7.1 The Path Integral

Note Title 2/3/2017

1.

Consider $(3\cos t, 3\sin t)$ for $0 \le t \le \pi$

and then for $\pi \le t \le 7\pi$, $(\frac{t}{\pi} - 4, 0)$

$\therefore$

$$c(t) = \begin{cases} (3\cos t, 3\sin t), & t \in [0, \pi] \\ (\frac{t}{\pi} - 4, 0), & t \in [\pi, 7\pi] \end{cases}$$

2.

For $y = x^2$: $c(t) = (t, t^2)$ for $t \in [0, 2]$

Line from $(2,4)$ to $(0,4)$: $(4-t, 4)$ for $t \in [2, 4]$

Line from $(0,4)$ to $(0,0)$: $(0, 8-t)$, $t \in [4, 8]$.

$\therefore$

$$c(t) = \begin{cases} (t, t^2), & t \in [0, 2] \\ (4-t, 4), & t \in [2, 4] \\ (0, 8-t), & t \in [4, 8] \end{cases}$$

3.

$$c(t) = \begin{cases} (t, \sin t), & t \in [0, \pi] \\ (2\pi - t, 0), & t \in [\pi, 2\pi] \end{cases}$$

4.

Center of ellipse at $(2,3)$, minor axis $=2$, major $=3$

Let $x = 2 + 2\cos\theta$, $y = 3 + 3\sin\theta$

$\therefore (x-2)^2 = 4\cos^2\theta \quad (y-3)^2 = 9\sin^2\theta$

$\therefore \dfrac{(x-2)^2}{4} + \dfrac{(y-3)^2}{9} = \dfrac{4\cos^2\theta}{4} + \dfrac{9\sin^2\theta}{9} = 1.$

$\therefore c(\theta) = (2 + 2\cos\theta, 3 + 3\sin\theta), \quad \theta \in [0, 2\pi]$

5.

$c(\theta) = (3\cos\theta, 4\sin\theta, 3), \quad \theta \in [0, 2\pi]$

6.

$(1,2,3)$ to $(0,-2,1)$: $(1,2,3) + t\left[(0,-2,1)-(1,2,3)\right]$
$= (1-t, 2-4t, 3-2t), \quad 0 \le t \le 1$

$(0,-2,1)$ to $(6,4,2)$: $(0,-2,1) + s\left[(6,4,2)-(0,-2,1)\right]$
$= (0,-2,1) + s(6,6,1), \quad 0 \le s \le 1$

Let $t-1=s$, so $t=s+1$, $\therefore 1 \le t \le 2$

$\therefore$ Segment $= (0,-2,1) + (t-1)(6,6,1)$

$= (0,-2,1) + (-6,-6,-1) + t(6,6,1)$

$= (-6,-8,0) + t(6,6,1), \quad 1 \le t \le 2$

$(6,4,2)$ to $(1,2,3)$: $(6,4,2) + s\left[(1,2,3)-(6,4,2)\right]$
$= (6,4,2) + s(-5,-2,1), \quad 0 \le s \le 1$

Let $t-2=s$, so $t=s+2$, $\therefore 2 \le t \le 3$

$\therefore$ segment $= (6,4,2) + (t-2)(-5,-2,1)$

$= (6,4,2) + (10,4,-2) + t(-5,-2,1)$

$= (16,8,0) + t(-5,-2,1), \quad 2 \le t \le 3$

$$\therefore \text{Triangle} = \begin{cases} (1-t, 2-4t, 3-2t), & 0 \le t \le 1 \\ (-6+6t, -8+6t, t), & 1 \le t \le 2 \\ (16-5t, 8-2t, t), & 2 \le t \le 3 \end{cases}$$

7. Should be $(2,2,8)$.

If look at $-3 \to 2$: $-3 + t(2-(-3)) = -3+5t$, $0 \le t \le 1$.

$\therefore [-3+5t, -3+5t, (-3+5t)^3]$, $0 \le t \le 1$

8.

The circle $y^2+z^2=1$ can be parametrized as

$(\ , \cos\theta, \sin\theta)$, $0 \le \theta \le 2\pi$

Since $x = z$, $\underline{c(\theta) = (\sin\theta, \cos\theta, \sin\theta)}$, $0 \le \theta \le 2\pi$

9.

$$\int_c f\,ds = \int_0^1 f(\vec{c}(t))\,\|\vec{c}\,'(t)\|\,dt$$

$$= \int_0^1 f(0,0,t)\,\|(0,0,1)\|\,dt = \int_0^1 0(1)\,dt = \underline{0}$$

10.

(a) $f(x(t), y(t), z(t)) = f(\sin t, \cos t, t) = \sin t + \cos t + t$
$\vec{c}\,'(t) = (\cos t, -\sin t, 1)$

$$\therefore \int_C f\,ds = \int_0^{2\pi} [\sin t + \cos t + t] \, \|(\cos t, -\sin t, 1)\| \, dt$$

$$= \int_0^{2\pi} [\sin t + \cos t + t] \sqrt{\cos^2 t + \sin^2 t + 1} \; dt$$

$$= \int_0^{2\pi} [\sin t + \cos t + t] \sqrt{2} \; dt$$

$$= \sqrt{2} \left[-\cos t + \sin t + \frac{t^2}{2} \right]_{t=0}^{t=2\pi}$$

$$= \sqrt{2} \left[-1 + 0 + \frac{(2\pi)^2}{2} - (-1 + 0 + 0) \right]$$

$$= \underline{2\sqrt{2}\,\pi^2}$$

(5)

$$f(x(t), y(t), z(t)) = \cos(z(t)) = \cos(t)$$

since $z(t) = t$, from $\vec{c}(t) = (\sin t, \cos t, t)$.

$$\therefore \int_C f\,ds = \int_0^{2\pi} [\cos t] \sqrt{\cos^2 t + \sin^2 t + 1} \; dt$$

$$= \sqrt{2} \int_0^{2\pi} \cos(t)\, dt = \sqrt{2} \left[\sin t \right]_{t=0}^{t=2\pi} = \underline{0}$$

11.

(a) $\vec{c}\,'(t) = (0, 0, 2t)$. $\therefore \|\vec{c}\,'(t)\| = \sqrt{4t^2} = 2t$, for $t \in [0,1]$

$$f(\vec{c}(t)) = \exp\left(\sqrt{t^2}\right) = e^t$$

$$\therefore \int_c f\, ds = \int_0^1 e^t (2t)\, dt = 2\left[t e^t - e^t \right]_0^1$$

$$= 2\left[e - e - (0 - 1) \right] = \underline{2}$$

(b) $\vec{c}\,'(t) = (1, 3, 2)$. $\therefore \|\vec{c}\,'(t)\| = \sqrt{14}$

$$f(\vec{c}(t)) = y(t)\, z(t) = (3t)(2t) = 6t^2$$

$$\therefore \int_c f\, ds = \int_1^3 (6t^2)(\sqrt{14})\, dt = \sqrt{14}\left[2t^3 \right]_1^3$$

$$= \sqrt{14}\,(54 - 2) = \underline{52\sqrt{14}}$$

12.

(a) $\vec{c}(t) = (t, t^2, 0)$, $\vec{c}\,'(t) = (1, 2t, 0)$ $\|\vec{c}\,'(t)\| = \sqrt{1+4t^2}$

$f(\vec{c}(t)) = (t)(\cos(0)) = t$

$\therefore \int_c f\,ds = \int_0^1 t\sqrt{1+4t^2}\,dt = \left(\frac{2}{3}\right)\left(\frac{1}{8}\right)(1+4t^2)^{\frac{3}{2}}\Big|_0^1$

$= \frac{1}{12}\left[1+4-(1+0)\right] = \underline{\frac{1}{3}}$

(b) $\vec{c}\,'(t) = (1, t^{\frac{1}{2}}, 1)$, $\|\vec{c}\,'(t)\| = \sqrt{2+t}$

$f(\vec{c}(t)) = \dfrac{t + \frac{2}{3}t^{3/2}}{\frac{2}{3}t^{3/2} + t} = 1$ for $1 \le t \le 2$

$\therefore \int_c f\,ds = \int_1^2 (1)\sqrt{2+t}\,dt = \left(\frac{2}{3}\right)(2+t)^{3/2}\Big|_1^2$

$= \frac{2}{3}\left[4^{3/2} - 3^{3/2}\right] = \frac{2}{3}\left[8 - 3\sqrt{3}\right] = \underline{\frac{16}{3} - 2\sqrt{3}}$

13.

$\vec{c}(t) = (\log t, t, 2)$, $\vec{c}\,'(t) = \left(\frac{1}{t}, 1, 0\right)$, $1 \le t \le e$

$$f(\vec{c}(t)) = \frac{1}{t^3} \qquad \|\vec{c}\,'(t)\| = \sqrt{\frac{1}{t^2}+1} = \frac{\sqrt{1+t^2}}{t}$$

$$\therefore \int_c f\,ds = \int_1^e \left(\frac{1}{t^3}\right)\left(\frac{\sqrt{1+t^2}}{t}\right) dt = \int_1^e \frac{\sqrt{1+t^2}}{t^4}\, dt$$

$$= -\frac{\sqrt{(t^2+1)^3}}{3t^3}\Bigg|_1^e$$ using text table of integrals # 70

$$= \left[-\frac{\sqrt{(e^2+1)^3}}{3e^3} - \left(\frac{-\sqrt{8}}{3}\right)\right] = \underline{\underline{\frac{2\sqrt{2}}{3} - \frac{\sqrt{(e^2+1)^3}}{3e^3}}}$$

14.

Let $\vec{c}(\theta) = (r\cos\theta, r\sin\theta)$ for $\theta_1 \le \theta \le \theta_2$

$$= (r(\theta)\cos\theta, r(\theta)\sin\theta) = [x(\theta), y(\theta)]$$

$$\therefore f(\vec{c}(\theta)) = f(r(\theta)\cos\theta, r(\theta)\sin\theta)$$

$$\frac{d}{d\theta}\vec{c}(\theta) = \vec{c}\,'(\theta) = \left[-r(\theta)\sin\theta + r'(\theta)\cos\theta,\ r(\theta)\cos\theta + r'(\theta)\sin\theta\right]$$

$$\therefore \|\vec{c}\,'(\theta)\| = \Big[r^2\sin^2\theta + (r')^2\cos^2\theta - 2rr'\sin\theta\cos\theta$$
$$+ r^2\cos^2\theta + (r')^2\sin^2\theta + 2rr'\cos\theta\sin\theta\Big]^{\frac{1}{2}}$$

$$= \left[r^2 + (r')^2\right]^{\frac{1}{2}} = \sqrt{r(\theta)^2 + \left(\frac{d\,r(\theta)}{d\theta}\right)^2}$$

$$\therefore \int_C f\,ds = \int_{\theta_1}^{\theta_2} f(r\cos\theta, r\sin\theta)\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

noting for simplicity that $r = r(\theta)$

(b) Here, $f(x,y) = 1$, so $f(r\cos\theta, r\sin\theta) = 1$

just like $\iiint_W f\,dv$ = volume of W when $f = 1$.

$\therefore$ Arc length $= \int_C ds = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + (r')^2}\,d\theta$

$$r^2(\theta) = (1+\cos\theta)^2, \quad \frac{dr}{d\theta} = -\sin\theta, \quad \left(\frac{dr}{d\theta}\right)^2 = \sin^2\theta$$

$$\therefore \int_C ds = \int_0^{2\pi} \sqrt{(1+\cos\theta)^2 + \sin^2\theta}\,d\theta$$

$$= \int_0^{2\pi} \sqrt{2 + 2\cos\theta}\,d\theta = \sqrt{2}\int_0^{2\pi} \sqrt{1+\cos\theta}\,d\theta$$

$$= 2\int_0^{2\pi} \sqrt{\frac{1+\cos\theta}{2}}\,d\theta$$

Use $\cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2}$

$$\cos\frac{\theta}{2} \geq 0 \text{ for } 0 \leq \theta \leq \pi$$

$$-\cos\frac{\theta}{2} \geq 0 \text{ for } \pi \leq \theta \leq 2\pi$$

$$= 2\int_0^{\pi} \cos\frac{\theta}{2}\, d\theta + 2\int_{\pi}^{2\pi} -\cos\frac{\theta}{2}\, d\theta$$

$$= 2(2)\sin\frac{\theta}{2}\Big|_0^{\pi} + 2(-2)\sin\frac{\theta}{2}\Big|_{\pi}^{2\pi}$$

$$= 4(1-0) + (-4)(0-1) = \underline{\underline{8}}$$

15.

(a) $\vec{c}(t) = (t^4, t^4)$, $\vec{c}\,'(t) = (4t^3, 4t^3)$

$$f(\vec{c}(t)) = 2(t^4) - (t^4) = t^4$$

$$\|\vec{c}\,'(t)\| = \sqrt{16t^6 + 16t^6} = 4\sqrt{2t^6}$$

Note that $\|\vec{c}\,'(t)\|$ is to be positive over the path.

The parametrization $\vec{c}(t) = (t^4, t^4)$ is basically the line $y=x$. For $t \in [-1,0]$, this from $(1,1)$ to $(0,0)$, and from $t \in [0,1]$, it is $(0,0)$ back to $(1,1)$, an equal path.

$$\therefore \int_C f\,ds = 2\int_0^1 f\,ds = 2\int_0^1 (t^4)\left(4\sqrt{2t^6}\right)dt$$

$$= 2\int_0^1 4\sqrt{2}\, t^7\,dt, \text{ since } \sqrt{2t^6} = \sqrt{2}\,t^3 \text{ for } t \in [0,1]$$

$$= \sqrt{2}\,t^8 \Big|_0^1 = \underline{\underline{\sqrt{2}}}$$

Since $f(\vec{c}(t)) = t^4$, the graph is $(x(t), y(t), z(t)) = (t^4, t^4, t^4)$, or a triangle from $(0,0,0)$ to $(1,1,1)$.
$\therefore$ Base is $\sqrt{2}$, height is 1, area is $\frac{\sqrt{2}}{2}$.

Going from $(0,0,0)$ to $(1,1,1)$, then back to $(0,0,0)$ gives twices the area of the triangle.

(6) From page 232 of text (section 4.2)

$$s(t) = \int_a^t \|\vec{c}\,'(u)\|\,du \quad , \quad \|\vec{c}\,'(u)\| = 4\sqrt{2u^6}$$
from above

For $a = -1$, $-1 \le u \le 0$, $\|\vec{c}\,'(u)\| = -4\sqrt{2}\,u^3$

$$s(t) = \int_{-1}^{t} -4\sqrt{2}\,u^3\,du = -\sqrt{2}\,u^4 \Big|_{-1}^{t}$$

$$= -\sqrt{2}\,t^4 - (-\sqrt{2}) = \sqrt{2} - \sqrt{2}\,t^4$$

when $t = 0$, $s(0) = \sqrt{2}$

For $0 \le u \le 1$, $\|\vec{c}\,'(u)\| = 4\sqrt{2}\,u^3$

Since $s(t)$ is supposed to represent total length from the start ($t = -1$), need to add length from $t = -1$ to $t = 0$ for $t \in [0, 1]$

$\therefore$

$$s(t) = s(0) + \int_{0}^{t} 4\sqrt{2}\,u^3\,du$$

$$= \sqrt{2} + \sqrt{2}\,u^4 \Big|_{0}^{t}$$

$$= \sqrt{2} + \sqrt{2}\,t^4$$

$$\therefore s(t) = \begin{cases} \sqrt{2} - \sqrt{2}\,t^4, & -1 \le t \le 0 \\ \sqrt{2} + \sqrt{2}\,t^4, & 0 \le t \le 1 \end{cases}$$

To find a parametrization of the path from (1,1) to (0,0), and then back to (1,1), note the length of the path from (1,1) to (0,0) is $\sqrt{2}$.

s is 0 at (1,1) and $\sqrt{2}$ at (0,0).

$$\therefore\ \vec{c}(s) = \left(1-\frac{s}{\sqrt{2}}\right)(1,1) + \frac{s}{\sqrt{2}}(0,0)$$

$$= \left(1-\frac{s}{\sqrt{2}}\right)(1,1), \quad 0 \le s \le \sqrt{2}$$

Coordinate parametrization going back is:

$$\vec{c}(t) = \left(1-\frac{t}{\sqrt{2}}\right)(0,0) + \frac{t}{\sqrt{2}}(1,1), \quad 0 \le t \le \sqrt{2}$$

or, $\sqrt{2} \le t + \sqrt{2} \le 2\sqrt{2}$, let $s = t + \sqrt{2}$

$$\vec{c}(s) = \left(1 - \frac{s-\sqrt{2}}{\sqrt{2}}\right)(0,0) + \frac{s-\sqrt{2}}{\sqrt{2}}(1,1)$$

$$= \left(\frac{s}{\sqrt{2}} - 1\right)(1,1), \quad \sqrt{2} \le s \le 2\sqrt{2}$$

using $(1-t)\vec{a} + t\vec{b}$, $0 \le t \le 1$, for $\vec{a}$ to $\vec{b}$

$$\therefore\ \vec{c}(s) = \begin{cases} \left(1-\frac{s}{\sqrt{2}}\right)(1,1), & 0 \le s \le \sqrt{2} \\ \left(\frac{s}{\sqrt{2}}-1\right)(1,1), & \sqrt{2} \le s \le 2\sqrt{2} \end{cases}$$

$$\therefore\ \vec{c}\,'(s) = \begin{cases} \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), & 0 \le s \le \sqrt{2} \\ \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), & \sqrt{2} \le s \le 2\sqrt{2} \end{cases}$$

$$\|\vec{c}\,'(s)\| = \sqrt{\tfrac{1}{2} + \tfrac{1}{2}} = 1.$$

$$f(x(s), y(s)) = \begin{cases} 2\left(1 - \frac{s}{\sqrt{2}}\right) - \left(1 - \frac{s}{\sqrt{2}}\right) \\ = \left(1 - \frac{s}{\sqrt{2}}\right), & 0 \le s \le \sqrt{2} \\ 2\left(\frac{s}{\sqrt{2}} - 1\right) - \left(\frac{s}{\sqrt{2}} - 1\right), & \sqrt{2} \le s \le 2\sqrt{2} \\ = \left(\frac{s}{\sqrt{2}} - 1\right) \end{cases}$$

$$\therefore \int_c f\,ds = \int_0^{\sqrt{2}} \left(1 - \frac{s}{\sqrt{2}}\right)(1)\,ds + \int_{\sqrt{2}}^{2\sqrt{2}} \left(\frac{s}{\sqrt{2}} - 1\right)(1)\,ds$$

$$= s - \frac{s^2}{2\sqrt{2}} \Big|_0^{\sqrt{2}} + \frac{s^2}{2\sqrt{2}} - s \Big|_{\sqrt{2}}^{2\sqrt{2}}$$

$$= \sqrt{2} - \frac{1}{\sqrt{2}} + \left[\frac{4 \cdot 2}{2\sqrt{2}} - 2\sqrt{2} - \left(\frac{2}{2\sqrt{2}} - \sqrt{2}\right)\right]$$

$$= \frac{1}{\sqrt{2}} + \left[\frac{3}{\sqrt{2}} - \sqrt{2}\right] = \frac{4}{\sqrt{2}} - \sqrt{2}$$

$$= 2\sqrt{2} - \sqrt{2} = \underline{\sqrt{2}}$$

Same answer as in (a) using length as a parametrization.

16.

(a) Let $[a,b]$ be an interval, subdivide it by a partition $a = t_0 < t_1 < t_2 < \dots < t_n = b$

Let $\vec{c}(t)$, $t \in [a,b]$ by C, and thereby breaking path $\vec{c}(t)$ into paths $\vec{c}_i$ defined on $[t_i, t_{i+1}]$, for $0 \le i \le n-1$.

When n is large, $[t_i, t_{i+1}]$ is small, and so is

$$\Delta s_i = \int_{t_i}^{t_{i+1}} \| \vec{c}_i\,'(t) \| \, dt$$

$\therefore$ $f(\vec{c}_i(t))$ is fairly constant on $[t_i, t_{i+1}]$

$\therefore$ For some $t \in [t_i, t_{i+1}]$,

consider the sum $S_n = \sum_{i=0}^{n-1} f(\vec{c}_i) \Delta s_i$

The average value f on the path would be:

$$f_{ave} = \frac{\sum_{i=0}^{n-1} f(\vec{c}_i)\,\Delta s_i}{\text{length of path}}$$

since $\Delta s_0 + \Delta s_1 + \cdots + \Delta s_{n-1} = \text{length of path}$

But by mean value theorem, $\Delta s_i = \|\vec{c}_i'(t^*)\|\,\Delta t_i$,

for some $t^* \in [t_i, t_{i+1}]$

$$\therefore \sum_{i=0}^{n-1} f(\vec{c}_i)\,\Delta s_i = \sum_{i=0}^{n-1} f(\vec{c}_i)\,\|\vec{c}_i'(t^*)\|\,\Delta t_i$$

$$= \frac{S_n}{\text{length of path}} = \frac{S_n}{\ell(\vec{c})}$$

$$= \frac{S_n}{\int_c ds}$$

As $n \to \infty$, $S_n \to \int_I f(x,y,z)\,\|\vec{c}\,'(t)\|\,dt$

$$= \int_c f\,ds$$

$$\therefore f_{ave} = \frac{\int_c f\,ds}{\int_c ds}$$

(b)

Here $\int_c ds = \int_0^{2\pi} \|\vec{c}\,'(t)\|\,dt$

and $\vec{c}\,'(t) = (-\sin t, \cos t, 1)$

$\therefore \|\vec{c}\,'(t)\| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$

$\therefore \int_0^{2\pi} \sqrt{2}\,dt = 2\sqrt{2}\,\pi$

From the text, $\int_c f\,ds = \frac{2\sqrt{2}\,\pi}{3}(3 + 4\pi^2)$

$$\therefore f_{ave} = \frac{\frac{2\sqrt{2}\,\pi\,(3+4\pi^2)}{3}}{2\sqrt{2}\,\pi} = \frac{(3+4\pi^2)}{3}$$

$$= 1 + \frac{4}{3}\pi^2$$

(c)

10(a): $\int_c f\,ds = 2\sqrt{2}\,\pi^2$

$\vec{c}(t) = (\sin t, \cos t, t)$, for $t \in [0, 2\pi]$

as in (b) above, $\int_c ds = 2\sqrt{2}\,\pi$

$\therefore\ \underline{f_{ave}} = \dfrac{2\sqrt{2}\,\pi^2}{2\sqrt{2}\,\pi} = \underline{\pi}$

10(b): $\int_c f\,ds = 0 \quad \therefore\ \underline{f_{ave} = 0}$

17.

