $\vec{y} \in \mathbb{R}^2$, $\exists\, \vec{x} \in \mathbb{R}^2$ s.t. $A\vec{x} = \vec{y}$

Let $\vec{y} = \vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{z} = \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\therefore \exists\, \vec{x}_1$ s.t. $A\vec{x}_1 = \vec{e}_1$ and $A\vec{x}_2 = \vec{e}_2$

$\therefore A[\vec{x}_1 \ \vec{x}_2] = [\vec{e}_1 \ \vec{e}_2] = I$

$\therefore [\vec{x}_1 \ \vec{x}_2] = A^{-1}$

$\therefore AA^{-1} = I$, $\det(AA^{-1}) = \det(I) = 1$

$\therefore \det(A)\det(A^{-1}) = 1$, so $\underline{\det A \neq 0}$

14.

(1) Let $P = \{\vec{q} : \vec{q} = \vec{p} + \lambda\vec{v} + \mu\vec{w},\ \lambda, \mu \in [0,1],$
$\vec{p}, \vec{v}, \vec{w} \in \mathbb{R}^2$, and $\vec{v}, \vec{w}$ not sclar multiples$\}$

Since T is linear, then

$$T(\vec{q}) = T(\vec{p} + \lambda\vec{v} + \mu\vec{w}) = T(\vec{p}) + \lambda T(\vec{v}) + \mu T(\vec{w})$$

Let $\vec{p}^* = T(\vec{p}) = A\vec{p}$, $\vec{v}^* = T(\vec{v}) = A\vec{v}$, $\vec{w}^* = T(\vec{w}) = A\vec{w}$

$\therefore \vec{p}^*, \vec{v}^*, \vec{w}^* \in \mathbb{R}^2$

$\therefore T(\vec{q}) = \vec{p}^* + \lambda \vec{v}^* + \mu \vec{w}^*$

Must now show $\vec{v}^*$ and $\vec{w}^*$ are not sclar multiples.

Suppose $\vec{v}^* = c\vec{w}^*$, $c \in \mathbb{R}$, $c \neq 0$.

Since $\det A \neq 0$, A^{-1} exists, so

$\vec{v} = A^{-1}\vec{v}^* = A^{-1}(c\vec{w}^*) = cA^{-1}\vec{w}^* = c\vec{w}$,

But $\vec{v}, \vec{w}$ are not scalar multiples,

so there is no $c \in \mathbb{R}$ s.t. $\vec{v}^* = c\vec{w}^*$

$\therefore \vec{p}^* + \lambda \vec{v}^* + \mu \vec{w}^*$ is a parallelogram

$\therefore T$ maps parallelograms to parallelograms.

(2) Suppose $\vec{q}$ is a parallelogram given by:

$\vec{q} = \vec{p} + \lambda \vec{v} + \mu \vec{w}$, $\vec{v}, \vec{w}$ not scalar multiples.

Since $\det A \neq 0$, then A^{-1} exists (and T is onto).

Let $\vec{q}^* = A^{-1}(\vec{p} + \lambda \vec{v} + \mu \vec{w})$

$= A^{-1}\vec{p} + \lambda A^{-1}\vec{v} + \mu A^{-1}\vec{w}$

Let $\vec{p}^* = A^{-1}\vec{p}$, $\vec{v}^* = A^{-1}\vec{v}$, $\vec{w}^* = A^{-1}\vec{w}$

$\therefore \vec{q}^* = \vec{p}^* + \lambda \vec{v}^* + \mu \vec{w}^*$

and $T(\vec{q}^*) = \vec{q}$

As in (1), $\vec{v}^*$ and $\vec{w}^*$ can't be scalar multiples.

$\therefore$ given any parallelogram $\vec{q} \in \mathbb{R}^2$, there is a parallelogram $\vec{q}^* \in \mathbb{R}^2$ s.t. $T(\vec{q}^*) = \vec{q}$.

$\therefore$ (1) + (2) $\Rightarrow$ T maps parallelograms <u>onto</u> parallelograms.

15.

(a)

(1) Suppose $\det A \neq 0$. If $T(\vec{x}) = T(\vec{y})$, then

$A\vec{x} + \vec{v} = A\vec{y} + \vec{v} \Rightarrow A\vec{x} = A\vec{y}$.

$\det A \neq 0 \Rightarrow A^{-1}$ exists. $\therefore A^{-1}A\vec{x} = A^{-1}A\vec{y} \Rightarrow$

$\vec{x} = \vec{y}$. $\therefore$ T is one-to-one.

(2) Suppose T is one-to-one.

$\therefore T(\vec{x}) = T(\vec{y}) \Rightarrow \vec{x} = \vec{y}$.

$\therefore A\vec{x} + \vec{v} = A\vec{y} + \vec{v} \Rightarrow \vec{x} = \vec{y}$,

or $A\vec{x} = A\vec{y} \Rightarrow \vec{x} = \vec{y}$, or

$A(\vec{x} - \vec{y}) = \vec{0} \Rightarrow \vec{x} - \vec{y} = \vec{0} \Rightarrow$

columns of A are independent $\Rightarrow$

A, which is 2×2 square, has an inverse

$\therefore AA^{-1} = I$, $\det(A)\det(A^{-1}) = \det(I)$

$\therefore \det(A)\det(A^{-1}) = 1 \Rightarrow \det(A) \neq 0$.

(b)

(1) Assume $\det A \neq 0$. $\therefore A^{-1}$ exists.

Let $\vec{y} \in R^2$, and $\vec{v}$ be the fixed vector.

Need to find $\vec{x} \in R^2$ s.t. $T(\vec{x}) = \vec{y}$

But $T(\vec{x}) = A\vec{x} + \vec{v}$.

$\therefore$ Let $\vec{x} = A^{-1}(\vec{y} - \vec{v})$

$\therefore T(\vec{x}) = A\left(A^{-1}(\vec{y}-\vec{v})\right) + \vec{v}$

$= (\vec{y}-\vec{v}) + \vec{v} = \vec{y}.$

$\therefore$ T is onto

(2) Assume T is onto. $\therefore$ Given any $\vec{y} \in R^2$, $\exists\, \vec{x} \in R^2$ s.t. $T(\vec{x}) = \vec{y}$, or $A\vec{x} + \vec{v} = \vec{y}$, where $\vec{v}$ is the fixed vector for T.

$\therefore$ Let $\vec{y} = \vec{e_1} + \vec{v}$, $\vec{e_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\therefore \exists\, \vec{x_1}$ s.t. $A\vec{x_1} + \vec{v} = \vec{e_1} + \vec{v}$, or

$A\vec{x_1} = \vec{e_1}$

Similarly, for $\vec{y} = \vec{e_2} + \vec{v}$, $\vec{e_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$,

$\exists\, \vec{x_2}$ s.t. $A\vec{x_2} = \vec{e_2}$

$\therefore A[\vec{x_1} \ \ \vec{x_2}] = [\vec{e_1} \ \ \vec{e_2}]$, or

$A[\vec{x_1} \ \ \vec{x_2}] = I.$

$\therefore [\vec{x_1} \ \ \vec{x_2}] = A^{-1}$

$\therefore T$ onto $\Rightarrow A^{-1}$ exists $\Rightarrow \det(A) \neq 0$.

(c)

(1) Suppose $\det A \neq 0$. $\therefore A^{-1}$ exists.

Let $\vec{q} = \vec{a} + \lambda \vec{b} + \mu \vec{c}$, $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^2$,

$\lambda, \mu \in [0,1]$, $\vec{b}, \vec{c}$ not scalar multiples.

$$\therefore T(\vec{q}) = T(\vec{a} + \lambda \vec{b} + \mu \vec{c})$$

$$= A(\vec{a} + \lambda \vec{b} + \mu \vec{c}) + \vec{v}$$

$$= A\vec{a} + \lambda A\vec{b} + \mu A\vec{c} + \vec{v}$$

$$= (A\vec{a} + \vec{v}) + \lambda A\vec{b} + \mu A\vec{c}$$

Let $\vec{a}^* = A\vec{a} + \vec{v}$, $b^* = A\vec{b}$, $\vec{c}^* = A\vec{c}$

$\therefore T(\vec{q}) = \vec{a}^* + \lambda \vec{b}^* + \mu \vec{c}^*$, which is the form for a parallelogram.

Must show $\vec{b}^*, \vec{c}^*$ are not scalar multiples.

Suppose they are scalar multiples,

i.e., $K \in R$ s.t. $\vec{b}^* = K\vec{c}^*$.

$\vec{b} = A^{-1}\vec{b}^* = K A^{-1}\vec{c}^* = K\vec{c}$,

a contradiction since there is no

K s.t. $\vec{b} = K\vec{c}$.

$\therefore$ $\det A \neq 0 \Rightarrow T$ maps parallelograms to parallelograms.

(2) Suppose $\vec{q}^*$ is a parallelogram.

$\therefore$ $\vec{q}^* = \vec{a}^* + \lambda\vec{b}^* + \mu\vec{c}^*$, $\vec{a}^*, \vec{b}^*, \vec{c}^* \in R^2$,

$\lambda, \mu \in [0,1]$, $\vec{b}^*, \vec{c}$ not scalar multiples.

To prove T is onto, need to find a parallelogram $\vec{q}$ s.t. $T(\vec{q}) = \vec{q}^*$,

or $A\vec{q} + \vec{v} = \vec{q}^* = \vec{a}^* + \lambda\vec{b}^* + \mu\vec{c}^*$,

or $A\vec{q} = (\vec{a}^* - \vec{v}) + \lambda\vec{b}^* + \mu\vec{c}^*$

$\therefore$ Let $\vec{q} = A^{-1}(\vec{a}^* - \vec{v}) + \lambda A^{-1}\vec{b}^* + \mu A^{-1}\vec{c}^*$

This is the form of a parallelogram

And $A^{-1}\vec{b}^*$ and $A^{-1}\vec{c}^*$ are not scalar multiples, for if they were, then $A^{-1}\vec{b}^* = k A^{-1}\vec{c}^*$, some $k \in \mathbb{R}$

$$\Rightarrow AA^{-1}\vec{b}^* = kAA^{-1}\vec{c}^*$$

$$\Rightarrow \vec{b} = k\vec{c}^*, \text{ a contradiction}$$

$\therefore$ (1) & (2) $\Rightarrow$ affine T maps parallelograms <u>onto</u> parallelograms.

16.

This was done in 14 (2)

17.

Assume $T(\vec{x}) = A\vec{x}$. For T to not be one-to-one, $\det(A) = 0$ by #12.

$\therefore$ Let $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$. $\therefore \det A = 0$

Let $\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$\therefore A\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $A\vec{y} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\therefore A\vec{x} = A\vec{y}$ but $\vec{x} \neq \vec{y}$,

$\therefore T$ is not one-to-one.

6.2 The Change of Variables Theorem

Note Title 12/14/2016

1.

(a) Make x-y one variable, and 3x+2y the other

$$\therefore \begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}, \quad T = \begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix}$$

$$\therefore T^{-1} = \frac{1}{-3-2}\begin{bmatrix} -1 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1/5 & 2/5 \\ 1/5 & -3/5 \end{bmatrix}$$

$$\begin{bmatrix} 1/5 & 2/5 \\ 1/5 & -3/5 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}, \text{ or } \quad \begin{aligned} x &= \tfrac{1}{5}(u+2v) \\ y &= 1/5(u-3v) \end{aligned}$$

$$|\det(T^{-1})| = \underline{\tfrac{1}{5} = \text{Jacobian}}$$

$$\therefore \iint_{R^*} \underline{u\sin(v)\left(\tfrac{1}{5}\right)} \, du\, dv$$

(b) Make $(-4x+7y) = u$, $(7x-2y) = v$

$$\therefore \begin{bmatrix} -4 & 7 \\ 7 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}, \quad T = \begin{bmatrix} -4 & 7 \\ 7 & -2 \end{bmatrix}$$

$$\therefore T^{-1} = -\frac{1}{41}\begin{bmatrix} -2 & -7 \\ -7 & -4 \end{bmatrix} \qquad \det(T^{-1}) = -\frac{1}{41}$$

$\therefore \quad x = \frac{1}{41}(2u + 7v) \quad \underline{\text{Jacobian}} = \left|-\frac{1}{41}\right| = \underline{\frac{1}{41}}$

$\quad\; y = \frac{1}{41}(7u + 4v)$

$$\therefore \iint_{R^*} e^u \cos(v)\left(\frac{1}{41}\right) du\,dv$$

2.

(a) Let $u = 5x + y \quad v = x + 9y$

$\therefore \begin{bmatrix} 5 & 1 \\ 1 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}, \quad T = \begin{bmatrix} 5 & 1 \\ 1 & 9 \end{bmatrix} \quad \det(T) = 46$

$\therefore T^{-1} = \frac{1}{46}\begin{bmatrix} 9 & -1 \\ -1 & 5 \end{bmatrix} \quad \det(T^{-1}) = \frac{1}{46}$

$\therefore \quad x = \frac{1}{46}(9u - v) \quad y = \frac{1}{46}(-u + 5v)$

$\text{Jacobian} = |\det(T^{-1})| = \frac{1}{46}$

$$\iint_{R^*} u^3 v^4 \left(\frac{1}{46}\right) du\,dv$$

(b) $\iint_R x\sin(6x+7y) - 3y\sin(6x+7y)\,dA =$

$\iint_R (x-3y)\sin(6x+7y)\,dA$

$\therefore$ Let $u = x-3y$, $v = 6x+7y$, $T = \begin{bmatrix} 1 & -3 \\ 6 & 7 \end{bmatrix}$

$\therefore \begin{bmatrix} 1 & -3 \\ 6 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$ $\det(T) = 25$

$\therefore T^{-1} = \frac{1}{25}\begin{bmatrix} 7 & 3 \\ -6 & 1 \end{bmatrix}$ $\therefore T^{-1}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$,

$x = \frac{1}{25}(7u+3v)$, $y = \frac{1}{25}(-6u+v)$

$\underline{\text{Jacobian}} = |\det(T^{-1})| = \underline{\frac{1}{25}}$

$\therefore \iint_{R^*} u\sin(v)\left(\frac{1}{25}\right)du\,dv$

3.

As $T(r,\theta) = (r\cos\theta, r\sin\theta)$, $\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$

$= r$

$$\iint_D \exp(x^2+y^2)\,dx\,dy = \iint_{D^*} \exp(r^2)\, r\,dr\,d\theta$$

$$= \int_0^1 \int_0^{2\pi} \exp(r^2)\, r\,dr\,d\theta = 2\pi \int_0^1 e^{r^2}\, r\,dr$$

$$= \pi \int_0^1 e^{r^2}\, d(r^2) = \pi e^{r^2}\Big|_0^1 = \underline{\pi(e-1)}$$

4.

(a) $x+y = (u+v)+(u-v) = 2u$

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad T = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

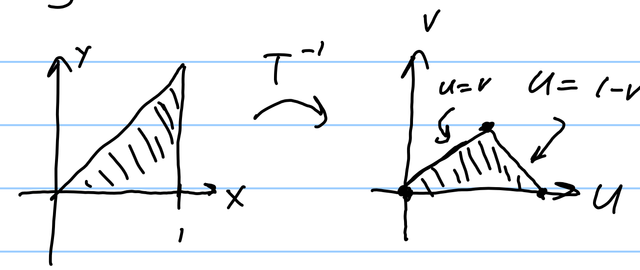
$$\therefore |\det(T)| = |-1-1| = 2$$

$$\therefore \iint_{D^*} 2u\,(2)\,du\,dv$$

What is D^*? $T^{-1} = \frac{1}{-2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$

$$\therefore \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$$

$\therefore T^{-1}(0,0) = (0,0)$
$T^{-1}(1,0) = (\frac{1}{2}, \frac{1}{2})$
$T^{-1}(1,1) = (1,0)$

$\therefore$ 

$\therefore$ a "flip" and rotation

This can be viewed as a "u-simple" region in uv-plane.

"Upper" boundary : $u = 1 - v$

"Lower" boundary : $u = v$

$$\therefore \iint_{D^*} (4u)\,du\,dv = \int_0^{\frac{1}{2}} \int_v^{1-v} (4u)\,du\,dv$$

$$= \int_0^{\frac{1}{2}} \left(2u^2 \Big|_{u=v}^{u=1-v} \right) dv = \int_0^{\frac{1}{2}} \left[2(1-v)^2 - 2v^2 \right] dv$$

$$= \int_0^{\frac{1}{2}} 2(v^2 - 2v + 1) - 2v^2\,dv = \int_0^{\frac{1}{2}} -4v + 2\,dv$$

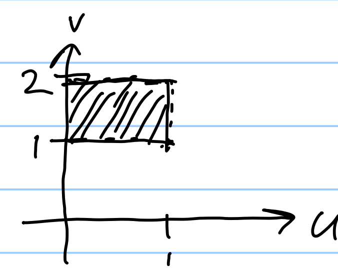
$$= -2v^2 + 2v \Big|_0^{\frac{1}{2}} = -2\left(\frac{1}{2}\right)^2 + 2\left(\frac{1}{2}\right) = \underline{\frac{1}{2}}$$

(b) $\iint_D (x+y)\,dx\,dy = \int_0^1 \int_0^x (x+y)\,dy\,dx$

$$= \int_0^1 \left[xy + \frac{y^2}{2} \Big|_{y=0}^{y=x} \right] dx = \int_0^1 \left(x^2 + \frac{x^2}{2}\right) dx$$

$$= \int_0^1 \frac{3x^2}{2}\,dx = \frac{x^3}{2}\Big|_0^1 = \underline{\frac{1}{2}}$$

5.

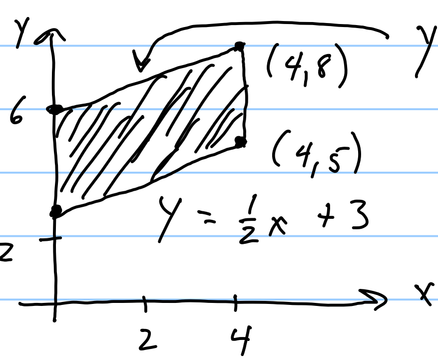


$T = \begin{bmatrix} 4 & 0 \\ 2 & 3 \end{bmatrix}$ as $\begin{bmatrix} 4 & 0 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 4u \\ 2u+3v \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

$T(0,1) = (0,3)$ $\quad T(1,2) = (4,8)$

$T(1,1) = (4,5)$ $\quad T(0,2) = (0,6)$

$\therefore D:$



$y = \frac{1}{2}x + 6$

$\underline{0 \le x \le 4}$

$\underline{\frac{1}{2}x + 3 \le y \le \frac{1}{2}x + 6}$

$\therefore$ Rectange $D^* \xrightarrow[T]{}$ parallelogram D

$$|\det(T)| = 12$$

(a) $$\iint_D xy\,dx\,dy = \iint_{D^*} (4u)(2u+3v)(12)\,du\,dv$$

$$= \int_1^2 \int_0^1 (96u^2 + 144uv)\,du\,dv$$

$$= \int_1^2 \left(32u^3 + 72u^2 v \Big|_{u=0}^{u=1} \right) dv$$

$$= \int_1^2 (32 + 72v)\,dv = 32v + 36v^2 \Big|_1^2$$

$$= 64 + 144 - (32 + 36) = \underline{\underline{140}}$$

(b) $$\iint_D (x-y)\,dx\,dy = \iint_{D^*} [4u - (2u+3v)](12)\,du\,dv$$

$$= \int_1^2 \int_0^1 (24u - 36v)\,du\,dv = \int_1^2 12u^2 - 36uv \Big|_{u=0}^{u=1} dv$$

$$= \int_1^2 (12 - 36v)\,dv = 12v - 18v^2 \Big|_1^2$$

$$= 24 - 72 - (12 - 18) = \underline{-42}$$

6.

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{bmatrix} 1 & 0 \\ v & 1 \end{bmatrix} \therefore \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = 1$$

(a) $$\iint_D xy\,dx\,dy = \iint_{D^*} (u)(v(1+u))\,(1)\,du\,dv$$

$$= \int_1^2 \int_0^1 (uv + u^2 v)\,du\,dv$$

$$= \int_1^2 \left(\frac{u^2 v}{2} + \frac{u^3 v}{3} \Big|_{u=0}^{u=1} \right) dv = \int_1^2 \frac{v}{2} + \frac{v}{3}\,dv$$

$$= \frac{v^2}{4} + \frac{v^2}{6} \Big|_1^2 = 1 + \frac{2}{3} - \left(\frac{1}{4} + \frac{1}{6}\right) = \underline{\frac{5}{4}}$$

(b) $$\iint_D (x-y)\,dx\,dy = \iint_{D^*} [u - (v + uv)]\,(1)\,du\,dv$$

$$= \int_1^2 \int_0^1 (u - v - uv)\,du\,dv$$

$$= \int_1^2 \left(\frac{u^2}{2} - uv - \frac{u^2 v}{2} \Big|_{u=0}^{u=1} \right) dv$$

$$= \int_1^2 \left(\frac{1}{2} - v - \frac{v}{2} \right) dv = \int_1^2 \left(\frac{1}{2} - \frac{3v}{2} \right) dv$$

$$= \frac{v}{2} - \frac{3v^2}{4} \Big|_1^2 = 1 - 3 - \left(\frac{1}{2} - \frac{3}{4} \right) = \underline{-\frac{7}{4}}$$

7.

$$T = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}, \text{ so } \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} u \\ v/2 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

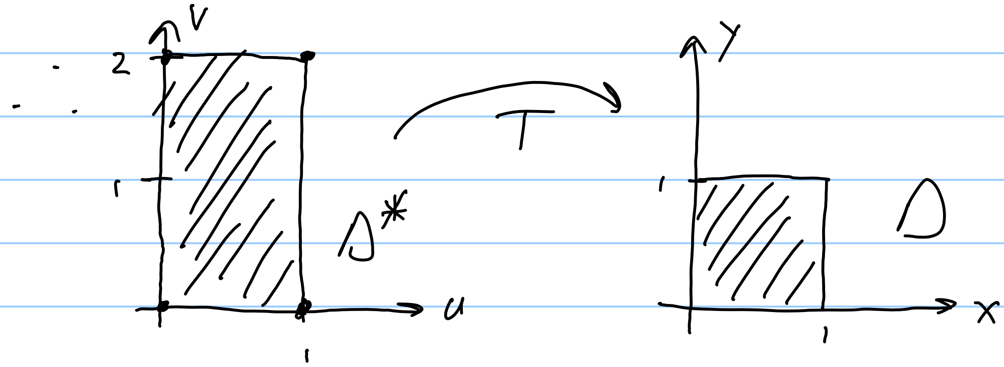
To find D^*, look at the 4 corners of D.

$$\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$



$D^* = [0,1] \times [0,2]$

$$|\det(T)| = \frac{1}{2}, \quad x = u, \quad y = \frac{v}{2}$$

$$\therefore \iint_D \frac{dx\,dy}{\sqrt{1+x+2y}} = \iint_{D^*} \frac{(\frac{1}{2})\,du\,dv}{\sqrt{1+u+v}}$$

$$= \int_0^2 \int_0^1 \frac{du\,dv}{2\sqrt{1+u+v}}$$

$$\frac{1}{2}\int_0^1 (1+u+v)^{-\frac{1}{2}}\,du = \frac{1}{2}(2)(1+u+v)^{\frac{1}{2}} \Big|_{u=0}^{u=1}$$

$$= (2+v)^{\frac{1}{2}} - (1+v)^{\frac{1}{2}}$$

$$\therefore \int_0^2 (2+v)^{\frac{1}{2}} - (1+v)^{\frac{1}{2}}\,dv$$

$$= \frac{2}{3}(2+v)^{3/2} - \frac{2}{3}(1+v)^{3/2} \Big|_{v=0}^{v=2}$$

$$= \frac{2}{3}(8) - \frac{2}{3}\sqrt{27} - \left(\frac{2}{3}\sqrt{8} - \frac{2}{3}\right)$$

$$= \frac{18}{3} - 2\sqrt{3} - \frac{4}{3}\sqrt{2}$$

It would have been easier to evaluate directly without making the change of variables!

8.

$u^2 + v^2 \le 1$ is a disc of radius 1, and with

$u \ge 0, v \ge 0$, This is the disc in quadrant I.

From $\iint_D dx\,dy$, $f(x,y) = 1$ $\therefore f(x(u,v), y(u,v)) = 1$

From $T(u,v) = (u^2 - v^2, 2uv)$, $\left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4u^2 + 4v^2$

$4u^2 + 4v^2 \ge 0$ for $u \ge 0, v \ge 0$.

$$\therefore \iint_D dx\,dy = \iint_{D^*} (4u^2 + 4v^2)\,du\,dv$$

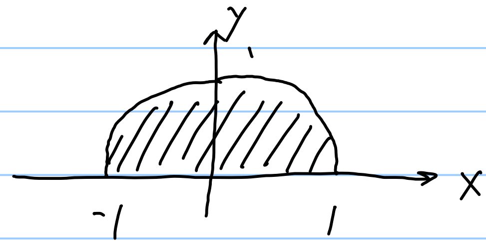
The problem is finding what D is.

Consider $u = \cos\theta$, $v = \sin\theta$ in Quadrant I, for $0 \le \theta \le \pi/2$. This satisfies $u^2 + v^2 \le 1$, and covers the disc one-to-one in Quadrant I.

Then $u^2 - v^2 = \cos^2\theta - \sin^2\theta = \cos 2\theta$
$2uv = 2\cos\theta\sin\theta = \sin 2\theta$

And for $0 \le \theta \le \frac{\pi}{2}$, $(\cos 2\theta, \sin 2\theta)$ is the unit disk in Quadrants I & II.

$\therefore$ D:



$$\therefore \iint_D dx\,dy = \int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dx\,dy$$

$$= \int_0^1 2\sqrt{1-y^2}\,dy = y\sqrt{1-y^2} + \arcsin(y) \Big|_0^1$$

$$= \arcsin(1) = \underline{\frac{\pi}{2}}$$

Perhaps a more elegant solution is to do another change of variables, this time on D^*.

From above, $\iint_D dx\,dy = \iint_{D^*} 4u^2 + 4v^2\,du\,dv$

Where D^* = unit disk in Quadrant I.

Consider $T^*(r,\theta) = (r\cos\theta, r\sin\theta)$

or $T^*(D^{**}) = D^*$, where

$D^{**} = [0,1] \times [0, \frac{\pi}{2}]$ for (r,θ).

$$\left| \frac{\partial T^*(u,v)}{\partial(r,\theta)} \right| = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

where $u = r\cos\theta$, $v = r\sin\theta$. This is one-to-one over D^{**}.

$$\therefore 4u^2 + 4v^2 = 4(r\cos\theta)^2 + 4(r\sin\theta)^2 = 4r^2$$

$$\therefore \iint_D dx\,dy = \iint_{D^*} 4u^2 + 4v^2 \,du\,dv = \iint_{D^{**}} (4r^2)(r)\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^1 4r^3 \,dr\,d\theta = \int_0^{\frac{\pi}{2}} r^4 \Big|_{r=0}^{r=1} d\theta$$

$$= \int_0^{\frac{\pi}{2}} d\theta = \frac{\pi}{2}$$

Here, you don't even need to know what D is.

9.

For $T(u,v) = (u^2 - v^2, 2uv)$, $\left|\dfrac{\partial(x,y)}{\partial(u,v)}\right| = 4u^2 + 4v^2$

as seen in #8.

Here, $f(x,y) = \dfrac{1}{\sqrt{x^2+y^2}}$, not 1 as in #8.

$\therefore f(x(u,v), y(u,v)) = f(u^2-v^2, 2uv) =$

$$\frac{1}{\sqrt{(u^2-v^2)^2 + (2uv)^2}} = \frac{1}{\sqrt{u^4 - 2u^2v^2 + v^4 + 4u^2v^2}}$$

$$= \frac{1}{\sqrt{u^4 + 2u^2v^2 + v^4}} = \frac{1}{\sqrt{(u^2+v^2)^2}}$$

$$= \frac{1}{u^2+v^2}, \text{ for } u \geq 0, v \geq 0, u^2+v^2 \leq 1$$

$$\therefore \iint_D \frac{dx\,dy}{\sqrt{x^2+y^2}} = \iint_{D^*} \frac{(4u^2+4v^2)}{u^2+v^2}\,du\,dv$$

$$= \iint_{D^*} 4\,du\,dv$$

where D^* = unit disk in Quadrant I from #8 above

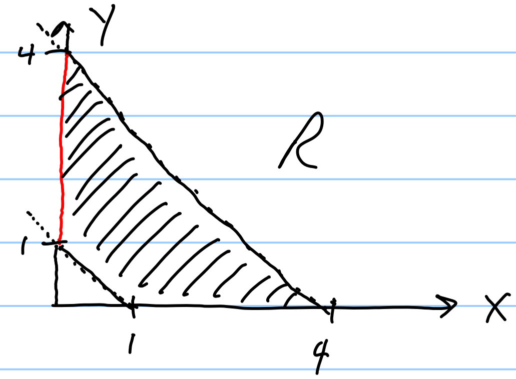
$$= \int_0^1 \int_0^{\sqrt{1-u^2}} 4 \, dv \, du = \int_0^1 4\sqrt{1-u^2} \, du$$

(see table of integrals #38 of text)

$$= 2u\sqrt{1-u^2} + 2\arcsin(u) \Big|_0^1$$

$$= 2\arcsin(1) = 2\left(\frac{\pi}{2}\right) = \underline{\underline{\pi}}$$

10.



(a) $\left| \dfrac{\partial(x,y)}{\partial(u,v)} \right| = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix}$

$$= u - uv + uv = u$$

Since $y = uv$, and $y \geq 0$ for $(x,y) \in R$,

then $u \geq 0$ and $v \geq 0$ or $u \leq 0$ and $v \leq 0$ [1]

For $x = u(1-v) \geq 0$ for $(x,y) \in R$,

then $u \geq 0$ and $1-v \geq 0$ or $u \leq 0$ and $1-v \leq 0$ [2]

[1] & [2] $\Rightarrow$ $u \geq 0$ and $0 \leq v \leq 1$, or

$u \leq 0$ and $1 \leq v$ and $v \leq 0$, a conflict

$\therefore\ u \geq 0$ and $0 \leq v \leq 1$

$$\therefore \left| \frac{\partial(x,y)}{\partial(u,v)} \right| = u$$

(b) $f(x(u,v), y(u,v)) = \dfrac{1}{x(u,v) + y(u,v)} = \dfrac{1}{(u-uv) + (uv)}$

$$= \frac{1}{u}$$

$$\therefore\ f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \frac{1}{u} \cdot u = 1$$

$$\therefore \iint_R \frac{1}{x+y}\,dx\,dy = \iint_{R^*} du\,dv = \text{Area}(R^*)$$

(c) Look at $T(R^*) = R$

By definition of R^*, T is onto

Now consider $T(u,v) = T(u',v')$ for $(u,v) \in R^*$, $(u',v') \in R^*$

$$\therefore\ (u-uv, uv) = (u'-u'v', u'v')$$

$\Rightarrow uv = u'v'$ (2nd coordinates equal)

$\therefore\ u - uv = u' - u'v'$, so $u - u'v' = u' - u'v'$

$\Rightarrow u = u'$. $\therefore uv = u'v' \Rightarrow v = v'$ for $u \neq 0$

But $u \neq 0$ for $(u,v) \in R^*$, because if $u = 0$, $T(0,v) = (0,0)$, and $(0,0) \notin R$.

$\therefore T(u,v) = T(u',v') \Rightarrow u = u',\ v = v'$

$\therefore T$ is one-to-one and onto so T^{-1} exists.

$\therefore$ Look at corners: $(1,0), (4,0), (0,4), (0,1)$

$$T(u,v) = (1,0) \Rightarrow (u - uv, uv) = (1,0)$$

$$\Rightarrow (u,v) = (1,0)$$

$$T(u,v) = (4,0) \Rightarrow (u - uv, uv) = (4,0)$$

$$\Rightarrow (u,v) = (4,0)$$

$$T(u,v) = (0,4) \Rightarrow (u - uv, uv) = (0,4)$$

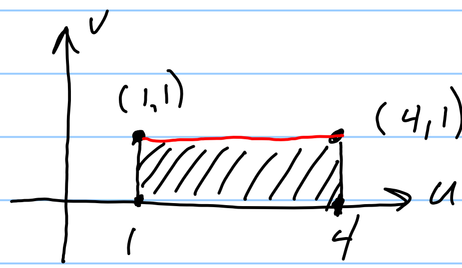
$$\Rightarrow u = 4,\ v = 1, \text{ so } (u,v) = (4,1)$$

$$T(u,v) = (0,1) \Rightarrow (u - uv, uv) = (0,1)$$

$$\Rightarrow u = 1,\ v = 1, \text{ so } (u,v) = (1,1)$$

Note $T(u,1) = (0,u)$, so the top horizontal border of R^* corresponds to left vertical border of R.

$\therefore R^*$:



$$\therefore \iint_{R^*} du\,dv = \int_0^1 \int_1^4 du\,dv = 3$$

$$\therefore \iint_R \frac{1}{x+y}\,dy\,dx = \iint_{R^*} du\,dv = \underline{3}$$

11.