Length of semicircle of radius $a = \pi a$

$f(x,y,z) = y = a\sin\theta$

$\therefore \int_c f\,ds = \int_0^{\pi} (a\sin\theta)\, \|\vec{c}\,'(\theta)\|\, d\theta$

$\vec{c}\,'(\theta) = (0, a\cos\theta, -a\sin\theta)$. $\therefore \|\vec{c}\,'(\theta)\| = a$

$\therefore \int_c f\,ds = \int_0^{\pi} (a\sin\theta)(a)\,d\theta = -a^2\cos\theta \Big|_0^{\pi}$

$= a^2 - (-a^2) = 2a^2$

$$\therefore\ y_{ave} = \frac{2a^2}{\pi a} = \underline{\underline{\frac{2a}{\pi}}}$$

18.

(a) Mass = (density)(length) = $(2)(\pi a)$ = $2\pi a$ grams

(b) By symmetry, since wire is in yz-plane and a semicircle with center $(0,0,0)$, extending from $(0,0,a)$ to $(0,0,-a)$, the center of mass will have coordinates $(0,y,0)$.

The wire has uniform density $= \sigma = 2$

$$\therefore \text{ center of mass } = \frac{\int_c \sigma y\, ds}{\int_c \sigma\, ds} = \frac{\int_c y\, ds}{\int_c ds}$$

This computation was made in #17, and so y-coordinate $= \frac{2a}{\pi}$.

$\therefore$ Center of mass $= \underline{(0, 2a/\pi, 0)}$.

19.

(a) $\vec{c}\,'(t) = (2t, 1, 0)$. $\therefore \|\vec{c}\,'(t)\| = \sqrt{4t^2+1+0} = \sqrt{4t^2+1}$

$$\ell(\vec{c}) = \int_c ds = \int_0^1 \sqrt{4t^2+1}\, dt$$

letting $u = 2t$
$\therefore du = 2dt$

$$= \frac{1}{2}\int_0^2 \sqrt{u^2+1}\, du = \frac{1}{2}\left[\frac{u}{2}\sqrt{u^2+1} + \frac{1}{2}\ln\left(u+\sqrt{u^2+1}\right)\right]_0^2$$

$$= \frac{1}{2}\left[\sqrt{5} + \frac{1}{2}\ln(2+\sqrt{5})\right] = \underline{\frac{\sqrt{5}}{2} + \frac{1}{4}\ln(2+\sqrt{5})}$$

(b) $$\int_c t\,\|\vec{c}\,'(t)\|\,dt = \int_0^1 t\sqrt{4t^2+1}\, dt$$

$$= \left(\frac{2}{3}\right)\left(\frac{1}{8}\right)\left(4t^2+1\right)^{3/2}\Big|_0^1 = \frac{1}{12}\left(5^{3/2}-1\right)$$

$$= \frac{5\sqrt{5}-1}{12}$$

$$\therefore \text{Ave of } y\text{-coord} = \frac{\dfrac{5\sqrt{5}-1}{12}}{\dfrac{2\sqrt{5}+\ln(2+\sqrt{5})}{4}}$$

$$= \frac{5\sqrt{5} - 1}{6\sqrt{5} - 3\ln(2+\sqrt{5})}$$

20.

(a) $\vec{c}(x) = (x, g(x))$, for $a \le x \le b$

$\therefore \frac{d}{dx}\vec{c}(x) = \vec{c}\,'(x) = (1, g'(x))$.

$\therefore \|\vec{c}\,'(x)\| = \sqrt{1 + (g'(x))^2}$

On $\vec{c}$, $f(x,y) = f(x, g(x))$.

$$\therefore \int_c f\,ds = \int_a^b f(x, g(x))\sqrt{1+[g'(x)]^2}\,dx$$

(b) For $f(x,y) = 1$, the above formula gives the length of $g(x)$

$\therefore$ length $g(x)$ on $[a,b]$ $= \displaystyle\int_a^b \sqrt{1+g'(x)^2}\,dx$

21.

As shown in #20 above, given $\vec{c}(t) = (t, g(t))$,

$\vec{c}\,'(t) = (1, g'(t))$. $\therefore \|\vec{c}\,'(t)\| = \sqrt{1 + [g'(t)]^2}$

$\int_c f\,ds$ where $f = 1$ becomes $\int_c ds$ and is the length

of the curve traced out by $\vec{c}(t)$.

$\therefore$ length of the graph (= length of curve from $\vec{c}(t)$)

is $$\int_c ds = \int_a^b \|\vec{c}\,'(t)\|\,dt = \int_a^b \sqrt{1 + [g'(t)]^2}\,dt$$

22.

$$\int_1^2 \sqrt{1 + \left(\frac{1}{x}\right)^2}\,dx = \int_1^2 \frac{\sqrt{x^2+1}}{x}\,dx$$

use table of integrals #56 from text.

$$= \sqrt{x^2+1} - \log\left[\frac{1 + \sqrt{x^2+1}}{x}\right]\Bigg|_{x=1}^{x=2}$$

$$= \left[\sqrt{5} - \log\left(\frac{1+\sqrt{5}}{2}\right) - \left(\sqrt{2} - \log(1+\sqrt{2}\right) \right]$$

$$= \sqrt{5} - \sqrt{2} + \log\left(\frac{2+2\sqrt{2}}{1+\sqrt{5}}\right)$$

23.

$$\int_{-1}^{1} f(x, g(x)) \sqrt{1+g'(x)^2}\, dx$$

Here, $f(x, g(x)) = g(x) = \sqrt{1-x^2}$

as $g(x) = \sqrt{1-x^2}$ and

$g'(x) = \frac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x) = -\frac{x}{\sqrt{1-x^2}}$

$$= \int_{-1}^{1} \sqrt{1-x^2} \sqrt{1+\frac{x^2}{1-x^2}}\, dx$$

$$= \lim_{a \to 1} \int_{-a}^{a} \sqrt{1-x^2} \sqrt{\frac{1}{1-x^2}}\, dx = \lim_{a \to 1} \int_{-a}^{a} (1)\, dx = \underline{2}$$

24.

$$f(x,y) = y^2 = (e^x)^2 = e^{2x}$$

$$\therefore \int_0^1 (e^{2x}) \sqrt{1+(e^x)^2}\, dx = \int_0^1 e^{2x} \sqrt{1+e^{2x}}\, dx$$

$$= \left(\frac{2}{3}\right)\left(\frac{1}{2}\right)\left(1+e^{2x}\right)^{3/2}\Bigg|_{x=0}^{x=1} = \frac{1}{3}\left[1+e^{2}-(1+1)\right]$$

$$= \underline{\frac{e^{2}-1}{3}}$$

25.

$\vec{c}\,'(t) = (-\sin t, \cos t, 1)$ $\quad \|\vec{c}\,'(t)\| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$

From $\vec{c}(t) = (x(t), y(t), z(t))$, $f(x,y,z) = t\cos t\sin t$

$$\therefore \int_c f\,ds = \int_0^{\frac{\pi}{2}} (t\cos t\sin t)(\sqrt{2})\,dt$$

$$= \frac{\sqrt{2}}{2}\int_0^{\frac{\pi}{2}} t\sin 2t\,dt$$

Let $u = t$ $\quad dv = \sin 2t\,dt$
$du = dt$ $\quad v = -\frac{1}{2}\cos 2t$
integration by parts

$$= \frac{\sqrt{2}}{2}\left[-\frac{t}{2}\cos 2t\Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} -\frac{1}{2}\cos 2t\,dt\right]$$

$$= \frac{\sqrt{2}}{2}\left[\frac{\pi}{4} - 0 + \frac{1}{4}\sin 2t\Big|_0^{\frac{\pi}{2}}\right]$$

$$= \underline{\frac{\sqrt{2}\,\pi}{8}}$$

26.

Mass = (density)(length). $\therefore \Delta m = \rho(x,y,z)\,ds$

Total mass $= \sum \Delta m = \sum \rho\, ds = \int_C \rho\, ds$

Need to find a parameterization for the curve.

From $z = -x-y$ of the plane,

$$x^2 + y^2 + (-x-y)^2 = 1 \Rightarrow 2x^2 + 2y^2 + 2xy = 1$$

From analytic geometry, can do a rotation transformation to get rid of xy term. Resulting curve likely an ellipse, but then need to find a parameterization for the ellipse. A lot of messy work.

Other idea: The intersection is a unit circle in the $x+y+z=0$ plane, since the plane goes through $(0,0,0)$.

$\therefore$ if can find orthogonal unit vectors in that plane, can use $x' = \cos\theta$ and $y' = \sin\theta$ to parametrize the circle.

$(1,-1,0)$ is in the plane $(1+(-1)+0=0)$

$(1,1,-2)$ is in the plane $(1+1-2=0)$

and $(1,-1,0)\cdot(1,1,-2) = 0$, so the two vectors are perpendicular.

Unit vectors are $\frac{1}{\sqrt{2}}(1,-1,0)$ and $\frac{1}{\sqrt{6}}(1,1,-2)$

$\therefore$ A parameterization of the circle is

$\vec{c}(\theta) = \frac{1}{\sqrt{2}}(1,-1,0)\cos\theta + \frac{1}{\sqrt{6}}(1,1,-2)\sin\theta, \quad 0 \le \theta \le 2\pi$

$\therefore \vec{c}\,'(\theta) = -\frac{1}{\sqrt{2}}(1,-1,0)\sin\theta + \frac{1}{\sqrt{6}}(1,1,-2)\cos\theta$

To find $\|\vec{c}\,'(\theta)\|$, note if $\vec{a} = \vec{b} + \vec{c}$,

$$\|\vec{a}\| = \sqrt{\vec{b}\cdot\vec{b} + 2\vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{c}}$$

In this case, $\vec{b}\cdot\vec{c} = 0$. $\therefore \|\vec{a}\| = \sqrt{\vec{b}\cdot\vec{b} + \vec{c}\cdot\vec{c}}$

$$\therefore \|\vec{c}\,'(\theta)\| = \sqrt{\frac{\cos^2\theta}{2}(1+1+0) + \frac{\sin^2\theta}{6}(1+1+4)}$$

$$= \sqrt{\cos^2\theta + \sin^2\theta} = 1.$$

Now, $\vec{c}(\theta) = (x(\theta), y(\theta), z(\theta))$

$$\therefore x(\theta) = \frac{\cos\theta}{\sqrt{2}} + \frac{\sin\theta}{\sqrt{6}}$$

$$\rho(x(\theta), y(\theta), z(\theta)) = x(\theta)^2 = \left(\frac{\cos\theta}{\sqrt{2}} + \frac{\sin\theta}{\sqrt{6}}\right)^2$$

$$= \frac{\cos^2\theta}{2} + \frac{\sin^2\theta}{6} + \frac{\sin 2\theta}{\sqrt{12}}$$

Note $\cos 2\theta = \cos^2\theta - \sin^2\theta \Rightarrow \cos^2\theta = \frac{1+\cos 2\theta}{2}$, $\sin^2\theta = \frac{1-\cos 2\theta}{2}$

$$\therefore \int_C f\,ds = \int_0^{2\pi} \left(\frac{\cos^2\theta}{2} + \frac{\sin^2\theta}{6} + \frac{\sin 2\theta}{\sqrt{12}} \right)(1)\,d\theta$$

$$= \int_0^{2\pi} \frac{1+\cos 2\theta}{4} + \frac{1-\cos 2\theta}{12} + \frac{\sin 2\theta}{\sqrt{12}} \quad d\theta$$

$$= \left[\frac{1}{4}\theta + \frac{\sin 2\theta}{8} + \frac{1}{12}\theta - \frac{\sin 2\theta}{24} - \frac{\cos 2\theta}{2\sqrt{12}} \right]_{\theta=0}^{\theta=2\pi}$$

$$= \frac{1}{4}(2\pi) + 0 + \frac{1}{12}(2\pi) - 0 - 0 = \underline{\underline{\frac{2}{3}\pi \text{ grams}}}$$

27.

$$\vec{c}\,'(t) = (\cos t - t\sin t,\ \sin t + t\cos t,\ 1)$$

$$\vec{c}\,' \cdot \vec{c}\,' = \cos^2 t - 2t\cos t\sin t + t^2\sin^2 t$$
$$+ \sin^2 t + 2t\cos t\sin t + t^2\cos^2 t + 1$$

$$= 1 + t^2 + 1 = t^2 + 2$$

$$\therefore \|\vec{c}\,'(t)\| = \sqrt{t^2+2}$$

$$f(x(t), y(t), z(t)) = z(t) = t$$

$$\therefore \int_C f\,ds = \int_0^{t_0} t\sqrt{t^2+2}\,dt = \left(\tfrac{2}{3}\right)\left(\tfrac{1}{2}\right)(t^2+2)^{3/2} \Big|_0^{t_0}$$

$$= \frac{1}{3}\left[(t_0^2+2)^{3/2} - 2^{3/2}\right] = \underline{\frac{(t_0^2+2)^{3/2} - 2\sqrt{2}}{3}}$$

28.

$$\lim_{N\to\infty} \sum_{i=1}^{N-1} t_i^2 (t_{i+1}^2 - t_i^2) = \lim_{N\to\infty} \sum_{i=1}^{N-1} t_i^2 (t_{i+1} + t_i)(t_{i+1} - t_i)$$

$f(x(t), y(t), z(t)) = x(t)\,y(t)$.

$\therefore$ If $\vec{c}(t) = (t, t, 0)$, $f(\vec{c}(t)) = t^2$,
but $\vec{c}\,'(t) = (1,1,0)$ which doesn't help with the $(t_{i+1} + t_i)(t_{i+1} - t_i)$.

If $\vec{c}(t) = (t^2, 1, 0)$, $f(\vec{c}(t)) = t^2$,
and $\vec{c}\,'(t) = (2t, 0, 0)$. $\therefore \|\vec{c}\,'(t)\| = 2t$
This can be $t_{i+1} + t_i \approx 2t^*$, where $t^* \in [t_i, t_{i+1}]$.

i.e., $\lim_{N\to\infty} (t_{i+1} + t_i) = 2t_i$

$$\therefore \lim_{N\to\infty} \sum_{i=1}^{N-1} t_i^2 (t_{i+1} + t_i)(t_{i+1} - t_i) = \lim_{N\to\infty} \sum_{i=1}^{N-1} t_i^2 (2t_i)\Delta t =$$

$$\lim_{N\to\infty} \sum_{i=1}^{N-1} f(\vec{c}(t_i)) \|\vec{c}\,'(t_i)\| \Delta t = \int_c f\, ds$$

where $f(x,y,z) = (x,y)$, and $\vec{c}(t) = (t^2, 1, 0)$

Actually, $\vec{c}(t)$ could be $(1, t^2, K)$ or $(t^2, 1, K)$,

where K = constant, since $\|\vec{c}\,'(t)\| = 2t$

and $f(\vec{c}(t)) = t^2$ for all these

parameterizations, $t \in [0,1]$.

$$\therefore \int_c f\, ds = \int_0^1 t^2 (2t)\, dt = \frac{1}{2} t^4 \Big|_0^1 = \frac{1}{2}$$

$$\therefore \underline{\lim_{N\to\infty} \sum_{i=1}^{N-1} t_i^2 (t_{i+1}^2 - t_i^2) = \frac{1}{2}}$$, t_i a partition on $[0,1]$

Note: index of i started at $i=1$

so $t_1 = 0 < t_2 < t_3 < \dots < t_N = 1$.

If chose $i=0$ so $t_0 = 0 < t_1 < t_2 \cdots < t_N = 1$,

you would get same result as first term

of $\sum_{i=1}^{N-1} t_i^2 (t_{i+1}^2 - t_i^2)$ is 0.

29.

From energy considerations,

starting at height h_0

$mgh_0 + \frac{1}{2}mv_0^2 = mgh_f + \frac{1}{2}mv_f^2$, f = final position.

$\therefore$ Assuming $V_0 = 0$, $\sqrt{-2g\Delta h} = V_f$

Note Δh is negative since $h_f < h_0$.

The velocity function is $\therefore$ $V(y) = \sqrt{2g(1-y)}$

Since time $= \frac{\text{distance}}{\text{velocity}}$, for a given

segment Δs, the time is $\frac{\Delta s}{V}$

If $\vec{c}(t)$ is the path, then $\Delta s = \|\vec{c}\,'(t^*)\|\Delta t$,
$t^* \in \Delta t$.

As the partition increases,

$$\text{total time} = \lim_{N\to\infty} \sum_{i=0}^{N-1} \frac{\Delta s}{v} = \lim_{N\to\infty} \sum_{i=0}^{N-1} \frac{\|\vec{c}\,'(t^*)\|\,\Delta t}{v}$$

$$= \int_0^1 \frac{\|\vec{c}\,'(t)\|\,dt}{\sqrt{2g(1-y(t))}}, \quad \vec{c}(t) \text{ defined on } [0,1]$$

(a) For $y = 1 - x$, $\vec{c}(t) = (t, 1-t)$, $t \in [0,1]$

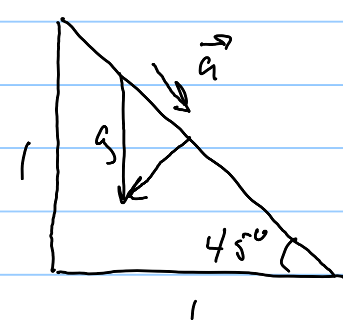
$$\therefore \vec{c}\,'(t) = (1,-1), \quad \|\vec{c}\,'(t)\| = \sqrt{2}$$

$$\sqrt{2g[1-y(t)]} = \sqrt{2g[1-(1-t)]} = \sqrt{2gt}$$

$$\therefore \int_0^1 \frac{\|\vec{c}\,'(t)\|}{\sqrt{2g[1-y(t)]}}\,dt = \int_0^1 \frac{\sqrt{2}}{\sqrt{2gt}}\,dt = \frac{1}{\sqrt{g}}\int_0^1 t^{-\frac{1}{2}}\,dt$$

$$= \frac{1}{\sqrt{g}}\, 2t^{\frac{1}{2}}\Big|_0^1 = \underline{\underline{\frac{2}{\sqrt{g}}}}$$

To check, from mechanics,



$d = \frac{1}{2}at^2$, $d = \sqrt{2}$

and $\vec{a} = \vec{g}\cos 45^\circ = \frac{\sqrt{2}}{2}g$

$\therefore \sqrt{2} = \frac{1}{2}\left(\frac{\sqrt{2}}{2}g\right)t^2, \quad \frac{4}{g} = t^2, \quad \underline{t = \frac{2}{\sqrt{g}}}$

(5)

A parameterization for the circular path is

$\vec{c}(\theta) = (1+\cos\theta, 1+\sin\theta), \quad \pi \le \theta \le \frac{3}{2}\pi$

$\therefore v = \sqrt{2g[1-y(\theta)]} = \sqrt{2g[1-(1+\sin\theta)]} = \sqrt{-2g\sin\theta}$

$\vec{c}\,'(\theta) = (-\sin\theta, \cos\theta), \quad \|\vec{c}\,'(\theta)\| = 1$

$$\therefore T = \underline{\int_{\pi}^{3/2\,\pi} \frac{d\theta}{\sqrt{-2g\sin\theta}}} = \underline{\frac{1}{\sqrt{2g}} \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{\sin\theta}}}$$

Alternatively, $(x-1)^2 + (y-1)^2 = 1 \Rightarrow (y-1)^2 = 1-(x-1)^2$

$\therefore y-1 = \pm\sqrt{1-(x-1)^2}$, choose the negative root

since dealing with $0 \le y \le 1$

$\therefore y = 1 - \sqrt{1-(x-1)^2}$

$\therefore \vec{c}(x) = \left(x, 1-\sqrt{1-(x-1)^2}\right) = \left(x, 1-\sqrt{2x-x^2}\right)$

$\therefore v = \sqrt{2g\sqrt{2x-x^2}} = \sqrt{2g}\,(2x-x^2)^{\frac{1}{4}}$

$$\vec{c}\,'(x) = \left[1, -\tfrac{1}{2}(2x-x^2)^{-\frac{1}{2}}(2-2x)\right]$$

$$= \left[1, \frac{x-1}{\sqrt{2x-x^2}}\right]$$

$$\therefore \|\vec{c}\,'(x)\| = \sqrt{1 + \frac{(x-1)^2}{(2x-x^2)}} = \sqrt{\frac{2x-x^2+x^2-2x+1}{2x-x^2}}$$

$$= \frac{1}{\sqrt{2x-x^2}} = \frac{1}{(2x-x^2)^{\frac{1}{2}}}$$

$$\therefore T = \int_0^1 \frac{\|\vec{c}\,'(x)\|\,dx}{\sqrt{2g[1-y(x)]}} = \int_0^1 \frac{1}{(2x-x^2)^{\frac{1}{2}}} \cdot \frac{1}{\sqrt{2g}\,(2x-x^2)^{1/4}}\,dx$$

$$= \frac{1}{\sqrt{2g}} \int_0^1 \frac{dx}{(2x-x^2)^{3/4}}$$

7.2 Line Integrals

Note Title 2/10/2017

1.

Let $\vec{c}(\theta) : [0, \frac{\pi}{2}] \to [\cos\theta, \sin\theta]$

$\therefore \quad \vec{c}\,'(\theta) = [-\sin\theta, \cos\theta]$

$\vec{F}(\vec{c}(\theta)) = [\sin^2\theta, -\cos\theta\sin\theta]$

$$\therefore \int_C \vec{F}\cdot ds = \int_0^{\frac{\pi}{2}} (\sin^2\theta, -\cos\theta\sin\theta)\cdot(-\sin\theta, \cos\theta)\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} -\sin^3\theta - \cos^2\theta\sin\theta \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} -\sin\theta\,(1-\cos^2\theta)\, d\theta \quad + \frac{\cos^3\theta}{3}\Big|_0^{\frac{\pi}{2}}$$

$$= \cos\theta\Big|_0^{\frac{\pi}{2}} - \frac{\cos^3\theta}{3}\Big|_0^{\frac{\pi}{2}} + \frac{\cos^3\theta}{3}\Big|_0^{\frac{\pi}{2}}$$

$$= 0 - 1 = \underline{-1}$$

2.

Let $\vec{c}(\theta) : [0, 2\pi] \to [\cos\theta, \sin\theta]$

$\therefore \vec{c}\,'(\theta) = (-\sin\theta, \cos\theta)$

$\vec{F}(\vec{c}(\theta)) = [\sin^2\theta, 2\cos\theta\sin\theta]$

$\vec{F}(\vec{c}(\theta)) \cdot \vec{c}\,'(\theta) = -\sin^3\theta + 2\cos^2\sin\theta$

$$\therefore \int_C \vec{F} \cdot ds = \int_0^{2\pi} -\sin^3\theta + 2\cos^2\sin\theta \, d\theta$$

$$= \int_0^{2\pi} -\sin\theta(1-\cos^2\theta) - \frac{2}{3}\cos^3\theta \Big|_0^{2\pi}$$

$$= \cos\theta \Big|_0^{2\pi} - \frac{\cos^3\theta}{3}\Big|_0^{2\pi} - \frac{2\cos^3\theta}{3}\Big|_0^{2\pi}$$

$$= 1 - 1 - \cos^3\theta\Big|_0^{2\pi} = 0 - (1-1) = \underline{0}$$

3.