Try a polar coordinate transfer:

$T(r, \theta) = (r\cos\theta, r\sin\theta)$

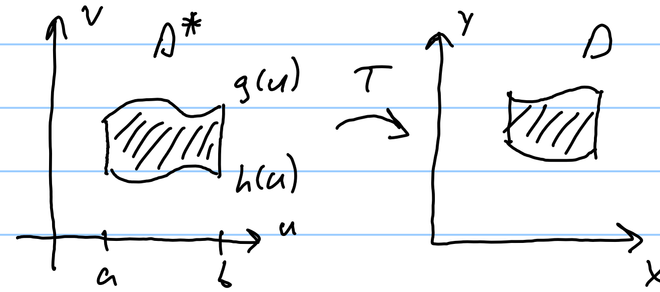
$\therefore 0 \le r \le 2, \quad 0 \le \theta \le 2\pi$

$$\left| \frac{\partial(x,y)}{\partial(r,\theta)} \right| = r, \quad (x^2+y^2)^{3/2} = (r^2)^{3/2} = r^3$$

$$\therefore \iint_D (x^2+y^2)^{3/2}\,dx\,dy = \int_0^{2\pi}\int_0^2 (r^3)\, r\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{r^5}{5}\Big|_{r=0}^{r=2}\, d\theta = \int_0^{2\pi} \frac{32}{5}\, d\theta = \underline{\underline{\frac{64}{5}\pi}}$$

12.



From chapter 5 of text, f is integrable over D.

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ \partial\psi/\partial u & \frac{\partial\psi}{\partial v} \end{vmatrix}$$

$= \left|\frac{\partial\psi}{\partial v}\right| =$ absolute value of $\partial\psi/\partial v \neq 0$ by assumption.

$\therefore$ By Theorem 2, p. 319 of text,

$$\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

$$= \underline{\underline{\iint_{D^*} f(u, \psi(u,v)) \left|\frac{\partial\psi}{\partial v}\right| du\,dv}}$$

Note: $T(u,v) = (u, \psi(u,v))$ is one-to-one.

$T(u,v) = T(u',v') \Rightarrow (u, \psi(u,v)) = (u', \psi(u',v'))$.

$\Rightarrow u = u'$. $\therefore \psi(u,v) = \psi(u,v')$. But $\frac{\partial \psi}{\partial v} \neq 0$

$\Rightarrow g(v) = \psi(u,v)$, for fixed u, is either increasing or decreasing on any v-axis of D^*.

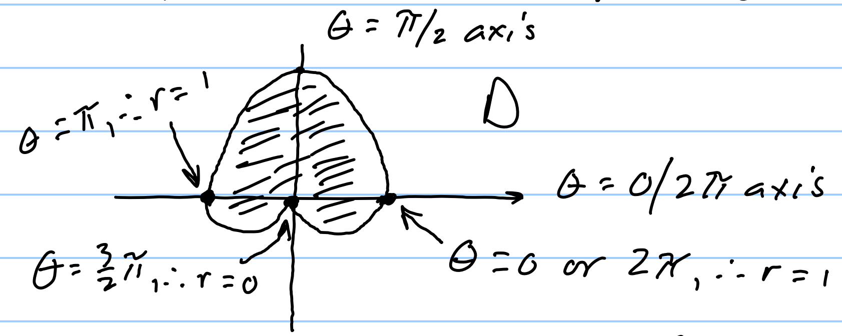
$\therefore g^{-1}$ exists, so $g(v) = g(v') \Rightarrow$

$g^{-1}(g(v)) = g^{-1}(g(v')) \Rightarrow v = v'$.

$\therefore T(u,v) = T(u',v') \Rightarrow u = u'$ and $v = v'$.

13.

Assume $0 \le \theta \le 2\pi$. $\therefore 0 \le r \le 2$ as $-1 \le \sin\theta \le 1$

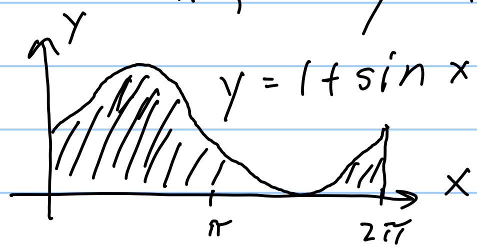


Curve is $r(\theta) = 1 + \sin\theta$

Let $f(r,\theta) = 1$. $\therefore$ Area $= \iint_D f(r,\theta)\, dr\, d\theta = \iint_D dr\, d\theta$

Now need a region D^* so that $T(D^*) = D$, making the integration easier. Use a polar transformation: $(r,\theta) = T(x,y) = (y\cos x, y\sin x)$, where $0 \le x \le 2\pi$, $0 \le y \le 1 + \sin x$

$\therefore D^*$:



Although D^* doesn't look simple, The integration will be familiar.

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{bmatrix} -y\sin x & \cos x \\ y\cos x & \sin x \end{bmatrix}$$

$$\therefore \left| \det \frac{\partial(r,\theta)}{\partial(x,y)} \right| = \left| -y\sin^2 x - y\cos^2 x \right| = y$$

$$\therefore \iint_D dr\,d\theta = \iint_{D^*} y\,dy\,dx = \int_0^{2\pi}\int_0^{1+\sin x} y\,dy\,dx$$

$$= \int_0^{2\pi} \frac{y^2}{2}\Bigg|_{y=0}^{y=1+\sin x} dx = \int_0^{2\pi} \frac{(1+\sin x)^2}{2}\,dx$$

$$= \frac{1}{2}\int_0^{2\pi} 1 + 2\sin x + \sin^2 x\,dx$$

using $\cos^2 x - \sin^2 x = \cos 2x$,
$\therefore (1-\sin^2 x) - \sin^2 x = \cos 2x$
$\therefore \dfrac{1-\cos 2x}{2} = \sin^2 x$

$$= \int_0^{2\pi} \frac{1}{2} + \sin x + \frac{1-\cos 2x}{4}\,dx$$

$$= \int_0^{2\pi} \frac{3}{4} + \sin x - \frac{\cos 2x}{2} \, dx$$

$$= \frac{3}{4}x - \cos x - \frac{\sin 2x}{4} \Big|_0^{2\pi} = \frac{3}{2}\pi - 1 - 0 - (0 - 1 - 0)$$

$$= \underline{\frac{3}{2}\pi}$$

Note: Could have originally set variables as $y = 1 + \sin x$, and then change using $(x,y) = T(r,\theta)$. However, the process is the same.

14.

(a) Let $T(u,v) = (u, v^2) = (x,y)$. For $0 \le u \le 1$, $0 \le v \le u$,

$$T(u,v) = T(u', v') \Rightarrow (u, v^2) = (u', v'^2)$$

$$\Rightarrow u = v' \text{ and } v = v'. \therefore T \text{ is one-to-one.}$$

$$\therefore \frac{\partial(x,y)}{\partial(u,v)} = \begin{bmatrix} 1 & 0 \\ 0 & 2v \end{bmatrix}. \therefore \left| \frac{\partial(x,y)}{\partial(u,v)} \right| = 2v \ge 0$$

$$\therefore \int_0^1 \int_0^{x^2} xy \, dy\, dx = \iint_{D^*} (u)(v^2)(2v) \, du\, dv$$

$$= \int_0^1 \int_0^u 2uv^3 \, dv \, du$$

(b) (1) $$\int_0^1 \int_0^u 2uv^3 \, dv \, du = \int_0^1 \left(\frac{1}{2} uv^4 \Big|_{v=0}^{v=u} \right) du$$

$$= \int_0^1 \frac{1}{2} u^5 \, du = \frac{1}{12} u^6 \Big|_0^1 = \frac{1}{12}$$

(2) $$\int_0^1 \int_0^{x^2} xy \, dy \, dx = \int_0^1 \left(x \frac{y^2}{2} \Big|_{y=0}^{y=x^2} \right) dx$$

$$= \int_0^1 \frac{x^5}{2} \, dx = \frac{x^6}{12} \Big|_0^1 = \frac{1}{12}$$

15.

$$\int_2^3 \int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} z e^{x^2+y^2} \, dy \, dx \, dz$$

Use cylindrical transformation :

$$T(r, \theta, z) = (r\cos\theta, r\sin\theta, z) = (x, y, z)$$

$$\therefore r^2 = x^2 + y^2 \qquad \left| \frac{\partial(x,y,z)}{\partial(r,\theta,z)} \right| = r$$

where $0 \le r \le 2,\ 0 \le \theta \le 2\pi,\ 2 \le z \le 3$

$$\therefore \int_2^3 \int_0^{2\pi} \int_0^2 z e^{r^2} (r)\, dr\, d\theta\, dz$$

$$= \int_2^3 \int_0^{2\pi} \frac{1}{2} z e^{r^2} \Big|_{r=0}^{r=2} d\theta\, dz$$

$$= \int_2^3 \int_0^{2\pi} \frac{1}{2} z (e^4 - 1)\, d\theta\, dz = \int_2^3 \frac{1}{2} z (e^4 - 1)\theta \Big|_{\theta=0}^{\theta=2\pi} dz$$

$$\int_2^3 \pi (e^4 - 1) z\, dz = \pi (e^4 - 1) \frac{z^2}{2} \Big|_{z=2}^{z=3} = \underline{\frac{5}{2}\pi (e^4 - 1)}$$

16.

Use the polar transformation: $T(r,\theta) = (r\cos\theta, r\sin\theta)$

$\left| \frac{\partial(x,y)}{\partial(r,\theta)} \right| = r$, and $x^2 + y^2 = r^2$

$$\therefore \iint_D (1 + x^2 + y^2)^{3/2} dx\, dy = \iint_{D^*} (1 + r^2)^{3/2} (r)\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^1 (1+r^2)^{3/2} r \, dr \, d\theta$$

$$= \int_0^{2\pi} \frac{1}{5} (1+r^2)^{5/2} \Big|_{r=0}^{r=1} d\theta$$

$$= \int_0^{2\pi} \frac{1}{5}(2)^{5/2} - \frac{1}{5} \, d\theta$$

$$= \int_0^{2\pi} \frac{4}{5}\sqrt{2} - \frac{1}{5} \, d\theta = \underline{\underline{\frac{2\pi}{5}(4\sqrt{2}-1)}}$$

17.

Note: xy-graph has symmetry with respect to x-axis and y-axis. Assume $a > 0$.

Let $T(r,\theta) = (r\cos\theta, r\sin\theta) = (x,y)$ $\therefore \left| \frac{\partial(x,y)}{\partial(r,\theta)} \right| = r$

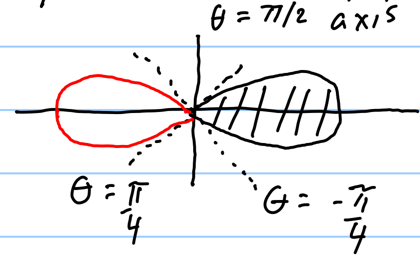
and $x^2+y^2 = r^2$, $x^2-y^2 = r^2(\cos^2\theta - \sin^2\theta) = r^2\cos 2\theta$

When $y=0$, $(x^2+y^2)^2 = 2a^2(x^2-y^2) \Rightarrow x^4 = 2a^2x^2$, $x = \pm\sqrt{2}\,a$

When $x=0$, $(x^2+y^2)^2 = 2a^2(x^2-y^2) \Rightarrow y^4 = 2a^2(-y^2) \Rightarrow y=0$.

Since $(x^2+y^2)^2 \geq 0$, $x^2-y^2 \geq 0$, $\therefore |y| \leq |x|$

$$(x^2+y^2)^2 = 2a^2(x^2-y^2) \Rightarrow r^4 = 2a^2 r^2 \cos 2\theta \Leftrightarrow r^2 = 2a^2 \cos 2\theta$$



As $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$, curve traces out loop shown at left, for $r \ge 0$.

For $r \le 0$, symmetric curve in — is traced out. Areas are equal.

Let D^* be just the loop for $r \ge 0$.

$$\therefore D^* = \left\{ (r,\theta) : 0 \le r \le \sqrt{2}\,a,\ -\frac{\pi}{4} \le \theta \le \frac{\pi}{4} \right\}$$

If $\theta > \frac{\pi}{4}$, $2a^2\cos 2\theta < 0$, and $r^2 \ge 0$.

$$\therefore \iint_D dx\,dy = 2\iint_{D^*} r\,dr\,d\theta = 2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^{f(\theta)} r\,dr\,d\theta$$

where $r = f(\theta) = \sqrt{2a^2\cos 2\theta}$

$$\therefore 2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^{\sqrt{2a^2\cos 2\theta}} r\,dr\,d\theta = 2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left. \frac{r^2}{2} \right|_{r=0}^{r=\sqrt{2a^2\cos 2\theta}} d\theta$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 2a^2\cos 2\theta\, d\theta = a^2 \sin 2\theta \Big|_{-\frac{\pi}{4}}^{\pi/4}$$

$$= a^2 - (-a^2) = \underline{\underline{2a^2}}$$

18.

Use a cylindrical coordinate conversion.

$$\text{Volume} = \iiint_{W^*} r\, dr\, d\theta\, dz$$

where $0 \le z \le 10$, $0 \le \theta \le 2\pi$, $0 \le r \le \sqrt{z}$

(from $z = x^2 + y^2 = r^2$)

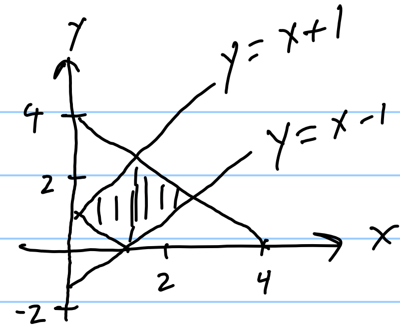
$$\therefore V = \int_0^{10} \int_0^{2\pi} \int_0^{\sqrt{z}} r\, dr\, d\theta\, dz$$

$$= \int_0^{10} \int_0^{2\pi} \frac{1}{2} r^2 \Big|_{r=0}^{r=\sqrt{z}} d\theta\, dz$$

$$= \int_0^{10} \int_0^{2\pi} \frac{z}{2}\, d\theta\, dz = \int_0^{10} \pi z\, dz$$

$$= \pi \frac{z^2}{2} \Big|_0^{10} = \underline{50\pi}$$

19.



Let $u = x+y$, $v = x-y$

$\therefore \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$, $T = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. $\det(T) = -2 \neq 0$

$\therefore T^{-1} = \frac{1}{-2} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}$ $\therefore T^{-1}(u,v) = (x,y)$

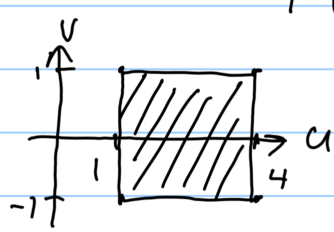
$$\left| \frac{\partial(x,y)}{\partial(u,v)} \right| = |\det(T^{-1})| = \left| -\frac{1}{2} \right| = \frac{1}{2}$$

Corners for R: $(0,1)$, $(1,0)$, $(\frac{5}{2}, \frac{3}{2})$, $(\frac{3}{2}, \frac{5}{2})$

$\therefore$ Corners for R^*: $T(0,1) = (1,-1)$ $T(\frac{5}{2}, \frac{3}{2}) = (4,1)$

$T(1,0) = (1,1)$ $T(\frac{3}{2}, 5/2) = (4,-1)$

$\therefore R^*$



$1 \leq u \leq 4$

$-1 \leq v \leq 1$

$$\therefore \iint_R (x+y)^2 e^{x-y} \, dx\,dy = \iint_{R^*} u^2 e^v \, |\det(T^{-1})| \, du\,dv$$

$$= \int_{-1}^{1} \int_{1}^{4} u^2 e^v \left(\tfrac{1}{2}\right) du\,dv$$

$$= \int_{-1}^{1} \left(\frac{e^v}{2} \frac{u^3}{3} \Big|_{u=1}^{u=4} \right) dv = \int_{-1}^{1} \frac{64}{6} e^v - \frac{e^v}{6} \, dv$$

$$= \frac{63}{6} e^v \Big|_{v=-1}^{v=1} = \underline{\frac{21}{2}\left(e - \frac{1}{e}\right)}$$

20.

(a) T is much like spherical coordinates.

Note $[u\cos(v)\cos(w)]^2 + [u\sin(v)\cos(w)]^2 + [u\sin(w)]^2$

$$= u^2\cos^2(w)[\cos^2(v) + \sin^2(v)] + u^2\sin^2(w)$$

$$= u^2\cos^2(w) + u^2\sin^2(w) = u^2$$

$$\therefore \|T(u,v,w)\|^2 = u^2 = \|(x,y,z)\|^2 = x^2+y^2+z^2 = 1$$

$\therefore$ Let $\underline{u = \sqrt{x^2+y^2+z^2}}$ (analogous to ρ)

Let w = radian measure of point from the xy-plane (analogous to $\frac{\pi}{2} - \phi$).

$\therefore$ Let $\sin(w) = \frac{z}{\sqrt{x^2+y^2+z^2}}$, or $\underline{w = \mathrm{Arcsin}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)}$

so $-\frac{\pi}{2} \le w \le \frac{\pi}{2}$ as $-1 \le z \le 1$

Let v = radian measure from x-axis of projected point onto the xy-plane ($\equiv \theta$).

$\therefore$ Let $\cos(v) = \frac{x}{\sqrt{x^2+y^2}}$, $\therefore v = \text{Arccos}\left(\frac{x}{\sqrt{x^2+y^2}}\right)$

To cover the entire circle,

Let $v = \text{Arccos}\left(\frac{x}{\sqrt{x^2+y^2}}\right)$ for $y \geq 0$

$v = -\text{Arccos}\left(\frac{x}{\sqrt{x^2+y^2}}\right)$ for $y < 0$

$\therefore -\pi < v \leq \pi$

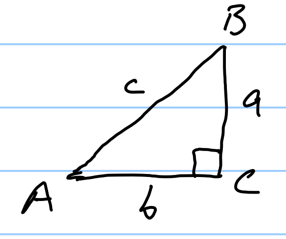
$\therefore$ Given any (x,y,z) s.t. $x^2+y^2+z^2=1$,

Choose $u = \sqrt{x^2+y^2+z^2}$

$v = \text{Arcos}\left(\frac{x}{\sqrt{x^2+y^2}}\right)$, $y \geq 0$

$= -\text{Arccos}\left(\frac{x}{\sqrt{x^2+y^2}}\right)$, $y < 0$

$w = \text{Arc}\sin\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$



Note: Given any right triangle ABC with sides a, b, c labelled opposite the angles $\angle A, \angle B, \angle C$, respectively.

$c^2 = a^2 + b^2$, $\cos(\angle A) = \frac{b}{c}$, so $\angle A = \text{Arccos}\left(\frac{b}{\sqrt{a^2+b^2}}\right)$

$\therefore \sin(\angle A) = \frac{a}{\sqrt{a^2+b^2}} = \sin\left(\text{Arccos} \frac{b}{\sqrt{a^2+b^2}}\right)$

i.e., you swith variables under the radical.

e.g., $\sin\left(\text{Arccos} \frac{r}{\sqrt{r^2+s^2}}\right) = \frac{s}{\sqrt{r^2+s^2}}$, $r, s > 0$.

$\therefore$ Given $[x, y, z]$, with $y \geq 0$

$$T(u,v,w) = [u\cos(v)\cos(w), u\sin(v)\cos(w), u\sin(w)]$$

$$= \left[\sqrt{x^2+y^2+z^2}\cos\left(\text{Arccos}\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right)\cos\left(\text{Arcsin}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)\right),\right.$$
$$\sqrt{x^2+y^2+z^2}\sin\left(\text{Arccos}\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right)\cos\left(\text{Arcsin}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)\right),$$
$$\left.\sqrt{x^2+y^2+z^2}\sin\left(\text{Arcsin}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)\right)\right]$$

$$= \left[\sqrt{x^2+y^2+z^2}\ \frac{x}{\sqrt{x^2+y^2}}\ \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}},\right.$$
$$\sqrt{x^2+y^2+z^2}\ \frac{y}{\sqrt{x^2+y^2}}\ \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}},$$
$$\left.\sqrt{x^2+y^2+z^2}\ \frac{z}{\sqrt{x^2+y^2+z^2}}\right]$$

$$= [x, y, z]$$

Note use of $\sin\left(\text{Arccos}\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right) = \frac{y}{\sqrt{x^2+y^2}}$ (assuming $y \geq 0$)

and $\cos\left(\text{Arcsin}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)\right) = \cos\left(\text{Arcsin}\left(\frac{z}{\sqrt{r^2+z^2}}\right)\right)$

$$= \frac{r}{\sqrt{r^2+z^2}} = \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}} \quad (\text{letting } r^2 = x^2+y^2)$$

using the note above.

Also note $T(u,-v,w) = [u\cos(v)\cos(w), -u\sin(v)\cos(w), u\sin(w)]$
so if $y<0$, the y-coordinate will have a negative contribution.

Thus, given a (x,y,z), $\exists\ (u,v,w)$ s.t.

$T(u,v,w) = (x,y,z)$, where $x^2+y^2+z^2=1$,

so T is <u>onto</u>.

(5) Since $u = 1 = \sqrt{x^2+y^2+z^2}$,

$T(u,v,w) = (\cos(v)\cos(w), \sin(v)\cos(w), \sin(w))$

Let $w = \frac{\pi}{2}$. $\therefore T(1,v,\frac{\pi}{2}) = (0,0,1)$.

$\therefore$ Even when $v_1 \neq v_2$, $T(1,v_1,\frac{\pi}{2}) = T(1,v_2,\frac{\pi}{2})$

$\therefore T$ is not one-to-one.

21.

$\iiint_R (x^2+y^2+z^2)\,dx\,dy\,dz$ Use a cylindrical transformation

$x = r\cos\theta \quad y = r\sin\theta \quad z = z$

$\therefore x^2+y^2+z^2 \Rightarrow r^2+z^2$

$$\therefore \iiint_{R^*} (r^2+z^2)\, r\,dr\,d\theta\,dz = \int_{-2}^{3}\int_{0}^{2\pi}\int_{0}^{\sqrt{2}} (r^2+z^2)\, r\,dr\,d\theta\,dz$$

$$= \int_{-2}^{3}\int_{0}^{2\pi}\int_{0}^{\sqrt{2}} r^3 + rz^2 \,dr\,d\theta\,dz = \int_{-2}^{3}\int_{0}^{2\pi} \frac{r^4}{4} + \frac{r^2}{2}z^2 \Big|_{r=0}^{r=\sqrt{2}} d\theta\,dz$$

$$= \int_{-2}^{3}\int_{0}^{2\pi} (1+z^2)\,d\theta\,dz = \int_{-2}^{3} 2\pi + 2\pi z^2\,dz$$

$$= 2\pi z + \frac{2\pi z^3}{3}\Big|_{z=-2}^{z=3} = 6\pi + 18\pi - \left(-4\pi - \frac{16}{3}\pi\right)$$

$$= 28\pi + \frac{16}{3}\pi = 33\tfrac{1}{3}\pi = \underline{\underline{\frac{100}{3}\pi}}$$

22.

$$\int_0^\infty e^{-4x^2}dx = \lim_{u\to\infty}\int_0^u e^{-4x^2}dx = \lim_{u\to\infty}\int_0^u e^{-(2x)^2}dx$$

Let $y=2x$. $\therefore$ when $x=u$, $y=2u$ $\quad dy=2dx$

$$\therefore \lim_{u\to\infty}\int_0^u e^{-(2x)^2}dx = \lim_{y\to\infty}\int_0^{2y} e^{-(y)^2}\left(\tfrac{1}{2}\right)dy$$

$$= \frac{1}{2}\lim_{y\to\infty}\int_0^{2y} e^{-y^2}dy = \frac{1}{2}\left[\frac{1}{2}\lim_{y\to\infty}\int_{-2y}^{2y} e^{-y^2}dy\right]$$

using symmetry

$$= \underline{\frac{1}{4}\sqrt{\pi}}$$

23.

Use spherical coordinates

$$\therefore f(x,y,z) = \frac{1}{\sqrt{2+x^2+y^2+z^2}} \Rightarrow \frac{1}{\sqrt{2+\rho^2}}$$

$$dx\,dy\,dz \Rightarrow \rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

$$\therefore \iiint_{B^*}\frac{\rho^2\sin\phi}{\sqrt{2+\rho^2}}\,d\rho\,d\theta\,d\phi$$

$$= \int_0^1\int_0^{\pi}\int_0^{2\pi}\frac{\rho^2\sin\phi}{\sqrt{2+\rho^2}}\,d\theta\,d\phi\,d\rho$$

$$= 2\pi\int_0^1\int_0^{\pi}\frac{\rho^2\sin\phi}{\sqrt{2+\rho^2}}\,d\phi\,d\rho$$

$$= 2\pi \int_0^1 \left(\frac{\rho^2}{\sqrt{2+\rho^2}} - \cos\phi \Big|_{\phi=0}^{\phi=\pi} \right) d\rho$$

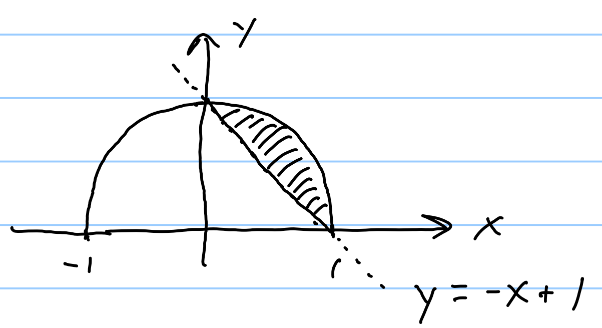
$$= 4\pi \int_0^1 \frac{\rho^2}{\sqrt{2+\rho^2}}\, d\rho$$ Use table of integrals #58 of text

$$= 4\pi \left[\frac{\rho\sqrt{2+\rho^2}}{2} - \log(\rho + \sqrt{2+\rho^2}) \right]_{\rho=0}^{\rho=1}$$

$$= 4\pi \left[\frac{\sqrt{3}}{2} - \log(1+\sqrt{3}) - (0 - \log\sqrt{2}) \right]$$

$$= 4\pi \left[\frac{\sqrt{3}}{2} + \log\sqrt{2} - \log(1+\sqrt{3}) \right]$$

24.



Use a polar conversion

$0 \le \theta \le \frac{\pi}{2}$ $\quad r^2 = x^2 + y^2 \quad x = r\cos\theta \quad y = r\sin\theta$

$x + y \ge 1 \Rightarrow r(\cos\theta + \sin\theta) \ge 1$

$\therefore\ r \ge \dfrac{1}{\cos\theta + \sin\theta}$, and $\cos\theta + \sin\theta \ne 0$ for $0 \le \theta \le \frac{\pi}{2}$

$x^2 + y^2 \le 1 \Rightarrow r^2 \le 1$, or $r \le 1$.

$$\therefore \iint_A \frac{1}{(x^2+y^2)^2}\,dx\,dy = \iint_{A^*} \frac{r\,dr\,d\theta}{r^4}$$

$$= \int_0^{\frac{\pi}{2}} \int_{\frac{1}{\cos\theta+\sin\theta}}^{1} \frac{1}{r^3}\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{r^{-2}}{-2} \Bigg|_{r=\frac{1}{\cos\theta+\sin\theta}}^{r=1} \right) d\theta$$

$$= -\frac{1}{2} \int_0^{\frac{\pi}{2}} 1 - (\cos\theta+\sin\theta)^2 \; d\theta$$

$$= -\frac{1}{2} \int_0^{\frac{\pi}{2}} -2\cos\theta\sin\theta\,d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin 2\theta\,d\theta$$

$$= -\frac{1}{4}\cos 2\theta \Big|_0^{\frac{\pi}{2}} = \frac{1}{4} - \left(-\frac{1}{4}\right) = \underline{\frac{1}{2}}$$

25.

Use spherical coordinates.

$$x^2+y^2+z^2 = \rho^2 \qquad z = \rho\cos\phi$$

$x = \rho \sin\phi \cos\theta \quad y = \rho \sin\phi \sin\theta$

$$\iiint_W \frac{dx\,dy\,dz}{(x^2+y^2+z^2)^{3/2}} = \iiint_{W^*} \frac{\rho^2 \sin\phi}{(\rho^2)^{3/2}}\, d\rho\, d\theta\, d\phi$$

$$= \int_0^{\pi} \int_0^{2\pi} \int_b^a \frac{\sin\phi}{\rho}\, d\rho\, d\theta\, d\phi$$

$$= \int_0^{\pi} \int_0^{2\pi} \left(\sin\phi \log\rho \Big|_{\rho=b}^{\rho=a} \right) d\theta\, d\phi$$

$$= \int_0^{\pi} \int_0^{2\pi} \sin\phi \log\left(\frac{a}{b}\right) d\theta\, d\phi = \int_0^{\pi} 2\pi \log\left(\frac{a}{b}\right) \sin\phi\, d\phi$$

$$= -2\pi \log\left(\frac{a}{b}\right) \cos\phi \Big|_{\phi=0}^{\phi=\pi} = 2\pi \log\left(\frac{a}{b}\right)\left[1-(-1)\right]$$

$$= \underline{4\pi \log\left(\frac{a}{b}\right)}$$

26.

$x^2 + y^2 + z^2 = \rho^2$

For the integration limits, the original region is a sphere, centred at $(0,0,0)$, of radius 3. But with $0 \leq x \leq 3$, $0 \leq y$, and $0 \leq z$, so sphere in quadrant I.

This is analogous to $0 \leq \rho \leq 3$, $0 \leq \phi \leq \frac{\pi}{2}$, and $0 \leq \theta \leq \pi/2$.

Jacobian is $\rho^2 \sin\phi$

$$\therefore \iiint_{R^*} \frac{\rho}{1+(\rho^2)^2}\, \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi =$$

$$\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^3 \frac{\rho^3 \sin\phi}{1+\rho^4}\, d\rho\, d\theta\, d\phi$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \left(\sin\phi \; \frac{1}{4} \log(1+\rho^4) \Big|_{\rho=0}^{\rho=3} \right) d\theta\, d\phi$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \sin\phi \, \frac{\log(82)}{4}\, d\theta\, d\phi$$

$$= \int_0^{\frac{\pi}{2}} \frac{\pi \log(82)}{8} \sin\phi \, d\phi = \frac{\pi \log(82)}{8} (-\cos\phi) \Big|_{\phi=0}^{\phi=\frac{\pi}{2}}$$

$$= \frac{\pi \log(82)}{8}$$

27.

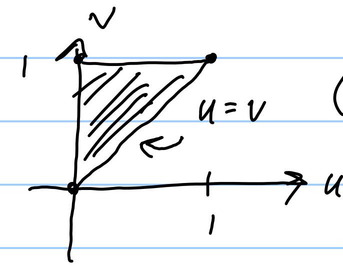
Let $u = x - y$, $v = x + y$ $\therefore \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$

$\therefore T = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, $\det(T) = 2$, $T^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

$\therefore \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$ $\quad \det(T^{-1}) = \frac{1}{2}$

$$\therefore \iint_{D^*} \cos \pi\left(\frac{u}{v}\right) \left(\frac{1}{2}\right) du\, dv$$

D^* : $T(0,0) = (0,0)$
$T(\frac{1}{2}, \frac{1}{2}) = (0,1)$
$T(1,0) = (1,1)$



(u-simple)

$\therefore 0 \le v \le 1 \quad 0 \le u \le v$

$$\therefore \int_0^1 \int_0^v \cos \pi\left(\frac{u}{v}\right) \left(\frac{1}{2}\right) du\, dv$$

$$\int_0^v \frac{1}{2}\cos\left(\frac{\pi u}{v}\right) du = \frac{v}{2\pi}\sin\left(\frac{\pi u}{v}\right)\Big|_{u=0}^{u=v}$$

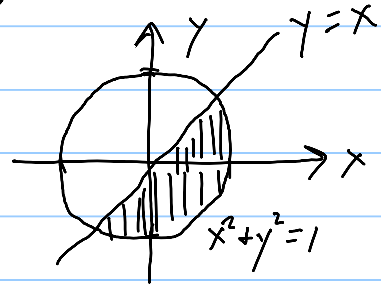
$$= 0 - 0 = 0$$

$$\therefore \int_0^1 0 \, dv = \underline{0}.$$

$$\therefore \iint_D \cos \pi\left(\frac{x-y}{x+y}\right) dx\,dy = \underline{\underline{0}}$$

Note: Integrating $\iint_{D^*}$ using u-simple regions is much easier (because of 0) than integrating using v-simple regions.

28.



A polar conversion using $0 \le r \le 1$ and $-\frac{3}{4}\pi \le \theta \le \frac{\pi}{4}$ covers D.

$$\therefore \iint_D x^2 \, dx\,dy = \iint_{D^*} (r\cos\theta)^2 \, r\,dr\,d\theta$$

$$= \int_{-\frac{3}{4}\pi}^{\frac{\pi}{4}} \int_0^1 r^3 \cos^2\theta \, dr\, d\theta = \int_{-\frac{3}{4}\pi}^{\frac{\pi}{4}} \left(\frac{r^4}{4} \cos^2\theta \Big|_{r=0}^{r=1} \right) d\theta$$

$$= \int_{-\frac{3\pi}{4}}^{\frac{\pi}{4}} \cos^2\theta \, d\theta = = \int_{-\frac{3\pi}{4}}^{\frac{\pi}{4}} \frac{1+\cos 2\theta}{2} \, d\theta$$

$$= \frac{\theta}{2} + \frac{\sin 2\theta}{4} \Big|_{\theta = -\frac{3\pi}{4}}^{\theta = \frac{\pi}{4}}$$

$$= \frac{\pi}{8} + \frac{1}{4} - \left(-\frac{3\pi}{8} + \frac{1}{4} \right) = \underline{\frac{\pi}{2}}$$

29.