(a) $\vec{c}\,'(t) = (1,1,1)$ $\vec{F}(\vec{c}(t)) = (t,t,t)$

$\therefore \vec{F}(\vec{c}(t)) \cdot \vec{c}\,'(t) = 3t$

$\therefore \int_c \vec{F} \cdot ds = \int_0^1 3t\, dt = \frac{3}{2}t^2 \Big|_0^1 = \underline{\frac{3}{2}}$

(b) $\vec{c}\,'(t) = (-\sin t, \cos t, 0)$ $\vec{F}(\vec{c}(t)) = (\cos t, \sin t, 0)$

$\therefore \vec{F}(\vec{c}(t)) \cdot \vec{c}\,'(t) = -\sin t \cos t + \cos t \sin t + 0 = 0$

$\therefore \int_c \vec{F} \cdot ds = \underline{0}$

(c) $\vec{c}\,'(t) = (\cos t, 0, -\sin t)$ $\vec{F}(\vec{c}(t)) = (\sin t, 0, \cos t)$

$\therefore \vec{F}(\vec{c}(t)) \cdot \vec{c}\,'(t) = \cos t \sin t + 0 - \sin t \cos t = 0$

$\therefore \int_c \vec{F} \cdot ds = 0$

(d) $\vec{c}\,'(t) = (2t, 3, 6t^2)$ $\vec{F}(\vec{c}(t)) = (t^2, 3t, 2t^3)$

$\therefore \vec{F}(\vec{c}(t)) \cdot \vec{c}\,'(t) = 2t^3 + 9t + 12t^5$

$\therefore \int_c \vec{F} \cdot ds = \int_{-1}^{2} 2t^3 + 9t + 12t^5\, dt$

$$= \frac{1}{2}t^4 + \frac{9}{2}t^2 + 2t^6 \Big|_{-1}^{2}$$

$$= \left(8 - \frac{1}{2}\right) + \left(18 - \frac{9}{2}\right) + \left(128 - 2\right) = 154 - 7$$

$$= \underline{147}$$

4.

(a) $\vec{f}(x,y) = (-y, x)$ $\quad \vec{c}\,'(t) = (-\sin t, \cos t)$

$\vec{f}(\vec{c}(t)) = (-\sin t, \cos t)$. $\therefore \vec{f} \cdot \vec{c}\,' = \sin^2 t + \cos^2 t = 1$

$$\therefore \int_C \vec{f} \cdot d\vec{s} = \int_0^{2\pi} 1 \, dt = \underline{2\pi}$$

(b) $\vec{f}(x,y) = (x, y)$ $\quad \vec{c}\,'(t) = (-\pi \sin \pi t, \pi \cos \pi t)$

$\vec{f}(\vec{c}(t)) = (\cos \pi t, \sin \pi t)$

$\therefore \vec{f} \cdot \vec{c}\,' = -\pi \cos \pi t \sin \pi t + \pi \sin \pi t \cos \pi t = 0$

$$\therefore \int_C \vec{f} \cdot d\vec{s} = \int_0^2 0 \, dt = \underline{0}$$

(c) $\vec{f}(x,y,z) = (yz, xz, xy)$

Note $\vec{f} = \nabla f$, where $f(x,y,z) = xyz$

$\therefore \int_c \vec{f} \cdot d\vec{s} = f(0,0,1) - f(1,0,0) = 0 - 0 = \underline{0}$

(d) $\vec{f}(x,y,z) = (x^2, -xy, 1) \quad \vec{c}(t) = (t, 0, t^2),\ t \in [-1,1]$

$\therefore \vec{f}(\vec{c}(t)) = (t^2, 0, 1) \quad \vec{c}\,'(t) = (1, 0, 2t)$

$\therefore \vec{f} \cdot \vec{c}\,' = t^2 + 0 + 2t = t^2 + 2t$

$\therefore \int_c \vec{f} \cdot d\vec{s} = \int_{-1}^{1} t^2 + 2t \, dt = \frac{t^3}{3} + t^2 \Big|_{-1}^{1}$

$= \frac{1}{3} + 1 - \left(-\frac{1}{3} + 1\right) = \underline{\frac{2}{3}}$

5.

$\vec{c}(t) = (t, t^2, 0), \quad \vec{c}\,'(t) = (1, 2t, 0)$

$\vec{F}(\vec{c}(t)) = (t, t^2, 0). \quad \therefore \vec{F}(\vec{c}) \cdot \vec{c}\,' = t + 2t^3$

$\therefore \int_{\vec{c}} \vec{F} \cdot d\vec{s} = \int_{-1}^{2} t + 2t^3 = \frac{t^2}{2} + \frac{t^4}{2} \Big|_{-1}^{2} =$

$$2 + 8 - \left(\frac{1}{2} + \frac{1}{2}\right) = \underline{9}$$

6.

Let $\vec{c}(t) : [a,b] \to R^n$
$\vec{F} : R^n \to R^n$

(a) $$\int_{\vec{c}} \vec{F} \cdot d\vec{s} = \int_a^b \vec{F}(\vec{c}(t)) \cdot \vec{c}\,'(t)\, dt = \int_a^b 0\, dt = 0$$

(b)

Let $\vec{c}(t) : [a,b] \to R^n$

$\vec{F} : R^n \to R^n$

$$\vec{F}(\vec{c}(t)) \cdot \vec{c}\,'(t) = \lambda(t)\, \vec{c}\,'(t) \cdot \vec{c}\,'(t) = \lambda(t) \|\vec{c}\,'(t)\|^2$$

$$= \lambda(t) \|\vec{c}\,'(t)\|\, \|\vec{c}\,'(t)\|$$

$$= \|\lambda(t)\, \vec{c}\,'(t)\|\, \|\vec{c}\,'(t)\|$$

$$= \|\vec{F}(\vec{c}(t))\|\, \|\vec{c}\,'(t)\|$$

$$\therefore \int_{\vec{c}} \vec{F} \cdot d\vec{s} = \int_a^b \vec{F}(\vec{c}(t)) \cdot \vec{c}\,'(t)\, dt$$

$$= \int_a^b \|\vec{F}(\vec{c}(t))\|\, \|\vec{c}\,'(t)\|\, dt$$

$$= \int_{\vec{c}} \|\vec{F}(\vec{c}(t))\| \, ds \quad \text{by definition}$$

7.

Since $\vec{F}(\vec{c}(t)) \cdot \vec{c}\,'(t) = \|\vec{F}(\vec{c}(t))\| \cdot \|\vec{c}\,'(t)\| \cos\theta$, for R^3

$$\therefore \left|\vec{F}(\vec{c}(t)) \cdot \vec{c}\,'(t)\right| \leq \|\vec{F}(\vec{c}(t))\| \, \|\vec{c}\,'(t)\|, \text{ by Cauchy-Schwartz } |\vec{a} \cdot \vec{b}| \leq \|\vec{a}\| \, \|\vec{b}\|$$

$$\leq M \|\vec{c}\,'(t)\|$$

$\therefore$ From $|a+b| \leq |a| + |b|$,

$$\left| \int_a^b \vec{F}(\vec{c}(t)) \cdot \vec{c}\,'(t) \, dt \right| \leq \int_a^b \left|\vec{F}(\vec{c}(t)) \cdot \vec{c}\,'(t)\right| dt$$

$$\leq \int_a^b \|\vec{F}(\vec{c}(t))\| \, \|\vec{c}\,'(t)\| \, dt$$

$$\leq \int_a^b M \|\vec{c}\,'(t)\| \, dt = M \int_a^b \|\vec{c}\,'(t)\| \, dt = ML$$

$$\therefore \left| \int_{\vec{c}} \vec{F} \cdot d\vec{s} \right| = \left| \int_a^b \vec{F}(\vec{c}(t)) \cdot \vec{c}\,'(t) \, dt \right| \leq ML$$

8.

$\vec{c}\,'(t) = (1, 2t, 3t^2)$. $F(\vec{c}(t)) = (t^2, 2t, t^2)$

$\therefore \vec{F}(\vec{c}(t)) \cdot \vec{c}\,'(t) = t^2 + 4t^2 + 3t^4 = 5t^2 + 3t^4$

$\therefore \int_0^1 5t^2 + 3t^4 \, dt = \frac{5}{3}t^3 + \frac{3}{5}t^5 \Big|_0^1 = \frac{5}{3} + \frac{3}{5} = \underline{\underline{\frac{34}{15}}}$

9.

$\vec{c}\,'(t) = (1, nt^{n-1}, 0)$

$\therefore (y, 3y^3 - x, z) \cdot (1, nt^{n-1}, 0) = (t^n, 3t^{3n} - t, 0) \cdot (1, nt^{n-1}, 0)$

$= t^n + 3nt^{3n+n-1} - nt^n$

$= (1-n)t^n + 3nt^{4n-1}$

$\therefore \int_0^1 (1-n)t^n + 3nt^{4n-1} \, dt = \frac{(1-n)t^{n+1}}{n+1} + \frac{3n}{4n} t^{4n} \Big|_{t=0}^{t=1}$

$= \frac{1-n}{n+1} + \frac{3}{4} - 0 = \underline{\frac{3}{4} - \frac{n-1}{n+1}}$

10.

Note $\|\vec{H}\| = H\|\vec{T}\| = H$

$$\int_C \vec{H} \cdot d\vec{s} = I$$ Let the wire be at $(x,y) = (0,0)$.

$\therefore$ C can be described as $(r\cos\theta, r\sin\theta)$, $0 \le \theta \le 2\pi$

$\therefore$ $\vec{c}\,'(t) = r(-\sin\theta, \cos\theta) = r\vec{T} = r\dfrac{\vec{H}}{H}$

$$\therefore \int_C \vec{H} \cdot d\vec{s} = \int_0^{2\pi} \vec{H} \cdot \vec{c}\,'(t)\,dt = \int_0^{2\pi} \vec{H} \cdot \left(\frac{r}{H}\vec{H}\right) dt$$

$$= \int_0^{2\pi} r\,\frac{\|\vec{H}\|^2}{H}\,dt = \int_0^{2\pi} rH\,dt = rH(2\pi)$$

$$\therefore 2\pi r H = \int_C \vec{H} \cdot d\vec{s} = I \Rightarrow H = \underline{\frac{I}{2\pi r}}$$

11.

$$\vec{F}(\vec{c}(t)) = (\cos^3 t, \sin^3 t)$$

$$\vec{c}\,'(t) = (-3\cos^2 t \sin t, 3\sin^2 t \cos t)$$

$$\vec{F}(\vec{c}) \cdot \vec{c}\,' = -3\cos^5 t \sin t + 3\sin^5 t \cos t$$

$$= 3\sin t \cos t (\sin^4 t - \cos^4 t)$$

$$= 3\sin t \cos t (\sin^2 t + \cos^2 t)(\sin^2 t - \cos^2 t)$$

$$= 3\sin t \cos t (\sin^2 t - \cos^2 t)$$

$$= 3\sin^3 t \cos t - 3\sin t \cos^3 t$$

$$\therefore \int_c \vec{F} \cdot d\vec{s} = \int_0^{2\pi} 3\sin^3 t \cos t - 3\sin t \cos^3 t \, dt$$

$$= \frac{3}{4}\sin^4 t + \frac{3}{4}\cos^4 t \Big|_0^{2\pi} = 0 + \frac{3}{4} - \left(0 + \frac{3}{4}\right) = \underline{0}$$

12.

$$\int_{\vec{C}_2} \vec{F} \cdot d\vec{s} = -\int_{\vec{C}_{2\,opp}} \vec{F} \cdot d\vec{s}$$ by Theorem 1, p. 364 of text.

$$\therefore \int_{\vec{C}_1} \vec{F} \cdot d\vec{s} = -\int_{\vec{C}_{2\,opp}} \vec{F} \cdot d\vec{s} \iff$$

$$\int_{\vec{C}_1} \vec{F} \cdot d\vec{s} + \int_{\vec{C}_{2\,opp}} \vec{F} \cdot d\vec{s} = 0 \iff$$

$$\int_C \vec{F} \cdot d\vec{s} = 0$$

13.

Let $\vec{c}(t) : [a,b] \to R^n$

$$\vec{T}(t) = \frac{\vec{c}\,'(t)}{\|\vec{c}\,'(t)\|} . \quad \therefore \int_{\vec{c}} \vec{T} \cdot d\vec{s} =$$

$$\int_{\vec{c}} \frac{\vec{c}\,'(t)}{\|\vec{c}\,'(t)\|} \cdot d\vec{s} = \int_a^b \frac{\vec{c}\,'(t) \cdot \vec{c}\,'(t)}{\|\vec{c}\,'(t)\|}\, dt$$

$$= \int_a^b \frac{\|\vec{c}\,'(t)\|^2}{\|\vec{c}\,'(t)\|}\, dt = \int_a^b \|\vec{c}\,'(t)\|\, dt$$

$= \underline{\text{length of path}}$ of $\vec{c}(t)$

14.

Let $f(x,y,z) = xz^3 + x^2 y$

$\therefore \vec{F}(x,y,z) = \nabla f(x,y,z) = (z^3 + 2xy,\ x^2,\ 3xz^2)$

$$\therefore \int_c \vec{F} \cdot d\vec{s} = \int_c \nabla f \cdot d\vec{s} = f(\vec{c}(b)) - f(\vec{c}(a))$$

where $\vec{c}(a)$ = start point, $\vec{c}(b)$ = end point

(i.e., $\vec{c} : [a,b] \to \mathbb{R}^3$)

But $\vec{c}(a) = \vec{c}(b)$. $\therefore f(\vec{c}(b)) - f(\vec{c}(a)) = 0$

$$\therefore \int_c \vec{F} \cdot d\vec{s} = 0.$$

15.

If $\vec{c}\,'(t) \neq 0$, Then $\vec{c}\,'(t)$ has a defined direction.

The zero vector has no defined direction.

$\therefore$ The "points" of the path in Exercise 11, which have no direction, will be avoided.

The pointed parts of the path are where $\vec{c}\,'(t)$ must be zero, as the zero vector is the only vector without direction.

16.

It is 0, since if $\vec{F} = \nabla f$, $\int_C \nabla f \cdot d\vec{s} =$

$F(\text{end point}) - F(\text{start point}) = 0$, since

end point = start point in a closed curve.

17.

Assume start point $= (1,1,1)$, endpoint $= (1,2,4)$

Note that if $f(x,y,z) = x^2yz$, then

$\nabla f = (2xyz, x^2z, x^2y)$, which is $\vec{F}$ in the line integral $\int_C \vec{F}\cdot d\vec{s}$, so the exact path of C doesn't matter, just the endpoints.

$$\therefore \int_C \vec{F}\cdot d\vec{s} = \int_C \nabla f \cdot d\vec{s} = f(1,2,4) - f(1,1,1)$$

$$= 8 - 1 = \underline{7}$$

18.

$$\frac{\partial f}{\partial x} = 2xyze^{x^2} \Rightarrow f(x,y,z) = yze^{x^2} + g(y,z)$$

$$\frac{\partial f}{\partial y} = ze^{x^2} + \frac{\partial g}{\partial y} = ze^{x^2} \Rightarrow f(x,y,z) = yze^{x^2} + h(z)$$

$$\frac{\partial f}{\partial z} = ye^{x^2} + \frac{\partial h}{\partial z} = ye^{x^2} \Rightarrow f(x,y,z) = yze^{x^2} + K,$$

K a constant

$$f(0,0,0) = 5 \Rightarrow f(x,y,z) = yze^{x^2} + 5$$

$$\therefore f(1,1,2) = \underline{2e + 5}$$

19.

Need to show $\vec{F} = \nabla f$ for some $f(x,y,z)$.

Then $\int_C \vec{F} \cdot d\vec{s}$ = work done along a path C =

f(endpoint) - f(startpoint)

Look at $f(x,y,z) = (x^2+y^2+z^2)^{-\frac{1}{2}}$

Then $\nabla f = -(x^2+y^2+z^2)^{-\frac{3}{2}}(x,y,z) = \vec{F}$

$$\therefore \int_C \vec{F} \cdot d\vec{s} = \int_C \nabla f \cdot d\vec{s} = f(\text{end point}) - f(\text{start point})$$

Since start point $= (x_1, y_1, z_1)$, then

$$f(x_1, y_1, z_1) = \frac{1}{\sqrt{x_1^2+y_1^2+z_1^2}} = \frac{1}{R_1}$$

and end point $= (x_2, y_2, z_2)$, so

$$f(x_2, y_2, z_2) = \frac{1}{\sqrt{x_2^2+y_2^2+z_2^2}} = \frac{1}{R_2}$$

$$\therefore \int_C \vec{F} \cdot d\vec{s} = \frac{1}{R_2} - \frac{1}{R_1}$$

20.

(a) $\vec{F} = \nabla f(x,y,z)$, where $f(x,y,z) = xy + z$

Work done $= \int_C \vec{F} \cdot d\vec{s} = \int_C \nabla f \cdot d\vec{s}$

$= f(\text{end point}) - f(\text{start point})$

Start point $= (\sqrt{2\pi}, 0, 0)$

end point $= (0, 0, 2\pi)$.

$\therefore W = f(0,0,2\pi) - f(\sqrt{2\pi}, 0, 0)$

$= 2\pi - 0 = \underline{2\pi}$ units

(b) Near the top, $x \approx 0$, $y \approx 0$, so that $\vec{F}$ is virtually straight upward ("all $\hat{k}$"), an orientation highly unlikely for a cyclist.

21.

(a) By the Fundamental Theorem of Calculus,

$$\frac{d}{dx} f(x) = \frac{d}{dx} \int_a^x \| \vec{c}\,'(t) \| \, dt = \underline{\| \vec{c}\,'(x) \|}$$

(b)

Since $f'(x) = \| \vec{c}\,'(x) \| > 0$, then f is increasing on $[a,b]$, and so is one-to-one on $[a,b]$.

Since $f(b) = \int_a^b \| \vec{c}(t) \| \, dt = L$, and $f(a) = 0$,

Then f is also onto

$\therefore$ An inverse function $g : [0,L] \to [a,b]$ exists

That is continuous, one-to-one and onto,

and $g'(s) = \frac{1}{f'(x)}$, $(f \circ g)(s) = f(g(s)) = s$,

$(g \circ f)(x) = g(f(x)) = x$.

(c)

$$\frac{dg}{ds} = \frac{1}{f'(x)} = \underline{\frac{1}{\| c'(x) \|}}$$

(d)

$$\vec{b}'(s) = \vec{c}'(g(s)) \cdot g'(s)$$

$$= \vec{c}'(x) \cdot \frac{1}{\|\vec{c}'(x)\|} \quad \text{since } g(s) = x$$

$$\therefore \|\vec{b}'(s)\| = \frac{\|\vec{c}'(x)\|}{\|\vec{c}'(x)\|} = 1.$$

(b) above showed a reparametrization function exists for $\vec{c}(t)$, if $\vec{c}'(t) \neq 0$, so that $\vec{b}(s)$ can always be created.

22.

(a) $\frac{dT}{dV} = -\frac{\Lambda_v}{K_v}$, $\therefore \Lambda_v \, dV + K_v \, dT = 0$

(i) Heat gained $= \int_C \Lambda_v \, dV + K_v \, dT = \int_C 0 = \underline{0}$

(2) Work done $= \int_C P\,dV = \int_C P \frac{dV}{dt}\,dt$

$P(V,T) = \frac{RT}{V-b} - \frac{a}{V^2}$, a function of V and T.

From $\frac{dT}{dt} = -\frac{\Lambda_V}{K_V}\frac{dV}{dt}$, and $\Lambda_V = \frac{RT}{J(V-b)}$,

$$\frac{dT}{dt} = -\frac{RT}{JK_V}\cdot\frac{1}{V-b}\frac{dV}{dt}, \text{ or}$$

$$\frac{1}{T}\frac{dT}{dt} = -\frac{R}{JK_V}\cdot\frac{1}{V-b}\frac{dV}{dt}$$

$$\therefore \int_{t_0}^{t}\frac{1}{T}\frac{dT}{dt} = -\frac{R}{JK_V}\int_{t_0}^{t}\frac{1}{V-b}\frac{dV}{dt}$$

$$\ln T\Big|_{t_0}^{t} = -\frac{R}{JK_V}\ln(V-b)\Big|_{t_0}^{t}$$

$$\therefore \ln\frac{T(t)}{T_0} = -\frac{R}{JK_V}\ln\left[\frac{V(t)-b}{V_0-b}\right]$$

where $T(t_0) = T_0$ = intial temperature

$V(t_0) = V_0$ = initial volume

Taking exponents, and using $T(t)=T$, $V(t)=V$

$$\frac{T}{T_0} = e^{-R/JK_v} \frac{V-b}{V_0-b}$$

$$\therefore \quad T = \frac{T_0 e^{-R/JK_v}}{V_0 - b}(V-b) \qquad [1]$$

Substituting this into $P(V,T)$:

$$P(V,T) = \frac{R}{V-b} \cdot \frac{T_0 e^{-R/JK_v}}{V_0-b}(V-b) - \frac{a}{V^2}$$

$$\frac{RT_0 e^{-R/JK_v}}{V_0-b} - \frac{a}{V^2} \qquad [2]$$

Thus P is just a function of V, given T_0 and V_0.

$$\therefore \int_C P\,dV = \int_{t_0}^{t_f} \left[\frac{RT_0 e^{-R/JK_v}}{V_0-b} - \frac{a}{V^2} \right] \frac{dV}{dt}\,dt$$

$$= \frac{RT_0 e^{-R/JK_v}}{V_0-b} V(t) + \frac{a}{V(t)} \Bigg|_{t_0}^{t_f}$$

and $V(t_0) = V_0$, $V(t_f) = 2V_0$

$$\therefore \int_C P\,dV = \frac{RT_0 e^{-R/JK_v}}{V_0 - b}(2V_0 - V_0) + \left(\frac{a}{2V_0} - \frac{a}{V_0}\right)$$

$$\therefore \text{Work done} = \frac{RT_0 V_0 e^{-R/JK_v}}{V_0 - b} - \frac{a}{2V_0}$$

(3) $V_{final} = 2V_0$

$$T_{final} = \frac{T_0 e^{-R/JK_v}}{V_0 - b}(2V_0 - b) \quad , \text{ from [1]}$$

$$P_{final} = \frac{RT_0 e^{-R/JK_v}}{V_0 - b} - \frac{a}{4V_0^2} \quad , \text{ from [2]}$$

(b)

(1) Heat gained $= \int_C \Lambda_v \, dV + K_v \, dT$

Here $\frac{dV}{dt} = 0$ since V is constant.

$\therefore$ Heat gained $= \int_C K_v \, dT = K_v(T_f - T_i)$

Here, $T_f = T_0$, and T_i is, from (a)

$$T_i = \frac{T_0 e^{-R/JK_v}}{V_0 - b}(2V_0 - b)$$

$\therefore$ Heat gained $= K_v \left[T_0 - \frac{T_0 (2V_0 - b)}{V_0 - b} e^{-R/JK_v} \right]$

$$= K_v T_0 \left[1 - \left(\frac{2V_0 - b}{V_0 - b} \right) e^{-R/JK_v} \right]$$

(2) Since $dV = 0$,

Work done $= \int_c P\,dV = \underline{0}$

(3) Final volume $= \underline{2V_0}$

Final temperature $= \underline{T_0}$

$$\text{Final pressure} = \frac{R(T_0)}{(2V_0) - b} - \frac{a}{(2V_0)^2}$$

$$= \frac{RT_0}{2V_0 - b} - \frac{a}{4V_0^2}$$

(c)

(1) Heat gained $= \int_c \Lambda_v \, dv + K_v \, dT = \int_c \Lambda_v \, dv$

since here $\frac{dT}{dt} = 0$ since temperature constant at T_0

$$\therefore \int_c \Lambda_v \, dV = \int_{2v_0}^{v_0} \frac{R}{J} \frac{T_0}{V-b} \, dv$$

$$\frac{RT_0}{J} \ln(v-b) \Big|_{v=2v_0}^{v=v_0} = \frac{RT_0}{J} \ln\left(\frac{v_0-b}{2v_0-b}\right)$$

(2) Work done $= \int_c P \, dV$

$$= \int_{2v_0}^{v_0} \frac{RT_0}{v-b} - \frac{a}{V^2} \, dv$$

$$= RT_0 \ln(v-b) + \frac{a}{v} \Big|_{v=2v_0}^{v=v_0}$$

$$= RT_0 \ln\left(\frac{v_0-b}{2v_0-b}\right) + \frac{a}{2v_0}$$

(3) Final volume = $\underline{V_0}$

Final temperature = $\underline{T_0}$

Final pressure = $\underline{\dfrac{RT_0}{V_0 - b} - \dfrac{a}{V_0^2}}$

$\therefore$ All 3 parameters back to intial state before cycle began

(d)

(1) Total heat gained = 0 from (a)

$$+ K_v T_0 \left[1 - \left(\frac{2V_0 - b}{V_0 - b}\right) e^{-R/JK_v}\right] \text{ from (b)}$$

$$+ \frac{RT_0}{J} \ln\left(\frac{V_0 - b}{2V_0 - b}\right) \text{ from (c)}$$

(2) Total work done = $\dfrac{RT_0 V_0 e^{-R/JK_v}}{V_0 - b} - \dfrac{a}{2V_0}$ (a)

$+ \; 0$ (b)

$+ \; RT_0 \ln\left(\dfrac{V_0 - b}{2V_0 - b}\right) + \dfrac{a}{2V_0}$ (c)

$$= RT_0 \left[\frac{V_0 \, e^{-R/JK_v}}{V_0 - b} + \ln\left(\frac{V_0 - b}{2V_0 - b} \right) \right]$$

7.3 Parametrized Surfaces

Note Title 2/20/2017

1.

Let $\phi(u,v) = (2u, u^2+v, v^2) = (0,1,1) \Rightarrow u=0,\ v=1$

$x_u = 2 \quad y_u = 2u \quad z_u = 0 \quad \therefore \phi_u(0,1) = (2,0,0)$

$x_v = 0 \quad y_v = 1 \quad z_v = 2v \quad \therefore \phi_v(0,1) = (0,1,2)$

$$\therefore \phi_u \times \phi_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & 0 \\ 0 & 1 & 2 \end{vmatrix} = (0,-4,2)$$

$\therefore (0,-4,2)\cdot(x-0, y-1, z-1) = 0$, or

$-4(y-1) + 2(z-1) = 0$, or $z-1 = 2(y-1)$

or $\underline{z = 2(y-1)+1}$ or $2y - z - 1 = 0$

2.

Let $\phi(u,v) = (u^2-v^2, u+v, u^2+4v) = (-\frac{1}{4}, \frac{1}{2}, 2)$

$\Rightarrow u+v = \frac{1}{2}$. $\therefore u^2-v^2 = (u+v)(u-v) = \frac{1}{2}(u-v) = -\frac{1}{4}$

$\Rightarrow u-v = -\frac{1}{2}$

$\therefore 2u = 0 \Rightarrow u=0,\ v = \frac{1}{2}$

check: $u^2+4v = 0 + 4(\frac{1}{2}) = 2 = z.$

$\phi_u = (2u, 1, 2u) \qquad \phi_u(0, \frac{1}{2}) = (0, 1, 0)$

$\phi_v = (-2v, 1, 4) \qquad \phi_v(0, \frac{1}{2}) = (-1, 1, 4)$

$$\therefore \phi_u \times \phi_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 0 \\ -1 & 1 & 4 \end{vmatrix} = (4, 0, 1)$$

$\therefore (4, 0, 1) \cdot (x + \frac{1}{4}, y - \frac{1}{2}, z - 2) = 0$

or $4x + 1 + z - 2 = 0$, or $4x + z - 1 = 0$

or $\underline{z = -4x + 1}$

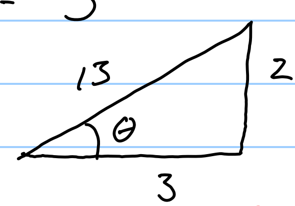
3.

Let $\phi(u,v) = (u^2, u \sin e^v, \frac{1}{3} u \cos e^v) = (13, -2, 1)$

$\Rightarrow u^2 = 13, \; u = \pm\sqrt{13}, \quad \pm\sqrt{13} \sin(e^v) = -2$

$\pm\sqrt{13} \cos(e^v) = 3$

$\therefore e^v = \pi - \text{Arcsin} \frac{2}{\sqrt{13}}$



$v = \ln(\pi - \text{Arcsin} \frac{2}{\sqrt{13}}) \qquad u = -\sqrt{13}$

$\theta = \text{Arcsin} \frac{2}{\sqrt{13}}$

$\phi_u = (2u, \sin(e^v), \frac{1}{3} \cos(e^v))$

$\phi_v = (0, u e^v \cos(e^v), -\frac{1}{3} u e^v \sin(e^v))$

Calculating ϕ_u, ϕ_v at $(u,v) = (-\sqrt{13}, \ln(\pi - \text{Arcsin}\frac{2}{\sqrt{13}}))$ could get messy.

$\therefore$ Try $\phi_u \times \phi_v$ before evaluating at point.

$$\phi_u \times \phi_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2u & \sin(e^v) & \frac{1}{3}\cos(e^v) \\ 0 & ue^v\cos(e^v) & -\frac{1}{3}ue^v\sin(e^v) \end{vmatrix}$$

$$= \Big[-\tfrac{1}{3}ue^v\sin^2(e^v) - \tfrac{1}{3}ue^v\cos^2(e^v),$$
$$\tfrac{2}{3}u^2e^v\sin(e^v),\ 2u^2e^v\cos(e^v) \Big]$$

$$= \Big[-\tfrac{1}{3}ue^v,\ \tfrac{2}{3}u^2e^v\sin(e^v),\ 2u^2e^v\cos(e^v) \Big]$$

$$= \frac{ue^v}{3}\Big[-1,\ 2u\sin(e^v),\ 6u\cos(e^v) \Big]$$

$$= \frac{ue^v}{3}\Big[-1,\ -4,\ 18 \Big]$$

using $u\sin(e^v) = -2$, $\frac{1}{3}u\cos(e^v) = 1$

Ignore $\frac{ue^v}{3}$ since that is a constant times the normal vector of $(-1, -4, 18)$.

$\therefore$ Plane $= (-1)(x-13) - 4(y+2) + 18(z-1) = 0$

or $-x - 4y + 18z + (13 - 8 - 18) = 0$

$$x + 4y - 18z + 13 = 0$$

4.

When $\phi_u \times \phi_v \neq 0$.

For #1: $\phi_u \times \phi_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 2u & 0 \\ 0 & 1 & 2v \end{vmatrix} = (4uv, -4v, 2)$

$\therefore \phi_u \times \phi_v \neq 0$ for any (u,v).

$\therefore$ Regular for all points

For #2: $\phi_u \times \phi_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2u & 1 & 2u \\ -2v & 1 & 4 \end{vmatrix} = (4-2u, -8u-4uv, 2u+2v)$

$4-2u = 0$ if $u=2$. $\therefore$ $-8(2) - 4(2)v = 0$ if $v = -2$

and $2(2) + 2(-2) = 0$.

$\therefore$ Regular for all points except $(u,v) = (2,-2)$,

or $(x,y,z) = (u^2 - v^2, u+v, u^2 + 4v)$

$= (4-4, 2-2, 4-8) = (0, 0, -4)$

$\therefore$ Regular everywhere except $(0,0,-4)$

5.

$\phi_u = (2u, 2u, 0) \quad \phi_v = (-2v, 2v, 1)$

$$\phi_u \times \phi_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2u & 2u & 0 \\ -2v & 2v & 1 \end{vmatrix} = (2u, -2u, 8uv)$$

$\phi_u \times \phi_v = \vec{0}$ only when $\underline{u = 0}$

6.

$\phi_u = (1, 1, 2v) \quad \phi_v = (-1, 1, 2u)$

$$\phi_u \times \phi_v = \begin{vmatrix} \hat{i} & \hat{j} & \vec{k} \\ 1 & 1 & 2v \\ -1 & 1 & 2u \end{vmatrix} = (2u - 2v, -2u - 2v, 2)$$

$\therefore$ $\phi_u \times \phi_v$ is never $\vec{0}$ due to $\hat{k}$ component

$\therefore$ S is <u>smooth for all points</u> (u, v)

7.

(a) Note $x^2+y^2 = 4(1+u^2)(\sin^2+\cos^2) = 4+4u^2$

$\therefore$ when $u=0$, $z=0$, and just a circle of radius 2 in xy-plane.

As u increases/decreases, circles are bigger by a factor of $4u^2$.

$\therefore$ (iii) fits this description

(b) Note $\phi(0,0)=(0,0,1)$. Only (i) satisfies this condition.

Note also $4x^2+9y^2 = 36\cos^2 u\sin^2 v + 36\sin^2 u \sin^2 v$
$= 36\sin^2 v$

And $\therefore$ $4x^2+9y^2+36z^2 = 36\sin^2 v + 36\cos^2 v = 36$

$\therefore$ an ellipsoid, $\therefore$ (i)

(c) Note for $y=0$ (from $v=0$), $(u,0,u^2)$ is a parabola, and also for any fixed value (or section from $y=$ constant) of v.

$\therefore$ paraboloid of one sheet : (ii)

(d) Note $x^2+y^2=u^2$, $\therefore$ a circle of radius $|u|$

As z changes linearly, a cone. $\therefore$ (iv)

8.

(a) $x^2+y^2=u^2$, a circle of radius u, $0\le u\le 1$

$z=4-x-y$, a plane, intersecting z-axis at $(0,0,4)$ with intercepts at $(4,0,0)$ and $(0,4,0)$

$\therefore$ Imagine the union of multiple cylinders (radii ≤ 1) perpendicular to xy-plane intersecting the angled plane of $z=4-x-y$.

$\therefore$ (i)

(b) $x=u\cos v$, $y=u\sin v$ is a circle of radius $|u|$
For a fixed u, the $z=4-u^2$ is a constant.
$\therefore$ level curves perpendicular to z-axis are circles.

$\therefore$ (iii).

(c) $z=\frac{1}{3}(12-8x-3y)$ is a plane with intercepts $(0,0,4)$, $(3/2,0,0)$, $(0,4,0)$
x and y are independent of each other.

$\therefore$ (ii)

(d) Note y is independent. For a fixed y (fixed u)

$x^2 + z^2 = (K+11)^2$, where $K = u^2 + u$

$\therefore$ level curves perpendicular to y-axis are circles, whose radii act like parabolas as K (or u) changes.

$\therefore$ (iv)

9.

(a) $\phi(u,v) = (\cos v \sin u, \sin v \sin u, \cos u)$

$\therefore \phi_u = (\cos v \cos u, \sin v \cos u, -\sin u)$

$\phi_v = (-\sin v \sin u, \cos v \sin u, 0)$

$$\therefore \text{Normal vector} = \phi_u \times \phi_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos v \cos u & \sin v \cos u & -\sin u \\ -\sin v \sin u & \cos v \sin u & 0 \end{vmatrix}$$

$$= (\sin^2 u \cos v, \sin^2 u \sin v, \cos^2 v \sin u \cos u + \sin^2 v \sin u \cos u)$$

$$= (\sin^2 u \cos v, \sin^2 u \sin v, \sin u \cos u)$$

$$\|\phi_u \times \phi_v\|^2 = \sin^4 u (\cos^2 v + \sin^2 v) + \sin^2 u \cos^2 u$$

$= \sin^4 u + \sin^2 u \cos^2 u = \sin^2 u(\sin^2 u + \cos^2 u)$

$= \sin^2 u$

$\therefore \|\phi_u \times \phi_v\| = \sin u$, and for $u \in (0, \pi)$, $\sin u > 0$.

$\therefore$ Vector in direction of $\phi_u \times \phi_v$ has length factor of $\sin u$. If factor out the $(\sin u)$, then vector has length $= 1$ for all $u \in [0, \pi]$, and so is never 0, making S regular for all points.

This parametrization does not assure regularity at poles: $u = 0, \pi$. A different parametrization could make the surface regular at poles.

A unit normal vector: $(\sin u \cos v, \sin u \sin v, \cos u)$

$= (x, y, z)$, where $x^2 + y^2 + z^2 = 1$.

(3) For a fixed $u \in [0, \pi]$, $0 \leq \sin u \leq 1$,

x, y are on a circle of radius $\sin u$.

These are level curves for $z = \cos u$, and for $u \in [0, \pi]$, $-1 \leq \cos u \leq 1$.

For $\cos u = 1$ or -1, level curves are a point ($\sin u = 0$).

$\therefore$ This parametrization is the spherical coordinate system, $\rho = 1$, $u \equiv \phi$, $v \equiv \theta$

$\therefore$ Surface is a unit sphere, center at $(0,0,0)$.

10.

(a) Let $c(\theta,\phi) = (3\cos\theta\sin\phi, 2\sin\theta\sin\phi, \cos\phi)$

$\therefore\ c_\theta = (-3\sin\theta\sin\phi, 2\cos\theta\sin\phi, \cos\phi)$

$c_\phi = (3\cos\theta\cos\phi, 2\sin\theta\cos\phi, -\sin\phi)$

$$\therefore\ c_\theta \times c_\phi = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3\sin\theta\sin\phi & 2\cos\theta\sin\phi & 0 \\ 3\cos\theta\cos\phi & 2\sin\theta\cos\phi & -\sin\phi \end{vmatrix}$$

$$= (-2\cos\theta\sin^2\phi, -3\sin\theta\sin^2\phi, -6\sin^2\theta\sin\phi\cos\phi - 6\cos^2\theta\sin\phi\cos\phi)$$

$$= (-2\cos\theta\sin^2\phi, -3\sin\theta\sin^2\phi, -6\sin\phi\cos\phi)$$

$$\|c_\theta \times c_\phi\|^2 = 4\cos^2\theta\sin^4\phi + 9\sin^2\theta\sin^4\phi + 36\sin^2\phi\cos^2\phi$$

$$= 4\sin^4\phi + 5\sin^2\theta\sin^4\phi + 36\sin^2\phi\cos^2\phi$$

$$= \sin^2\phi \left[4\sin^2\phi + 5\sin^2\theta\sin^2\phi + 36\cos^2\phi\right]$$

$$= \sin^2\phi \left[4 + 5\sin^2\theta\sin^2\phi + 32\cos^2\phi\right]$$

$$\therefore \|C_\theta \times C_\phi\| = \sin\phi\sqrt{4 + 5\sin^2\theta\sin^2\phi + 32\cos^2\phi}$$

$$\therefore \frac{C_\theta \times C_\phi}{\|C_\theta \times C_\phi\|} = \frac{(-2\cos\theta\sin\phi, -3\sin\theta\sin\phi, -6\cos\phi)}{\sqrt{4 + 5\sin^2\theta\sin^2\phi + 32\cos^2\phi}}$$

for $\phi \neq 0, \pi$

$$= \frac{\left(-\frac{2}{3}x, -\frac{3}{2}y, -6z\right)}{\sqrt{4 + \frac{5}{4}y^2 + 32z^2}}$$

or,
$$\underline{\frac{(4x, 9y, 36z)}{6\sqrt{4 + \frac{5}{4}y^2 + 32z^2}}}$$

(5) $4x^2 + 9y^2 = 36\cos^2\theta\sin^2\phi + 36\sin^2\theta\sin^2\phi = 36\sin^2\phi$

$\therefore 4x^2 + 9y^2 + 36z^2 = 36\sin^2\phi + 36\cos^2\phi = 36$

$\therefore \frac{x^2}{9} + \frac{y^2}{4} + z^2 = 1$, an <u>ellipsoid</u>.

11.

(a) Let $\phi(u,v) = (\sin v, u, \cos v)$

$\phi_u = (0,1,0) \quad \phi_v = (\cos v, 0, -\sin v)$

$$\therefore \phi_u \times \phi_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 0 \\ \cos v & 0 & -\sin v \end{vmatrix} = (-\sin v, 0, -\cos v)$$

$\therefore$ unit normal is $\underline{(\sin v, 0, \cos v)} = \underline{\dfrac{(x,0,z)}{\sqrt{x^2+z^2}}}$

(b) x and z describe a circle, parallel to y-axis, of radius 1.

$-1 \leq y \leq 3$. $\therefore$ A $\underline{\text{cylinder}}$ parallel to y-axis, from $y=-1$ to $y=3$, with y-axis the axis.

12.

(a) Let $\phi(u,v) = [(2-\cos v)\cos u, (2-\cos v)\sin u, \sin v]$

$\therefore \phi_u = [-(2-\cos v)\sin u, (2-\cos v)\cos u, 0]$

$\phi_v = [\sin v \cos u, \sin v \sin u, \cos v]$

$$\therefore \phi_u \times \phi_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -(2-\cos v)\sin u & (2-\cos v)\cos u & 0 \\ \sin v \cos u & \sin v \sin u & \cos v \end{vmatrix}$$

$$= \left[(2\cos v - \cos^2 v)\cos u,\ (2\cos v - \cos^2 v)\sin u,\right.$$

$$\left. -(2-\cos v)\sin v(\sin^2 u + \cos^2 u) \right]$$

$$= (2-\cos v)\left[\cos v \cos u,\ \cos v \sin u,\ -\sin v\right]$$

$2-\cos v > 0$, and so is just a magnitude factor.

$\therefore$ Let $\vec{N} = \underline{(\cos v \cos u,\ \cos v \sin u,\ -\sin v)}$

and $\|\vec{N}\| = 1$

(b) $\sin(v) = 0$ for $v = -\pi, 0, \pi$

For those values, $\cos v = \pm 1$.

$\therefore \vec{N} = \pm(\cos u, \sin u, 0)$ for $v = -\pi, 0, \pi$.

But $\cos u$ and $\sin u$ are never simultaneously zero.

$\therefore \vec{N} \neq \vec{0}$. $\therefore$ Surface is regular

(c) For a fixed v, $x = (2-\cos v)\cos u$, $y = (2-\cos v)\sin u$ describes a circle of positive radius $(2-\cos v)$.

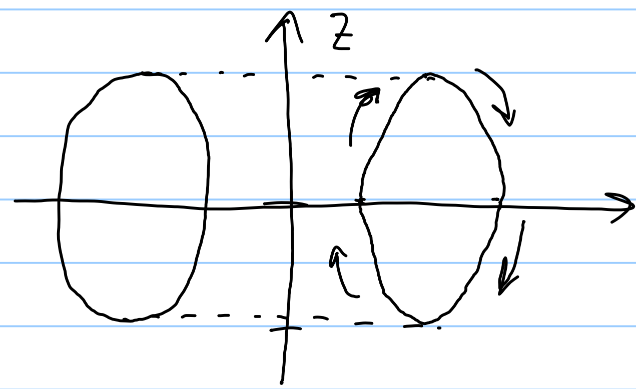
$\therefore$ Level curves perpendicular to z axis are circles.

When $z = 0$ $(v = -\pi, 0, \pi)$, $(2-\cos v) = 3, 1, 3$

$$z = 1 \left(v = \frac{\pi}{2}\right), \ (2-\cos v) = 2$$
$$z = -1 \left(v = -\frac{\pi}{2}\right), \ (2-\cos v) = 2$$

$\therefore$ v : $-\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi$ circle radius: 3, 2, 1, 2, 3

z : 0, -1, 0, 1, 0 circle radius : 3, 2, 1, 2, 3



$\therefore$ a torus or doughnut perpendicular to z-axis.

13.

(a) Let $\phi(u,v) = (h(u,v), u, v)$

$\therefore$ $\phi_u = (h_u, 1, 0)$ $\phi_v = (h_v, 0, 1)$

$$\therefore \phi_u \times \phi_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ h_u & 1 & 0 \\ h_v & 0 & 1 \end{vmatrix} = (1, -h_u, -h_v)$$

This is the same as $(1, -h_y, -h_z)$

Let $x_0 = h(y_0, z_0)$.

$\therefore (1, -h_y(y_0,z_0), -h_z(y_0,z_0)) \cdot (x-x_0, y-y_0, z-z_0) = 0$

or $(x-x_0) - h_y(y_0,z_0)(y-y_0) - h_z(y_0,z_0)(z-z_0) = 0$

or $\underline{x = x_0 + h_y(y_0,z_0)(y-y_0) + h_z(y_0,z_0)(z-z_0)}$

(b) $\underline{y = y_0 + K_x(x_0,z_0)(x-x_0) + K_z(x_0,z_0)(z-z_0)}$

where $y_0 = K(x_0, z_0)$

14.

Let $\phi(u,v) = (u^2, v^2, u^2+v^2)$

$\therefore \phi_u = (2u, 0, 2u) \quad \phi_v = (0, 2v, 2v)$

$$\therefore \phi_u \times \phi_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2u & 0 & 2u \\ 0 & 2v & 2v \end{vmatrix} = (-4uv, -4uv, 4uv)$$

$\therefore \phi_u \times \phi_v(1,1) = (-4,-4,4)$. Let $\vec{N} = (-1,-1,1)$

$\phi(1,1) = (1,1,2)$.

$\therefore (-1,-1,1) \cdot (x-1, y-1, z-2) = 0$, or

$(1-x)+(1-y)+z-2=0$, or $\underline{z=x+y}$

15.