Use spherical coordinates: $\rho^2 = x^2 + y^2 + z^2$

Here, $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi$

$$\therefore \int_0^{2\pi} \int_0^{\pi} \int_b^a \rho e^{-\rho^2} (\rho^2 \sin\phi) \, d\rho \, d\phi \, d\theta$$

Look at $\int_b^a \rho^3 e^{-\rho^2} d\rho$ Let $u = \rho^2$. $\therefore du = 2\rho \, d\rho$

When $\rho = b$, $u = b^2$, when $\rho = a$, $u = a^2$

$$\therefore \int_b^a \rho^3 e^{-\rho^2} d\rho = \frac{1}{2}\int_{b^2}^{a^2} u e^{-u} du$$ integrate by parts

$$= -\frac{1}{2} u e^{-u}\Big|_{b^2}^{a^2} + \frac{1}{2}\int_{b^2}^{a^2} e^{-u} du$$

$$= -\frac{1}{2} a^2 e^{-a^2} + \frac{1}{2} b^2 e^{-b^2} - \frac{1}{2} e^{-u}\Big|_{b^2}^{a^2}$$

$$= -\frac{1}{2} e^{-a^2}(a^2+1) + \frac{1}{2} e^{-b^2}(b^2+1)$$

$$\int_0^\pi \sin\phi \, d\phi = -\cos\phi\Big|_0^\pi = 2$$

$$\int_0^{2\pi} d\theta = 2\pi$$

$$\therefore \underline{-2\pi e^{-a^2}(a^2+1) + 2\pi e^{-b^2}(b^2+1)}$$

30.

(a) since $r^2 = x^2 + y^2 = 1$, $0 \le r \le 1$

$0 \le \theta \le 2\pi \qquad 0 \le z \le (r^2)^{\frac{1}{2}} = r$

$$\therefore \iiint_{B^*} (z)\, r\, dr\, d\theta\, dz = \int_0^{2\pi}\int_0^1\int_0^r zr\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 \left(\frac{z^2}{2}\Big|_{z=0}^{z=r}\right) r\, dr\, d\theta = \int_0^{2\pi}\int_0^1 \frac{r^3}{2}\, dr\, d\theta$$

$$= \int_0^{2\pi} \frac{r^4}{8}\Big|_{r=0}^{r=1} d\theta = \int_0^{2\pi} \frac{1}{8}\, d\theta = \frac{2\pi}{8} = \underline{\frac{\pi}{4}}$$

(b)

$x^2 + y^2 = r^2$, $x^2 + y^2 + z^2 \le 1 \Rightarrow r^2 + z^2 \le 1$

$\Rightarrow r \le \sqrt{1 - z^2}$

$\therefore 0 \le r \le \sqrt{1 - z^2}$

remember: r is in a plane parallel to xy-plane.

$0 \le \theta \le 2\pi$

$$\therefore \iiint_{W^*} \frac{1}{\sqrt{r^2+z^2}}\, r\, dr\, d\theta\, dz = \int_0^{2\pi}\int_{\frac{1}{2}}^{1}\int_0^{\sqrt{1-z^2}} \frac{r}{\sqrt{r^2+z^2}}\, dr\, dz\, d\theta$$

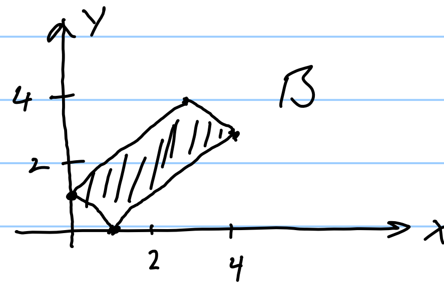
$$= \int_0^{2\pi}\int_{\frac{1}{2}}^{1} \left(\sqrt{r^2+z^2} \Big|_{r=0}^{r=\sqrt{1-z^2}} \right) dz\, d\theta$$

$$= \int_0^{2\pi}\int_{\frac{1}{2}}^{1} (1-z)\, dz\, d\theta = \int_0^{2\pi} \left(z - \frac{z^2}{2} \Big|_{z=\frac{1}{2}}^{z=1} \right) d\theta$$

$$= \int_0^{2\pi} 1 - \frac{1}{2} - \left(\frac{1}{2} - \frac{1}{8}\right) d\theta = \int_0^{2\pi} \frac{1}{8}\, d\theta$$

$$= \frac{2\pi}{8} = \underline{\underline{\frac{\pi}{4}}}$$

31.



Note the lines of two sides of B have form $y = x + k_1$, two sides have $y = -x + k_2$ or $x + y = k_2$, $x - y = -k_2$.

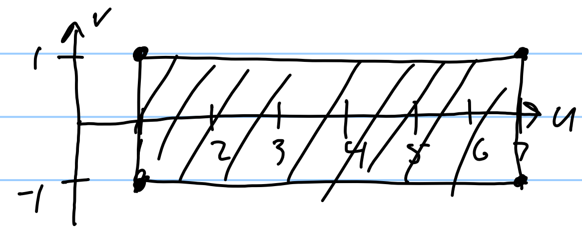
$x + y$ is in the integrand, $\therefore$ make a change of variable of form $u = x + y$, $v = x - y$.

$$\therefore T = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \text{ so } \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ x-y \end{bmatrix}$$

and $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$, where $\begin{bmatrix} u \\ v \end{bmatrix}$ = new corner for B^*

The new corners determine new limits of integration.

$$\therefore \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 4 & 3 \\ 1 & 0 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 7 & 7 \\ -1 & 1 & 1 & -1 \end{bmatrix}$$

$\therefore$ 

$$\det(T) = -2$$

$$\therefore \det(T^{-1}) = -\frac{1}{2}$$

It is $T^{-1} : (u,v) \to (x,y)$, or $T^{-1} : B^* \to B$

$$T^{-1} = -\frac{1}{2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$$

$$\therefore \left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \left| -\frac{1}{4} - \frac{1}{4} \right| = \frac{1}{2}$$

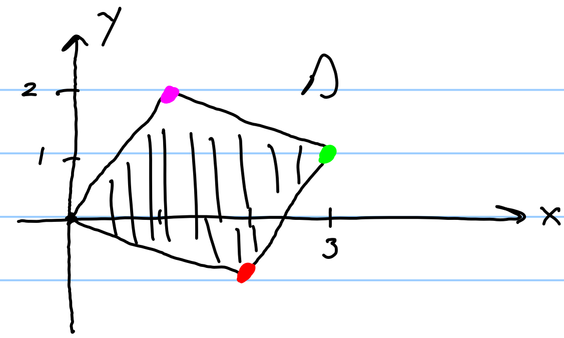
$$\therefore \iint_B (x+y)\,dx\,dy = \iint_{B^*} u\left(\frac{1}{2}\right) du\,dv$$

using
$u = x+y$
$v = x-y$

$$= \int_{-1}^{1}\int_{1}^{7} \frac{u}{2}\,du\,dv = \int_{-1}^{1} \left.\frac{u^2}{4}\right|_{u=1}^{u=7} dv$$

$$= \int_{-1}^{1} \left(\frac{49}{4} - \frac{1}{4}\right) dv = 12v\Big|_{-1}^{1} = \underline{24}$$

32.



Border lines are: $y = 2x$
$y = 2x - 5$

$y = -\frac{1}{2}x$, $y = -\frac{1}{2}x + \frac{5}{2}$

or $y - 2x = 0$ $\quad 2y + x = 0$
$y - 2x = -5$ $\quad 2y + x = 5$

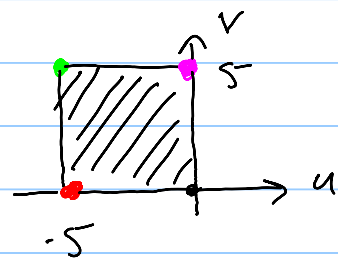
If $u = y - 2x$ Then $u + 2v = 5y$
$v = 2y + x$ $\quad v - 2u = 5x$ $\Big\}$ $y = \frac{u+2v}{5}$ [1]
$x = \frac{v-2u}{5}$ [2]

$\therefore \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$ $\quad \therefore 3v - u = 5y + 5x$, $x + y = \frac{3v - u}{5}$

Look at corners for new limits of integration (uv-plane)

$$\begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 2 & 3 & 1 \\ 0 & -1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -5 & -5 & 0 \\ 0 & 0 & 5 & 5 \end{bmatrix}$$

$\therefore D^*$



From [1], [2], $\left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \left| \begin{matrix} -2/5 & 1/5 \\ 1/5 & 2/5 \end{matrix} \right| = \frac{1}{5}$

$$\therefore \iint_D (x+y)\,dx\,dy = \iint_{D^*} \left(\frac{3v-u}{5}\right)\left(\frac{1}{5}\right) du\,dv$$

$$= \int_0^5 \int_{-5}^0 \frac{3v-u}{25}\, du\, dv = \int_0^5 \left(\frac{3vu}{25} - \frac{u^2}{50} \Big|_{u=-5}^{u=0} \right) dv$$

$$= \int_0^5 0 - \left(-\frac{3v}{5} - \frac{1}{2} \right) dv = \int_0^5 \frac{3v}{5} + \frac{1}{2}\, dv$$

$$= \frac{3v^2}{10} + \frac{v}{2} \Big|_0^5 = \frac{75}{10} + \frac{5}{2} - 0 = \underline{10}$$

Note: The strategy above is to look at the <u>sides</u> of D

(1) Assign $u =$ side 1, $v =$ side 2 ($ax + by$ form)
(2) Solve x, y in terms of u, v; i.e., get T^{-1}.

$\therefore \quad T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}, \quad T^{-1} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

(3) Convert $f(x,y)$ to $f(x(u,v), y(u,v))$

(4) Jacobian $= |\det(T^{-1})|$

(5) $T\begin{bmatrix} x \\ y \end{bmatrix}$ gives new corners $\begin{bmatrix} u \\ v \end{bmatrix}$ from old corners $\begin{bmatrix} x \\ y \end{bmatrix}$. New corners $\begin{bmatrix} u \\ v \end{bmatrix}$ yield new limits of integration for $\iint_{D^*}$

Method 2: Use given corners and assign new "rectangular" corners, respecting orientation.

(1) $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$, $\begin{bmatrix} x \\ y \end{bmatrix}$ = old corners,

$\begin{bmatrix} u \\ v \end{bmatrix}$ = new, assigned corners,

$T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and solve for $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

(2) Find T^{-1}, Jacobian is $|\det(T^{-1})|$

$= \dfrac{1}{|\det(T)|}$

(3) $T^{-1}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$, so now have

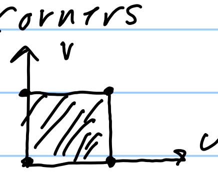
$x(u,v)$, $y(u,v)$ $[x = n_1 u + n_2 v,\ y = m_1 u + m_2 v]$

(4) Convert $f(x,y)$ into $f(x(u,v), y(u,v))$

(5) $\iint_D f(x,y)\,dx\,dy = \int_{u_1}^{u_2}\int_{v_1}^{v_2} f(u,v)\,|\det(T^{-1})|\,du\,dv$

In the problem above, make new corners

$(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$



(1) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 2 & 3 & 1 \\ 0 & -1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

Taking transposes, $\begin{bmatrix} 0 & 0 \\ 2 & -1 \\ 3 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$

$$\therefore \begin{bmatrix} 0 & 0 \\ 0 & -5 \\ 0 & -5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & -2 \\ 1 & -2 \\ 0 & 1 \end{bmatrix} \qquad \begin{array}{l} -5b = 1,\ b = -\frac{1}{5} \\ a + 2b = 0,\ a = 2/5 \\ -5d = -2,\ d = 2/5 \\ c + 2d = 1,\ c = 1/5 \end{array}$$

$$\therefore T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2/5 & -1/5 \\ 1/5 & 2/5 \end{bmatrix} \quad \det(T) = \frac{1}{5}$$

(2) $T^{-1} = 5 \begin{bmatrix} 2/5 & 1/5 \\ -1/5 & 2/5 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \quad \det(T^{-1}) = 5$

(3) $T^{-1} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 2u + v \\ -u + 2v \end{bmatrix}$

$$\therefore \begin{array}{l} x = 2u + v \\ y = -u + 2v \end{array}$$

(4) $f(x,y) = x + y = (2u + v) + (-u + 2v) = u + 3v$

(5) $\therefore \iint_D x + y \, dx dy = \iint_{D^*} (u + 3v)(5) \, du dv$

$$= \int_0^1 \int_0^1 (5u + 15v) \, du dv = \int_0^1 5\frac{u^2}{2} + 15uv \Big|_{u=0}^{u=1} dv$$

$$= \int_0^1 \left(\frac{5}{2} + 15v\right) dv = \frac{5}{2}v + \frac{15v^2}{2} \Big|_0^1 = \underline{10}$$

33.

(a) $\text{Vol } E = \iiint_D dx\,dy\,dz$, D = ellipsoid

$$= \int_{-a}^{a} \int_{-b\sqrt{1-x^2/a^2}}^{b\sqrt{1-x^2/a^2}} \int_{-c\sqrt{1-x^2/a^2-y^2/b^2}}^{c\sqrt{1-x^2/a^2-y^2/b^2}} dz\,dy\,dx$$

Let $x = au$, $y = bv$, $z = cw$ (assume $a, b, c > 0$).

$\therefore$ Jacobian $= \left|\dfrac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$

$$\therefore \iiint_D dx\,dy\,dz = \iiint_{D^*} abc\,du\,dv\,dw$$

and D^* becomes a unit ball.

Now use a spherical coordinate transformation.

$$\therefore \iiint_{D^*} abc\,du\,dv\,dw = \iiint_{D^{**}} abc\,\rho^2 \sin\phi\,d\rho\,d\theta\,d\phi$$

As D^{**} is a unit ball, limits of integration are
$0 \le \rho \le 1$, $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi$.

$$= abc \int_0^{\pi} \int_0^{2\pi} \int_0^{1} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

$$= abc \int_0^{\pi} \int_0^{2\pi} \left(\frac{\rho^3}{3} \Big|_{\rho=0}^{\rho=1} \right) \sin\phi \, d\theta \, d\phi$$

$$= \frac{abc}{3} \int_0^{\pi} \int_0^{2\pi} \sin\phi \, d\theta \, d\phi$$

$$= \frac{2\pi abc}{3} \int_0^{\pi} \sin\phi \, d\phi = \frac{2\pi abc}{3} \Big[-\cos\phi \Big]_0^{\pi}$$

$$= \underline{\frac{4\pi}{3} abc}$$

(b) Using the above substitution,

$$\iiint_D \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \right) dx\,dy\,dz =$$

$$\iiint_{D^*} (u^2 + v^2 + w^2)(abc) \, du\,dv\,dw =$$

$$abc\int_0^{\pi}\int_0^{2\pi}\int_0^{1} \rho^2 (\rho^2 \sin\phi)\, d\rho\, d\theta\, d\phi$$

$$= abc\int_0^{\pi}\int_0^{2\pi}\left(\sin\phi \frac{\rho^5}{5}\Big|_{\rho=0}^{\rho=1}\right) d\theta\, d\phi$$

$$= \frac{abc}{5}\int_0^{\pi}\int_0^{2\pi} \sin\phi\, d\theta\, d\phi = \frac{2\pi}{5} abc \int_0^{\pi} \sin\phi\, d\phi$$

$$= \frac{2\pi\, abc}{5}\left[-\cos\phi\right]\Big|_0^{\pi} = \underline{\underline{\frac{4\pi\, abc}{5}}}$$

34.

$0 \le \rho \le \sqrt{6}$, $0 \le \theta \le \frac{\pi}{2}$, $\frac{\pi}{4} \le \phi \le \arctan 2$

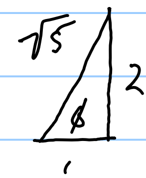
Note: We are changing variables here from xyz (problem states $\mathbb{R}^3$ 1st octant).

$$\therefore \int_{\frac{\pi}{4}}^{\arctan 2}\int_0^{\frac{\pi}{2}}\int_0^{\sqrt{6}} \left(\frac{1}{\rho}\right)(\rho^2 \sin\phi)\, d\rho\, d\theta\, d\phi$$

$$= \int_{\frac{\pi}{4}}^{\arctan 2} \int_0^{\frac{\pi}{2}} \left(\sin\phi \, \frac{\rho^2}{2} \Big|_{\rho=0}^{\rho=\sqrt{6}} \right) d\theta \, d\phi$$

$$= \int_{\frac{\pi}{4}}^{\arctan 2} \int_0^{\frac{\pi}{2}} 3\sin\phi \, d\theta \, d\phi = \int_{\frac{\pi}{4}}^{\arctan 2} \frac{3\pi}{2} \sin\phi \, d\phi$$

$$= -\frac{3\pi}{2} \cos\phi \Big|_{\phi=\frac{\pi}{4}}^{\phi=\arctan 2}$$



$\phi = \arctan 2$

$\cos\phi = \frac{1}{\sqrt{5}}$

$$= -\frac{3\pi}{2} \cos(\arctan 2) - \left(-\frac{3\pi}{2} \frac{\sqrt{2}}{2} \right)$$

$$= -\frac{3\pi}{2} \left(\frac{\sqrt{5}}{5} \right) + \frac{3\sqrt{2}}{4} \pi = \underline{3\pi \left(\frac{\sqrt{2}}{4} - \frac{\sqrt{5}}{10} \right)}$$

35.

(a) Let $T(u,v) = T(x,y) = (a,b)$ where $1 \le u \le 2, \ 1 \le v \le 3$

$1 \le x \le 2, \ 1 \le y \le 3$

$\therefore \ u^2 - v^2 = a$

$2uv = b$

(1) $\therefore\ v = \frac{b}{2u}$ as $u \neq 0$, $\therefore\ u^2 - \left(\frac{b}{2u}\right)^2 = a$

or $u^2 - \frac{b^2}{4u^2} = a$, $4u^4 - 4au^2 - b^2 = 0$

$$\therefore\ u^2 = \frac{4a \pm \sqrt{16a^2 - 4(4)(-b^2}}{8}$$

$$= \frac{4a \pm 4\sqrt{a^2+b^2}}{8} = \frac{a \pm \sqrt{a^2+b^2}}{2}$$

Since $u^2 > 0$ for $1 \le u \le 2$, and $\sqrt{a^2+b^2} > a^2$

reject the root $\frac{a - \sqrt{a^2+b^2}}{2}$

$$\therefore\ u^2 = \frac{a + \sqrt{a^2+b^2}}{2},\quad u = \sqrt{\frac{a+\sqrt{a^2+b^2}}{2}}$$

(2) Similarly, $u = \frac{b}{2v}$ as $v \neq 0$ for $1 \le v \le 3$

$\therefore\ \left(\frac{b}{2v}\right)^2 - v^2 = a$, $b^2 - 4v^4 = 4av^2$

Or, $4v^4 + 4av^2 - b^2 = 0$

$$\therefore\ v^2 = \frac{-4a \pm \sqrt{16a^2 - 4(4)(-b^2)}}{8}$$

$$= \frac{-a \pm \sqrt{a^2+b^2}}{2}$$

Since $\sqrt{a^2+b^2} > a > 0$, reject $\frac{-a-\sqrt{a^2+b^2}}{2}$

as $v^2 > 0$ for $1 \le v \le 3$

$\therefore v^2 = \frac{-a+\sqrt{a^2+b^2}}{2}$, $v = \sqrt{\frac{-a+\sqrt{a^2+b^2}}{2}}$

$\therefore T(u,v) = (a,b) \Rightarrow u = \sqrt{\frac{a+\sqrt{a^2+b^2}}{2}}$, $v = \sqrt{\frac{-a+\sqrt{a^2+b^2}}{2}}$

Similarly, $T(x,y) = (a,b) \Rightarrow x = \sqrt{\frac{a+\sqrt{a^2+b^2}}{2}}$, $y = \sqrt{\frac{-a+\sqrt{a^2+b^2}}{2}}$

$\therefore T(u,v) = T(x,y) \Rightarrow u = x$ and $v = y$.

$\therefore$ <u>T is one-to-one</u>

(b)

$$\iint_D dx\,dy = \iint_{D^*} \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

where $D^* = \{(u,v) : 1 \le u \le 2,\ 1 \le v \le 3\}$

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4u^2 + 4v^2$$

$$\therefore \iint_{D^*} \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv = \int_1^3 \int_1^2 (4u^2 + 4v^2)\, du\,dv$$

$$= \int_1^3 \left(\frac{4}{3}u^3 + 4uv^2 \Big|_{u=1}^{u=2} \right) dv$$

$$= \int_1^3 \frac{4}{3}(8) + 8v^2 - \frac{4}{3} - 4v^2 \, dv$$

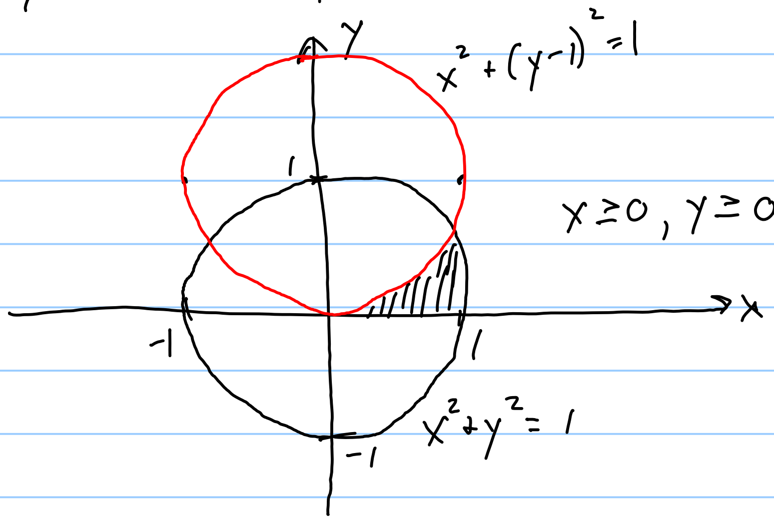
$$= \int_1^3 \left(4v^2 + \frac{28}{3} \right) dv = \frac{4}{3}v^3 + \frac{28}{3}v \Big|_1^3$$

$$= 36 + 28 - \left(\frac{4}{3} + \frac{28}{3} \right) = 64 - \frac{32}{3} = \frac{192}{3} - \frac{32}{3} = \underline{\underline{\frac{160}{3}}}$$

36.

(a) $x^2 + y^2 = 2y \Leftrightarrow x^2 + (y-1)^2 = 1$

$\therefore$



(6) (b) Let $u = x^2 + y^2$, $v = x^2 + y^2 - 2y$. Sketch the region D in the uv plane, which corresponds to R under this change of coordinates.

Since $0 \le x^2+y^2 \le 1$, $0 \le u \le 1$

For $x^2+(y-1)^2 \ge 1$, $v+1 \ge 1$, or $v \ge 0$

$T(x,y) = (x^2+y^2,\ x^2+y^2-2y) = (u,v)$

What is T^{-1}?

$x^2+y^2 = u$ so $x^2+y^2-2y = u-2y = v \Rightarrow y = \frac{u-v}{2}$

$\therefore x^2 + \left(\frac{u-v}{2}\right)^2 = u$, $4x^2 = 4u - (u-v)^2$
$= 4u - u^2 + 2uv - v^2$

$\therefore x^2 = u - \frac{u^2}{4} + \frac{uv}{2} - \frac{v^2}{4}$

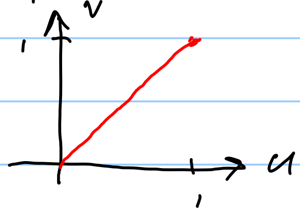
$\therefore T^{-1}(u,v) = \left(u + \frac{uv}{2} - \frac{u^2}{4} - \frac{v^2}{4},\ \frac{u-v}{2}\right)$

Not very helpful, except for $\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$

$\therefore$ Look at borders in x-y plane and graph borders in uv-plane using $T(x,y)$

With $(x,y) = (x,0)$, $T(x,0) = (x^2, x^2)$, $0 \le x \le 1$

$\therefore$



With $x^2+y^2=1$, $T(x,y)=(1,1-2y)$, $0\le y$

But since $x^2+y^2-2y\ge 0$, $1-2y\ge 0$, $y\le\frac{1}{2}$

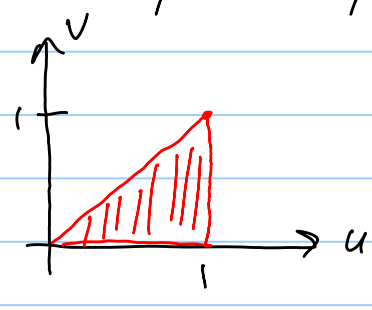
$\therefore$ $0\le y\le\frac{1}{2}$, $0\ge -2y\ge -1$, $0\le 1-2y\le 1$

$\therefore$

The top border in xy-plane is $x^2+y^2-2y=0$, with $0\le y\le\frac{1}{2}$, so $x^2+y^2\le 1 \Rightarrow x^2\le\frac{3}{4}$, $0\le x\le\sqrt{3}/2$.

$\therefore$ $T(x,y)=(2y,0)$, so $0\le 2y\le 1$.

$\therefore$



$\therefore$ $0\le u\le 1$, $0\le v\le u$

$$J=\left|\frac{\partial(x,y)}{\partial(u,v)}\right| \text{ with } x=u+\frac{uv}{2}-\frac{u^2}{4}-\frac{v^2}{4}$$
$$y=\frac{u-v}{2}$$

$$=\begin{vmatrix} 1+\frac{v}{2}-\frac{u}{2} & \frac{u}{2}-\frac{v}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix}$$

$$=\left|-\frac{1}{2}-\frac{v}{4}+\frac{u}{4}-\frac{u}{4}+\frac{v}{4}\right|=\frac{1}{2}$$

$$\therefore \iint_R x e^y \, dx\,dy = \iint_D \left(u + \frac{uv}{2} - \frac{u^2}{4} - \frac{v^2}{4}\right) e^{\frac{u-v}{2}} \left(\frac{1}{2}\right) du\,dv$$

$$= \int_0^1 \int_0^u \left(\frac{u}{2} + \frac{uv}{4} - \frac{u^2}{8} - \frac{v^2}{8}\right) e^{\frac{u-v}{2}} \, dv\,du$$

$$\int_0^u \left(\frac{u}{2} + \frac{uv}{4} - \frac{u^2}{8} - \frac{v^2}{8}\right) e^{\frac{u}{2}} e^{-\frac{v}{2}} \, dv$$

$$= \underbrace{\int_0^u \frac{u}{2} e^{\frac{u}{2}} e^{-\frac{v}{2}} dv}_{(1)} + \underbrace{\int_0^u e^{\frac{u}{2}} \frac{uv}{4} e^{-\frac{v}{2}} dv}_{(2)} - \underbrace{\int_0^u \frac{u^2}{8} e^{\frac{u}{2}} e^{-\frac{v}{2}} dv}_{(3)}$$

$$- \underbrace{\int_0^u e^{\frac{u}{2}} \frac{v^2}{8} e^{-\frac{v}{2}} dv}_{(4)}$$

$$= (-2) \frac{u}{2} e^{\frac{u}{2}} e^{-\frac{v}{2}} \Big|_{v=0}^{v=u} \qquad (1)$$

$$+ \frac{u e^{\frac{u}{2}}}{4} \left[\frac{v e^{-\frac{v}{2}}}{\left(-\frac{1}{2}\right)} \Bigg|_{v=0}^{v=u} \right] - \frac{u e^{\frac{u}{2}}}{4} \frac{1}{\left(-\frac{1}{2}\right)} \int_0^u e^{-\frac{v}{2}} dv \qquad (2)$$

$$- \left[\frac{u^2 e^{\frac{u}{2}}}{8} (-2) e^{-\frac{v}{2}} \Bigg|_{v=0}^{v=u} \right] \qquad (3)$$

$$-\frac{e^{\frac{u}{2}}}{8}\left[\frac{v^2 e^{-\frac{v}{2}}}{\left(-\frac{1}{2}\right)}\Bigg|_{v=0}^{v=u} - \frac{2}{\left(-\frac{1}{2}\right)}\int_0^u v e^{-\frac{v}{2}}\,dv\right] \qquad (4)$$

$$= \quad -u - \left(-u e^{\frac{u}{2}}\right) \qquad (1)$$

$$+ \frac{u e^{\frac{u}{2}}}{4}\left[-2u e^{-\frac{u}{2}}\right] + \frac{u e^{\frac{u}{2}}}{2}\left[(-2)\, e^{-\frac{v}{2}}\Bigg|_{v=0}^{v=u}\right] \qquad (2)$$

$$-\left[\frac{u^2 e^{\frac{u}{2}}}{-4} e^{-\frac{u}{2}} - \left(\frac{u^2 e^{\frac{u}{2}}}{-4}\right)\right] \qquad (3)$$

$$-\frac{e^{\frac{u}{2}}}{8}\left[-2u^2 e^{-\frac{u}{2}}\right] + 4\left[\frac{v e^{-\frac{v}{2}}}{\left(-\frac{1}{2}\right)}\Bigg|_{v=0}^{v=u} - \frac{1}{\left(-\frac{1}{2}\right)}\int_0^u e^{-\frac{v}{2}}\,dv\right] \qquad (4)$$

$$= -u + u e^{\frac{u}{2}} - \frac{u^2}{2} + \frac{u e^{\frac{u}{2}}}{2}\left[-2e^{-\frac{u}{2}} - (-2)\right] \qquad (1)+(2)$$

$$+ \frac{u^2}{4} - \frac{u^2 e^{\frac{u}{2}}}{4} \qquad (3)$$

$$+ \frac{u^2}{4} - 8u e^{-\frac{u}{2}} + 2\left[(-2)\, e^{-\frac{v}{2}}\Bigg|_{v=0}^{v=u}\right] \qquad (4)$$

$$= -u + u e^{\frac{u}{2}} - \frac{u^2}{2} - u + u e^{\frac{u}{2}} \qquad (1)+(2)$$

$$+ \frac{u^2}{4} - \frac{u^2 e^{\frac{u}{2}}}{4} \qquad (3)$$

$$+\frac{u^2}{4} - 8ue^{-\frac{u}{2}} + 2\left[-2e^{-\frac{u}{2}} - (-2)\right] \qquad (4)$$

$$= -2u + 2ue^{\frac{u}{2}} - \frac{u^2}{4} - \frac{u^2 e^{\frac{u}{2}}}{4} \qquad (1),(2),(3)$$

$$+\frac{u^2}{4} - 8ue^{-\frac{u}{2}} - 4e^{-\frac{u}{2}} + 4 \qquad (4)$$

$$= -2u + 2ue^{\frac{u}{2}} - \frac{u^2 e^{\frac{u}{2}}}{4} - 8ue^{-\frac{u}{2}} - 4e^{-\frac{u}{2}} + 4$$

$$= 4 - 2u - 4e^{-\frac{u}{2}} + 2ue^{\frac{u}{2}} - 8ue^{-\frac{u}{2}} - \frac{u^2 e^{\frac{u}{2}}}{4} \qquad (5)$$

$$\therefore \int_0^1 \overbrace{4 - 2u - 4e^{-\frac{u}{2}}}^{[6]} + \overbrace{2ue^{\frac{u}{2}}}^{[7]} - \overbrace{8ue^{-\frac{u}{2}}}^{[8]} - \overbrace{\frac{u^2}{4}e^{\frac{u}{2}}}^{[9]}\, du$$

$$= \left(4u - u^2 + 8e^{-\frac{u}{2}}\right)\Big|_0^1 \qquad [6]$$

$$+ 2\left[\frac{ue^{\frac{u}{2}}}{(\frac{1}{2})}\Big|_0^1 - \frac{1}{(\frac{1}{2})}\int_0^1 e^{\frac{u}{2}}\,du\right] \qquad [7]$$

$$- 8\left[\frac{ue^{-\frac{u}{2}}}{(-\frac{1}{2})}\Big|_0^1 - \frac{1}{(-\frac{1}{2})}\int_0^1 e^{-\frac{u}{2}}\,du\right] \qquad [8]$$

$$- \frac{1}{4}\left[\frac{u^2 e^{\frac{u}{2}}}{(\frac{1}{2})}\Big|_0^1 - \frac{2}{(\frac{1}{2})}\int_0^1 ue^{\frac{u}{2}}\,du\right] \qquad [9]$$

$$= 4 - 1 + 8e^{-\frac{1}{2}} - (8) \qquad [6]$$

$$+ 2\left[2e^{\frac{1}{2}} - 2\left(2e^{\frac{u}{2}}\Big|_0^1\right)\right] \qquad [7]$$

$$- 8\left[-2e^{-\frac{1}{2}} + 2\left(-2e^{-\frac{u}{2}}\Big|_0^1\right)\right] \qquad [8]$$

$$- \frac{1}{4}\left[2e^{\frac{1}{2}} - 4\left[\frac{ue^{\frac{u}{2}}}{(\frac{1}{2})}\Big|_0^1 - \frac{1}{(\frac{1}{2})}\int_0^1 e^{\frac{u}{2}}\,du\right]\right] \qquad [9]$$

$$= -5 + 8e^{-\frac{1}{2}} + 4e^{\frac{1}{2}} - 4\left[2e^{\frac{1}{2}} - 2\right] \qquad [6]+[7]$$

$$+ 16e^{-\frac{1}{2}} - 16\left[-2e^{-\frac{1}{2}} - (-2)\right] \qquad [8]$$

$$- \frac{1}{2}e^{\frac{1}{2}} + \left[2e^{\frac{1}{2}} - 2\left(2e^{\frac{u}{2}}\Big|_0^1\right)\right] \qquad [9]$$

$$= -5 + 8e^{-\frac{1}{2}} + 4e^{\frac{1}{2}} - 8e^{\frac{1}{2}} + 8 \qquad [6]+[7]$$

$$+ 16e^{-\frac{1}{2}} + 32e^{-\frac{1}{2}} - 32 \qquad [8]$$

$$- \frac{1}{2}e^{\frac{1}{2}} + 2e^{\frac{1}{2}} - 2\left(2e^{\frac{1}{2}} - 2\right) \qquad [9]$$

$$= -29 + 56e^{-\frac{1}{2}} - 4e^{\frac{1}{2}} \qquad [6],[7],[8]$$

$$- \frac{1}{2}e^{\frac{1}{2}} + 2e^{\frac{1}{2}} - 4e^{\frac{1}{2}} + 4 \qquad [9]$$

$$= -25 + 56e^{-\frac{1}{2}} - \frac{13}{2}e^{\frac{1}{2}}$$

37.