Let $\phi(u,v)=(u,v,3u^2+8uv)$

$\phi_u=(1,0,6u+8v) \qquad \phi_v=(0,1,8u)$

$\therefore \phi_u \times \phi_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 6u+8v \\ 0 & 1 & 8u \end{vmatrix} = (-6u-8v,-8u,1)$

$\therefore \phi_u \times \phi_v(1,0) = (-6,-8,1) \qquad \phi(1,0)=(1,0,3)$

$\therefore (-6,-8,1)\cdot(x-1,y,z-3)=0$, or

$-6(x-1)-8y+z-3=0$, or $\underline{z-6x-8y+3=0}$

16.

Why stipulate $z>0$ when asking at point $z=0$?

(a) $z=\sqrt{2-x^3-3xy}$. If use $\phi(u,v)=(u,v,\sqrt{2-u^3-3uv})$,

Then ϕ_u and ϕ_v will have $(2-u^3-3uv)^{-\frac{1}{2}}$ in 3rd component, and can't divide by 0.

$\therefore$ Try $y = \dfrac{2 - z^2 - x^3}{3x}$.

Let $\phi(u,v) = \left(u, \dfrac{2 - v^2 - u^3}{3u}, v\right)$

$\therefore \phi(1,0) = \left(1, \frac{1}{3}, 0\right)$

$$\phi_u = \left[1, \frac{3u(-3u^2) - (2 - v^2 - u^3)(3)}{9u^2}, 0\right]$$

$$\phi_v = \left[0, \frac{-2v}{3u}, 1\right]$$

$$\therefore \phi_u(1,0) = \left[1, \frac{3(-3) - (1)(3)}{9}, 0\right] = \left[1, -\frac{4}{3}, 0\right]$$

$$\phi_v(1,0) = [0, 0, 1]$$

$$\therefore \phi_u \times \phi_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -4/3 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \left(-\frac{4}{3}, -1, 0\right)$$

$\therefore \left(-\frac{4}{3}, -1, 0\right) \cdot \left(x - 1, y - \frac{1}{3}, z - 0\right) = 0$, or

$-\frac{4}{3}(x-1) - \left(y - \frac{1}{3}\right) = 0$, or

$-4(x-1) - 3\left(y - \frac{1}{3}\right) = 0$, or $\underline{4x + 3y = 5}$

(b) Using level sets, $f(x,y,z) = x^3 + 3xy + z^2 = 2$

Gradient is perpendicular to level set.

$$\nabla f = (3x^2+3y,\ 3x,\ 2z). \quad \therefore \nabla(1, 1/3, 0) = (4,3,0)$$

$$\therefore (4,3,0)\cdot(x-1,\ y-1/3,\ z-0) = 0, \text{ or}$$

$$4(x-1) + 3(y-\tfrac{1}{3}) + 0 = 0, \text{ or}$$

$$4x - 4 + 3y - 1 = 0, \text{ or } \underline{\underline{4x+3y = 5}}$$

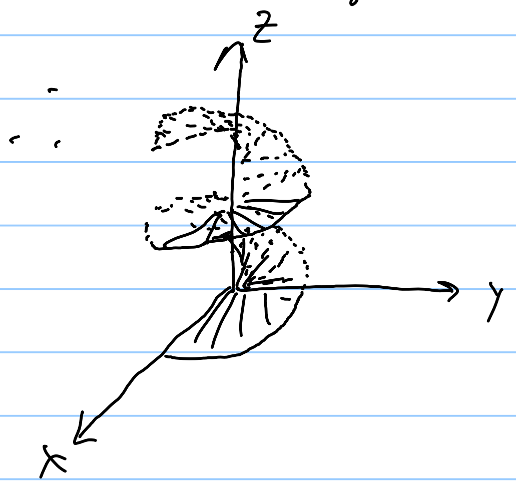
17.

(a)

$(r\cos\theta, r\sin\theta)$ is a circle of radius r.

$\therefore$ as r varies from $0 \le r \le 1$, a line sweeps out a circle, as θ increases, and the "ray" elevates according to θ, and so sweeps out a sheet, like a screw, going around the start point $(1,0,0)$ twice, once at $\theta = 2\pi$, and again at $\theta = 4\pi$.

$\therefore$



$$\phi(r,\theta) = (r\cos\theta,\ r\sin\theta,\ \theta)$$

$$0 \le r \le 1$$
$$0 \le \theta \le 4\pi$$

(b)

$$\phi_r = (\cos\theta, \sin\theta, 0) \quad \phi_\theta = (-r\sin\theta, r\cos\theta, 1)$$

$$\therefore \phi_r \times \phi_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 1 \end{vmatrix}$$

$$= (\sin\theta, -\cos\theta, r\cos^2\theta + r\sin^2\theta)$$

$$= (\sin\theta, -\cos\theta, r)$$

$$\|(\sin\theta, \cos\theta, r)\| = \sqrt{1+r^2}$$

$$\therefore \vec{N} = \frac{1}{\sqrt{1+r^2}}(\sin\theta, -\cos\theta, r)$$

(c)

$$x_0 = r\cos\theta, \quad y_0 = r\sin\theta, \quad z_0 = \theta$$

$$\therefore x_0^2 + y_0^2 = r^2, \quad r = \sqrt{x_0^2 + y_0^2}$$

$$\therefore \vec{N} = \frac{1}{\sqrt{1+x_0^2+y_0^2}}\left(\sin(z_0), -\cos(z_0), \sqrt{x_0^2+y_0^2}\right)$$

$$\therefore \left(\sin(z_0), -\cos(z_0), \sqrt{x_0^2+y_0^2}\right) \cdot (x-x_0, y-y_0, z-z_0) = 0$$

or, $\sin(z_0)(x-x_0) - \cos(z_0)(y-y_0) + \sqrt{x_0^2+y_0^2}\,(z-z_0) = 0$

or, multiplying by $\sqrt{x_0^2+y_0^2}$ and noting

$\sin(z_0)\sqrt{x_0^2+y_0^2} = y_0$ $(y = r\sin\theta = r\sin z)$

and $\cos(z_0)\sqrt{x_0^2+y_0^2} = x_0$ $(x = r\cos\theta = r\cos z)$

$\therefore$ $\underline{y_0(x-x_0) - x_0(y-y_0) + (x_0^2+y_0^2)(z-z_0) = 0}$

(d)

(1) Let $\theta_0 = z_0$, $r_0 = \sqrt{x_0^2+y_0^2}$ $\therefore \phi(r_0, \theta_0) = (x_0, y_0, z_0)$

$= (r_0\cos\theta_0, r_0\sin\theta_0, \theta_0)$ is on the surface,

and by definition, all points $0 \le r_0$ at $z = \theta_0$

are on the surface. $\therefore$ The segment from

$(0, 0, \theta_0)$ to $(r_0\cos\theta_0, r_0\sin\theta_0, \theta_0) = (x_0, y_0, z_0)$

is on the surface.

(2) A vector parallel to segment is

$(r_0\cos\theta_0, r_0\sin\theta_0, \theta_0) - (0, 0, \theta_0) =$

$(r_0 \cos\theta_0, r_0 \sin\theta_0, 0)$, or $(\cos\theta_0, \sin\theta_0, 0)$

A vector parallel to $\vec{N}$ from (c) is

$(\sin\theta_0, -\cos\theta_0, r_0)$.

$\therefore (\cos\theta_0, \sin\theta_0, 0) \cdot (\sin\theta_0, -\cos\theta_0, 0)$

$= \cos\theta_0 \sin\theta_0 - \sin\theta_0 \cos\theta_0 = 0$

$\therefore$ All points on segment from z-axis to (x_0, y_0, z_0) are perpendicular to $\vec{N}$, and $\therefore$ are in the tangent plane.

18.

(a) $\Phi_\theta = (-2\sin\theta \sin\phi, 2\cos\theta \sin\phi, 0)$

$\Phi_\phi = (2\cos\theta\cos\phi, 2\sin\theta\cos\phi, -2\sin\phi)$

For the point $(1, 1, \sqrt{2})$, $\theta = \text{Arctan}\,\frac{y}{x} = \text{Arctan}\,1 = \frac{\pi}{4}$

$\phi = \text{Arccos}\,\frac{z}{\rho} = \text{Arccos}\,\frac{\sqrt{2}}{2} = \frac{\pi}{4}$

$$\therefore \phi_\theta\left(\frac{\pi}{4}, \frac{\pi}{4}\right) = \left(-2\,\frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2},\ 2\,\frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2},\ 0\right) = (-1, 1, 0)$$

$$\phi_\varphi\left(\frac{\pi}{4}, \frac{\pi}{4}\right) = \left(2\,\frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2},\ 2\,\frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2},\ -2\,\frac{\sqrt{2}}{2}\right) = (1, 1, -\sqrt{2})$$

$$\therefore \phi_\theta \times \phi_\varphi = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & 0 \\ 1 & 1 & -\sqrt{2} \end{vmatrix} = (-\sqrt{2}, -\sqrt{2}, -2)$$

$$\therefore (-\sqrt{2}, -\sqrt{2}, -2)\cdot(x-1, y-1, z-\sqrt{2}) = 0, \text{ or}$$

$$-\sqrt{2}(x-1) - \sqrt{2}(y-1) - 2(z-\sqrt{2}) = 0, \text{ or}$$

$$\sqrt{2}x + \sqrt{2}y + 2z = 4\sqrt{2}, \text{ or}$$

$$\underline{x + y + \sqrt{2}z = 4}$$

(5)

Gradients are perpendicular to level sets.

The point $(1,1,\sqrt{2})$ means $f(1,1,\sqrt{2}) = 4$

$\nabla f = (2x, 2y, 2z)$, so that $\nabla f(1,1,\sqrt{2}) = (2,2,2\sqrt{2})$

$$\therefore (2,2,\sqrt{2})\cdot(x-1, y-1, z-\sqrt{2}) = 0, \text{ or}$$

$$2(x-1) + 2(y-1) + 2\sqrt{2}(z-\sqrt{2}) = 0, \text{ or}$$

$$2x + 2y + 2\sqrt{2}z = 8, \text{ or } \underline{x + y + \sqrt{2}z = 4}$$

(c)

$$g(1,1) = \sqrt{2} = z_0$$

Tangent plane: $z = z_0 + g_x(x-x_0) + g_y(y-y_0)$

$$g_x = \tfrac{1}{2}(4-x^2-y^2)^{-\frac{1}{2}}(-2x)$$

$$\therefore g_x(1,1) = \tfrac{1}{2}(4-1-1)^{-\frac{1}{2}}(-2) = -\tfrac{1}{\sqrt{2}}$$

$$g_y = \tfrac{1}{2}(4-x^2-y^2)^{-\frac{1}{2}}(-2y)$$

$$\therefore g_y(1,1) = \tfrac{1}{2}(4-1-1)^{-\frac{1}{2}}(-2) = -\tfrac{1}{\sqrt{2}}$$

$$\therefore z = \sqrt{2} - \tfrac{1}{\sqrt{2}}(x-1) - \tfrac{1}{\sqrt{2}}(y-1), \text{ or}$$

$$\sqrt{2}\,z = 2 - x + 1 - y + 1, \text{ or}$$

$$\underline{x + y + \sqrt{2}\,z = 4}$$

19.

(a) Let $x = r\cos\theta$, $y = r\sin\theta$. $\therefore x^2+y^2 = r^2$

Looking at $r^2 - z^2 = 25$ and noting

$$\cosh^2(u) - \sinh^2(u) = 1$$

Let $r = 5\cosh(u)$ $\quad z = 5\sinh(u)$

$\therefore$ $x = 5\cosh(u)\cos\theta$, $y = 5\cosh(u)\sin\theta$,

$\therefore$ Let $\phi(u,\theta) = \big(5\cosh(u)\cos\theta, 5\cosh(u)\sin\theta, 5\sinh(u)\big)$

for $-\infty < u < \infty$, $0 \le \theta < 2\pi$

Another, noting $z^2 + 25 = x^2 + y^2$

$\therefore$ $x = r\cos\theta$, $y = r\sin\theta$, $x^2 + y^2 = r^2$

$\therefore$ $r = \sqrt{z^2 + 25}$

$\therefore$ $\phi(\theta, z) = \left[\sqrt{z^2+25}\cos\theta, \sqrt{z^2+25}\sin\theta, z\right]$

(b) Use level sets. Let $f(x,y,z) = x^2 + y^2 - z^2 = 25$

$\nabla f = (2x, 2y, -2z)$ which is parallel to $(x,y,-z)$

$\therefore$ unit normal $= \dfrac{(x,y,-z)}{\sqrt{x^2+y^2+z^2}} = \dfrac{(x,y,-z)}{\sqrt{2z^2+25}}$

(c)

From (b), normal to plane is $(x_0, y_0, 0)$.

$\therefore$ $(x_0, y_0, 0) \cdot (x - x_0, y - y_0, z) = 0$, or

$$x_0(x-x_0) + y_0(y-y_0) = 0, \text{ or}$$

$$x_0 x + y_0 y = x_0^2 + y_0^2 = 25$$

$$\therefore \underline{x_0 x + y_0 y = 25}$$

(d)

(1) From (c), normal to tangent plane at $(x_0, y_0, 0)$ is $(x_0, y_0, 0) = \vec{N}$.

Note $(-y_0, x_0, 5)\cdot\vec{N} = -x_0y_0 + x_0y_0 + 0 = 0$

and $(y_0, -x_0, 5)\cdot\vec{N} = x_0y_0 - x_0y_0 + 0 = 0$.

$\therefore$ Both vectors $(-y_0, x_0, 5)$ and $(y_0, -x_0, 5)$ are $\perp \vec{N} \Rightarrow$ parallel to tangent plane.

Since $(x_0, y_0, 0)$ is in the plane, then the line $(x_0, y_0, 0) + t(-y_0, x_0, 5)$, $t \in \mathbb{R}$, goes through $(x_0, y_0, 0)$ $(t=0)$ and is parallel to the tangent plane, and so is in the tangent plane.

Similarly for the other line,

$$(x_0, y_0, 0) + t(y_0, -x_0, 5)$$

(2) Since $(x_0, y_0, 0)$ is on the surface, $x_0^2 + y_0^2 - 0 = 25$,

or $x_0^2 + y_0^2 = 25$ [1]

$$(x_0, y_0, 0) + t(-y_0, x_0, 5) = (x_0 - ty_0, y_0 + tx_0, 5t)$$

$\therefore$ to show every point on the line is on the surface, must show each point on line satisfies $x^2 + y^2 - z^2 = 25$

Since $(x_0 - ty_0)^2 + (y_0 + tx_0)^2 - (5t)^2 =$

$x_0^2 - 2x_0y_0t + t^2y_0^2 + y_0^2 + 2x_0y_0t + t^2x_0^2 + 25t^2 =$

$(x_0^2 + y_0^2) + t^2(x_0^2 + y_0^2 - 25) =$

$25 + t^2(0)$ from [1]

$= 25.$

$\therefore$ Each point on the line $(x_0, y_0, 0) + t(-y_0, x_0, 5)$

is on the surface.

Similarly for the line $(x_0, y_0, 0) + t(y_0, -x_0, 5)$

$= (x_0 + t y_0,\ y_0 - t x_0,\ 5t)$,

$(x_0 + t y_0)^2 + (y_0 - t x_0)^2 - (5t)^2 =$

$x_0^2 + y_0^2 + t^2(x_0^2 + y_0^2 - 25) =$

$25 + t^2(0)$ from [1]

$= 25$

$\therefore$ The line $(x_0, y_0, 0) + t(y_0, -x_0, 5)$ is also on the surface.

20.

(a)

$D\phi(u_0, v_0)$ is a 3×2 matrix of the partial derivatives of ϕ.

Let $\phi(u,v) = (f(u,v), g(u,v), h(u,v))$

$\therefore \vec{T}_u = (f_u, g_u, h_u) \quad \vec{T}_v = (f_v, g_v, h_v)$

$$\therefore D\phi = \begin{bmatrix} f_u & f_v \\ g_u & g_v \\ h_u & h_v \end{bmatrix} = \begin{bmatrix} f_u(u_0,v_0) & f_v(u_0,v_0) \\ g_u(u_0,v_0) & g_v(u_0,v_0) \\ h_u(u_0,v_0) & h_v(u_0,v_0) \end{bmatrix}$$

$$\therefore \text{Range of } D\phi = \begin{bmatrix} f_u & f_v \\ g_u & g_v \\ h_u & h_v \end{bmatrix} \begin{bmatrix} u-u_0 \\ v-v_0 \end{bmatrix}, \text{ for all } (u,v)$$

$$= \begin{bmatrix} f_u(u_0,v_0)(u-u_0) + f_v(u_0,v_0)(v-v_0) \\ g_u(u_0,v_0)(u-u_0) + g_v(u_0,v_0)(v-v_0) \\ h_u(u_0,v_0)(u-u_0) + h_v(u_0,v_0)(v-v_0) \end{bmatrix}$$

$$= (u-u_0)\begin{bmatrix} f_u \\ g_u \\ h_u \end{bmatrix} + (v-v_0)\begin{bmatrix} f_v \\ g_v \\ h_v \end{bmatrix}$$

$$= (u-u_0)\vec{T}_u + (v-v_0)\vec{T}_v$$

($\vec{T}_u$, $\vec{T}_v$ viewed as column vectors).

= plane spanned by $\vec{T}_u$ and $\vec{T}_v$

(b)

(1) Suppose $\vec{w}$ is in the range of $D\phi(u_0,v_0)$

From (a), this means that $\exists (u,v)$

s.t. $\vec{w} = (u-u_0)\vec{T}_u + (v-v_0)\vec{T}_v$

$$\therefore \vec{w} \cdot (\vec{T}_u \times \vec{T}_v) = [(u-u_0)\vec{T}_u + (v-v_0)\vec{T}_v] \cdot (\vec{T}_u \times \vec{T}_v)$$

$$= (u-u_0)\vec{T}_u \cdot (\vec{T}_u \times \vec{T}_v) + (v-v_0)\vec{T}_v \cdot (\vec{T}_u \times \vec{T}_v)$$

$$= 0 + 0 = 0,$$

since $\vec{T}_u \cdot (\vec{T}_u \times \vec{T}_v) = 0$ and $\vec{T}_v \cdot (\vec{T}_u \times \vec{T}_v) = 0$

$$[\vec{a} \cdot (\vec{a} \times \vec{b}) = 0]$$

(2) Suppose $\vec{w} \perp (\vec{T}_u \times \vec{T}_v)$

$\therefore \vec{w} \cdot (\vec{T}_u \times \vec{T}_v) = 0$, and so $\vec{w}$ is in the plane spanned by $\vec{T}_u$ and $\vec{T}_v$ through (u_0, v_0). $\therefore \exists (u,v)$ s.t.

$$\vec{w} = (u-u_0)\vec{T}_u + (v-v_0)\vec{T}_v$$

From (a), this means $\vec{w}$ is in the range of $D\phi(u_0, v_0)$

(c)

As shown in (a), $D\phi(u_0,v_0)\begin{bmatrix} u-u_0 \\ v-v_0 \end{bmatrix}$ represents

The range spanned by $\vec{T}_u(u_0,v_0)$ and $\vec{T}_v(u_0,v_0)$.

$\therefore$ This is the same plane referenced above, as both planes contain the point $\phi(u_0,v_0)$.

21.

(a)

The xy-plane is characterized by all points with z-component 0: $(x,y,0)$, $x,y \in \mathbb{R}$.

(1) $\therefore$ For ϕ_1, Given any x,y, let $u=x$, $v=y$.
$\therefore$ Any point in xy-plane can be represented by ϕ_1, so xy-plane $\subseteq$ Image ϕ_1

And every $\phi(u,v)$ is in the xy-plane, as $\phi(u,v) = (u,v,0)$. $\therefore$ Image $\phi_2 \subseteq$ xy-plane

$\therefore$ Image of ϕ_1 = xy-plane.

(2) For ϕ_2, given any x,y, let $u=\sqrt[3]{x}$, $v=\sqrt[3]{y}$

$\therefore \phi_2(u,v) = (x,y,0)$. $\therefore$ xy-plane $\subseteq$ Image ϕ_2

And every $\phi_2(u,v)$ is in xy-plane since z-component of $\phi_2(u,v)$ is zero.

$\therefore$ Image $\phi_2 \subseteq$ xy-plane.

$\therefore$ Image $\phi_2 =$ xy-plane.

(b)

(1) $\phi_{1_u} = (1,0,0) \quad \phi_{1_v} = (0,1,0)$.

$$\therefore \phi_{1_u} \times \phi_{1_v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = (0,0,1) \neq \vec{0}$$

$\therefore$ By def., ϕ_1 is regular.

(2) $\phi_{2_u} = (3u^2, 0, 0) \quad \phi_{2_v} = (0, 3v^2, 0)$.

For $u=0, v=0$, $\phi_{2u} \times \phi_{2v}(0,0) = (0,0,0)$.

$\therefore \phi_2$ not regular at $(u,v) = (0,0)$.

(c)

Problem is not sufficiently stated to understand what is being asked. Inverse function Theorem uses $R^n \to R^n$. ϕ_1 and ϕ_2 above are $R^2 \to R^3$. I don't see where enough information has been given to state a Jacobian determinant is nonzero.

(d) — you mean 7.3.8

No. A regular parametrization can't be found making cone smooth at origin.

22.

(a) $x^2 = a^2\sin^2 u\cos^2 v$
$y^2 = b^2\sin^2 u\sin^2 v$
$z^2 = c^2\cos^2 u$

$$\therefore \frac{x^2}{a^2}+\frac{y^2}{b^2} = \frac{a^2\sin^2 u\cos^2 v}{a^2}+\frac{b^2\sin^2 u\sin^2 v}{b^2}$$

$$= \sin^2 u\left(\cos^2 v+\sin^2 v\right) = \sin^2 u$$

$$\therefore \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = \sin^2 u+\frac{c^2\cos^2 u}{c^2} = \sin^2 u+\cos^2 u = 1$$

(b) Assume $a, b, c \neq 0$.

$$\phi_u = (a\cos u\cos v,\ b\cos u\sin v,\ -c\sin u)$$

$$\phi_v = (-a\sin u\sin v,\ b\sin u\cos v,\ 0)$$

$$\therefore \phi_u \times \phi_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a\cos u\cos v & b\cos u\sin v & -c\sin u \\ -a\sin u\sin v & b\sin u\cos v & 0 \end{vmatrix}$$

$$= (cb\sin^2 u\cos v,\ ac\sin^2 u\sin v,$$

$$ab\sin u\cos u\cos^2 v + ab\sin u\cos u\sin^2 v)$$

$$= (cb\sin^2 u\cos v,\ ac\sin^2 u\sin v,\ ab\sin u\cos u)$$

Exclude $u = 0, \pi$, so $\sin u \neq 0$.