Note for $x^{3/2} + y^{3/2} = a^{3/2}$ and $x, y \geq 0$, Then
$0 \leq x \leq a$ and $0 \leq y \leq a$.

The problem wants the transformation to be a triangle with sides a, not $a^{3/2}$

$\therefore$ If $u = a^{-\frac{1}{2}} x^{3/2}$, then $a^{\frac{1}{2}} u = x^{3/2}$, so $x = a^{\frac{1}{3}} u^{2/3}$

$\therefore$ When $0 \leq x \leq a$, at $x = a$, $u = a^{-\frac{1}{2}} (a)^{3/2} = a$

$\therefore$ $u = a^{-\frac{1}{2}} x^{3/2}$ and $v = a^{-\frac{1}{2}} y^{3/2}$ is the correct

transformation so D^* is a triangle with sides a.

$\therefore$ $x = a^{\frac{1}{3}} u^{2/3}$, $y = a^{\frac{1}{3}} v^{2/3}$

$$\therefore \frac{\partial(x,y)}{\partial(u,v)} = \begin{bmatrix} \frac{2}{3} a^{\frac{1}{3}} u^{-\frac{1}{3}} & 0 \\ 0 & \frac{2}{3} a^{\frac{1}{3}} v^{-\frac{1}{3}} \end{bmatrix}$$

$$\therefore J = \frac{4}{9} a^{2/3} u^{-\frac{1}{3}} v^{-\frac{1}{3}}$$

$$\therefore \iint_D f(x,y)\,dx\,dy = \iint_{D^*} f\left(a^{\frac{1}{3}} u^{2/3}, a^{\frac{1}{3}} v^{2/3}\right)\left(\frac{4}{9} a^{2/3} u^{-\frac{1}{3}} v^{-\frac{1}{3}}\right) du\,dv$$

$$= \frac{4}{9} a^{2/3} \iint_{D^*} f(a^{\frac{1}{3}} u^{2/3}, a^{\frac{1}{3}} v^{2/3})\, u^{-\frac{1}{3}} v^{-\frac{1}{3}}\, du\, dv$$

38.

Let $S(\rho_1, \theta_1, \phi_1) = S(\rho_2, \theta_2, \phi_2) = (x, y, z)$ [1]

and assume $\rho_1 > 0$, $\rho_2 > 0$.

(a) $\therefore x^2 + y^2 + z^2 = (\rho_1^2 \sin^2\phi_1 \cos^2\theta_1) + (\rho_1^2 \sin^2\phi_1 \sin^2\theta_1) + (\rho_1^2 \cos^2\phi_1)$

$= \rho_1^2 \sin^2\phi_1 [\cos^2\theta_1 + \sin^2\theta_1] + \rho_1^2 \cos^2\phi_1$

$= \rho_1^2 \sin^2\phi_1 + \rho_1^2 \cos^2\phi_1 = \rho_1^2$

Similarly, $x^2 + y^2 + z^2 = \rho_2^2$. $\therefore \rho_1^2 = \rho_2^2 \Rightarrow \rho_1 = \rho_2$

(b) $\therefore$ [1] $\Rightarrow (\rho_1 \sin\phi_1 \cos\theta_1, \rho_1 \sin\phi_1 \sin\theta_1, \rho_1 \cos\phi_1) =$

$(\rho_2 \sin\phi_2 \cos\theta_2, \rho_2 \sin\phi_2 \sin\theta_2, \rho_2 \cos\phi_2)$

$\Rightarrow (\sin\phi_1 \cos\theta_1, \sin\phi_1 \sin\theta_1, \cos\phi_1) =$

$(\sin\phi_2 \cos\theta_2, \sin\phi_2 \sin\theta_2, \cos\phi_2)$

$\therefore \cos\phi_1 = \cos\phi_2$ But $0 \le \phi_1 \le \pi$, $0 \le \phi_2 \le \pi$

and on $0 \le w \le \pi$, $\cos(w)$ is one-to-one

$\therefore \cos\phi_1 = \cos\phi_2 \Rightarrow \phi_1 = \phi_2$

(c) $\therefore$ [1] $\Rightarrow \sin\phi \sin\theta_1 = \sin\phi \sin\theta_2$ (y-coordinates)

If $\phi \neq 0, \pi$, then $\sin\phi \neq 0$.

$\therefore \sin\theta_1 = \sin\theta_2$

Also, $\cos\theta_1 = \cos\theta_2$ (x-coordinates).

Over $0 \le \theta < 2\pi$, $\sin\theta_1 = \sin\theta_2 \Rightarrow$

$\theta_1 = \theta_2$ or $\pi - \theta_1 = \theta_2$ (and $\theta_1 \neq \theta_2$)

But $\pi - \theta_1 = \theta_2$ is not possible since

then $\cos(\pi - \theta_1) = \cos(\theta_2)$, and

$$\begin{aligned} \cos(\pi-\theta_1) &= \cos(\pi)\cos(\theta_1) + \sin(\pi)\sin(\theta_1) \\ &= -\cos(\theta_1) \end{aligned}$$

$\therefore \pi - \theta_1 = \theta_2 \Rightarrow -\cos(\theta_1) = \cos(\theta_2)$

contradicting $\cos(\theta_1) = \cos(\theta_2)$

$\therefore$ Over $0 \le \theta < 2\pi$, [1] $\Rightarrow \theta_1 = \theta_2$

$\therefore$ [1] $\Rightarrow$ (a) $\rho_1 = \rho_2$ (assuming $\rho \neq 0$)

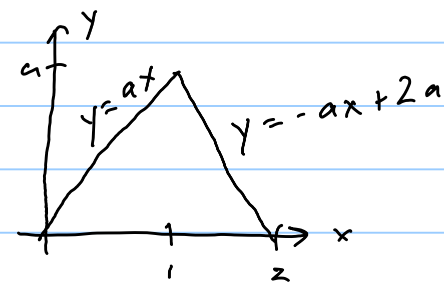
(b) $\phi_1 = \phi_2$ (assuming $0 < \phi < \pi$)

(c) $\theta_1 = \theta_2$ (assuming $0 \leq \theta < 2\pi$)

$\therefore$ $S(\rho, \theta, \phi)$ is one-to-one with the finite limitations mentioned above.

6.3 Applications

1.



$$\bar{x} = \frac{\iint_D x\,\delta\,dx\,dy}{\iint_D \delta\,dx\,dy} = \frac{\iint_D x\,dx\,dy}{\iint_D dx\,dy}$$

Similarly, $\bar{y} = \dfrac{\iint_D y\,dx\,dy}{\iint_D dx\,dy}$

Consider D to be x-simple

$\therefore\ x = \frac{y}{a}$ and $x = \frac{2a-y}{a} = 2 - \frac{y}{a}$

$$\therefore \iint_D dx\,dy = \int_0^a \int_{\frac{y}{a}}^{2-\frac{y}{a}} dx\,dy = \int_0^a 2 - \frac{2y}{a}\,dy$$

$$= 2y - \frac{y^2}{a}\Big|_{y=0}^{a} = 2a - a = a$$

consistent with (area of triangle)(density) = $(\frac{1}{2}2a)\delta = \delta a$.

$$\therefore\ \bar{x} = \frac{1}{a}\int_0^a \int_{\frac{y}{a}}^{2-\frac{y}{a}} x\,dx\,dy = \frac{1}{a}\int_0^a \left(\frac{x^2}{2}\Big|_{x=\frac{y}{a}}^{x=2-\frac{y}{a}}\right) dy$$

$$= \frac{1}{a} \int_0^a \frac{1}{2} \left[\left(2 - \frac{y}{a}\right)^2 - \left(\frac{y}{a}\right)^2 \right] dy$$

$$= \frac{1}{a} \int_0^a \frac{1}{2} \left[4 - \frac{4y}{a} \right] dy = \frac{1}{2a} \left[4y - \frac{2y^2}{a} \right]_{y=0}^{y=a}$$

$$= \frac{1}{2a} \left[4a - 2a \right] = \underline{1}$$

$$\bar{y} = \frac{1}{a} \int_0^a \int_{\frac{y}{a}}^{2-\frac{y}{a}} y \, dx \, dy = \frac{1}{a} \int_0^a y \left(2 - \frac{2y}{a}\right) dy$$

$$= \frac{1}{a} \int_0^a 2y - \frac{2y^2}{a} \, dy = \frac{1}{a} \left[y^2 - \frac{2}{3} \frac{y^3}{a} \right]_{y=0}^{y=a}$$

$$= \frac{1}{a} \left[a^2 - \frac{2}{3} a^2 \right] = \underline{\frac{a}{3}}$$

$\therefore$ Center of mass $= \underline{\left(1, \frac{a}{3}\right)}$

2.

x-coordinate will be $x = 0$ from symmetry.

$$\bar{y} = \frac{\iint_D y \, dx \, dy}{\text{Area of } D} = \frac{1}{\frac{1}{2}\pi r^2} \int_{-r}^{r} \int_0^{\sqrt{r^2 - x^2}} y \, dy \, dx$$

$$= \frac{2}{\pi r^2} \int_{-r}^{r} \left[\frac{y^2}{2} \Big|_{y=0}^{y=\sqrt{r^2-x^2}} \right] dx = \frac{1}{\pi r^2} \int_{-r}^{r} (r^2 - x^2)\, dx$$

$$= \frac{1}{\pi r^2} \left[r^2 x - \frac{x^3}{3} \right]_{x=-r}^{x=r}$$

$$= \frac{1}{\pi r^2} \left[r^3 - \frac{r^3}{3} - \left(-r^3 + \frac{r^3}{3} \right) \right] = \frac{1}{\pi r^2} \left[\frac{4}{3} r^3 \right]$$

$$= \frac{4r}{3\pi}$$

$\therefore$ Center of mass $= \underline{\left(0, \frac{4r}{3\pi}\right)}$

3.

$$\iint_D f(x,y)\, dx\, dy = \int_0^{\pi} \int_0^{\pi} y \sin(xy)\, dx\, dy$$

$$= \int_0^{\pi} \left[-\cos(xy) \Big|_{x=0}^{x=\pi} \right] dy = \int_0^{\pi} -\cos(\pi y) + 1 \; dy$$

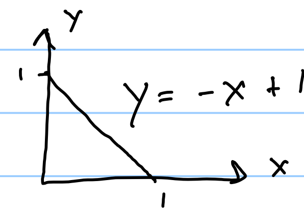
$$= -\frac{1}{\pi} \sin(\pi y) + y \Big|_0^{\pi} = \pi - \frac{\sin(\pi^2)}{\pi}$$

But $\sin(\pi^2) = \sin(\pi)\cos(\pi) + \cos(\pi)\sin(\pi) = 0 + 0 = 0$

$\therefore \iint_D f(x,y)\, dx\, dy = \pi$

$$\text{Area of } D = \pi^2. \quad \therefore \text{Average} = \frac{\pi}{\pi^2} = \underline{\underline{\frac{1}{\pi}}}$$

4.



$$\text{Area of triangle} = \frac{1}{2}$$

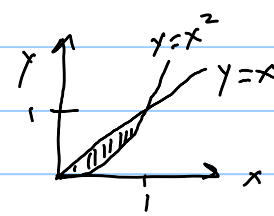
$$\therefore \iint_D f(x,y)\,dx\,dy = \int_0^1 \int_0^{-x+1} e^{x+y}\,dy\,dx$$

$$= \int_0^1 \left(e^{x+y} \Big|_{y=0}^{y=-x+1} \right) dx = \int_0^1 e - e^x\,dx$$

$$= ex - e^x \Big|_{x=0}^{x=1} = e - e - (0 - 1) = 1$$

$$\therefore \text{Average} = \frac{1}{1/2} = \underline{2}$$

5.



$$\text{Mass of } D = \iint_D \delta(x,y)\,dx\,dy = \int_0^1 \int_{x^2}^{x} (x+y)\,dy\,dx$$

$$= \int_0^1 \left(xy + \frac{y^2}{2} \Big|_{y=x^2}^{y=x} \right) dx = \int_0^1 \left[x^2 + \frac{x^2}{2} - \left(x^3 + \frac{x^4}{2} \right) \right] dx$$

$$= \int_0^1 -\frac{x^4}{2} - x^3 + \frac{3}{2}x^2\, dx = \frac{-x^5}{10} - \frac{x^4}{4} + \frac{x^3}{2} \Big|_{x=0}^{1}$$

$$= -\frac{1}{10} - \frac{1}{4} + \frac{1}{2} = \frac{-2-5+10}{20} = \frac{3}{20}$$

$$\therefore \quad \bar{x} = \frac{20}{3} \int_0^1 \int_{x^2}^{x} x(x+y)\, dy\, dx = \frac{20}{3} \int_0^1 x^2 y + \frac{xy^2}{2} \Big|_{y=x^2}^{y=x} dx$$

$$= \frac{20}{3} \int_0^1 \left[x^3 + \frac{x^3}{2} - \left(x^4 + \frac{x^5}{2} \right) \right] dx$$

$$= \frac{20}{3} \left(\frac{3}{8}x^4 - \frac{x^5}{5} - \frac{x^6}{12} \right) \Big|_{x=0}^{x=1}$$

$$= \frac{20}{3} \left(\frac{3}{8} - \frac{1}{5} - \frac{1}{12} \right) = \frac{20}{3} \left(\frac{45-24-10}{120} \right) = \frac{20}{3} \cdot \frac{11}{120}$$

$$= \underline{\frac{11}{18}}$$

$$\bar{y} = \frac{20}{3} \int_0^1 \int_{x^2}^{x} y(x+y)\, dy\, dx = \frac{20}{3} \int_0^1 \frac{xy^2}{2} + \frac{y^3}{3} \Big|_{y=x^2}^{y=x} dx$$

$$= \frac{20}{3} \int_0^1 \frac{x^3}{2} + \frac{x^3}{3} - \left(\frac{x^5}{2} + \frac{x^6}{3} \right) dx$$

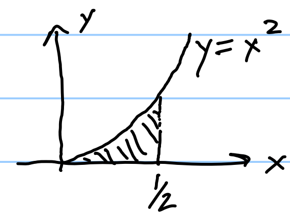
$$= \frac{20}{3} \left[\frac{5}{6} \cdot \frac{x^4}{4} - \frac{x^6}{12} - \frac{x^7}{21} \right]_{x=0}^{x=1}$$

$$= \frac{20}{3}\left[\frac{5}{24} - \frac{1}{12} - \frac{1}{21}\right] = \frac{20}{3}\left[\frac{35-14-8}{2^3\cdot 3\cdot 7}\right]$$

$$= \frac{5}{3}\left(\frac{13}{2\cdot 3\cdot 7}\right) = \frac{65}{126}$$

$$\therefore \text{Center of mass} = \left(\frac{11}{18}, \underline{\frac{65}{126}}\right)$$

6.



Assume uniform density $= \delta$

$$\text{Mass of area} = \delta \iint_D dx\,dy = \delta \int_0^{\frac{1}{2}} \int_0^{x^2} dy\,dx = \delta \int_0^{\frac{1}{2}} x^2\,dx$$

$$= \delta \left.\frac{x^3}{3}\right|_0^{\frac{1}{2}} = \delta \frac{1}{24}$$

$$\therefore \bar{x} = \frac{24}{\delta}\iint_D \delta x\,dx\,dy = 24\int_0^{\frac{1}{2}}\int_0^{x^2} x\,dy\,dx$$

$$= 24\int_0^{1/2} x^3\,dx = 24\left[\frac{x^4}{4}\right]_{x=0}^{x=\frac{1}{2}} = \frac{24}{64} = \underline{\frac{3}{8}}$$

$$\bar{y} = \frac{24}{\delta}\iint_D \delta y\,dx\,dy = 24\int_0^{\frac{1}{2}}\int_0^{x^2} y\,dy\,dx$$

$$= 24\int_0^{\frac{1}{2}} \left.\frac{y^2}{2}\right|_{y=0}^{y=x^2} dx = 24\int_0^{1/2}\frac{x^4}{2}\,dx = 12\left[\frac{x^5}{5}\right]_0^{\frac{1}{2}}$$

$$= \frac{12}{32(5)} = \frac{3}{40}$$

$$\therefore \text{Center of mass} = \left(\frac{3}{8}, \frac{3}{40}\right)$$

7.

$$\text{Mass} = \iint_D \delta(x,y)\,dx\,dy = \int_0^{2\pi}\int_0^{\pi} y^2 \sin^2(4x) + 2 \; dy\,dx$$

$$= \int_0^{2\pi} \left[\frac{y^3}{3}\sin^2(4x) + 2y\right]_{y=0}^{y=\pi} dx$$

$$= \int_0^{2\pi} \frac{\pi^3}{3}\sin^2(4x) + 2\pi \; dx$$

use $\sin^2\theta = \frac{1-\cos 2\theta}{2}$

$$= \int_0^{2\pi} \frac{\pi^3}{3}\left(\frac{1-\cos 8x}{2}\right) + 2\pi \; dx$$

$$= \frac{\pi^3}{6}x - \frac{\pi^3}{48}\sin 8x + 2\pi x \Big|_0^{2\pi}$$

$$= \frac{\pi^4}{3} - 0 + 4\pi^2 = \frac{\pi^4}{3} + 4\pi^2 \text{ grams}$$

$\therefore$ Cost $= 7\left(\frac{\pi^4}{3} + 4\pi^2\right) = \$\underline{\underline{503.64}}$

8.

From #7, total mass $= \frac{\pi^4}{3} + 4\pi^2$ grams

Total area of plate $D = (2\pi)(\pi) = 2\pi^2$

$\therefore$ Average density $= \dfrac{\frac{\pi^4}{3} + 4\pi^2}{2\pi^2} = \underline{\frac{\pi^2}{6} + 2}$

9.

(a) Mass = (Density)(Volume) $= \delta\left(\frac{1}{2} \times 1 \times 2\right) = \underline{\delta}$

where δ = uniform density of the mass.

(b) Mass $= \iiint_V \delta(x,y,z)\,dx\,dy\,dz$

$$= \int_0^{1/2}\int_0^1\int_0^2 (x^2 + 3y^2 + z + 1)\,dz\,dy\,dx$$

assuming mass orientation is $0 \le x \le \frac{1}{2}$, $0 \le y \le 1$, $0 \le z \le 2$

$$= \int_0^{\frac{1}{2}} \int_0^1 \left(x^2 z + 3y^2 z + \frac{z^2}{2} + z \Big|_{z=0}^{2} \right) dy\,dx$$

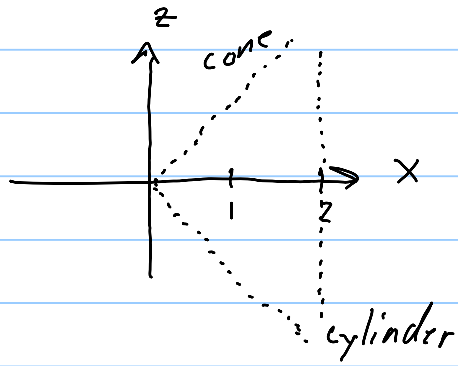
$$= \int_0^{1/2} \int_0^1 2x^2 + 6y^2 + 4 \; dy\,dx$$

$$= \int_0^{\frac{1}{2}} \left(2x^2 y + 2y^3 + 4y \Big|_{y=0}^{1} \right) dx$$

$$= \int_0^{\frac{1}{2}} 2x^2 + 6 \; dx = \frac{2}{3}x^3 + 6x \Big|_{x=0}^{\frac{1}{2}}$$

$$= \frac{1}{12} + 3 = \underline{3\tfrac{1}{12}}$$

10.

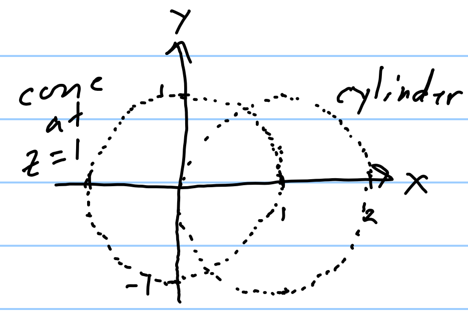


Cylinder $x^2 + y^2 = 2x \Leftrightarrow (x-1)^2 + y^2 = 1$

V is described as $0 \le x \le 2$,

$-\sqrt{2x - x^2} \le y \le \sqrt{2x - x^2}$,

$-\sqrt{x^2+y^2} \le z \le \sqrt{x^2+y^2}$



$\therefore$ Mass $= \iiint_V \delta(x,y,z)\,dx\,dy\,dz$

$$= \int_0^2 \int_{-\sqrt{2x-x^2}}^{\sqrt{2x-x^2}} \int_{-\sqrt{x^2+y^2}}^{\sqrt{x^2+y^2}} \sqrt{x^2+y^2}\,dz\,dy\,dx$$

It likely will be easier to use cylindrical coordinates

$$\text{Mass} = \iiint_V \delta(r,\theta,z)\,r\,dr\,d\theta\,dz$$

For $\delta(x,y,z) = \sqrt{x^2+y^2}$, $\delta(r,\theta,z) = r$

For V, $(x-1)^2+y^2=1$ translates to $r = 2\cos\theta$:

$$(r\cos\theta-1)^2 + r^2\sin^2\theta = 1 \iff r^2\cos^2\theta - 2r\cos\theta + 1 + r^2\sin^2\theta = 1$$
$$\iff r^2 - 2r\cos\theta = 0$$
$$\iff r = 2\cos\theta$$

$\therefore V: -\frac{\pi}{2} \le \theta \le \frac{\pi}{2},\ 0 \le r \le 2\cos\theta,\ -r \le z \le r$

$$\therefore \text{Mass} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} \int_{-r}^{r} r^2\,dz\,dr\,d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} \left(r^2 z \Big|_{z=-r}^{z=r} \right) dr\,d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} 2r^3\,dr\,d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{r^4}{2} \Bigg|_{r=0}^{r=2\cos\theta} \right) d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 8\cos^4\theta \, d\theta$$ use $\cos^2 x = \frac{1+\cos 2x}{2}$

$$= 8\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta = 2\int_{-\pi/2}^{\pi/2} 1 + 2\cos 2\theta + \cos^2 2\theta \, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} 2\, d\theta + 4\int_{-\pi/2}^{\pi/2} \cos 2\theta \, d\theta + 2\int_{-\pi/2}^{\pi/2} \cos^2 2\theta \, d\theta$$

$$= 2\theta \Big|_{-\pi/2}^{\pi/2} + 2\sin 2\theta \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1+\cos 4\theta}{2} \, d\theta$$

$$= (2\pi) + (0) + \theta \Big|_{-\pi/2}^{\pi/2} + \frac{\sin 4\theta}{4} \Big|_{-\pi/2}^{\pi/2}$$

$$= 2\pi + \pi + 0 = 3\pi$$

$\therefore$ Mass $= \underline{3\pi}$

11.

Use spherical coordinates. $\delta(\rho, \theta, \phi) = 2\rho^2 + 1$

$$\text{Mass} = \iiint_V \delta(\rho, \theta, \phi)\, \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

$$= \int_0^{2\pi} \int_0^{\pi} \int_0^{5} (2\rho^2+1)\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi} \int_0^{5} \sin\phi \, (2\rho^4+\rho^2) \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi} \sin\phi \left[\frac{2}{5}\rho^5 + \frac{\rho^3}{3} \right]_{\rho=0}^{5} d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi} \sin\phi \left(1250 + \frac{125}{3} \right) d\phi \, d\theta$$

$$= \int_0^{2\pi} \frac{3875}{3} \left[-\cos\phi \right]_{\phi=0}^{\pi} d\theta = \int_0^{2\pi} \frac{7750}{3} \, d\theta$$

$$= \underline{\frac{15500\pi}{3}}$$

12.

Use cylindrical coordinates. $\delta(r,\theta,z) = 2r^2 + 2z^2 + 1$

Mass $= \iiint_V \delta(r,\theta,z) \, r \, dr \, d\theta \, dz$

$$= \int_0^2 \int_0^{2\pi} \int_0^9 (2r^2+2z^2+1)\, r\, dr\, d\theta\, dz$$

$$= \int_0^2 \int_0^{2\pi} \int_0^9 2r^3 + 2rz^2 + r\, dr\, d\theta\, dz$$

$$= \int_0^2 \int_0^{2\pi} \frac{r^4}{2} + r^2 z^2 + \frac{r^2}{2} \Big|_{r=0}^{9} d\theta\, dz$$

$$= \int_0^2 \int_0^{2\pi} \left(\frac{6561}{2} + 81z^2 + \frac{81}{2}\right) d\theta\, dz = \int_0^2 (3321 + 81z^2)\, 2\pi\, dz$$

$$= 6642\pi z + 27z^3 \Big|_{z=0}^{z=2}$$

$$= \underline{13{,}284\pi + 216}$$

13.

$z = 2 - x - y$. At $z = 0$, $x + y = 2$. x-intercept: $(2,0,0)$

$\therefore$ Volume $= \displaystyle\int_0^2 \int_0^{2-x} \int_0^{2-x-y} dz\, dy\, dx$

$$= \int_0^2 \int_0^{2-x} (2-x-y)\,dy\,dx = \int_0^2 2y - xy - \frac{y^2}{2} \Big|_{y=0}^{y=2-x} dx$$

$$= \int_0^2 2(2-x) - x(2-x) - \frac{(2-x)^2}{2}\,dx$$

$$= \int_0^2 \left(2 - 2x + \frac{x^2}{2}\right) dx = 2x - x^2 + \frac{x^3}{6}\Big|_0^2 = \underline{\frac{4}{3}}$$

$$\bar{x} = \frac{1}{4/3} \int_0^2 \int_0^{2-x} \int_0^{2-x-y} x\,dz\,dy\,dx$$

$$= \frac{3}{4} \int_0^2 \int_0^{2-x} (2x - x^2 - xy)\,dy\,dx = \frac{3}{4} \int_0^2 2xy - x^2 y - \frac{xy^2}{2} \Big|_{y=0}^{y=2-x} dx$$

$$= \frac{3}{4} \int_0^2 2x(2-x) - x^2(2-x) - x\frac{(2-x)^2}{2}\,dx$$

$$x^2 - 4x + 4$$

$$= \frac{3}{4} \int_0^2 4x - 2x^2 - 2x^2 + x^3 - \frac{(x^3 - 4x^2 + 4x)}{2}\,dx$$

$$= \frac{3}{4} \int_0^2 \frac{x^3}{2} - 2x^2 + 2x\,dx = \frac{3}{4}\left[\frac{x^4}{8} - \frac{2}{3}x^3 + x^2\right]_0^2$$

$$= \frac{3}{4}\left(2 - \frac{16}{3} + 4\right) = \underline{\frac{1}{2}}$$

$$\bar{y} = \frac{3}{4}\int_0^2\int_0^{2-x}\int_0^{2-x-y} y\,dz\,dy\,dx = \frac{3}{4}\int_0^2\int_0^{2-x} y(2-x-y)\,dy\,dx$$

$$= \frac{3}{4}\int_0^2\int_0^{2-x}(2y - xy - y^2)\,dy\,dx = \frac{3}{4}\int_0^2\left(y^2 - \frac{xy^2}{2} - \frac{y^3}{3}\Big|_{y=0}^{y=2-x}\right)dx$$

$$= \frac{3}{4}\int_0^2 (2-x)^2 - \frac{x(2-x)^2}{2} - \frac{(2-x)^3}{3}\,dx$$

$$= \frac{3}{4}\int_0^2 x^2 - 4x + 4 - \frac{(x^3 - 4x^2 + 4x)}{2} + \frac{(x^3 - 6x^2 + 12x - 8)}{3}\,dx$$

$$6x^2 - 24x + 24 - 3x^3 + 12x^2 - 12x + 2x^3 - 12x^2 + 24x - 16$$

$$= \frac{3}{4}\int_0^2 \frac{-x^3 + 6x^2 - 12x + 8}{6}\,dx = \frac{1}{8}\left[-\frac{x^4}{4} + 2x^3 - 6x^2 + 8x\right]_{x=0}^{2}$$

$$= \frac{1}{8}(-4 + 16 - 24 + 16) = \frac{1}{8}(4) = \underline{\frac{1}{2}}$$

$$\bar{z} = \frac{3}{4}\int_0^2\int_0^{2-x}\int_0^{2-x-y} z\,dz\,dy\,dx = \frac{3}{4}\int_0^2\int_0^{2-x} \frac{z^2}{2}\Big|_{z=0}^{z=2-x-y} dy\,dx$$

$$= \frac{3}{4}\int_0^2\int_0^{2-x} \frac{4 - 4x - 4y + x^2 + 2xy + y^2}{2}\,dy\,dx$$

$$= \frac{3}{8}\int_0^2 4y - 4xy - 2y^2 + x^2y + xy^2 + \frac{y^3}{3}\Big|_{y=0}^{y=(2-x)}\,dx$$

$$= \frac{3}{8}\int_0^2 4(2-x) - 4x(2-x) - 2(2-x)^2 + x^2(2-x) + x(2-x)^2 + \frac{(2-x)^3}{3}\,dx$$

$$8 - 4x - 8x + 4x^2 - 2x^2 + 8x - 8 + 2x^2 - x^3 + x^3 - 4x^2 + 4x + \frac{(2-x)^3}{3}$$

$$= \frac{3}{8}\int_0^2 \frac{(2-x)^3}{3}\,dx = \frac{1}{8}(-1)\left.\frac{(2-x)^4}{4}\right|_{x=0}^{x=2}$$

$$= -\frac{1}{32}(0 - 16) = \frac{16}{32} = \underline{\frac{1}{2}}$$

$\therefore$ Center of Mass $= (\bar{x}, \bar{y}, \bar{z}) = \underline{\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)}$

14.

Mass $= \iiint_V \delta \, dV$, use cylindrical coordinates

$$\therefore \delta = (r^2\cos^2\theta + r^2\sin^2\theta)z^2 = r^2z^2$$

$$\therefore \text{Mass} = \int_1^2\int_0^1\int_0^{2\pi}(r^2z^2)\,r\,d\theta\,dr\,dz$$

$$= \int_1^2\int_0^1 2\pi r^3 z^2\,dr\,dz = \int_1 \pi z^2 \left.\frac{r^4}{2}\right|_{r=0}^{1} dz$$

$$= \int_1^2 \frac{\pi}{2} z^2\,dz = \frac{\pi}{2}\left[\frac{z^3}{3}\right]_{z=1}^{2}$$

$$= \frac{\pi}{2}\left[\frac{8}{3} - \frac{1}{3}\right] = \underline{\frac{7\pi}{6}}$$

$$\bar{x} = \frac{\iiint_V x\,\delta\,dV}{\text{Mass}}$$

Again, use cylindrical coordinates $\delta = r^2 z^2 \quad x = r\cos\theta$

$$= \frac{6}{7\pi}\int_1^2\int_0^1\int_0^{2\pi} (r\cos\theta)(r^2 z^2)\, r\, d\theta\, dr\, dz$$

$\sin(2\pi) = \sin(0) = 0$

$$= \frac{6}{7\pi}\int_1^2\int_0^1 r^4 z^2 \sin\theta \Big|_{\theta=0}^{\theta=2\pi} = \underline{0}$$

which makes sense since δ is symmetric with respect to z-axis : $\delta(x,y,z) = \delta(-x,y,z) = \delta(x,-y,z)$

$$\bar{y} = \frac{\iiint_V y\,\delta\,dV}{\text{Mass}}$$

Use cylindrical coordinates with $\delta = r^2 z^2 \quad y = r\sin\theta$

$$= \frac{6}{7\pi}\int_1^2\int_0^1\int_0^{2\pi} (r\sin\theta)(r^2 z^2)\, r\, d\theta\, dr\, dz$$

$$= \frac{6}{7\pi}\int_1^2\int_0^1 r^4 z^2 \Big[-\cos\theta\Big]_{\theta=0}^{\theta=2\pi} dr\, dz$$

$$= \frac{6}{7\pi}\int_1^2\int_0^1 r^4 z^2 \big[-1-(-1)\big]\, dr\, dz = \underline{0}$$

$$\bar{z} = \frac{\iiint_V z\,\delta\, dV}{mass}$$ Use cylindrical coordinates $\delta = r^2 z^2$

$$= \frac{6}{7\pi}\int_1^2\int_0^1\int_0^{2\pi} (z)(r^2z^2)\, r\, d\theta\, dr\, dz$$

$$= \frac{6}{7\pi}(2\pi)\int_1^2\int_0^1 z^3 r^3\, dr\, dz = \frac{12}{7}\int_1^2 z^3 \left.\frac{r^4}{4}\right|_0^1 dz$$

$$= \frac{12}{7}\cdot\frac{1}{4}\left[\frac{z^4}{4}\right]_{z=1}^{z=2} = \frac{3}{7}\left(\frac{16}{4}-\frac{1}{4}\right) = \underline{\frac{45}{28}}$$

$\therefore$ Center of mass $= \underline{\left(0, 0, \frac{45}{28}\right)}$

15.

Volume of cube $= (2)(4)(6) = 48$

Assume $0 \le x \le 2$, $0 \le y \le 4$, $0 \le z \le 6$

$$\therefore \text{Ave} = \frac{1}{48}\int_0^2\int_0^6\int_0^4 \sin^2(\pi z)\cos^2(\pi x)\, dy\, dz\, dx$$

$$= \frac{4}{48}\int_0^2 \cos^2(\pi x)\, dx \int_0^6 \sin^2(\pi z)\, dz$$

$\cos^2\theta - \sin^2\theta = \cos 2\theta$, $2\cos^2\theta - 1 = \cos 2\theta$, $1 - 2\sin^2\theta = \cos 2\theta$

$$= \frac{1}{12}\int_0^2 \frac{1+\cos 2\pi x}{2}\,dx \int_0^6 \frac{1-\cos 2\pi z}{2}\,dz$$

$$= \frac{1}{48}\left[x + \frac{\sin 2\pi x}{2\pi}\right]_0^2 \left[z - \frac{\sin 2\pi z}{2\pi}\right]_0^6$$

$$= \frac{1}{48}\left[2+0-(0+0)\right]\left[6-0-(0-0)\right]$$

$$= \frac{12}{48} = \underline{\underline{\frac{1}{4}}}$$

16.

Volume of ball $=1$. Use spherical coordinates

$z = \rho\cos\phi$

$$\therefore \text{Ave} = \int_0^1\int_0^{\pi}\int_0^{2\pi} e^{-\rho\cos\phi}\rho^2\sin\phi\, d\theta\, d\phi\, d\rho$$

$$= 2\pi\int_0^1\int_0^{\pi} e^{-\rho\cos\phi}\rho^2\sin\phi\, d\phi\, d\rho$$

$$= 2\pi\int_0^1 \rho^2\left[\frac{e^{-\rho\cos\phi}}{\rho}\right]_{\phi=0}^{\phi=\pi} d\rho$$

$$= 2\pi\int_0^1 \rho\left[e^{\rho} - e^{-\rho}\right] d\rho$$

$$= 2\pi \int_0^1 \rho e^{\rho}\, d\rho - 2\pi \int_0^1 \rho e^{-\rho}\, d\rho$$

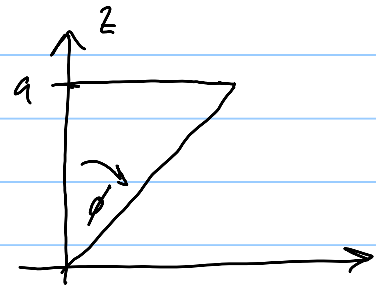
$$= 2\pi \left[\rho e^{\rho} - e^{\rho}\right]_{\rho=0}^{\rho=1} - 2\pi \left[-\rho e^{-\rho} - e^{-\rho}\right]_{\rho=0}^{\rho=1}$$

$$= 2\pi \left[e - e - (0-1)\right] - 2\pi \left[-e^{-1} - e^{-1} - (0-1)\right]$$

$$= 2\pi - 2\pi(-2e^{-1} + 1)$$

$$= \underline{\underline{\frac{4\pi}{e}}}$$

17.



Let δ = density, a constant.

Use cylindrical coordinates. An elemental mass is
$\delta dV = \delta\, r\, dr\, d\theta\, dz$, $0 \le z \le a$, $0 \le \theta \le 2\pi$, $0 \le r \le z\tan(\kappa)$

The distance^2 from the z-axis of the elemental
mass is $(= x^2+y^2)$ $(r\cos\theta)^2 + (r\sin\theta)^2 = r^2$

$$\therefore \quad \underline{I_z = \int_0^a \int_0^{2\pi} \int_0^{z\tan(\kappa)} \delta\, r^3\, dr\, d\theta\, dz}$$

With spherical coordinates, an elemental mass is
$\delta\, dV = \delta \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi$, $0 \le \theta \le 2\pi$, $0 \le \phi \le K$,
$0 \le \rho \le \frac{a}{\cos\phi} = a\sec\phi$

The distance^2 from the z-axis of the elemental mass is $(\equiv x^2+y^2)$ $(\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\sin\theta)^2$
$= \rho^2 \sin^2\phi$

$$\therefore I_z = \int_0^K \int_0^{2\pi} \int_0^{a\sec\phi} (\rho^2 \sin^2\phi)\, \delta \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi$$

$$= \underline{\underline{\int_0^K \int_0^{2\pi} \int_0^{a\sec\phi} \delta \rho^4 \sin^3\phi \, d\rho\, d\theta\, d\phi}}$$

18.

$I_y = \iiint_V (x^2+z^2)\, \delta\, dV$ Use spherical coordinates.

$dV = \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi$ $\quad x^2+z^2 = (\rho\sin\phi\cos\theta)^2 + (\rho\cos\phi)^2$
$= \rho^2\sin^2\phi\cos^2\theta + \rho^2\cos^2\phi$

$$\therefore I_y = \int_0^{\pi} \int_0^{2\pi} \int_0^{R} \delta\left[\rho^2\sin^2\phi\cos^2\theta + \rho^2\cos^2\phi\right] \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi$$

$$= \delta \int_0^{\pi} \int_0^{2\pi} \int_0^{R} \left[\sin^3\phi \cos^2\theta + \cos^2\phi \sin\phi \right] \rho^4 \, d\rho \, d\theta \, d\phi$$

$$= \delta \int_0^{\pi} \int_0^{2\pi} \left[\sin^3\phi \cos^2\theta + \cos^2\phi \sin\phi \right] \frac{\rho^5}{5} \Big|_{\rho=0}^{\rho=R} d\theta \, d\phi$$

$$= \frac{\delta R^5}{5} \int_0^{\pi} \int_0^{2\pi} \sin^3\phi \left(\frac{1+\cos 2\theta}{2} \right) + \cos^2\phi \sin\phi \, d\theta \, d\phi$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$= \frac{\delta R^5}{5} \int_0^{\pi} \sin^3\phi \left[\frac{\theta}{2} + \frac{\sin 2\theta}{4} \right]_{\theta=0}^{\theta=2\pi} + \cos^2\phi \sin\phi \left[\theta \right]_{\theta=0}^{\theta=2\pi} d\phi$$

$$= \frac{\delta R^5}{5} \int_0^{\pi} \pi \sin^3\phi + 2\pi \cos^2\phi \sin\phi \, d\phi$$

$$= \frac{\delta \pi R^5}{5} \int_0^{\pi} \sin\phi \, (1-\cos^2\phi) + 2\cos^2\phi \sin\phi \, d\phi$$

$$= \frac{\delta \pi R^5}{5} \int_0^{\pi} (\sin\phi + \cos^2\phi \sin\phi) \, d\phi$$

$$= \frac{\delta \pi R^5}{5} \left[-\cos\phi - \frac{\cos^3\phi}{3} \right]_{\phi=0}^{\phi=\pi}$$

$$= \frac{\delta \tilde{\pi} R^5}{5}\left[1 + \frac{1}{3} - \left(-1 - \frac{1}{3}\right)\right] = \frac{\delta \tilde{\pi} R^5}{5}\left[\frac{8}{3}\right]$$

$$= \underline{\frac{8 \delta \tilde{\pi} R^5}{15}}$$

19.

$$V = \frac{-GMm}{r} = -\frac{(6.67\times10^{-11})(3\times10^{26})}{2\times10^{8}}\,m$$

$$= -10\times10^{7}\,m = \underline{-1.0\times10^{8}\,m}\ \ N\cdot m$$

20.

$$F = \frac{GMm}{r^2} = \frac{(6.67\times10^{-11})(3\times10^{26})(70)}{(2\times10^{8})^2}$$

$$= \frac{(1.0\times10^{8})(70)}{(2\times10^{8})} = \underline{35 \text{ Newtons}}$$

21.

(a) The vertical plane down the middle dividing the car in half (left from right) is the only plane of symmetry.

(6)

(1) By definition,

$$\bar{z} = \frac{\iiint_W z\,\delta(x,y,z)\,dx\,dy\,dz}{\iiint_W \delta(x,y,z)\,dx\,dy\,dz}$$

as the Mass $= \iiint_W \delta(x,y,z)\,dx\,dy\,dz$

$$\therefore \bar{z}\cdot \iiint_W \delta(x,y,z)\,dx\,dy\,dz = \iiint_W z\,\delta(x,y,z)\,dx\,dy\,dz$$

(2) $\iiint_W = \iiint_{W^+} + \iiint_{W^-}$ is just additivity

of integrals (property (iv) on p. 275 of text).

(3) By symmetry, $\delta(x,y,z) = \delta(x,y,-z)$

and again by symmetry, integrating

$-a \le z \le -b \le 0$ on W^- is the same as integrating $0 \le b \le -z \le a$ on W^+

$$\therefore \int_{-a}^{-b} z \, dz = \int_{a}^{b} -z \, dz, \text{ or } \int_{W^-} z = \int_{W^+} -z$$

In the formula, the u, v, w are dummy variables

$$\therefore \iiint_{W^-} z \, \delta(x,y,z) \, dx\,dy\,dz = \iiint_{W^+} -z \, \delta(x,y,-z) \, dx\,dy\,dz$$

(4) $$\iiint_{W^+} -z \, \delta(x,y,-z) = \iiint_{W^+} -w \, \delta(u,v,-w) \, du\,dv\,dw$$

as the x,y,z and u,v,w are dummy variables.

$\therefore$ Last step is:

$$\iiint_{W^+} z \, \delta(x,y,z) \, dx\,dy\,dz - \iiint_{W^+} z \, \delta(x,y,z) \, dx\,dy\,dz$$

and this is $\underline{0}$.

(c)

(6) proves $\bar{z} = 0$, as the mass in not zero.

$\therefore$ z-coordinate of center of mass is in the plane of symmetry (the xy-plane).

(d)

From (c), one coordinate lies on one plane, another lies on the other plane. To lie on both planes, the center of mass must lie on the common aspect of each plane – i.e., the line of intersection.

22.

(a) Given constant density σ, an element of mass is $\sigma \Delta \, dx\,dy$, where Δ = the thickness of the plate, so $\Delta \, dx\,dy$ is an element of volume.
Let $\delta = \sigma \Delta$, a constant. $\therefore$ An element of mass $= \delta \, dx\,dy$ (dx = width, dy = length).

If the position of the element of mass is (x, y), its distance from the origin (center of mass) is $\sqrt{x^2+y^2}$.

A unit of mass rotating about an axis at angular velocity ω moves at speed $r\omega$, where r = distance from axis of rotation.

$\therefore r\omega = \sqrt{x^2+y^2}\,\omega$.

Kinetic energy of a unit mass m is $\frac{1}{2}mv^2$.

$$\therefore \frac{1}{2}mv^2 = \frac{1}{2}(\delta\, dx\, dy)\left(\sqrt{x^2+y^2}\,\omega\right)^2$$

$$= \delta \frac{\omega^2}{2}(x^2+y^2)\, dx\, dy$$

(b)

A Riemann sum of the kinetic energy of all the elemental masses yields the total kinetic energy, summing over the region of "plate"

$$\therefore K.E. = \iint_{plate} \delta \frac{\omega^2}{2}(x^2+y^2)\, dx\, dy$$

(c)

$$\int_{-\frac{a}{2}}^{\frac{a}{2}} \int_{-\frac{b}{2}}^{\frac{b}{2}} \delta \frac{\omega^2}{2}(x^2+y^2)\, dy\, dx = \frac{\delta\omega^2}{2}\int_{-a/2}^{a/2}\int_{-b/2}^{b/2}(x^2+y^2)\, dy\, dx$$

$$= \frac{\delta\omega^2}{2}\int_{-a/2}^{a/2} x^2y + \frac{y^3}{3}\Big|_{y=-b/2}^{y=b/2}\, dx$$

$$= \frac{\delta \omega^2}{2} \int_{-a/2}^{a/2} \frac{b}{2}x^2 + \frac{b^3}{24} - \left(-\frac{b}{2}x^2 - \frac{b^3}{24}\right) dx$$

$$= \frac{\delta \omega^2}{2} \int_{-a/2}^{a/2} bx^2 + \frac{b^3}{12}\, dx = \frac{\delta \omega^2}{2} \left[\frac{bx^3}{3} + \frac{b^3}{12} x \right]_{x=-\frac{a}{2}}^{x=a/2}$$

$$= \frac{\delta \omega^2}{2} \left[\frac{ba^3}{24} + \frac{b^3}{12} \cdot \frac{a}{2} - \left(-\frac{ba^3}{24} - \frac{b^3}{12}\frac{a}{2} \right) \right]$$

$$= \frac{\delta \omega^2}{2} \left\{ \frac{ba^3}{12} + \frac{b^3 a}{12} \right\} = \underline{\frac{\delta \omega^2}{24} \left(a^3 b + a b^3 \right)}$$

23.

(a) Mass of the constant density portion is:

$$\rho \frac{4}{3} \pi (10^4 \text{ cm})^3 = (3) \frac{4}{3} \pi 10^{12} = 4\pi 10^{12} \text{ g}$$

(b) Need to find the mass of the "shell" between 10^4 cm and 5×10^8 cm.

Use spherical coordinates. A unit of volume of the shell is $\rho^2 \sin\phi \, d\rho d\theta d\phi$.

$\therefore$ Mass of the unit volume is $\frac{3\times10^4}{\rho} \rho^2 \sin\phi \, d\rho d\theta d\phi$

$= (3\times10^4 \text{cm}) \rho \sin\phi \, d\rho d\theta d\phi$.

$\therefore$ Total mass $= (3\times10^4) \int_0^{\pi} \int_0^{2\pi} \int_{10^2}^{5\times10^6} \rho \sin\phi \, d\rho d\theta d\phi$

$= 3\times10^4 \int_0^{\pi} \int_0^{2\pi} \sin\phi \left. \frac{\rho^2}{2} \right|_{\rho=10^4}^{\rho=5\times10^8} d\theta d\phi$

$= 3\times10^4 \int_0^{\pi} \int_0^{2\pi} (12.5\times10^{16}) \sin\phi \, d\theta d\phi$

$= 3.75\times10^{21} \int_0^{\pi} \int_0^{2\pi} \sin\phi \, d\theta d\phi = 7.5\times10^{21}\pi \int_0^{\pi} \sin\phi \, d\phi$

$= (7.5\times10^{21}\pi) \left[-\cos\phi\right]_{\phi=0}^{\phi=\pi}$

$= 15\pi\times10^{21}$ g

(c) Total mass of planet, as seen from a point "outside" C.M.W. is:

$4\pi\times10^{12}\text{ g} + 15\pi\times10^{21}\text{ g} \simeq 15\pi\times10^{21}\text{ g}$

$= 15\pi\times10^{18}$ Kg

(d) $\therefore$ Potential "outside" C.M.W is $-\dfrac{GMm}{R}$

$$= -\left(6.67\times10^{-11}\ \frac{N\cdot m^2}{Kg^2}\right)\left(15\pi\times10^{18}\ Kg\right)\frac{m}{R}\ ,\quad R > 5\times10^6\ m$$

$$= -3.14\times10^{9}\ \frac{m}{R}\ N\cdot m \qquad \left(N\cdot m \equiv Kg\left(\tfrac{m}{sec}\right)^2\right)$$

24.

Consider an element of $A(D)$, call it dA. The solid swept out by dA around the y-axis is $2\pi x\, dA$, where x is the distance of dA from the y-axis. That is, the circumference, $2\pi x$, time the area of the element, dA.

$\therefore$ Total volume swept out by D is:

$$Vol(W) = \iint_D 2\pi x\, dA$$

By definition, $\bar{x} = \dfrac{\iint_D x\, dx dy}{\iint_D dx dy} = \dfrac{\iint_D x\, dx dy}{A(D)}$

$$\therefore\ \bar{x}\, A(D) = \iint_D x\, dx dy$$

$\therefore\ 2\pi\bar{x}A(D) = 2\pi\iint_D x\,dx\,dy = \iint_D 2\pi x\,dx\,dy$

But $dA = dx\,dy$ (an element of $A(D)$).

$\therefore\ 2\pi\bar{x}A(D) = \iint_D 2\pi x\,dx\,dy = \iint_D 2\pi x\,dA = Vol(W)$

$\therefore\ \underline{\underline{Vol(W) = 2\pi\bar{x}A(D)}}$

25.

Assume $a > r$

The center of mass of the circle is at $(a, 0)$, or a units from the y-axis. $\therefore\ \bar{x} = a$.

The area of the circle of radius r is πr^2

$\therefore\ Vol(W) = 2\pi\bar{x}A(D) = 2\pi(a)(\pi r^2) = \underline{2\pi^2 a r^2}$

6.4 Improper Integrals

Note Title 1/9/2017

1.

Choose $0<\delta$, $0<\epsilon$ s.t. $D_{\delta,\epsilon} \subset D$

By Fubini's Theorem,

$$\iint_{D_{\delta,\epsilon}} \frac{1}{\sqrt{xy}}\,dx\,dy = \int_{\delta}^{1-\delta}\int_{\epsilon}^{1-\epsilon} \frac{1}{\sqrt{xy}}\,dx\,dy$$

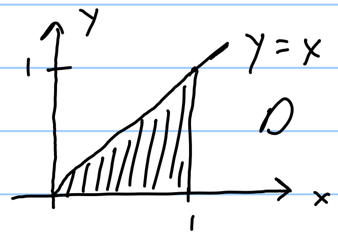
$$= \int_{\delta}^{1-\delta} \frac{1}{\sqrt{y}}\,dy \int_{\epsilon}^{1-\epsilon} \frac{1}{\sqrt{x}}\,dx = \left(2\sqrt{y}\,\Big|_{y=\delta}^{y=1-\delta}\right)\left(2\sqrt{x}\,\Big|_{x=\epsilon}^{x=1-\epsilon}\right)$$

$$= 4\left(\sqrt{1-\delta} - \sqrt{\delta}\right)\left(\sqrt{1-\epsilon} - \sqrt{\epsilon}\right)$$

$$\therefore \lim_{(\delta,\epsilon)\to(0,0)} \iint_{D_{\delta,\epsilon}} \frac{1}{\sqrt{xy}}\,dA = \lim_{(\delta,\epsilon)\to(0,0)} 4\left(\sqrt{1-\delta}-\sqrt{\delta}\right)\left(\sqrt{1-\epsilon}-\sqrt{\epsilon}\right)$$

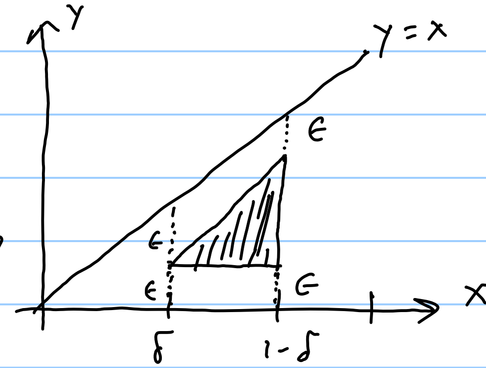
$$= \underline{4}$$

2.



For $y \leq x$, $x - y \geq 0$ so $|x-y| = x-y$.

Let $\delta > 0$, $\epsilon > 0$ and consider $\delta \leq x \leq 1-\delta$ and $\epsilon \leq y \leq 1-\epsilon$



The segment from δ to $1-\delta$ must be >0, so $1-\delta-\delta > 0 \Rightarrow 2\delta < 1$, or $\delta < \frac{1}{2}$. $\therefore$ Choose δ s.t. $0 < \delta < \frac{1}{2}$.

Similarly, the height of the triangle must be >0. The smallest height is at $x = \delta$ (height $=0$). So $2\epsilon < \delta$, so choose ϵ s.t. $\epsilon < \frac{\delta}{2}$.

$\therefore \Delta_{\delta,\epsilon} \subset \Delta$ and on $\Delta_{\delta,\epsilon}$ $x - y > 0$.

$$\therefore \iint_{\Delta_{\delta,\epsilon}} \frac{1}{\sqrt{|x-y|}}\,dx\,dy = \iint_{\Delta_{\delta,\epsilon}} \frac{1}{\sqrt{x-y}}\,dx\,dy$$

$$= \int_{\delta}^{1-\delta} \int_{\epsilon}^{x-\epsilon} \frac{1}{\sqrt{x-y}}\,dy\,dx \qquad \text{by Fubini's Theorem}$$

$$= \int_{\delta}^{1-\delta} \left(-2\sqrt{x-y}\,\Big|_{y=\epsilon}^{y=x-\epsilon} \right) dx$$

$$= \int_{\delta}^{1-\delta} \left(2\sqrt{x-\epsilon} - 2\sqrt{\epsilon} \right) dx$$

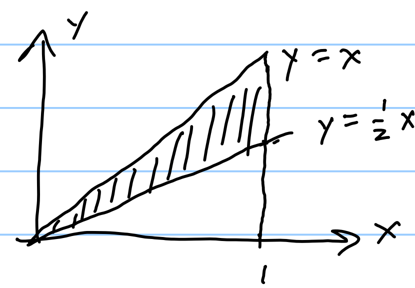
$$= (2)\left(\frac{2}{3}\right)(x-\epsilon)^{3/2}\Big|_{x=\delta}^{x=1-\delta} - 2\sqrt{\epsilon}\, x\Big|_{x=\delta}^{x=1-\delta}$$

$$= \frac{4}{3}(1-\delta-\epsilon)^{3/2} - \frac{4}{3}(\delta-\epsilon)^{3/2} - 2\sqrt{\epsilon}(1-\delta) + 2\sqrt{\epsilon}\,\delta$$

$$\therefore \lim_{(\delta,\epsilon)\to(0,0)} \iint_{D_{\delta,\epsilon}} \frac{1}{\sqrt{|x-y|}}\,dx\,dy = \frac{4}{3} - 0 - 0 + 0 = \frac{4}{3}$$

$$\therefore \iint_{D} \frac{1}{\sqrt{|x-y|}}\,dx\,dy = \underline{\frac{4}{3}}$$

3.



Consider D_δ : $0<\delta<1$,
$\delta \le x \le 1$

$\therefore \frac{y}{x}$ is defined over all of D_δ.

$$\therefore \iint_{D_\delta} \frac{y}{x}\,dx\,dy = \int_\delta^1 \int_{\frac{x}{2}}^{x} \frac{y}{x}\,dy\,dx$$

$$= \int_\delta^1 \left(\frac{y^2}{2x}\Big|_{y=\frac{x}{2}}^{y=x} \right) dx = \int_\delta^1 \left(\frac{x}{2} - \frac{x}{8}\right) dx$$

$$= \frac{3}{8}\,\frac{x^2}{2}\Big|_{x=\delta}^{x=1} = \frac{3}{16}\left[1-\delta^2\right]$$

$$\therefore \iint_D \frac{y}{x}\,dx\,dy = \lim_{\delta \to 0} \iint_{D_\delta} \frac{y}{x}\,dy\,dx = \lim_{\delta \to 0} \frac{3}{16}\left[1-\delta^2\right]$$

$$= \underline{\frac{3}{16}}$$

4.

Let $D_\delta : 0 \le y \le 1,\ 0 < \delta \le x \le e^v$

$$\therefore \int_0^1 \int_0^{e^v} \log x \,dx\,dy = \lim_{\delta \to 0^+} \int_0^1 \int_\delta^{e^v} \log x\,dx\,dy$$

$$= \lim_{\delta \to 0^+} \int_\delta^{e^v} \log x\,dx = \lim_{\delta \to 0^+} \Big[x \log x - x\Big]_{x=\delta}^{x=e^v}$$

$$= \lim_{\delta \to 0^+} \left[e^v \log e^v - e^v - \delta \log \delta + \delta\right]$$

$$= ve^v - e^v - \lim_{\delta \to 0^+} \delta \log \delta = ve^v - e^v - \lim_{\delta \to 0^+} \frac{\log \delta}{\frac{1}{\delta}}$$

$$= ve^v - e^v - \lim_{\delta \to 0^+} \frac{\frac{1}{\delta}}{-\frac{1}{\delta^2}} \quad \text{(L'Hopital's)}$$

$$= ve^v - e^v - \lim_{\delta \to 0^+} (-\delta) = \underline{ve^v - e^v}$$

5.

Let $0 < \delta < 1$, $0 < \epsilon < 1$, $D_{\delta,\epsilon}$: $\delta \le x \le 1$, $\epsilon \le y \le 1$

$$\therefore \iint_D \frac{dx\,dy}{x^\alpha y^\beta} = \lim_{(\delta,\epsilon)\to(0,0)} \iint_{D_{\delta,\epsilon}} \frac{dx\,dy}{x^\alpha y^\beta}$$

$$= \lim_{(\delta,\epsilon)\to(0,0)} \int_\delta^1 \int_\epsilon^1 \frac{1}{x^\alpha y^\beta}\,dy\,dx$$

$$= \lim_{(\delta,\epsilon)\to(0,0)} \left[\frac{x^{1-\alpha}}{1-\alpha}\right]_{x=\delta}^{x=1} \left[\frac{y^{1-\beta}}{1-\beta}\right]_{y=\epsilon}^{y=1}$$

$$= \lim_{(\delta,\epsilon)\to(0,0)} \left[\frac{1}{1-\alpha} - \frac{\delta^{1-\alpha}}{1-\alpha}\right]\left[\frac{1}{1-\beta} - \frac{\epsilon^{1-\beta}}{1-\beta}\right]$$

$$= \underline{\left(\frac{1}{1-\alpha}\right)\left(\frac{1}{1-\beta}\right)} \quad \text{as} \quad \lim_{x\to 0} x^r = 0 \text{ for } r > 0.$$

c.

Let $\delta > 1$, $\epsilon > 1$. Define $D_{\delta,\epsilon} : 1 \leq x \leq \delta,\ 1 \leq y \leq \epsilon$

$$\therefore \iint_D \frac{dx\,dy}{x^\gamma y^\rho} = \lim_{\substack{\delta \to \infty^+ \\ \epsilon \to \infty^+}} \iint_{D_{\delta,\epsilon}} \frac{dx\,dy}{x^\gamma y^\rho}$$

$$= \lim_{\substack{\delta \to \infty^+ \\ \epsilon \to \infty^+}} \int_1^\delta \int_1^\epsilon \frac{dy\,dx}{x^\gamma y^\rho} = \lim_{\substack{\delta \to \infty^+ \\ \epsilon \to \infty^+}} \left[\frac{x^{1-\gamma}}{1-\gamma}\right]_{x=1}^{x=\delta} \left[\frac{y^{1-\rho}}{1-\rho}\right]_{y=1}^{y=\epsilon}$$

$$= \lim_{\substack{\delta \to \infty^+ \\ \epsilon \to \infty^+}} \left[\frac{\delta^{1-\gamma}}{1-\gamma} - \frac{1}{1-\gamma}\right]\left[\frac{\epsilon^{1-\rho}}{1-\rho} - \frac{1}{1-\rho}\right]$$

$$= \lim_{\substack{\delta \to \infty^+ \\ \epsilon \to \infty^+}} \left[\frac{1}{(1-\gamma)\,\delta^{\gamma-1}} + \frac{1}{\gamma-1}\right]\left[\frac{1}{(1-\rho)\,\epsilon^{\rho-1}} + \frac{1}{\rho-1}\right]$$

$$= \left[0 + \frac{1}{\gamma-1}\right]\left[0 + \frac{1}{\rho-1}\right] = \underline{\underline{\frac{1}{(\gamma-1)(\rho-1)}}}$$

7.

(a) Use polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, $0 \le \theta \le 2\pi$
and $0 < \delta \le r \le 1$ $dA = r\,dr\,d\theta$

$$\therefore \iint_D \frac{dA}{(x^2+y^2)^{2/3}} = \lim_{\delta \to 0} \iint_{D_\delta} \frac{r\,dr\,d\theta}{(r^2)^{2/3}}$$

$$= \lim_{\delta \to 0} \int_\delta^1 \int_0^{2\pi} r^{-\frac{1}{3}}\,dr\,d\theta = 2\pi \lim_{\delta \to 0} \int_\delta^1 r^{-\frac{1}{3}}\,dr$$

$$= 2\pi \lim_{\delta \to 0} \left(\frac{3}{2} r^{\frac{2}{3}} \Big|_{r=\delta}^{r=1} \right) = 2\pi \lim_{\delta \to 0} \left(\frac{3}{2} - \frac{3}{2}\delta^{2/3} \right)$$

$$= 2\pi \left(\frac{3}{2} \right) = \underline{3\pi}$$

(b)

Using polar coordinates as in (a),

$$\iint_D \frac{dA}{(x^2+y^2)^\lambda} = \lim_{\delta \to 0} \iint_{D_\delta} \frac{r\,dr\,d\theta}{(r^2)^\lambda}$$, where D_δ is $0 \le \theta \le 2\pi$, $0 < \delta \le r \le 1$

$$= \lim_{\delta \to 0} \int_{\delta}^{1} \int_{0}^{2\pi} r^{1-2\lambda} \, dr\, d\theta = 2\pi \lim_{\delta \to 0} \int_{\delta}^{1} r^{1-2\lambda}\, dr$$

$$= 2\pi \lim_{\delta \to 0} \left(\frac{r^{2-2\lambda}}{2-2\lambda} \Bigg|_{r=\delta}^{r=1} \right), \quad 2-2\lambda \neq 0$$

$$= 2\pi \lim_{\delta \to 0} \left(\frac{1}{2-2\lambda} - \frac{\delta^{2-2\lambda}}{2-2\lambda} \right)$$

$\lim_{\delta \to 0} \delta^{n} = 0$ only if $n > 0$

$$= 2\pi \left(\frac{1}{2-2\lambda} - 0 \right), \quad \text{for } 2-2\lambda > 0$$

$$= \frac{2\pi}{2-2\lambda}, \quad \text{for } 2-2\lambda > 0, \text{ or } \underline{\underline{\lambda < 1}}$$

8.

(a) Let D_δ be $a \le x \le \delta$,
$\phi_1(x) \le y \le \phi_2(x)$

$$\iint_D f\, dA = \lim_{\delta \to +\infty} \iint_{D_\delta} f\, dA, \text{ if limit exists.}$$

$$= \lim_{\delta \to +\infty} \int_a^{\delta} \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\, dy\, dx$$

(b) Let D_δ be as in (a)

$$\lim_{\delta \to \infty} \int_0^{\delta} \int_0^{1} xy\, e^{-(x^2+y^2)}\, dy\, dx$$

$$= \lim_{\delta \to \infty} \int_0^1 y e^{-y^2} dy \int_0^{\delta} x e^{-x^2} dx$$

$$= \left(\frac{e^{-y^2}}{-2} \Bigg|_{y=0}^{y=1} \right) \left(\lim_{\delta \to \infty} \frac{e^{-x^2}}{-2} \Bigg|_{x=0}^{x=\delta} \right)$$

$$= \left(-\frac{1}{2e} + \frac{1}{2} \right) \left(\lim_{\delta \to \infty} -\frac{1}{2e^{\delta^2}} + \frac{1}{2} \right)$$

$$= \left(\frac{1}{2} - \frac{1}{2e} \right) \left(\frac{1}{2} \right) = \underline{\frac{1}{4}\left(1 - \frac{1}{e}\right)}$$

9.

(a) Let D_δ be $1 \le y \le 2$, $0 < a \le x \le b$

$$\therefore \lim_{\substack{a \to 0 \\ b \to \infty}} \int_a^b \int_1^2 e^{-xy}\, dy\, dx = \lim_{\substack{a \to 0 \\ b \to \infty}} \int_1^2 \int_a^b e^{-xy}\, dx\, dy$$

as $e^{-xy} > 0$

$$= \lim_{\substack{a \to 0 \\ b \to \infty}} \int_1^2 \left(\frac{e^{-xy}}{-y} \Bigg|_{x=a}^{x=b} \right) dy$$

$$= \lim_{\substack{a\to 0\\ b\to\infty}} \int_1^2 \left(\frac{1}{ye^{ay}} - \frac{1}{ye^{by}}\right) dy = \int_1^2 \left(\frac{1}{y} - 0\right) dy$$

$$= \int_1^2 \frac{dy}{y} = \log y \Big|_{y=1}^{y=2} = \underline{\log 2}$$

(b) Integrating by dy first in (a),

$$\lim_{\substack{a\to 0\\ b\to\infty}} \int_a^b \int_1^2 e^{-xy}\, dy\, dx = \lim_{\substack{a\to 0\\ b\to\infty}} \int_a^b \left(\frac{e^{-xy}}{-x}\Big|_{y=1}^{y=2}\right) dx$$

$$= \lim_{\substack{a\to 0\\ b\to\infty}} \int_a^b \left(\frac{e^{-x}}{x} - \frac{e^{-2x}}{x}\right) dx$$

$$= \int_0^\infty \frac{e^{-x} - e^{-2x}}{x}\, dx = \log 2 \quad \text{from (a)}$$

10.

Define Δ_δ to be $0 \le x \le 1,\ 0 \le y \le \delta < a$

$$\therefore \int_0^1 \int_0^a \frac{x}{\sqrt{a^2 - y^2}}\, dy\, dx = \lim_{\delta\to a} \int_0^1 \int_0^\delta \frac{x}{\sqrt{a^2-y^2}}\, dy\, dx$$

$$= \lim_{\delta \to a} \int_0^1 x\,dx \int_0^{\delta} \frac{1}{\sqrt{a^2 - y^2}}\,dy$$

$$= \frac{1}{2} \lim_{\delta \to a} \left(\operatorname{Arcsin} \frac{y}{a} \Big|_{y=0}^{y=\delta} \right)$$

$$= \frac{1}{2} \lim_{\delta \to a} \left(\operatorname{Arcsin} \frac{\delta}{a} - 0 \right) = \frac{1}{2} \operatorname{Arcsin}(1)$$

$$= \underline{\frac{\pi}{4}}$$

11.

$$\frac{x+y}{x^2+2xy+y^2} = \frac{x+y}{(x+y)^2} = \frac{1}{x+y}, \quad x+y \neq 0.$$

For this Δ, $x+y=0$ at $(0,0)$.

$$\therefore \iint_{\Delta} \frac{x+y}{x^2+2xy+y^2}\,dx\,dy = \int_0^1 \int_0^1 \frac{1}{x+y}\,dx\,dy$$

$$= \int_0^1 \log(x+y) \Big|_{x=0}^{x=1} dy$$

$$= \int_0^1 \log(1+y) - \log(y) \, dy \qquad \int \ln x = x\ln x - x$$

$$= (1+y)\ln(1+y)\Big|_{y=0}^{y=1} - (1+y)\Big|_{y=0}^{y=1} - \int_0^1 \ln(y)\,dy$$

$$= 2\ln(2) - (2-1) - \lim_{a\to 0^+} \int_a^1 \ln(y)\,dy$$

$$= 2\ln(2) - 1 - \lim_{a\to 0^+} \Big[y\ln(y) - y \Big]_{y=a}^{y=1}$$

$$= 2\ln(2) - 1 - \lim_{a\to 0^+} y\ln(y)\Big|_{y=a}^{y=1} + \lim_{a\to 0^+} (1-a)$$

$$= 2\ln(2) - \lim_{a\to 0^+} (0 - a\ln(a))$$

$$= 2\ln(2) + \lim_{a\to 0^+} \frac{\ln(a)}{\frac{1}{a}} = 2\ln(2) + \lim_{a\to 0^+} \frac{\frac{1}{a}}{-\frac{1}{a^2}}$$

$$= 2\ln(2) + \lim_{a\to 0^+} (-a) = \underline{2\ln(2)}$$

12.