$\therefore$ $cb\sin^2 u \neq 0$, $ac\sin^2 u \neq 0$ (x, y components).

$\cos v$ and $\sin v$ can't be be 0, so either the x-component or y-component $\neq 0$.

$\therefore$ $\phi_u \times \phi_v \neq 0$ for $u \neq 0, \pi$.

$\therefore$ Ellipsoid regular at all points not equal to $(0,0,c)$ and $(0,0,-c)$.

For the two z-polar points, use the parametrization

$$\phi(u,v) = (a\sin u\cos v,\ b\cos u,\ c\sin u\sin v),$$

$$0 \leq u \leq \pi,\ 0 \leq v \leq 2\pi$$

$\therefore$ each point satisfies $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

This time, the parametrization shows all points regular (including $(0,0,\pm c)$) except possibly $(0,\pm b,0)$. But these two polar points were shown to be regular with the first parametrization.

$\therefore$ All points are regular.

23.

$R \geq r$, where
R = distance from center of torus to center of ring,
r = radius of ring.

Problem fails to specify R.

(a) $x^2 + y^2 = (R + r\cos u)^2 \cos^2 v + (R + r\cos u)^2 \sin^2 v$

$= (R + r\cos u)^2$

$\therefore \sqrt{x^2+y^2} = |R + r\cos u| = R + r\cos u$
as $R \geq r\cos u$

$\therefore \sqrt{x^2+y^2} - R = r\cos u$

$$\therefore \left(\sqrt{x^2+y^2}-R\right)^2+z^2=(r\cos u)^2+(r\sin u)^2$$
$$=r^2$$

(5) $\phi_u = [-r\sin u\cos v,\ -r\sin u\sin v,\ r\cos u]$

$\phi_v = [-(R+r\cos u)\sin v,\ (R+r\cos u)\cos v,\ 0]$

$$\therefore \phi_u\times\phi_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -r\sin u\cos v & -r\sin u\sin v & r\cos u \\ -(R+r\cos u)\sin v & (R+r\cos u)\cos v & 0 \end{vmatrix}$$

$$= [-r(R+r\cos u)\cos u\cos v,\ -r(R+r\cos u)\cos u\sin v,$$
$$-r(R+r\cos u)\sin u\cos^2 v - r(R+r\cos u)\sin u\sin^2 v]$$

$$= -r(R+r\cos u)[\cos u\cos v,\ \cos u\sin v,\ \sin u]$$

If $\sin u = 0$, then $\cos u \neq 0$, and $\cos v$ and $\sin v$ can't both be 0.

$\therefore [\cos u\cos v, \cos u\sin v, \sin u] \neq [0,0,0]$.

Also $r>0$.

If $R=r$, then $R+r\cos u$ could be 0.
(i.e., torus has no hole).

$\therefore$ Specify $R>r$, and $\therefore \phi_u\times\phi_v \neq 0$.

$\therefore$ Torus regular everywhere.

Really, should state: $0 \leq u \leq 2\pi$
$0 \leq v \leq 2\pi$
$0 < r < R$

24.

$$\phi(u,v) = [x(u,v), y(u,v), z(u,v)]$$

should be (b)

(a) $\vec{T}_u = (x_u, y_u, z_u)$ $\quad \vec{T}_v = (x_v, y_v, z_v)$.

$$\vec{T}_u \times \vec{T}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} = \begin{bmatrix} y_u z_v - y_v z_u \\ x_v z_u - x_u z_v \\ x_u y_v - x_v y_u \end{bmatrix}$$

$\vec{T}_u \times \vec{T}_v \neq \vec{0} \Rightarrow$ at least one of the components is non-zero. Suppose, for simplicity, it is the z-component: $x_u y_v - x_v y_u \neq 0$.

$\therefore$ The determinant $\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} \neq 0$ at (u_0, v_0)

$\therefore$ Consider $F(u,v) = (x(u,v), y(u,v)) = (x,y)$

where $F_1(u,v) = x(u,v)$, $F_2(u,v) = y(u,v)$

Near $F(u_0,v_0) = (x_0,y_0)$, the Jacobian determinant of F at (u_0,v_0) is $\begin{vmatrix} x_u(u_0,v_0) & x_v(u_0,v_0) \\ y_u(u_0,v_0) & y_v(u_0,v_0) \end{vmatrix} \neq 0$

$\therefore$ By the Inverse Function Theorem (a special case of the Implicit Function Theorem) as stated on p. 209 of the text, there is a C^1 function $g: R^2 \to R^2$ s.t.

$g(x,y) = (u,v)$ for x near x_0, y near y_0.

$\therefore$ The graph is $(x,y,g(x,y))$, where $\underline{z = g(x,y)}$

If the y-component of $\vec{T}_u \times \vec{T}_v \neq 0$, then the graph becomes $(x,z,g(x,z))$, $y = g(x,y)$

If x-component of $\vec{T}_u \times \vec{T}_z \neq 0$, then $(y,z,g(y,z))$, $x = g(y,z)$

(5)

The tangent plane at $\phi(u_0,v_0)$ is

$$\phi(u_0,v_0) + \begin{bmatrix} x_u & x_v \\ y_u & y_v \\ z_u & z_v \end{bmatrix} \begin{bmatrix} u-u_0 \\ v-v_0 \end{bmatrix}$$

$$= \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + \begin{bmatrix} x_u \\ y_u \\ z_u \end{bmatrix}(u-u_0) + \begin{bmatrix} x_v \\ y_v \\ z_v \end{bmatrix}(v-v_0)$$

$$= \phi(u_0, v_0) + \vec{T}_u (u-u_0) + \vec{T}_v (v-v_0)$$

For the graph $(x, y, g(x,y))$, the tangent plane at $\phi(u_0, v_0) = (x_0, y_0, z_0)$ is

$$\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ g_x(x_0,y_0) \end{bmatrix}(x-x_0) + \begin{bmatrix} 0 \\ 1 \\ g_y(x_0,y_0) \end{bmatrix} y - y_0$$

The normal to this plane is

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & g_x(x_0,y_0) \\ 0 & 1 & g_y(x_0,y_0) \end{vmatrix} = [-g_x(x_0,y_0), -g_y(x_0,y_0), 1]$$

From (a), $F(g(x,y)) = (x,y)$ and $g(F(u,v)) = (u,v)$

$$g(F(u,v)) = g(F_1(u,v), F_2(u,v)) = g(x(u,v), y(u,v))$$

$$\therefore g_u = \frac{\partial g}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial g}{\partial y} \cdot \frac{\partial y}{\partial u} = g_x x_u + g_y y_u \quad [1]$$

$$g_v = \frac{\partial g}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial g}{\partial y} \cdot \frac{\partial y}{\partial v} = g_x x_v + g_y y_v \quad [2]$$

But $z = g(x,y) = g(x(u,v), y(u,v))$

$$\therefore z_u = g_u \ , \ z_v = g_v$$

$\therefore$ [1], [2] become $\begin{bmatrix} x_u & y_u \\ x_v & y_v \end{bmatrix} \begin{bmatrix} g_x \\ g_y \end{bmatrix} = \begin{bmatrix} z_u \\ z_v \end{bmatrix}$

Since $x_u y_v - x_v y_u \neq 0$ (From (a)), Then

$$\begin{bmatrix} g_x \\ g_y \end{bmatrix} = \frac{1}{x_u y_v - x_v y_u} \begin{bmatrix} y_v & -y_u \\ -x_v & x_u \end{bmatrix} \begin{bmatrix} z_u \\ z_v \end{bmatrix}$$

$$= \begin{bmatrix} z_u y_v - z_v y_u \\ z_v x_u - z_u x_v \end{bmatrix} \cdot \frac{1}{x_u y_v - x_v y_u}$$

$$\therefore \begin{bmatrix} -g_x \\ -g_y \\ 1 \end{bmatrix} = \frac{1}{x_u y_v - x_v y_u} \begin{bmatrix} z_v y_u - z_u y_v \\ z_v x_u - z_u x_v \\ x_u y_v - x_v y_u \end{bmatrix}$$

i.e., by ignoring the scaling factor, the normal to the plane is parallel to:

$$\begin{bmatrix} y_u z_v - y_v z_u \\ x_u z_v - x_v z_u \\ x_u y_v - x_v y_u \end{bmatrix}$$ which is the same vector as $\vec{T}_u \times \vec{T}_v$

$\therefore$ Near $\phi(u_0, v_0) = (x_0, y_0, z_0)$, the plane spanned by $\vec{T_u}$ and $\vec{T_v}$ that includes (x_0, y_0, z_0) is the same plane with normal $\begin{bmatrix} -g_x \\ -g_y \\ 1 \end{bmatrix}$ that also passes through (x_0, y_0, z_0),

where $z = g(x, y)$ was defined by the inverse function theorem near (x_0, y_0, z_0).

1.

(1) D is an elementary region

(2) $\Phi(\theta,\phi)$ is C^1 and one-to-one except at $\theta = 2\pi$ (the boundary)

(3) The image is regular, under this parametrization, except at $\phi = 0, \frac{\pi}{2}$ (finite # of points).

$$\vec{T}_\theta = (-\sin\theta\sin\phi, \cos\theta\sin\phi, 0) = (x_\theta, y_\theta, z_\theta)$$

$$\vec{T}_\phi = (\cos\theta\cos\phi, \sin\theta\cos\phi, -\sin\phi) = (x_\phi, y_\phi, z_\phi)$$

$$\vec{T}_\theta \times \vec{T}_\phi = (-\cos\theta\sin^2\phi, -\sin\theta\sin^2\phi, -\sin^2\theta\sin\phi\cos\phi - \cos^2\theta\sin\phi\cos\phi)$$

$$= (-\cos\theta\sin^2\phi, -\sin\theta\sin^2\phi, -\sin\phi\cos\phi)$$

(Note for $\phi = 0, \pi$, $\vec{T}_\theta \times \vec{T}_\phi = \vec{0}$)

$$\therefore \left|\frac{\partial(x,y)}{\partial(\theta,\phi)}\right|^2 = \sin^2\phi\cos^2\phi \quad \text{(from the } z\text{-coordinate)}$$

$$\left|\frac{\partial(x,z)}{\partial(\theta,\phi)}\right|^2 = \sin^2\theta \sin^4\phi \quad \text{(from the y-coordinate)}$$

$$\left|\frac{\partial(y,z)}{\partial(\theta,\phi)}\right|^2 = \cos^2\theta \sin^4\phi \quad \text{(from the x-coordinate)}$$

$$\therefore \|\vec{T}_\theta \times \vec{T}_\phi\|^2 = \sin^2\phi\cos^2\phi + (\sin^2\theta + \cos^2\theta)\sin^4\phi$$

$$= \sin^2\phi(\cos^2\phi + \sin^2\phi)$$

$$= \sin^2\phi$$

$$\therefore \|\vec{T}_\theta \times \vec{T}_\phi\| = \sin\phi, \quad \text{and } \sin\phi \geq 0 \text{ for } 0 \leq \phi \leq \pi$$

$$\therefore A(S) = \iint_D \sin\phi \, d\theta \, d\phi = \int_0^{\pi}\int_0^{2\pi} \sin\phi \, d\theta \, d\phi$$

$$= 2\pi \int_0^{\pi} \sin\phi \, d\phi = 2\pi \left[-\cos\phi\right]\Big|_{\phi=0}^{\phi=\pi}$$

$$= 2\pi[1-(-1)] = \underline{4\pi}$$

2.

(a) (1) D is still an elementary region
(2) ϕ is still C^1 but is not one-to-one for points in D and not on the boundary

$\{$e.g., $\phi = \pm \frac{\pi}{6}$, $\cos(\frac{\pi}{6}) = \cos(-\frac{\pi}{6})$ and

$$\cos\theta \sin(\tfrac{\pi}{6}) = \cos(\pi+\theta)\sin(-\tfrac{\pi}{6})$$

$$\sin\theta \sin(\tfrac{\pi}{6}) = \sin(\pi+\theta)\sin(-\tfrac{\pi}{6})$$

$\therefore$ an infinite # of non-boundary points for which $\Phi(\theta,\phi)$ is not one-to-one. $\}$

However, this can be addressed by breaking up $\Phi(\theta,\phi)$ to be Φ_1 for $-\pi/2 \le \phi \le 0$, and Φ_2 for $0 \le \phi \le \pi/2$.

(3) The image is regular except at $\phi = 0$ (finite # points).

for $-\pi/2 \le \phi \le 0$, $\|\vec{T}_\theta \times \vec{T}_\phi\| = -\sin\phi$ since $\sin\phi \le 0$.

for $0 \le \phi \le \pi/2$, $\|\vec{T}_\theta \times \vec{T}_\phi\| = \sin\phi$

$$\therefore A(S) = \iint_D \|\vec{T}_\theta \times \vec{T}_\phi\| \, d\theta \, d\phi$$

$$= \int_0^{2\pi}\int_{-\pi/2}^{0} -\sin\phi \, d\theta \, d\phi + \int_0^{2\pi}\int_0^{\pi/2} \sin\phi \, d\theta \, d\phi$$

$$= -2\pi \int_{-\pi/2}^{0} \sin\phi \, d\phi + 2\pi \int_{0}^{\pi/2} \sin\phi \, d\phi$$

$$= -2\pi \Big[-\cos\phi \Big]_{-\pi/2}^{0} + 2\pi \Big[-\cos\phi \Big]_{0}^{\pi/2}$$

$$= -2\pi \big[-1 - (0) \big] + 2\pi \big[0 - (-1) \big]$$

$$= \underline{4\pi}$$

(b) $\phi(\theta, \phi)$ is not one-to-one for an infinite # of non-boundary points, as $0 \leq \phi \leq \pi$ covers the sphere once, and $\pi \leq \phi \leq 2\pi$ cover it again.

This can be addressed by letting

$$\phi(\theta,\phi) = \begin{cases} \phi_1(\theta,\phi) : 0 \leq \phi \leq \pi, \; 0 \leq \theta \leq 2\pi \\ \phi_2(\theta,\phi) : \pi \leq \phi \leq 2\pi, \; 0 \leq \theta \leq 2\pi \end{cases}$$

Note $\|\vec{T}_\theta \times \vec{T}_\phi\| = \sin\phi, \quad 0 \leq \phi \leq \pi$

$$= -\sin\phi, \quad \pi \leq \phi \leq 2\pi$$

$$\therefore A(\sigma) = \int_0^{\pi} \int_0^{2\pi} \sin\phi \, d\theta \, d\phi + \int_{\pi}^{2\pi} -\sin\phi \, d\theta \, d\phi$$

$$= 2\pi \int_0^{\pi} \sin\phi \, d\phi - 2\pi \int_{\pi}^{2\pi} \sin\phi \, d\phi$$

$$= 2\pi \Big[-\cos\phi \Big]_0^{\pi} - 2\pi \Big[-\cos\phi \Big]_{\pi}^{2\pi}$$

$$= 2\pi \big[1 - (-1) \big] - 2\pi \big[-1 - (1) \big]$$

$$= 4\pi + 4\pi = \underline{8\pi}$$

Different answers because (a) covers the sphere once, and (b) covers it twice.

3.

As in Example 2, $A(S) = \iint_D \| \vec{T}_r \times \vec{T}_\theta \| \, dr \, d\theta$

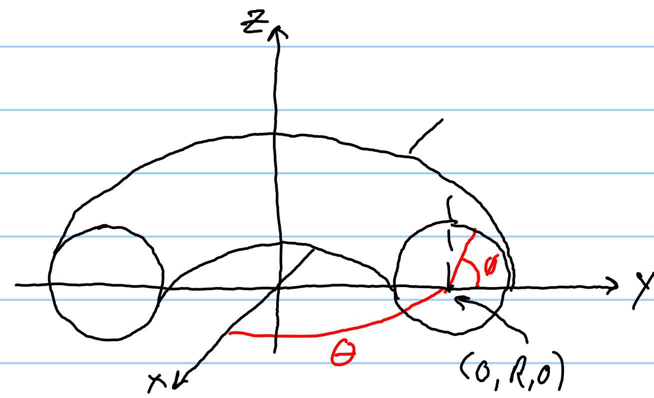
$$= \int_0^{3\pi} \int_0^1 \sqrt{r^2+1} \, dr \, d\theta = 3\pi \int_0^1 \sqrt{r^2+1} \, dr$$

From Table of Integrals # 43 from text,

$$= 3\pi \left[\frac{r}{2} \sqrt{r^2+1} + \frac{1}{2} \log \left| r + \sqrt{r^2+1} \right| \right]_{r=0}^{r=1}$$

$$= \underline{3\pi \left[\frac{1}{2} \sqrt{2} + \frac{1}{2} \log (1+\sqrt{2}) \right]}$$

4.



Radius of cross sectional circle is 1, $\therefore R > 1$.

(a) Using $\displaystyle\iint_D \sqrt{\frac{\partial(x,y)}{\partial(\theta,\phi)}^2 + \frac{\partial(y,z)}{\partial(\theta,\phi)}^2 + \frac{\partial(x,z)}{\partial(\theta,\phi)}^2}$

$$\frac{\partial(x,y)}{\partial(\theta,\phi)}^2 = \begin{vmatrix} -(R+\cos\phi)\sin\theta & -\sin\phi\cos\theta \\ (R+\cos\phi)\cos\theta & -\sin\phi\sin\theta \end{vmatrix}^2$$

$$= \left[(R+\cos\phi)\sin\phi\sin^2\theta + (R+\cos\phi)\sin\phi\cos^2\theta\right]^2$$

$$= (R+\cos\phi)^2\sin^2\phi$$

$$\frac{\partial(y,z)}{\partial(\theta,\phi)}^2 = \begin{vmatrix} (R+\cos\phi)\cos\theta & -\sin\phi\sin\theta \\ 0 & \cos\phi \end{vmatrix}^2$$

$$= (R+\cos\phi)^2\cos^2\theta\cos^2\phi$$

$$\frac{\partial(x,z)}{\partial(\theta,\phi)}^2 = \begin{vmatrix} -(R+\cos\phi)\sin\theta & -\sin\phi\sin\theta \\ 0 & \cos\phi \end{vmatrix}^2$$

$$= (R+\cos\phi)^2\sin^2\theta\cos^2\phi$$

$$\therefore \frac{\partial(x,y)}{\partial(\theta,\phi)}^2 + \frac{\partial(y,z)}{\partial(\theta,\phi)}^2 + \frac{\partial(x,z)}{\partial(\theta,\phi)}^2 =$$

$$(R+\cos\phi)^2\sin^2\phi + (R+\cos\phi)^2\cos^2\theta\cos^2\phi + (R+\cos\phi)^2\sin^2\theta\cos^2\phi$$

$$= (R+\cos\phi)^2\sin^2\phi + (R+\cos\phi)^2\cos^2\phi$$

$$= (R+\cos\phi)^2$$

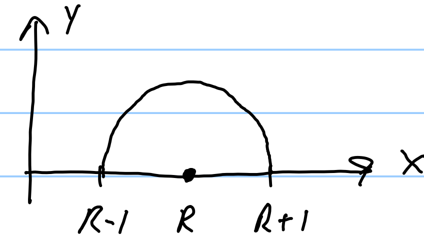
Since $R>1$, $R+\cos\phi>0$, $\therefore \sqrt{(R+\cos\phi)^2} = R+\cos\phi$

$$\therefore A(T) = \int_0^{2\pi}\int_0^{2\pi} (R+\cos\phi)\,d\theta\,d\phi$$

$$= 2\pi\int_0^{2\pi}(R+\cos\phi)\,d\phi = 2\pi\Big[R\phi + \sin\phi\Big]_{\phi=0}^{\phi=2\pi}$$

$$= \underline{\underline{4\pi^2 R}}$$

(b) Using $A = 2\pi\int_a^b |x|\sqrt{1+f'(x)^2}\,dx$



Here, $y=f(x) = \sqrt{1-(x-R)^2}$, $R>1$

$a = R-1$, $b = R+1$

$\therefore x>0$ for $x\in[R-1, R+1]$

This will give area of top half of torus.

$$f'(x) = \frac{1}{2}\left(1-(x-R)^2\right)^{-\frac{1}{2}}\left(-2(x-R)\right) = -\frac{x-R}{\sqrt{1-(x-R)^2}}$$

$$\therefore 1+f'(x)^2 = 1 + \frac{(x-R)^2}{1-(x-R)^2} = \frac{1}{1-(x-R)^2}$$

$$\therefore A(T) = 4\pi \int_{R-1}^{R+1} \frac{x}{\sqrt{1-(x-R)^2}}\,dx$$

Let $u = x-R$, $\therefore du = dx$, $x = u+R$

$x = R-1 \Rightarrow u = -1$, $x = R+1 \Rightarrow u = 1$

$$= 4\pi \int_{-1}^{1} \frac{u+R}{\sqrt{1-u^2}}\,du = 4\pi \int_{-1}^{1} \frac{u\,du}{\sqrt{1-u^2}} + 4\pi R \int_{-1}^{1} \frac{du}{\sqrt{1-u^2}}$$

$$= 4\pi \left[-\sqrt{1-u^2}\right]_{-1}^{1} + 4\pi R \left[\operatorname{Arcsin} u\right]_{-1}^{1}$$

$$= 0 + 4\pi R\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right]$$

$$= \underline{\underline{4\pi^2 R}}$$

5.

(a) $\vec{T}_u = (e^u \cos v, e^u \sin v, 0)$

$$\vec{T}_v = (-e^u \sin v,\ e^u \cos v,\ 1)$$

$$\vec{T}_u \times \vec{T}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ e^u \cos v & e^u \sin v & 0 \\ -e^u \sin v & e^u \cos v & 1 \end{vmatrix}$$

$$= \underline{(e^u \sin v,\ -e^u \cos v,\ e^{2u})}$$

(b)

$$\phi(0, \pi/2) = (0, 1, \pi/2),\quad \vec{T}_u(0,\pi/2) \times \vec{T}_v(0,\pi/2) = (1,0,1)$$

$$\therefore\ \vec{T}_u \times \vec{T}_v \cdot (x-x_0, y-y_0, z-z_0) = 0 \Rightarrow$$

$$(1,0,1)\cdot(x-0, y-1, z-\pi/2) = 0 \Rightarrow$$

$$x + z - \frac{\pi}{2} = 0$$

(c)

Note $\phi(u,v)$ is one-to-one and regular on $[0,1]\times[0,\pi]$

$$\|\vec{T}_u \times \vec{T}_v\| = \sqrt{e^{2u}\sin^2 v + e^{2u}\cos^2 v + e^{4u}} = \sqrt{e^{2u} + e^{4u}}$$

$$\therefore\ A(\phi(D)) = \int_0^1 \int_0^{\pi} \sqrt{e^{2u} + e^{4u}}\, dv\, du$$

$$= \underline{\pi \int_0^1 \sqrt{e^{2u} + e^{4u}}\, du}$$

From Wolfram: $\int_0^1 \sqrt{\exp(2x)+\exp(4x)}\,dx = \frac{1}{2}\left(-\sqrt{2} + e\sqrt{1+e^2} - \sinh^{-1}(1) + \sinh^{-1}(e)\right) \approx 3.6515$

$\therefore A(\phi(D)) \doteq \pi(3.6515)$

Note: Answer in back of text says

$\|T_u \times T_v\| = (e^u \sin v, -e^u \cos v, e^u)$

This makes $A(\phi(D)) = \int_0^1 \int_0^\pi \sqrt{e^{2u}+e^{2u}} = \sqrt{2}\pi(e-1)$

But the z-component is e^{2u}, not e^u

6.