$|x-y| = 0$ when $y = x$. $\therefore$ Break Δ up into 2 regions, one above $y=x$, one below $y=x$.

$$D^+ = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le 1, y \ge x\}$$

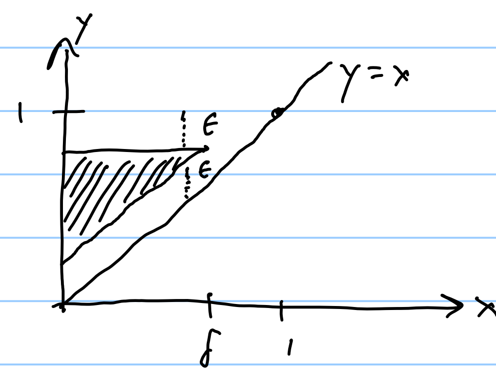
$$D^- = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le 1, y \le x\}$$

$D = D^+ \cup D^-$. $\therefore \iint_D \frac{dx\,dy}{\sqrt{|x-y|}} = \iint_{D^+} \frac{dx\,dy}{\sqrt{|x-y|}} + \iint_{D^-} \frac{dx\,dy}{\sqrt{|x-y|}}$

From problem #2 above, $\iint_{D^-} \frac{dx\,dy}{\sqrt{|x-y|}} = \frac{4}{3}$

For D^+, $\sqrt{|x-y|} = \sqrt{y-x}$, and similar to #2,

$$\iint_{D^+} \frac{dx\,dy}{\sqrt{|x-y|}} =$$



$$\lim_{\substack{\delta \to 1^- \\ \epsilon \to 0^+}} \int_0^\delta \int_{x+\epsilon}^{1-\epsilon} \frac{dy\,dx}{\sqrt{y-x}}$$

$$= \lim_{\substack{\delta \to 1^- \\ \epsilon \to 0^+}} \int_0^\delta \left(2\sqrt{y-x} \Big|_{y=x+\epsilon}^{y=1-\epsilon} \right) dx$$

$$= \lim_{\substack{\delta \to 1^- \\ \epsilon \to 0^+}} \int_0^{\delta} \left(2\sqrt{1-x-\epsilon} - 2\sqrt{\epsilon}\right) dx = \lim_{\delta \to 1^-} \int_0^{\delta} 2\sqrt{1-x}\, dx$$

$$= \lim_{\delta \to 1^-} 2\left(-\frac{2}{3}\right)(1-x)^{3/2} \Big|_0^{\delta} = -\frac{4}{3} \lim_{\delta \to 1^-} \left[(1-\delta)^{3/2} - 1\right]$$

$$= \underline{\frac{4}{3}}.$$

$$\therefore \iint_0 \frac{1}{\sqrt{|x-y|}}\, dx\, dy = \frac{4}{3} + \frac{4}{3} = \underline{\frac{8}{3}}$$

13.

Use spherical coordinates: $x^2+y^2+z^2 = \rho^2$, $z = \rho\cos\phi$, $0 \le \rho \le a$, $0 \le \theta \le \frac{\pi}{2}$, $0 \le \phi \le \frac{\pi}{2}$

$$\therefore \int_0^a \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{(\rho^2)^{\frac{1}{4}}}{\sqrt{\rho\cos\phi + \rho^4}}\, \rho^2 \sin\phi\, d\theta\, d\phi\, d\rho$$

$$= \frac{\pi}{2} \int_0^a \int_0^{\frac{\pi}{2}} \rho^{5/2} \sin\phi \left(\rho\cos\phi + \rho^4\right)^{-\frac{1}{2}} d\phi\, d\rho$$

$$= \frac{\pi}{2} \int_0^a \rho^{5/2} \left(-\frac{2}{\rho}\right) \left(\rho\cos\phi + \rho^4\right)^{\frac{1}{2}} \Big|_{\phi=0}^{\phi=\frac{\pi}{2}} d\rho$$

$$= -\pi \int_0^a \rho^{3/2}\left[(\rho^4)^{\frac{1}{2}} - (\rho+\rho^4)^{\frac{1}{2}} \right] d\rho$$

$$= -\pi \int_0^a \rho^{7/2} - \rho^2(1+\rho^3)^{\frac{1}{2}} \, d\rho$$

$$= -\pi \left[\frac{2}{9}\rho^{9/2}\Big|_0^a - \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)(1+\rho^3)^{\frac{3}{2}}\Big|_0^a \right]$$

$$= -\pi \left[\frac{2}{9}a^{9/2} - \frac{2}{9}(1+a^3)^{3/2} + \frac{2}{9} \right]$$

$$= -\frac{2\pi}{9}\left[\underline{a^{9/2} - (1+a^3)^{3/2} + 1} \right]$$

14.

This is somewhat analogous to a bounded series composed of positive terms: a montonically increasing series that is bounded above has a limit. The upper bound in this case is $\iint_D g(x,y)\,dA$! However, "somewhat" is used

because increasing the graniness of a partition doesn't guarantee that $S_{n+1} > S_n$, i.e., the Riemann sum of a finer partition may not be larger than a coarser partition.

$f(x,y)$ is nonnegative.

$$\therefore 0 \le \lim_{n\to\infty} \sum_{j,k}^{n-1} f(\vec{c}_{jk})\,\Delta x\,\Delta y \le \iint_D g(x,y)\,dA$$

15.

$\dfrac{\sin^2(x-y)}{\sqrt{1-x^2-y^2}}$ is nonnegative on D.

$$\frac{\sin^2(x-y)}{\sqrt{1-x^2-y^2}} \le \frac{1}{\sqrt{1-x^2-y^2}} \quad \text{as } |\sin\theta| \le 1 \text{ for all } \theta.$$

From Example #2 in text (p. 343), $\displaystyle\iint_D \frac{1}{\sqrt{1-x^2-y^2}}\,dA = 2\pi$

$\therefore$ From #14 above, $\displaystyle\iint_D \frac{\sin^2(x-y)}{\sqrt{1-x^2-y^2}}\,dA$ exists.

16.

By #14, if $\iint_D f\,dA$ did exist, then $\iint_D g\,dA$ would exist.

More directly, since $g(x,y)$ is nonnegative, the Riemann sum of $g(x,y)$ over D is like an unbounded nonnegative sequence.

17.

For all $0 \le x^2$, $e^0 \le e^{x^2} \Rightarrow 1 \le e^{x^2}$. Similarl for $0 \le y^2$, $1 \le e^{y^2}$. $\therefore$ $1 \le e^{x^2} \cdot e^{y^2} = e^{x^2+y^2}$

By Example #3 of text, p. 343, $\iint_D \frac{1}{x-y}\, dA$ does not exist, where D is $0 \le x \le 1$, $0 \le y \le x$.

$\therefore$ By #16 above, $\iint_D \frac{1}{x-y}\, dA \le \iint_D \frac{e^{x^2+y^2}}{x-y}\, dA$

does not exist.

18.

Use spherical coordinates

$$\iiint_D \frac{dx\,dy\,dz}{(x^2+y^2+z^2)^2} = \iiint_D \frac{\rho^2 \sin\phi}{\rho^4}\, d\rho\, d\phi\, d\theta$$

$$= \int_1^\infty \int_0^\pi \int_0^{2\pi} \frac{\sin\phi}{\rho^2} \, d\theta \, d\phi \, d\rho$$

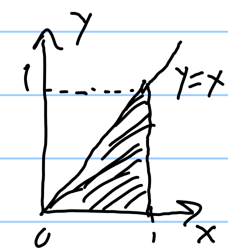
$$= 2\pi \int_1^\infty \int_0^\pi \frac{\sin\phi}{\rho^2} \, d\phi \, d\rho = -2\pi \int_1^\infty \left. \frac{\cos\phi}{\rho^2} \right|_{\phi=0}^{\phi=\pi} d\rho$$

$$= 4\pi \int_1^\infty \frac{d\rho}{\rho^2} = 4\pi \lim_{a \to \infty} \left(-\rho^{-1}\right) \Big|_{\rho=1}^{\rho=a}$$

$$= 4\pi \lim_{a \to \infty} \left[-\frac{1}{a} + 1 \right] = 4\pi$$

19.

(a) $$\int_0^1 \int_0^y \frac{x}{y} \, dx \, dy = \lim_{a \to 0} \int_a^1 \int_0^y \frac{x}{y} \, dx \, dy$$



$$= \lim_{a \to 0} \int_a^1 \left(\left. \frac{x^2}{2y} \right|_{x=0}^{x=y} \right) dy = \lim_{a \to 0} \int_a^1 \frac{y^2}{2y} \, dy$$

$$= \lim_{a \to 0} \left. \frac{y^2}{4} \right|_{y=a}^{y=1} = \lim_{a \to 0} \left[\frac{1}{4} - \frac{a^2}{4} \right] = \underline{\frac{1}{4}}$$

Theorem 3 (Fubini's Theorem) on p. 343 of text does apply, since x/y is nonnegative on D, continuous on D except at $(x, 0)$, a boundary of D.

$$\therefore \iint_D \frac{x}{y}\, dA = \int_0^1 \int_0^y \frac{x}{y}\, dx\, dy = \int_0^1 \int_x^1 \frac{x}{y}\, dy\, dx$$

(b) $$\int_0^1 \int_x^1 \frac{x}{y}\, dy\, dx = \int_0^1 \left(x \ln y \Big|_{y=x}^{y=1} \right) dx$$

$$= \int_0^1 -x \ln x \, dx = -x^2 \left[\frac{\ln x}{2} - \frac{1}{4} \right] \Bigg|_{x=0}^{x=1}$$

$$= \frac{1}{4} + \frac{1}{2} \lim_{a \to 0^+} x^2 \ln x = \frac{1}{4} + \frac{1}{2} \lim_{a \to 0^+} \frac{\ln x}{\frac{1}{x^2}}$$

$$= \frac{1}{4} + \frac{1}{2} \lim_{a \to 0^+} \frac{\frac{1}{x}}{\frac{-2}{x^3}} = \frac{1}{4} + \frac{1}{2} \lim_{a \to 0^+} -\frac{x^2}{2} = \frac{1}{4} + 0$$

$$= \underline{\frac{1}{4}}$$

20.

First, $\frac{x^2-y^2}{(x^2+y^2)^2}$ is not defined at $(0,0)$. This in itself does not violate Theorem 3, p. 343 of the text (Fubini's Theorem).

However, because of the term x^2-y^2, the function is not nonnegative in every neighborhood around $(0,0)$: if $r>0$, $D_r(0,0)$ contains points (x,y) for which $x^2-y^2>0$ (i.e., $x>y$), and points (x,y) for which $x^2-y^2<0$ (i.e., $x<y$). This does violate the premise of Fubini's Theorem.

21.

It really should state $0 \le f(x,y) \le g(x,y)$ for all $(x,y) \in D$ except at boundary points of D for which f and g may not be defined.
Better worded in exercise #14.

First consider $D_1 = [0,1] \times [0,1]$ for $(x,y) \in D_1$.

For $(x,y) \neq 0$, $\frac{1}{x^\alpha y^\beta + x^\gamma y^\rho} < \frac{1}{x^\alpha y^\beta}$

since $0 < x^\alpha y^\beta < x^\alpha y^\beta + x^\gamma y^\rho$ and $0 < x^\gamma y^\rho$

Since, by exercise #5, $\iint_{D_1} \frac{1}{x^\alpha y^\beta}\,dxdy$ exists,

Then by the premise $\iint_{D_1} \frac{1}{x^\alpha y^\beta + x^\gamma y^\rho}\,dxdy$ exists.

Now consider $D_2 = [1,\infty) \times [1,\infty)$ for $(x,y) \in D_2$

For every $(x,y) \in D_2$, $\frac{1}{x^\alpha y^\beta + x^\gamma y^\rho} < \frac{1}{x^\gamma y^\rho}$

since $x^\gamma y^\rho < x^\alpha y^\beta + x^\gamma y^\rho$ and $0 < x^\alpha y^\beta$

Since, by exercise #6, $\iint_{D_2} \frac{1}{x^\gamma y^\rho}\,dxdy$ exists,

then by the premise $\iint_{D_2} \frac{1}{x^\alpha y^\beta + x^\gamma y^\rho}\,dxdy$ exists.

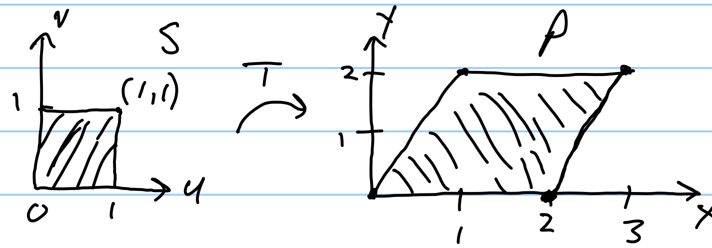
$\therefore \iint_{D_1} \frac{1}{x^\alpha y^\beta + x^\gamma y^\rho}\,dxdy + \iint_{D_2} \frac{1}{x^\alpha y^\beta + x^\gamma y^\rho}\,dxdy =$

$\iint_{D} \frac{1}{x^\alpha y^\beta + x^\gamma y^\rho}\,dxdy$ exists since $D = D_1 \cup D_2$.

Review Exercises for Chapter 6

Note Title 1/15/2017

1.



(a) Let $T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\therefore$ $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 2 & 2 \end{bmatrix}$

Taking transposes: $\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 3 & 2 \\ 1 & 2 \end{bmatrix}$

$\Rightarrow$ $a(1) + b(0) = 2$, $a = 2$ $\quad$ $c(1) + d(0) = 0$, $c = 0$
$a(0) + b(1) = 1$, $b = 1$ $\quad$ $c(0) + d(1) = 2$, $d = 2$

$\Rightarrow$ $T = \underline{\underline{\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}}}$

(b) $\det(T) = 4$. Note the $T: (u,v) \to (x,y)$.

So, $\iint_S f \left| \dfrac{\partial(x,y)}{\partial(u,v)} \right| du\,dv = \iint_P f \, dx\,dy$

Here, $T(u,v) = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 2u + v \\ 2v \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

$\therefore$ $\left| \dfrac{\partial(x,y)}{\partial(u,v)} \right| = \begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix} = \det(T) = 4$

$$\therefore \iint_{D^*} f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \iint_D f(x,y)\,dx\,dy$$

or, $$\iint_{D^*} f(T(u,v))\det(T)\,du\,dv = \iint_{T(D^*)} f(x,y)\,dx\,dy$$

$\therefore$ In this example, $T(S) = P$

$$\iint_S f(x(u,v), y(u,v))\det(T)\,du\,dv = \iint_P f(x,y)\,dx\,dy$$

or $$\int_0^1\int_0^1 f(2u+v, 2v)(4)\,du\,dv = \iint_P f(x,y)\,dx\,dy$$

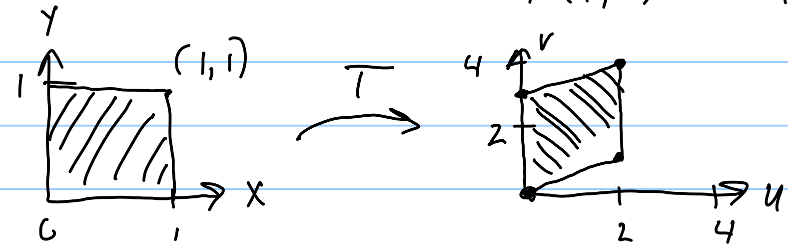
(simple region, complex integrand $\rightarrow$ complex region, simple integrand)

2.

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ x+3y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$$

(a) Look at corners : $T(0,0)=(0,0) \qquad T(1,1)=(2,4)$

$T(1,0)=(2,1) \qquad T(0,1)=(0,3)$



parallelogram

$$\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$$

(b) S = square P = parallelogram

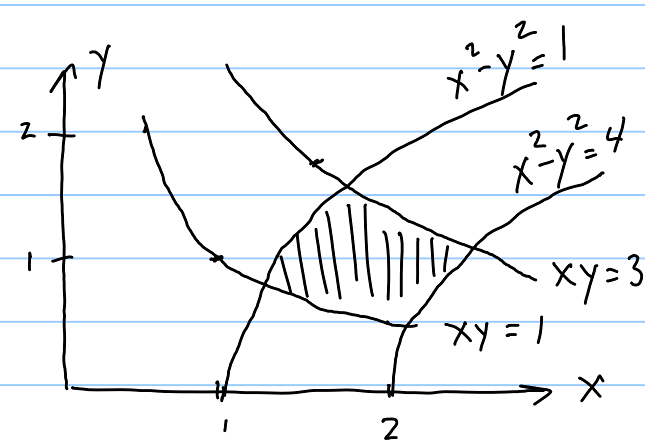
$$\iint_S f(u(x,y), v(x,y)) \left| \frac{\partial(u,v)}{\partial(x,y)} \right| dx\,dy = \iint_P f(u,v)\,du\,dv$$

$$\left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \det(T) = \det \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} = 6$$

So,

$$6 \iint_S f(2x, x+3y)\,dx\,dy = \iint_P f(u,v)\,du\,dv$$

3.



$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x^2-y^2 \\ xy \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$$

$$\therefore \left| \frac{\partial(u,v)}{\partial(x,y)} \right| = \begin{vmatrix} 2x & -2y \\ y & x \end{vmatrix} = 2x^2+2y^2 = 2(x^2+y^2)$$

Note the given integrand already contains a form of the Jacobian determinant.

$$\therefore \frac{1}{2}\iint_B 2(x^2+y^2)\,dx\,dy = \iint_B \frac{1}{2}\left|\frac{\partial(u,v)}{\partial(x,y)}\right| dx\,dy = \iint_{B^*} \frac{1}{2}\,du\,dv$$

where $f(x,y) = \frac{1}{2} = f(u,v)$

[using Theorem 2, p. 319 of text, setting $B = D^*$, $B^* = D$, and swapping the dummy variables x, y and u, v. Also assuming T is one-to-one and onto].

So what is $B^* = T(B)$?

Note that the borders of B are $xy = 1$, $xy = 3$, (so $v = 1$, $v = 3$) and $x^2 - y^2 = 1$, $x^2 - y^2 = 4$ (so $u = 1$, $u = 4$)

$$\therefore B^* = \{(u,v) : 1 \le u \le 4,\ 1 \le v \le 3\}$$

$$\therefore \iint_{B^*} \frac{1}{2}\,du\,dv = \frac{1}{2}\int_1^4 \int_1^3 dv\,du = \frac{1}{2}\int_1^4 \left(v\Big|_1^3\right) du$$

$$= \frac{1}{2}(2)\int_1^4 du = \frac{1}{2}(2)\left(u\Big|_1^4\right) = \frac{1}{2}(2)(3) = \underline{\underline{3}}$$

Note: T is one-to-one: Suppose $T\begin{bmatrix} x \\ y \end{bmatrix} = T\begin{bmatrix} a \\ b \end{bmatrix}$

Then $x^2 - y^2 = a^2 - b^2$ and $x \neq 0, y \neq 0$
$xy = ab$ $\qquad a \neq 0, b \neq 0$

$\therefore x = \frac{ab}{y} \Rightarrow \left(\frac{ab}{y}\right)^2 + b^2 = a^2 + y^2$

$\therefore a^2 b^2 + y^2 b^2 = a^2 y^2 + y^4$

$\Rightarrow (a^2 + y^2) b^2 = (a^2 + y^2) y^2$

$\Rightarrow b^2 = y^2 \Rightarrow y = b$

and from $xy = a(y)$, $x = a$

T is onto: Given any $(u,v) \in \{1 \le u \le 4, 1 \le v \le 3\}$

Need to find an $(x,y) \in B$ s.t. $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$
i.e., $x^2 - y^2 = u$, $xy = v$

Let $y = \frac{v}{x}$. $\therefore x^2 - \left(\frac{v^2}{x^2}\right) = u$

$\therefore x^4 - ux^2 - v^2 = 0$, $x^2 = \frac{u \pm \sqrt{u^2 - 4(-v^2)}}{2}$

$x^2 = \frac{u + \sqrt{u^2 + 4v^2}}{2}$ since $x^2 > 0$

$\therefore x = \sqrt{\frac{u + \sqrt{u^2 + 4v^2}}{2}}$, and $y = \frac{v}{x}$

and these satisfy $1 \le x^2 - y^2 \le 4$, $1 \le xy \le 3$
under the above construction.

4.

(a)

$$x = r\cos\theta \qquad \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = r$$
$$y = r\sin\theta$$
$$z = z$$

Here, $0 \le r \le 1$, $0 \le \theta \le 2\pi$, $(x^2+y^2)^{\frac{1}{2}} = r$

$$\therefore \int_0^1 \int_0^{2\pi} \int_0^1 \underline{r^2 \, dr\, d\theta\, dz}$$

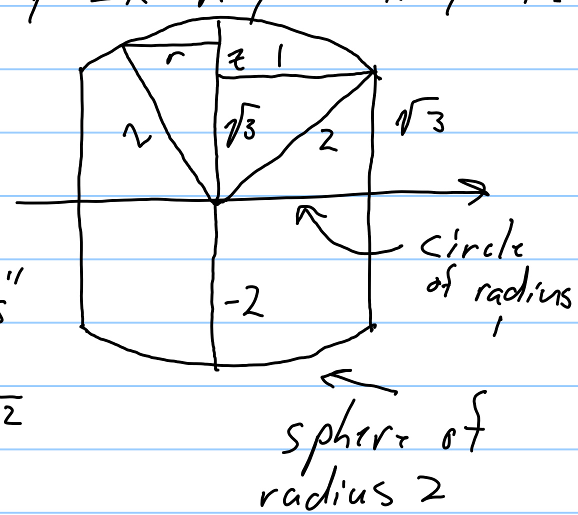
(b)

$$x = r\cos\theta \qquad J = r$$
$$y = r\sin\theta$$
$$z = z$$

A sphere of radius 2: $-\sqrt{4-x^2-y^2} \le z \le \sqrt{4-x^2-y^2}$
$\Rightarrow \quad z^2 \le 4-x^2-y^2 \Rightarrow x^2+y^2+z^2 \le 2$

"caps" top & bottom of a cylinder of radius 1: $-1 \le y \le 1$, $-\sqrt{1-y^2} \le x \le \sqrt{1-y^2} \Rightarrow x^2+y^2 = 1$.

$\therefore \; 0 \le r \le 1$
$0 \le \theta \le 2\pi$
$-\sqrt{3} \le z \le \sqrt{3}$ for cylinder minus top & bottom "caps"



For caps: $r^2 + z^2 = 4$, $z = \pm\sqrt{4-r^2}$

$$\therefore \int_0^{2\pi} \int_0^1 \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} (r\cos\theta)(r\sin\theta)\, z\, (r)\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^1 \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r^3 \cos\theta \sin\theta \, z \, dz\, dr\, d\theta$$

(c)

$z = \rho\cos\phi \qquad J = \rho^2 \sin\phi$

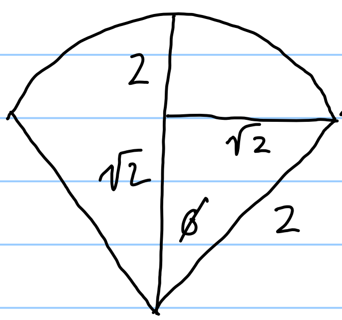
$y = \rho \sin\phi \sin\theta$

$x = \rho \sin\phi \cos\theta$

$-\sqrt{2} \le y \le \sqrt{2}$, $-\sqrt{2-y^2} \le x \le \sqrt{2-y^2}$ is a circle of radius $\sqrt{2}$

$z^2 = x^2+y^2$ is a cone, $z \ge \sqrt{x^2+y^2}$ a cone above xy-plane.

$z = \sqrt{4-x^2-y^2}$ is a sphere of radius 2.



$\sqrt{x^2+y^2} = \sqrt{4-x^2-y^2}$

$\cos\phi = \frac{\sqrt{2}}{2}$, $\phi = \frac{\pi}{4}$

$\therefore \; 0 \le \rho \le 2$

$0 \le \theta \le 2\pi$

$0 \le \phi \le \pi/4$

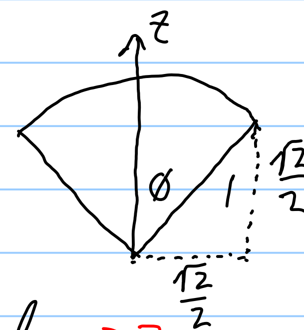
$$\therefore \int_0^{\pi/4} \int_0^{2\pi} \int_0^2 (\rho\cos\phi)^2 (\rho^2 \sin\phi)\, d\rho\, d\theta\, d\phi$$

$$= \int_0^{\frac{\pi}{4}} \int_0^{2\pi} \int_0^2 \rho^4 \cos^2\phi \sin\phi \, d\rho\, d\theta\, d\phi$$

(d)

Note $\sin 2\phi = 2\sin\phi\cos\phi$, so the Jacobian of $T\begin{bmatrix}\rho\\\theta\\\phi\end{bmatrix} = \begin{bmatrix}x\\y\\z\end{bmatrix}$ seems to be embedded in the integrand. This seems easier than finding the Jacobian of $T\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}\rho\\\theta\\\phi\end{bmatrix} = \begin{bmatrix}\sqrt{x^2+y^2+z^2}\\ \arctan\frac{y}{x}\\ \arccos\frac{z}{\sqrt{x^2+y^2+z^2}}\end{bmatrix}$

$$\therefore \int_0^1\int_0^{\frac{\pi}{4}}\int_0^{2\pi} \rho^3\, 2\sin\phi\cos\phi \, d\theta\, d\phi\, d\rho$$



$$= \int_0^1\int_0^{\frac{\pi}{4}}\int_0^{2\pi} 2\rho\cos\phi\,(\rho^2\sin\phi)\, d\theta\, d\phi\, d\rho \quad [1]$$

This cone projects to the xy-plane as a disc of radius $\sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2}$ $\left(\sqrt{\frac{1}{2}}^2 + \sqrt{\frac{1}{2}}^2 = 1^2\right)$

$\therefore$ For x: $-\frac{\sqrt{2}}{2} \le x \le \frac{\sqrt{2}}{2}$

For y: $x^2+y^2 = \frac{1}{2}$, $-\sqrt{\frac{1}{2}-x^2} \le y \le \sqrt{\frac{1}{2}-x^2}$

For z, a sphere of radius 1, $x^2+y^2+z^2 = 1$,

The upper limit is $z \le \sqrt{1-x^2-y^2}$

The lower bound for z is a cone: $z^2 = x^2 + y^2$,

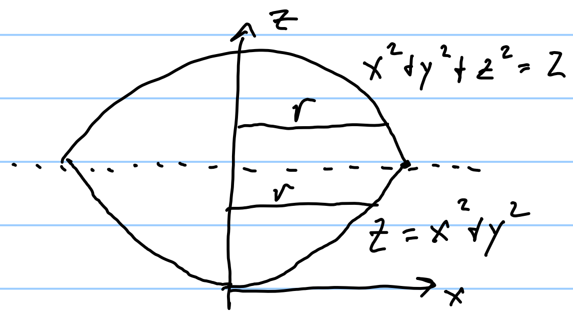
or $\sqrt{x^2+y^2} \le z$ (upper half of cone).

From [1], $\displaystyle\int_0^1 \int_0^{\frac{\pi}{4}} \int_0^{2\pi} 2\rho\cos\phi \left| \frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} \right| d\theta\, d\phi\, d\rho$

$$= \int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} \int_{-\sqrt{\frac{1}{2}-x^2}}^{\sqrt{\frac{1}{2}-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} 2\sqrt{x^2+y^2+z^2}\, \frac{z}{\sqrt{x^2+y^2+z^2}}\, dz\, dy\, dx$$

$$= \int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} \int_{-\sqrt{\frac{1}{2}-x^2}}^{\sqrt{\frac{1}{2}-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} 2z\, dz\, dy\, dx$$

5.



$z = x^2+y^2$ is a paraboloid

$x^2+y^2+z^2 = 2$ is a sphere of radius $\sqrt{2}$

The point of intersection is: $x^2+y^2 = z \Rightarrow (z) + z^2 = 2$,

$(z-1)(z+2) = 0$, $z = 1$.

$\therefore$ projection onto xy-plane is a unit disc as $z = 1 = x^2+y^2$.

$\therefore$ Using cylindrical coordinates, $0 \le \theta \le 2\pi$,

$0 \le r \le 1$.

z extends from $z = x^2 + y^2$ to $z = \sqrt{2 - x^2 - y^2}$

But $x^2 + y^2 = r^2$, $\therefore r^2 \le z \le \sqrt{2 - r^2}$

$$\therefore \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{x^2+y^2}^{\sqrt{2-x^2-y^2}} dz\,dy\,dx = \iiint_{V^*} r\,dr\,d\theta\,dz$$

$$= \int_0^{2\pi} \int_0^1 \int_{r^2}^{\sqrt{2-r^2}} r\,dz\,dr\,d\theta$$

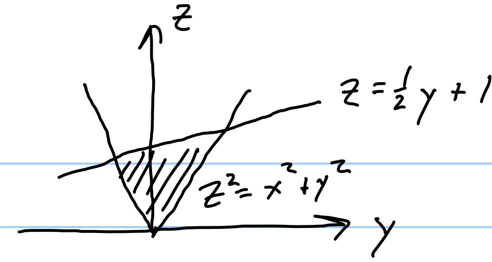
$$= \int_0^{2\pi} \int_0^1 \left(r z \Big|_{z=r^2}^{z=\sqrt{2-r^2}} \right) dr\,d\theta = \int_0^{2\pi} \int_0^1 r(2-r^2)^{\frac{1}{2}} - r^3\,dr\,d\theta$$

$$= \int_0^{2\pi} -\frac{1}{2}\left(\frac{2}{3}\right)(2-r^2)^{\frac{3}{2}} \Big|_{r=0}^{1} - \frac{r^4}{4} \Big|_{r=0}^{1} \, d\theta$$

$$= \int_0^{2\pi} \left[-\frac{1}{3} - \left(-\frac{2\sqrt{2}}{3} \right) - \frac{1}{4} \right] d\theta = \left(\frac{2\sqrt{2}}{3} - \frac{7}{12} \right) 2\pi$$

$$= \pi \underline{\left(\frac{4\sqrt{2}}{3} - \frac{7}{6} \right)}$$

c.



The object projects onto the xy-plane as an ellipse.

Find the equation of the projection.

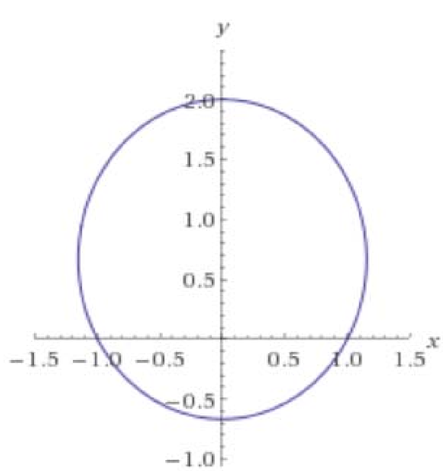
$$x^2 + y^2 = z^2 = \left(\tfrac{1}{2}y + 1\right)^2 = \tfrac{1}{4}y^2 + y + 1$$

$$\therefore \quad x^2 + \tfrac{3}{4}y^2 - y = 1 \implies \frac{x^2}{\frac{3}{4}} + y^2 - \tfrac{4}{3}y = \tfrac{4}{3}$$

$$\implies \frac{x^2}{\frac{3}{4}} + \left(y - \tfrac{2}{3}\right)^2 = \tfrac{4}{3} + \tfrac{4}{9} = \tfrac{16}{9}$$

$$\implies \frac{x^2}{\frac{4}{3}} + \frac{\left(y - \frac{2}{3}\right)^2}{16/9} = 1, \text{ or } \frac{x^2}{\left(\sqrt{4/3}\right)^2} + \frac{\left(y - \frac{2}{3}\right)^2}{(4/3)^2} = 1$$

This is an ellipse centered at $(0, \frac{2}{3})$
major axis of $4/3$ along y-axis,
minor axis of $\sqrt{4/3}$ along x-axis.



From analytic geometry,

$$\frac{x^2}{a^2(1-e^2)} + \frac{y^2}{a^2} = 1$$

with focus at $(0, \pm ae)$
e = eccentricity, assuming no translation.

$\therefore\ a^2(1-e^2) = \frac{16}{9}(1-e^2) = \frac{4}{3}$, $1-e^2 = \frac{3}{4}$, $e = \frac{1}{2}$

$f = \pm ae = \pm \frac{4}{3}\left(\frac{1}{2}\right) = \pm \frac{2}{3}$

The equation of the ellipse above, if a focus is at the pole, and directrix parallel and below the polar axis (the x-axis) is:

$$r = \frac{ed}{1-e\sin\theta}. \quad d = \frac{a}{e} - ae = \frac{a(1-e^2)}{e}$$

In this case, $d = \dfrac{\frac{4}{3}\left(1-\frac{1}{4}\right)}{\frac{1}{2}} = 2$

$$\therefore\ r = \frac{\frac{1}{2}(2)}{1-\frac{1}{2}\sin\theta} = \frac{2}{2-\sin\theta}$$

That is, $r=0$ corresponds to the lower focus at $(0, -2/3)$.

But the above ellipse has the lower focus translated upward by $2/3$, and $\therefore$ is at the origin

$\therefore\ r = \dfrac{2}{2-\sin\theta}$ describes the ellipse in the xy-plane in polar coordinates

(a) Cylindrical coordinates:

The solid can be described as (r, θ, z) such that

$$0 \le \theta \le 2\pi, \quad 0 \le r \le \frac{2}{2-\sin\theta}$$

$\sqrt{x^2+y^2} \le z \le \frac{1}{2}y+1$. Since $\sqrt{x^2+y^2} = r$, $y = r\sin\theta$

$\therefore\ r \le z \le \frac{r\sin\theta}{2} + 1$

$$\therefore\ V = \int_0^{2\pi} \int_0^{\frac{2}{2-\sin\theta}} \int_r^{\frac{r\sin\theta}{2}+1} r \, dz \, dr \, d\theta$$

From Wolfram:

Definite integral:

$$\int_0^{2\pi} \int_0^{\frac{2}{2-\sin(x)}} \int_r^{\frac{1}{2} r\sin(x)+1} r \, dz \, dr \, dx = \frac{8\pi}{9\sqrt{3}} \approx 1.61227$$

The integration can be performed up to a point:

$$= \int_0^{2\pi} \int_0^{\frac{2}{2-\sin\theta}} \left(r z \Big|_{z=r}^{z=\frac{r\sin\theta}{2}+1} \right) dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\frac{2}{2-\sin\theta}} \left(\frac{r^2\sin\theta}{2} + r - r^2 \right) dr \, d\theta$$

$$= \int_0^{2\pi} \frac{r^3}{3}\left(\frac{\sin\theta}{2} - 1 \right) + \frac{r^2}{2} \Bigg|_{r=0}^{r=\frac{2}{2-\sin\theta}} d\theta$$

$$= \int_0^{2\pi} \frac{8}{3(2-\sin\theta)^3} \frac{(\sin\theta - 2)}{2} + \frac{2}{(2-\sin\theta)^2} \, d\theta$$

$$= \int_0^{2\pi} -\frac{4}{3}\frac{1}{(2-\sin\theta)^2} + \frac{2}{(2-\sin\theta)^2}\, d\theta$$

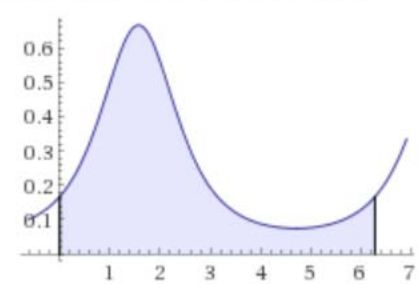
$$= \frac{2}{3}\int_0^{2\pi} \frac{1}{(2-\sin\theta)^2}\, d\theta$$

Definite integral:

$$\int_0^{2\pi} \frac{2}{3\,(2-\sin(x))^2}\, dx = \frac{8\pi}{9\sqrt{3}} \approx 1.6123$$

From Wolfrom:

Visual representation of the integral:



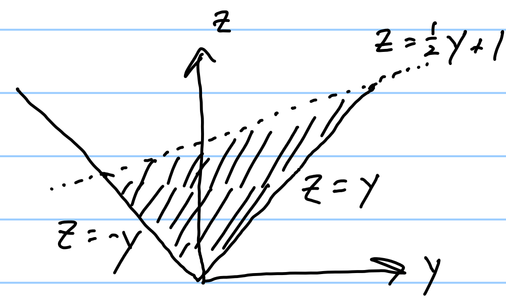
which agrees with the above. The indefinite integral is:

$$\int \frac{2}{3\,(2-\sin(x))^2}\, dx = \frac{2\left(3\cos(x) - 4\sqrt{3}\,(\sin(x)-2)\tan^{-1}\left(\frac{1-2\tan\left(\frac{x}{2}\right)}{\sqrt{3}}\right)\right)}{27\,(\sin(x)-2)} + \text{constant}$$

Such rational functions of sine can be evaluated using the substitution $u = \tan\frac{\theta}{2}$ so that $\sin\theta = \frac{2u}{1+u^2}$, Then using partial fractions. Too much work.

$\therefore$ Volume $= \underline{\frac{8\pi}{9\sqrt{3}}}$

(b) Spherical coordinates:

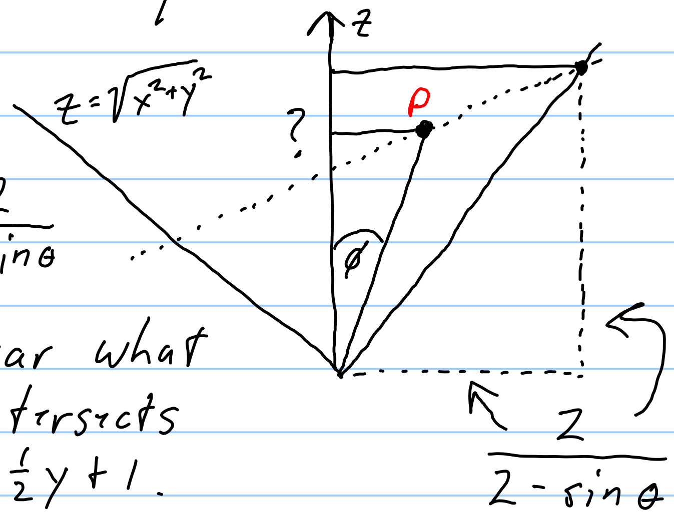


The solid can be described as

$0 \le \theta \le 2\pi$, $0 \le \phi \le \pi/4$

But for a fixed θ and ϕ, it is not immediately clear what the extent of ρ is.

From above, the outer edge of the cone has height $= \frac{2}{2-\sin\theta}$



However, it is not clear what height the ray ϕ intersects the angled plane $z = \frac{1}{2}y+1$.

Viewed in profile, the plane $z = \frac{1}{2}y+1$ will intersect the z-axis at different angles depending on θ. Thus, the extent of ρ from 0 to ρ is not clear.

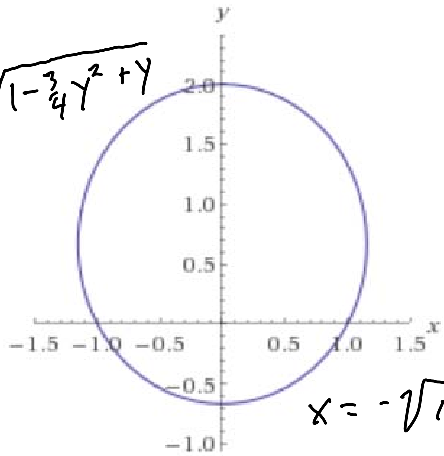
A height of $\frac{2}{\cos\phi\,(2-\sin\theta)}$ works fine for $\phi = \frac{\pi}{4}$, but this formula doesn't work for other values of ϕ.

As a result, this problem does not lend itself easily to computation via spherical coordinates.

(c) Rectangular coordinates

From $x^2 + \frac{3}{4}y^2 - y = 1$, $\quad x = \pm\sqrt{1-\frac{3}{4}y^2+y}$

$x = \sqrt{1 - \frac{3}{4}y^2 + y}$



$x = -\sqrt{1 - \frac{3}{4}y^2 + y}$

Use x-simple curves.

Limits for y: $-\frac{2}{3} \le y \le 2$

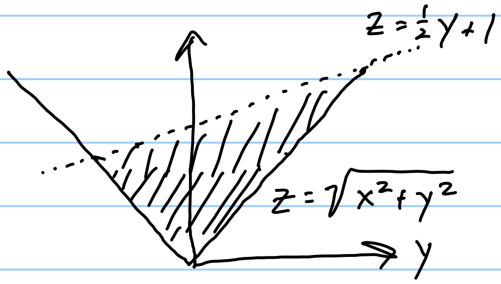
obtained by setting $x = 0$

so that

$\frac{3}{4}y^2 - y = 1 \Rightarrow y^2 - \frac{4}{3}y = \frac{4}{3}$

$\Rightarrow \left(y - \frac{2}{3}\right)^2 = \frac{16}{9} \Rightarrow y = \frac{2}{3} \pm \frac{4}{3}$

$\Rightarrow y = -2/3,\ 2$



$$\therefore \quad V = \int_{-\frac{2}{3}}^{2} \int_{-\sqrt{1-\frac{3}{4}y^2+y}}^{\sqrt{1-\frac{3}{4}y^2+y}} \int_{\sqrt{x^2+y^2}}^{\frac{1}{2}y+1} dz\,dx\,dy$$

From Wolfram:

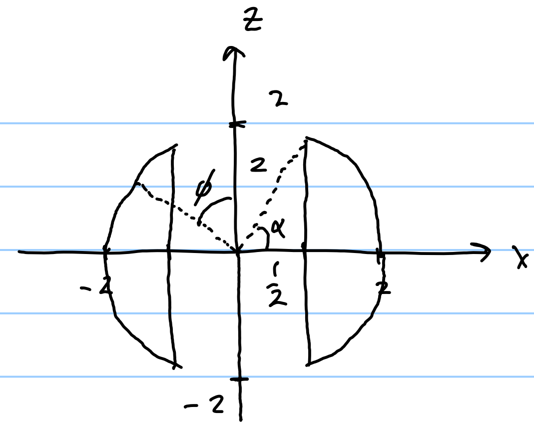
Definite integral:

$$\int_{-\frac{2}{3}}^{2} \int_{-\sqrt{1-\frac{3y^2}{4}+y}}^{\sqrt{1-\frac{3y^2}{4}+y}} \int_{\sqrt{x^2+y^2}}^{\frac{y}{2}+1} 1\,dz\,dx\,dy = 1.61227$$

This agrees with cylindrical coordinates of

$$\underline{\frac{8\pi}{9\sqrt{3}}} \approx 1.61$$

7.

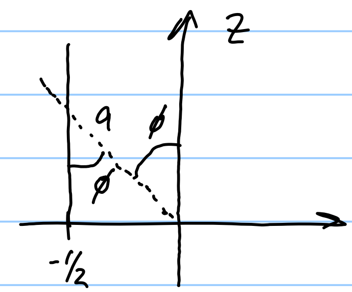


Use spherical coordinates

$\cos\alpha = \frac{1}{4}$, $\sin\phi = \frac{1}{4}$

$\therefore \operatorname{Arcsin}(\frac{1}{4}) \le \phi \le \pi - \operatorname{Arcsin}(\frac{1}{4})$, $0 \le \theta \le 2\pi$

ρ is a function of ϕ, not θ.



Note that $|a \sin\phi| = \frac{1}{2}$

where a = hypotenuse of triangle
= minimum of ρ given ϕ.

$\therefore \frac{1}{2\sin\phi} \le \rho \le 2$.

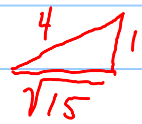
$$\therefore \text{Volume} = \int_0^{2\pi}\int_{\operatorname{Arcsin}(\frac{1}{4})}^{\pi - \operatorname{Arcsin}(\frac{1}{4})}\int_{\frac{1}{2\sin\phi}}^{2} \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_{\sin^{-1}(\frac{1}{4})}^{\pi - \sin^{-1}(\frac{1}{4})} \sin\phi \left.\frac{\rho^3}{3}\right|_{\rho = \frac{1}{2\sin\phi}}^{\rho=2} d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_{\sin^{-1}(\frac{1}{4})}^{\pi - \sin^{-1}(\frac{1}{4})} \frac{8}{3}\sin\phi - \frac{1}{24}\frac{1}{\sin^2\phi}\, d\phi\, d\theta$$

$\frac{1}{\sin^2\phi} = \csc^2\phi$

$$= \int_0^{2\pi} \frac{8}{3}\Big[-\cos\phi\Big]_{\phi=\sin^{-1}(\frac{1}{4})}^{\pi-\sin^{-1}(\frac{1}{4})} - \frac{1}{24}\Big[-\cot\phi\Big]_{\sin^{-1}(\frac{1}{4})}^{\pi-\sin^{-1}(\frac{1}{4})}\, d\theta$$



If $\sin\phi = \frac{1}{4}$, $\cos\phi = \frac{\sqrt{15}}{4}$, $\tan\phi = \frac{1}{\sqrt{15}}$

$$\therefore \sin^{-1}\left(\tfrac{1}{4}\right) \Leftrightarrow \cos^{-1}\left(\tfrac{\sqrt{15}}{4}\right) \Leftrightarrow \tan^{-1}\left(\tfrac{1}{\sqrt{15}}\right) \Leftrightarrow \cot^{-1}(\sqrt{15})$$

$$= \int_0^{2\pi} \frac{8}{3}\Big[-\cos\phi\Big]_{\phi=\cos^{-1}(\frac{\sqrt{15}}{4})}^{\pi-\cos^{-1}(\frac{\sqrt{15}}{4})} + \frac{1}{24}\Big[\cot\phi\Big]_{\cot^{-1}(\sqrt{15})}^{\pi-\tan^{-1}(\frac{1}{\sqrt{15}})}\, d\theta$$

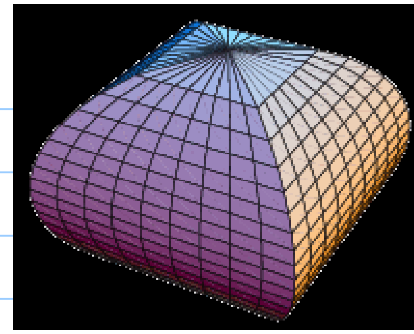
$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ $\qquad$ $\cot(\alpha-\beta) = \dfrac{1+\tan\alpha\tan\beta}{\tan\alpha - \tan\beta}$

$$= \int_0^{2\pi} \frac{8}{3}\left[\frac{\sqrt{15}}{4} - \left(-\frac{\sqrt{15}}{4}\right)\right] + \frac{1}{24}\left[\frac{1}{-\frac{1}{\sqrt{15}}} - \sqrt{15}\right] d\theta$$

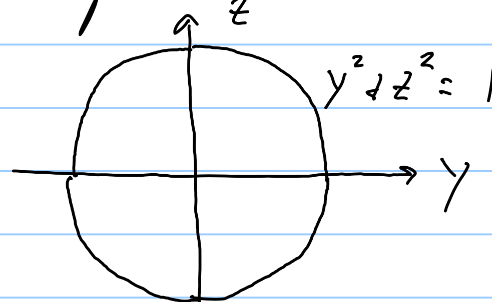
$$= \int_0^{2\pi} \frac{4}{3}\sqrt{15} - \frac{1}{12}\sqrt{15}\, d\theta = \int_0^{2\pi} \frac{5}{4}\sqrt{15}\, d\theta$$

$$= \underline{\underline{\frac{5\sqrt{15}}{2}\pi}}$$

8.



Using above figure, place the two "points" on the z-axis. ∴ One cylinder (purple) is parallel to the y-axis, the other (yellow) parallel to the x-axis.



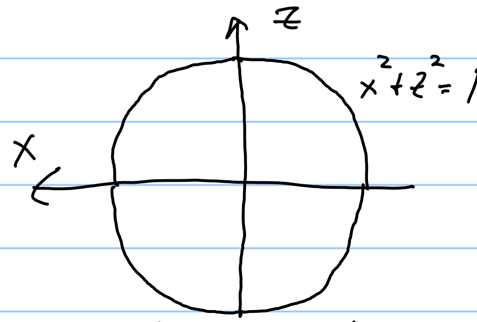
looking down x-axis

The solid projects onto the yz-plane as a circle of radius 1: $y^2+z^2=1$

$\therefore\ -1 \le z \le 1,\ -\sqrt{1-z^2} \le y \le \sqrt{1-z^2}$

The extent of x depends on z

$$-\sqrt{1-z^2} \le x \le \sqrt{1-z^2}$$



looking down y-axis

∴

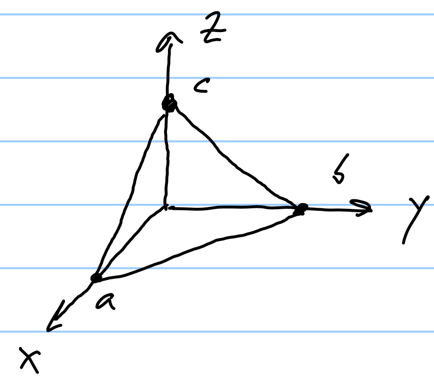
$$\text{Volume} = \int_{-1}^{1}\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}}\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} dx\,dy\,dz$$

$$= \int_{-1}^{1}\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} 2\sqrt{1-z^2}\,dy\,dz$$

$$= \int_{-1}^{1} 4(1-z^2)\,dz = \int_{-1}^{1} 4-4z^2\,dz = 4z - \frac{4}{3}z^3 \Big|_{-1}^{1}$$

$$= 4 - \frac{4}{3} - \left(-4 + \frac{4}{3}\right) = 8 - \frac{8}{3} = \underline{\frac{16}{3}}$$

9.



A plane with axis intercepts of $(a,0,0)$, $(0,b,0)$, $(0,0,c)$

Unknown if a, b, c are positive or negative.

Note that $\int_0^n = -\int_n^0$, so will take absolute value of final formula as "volume" ≥ 0.

$\therefore$ Assume $0 \leq x \leq a$ (it would be $a \leq x \leq 0$ if $a \leq 0$).

Note plane intersects xy-plane in a line,

$y = -\frac{b}{a}x + b$. $\therefore$ $0 \leq y \leq -\frac{b}{a}x + b$

$\frac{z}{c} = 1 - \frac{x}{a} - \frac{y}{b}$, $\therefore$ $z = c - \frac{c}{a}x - \frac{c}{b}y$

$\therefore$ $0 \leq z \leq c - \frac{c}{a}x - \frac{c}{b}y$

Assuming $a, b, c \neq 0$. If one is 0, volume $= 0$.

$$\therefore \text{Volume} = \int_0^a \int_0^{-\frac{b}{a}x+b} \int_0^{c-\frac{c}{a}x-\frac{c}{b}y} dz\,dy\,dx$$

$$= \int_0^a \int_0^{-\frac{b}{a}x+b} \left(c - \frac{c}{a}x - \frac{c}{b}y\right) dy\,dx$$

$$= \int_0^a cy - \frac{c}{a}xy - \frac{c}{2b}y^2 \Big|_{y=0}^{y=-\frac{b}{a}x+b} dx$$

$$= \int_0^a c\left(-\frac{b}{a}x+b\right) - \frac{c}{a}x\left(-\frac{b}{a}x+b\right) - \frac{c}{2b}\left(-\frac{b}{a}x+b\right)^2 dx$$

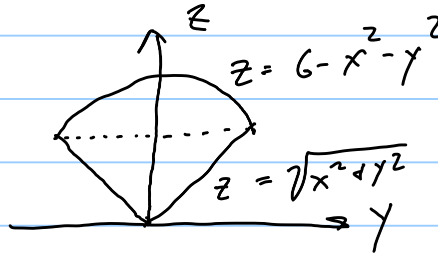
$$-\frac{bc}{a}x + bc + \frac{bc}{a^2}x^2 - \frac{bc}{a}x - \frac{bc}{2a^2}x^2 + \frac{bc}{a}x - \frac{bc}{2}$$

$$= \int_0^a \frac{bc}{2} - \frac{bc}{a}x + \frac{bc}{2a^2}x^2\,dx = \frac{bcx}{2} - \frac{bcx^2}{2a} + \frac{bcx^3}{6a^2}\Big|_{x=0}^{x=a}$$

$$= \frac{abc}{2} - \frac{abc}{2} + \frac{abc}{6} = \frac{abc}{6}. \quad \therefore \text{Volume} = \frac{|abc|}{6}$$

10.

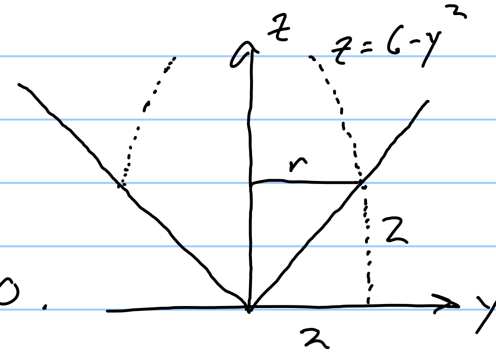
Basically, an ice cream cone.



Use cylindrical coordinates: $0 \le \theta \le 2\pi$

Since $r^2 = x^2 + y^2$, $\therefore z = 6 - r^2$ at top.

$\therefore \; r \le z \le 6 - r^2$.



What is extent of r?

From symmetry, let $x = 0$.

From paraboloid, $z = 6 - y^2$

From cone, $z = y$

$\therefore 6 - y^2 = y$, or $y^2 + y - 6 = 0$, $(y+3)(y-2) = 0$

$\therefore \underline{y = 2}$, since $y \ge 0$.

$\therefore \; 0 \le r \le 2$

$$\therefore \text{Volume} = \int_0^{2\pi} \int_0^2 \int_r^{6-r^2} r \, dz \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^2 rz \Big|_{z=r}^{z=6-r^2} dr\,d\theta = \int_0^{2\pi} \int_0^2 6r - r^3 - r^2 \, dr\,d\theta$$

$$= \int_0^{2\pi} 3r^2 - \frac{r^4}{4} - \frac{r^3}{3} \Big|_{r=0}^{r=2} d\theta = \int_0^{2\pi} \left(12 - 4 - \frac{8}{3}\right) d\theta$$

$$= \int_0^{2\pi} \frac{16}{3}\, d\theta = \underline{\frac{32}{3}\pi}$$

11.

From problem # 9, Volume of tetrahedron bounded by $\frac{x}{a} + \frac{y}{a} + \frac{z}{a} = 1$ is $\frac{a^3}{6}$.

Volume of $x + y + z = 1$ is $\therefore \frac{1}{6}$

$\therefore$ Each slice is to have volume $\frac{1}{6n}$

Let the slices be made at:

$$x = y = z = a_1$$
$$x = y = z = a_2$$
$$\vdots$$
$$x = y = z = a_n = 1$$

$$\therefore \frac{a_1^3}{6} + \left[\frac{a_2^3}{6} - \frac{a_1^3}{6}\right] + \left[\frac{a_3^3}{6} - \frac{a_2^3}{6}\right] + \cdots + \left[\frac{a_n^3}{6} - \frac{a_{n-1}^3}{6}\right] = \frac{1}{6}$$

$$\frac{a_1^3}{6} = \frac{1}{6n}, \quad \frac{a_2^3}{6} - \frac{a_1^3}{6} = \frac{1}{6n}, \quad \cdots\cdots \quad \frac{a_n^3}{6} - \frac{a_{n-1}^3}{6} = \frac{1}{6n}$$

$$a_1^3 = \frac{1}{n}, \quad a_2^3 = \frac{2}{n}, \quad a_3^3 = \frac{3}{n}, \ldots, \quad a_n^3 = \frac{n}{n}$$

$\therefore\ a_1 = \sqrt[3]{\frac{1}{n}},\ a_2 = \sqrt[3]{\frac{2}{n}},\ a_3 = \sqrt[3]{\frac{3}{n}},\ \ldots,\ a_n = 1$

$\therefore$ Each plane is of form: $\frac{x}{a_K} + \frac{y}{a_K} + \frac{z}{a_K} = 1$

where $\underline{a_K = \sqrt[3]{\frac{K}{n}}}$, $K = 1, 2, \ldots, n$

12.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

(a)

Projection of ellipsoid on xy-plane yields:

$-a \le x \le a,\ -b\sqrt{1-\frac{x^2}{a^2}} \le y \le b\sqrt{1 - x^2/a^2}$

$$\therefore \int_{-a}^{a}\int_{-b\sqrt{1-x^2/a^2}}^{b\sqrt{1-x^2/a^2}}\int_{-c\sqrt{1-x^2/a^2-y^2/b^2}}^{c\sqrt{1-x^2/a^2-y^2/b^2}} xyz\,dz\,dy\,dx$$

$$= \int_{-a}^{a}\int_{-b\sqrt{1-x^2/a^2}}^{b\sqrt{1-x^2/a^2}} xy\frac{z^2}{2}\Bigg|_{z=-c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}^{z=c\sqrt{1-x^2/a^2-y^2/b^2}} dy\,dx$$

$$= \int_{-a}^{a} \int_{-b\sqrt{1-x^2/a^2}}^{b\sqrt{1-x^2/a^2}} \frac{xy}{2} \left[c^2\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) - (-c)^2\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) \right] dy\,dx$$

$$= \int_{-a}^{a} \int_{-b\sqrt{1-x^2/a^2}}^{b\sqrt{1-x^2/a^2}} \frac{xy}{2}(0)\, dy\,dx = \underline{0}$$

Same result if integrate $dz\,dx\,dy$, $dx\,dy\,dz$, ...etc. because of symmetry of xyz.

(b)

$$\int_0^a \int_0^{b\sqrt{1-x^2/a^2}} \int_0^{c\sqrt{1-x^2/a^2-y^2/b^2}} xyz\, dz\,dy\,dx$$

$$= \int_0^a \int_0^{b\sqrt{1-x^2/a^2}} xy\frac{z^2}{2} \Bigg|_{z=0}^{z=c\sqrt{1-x^2/a^2-y^2/b^2}} dy\,dx$$

$$= \int_0^a \int_0^{b\sqrt{1-x^2/a^2}} \frac{c^2}{2}\left(xy - \frac{x^3y}{a^2} - \frac{xy^3}{b^2}\right) dy\,dx$$

$$= \frac{c^2}{2}\int_0^a \frac{xy^2}{2} - \frac{x^3y^2}{2a^2} - \frac{xy^4}{4b^2} \Bigg|_{y=0}^{y=b\sqrt{1-x^2/a^2}} dx$$

$$= \frac{c^2}{2}\int_0^a \frac{b^2}{2}\left(x - \frac{x^3}{a^2}\right) - \frac{b^2}{2a^2}\left(x^3 - \frac{x^5}{a^2}\right) - \frac{b^2}{4}x\left(1-\frac{x^2}{a^2}\right)^2 dx$$

$$-\frac{x}{2}\left(1 - \frac{2x^2}{a^2} + \frac{x^4}{a^4}\right)$$

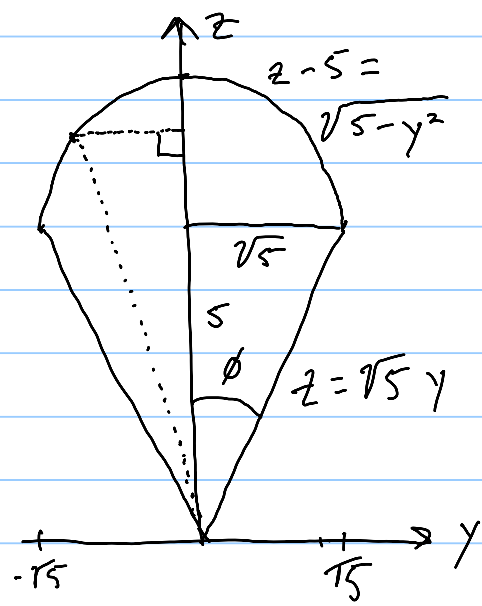
$$= \frac{b^2c^2}{4}\int_0^a x - \frac{x^3}{a^2} - \frac{x^3}{a^2} + \frac{x^5}{a^4} - \frac{x}{2} + \frac{x^3}{a^2} - \frac{x^5}{2a^4} \quad dx$$

$$= \frac{b^2c^2}{4}\int_0^a \frac{x}{2} - \frac{x^3}{a^2} + \frac{x^5}{2a^4} \quad dx$$

$$= \frac{b^2c^2}{4}\left[\frac{x^2}{4} - \frac{x^4}{4a^2} + \frac{x^6}{12a^4}\right]_{x=0}^{x=a}$$

$$= \frac{b^2c^2}{4}\left[\frac{a^2}{4} - \frac{a^2}{4} + \frac{a^2}{12}\right] = \underline{\underline{\frac{a^2b^2c^2}{48}}}$$

13.



$x^2+y^2 = \frac{1}{5}z^2 \iff z = \sqrt{5}\sqrt{x^2+y^2}$

For $x=0$ (or $y=0$) this is equivalent to $z = \pm\sqrt{5}\,y$

$z = 5+\sqrt{5-x^2-y^2} \iff z-5 = \sqrt{5-x^2-y^2}$

For $x=0$ (or $y=0$) this is equivalent to $(z-5)^2+y^2=5$, a circle of radius $\sqrt{5}$, centered at $(y,z)=(0,5)$

Try spherical coordinates.

For this problem, $0 \le \theta \le 2\pi$, $0 \le \phi \le \arctan\left(\frac{\sqrt{5}}{5}\right)$

as the "cone" meets the sphere at $z=5$,
$y = \sqrt{5}$ $(x=0)$.

For a given θ and ϕ, ρ ranges from 0 to the edge of the sphere (top of ice cream cone).
ρ is independent of θ, but depends on ϕ.

$$\rho\cos\phi = z = 5 + \sqrt{5 - x^2 - y^2}$$
$$= 5 + \sqrt{5 - (\rho\sin\phi\cos\theta)^2 - (\rho\sin\phi\sin\theta)^2}$$
$$= 5 + \sqrt{5 - \rho^2\sin^2\phi}$$

$$\therefore (\rho\cos\phi - 5)^2 = 5 - \rho^2\sin^2\phi$$

$$\rho^2\cos^2\phi - 10\rho\cos\phi + 25 = 5 - \rho^2\sin^2\phi$$

$$\rho^2 - 10\rho\cos\phi + 20 = 0$$

This looks messy.

Try cylindrical coordinates

$0 \le \theta \le 2\pi$, $0 \le r \le \sqrt{5}$ (from the above figure)

$\sqrt{5(x^2+y^2)} \le z \le 5 + \sqrt{5 - x^2 - y^2}$, or

$$\sqrt{5}\, r \le z \le 5 + \sqrt{5 - r^2}$$

$$\therefore \text{Volume} = \int_0^{2\pi} \int_0^{\sqrt{5}} \int_{\sqrt{5}r}^{5+\sqrt{5-r^2}} r \, dz \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\sqrt{5}} r\left(5 + \sqrt{5 - r^2}\right) - \sqrt{5}\, r^2 \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\sqrt{5}} 5r - \sqrt{5}\, r^2 + r\sqrt{5 - r^2} \, dr \, d\theta$$

$$= \int_0^{2\pi} \frac{5}{2} r^2 - \frac{\sqrt{5}}{3} r^3 - \left(\frac{1}{2}\right)\left(\frac{2}{3}\right)\left(5 - r^2\right)^{3/2} \Bigg|_{r=0}^{r=\sqrt{5}} d\theta$$

$$= \int_0^{2\pi} \frac{25}{2} - \frac{25}{3} + \frac{1}{3} 5^{\frac{3}{2}} \, d\theta = 2\pi\left(\frac{25}{6} + \frac{5\sqrt{5}}{3}\right)$$

$$= \frac{\pi}{3}\left(25 + 10\sqrt{5}\right)$$

14.

A surface surrounding the origin must have points in all 8 quadrants.

$\therefore\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi,\ 0 \le \rho \le f(\theta,\phi)$

describes the solid. The Jacobian $= \rho^2 \sin\phi$

$$\therefore\quad V = \int_0^{2\pi}\int_0^{\pi}\int_0^{f(\theta,\phi)} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{\rho^3}{3}\sin\phi \Big|_{\rho=0}^{\rho=f(\theta,\phi)} d\phi \, d\theta$$

$$= \underline{\int_0^{2\pi}\int_0^{\pi} \frac{[f(\theta,\phi)]^3}{3}\sin\phi \, d\phi \, d\theta}$$

15.

The structure of exp( ) suggests letting

$u = y - x$, $v = y + x$, or $T\begin{bmatrix}u\\v\end{bmatrix} = \begin{bmatrix}x\\y\end{bmatrix}$

or $T^{-1}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}u\\v\end{bmatrix} = \begin{bmatrix}y-x\\y+x\end{bmatrix}$ $\therefore T^{-1} = \begin{bmatrix}-1 & 1\\1 & 1\end{bmatrix}$

$\therefore T = -\frac{1}{2}\begin{bmatrix}1 & -1\\-1 & -1\end{bmatrix} = \begin{bmatrix}-1/2 & 1/2\\1/2 & 1/2\end{bmatrix}$

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = |\det(T)| = \left|-\frac{1}{4} - \frac{1}{4}\right| = \frac{1}{2}$$

$$\therefore \iint_{B^*} f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \iint_B f(x,y)\,dx\,dy$$

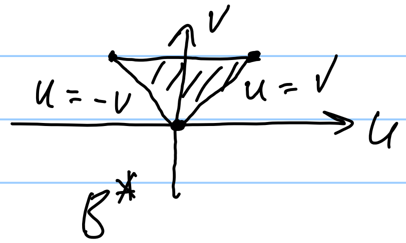
$$T\begin{bmatrix}u\\v\end{bmatrix} = \begin{bmatrix}-1/2 & 1/2\\1/2 & 1/2\end{bmatrix}\begin{bmatrix}u\\v\end{bmatrix} = \begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}\frac{-u+v}{2}\\ \frac{u+v}{2}\end{bmatrix}$$

or $x = \frac{v-u}{2}$, $y = \frac{u+v}{2}$

$T^{-1}(0,0) = (0,0)$, $T^{-1}(0,1) = \begin{bmatrix}-1 & 1\\1 & 1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$

$T^{-1}(1,0) = \begin{bmatrix}-1 & 1\\1 & 1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}-1\\1\end{bmatrix}$

$\therefore B^*$ has corners $(0,0)$, $(1,1)$, $(-1,1)$ $= (u,v)$



A u-simple region

$\therefore 0 \le v \le 1$, from $u = -v$ to $u = v$

$$\therefore \iint_{B^*} f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

$$= \int_0^1 \int_{-v}^{v} e^{\frac{u}{v}} \left(\frac{1}{2}\right) du\,dv = \frac{1}{2} \int_0^1 v e^{\frac{u}{v}} \Big|_{u=-v}^{u=v} dv$$

$$= \frac{1}{2} \int_0^1 (ve - ve^{-1}) dv = \frac{1}{2}(e - e^{-1}) \frac{v^2}{2} \Big|_{v=0}^{v=1}$$

$$= \underline{\frac{1}{4}\left(e - \frac{1}{e}\right)}$$

16.