Use the parametrization $x = r\cos\theta$, $y = r\sin\theta$, $z = r^2\cos\theta\sin\theta = \frac{1}{2}r^2\sin 2\theta$

$0 \le r \le \sqrt{2}$, $0 \le \theta \le 2\pi$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

$$\frac{\partial(x,z)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ r\sin 2\theta & r^2\cos 2\theta \end{vmatrix} = r^2(\cos\theta\cos 2\theta + \sin 2\theta\sin\theta)$$
$$= r^2(\cos^3\theta - \cos\theta\sin^2\theta + 2\sin^2\theta\cos\theta)$$
$$= r^2(\cos^3\theta + \sin^2\theta\cos\theta)$$
$$= r^2\cos\theta$$

$$\frac{\partial(y,z)}{\partial(r,\theta)} = \begin{vmatrix} \sin\theta & r\cos\theta \\ r\sin 2\theta & r^2\cos 2\theta \end{vmatrix} = r^2(\sin\theta\cos 2\theta - \cos\theta\sin 2\theta)$$

$$= r^2(\sin\theta\cos^2\theta - \sin^3\theta - 2\sin\theta\cos^2\theta)$$

$$= -r^2(\sin\theta\cos^2\theta + \sin^3\theta)$$

$$= -r^2\sin\theta$$

$$\therefore A(S) = \int_0^{2\pi}\int_0^{\sqrt{2}} \sqrt{r^2 + r^4\cos^2\theta + r^4\sin^2\theta}\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\sqrt{2}} \sqrt{r^2 + r^4}\, dr\, d\theta = 2\pi\int_0^{\sqrt{2}} r\sqrt{1+r^2}\, dr$$

$$= 2\pi\left[\left(\frac{1}{2}\right)\left(\frac{2}{3}\right)(1+r^2)^{3/2}\right]_{r=0}^{r=\sqrt{2}}$$

$$= \frac{2\pi}{3}\left[3^{3/2} - 1\right] = \underline{\frac{2\pi}{3}\left[3\sqrt{3} - 1\right]}$$

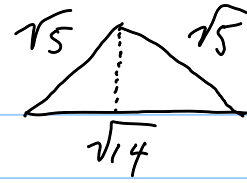
7.

(a) Classical geometry

Sides : $\sqrt{(2-1)^2 + (1-1)^2 + (2-0)^2} = \sqrt{5}$

$\sqrt{(2-2)^2 + (3-1)^2 + (3-2)^2} = \sqrt{5}$

$$\sqrt{(2-1)^2+(3-1)^2+(3-0)^2}=\sqrt{14} \quad \therefore$$



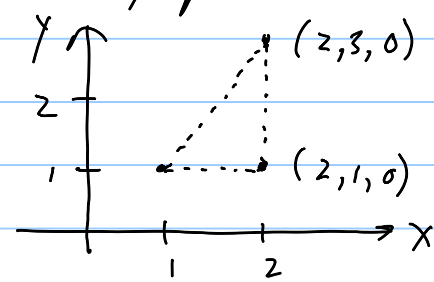
Isosceles, $\therefore$ height $=\sqrt{(\sqrt{5})^2-\left(\frac{\sqrt{14}}{2}\right)^2}=\sqrt{5-7/2}=\sqrt{3/2}$

$$\therefore \ \frac{1}{2}\sqrt{14}\left(\sqrt{3/2}\right)=\frac{1}{2}\sqrt{42/2}=\underline{\frac{1}{2}\sqrt{21}}$$

(b) The triangle projects onto the xy-plane as

$(1,1,0)$, $(2,1,0)$, $(2,3,0)$

a right triangle



$\therefore$ Can use orthogonal (independent) parameters

(u,v), $\quad 1\le u\le 2$, $\quad 1\le v\le 2u-1$

as $y=2x-1$ is the line connecting $(1,1)$ to $(2,3)$.

The surface of the triangle in R^3 is a plane connecting the 3 points. $\therefore$ $z=g(x,y)$ will be equation of this plane.

Normal to plane: $[(2,1,2)-(1,1,0)]\times[(2,3,3)-(1,1,0)]$

$$=\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 2 \\ 1 & 2 & 3 \end{vmatrix}=(-4,-1,2)$$

$\therefore$ plane is $(4,1,2)\cdot(x-1,y-1,z-0)=0$, or

$$4x + y + 2z - 5 = 0, \text{ or } z = -2x - \tfrac{1}{2}y + \tfrac{5}{2}$$

$$\therefore g(x,y) = -2x - \tfrac{1}{2}y + \tfrac{5}{2}$$

$\therefore$ Parametrization of plane is:

$$\phi(u,v) = \left(u, v, -2u - \tfrac{v}{2} + \tfrac{5}{2}\right).$$

$$\phi_u = (1, 0, -2) \qquad \phi_v = (0, 1, -\tfrac{1}{2})$$

$$\therefore \phi_u \times \phi_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -2 \\ 0 & 1 & -\frac{1}{2} \end{vmatrix} = \left(2, \tfrac{1}{2}, 1\right)$$

$$\therefore \|\phi_u \times \phi_v\| = \sqrt{2^2 + (\tfrac{1}{2})^2 + 1^2} = \sqrt{21/4} = \frac{\sqrt{21}}{2}$$

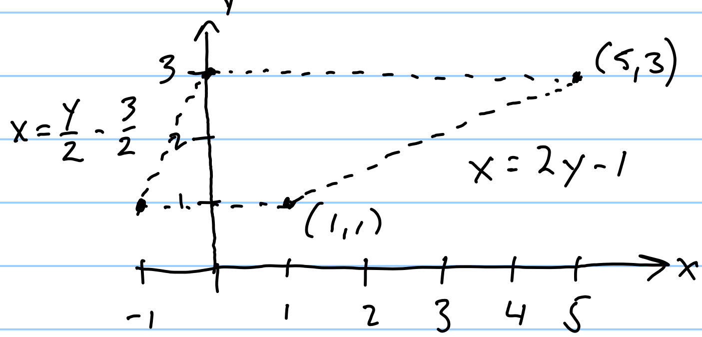
$$\therefore A(S) = \iint_D \frac{\sqrt{21}}{2}\, du\, dv = \int_1^2 \int_1^{2u-1} \frac{\sqrt{21}}{2}\, dv\, du$$

$$= \frac{\sqrt{21}}{2} \int_1^2 v \Big|_1^{2u-1} du = \frac{\sqrt{21}}{2} \int_1^2 2u - 2\, du$$

$$= \sqrt{21} \left[\tfrac{1}{2}(u-1)^2\right] \Big|_{u=1}^{u=2} = \underline{\frac{\sqrt{21}}{2}}$$

8.

(a) Projection onto xy-plane is: $(0,3,0)$ $(5,3,0)$
$(-1,1,0)$ $(1,1,0)$



Let $y=v$, $1 \le v \le 3$
$x=u$, $\frac{v}{2}-\frac{3}{2} \le u \le 2v-1$

Making x a function of y due to shape

Need expression of plane containing the quadrilateral.

Normal to plane: $[(1,1,2)-(-1,1,2)] \times [(0,3,5)-(-1,1,2)]$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & 0 \\ 1 & 2 & 3 \end{vmatrix} = (0,-6,4), \text{ or } (0,-3,2)$$

$\therefore (0,-3,2)\cdot(x-(-1), y-1, z-2) = 0$, or

$-3y + 2z - 1 = 0$, or $z = \frac{3}{2}y + \frac{1}{2}$ (parallel to x-axis)

Parametrization of surface plane $z = g(x,y) = \frac{3}{2}y + \frac{1}{2}$

is $\phi(u,v) = (u, v, \frac{3}{2}v + \frac{1}{2})$ for

$$\frac{v}{2}-\frac{3}{2} \le u \le 2v-1, \quad 1 \le v \le 3$$

$$\phi_u = (1,0,0) \qquad \phi_v = (0,1,\tfrac{3}{2})$$

$$\phi_u \times \phi_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ 0 & 1 & 3/2 \end{vmatrix} = (0, -\tfrac{3}{2}, 1)$$

$$\therefore \|\phi_u \times \phi_v\| = \sqrt{\tfrac{9}{4}+1} = \frac{\sqrt{13}}{2}$$

$$\therefore A(S) = \iint_D \|\phi_u \times \phi_v\| \, du\, dv = \int_1^3 \int_{\frac{v}{2}-\frac{3}{2}}^{2v-1} \frac{\sqrt{13}}{2} \, du\, dv$$

$$= \frac{\sqrt{13}}{2} \int_1^3 (2v-1) - \left(\frac{v}{2}-\frac{3}{2}\right) du = \frac{\sqrt{13}}{2} \int_1^3 \frac{3}{2}v + \frac{1}{2} \; dv$$

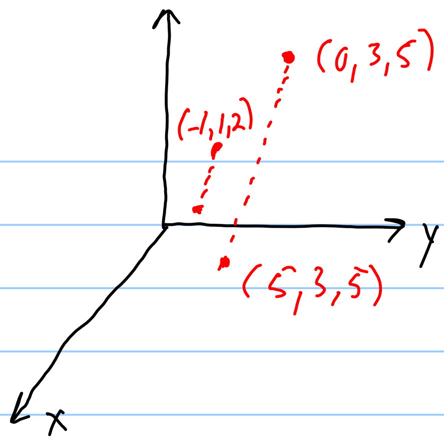
$$= \frac{\sqrt{13}}{2} \left[\frac{3}{4}v^2 + \frac{v}{2} \right]_{v=1}^{v=3}$$

$$= \frac{\sqrt{13}}{2} \left[\frac{27}{4} + \frac{3}{2} - \left(\frac{3}{4}+\frac{1}{2}\right) \right] = \frac{\sqrt{13}}{2}\left[6+1\right]$$

$$= \underline{\underline{\frac{7}{2}\sqrt{13}}}$$

(b) Classical geometry

Side lengths: $(-1,1,2)$ to $(1,1,2)$: $\sqrt{2^2+0^2+0^2} = 2$



$(1,1,2)$ to $(0,3,5)$: $\sqrt{1^2+2^2+3^2} = \sqrt{14}$

$(0,3,5)$ to $(5,3,5)$: $\sqrt{5^2+0^2+0^2} = 5$

$(5,3,5)$ to $(-1,1,2)$: $\sqrt{6^2+2^2+3^2} = 7$

Sides of length 2 & 5 are parallel:

vector $(-1,1,2)$ to $(1,1,2) = (2,0,0)$

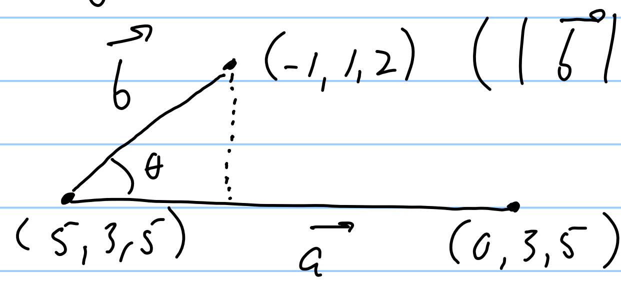
vector $(0,3,5)$ to $(5,3,5) = (5,0,0)$

$\therefore$ One is a multiple of the other.

Average of lengths $= \frac{1}{2}(2+5) = \underline{\frac{7}{2}}$

Height = distance from one side to other

Find projection of one point to other line:



$$\left(|\vec{b}|\cos\theta\right)\frac{\vec{a}}{|\vec{a}|} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\left(\frac{\vec{a}}{|\vec{a}|}\right) = \frac{\vec{a}\cdot\vec{b}}{\vec{a}\cdot\vec{a}}\,\vec{a} = \vec{p}$$

$\vec{b} = (-1,1,2)-(5,3,5) = (-6,-2,-3)$ $\quad\vec{a} = (-5,0,0)$

$$\therefore \frac{(-5,0,0)\cdot(-6,-2,-3)}{(-5,0,0)\cdot(-5,0,0)}(-5,0,0) = \frac{30}{25}(-5,0,0) =$$

$(-6, 0, 0) = \vec{p}.$

$\therefore\ \vec{b} - \vec{p}$ = perpendicular vector from point to line

$= (-6, -2, -3) - (-6, 0, 0) = (0, -2, -3)$

$\therefore$ height $= \|\vec{b} - \vec{p}\| = \sqrt{2^2 + 3^2} = \underline{\sqrt{13}}$

$\therefore$ Area = (average of parallel bases)(height)

$= \underline{\frac{7}{2}\sqrt{13}}$

9.

$D = \{(u, v) : -1 \leq u \leq 1,\ -\sqrt{1-u^2} \leq v \leq \sqrt{1-u^2}\}$

$\phi_u = (1, 1, v) \qquad \phi_v = (-1, 1, u)$

$$\phi_u \times \phi_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & v \\ -1 & 1 & u \end{vmatrix} = (u - v, -u - v, 2)$$

$$\|\phi_u \times \phi_v\| = \sqrt{(u-v)^2 + (-u-v)^2 + 2^2} = \sqrt{2u^2 + 2v^2 + 4}$$

$\therefore$

$$A(S) = \iint_D \sqrt{2u^2 + 2v^2 + 4}\ du\,dv$$

$$= \int_{-1}^{1} \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} \sqrt{2u^2+2v^2+4}\, dv\, du$$

As this is messy, let $u = r\cos\theta$, $v = r\sin\theta$,
$0 \leq \theta \leq 2\pi$, $0 \leq r \leq 1$.

Jacobian of $T(r,\theta) = (r\cos\theta, r\sin\theta)$ is $(r\,dr\,d\theta)$,

and $f(u,v) = \sqrt{2u^2+2v^2+4}$ becomes

$$f(r\cos\theta, r\sin\theta) = \sqrt{2r^2\cos^2\theta + 2r^2\sin^2\theta + 4}$$

$$= \sqrt{2r^2+4}$$

$$\therefore A(S) = \int_0^{2\pi} \int_0^1 \sqrt{2r^2+4}\; r\, dr\, d\theta$$

$$= \sqrt{2}(2\pi) \int_0^1 r\sqrt{r^2+2}\, dr$$

$$= 2\sqrt{2}\pi \left[\left(\tfrac{2}{3}\right)\left(\tfrac{1}{2}\right)(r^2+2)^{3/2} \right]_{r=0}^{r=1}$$

$$= \tfrac{2}{3}\sqrt{2}\pi \left[3\sqrt{3} - 2\sqrt{2} \right]$$

$$= 2\sqrt{6}\pi - \tfrac{8}{3}\pi$$

10.

Using spherical coordinates $T(r,\theta) = \begin{bmatrix} \cos\theta \sin\phi \\ \sin\theta \sin\phi \\ \cos\theta \end{bmatrix}$

$0 \le \theta \le 2\pi, \quad 0 \le \phi \le \frac{\pi}{4}$

as $\phi = \frac{\pi}{4}$ represents $z = \sqrt{x^2+y^2}$

From problem #1, $\| \vec{T}_\theta \times \vec{T}_\phi \| = \sin\phi$

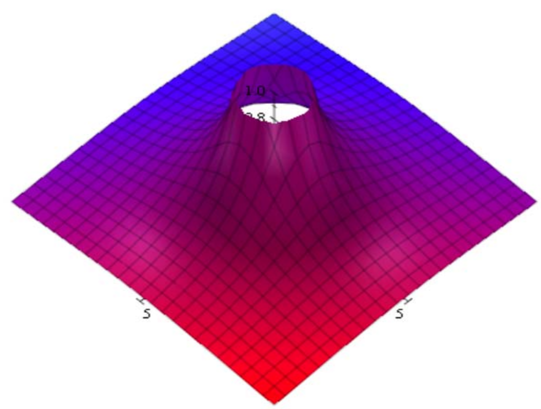
$$\therefore A(S) = \int_0^{\frac{\pi}{4}} \int_0^{2\pi} \sin\phi \, d\theta \, d\phi$$

$$= 2\pi \int_0^{\pi/4} \sin\phi \, d\phi = 2\pi \Big[-\cos\phi \Big]_0^{\pi/4}$$

$$= 2\pi \left[-\frac{\sqrt{2}}{2} - (-1) \right] = \underline{\pi (2 - \sqrt{2})}$$

11.

Basically, a horn going to ∞.
(x-axis is pointing up)



Need to show volume (surface of revolution) is finite, but surface area is infinite.

The surface of revolution is obtained by letting $y = f(x) = \frac{1}{x}$. $y^2 + z^2$ is a unit circle, and the rotation of $f(x)$ about the x-axis yields the surface $x = 1/\sqrt{y^2 + z^2}$

(a) $A(S) = 2\pi \int_1^\infty |f(x)| \sqrt{1 + f'(x)^2}\, dx$

$$= 2\pi \int_1^\infty \frac{1}{x} \sqrt{1 + \left(-\frac{1}{x^2}\right)^2}\, dx = 2\pi \int_1^\infty \sqrt{\frac{x^4+1}{x^6}}\, dx$$

$$= 2\pi \lim_{b \to \infty} \int_1^b \sqrt{\frac{x^4+1}{x^6}}\, dx$$

Note since $x > 0$, $\sqrt{\frac{x^4+1}{x^6}} > \sqrt{\frac{x^4}{x^6}} = \frac{1}{x}$

$$\therefore 2\pi \lim_{b \to \infty} \int_1^b \sqrt{\frac{x^4+1}{x^6}}\, dx \geq 2\pi \lim_{b \to \infty} \int_1^b \frac{1}{x}\, dx$$

$$= 2\pi \lim_{b \to \infty} \ln(b) = \underline{\infty}$$

$\therefore A(S) \geq 2\pi \lim_{b\to\infty} \ln(b) = \infty$, so

$A(S)$ doesn't exist

(b) Volume of revolution of $y = \frac{1}{x}$ is:

$$V = \pi \int_1^\infty f(x)^2 \, dx = \pi \int_1^\infty \frac{dx}{x^2}$$

$$= \pi \lim_{b\to\infty} \int_1^b \frac{dx}{x^2} = \pi \lim_{b\to\infty} \left[-\frac{1}{b} + 1\right] = \underline{\underline{\pi}}$$

$\therefore$ Volume exists

12.

(a) From $\cosh^2 u - \sinh^2 u = 1$, let $x = \cosh(u)$, $y = \sinh(u)$

$\therefore \phi(u,v) = (\cosh(u), \sinh(u), v)$, $0 \leq v \leq 1$.

For $-1 \leq y \leq 1$, $-1 \leq \sinh(u) \leq 1$, $-1 \leq \frac{e^u - e^{-u}}{2} \leq 1$,

$-2 \leq e^u - e^{-u} \leq 2$. Let $x = e^u \therefore e^{-u} = \frac{1}{x}$

For $e^u - e^{-u} = 2$, $x = 2 + \frac{1}{x}$, $x^2 - 2x + 1$,
$x = 1 \pm \sqrt{2}$, so $x = 1 + \sqrt{2}$ as $e^u > 0$.

$\therefore e^u = 1+\sqrt{2}, \; u = \log(1+\sqrt{2})$

For $-2 = e^u - e^{-u}$, $\frac{1}{x} - 2 = x$, $x^2 + 2x - 1$, $x = -1 \pm \sqrt{2}$,
$\therefore e^u = -1+\sqrt{2}$, $u = \log(-1+\sqrt{2})$.

$\therefore \log(-1+\sqrt{2}) \le u \le \log(1+\sqrt{2})$

$\therefore \phi(u,v) = (\cosh(u), \sinh(u), v)$, for

$\log(-1+\sqrt{2}) \le u \le \log(1+\sqrt{2})$, $0 \le v < 1$

(b) $\phi_u = (\sinh(u), \cosh(u), 0) \qquad \phi_v = (0,0,1)$

$$\phi_u \times \phi_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \sinh(u) & \cosh(u) & 0 \\ 0 & 0 & 1 \end{vmatrix} = (\cosh(u), -\sinh(u), 0)$$

$\therefore \|\phi_u \times \phi_v\| = \sqrt{\cosh^2 u + \sinh^2 u}$

$\therefore$

$$A(S) = \int_0^1 \int_{\log(-1+\sqrt{2})}^{\log(1+\sqrt{2})} \sqrt{\cosh^2 u + \sinh^2 u}\, du\, dv$$

$$= \int_{\log(-1+\sqrt{2}}^{\log(1+\sqrt{2})} \sqrt{\cosh^2 u + \sinh^2 u}\, du$$

13.

Use spherical coordinates

$T(\theta,\phi) = (a\cos\theta\sin\phi,\ b\sin\theta\sin\phi,\ c\cos\phi)$,

for $0 \le \phi \le \pi$, $0 \le \theta \le 2\pi$

Check: $\frac{(a\cos\theta\sin\phi)^2}{a^2} + \frac{(b\sin\theta\sin\phi)^2}{b^2} + \frac{(c\cos\phi)^2}{c^2}$

$= (\cos^2\theta + \sin^2\theta)\sin^2\phi + \cos^2\phi = 1$

$$\frac{\partial(x,y)}{\partial(\theta,\phi)} = \begin{vmatrix} -a\sin\theta\sin\phi & a\cos\theta\cos\phi \\ b\cos\theta\sin\phi & b\sin\theta\cos\phi \end{vmatrix}$$

$= -ab\sin^2\theta\sin\phi\cos\phi - ab\cos^2\theta\sin\phi\cos\phi$

$= -ab\sin\phi\cos\phi$

$$\frac{\partial(x,z)}{\partial(\theta,\phi)} = \begin{vmatrix} -a\sin\theta\sin\phi & a\cos\theta\cos\phi \\ 0 & -c\sin\phi \end{vmatrix}$$

$= ac\sin\theta\sin^2\phi$

$$\frac{\partial(y,z)}{\partial(\theta,\phi)} = \begin{vmatrix} b\cos\theta\sin\phi & b\sin\theta\cos\phi \\ 0 & -c\sin\phi \end{vmatrix}$$

$$= -bc\cos\theta\sin^2\phi$$

$$\therefore A(s) = \iint_D \sqrt{a^2b^2\sin^2\phi\cos^2\phi + a^2c^2\sin^2\theta\sin^4\phi + b^2c^2\cos^2\theta\sin^4\phi}$$

$$= \int_0^{\pi}\int_0^{2\pi} \sqrt{a^2b^2\sin^2\phi\cos^2\phi + a^2c^2\sin^2\theta\sin^4\phi + b^2c^2\cos^2\theta\sin^4\phi}\, d\theta\, d\phi$$

14.