(Density)(Volume) = mass.

Use spherical coordinates on a sphere.

Volume unit = $\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

$\therefore$ Mass of unit = $\left(\frac{1}{1+\rho^3}\right)(\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta)$

$$\therefore \text{Mass} = \int_0^{2\pi} \int_0^{\pi} \int_0^{R} \frac{1}{1+\rho^3} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{1}{3}\sin\phi \log(1+\rho^3)\Big|_{\rho=0}^{\rho=R} \, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{\log(1+R^3)}{3} \sin\phi \, d\phi\, d\theta$$

$$= \frac{\log(1+R^3)}{3}\int_0^{2\pi} -\cos\phi \Big|_{\phi=0}^{\phi=\pi} \, d\theta$$

$$= \frac{\log(1+R^3)}{3}\int_0^{2\pi} 2 \, d\theta = \underline{\frac{4\pi}{3}\log(1+R^3)}$$

17.

Use spherical coordinates on the spherical shell.

$$\text{Mass} = (\text{Density})(\text{Volume}) = (0.4\rho^2)(\rho^2 \sin\phi \, d\rho\, d\phi\, d\theta)$$

$$0.4\,g/cm^3 = 0.4\,g/(0.01m)^3 = 4\times10^5\, g/m^3$$

$$\therefore \text{Mass} = \int_0^{2\pi}\int_0^{\pi}\int_1^{2} (4\times10^5)\,\rho^4 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= (4\times 10^5)\int_0^{2\pi}\int_0^{\pi} \sin\phi \, \frac{\rho^5}{5}\Big|_{\rho=1}^{\rho=2} \, d\phi\, d\theta$$

$$= (8\times 10^4)\int_0^{2\pi}\int_0^{\pi} \sin\phi \,[32-1]\, d\phi\, d\theta$$

$$= (8\times 10^4)(31)\int_0^{2\pi} -\cos\phi\Big|_{\phi=0}^{\phi=\pi} \, d\phi\, d\theta$$

$$= 2.48\times 10^6\int_0^{2\pi} 2\, d\theta = (2.48\times 10^6)(4\pi)\ g$$

$$= \underline{(9.92\times 10^6)\pi} \text{ grams}$$

18.

(a) It's volume is: $\frac{4}{3}\pi\,(200\ cm)^3 = \left(\frac{32}{3}\times 10^6\right)\pi\ cm^3$

$$\text{Density} = \frac{\text{Mass}}{\text{Volume}} = \frac{(9.92\times 10^6)\pi \text{ grams}}{(32/3\times 10^6)\pi\ cm^3}$$

$$= 0.93\ g/cm^3$$

As it is less dense than water, it will <u>float</u>.

(b) If the shell leaked, the inside would fill with water. The inside of the shell has

$$\text{volume} = \tfrac{4}{3}\pi(100\,\text{cm})^3 = \left(\tfrac{4}{3}\times 10^6\right)\pi\ \text{cm}^3$$

$\therefore$ Mass of water inside shell is:

$$\left[\left(\tfrac{4}{3}\times 10^6\right)\pi\ \text{cm}^3\right](1\,\text{g}/\text{cm}^3) = \left(\tfrac{4}{3}\times 10^6\right)\pi \text{ grams.}$$

$\therefore$ Mass shell + inside water is:

$$(9.92\times 10^6)\pi + \left(\tfrac{4}{3}\times 10^6\right)\pi = (11.25\times 10^6)\pi\ \text{g}$$

$\therefore$ Density of leaked shell is:

$$\frac{\text{Mass}}{\text{Volume}} = \frac{(11.25\times 10^6)\pi\ \text{g}}{\left(\frac{32}{3}\times 10^6\right)\pi\ \text{cm}^3} = 1.06\ \text{g}/\text{cm}^3$$

Density of leaked shell is greater than water, and so it will sink.

19.

(a) Sum the temperature in all the volume "elements" of the sphere, then divide by total volume.

$$\text{Total volume} = 2^3 = 8$$

$$\sum \text{Temp} = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} 32(x^2+y^2+z^2)\,dx\,dy\,dz$$

$$= 32\int_{-1}^{1}\int_{-1}^{1} \frac{x^3}{3} + y^2x + z^2x \Big|_{x=-1}^{x=1} dy\,dz$$

$$= 32\int_{-1}^{1}\int_{-1}^{1} \frac{2}{3} + 2y^2 + 2z^2 \; dy\,dz$$

$$= 32\int_{-1}^{1} \frac{2}{3}y + \frac{2}{3}y^3 + 2z^2y \Big|_{y=-1}^{y=1} dz$$

$$= 32\int_{-1}^{1} \left(\frac{4}{3} + \frac{4}{3} + 4z^2\right) dz$$

$$= 32\left[\frac{8}{3}z + \frac{4}{3}z^3\right]_{z=-1}^{z=1} = 32\left[\frac{16}{3} + \frac{8}{3}\right]$$

$$= 32\left(\frac{24}{3}\right) = 32(8)$$

$\therefore$ Ave Temp $= \dfrac{32(8)}{8} = \underline{32}$

(5) $T(D) = 32d^2$. $\therefore$ $32d^2 = 32$, or $d = 1$.

$\therefore$ Points where ave. temp $= 32$ are on the <u>sphere of radius 1.</u>

20.

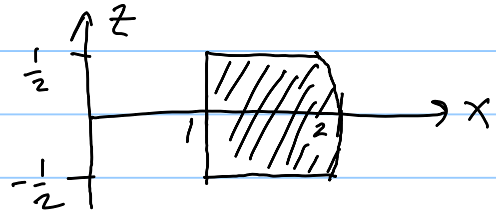
Assume uniform density.

$y^2 + z^2 \le \frac{1}{4}$ is a circular pipe extending to $\pm\infty$ along x-axis, radius $\frac{1}{2}$.

$(x-1)^2 + y^2 + z^2 \le 1$ is a unit sphere, center at $(1,0,0)$.

$x \ge 1$ restricts sphere to hemisphere, pipe to $+\infty$.

$\therefore$ A pipe of radius $\frac{1}{2}$, along x-axis from $x=1$ with a unit spherical cap at $x=2$



Make x-axis the traditional "z-axis" of cylindrical coordinates.

$\therefore 0 \le \theta \le 2\pi, \quad 0 \le r \le \frac{1}{2}$, where $r^2 = y^2 + z^2$

From $(x-1)^2 = 1 - y^2 - z^2 = 1 - r^2$,

$$x - 1 = \sqrt{1-r^2}, \quad x = 1 + \sqrt{1-r^2}$$

$$\therefore 1 \le x \le 1 + \sqrt{1-r^2}$$

$$\therefore \text{ Volume} = \int_0^{2\pi}\int_0^{\frac{1}{2}}\int_1^{1+\sqrt{1-r^2}} r\,dx\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\frac{1}{2}} rx\Big|_{x=1}^{x=1+\sqrt{1-r^2}}\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\frac{1}{2}} r\sqrt{1-r^2}\,dr\,d\theta = \int_0^{2\pi}\left(-\tfrac{1}{2}\right)\left(\tfrac{2}{3}\right)\left(1-r^2\right)^{\frac{3}{2}}\Big|_{r=0}^{r=\frac{1}{2}}\,d\theta$$

$$= -\frac{1}{3}\int_0^{2\pi}\left[\left(\tfrac{3}{4}\right)^{\frac{3}{2}}-1\right]d\theta = \int_0^{2\pi}\left(\tfrac{1}{3}-\tfrac{\sqrt{3}}{8}\right)d\theta = \underline{\left(\tfrac{1}{3}-\tfrac{\sqrt{3}}{8}\right)2\pi}$$

From symmetry, center of mass is on the x-axis.
$\therefore$ Just compute $\bar{x}$.

$$\therefore \bar{x} = \frac{\int_0^{2\pi}\int_0^{\frac{1}{2}}\int_1^{1+\sqrt{1-r^2}} xr\,dx\,dr\,d\theta}{2\pi\left(\frac{1}{3}-\frac{\sqrt{3}}{8}\right)}$$

$$\int_0^{2\pi}\int_0^{\frac{1}{2}}\int_1^{1+\sqrt{1-r^2}} xr\,dx\,dr\,d\theta = \int_0^{2\pi}\int_0^{\frac{1}{2}} r\frac{x^2}{2}\Big|_{x=1}^{x=1+\sqrt{1-r^2}}\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\frac{1}{2}} \frac{r\left(1+\sqrt{1-r^2}\right)^2}{2} - \frac{r}{2}\,dr\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\int_0^{\frac{1}{2}} r\left(1 + 2\sqrt{1-r^2} + 1 - r^2\right) - r \quad dr\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\int_0^{\frac{1}{2}} 2r\sqrt{1-r^2} + r - r^3 \quad dr\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \left(-\frac{2}{3}\right)\left(1-r^2\right)^{3/2} + \frac{r^2}{2} - \frac{r^4}{4}\Bigg|_{r=0}^{r=\frac{1}{2}} \quad d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\left[-\frac{\sqrt{3}}{4} + \frac{1}{8} - \frac{1}{64} - \left(-\frac{2}{3}\right)\right] d\theta$$

$$= \frac{2\pi}{2}\left[\frac{7}{64} + \frac{2}{3} - \frac{\sqrt{3}}{4}\right] = \pi\left(\frac{7}{64} + \frac{2}{3} - \frac{\sqrt{3}}{4}\right)$$

$$\therefore \bar{x} = \frac{\pi\left(\frac{7}{64} + \frac{2}{3} - \frac{\sqrt{3}}{4}\right)}{2\pi\left(\frac{1}{3} - \frac{\sqrt{3}}{8}\right)} = \underline{\frac{\left(\frac{7}{64} + \frac{2}{3} - \frac{\sqrt{3}}{4}\right)}{2\left(\frac{1}{3} - \frac{\sqrt{3}}{8}\right)}}$$

$$\simeq \underline{1.468}$$

Note: solution manual states "shifted down $\frac{1}{2}$ unit", but actually was shifted down 1 unit. Using 1 unit, the answer is the same as above: 1.468.

21.

Volume $= \frac{1}{2}\left(\frac{4}{3}\pi a^3\right) = \frac{2}{3}\pi a^3$

By symmetry, center of mass on z-axis.

Use spherical coordinates. Distance from xy-plane is $\rho\cos\phi$.

$$\therefore \quad \bar{z} = \frac{\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^a (\rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta}{\frac{2}{3}\pi a^3}$$

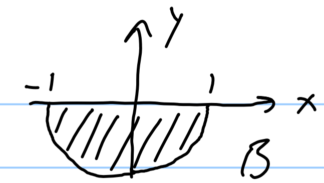
$$= 2\pi\left(\frac{3}{2\pi a^3}\right)\int_0^{\frac{\pi}{2}}\int_0^a \rho^3\left(\tfrac{1}{2}\right)(2\cos\phi\sin\phi)\,d\rho\,d\phi$$

$$= \frac{3}{2a^3}\int_0^{\frac{\pi}{2}}\int_0^a \sin 2\phi\,\rho^3\,d\rho\,d\phi$$

$$= \frac{3}{2a^3}\left[\frac{\rho^4}{4}\right]_{\rho=0}^{\rho=a}\left[-\frac{1}{2}\cos 2\phi\right]_{\phi=0}^{\phi=\frac{\pi}{2}}$$

$$= \frac{3}{2a^3}\left[\frac{a^4}{4}\right]\left[1\right] = \frac{3}{8}a. \quad \therefore \underline{\left(0, 0, \frac{3}{8}a\right)}$$

22.



$$\iint_B e^{-x^2-y^2}\,dx\,dy = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{0} e^{-x^2-y^2}\,dy\,dx$$

Use a polar coordinates conversion.

$r = x^2+y^2$, $\therefore$ $0 \le r \le 1$, $-\pi \le \theta \le 0$, Jacobian $= r$

$$\therefore \int_{-\pi}^{0}\int_{0}^{1} e^{-r^2}\,r\,dr\,d\theta = \int_{-\pi}^{0} \left(-\tfrac{1}{2}\right) e^{-r^2}\Big|_{r=0}^{r=1}\,d\theta$$

$$= -\frac{1}{2}\int_{-\pi}^{0} \left(e^{-1}-1\right)d\theta = \frac{1-e^{-1}}{2}\left[\theta\right]_{-\pi}^{0} = \underline{\frac{\pi}{2}\left(1-\frac{1}{e}\right)}$$

23.

Use spherical coordinates: $x^2+y^2+z^2=\rho^2$, $b \le \rho \le a$

$$\therefore \int_{0}^{2\pi}\int_{0}^{\pi}\int_{b}^{a} \frac{\rho^2 \sin\phi \,d\rho\,d\phi\,d\theta}{(\rho^2)^{3/2}} = \int_{0}^{2\pi}\int_{0}^{\pi}\int_{b}^{a} \frac{\sin\phi}{\rho}\,d\rho\,d\phi\,d\theta$$

$$= \left[\log \rho\right]_{\rho=6}^{\rho=9} \left[-\cos\phi\right]_{\phi=0}^{\phi=\pi} \left[\theta\right]_{\theta=0}^{\theta=2\pi}$$

$$= \left[\ln \tfrac{9}{6}\right]\left[2\right]\left[2\pi\right] = \underline{4\pi \ln \tfrac{9}{6}}$$

24.

(a)

Use spherical coordinates : $x^2+y^2+z^2=\rho^2$

$$xyz = (\rho\sin\phi\cos\theta)(\rho\sin\phi\sin\theta)(\rho\cos\phi)$$

$$= \rho^3(\sin^2\phi\cos\phi)(\sin\theta\cos\theta)$$

$$= \rho^3\left(\frac{\sin 2\theta}{2}\right)(\sin^2\phi\cos\phi)$$

Sphere: $0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi,\ 0 \le \rho \le R$

Jacobian $= \rho^2 \sin\phi$

$$\therefore \int_0^{2\pi}\int_0^{\pi}\int_0^{R} (\rho^2)\left(\frac{\rho^3}{2}\sin 2\theta \sin^2\phi\cos\phi\right)(\rho^2\sin\phi)\,d\rho\,d\phi\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\int_0^{\pi}\int_0^{R} (\rho^7)(\sin 2\theta)(\sin^3\phi\cos\phi)\,d\rho\,d\phi\,d\theta$$

$$= \frac{1}{2}\left[\frac{\rho^8}{8}\right]_{\rho=0}^{\rho=R}\left[-\frac{\cos 2\theta}{2}\right]_{\theta=0}^{\theta=2\pi}\left[\frac{\sin^4\phi}{4}\right]_{\phi=0}^{\phi=\pi}$$

$$= \frac{1}{2}\left[\frac{R^8}{8}\right]\left[-\frac{1}{2}-\left(-\frac{1}{2}\right)\right]\left[0-0\right] = \underline{\underline{0}}$$

(b)

Use the above spherical coordinates, but here,

$0 \le \phi \le \frac{\pi}{2}$

$$\therefore \frac{1}{2}\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{R}(\rho^7)(\sin 2\theta)(\sin^3\phi\cos\phi)\,d\rho\,d\phi\,d\theta$$

$$= \frac{1}{2}\left[\frac{\rho^8}{8}\right]_{\rho=0}^{\rho=R}\left[-\frac{\cos 2\theta}{2}\right]_{\theta=0}^{\theta=2\pi}\left[\frac{\sin^4\phi}{4}\right]_{\phi=0}^{\phi=\frac{\pi}{2}}$$

$$= \frac{1}{2}\left[\frac{R^8}{8}\right]\underbrace{\left[-\frac{1}{2}-\left(-\frac{1}{2}\right)\right]}_{=0}\left[\frac{1}{4}-0\right] = \underline{\underline{0}}$$

(c)

Use the above spherical coordinates, but here,

$0 \le \theta \le \frac{\pi}{2}$, $0 \le \phi \le \frac{\pi}{2}$, $0 \le \rho \le R$

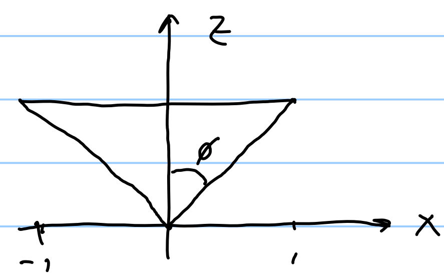
$$\therefore \frac{1}{2}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^{R}(\rho^7)(\sin 2\theta)(\sin^3\phi\cos\phi)\,d\rho\,d\phi\,d\theta$$

$$= \frac{1}{2}\left[\frac{\rho^8}{8}\right]_{\rho=0}^{\rho=R}\left[-\frac{\cos 2\theta}{2}\right]_{\theta=0}^{\theta=\frac{\pi}{2}}\left[\frac{\sin^4\phi}{4}\right]_{\phi=0}^{\phi=\frac{\pi}{2}}$$

$$= \frac{1}{2}\left[\frac{R^8}{8}\right]\left[\frac{1}{2}-\left(-\frac{1}{2}\right)\right]\left[\frac{1}{4}-0\right]$$

$$= \underline{\frac{R^8}{64}}$$

25.



Convert to spherical coordinates. $0 \le \theta \le 2\pi$, $0 \le \phi \le \frac{\pi}{4}$

Note $z = \rho\cos\phi$, so $0 \le \rho \le \frac{1}{\cos\phi}$ as $0 \le z \le 1$.

$$x^2+y^2 = (\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\sin\theta)^2$$

$$= \rho^2\sin^2\phi$$

$$\therefore 1+\sqrt{x^2+y^2} = 1+\rho\sin\phi$$

Jacobian $= \rho^2\sin\phi$

$$\therefore \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\frac{1}{\cos\phi}} (1+\rho\sin\phi)\,\rho^2\sin\phi\; d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\frac{1}{\cos\phi}} \rho^2\sin\phi + \rho^3\sin^2\phi\; d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{4}} \sin\phi\,\frac{\rho^3}{3} + \sin^2\phi\,\frac{\rho^4}{4}\Bigg|_{\rho=0}^{\rho=\frac{1}{\cos\phi}} d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{4}} \frac{\sin\phi}{3\cos^3\phi} + \frac{\sin^2\phi}{4\cos^4\phi}\; d\phi\, d\theta$$

$$= \int_0^{2\pi} -\frac{1}{3}\left(-\frac{1}{2}\right)\cos^{-2}\phi\Bigg|_{\phi=0}^{\phi=\frac{\pi}{4}} d\theta + \int_0^{2\pi}\int_0^{\frac{\pi}{4}} \frac{1}{4}\tan^2\phi\sec^2\phi\; d\phi\, d\theta$$

$$= \int_0^{2\pi} \frac{1}{6}\left[\frac{1}{\left(\frac{1}{\sqrt{2}}\right)^2} - 1\right] d\theta + \int_0^{2\pi} \frac{1}{4}\left[\frac{\tan^3\phi}{3}\right]_{\phi=0}^{\phi=\frac{\pi}{4}} d\theta$$

$$= \int_0^{2\pi} \frac{1}{6}\, d\theta + \int_0^{2\pi} \frac{1}{4}\left(\frac{1}{3}\right) d\theta = \frac{2\pi}{6} + \frac{2\pi}{12}$$

$$= \underline{\frac{\pi}{2}}$$

26.

Use spherical coordinates. $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi$

$x^2 + y^2 + z^2 = \rho^2$, $0 \le \rho < \infty$

$$\therefore \lim_{a\to\infty} \int_0^{2\pi}\int_0^{\pi}\int_0^{a} e^{-\rho^3}\rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

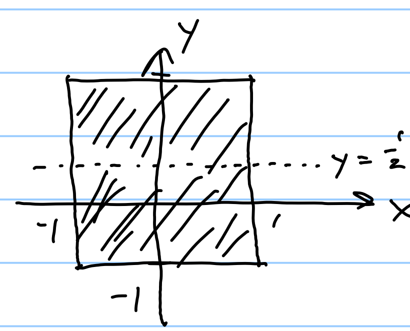
$$= \lim_{a\to\infty} \int_0^{2\pi} d\theta \int_0^{\pi} \sin\phi\, d\phi \int_0^{a} e^{-\rho^3}\rho^2\, d\rho$$

$$= \lim_{a\to\infty} (2\pi)\Big[-\cos\phi\Big]_0^{\pi} \Big[-\tfrac{1}{3} e^{-\rho^3}\Big]_0^{a}$$

$$= 4\pi \lim_{a\to\infty}\left[-\frac{1}{3e^{a^3}} + \frac{1}{3}\right] = \underline{\underline{\frac{4\pi}{3}}}$$

27.

(a)

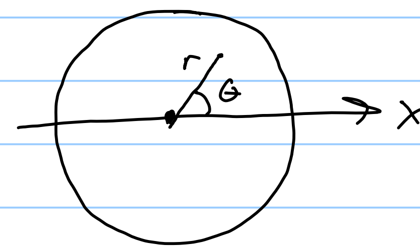


$$d(x,y) = \sqrt{(y-\tfrac{1}{2})^2} = |y - \tfrac{1}{2}|, \quad \therefore d(x,y) = (y-\tfrac{1}{2})^2$$

$$\therefore I = \int_{-1}^{1}\int_{-1}^{2} (y-\tfrac{1}{2})^2 \, dy\, dx = \int_{-1}^{1} \tfrac{1}{3}(y-\tfrac{1}{2})^3 \Big|_{y=-1}^{y=2} dx$$

$$\int_{-1}^{1} \tfrac{1}{3}\left[\left(\tfrac{3}{2}\right)^3 - \left(-\tfrac{3}{2}\right)^3\right] dx = \int_{-1}^{1} \tfrac{9}{4}\, dx = \underline{\tfrac{9}{2}}$$

(b)



$$d = r\sin\theta, \quad d^2 = r^2\sin^2\theta$$

$$\therefore I = \int_{0}^{2\pi}\int_{0}^{4} (r^2\sin^2\theta)\, r\, dr\, d\theta$$

$$= \int_{0}^{2\pi} \frac{r^3}{3}\sin^2\theta \Big|_{r=0}^{r=4} d\theta = \frac{64}{3}\int_{0}^{2\pi} \sin^2\theta\, d\theta$$

$\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta$

$$= \frac{64}{3}\int_{0}^{2\pi} \frac{1-\cos 2\theta}{2}\, d\theta = \frac{64}{3}\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_{\theta=0}^{\theta=2\pi}$$

$$= \underline{\frac{64}{3}\pi}$$

28.

Use spherical coordinates. $\rho^2 = x^2 + y^2 + z^2$

$$\therefore f(x(\rho,\theta,\phi), y(\rho,\theta,\phi), z(\rho,\theta,\phi)) = \frac{1}{(1+\rho^3)^{3/2}}$$

$0 \le \theta \le 2\pi$, $0 \le \phi \le \pi$, $0 \le \rho \le a$, $\lim_{a\to\infty} \rho$

$$\therefore \iiint_{R^3} f \, dx\,dy\,dz = \lim_{a\to\infty} \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \frac{\rho^2 \sin\phi}{(1+\rho^3)^{3/2}} \, d\rho\, d\phi\, d\theta$$

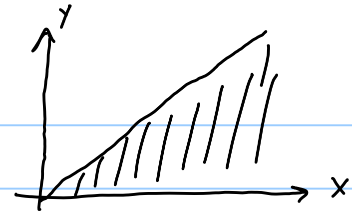
$$= \lim_{a\to\infty} \int_0^{2\pi}\int_0^{\pi} \left(\tfrac{1}{3}\right)(-2)(1+\rho^3)^{-\frac{1}{2}} \sin\phi \Big|_{\rho=0}^{\rho=a} d\phi\, d\theta$$

$$= -\frac{2}{3} \lim_{a\to\infty} \int_0^{2\pi}\int_0^{\pi} \left[\frac{1}{\sqrt{1+a^3}} - 1\right] \sin\phi \, d\phi\, d\theta$$

$$= -\frac{2}{3} \left[\lim_{a\to\infty}\left(\frac{1}{\sqrt{1+a^3}} - 1\right)\right]\left[\int_0^{2\pi} d\theta\right]\left[\int_0^{\pi} \sin\phi\, d\phi\right]$$

$$= -\frac{2}{3}[-1][2\pi][2] = \frac{8\pi}{3}$$

29.



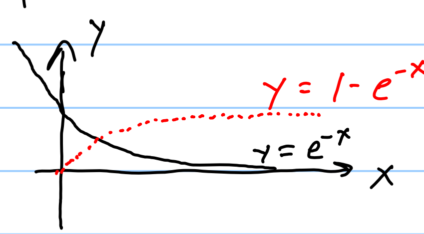
$$\iint_D f(x,y)\,dx\,dy = \lim_{\substack{a\to\infty \\ b\to 0^+}} \int_b^a \int_0^x x^{-3/2} e^{y-x}\,dy\,dx$$

$$= \lim_{\substack{a\to\infty \\ b\to 0^+}} \int_b^a x^{-\frac{3}{2}} e^{y-x} \Big|_{y=0}^{y=x}\,dx$$

$$= \lim_{\substack{a\to\infty \\ b\to 0^+}} \int_b^a x^{-\frac{3}{2}} - x^{-\frac{3}{2}} e^{-x}\,dx$$

$$= \lim_{\substack{a\to\infty \\ b\to 0^+}} \int_b^a x^{-\frac{3}{2}}\left(1-e^{-x}\right)dx$$

$$= \lim_{b\to 0^+} \int_b^1 x^{-\frac{3}{2}}\left(1-e^{-x}\right)dx + \lim_{a\to\infty} \int_1^a x^{-\frac{3}{2}}\left(1-e^{-x}\right)dx$$



(a) $1-e^{-x} < 1$ for $1 \le x < \infty$

$\therefore\ x^{-\frac{3}{2}}\left(1-e^{-x}\right) < x^{-\frac{3}{2}}$ for $1 \le x < \infty$

$$\therefore \int_1^a x^{-\frac{3}{2}}\left(1-e^{-x}\right)dx \le \int_1^a x^{-\frac{3}{2}}\,dx$$

Since $\lim\limits_{a\to\infty}\int_1^a x^{-\frac{3}{2}}\,dx = \lim\limits_{a\to\infty}\left[-2x^{-\frac{1}{2}}\right]_1^a$

$$= \lim_{a\to\infty}\left[\frac{-2}{\sqrt{a}} + 2\right] = 2$$

Then since $x^{-\frac{3}{2}}(1-e^{-x}) > 0$ for $x \geq 1$,

and is bounded above by $x^{-3/2}$,

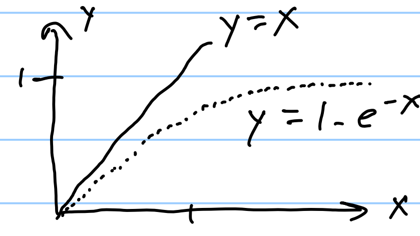
and $\int_1^\infty x^{-\frac{3}{2}}\,dx$ exists,

Then $\int_1^\infty x^{-\frac{3}{2}}(1-e^{-x})\,dx$ exists

(b) On $0 < x \leq 1$, $1 - e^{-x} < x$

$\frac{d}{dx}(1-e^{-x}) = e^{-x}$, so max slope is at $x=0$,

$e^{-0} = 1$, so $1 - e^{-x} < x$



$\therefore\ x^{-\frac{3}{2}}(1-e^{-x}) < x^{-\frac{3}{2}}x = x^{-\frac{1}{2}}$ for $0 < x < 1$.

$\therefore\ \int_b^1 x^{-\frac{3}{2}}(1-e^{-x}) < \int_b^1 x^{-\frac{1}{2}}\,dx$ for $b > 0$

since $0 < x^{-\frac{3}{2}}(1-e^{-x})$ for $0 < x < 1$.

$$\text{But } \lim_{b\to 0^+} \int_b^1 x^{-\frac{1}{2}}\,dx = \lim_{b\to 0^+} \left[2x^{\frac{1}{2}}\right]_b^1$$

$$= \lim_{b\to 0^+} \left[2 - 2\sqrt{b}\right] = 2.$$

$$\therefore \lim_{b\to 0^+} \int_b^1 x^{-\frac{3}{2}}(1-e^{-x}) \text{ exists.}$$

$$\therefore \int_0^1 x^{-3/2}(1-e^{-x}) \text{ exists.}$$

$$(a) \,\&\, (b) \Rightarrow \int_0^{\infty} x^{-3/2}(1-e^{-x})\,dx \text{ exists.}$$

$$\therefore \iint_D x^{-3/2} e^{y-x}\,dx\,dy \ \underline{\text{exists}}.$$

30.

$V(x_1,y_1) = K\sigma dV \ln\sqrt{(x-x_1)^2+(y-y_1)^2}$, potential at (x_1,y_1)

σdV = mass of "point mass" of volume dV,

K = a constant, (x,y) = point in disk

Disk of density σ, radius R

center at (x_0,y_0).

$$\therefore \iint_U k\sigma \ln\sqrt{(x-x_1)^2+(y-y_1)^2}\,dx\,dy \quad , \quad U = \text{disk}$$

Let $(x_0, y_0) = (0,0)$ for ease.

$$\therefore V(x_1,y_1) = k\sigma \int_0^{2\pi}\int_0^{R} \ln\sqrt{(r\cos\theta - x_1)^2 + (r\sin\theta - y_1)^2}\; r\,dr\,d\theta$$

31.

(a) $$\int_0^{\infty} e^{-y^3} \frac{x^2}{2}\Big|_{x=0}^{x=y} dy = \int_0^{\infty} e^{-y^3}\frac{y^2}{2}\,dy$$

$$= \lim_{a\to\infty} -\frac{1}{6}e^{-y^3}\Big|_{y=0}^{y=a} = \lim_{a\to\infty}\left[-\frac{1}{6e^{a^3}} + \frac{1}{6}\right] = \frac{1}{6}$$

(b)

Use polar coordinates. $x = r\cos\theta$, $y = r\sin\theta$

$0 \le r \le 2, \quad 0 \le \theta \le \frac{\pi}{2} \qquad \text{Jacobian} = r$

$x^4 + 2x^2y^2 + y^4 \Rightarrow (x^2+y^2)^2 \Rightarrow r^4$

$$\therefore \int_0^{\frac{\pi}{2}} \int_0^2 (r^4)\, r\,dr\,d\theta = \int_0^{\frac{\pi}{2}} \frac{r^6}{6} \Big|_{r=0}^{r=2} d\theta$$

$$= \frac{64}{6}\left(\frac{\pi}{2}\right) = \underline{\frac{16}{3}\pi}$$

32.

If D is y-simple, define $D_{\delta,\epsilon}$ as

$$D_{\delta,\epsilon} = \{(x,y) : a+\delta \le x \le b-\delta,\ \phi_1(x)+\epsilon \le y \le \phi_2(x)-\epsilon\}$$

If D is x-simple, define $D_{\delta,\epsilon}$ as

$$D_{\delta,\epsilon} = \{(x,y) : \phi_1(y)+\delta \le x \le \phi_2(y)-\delta,\ c+\epsilon \le y \le d-\epsilon\}$$

Suppose there are n points of discontinuity in D, whose coordinates are (x_i, y_i), $i = 1, \dots, n$.

Let $B^i_{\delta,\epsilon} = \{(x,y) : x_i-\delta < x < x_i+\delta,\ y_i-\epsilon < y < y_i+\epsilon\}$

(so $B^i_{\delta,\epsilon}$ is open)

Let $D^*_{\delta,\epsilon} = D_{\delta,\epsilon} - B^1_{\delta,\epsilon} - B^2_{\delta,\epsilon} - \dots - B^n_{\delta,\epsilon}$

choosing δ, ϵ small enough so that $B^i_{\delta,\epsilon} \subset D_{\delta,\epsilon}$

Define $\iint_D f\,dA = \lim\limits_{\substack{\delta\to 0\\ \epsilon\to 0}} \iint_{D^*_{\delta,\epsilon}} f\,dA$

33.

Use a polar conversion : $x^2+y^2=r^2$

$\therefore \dfrac{1}{(1+x^2+y^2)^{3/2}} \Rightarrow \dfrac{1}{(1+r^2)^{3/2}}$ Jacobian $= r$

$0 \le \theta \le 2\pi, \quad 0 \le r \le a < \infty$

$$\therefore \lim_{a\to\infty} \int_0^{2\pi}\int_0^a \frac{r\,dr\,d\theta}{(1+r^2)^{3/2}} = \lim_{a\to\infty} \int_0^{2\pi} -(1+r^2)^{-\frac{1}{2}} \Big|_{r=0}^{r=a} d\theta$$

$$= \lim_{a\to\infty} \int_0^{2\pi} -\frac{1}{\sqrt{1+a^2}} + 1\,d\theta = \int_0^{2\pi} d\theta = \underline{2\pi}$$

as $\lim\limits_{a\to\infty} \dfrac{1}{\sqrt{1+a^2}} = 0$