Assuming $f(x)$ is C^1, for a given $\Delta x = x_{i+1} - x_i$,

by the mean value theorem, $f(x_{i+1}) - f(x_i) = f'(x^*)\cdot\Delta x$,
$x_i \le x_i^* \le x_{i+1}$, or $\Delta y = f'(x_i^*)\Delta x$

$\therefore$ The length of the curve from $f(x_i)$ to $f(x_{i+1})$

is about $\sqrt{\Delta y^2 + \Delta x^2}$ by Pythagorean Theorem.

$$\sqrt{\Delta y^2 + \Delta x^2} = \sqrt{f'(x_i^*)^2\Delta x^2 + \Delta x^2} = \left(\sqrt{f'(x_i^*)^2 + 1}\right)\Delta x$$

If revolved around the y-axis, the area of this rim is $2\pi r \ell = 2\pi |x_i^*| \sqrt{1+f'(x_i^*)^2}\,\Delta x$

Taking a Riemann sum of all of these rims,

$$\sum_{i=0}^{n-1} 2\pi |x_i^*| \sqrt{1+f'(x_i^*)^2}\,\Delta x =$$

$$\underline{\underline{2\pi \int_a^b |x| \sqrt{1+f'(x)^2}\,dx}}$$

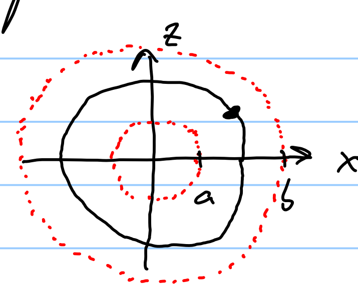
Interpreting using arc length, $ds = \sqrt{\Delta x^2 + \Delta y^2}$ above, so $ds = \sqrt{1+f'(x)^2}\,dx$. Each rim has surface area of $2\pi |x|\,ds$, where $|x|$ = radius of rim, and $2\pi|x|$ is the circumference.

$$\therefore\ 2\pi \int_a^b |x| \sqrt{1+f'(x)^2}\,dx = 2\pi \int_C |x|\,ds$$

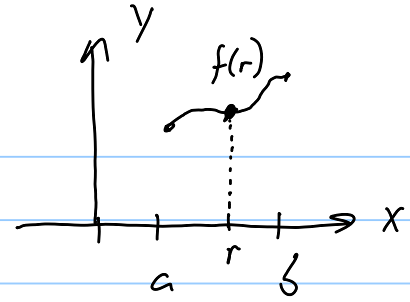
For a parametrization of the surface, rotating about y-axis creates circles parallel to xz-plane.

$\therefore$ Let $x = r\cos\theta \quad z = r\sin\theta$

$a \le r \le b, \quad 0 \le \theta \le 2\pi$



Here, $y = f(r)$



$$\therefore\ \phi(r,\theta) = (r\cos\theta,\ f(r),\ r\sin\theta)$$

$$\phi_r = (\cos\theta,\ f'(r),\ \sin\theta)$$

$$\phi_\theta = (-r\sin\theta,\ 0,\ r\cos\theta)$$

$$\phi_r \times \phi_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & f'(r) & \sin\theta \\ -r\sin\theta & 0 & r\cos\theta \end{vmatrix}$$

$$= \left(r\cos\theta\, f'(r),\ -r\sin^2\theta - r\cos^2\theta,\ r\sin\theta\, f'(r)\right)$$

$$= \left(r\cos\theta\, f'(r),\ -r,\ r\sin\theta\, f'(r)\right)$$

$$\therefore\ \|\phi_r \times \phi_\theta\| = \sqrt{r^2\cos^2\theta\, f'(r)^2 + r^2 + r^2\sin^2\theta\, f'(r)^2}$$

$$= \sqrt{r^2 + r^2 f'(r)^2} = |r|\sqrt{1 + f'(r)^2}$$

$$\therefore\ A(s) = \iint_D \|\phi_r \times \phi_\theta\|\, dr\, d\theta = \int_0^{2\pi}\int_a^b |r|\sqrt{1+f'(r)^2}\, dr\, d\theta$$

$$= 2\pi \int_a^b |r|\sqrt{1 + f'(r)^2}\, dr$$

15.

$$A(S) = 2\pi \int_0^1 x\sqrt{1+(2x)^2}\,dx \quad , \text{ as } f'(x) = 2x,\ |x| = x$$

$$= \frac{2\pi}{8}\int_0^1 8x\sqrt{1+4x^2}\,dx = \frac{\pi}{4}\left[\frac{2}{3}\left(1+4x^2\right)^{\frac{3}{2}}\right]_{x=0}^{x=1}$$

$$= \frac{\pi}{6}\left\{5^{3/2} - 1\right\} = \underline{\frac{\pi}{6}\left(5\sqrt{5}-1\right)}$$

16.

$$\phi(x,y) = \left(x, y, \sqrt{x^2+y^2}\right)$$

$$\frac{\partial z}{\partial x} = x\left(x^2+y^2\right)^{-\frac{1}{2}}$$

$$\frac{\partial z}{\partial y} = y\left(x^2+y^2\right)^{-\frac{1}{2}}$$

Note $-1 \le x \le 1$, $-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$ from Example 1, p.385 of text

$$A(S) = \iint_D \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\ dA$$

$$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} + 1}\ dy\,dx$$

$$= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{2}\, dy\, dx = \int_{-1}^{1} 2\sqrt{2}\sqrt{1-x^2}\, dx$$

$$= 2\sqrt{2} \left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\arcsin x \right]_{x=-1}^{x=1}$$

$$= 2\sqrt{2} \left[\frac{1}{2}\arcsin(1) - \frac{1}{2}\arcsin(-1) \right]$$

$$= 2\sqrt{2} \left[\frac{\pi}{4} + \frac{\pi}{4} \right] = \underline{\sqrt{2}\pi}$$

17.

An ellipse in xy-plane : $x^2 + \frac{y^2}{1/2} \le 1$

$\therefore$ let $x = r\cos\theta$, $y = \frac{r\sin\theta}{\sqrt{2}}$, $0 \le r \le 1$, $0 \le \theta \le 2\pi$

$$\therefore r^2\cos^2\theta + 2\left(\frac{r^2\sin^2\theta}{2}\right) = r^2$$

$z = 1 - x - y$.

$$\therefore \phi(r, \theta) = \left(r\cos\theta, \frac{r\sin\theta}{\sqrt{2}}, 1 - r\cos\theta - \frac{r\sin\theta}{\sqrt{2}} \right)$$

$$\phi_r = \left(\cos\theta, \frac{\sin\theta}{\sqrt{2}}, -\cos\theta - \frac{\sin\theta}{\sqrt{2}} \right)$$

$$\phi_\theta = \left(-r\sin\theta, \frac{r\cos\theta}{\sqrt{2}}, r\sin\theta - \frac{r\cos\theta}{\sqrt{2}} \right)$$

$$\phi_r \times \phi_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \frac{\sin\theta}{\sqrt{2}} & -\cos\theta - \frac{\sin\theta}{\sqrt{2}} \\ -r\sin\theta & \frac{r\cos\theta}{\sqrt{2}} & r\sin\theta - \frac{r\cos\theta}{\sqrt{2}} \end{vmatrix}$$

$$= \left(\frac{r\sin^2\theta}{\sqrt{2}} - \frac{r\sin\theta\cos\theta}{2} + \frac{r\cos^2\theta}{\sqrt{2}} + \frac{r\cos\theta\sin\theta}{2}, \right.$$

$$r\cos\theta\sin\theta + \frac{r\sin^2\theta}{\sqrt{2}} - r\cos\theta\sin\theta + \frac{r\cos^2\theta}{\sqrt{2}},$$

$$\left. \frac{r\cos^2\theta}{\sqrt{2}} + \frac{r\sin^2\theta}{\sqrt{2}} \right)$$

$$= \left(\frac{r}{\sqrt{2}}, \frac{r}{\sqrt{2}}, \frac{r}{\sqrt{2}} \right)$$

$$\therefore \|\phi_r \times \phi_\theta\| = \sqrt{\frac{r^2}{2} + \frac{r^2}{2} + \frac{r^2}{2}} = r\sqrt{\frac{3}{2}}$$

$$\therefore A(S) = \int_0^{2\pi}\int_0^1 \sqrt{\frac{3}{2}}\, r \, dr\, d\theta = \sqrt{\frac{3}{2}}\, 2\pi \int_0^1 r\, dr$$

$$= \sqrt{\frac{3}{2}}(2\pi)\left[\frac{r^2}{2}\right]_{r=0}^{r=1} = \pi\sqrt{\frac{3}{2}} = \underline{\frac{\pi}{2}\sqrt{6}}$$

18.

$$T(u,v) = (x(u,v), y(u,v), z(u,v))$$

$$\vec{T}_u = (x_u, y_u, z_u), \quad \vec{T}_v = (x_v, y_v, z_v).$$

$$\vec{T}_u \times \vec{T}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix}$$

$$= (y_u z_v - y_v z_u,\ x_v z_u - x_u z_v,\ x_u y_v - x_v y_u)$$

$$= \left(\begin{vmatrix} y_u & y_v \\ z_u & z_v \end{vmatrix},\ - \begin{vmatrix} x_u & x_v \\ z_u & z_v \end{vmatrix},\ \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} \right)$$

$$= \left(\frac{\partial(y,z)}{\partial(u,v)},\ -\frac{\partial(x,z)}{\partial(u,v)},\ \frac{\partial(x,y)}{\partial(u,v)} \right)$$

$$\therefore \|\vec{T}_u \times \vec{T}_v\| = \sqrt{\frac{\partial(x,y)^2}{\partial(u,v)^2} + \frac{\partial(x,z)^2}{\partial(u,v)} + \frac{\partial(y,z)^2}{\partial(u,v)}}$$

19.

For x and y, this is $y = 2x$. Projection onto xy-plane is the segment $-1 \le x \le 1$, $-2 \le y \le 2$ with slope $= 2$. Initial point $(\theta = 0, r = 1)$ is $(1, 2, 0)$. Point at $\theta = \frac{\pi}{2}$ is $(0, 0, \frac{\pi}{2})$. A straight line equation would be $(1,2,0) + t\left(-1,-2,\frac{\pi}{2}\right)$, $0 \le t \le 1$

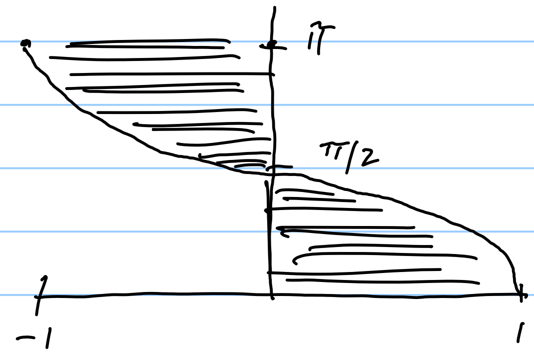
For $\theta = \frac{\pi}{3}$ $(r=1)$, the point is $\left(\frac{1}{2}, 1, \frac{\pi}{3}\right)$

But for $t = \frac{1}{2}$, the point on the line is $\left(\frac{1}{2}, 1, \frac{\pi}{4}\right)$.

$\therefore$ The actual image is a curve.

An en-face view of the surface for $0 \le \theta \le \pi$

is:

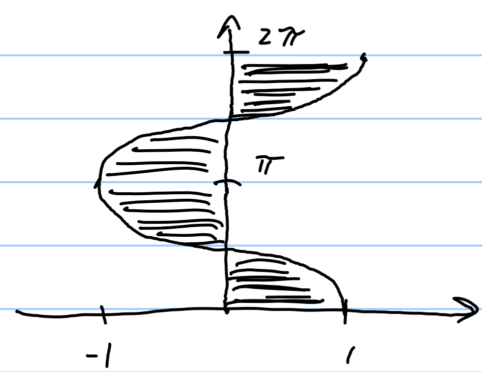


En-face view, looking at plane of $y = 2x$.

Note that $0 \le r \le 1$ means for a given $z = \theta$, x and y vary between the z-axis and the respective values for x and y.

i.e., the "rays" fan out from the z-axis, not the point $(0,0,0)$.

$\therefore$ For $0 \le \theta \le 2\pi$:



$$\vec{T}(r,\theta) = (r\cos\theta, 2r\cos\theta, \theta)$$

$\therefore \vec{T}_r = (\cos\theta, 2\cos\theta, 0) \quad \vec{T}_\theta = (-r\sin\theta, -2r\sin\theta, 1)$

$$\therefore \vec{T}_r \times \vec{T}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & 2\cos\theta & 0 \\ -r\sin\theta & -2r\sin\theta & 1 \end{vmatrix}$$

$$= (2\cos\theta, -\cos\theta, 0)$$

$$\therefore \|\vec{T}_r \times \vec{T}_\theta\| = \sqrt{5\cos^2\theta} = \sqrt{5}\,|\cos\theta|$$

$$\therefore A(S) = \int_0^1 \int_0^{2\pi} \sqrt{5}\,|\cos\theta|\, d\theta\, dr$$

$\cos\theta \geq 0: \ 0 \leq \theta \leq \frac{\pi}{2}$

$\frac{3}{2}\pi \leq \theta \leq 2\pi$

$\cos\theta \leq 0: \ \frac{\pi}{2} \leq \theta \leq \frac{3}{2}\pi$

$$= \sqrt{5}\int_0^{\pi/2} \cos\theta\, d\theta - \sqrt{5}\int_{\frac{\pi}{2}}^{\frac{3}{2}\pi} \cos\theta\, d\theta + \sqrt{5}\int_{\frac{3}{2}\pi}^{2\pi} \cos\theta\, d\theta$$

$$= \sqrt{5}\left[\sin\theta\Big|_0^{\frac{\pi}{2}} - \sin\theta\Big|_{\frac{\pi}{2}}^{\frac{3}{2}\pi} + \sin\theta\Big|_{\frac{3}{2}\pi}^{2\pi} \right]$$

$$= \sqrt{5}\left[1 - (-1-1) + (0-(-1)) \right]$$

$$= \sqrt{5}\left[1 + 2 + 1\right] = \underline{4\sqrt{5}}$$

Note: $D = \{r, \theta : 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi\}$ is elementary

$T(r,\theta)$ is one-to-one and C^1

$T(D)$ is regular except at $\theta = \frac{\pi}{2}, \frac{3}{2}\pi$

20.

Let $f(x,y,z) = x$ (i.e., the x-coordinate of a point).

From section 7.1, Exercises 16-19, the average value of $f(x,y,z)$ along $\vec{c}$ is $\dfrac{\int_{\vec{c}} f(x,y,z)\,ds}{\ell(\vec{c})}$

and $\ell(\vec{c}) = \int_{\vec{c}} \|\vec{c}\,'(t)\|\,dt$

Let $\vec{c}(t) = [x(t), y(t)]$, and making $f(x,y,z) = x$ be the x-coordinates of $\vec{c}(t)$, then

$$\int_{\vec{c}} f(x,y,z)\,ds = \int_a^b x(t)\,\|\vec{c}\,'(t)\|\,dt$$ from section 7.1.

Note $x(t) \geq 0$ for $\vec{c}(t)$ since $\vec{c}$ is in the right half of the xy-plane.

$$\therefore\ \bar{x} = \frac{\int_{\vec{c}} x\,ds}{\int_{\vec{c}} \|\vec{c}\,'(t)\|\,dt} = \frac{\int_a^b x(t)\,\|\vec{c}\,'(t)\|\,dt}{\int_a^b \|\vec{c}\,'(t)\|\,dt}$$

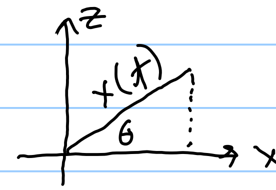
$$\therefore \bar{x} \int_a^b \|\vec{c}\,'(t)\|\,dt = \int_a^b x(t)\|\vec{c}\,'(t)\|\,dt$$

$$\therefore 2\pi \bar{x} \int_a^b \|\vec{c}\,'(t)\|\,dt = 2\pi \int_a^b x(t)\,\|\vec{c}\,'(t)\|\,dt, \text{ or}$$

$$2\pi \bar{x}\, \ell(\vec{c}) = 2\pi \int_a^b x(t)\,\|\vec{c}\,'(t)\|\,dt \qquad [1]$$

A parametrization of $\vec{c}(t)$ rotating around the y-axis is $\phi(t,\theta) = (x(t)\cos\theta, y(t), x(t)\sin\theta)$

$a \le t \le b$, $0 \le \theta \le 2\pi$,



θ rotates in xz-plane.

$$\therefore \phi_t = (x'\cos\theta, y', x'\sin\theta) \quad \phi_\theta = (-x\sin\theta, 0, x\cos\theta)$$

$$\therefore \phi_t \times \phi_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x'\cos\theta & y' & x'\sin\theta \\ -x\sin\theta & 0 & x\cos\theta \end{vmatrix}$$

$$= (xy'\cos\theta, -xx'\sin^2\theta - xx'\cos^2\theta, xy'\sin\theta)$$

$$= (xy'\cos\theta, -xx', xy'\sin\theta)$$

$$\therefore \|\phi_t \times \phi_\theta\| = \sqrt{x^2(y')^2\cos^2\theta + x^2(x')^2 + x^2(y')^2\sin^2\theta}$$

$$= \sqrt{x(t)^2 y'(t)^2 + x(t)^2 x'(t)^2}$$

$$= x(t)\sqrt{x'(t)^2 + y'(t)^2} = x(t)\,\|\vec{c}\,'(t)\|$$

Note $\sqrt{x(t)^2} = x(t)$ since in right half of xy-plane

$$\therefore A(S) = \iint_D \|\phi_t \times \phi_\theta\|\, dA = \int_a^b \int_0^{2\pi} x(t)\,\|\vec{c}\,'(t)\|\, dt$$

$$= 2\pi \int_a^b x(t)\,\|\vec{c}\,'(t)\|\, dt$$

$\therefore$ [1] becomes

$$\underline{2\pi \bar{x}\, l(\vec{c})} = \underline{2\pi \int_a^b x(t)\,\|\vec{c}\,'(t)\|\, dt} = \underline{A(S)}$$

21.

$x^2 + y^2 = x \Rightarrow x^2 - x + y^2 = 0 \Rightarrow (x - \frac{1}{2})^2 + y^2 = \frac{1}{4}$, a circle

centered at $(\frac{1}{2}, 0)$, radius $\frac{1}{2}$

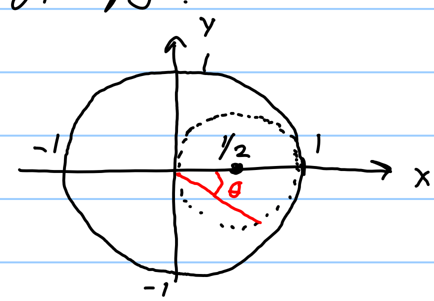
Unit sphere has surface area 4π.

$\therefore A(S_1) + A(S_2) = 4\pi$, $\therefore \dfrac{A(S_2)}{A(S_1)} = \dfrac{4\pi - A(S_1)}{A(S_1)}$

$\therefore$ Just find $A(S_1)$. Using symmetry, just find top half of $A(S_1)$.

Top half of sphere is $z = \sqrt{1-x^2-y^2}$

Need to find a parametrization of D.



Let $x = r\cos\theta$, $y = r\sin\theta$

$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

From $x^2+y^2 = x$, $r^2\cos^2\theta + r^2\sin^2\theta = r\cos\theta$,

or $r = \cos\theta$ at the rim of the inscribed circle.

$\therefore$ $0 \le r \le \cos\theta$.

$$z = \sqrt{1-(r\cos\theta)^2-(r\sin\theta)^2} = \sqrt{1-r^2}$$

$$\therefore \phi(r,\theta) = \left(r\cos\theta,\ r\sin\theta,\ \sqrt{1-r^2}\right)$$

$-\pi/2 \le \theta \le \pi/2$, $0 \le r \le \cos\theta$

$$\therefore \phi_r = \left[\cos\theta,\ \sin\theta,\ -r(1-r^2)^{-\frac{1}{2}}\right]$$

$$\phi_\theta = \left[-r\sin\theta,\ r\cos\theta,\ 0\right]$$

$$\therefore \phi_r \times \phi_\theta = \left[r^2(1-r^2)^{-\frac{1}{2}}\cos\theta,\ r^2(1-r^2)^{-\frac{1}{2}}\sin\theta,\ r\right]$$

$$\therefore \|\phi_r \times \phi_\theta\| = \left[\frac{r^4\cos^2\theta}{(1-r^2)} + \frac{r^4\sin^2\theta}{(1-r^2)} + r^2 \right]^{1/2}$$

$$= \left[\frac{r^4 + r^2 - r^4}{(1-r^2)} \right]^{1/2} = \frac{r}{\sqrt{1-r^2}}$$

$$\therefore A(S) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\cos\theta} \frac{r}{\sqrt{1-r^2}} \, dr \, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} -(1-r^2)^{\frac{1}{2}} \Big|_0^{\cos\theta} d\theta$$

$\sqrt{1-\cos^2\theta} = \sin\theta, \ 0 \le \theta \le \pi/2$

$\sqrt{1-\cos^2\theta} = -\sin\theta \quad -\pi/2 \le \theta \le 0$

$$= \int_0^{\pi/2} -\sin\theta - (-1) \, d\theta + \int_{-\pi/2}^{0} \sin\theta - (-1) \, d\theta$$

$$= \cos\theta + \theta \Big|_0^{\frac{\pi}{2}} + \left(-\cos\theta + \theta \Big|_{-\pi/2}^{0} \right)$$

$$= \left(0 + \frac{\pi}{2}\right) - (1+0) + \left[(-1+0) - \left(0 - \frac{\pi}{2}\right) \right]$$

$$= \frac{\pi}{2} - 1 + \left(-1 + \frac{\pi}{2}\right) = \pi - 2$$

$$\therefore A(S_1) = 2(\pi - 2) = 2\pi - 4 \qquad 4\pi - A(S_1) = 2\pi + 4$$

$$\therefore \frac{A(S_2)}{A(S_1)} = \frac{4\pi - A(S_1)}{A(S_1)} = \frac{2\pi + 4}{2\pi - 4} = \underline{\frac{\pi + 2}{\pi - 2}}$$

22.

$$F(x,y,z) = z - f(x,y) = 0.$$

$$\therefore F_x = -f_x \quad F_y = -f_y \quad F_z = 1$$

$$\therefore (F_x)^2 = (-f_x)^2, \quad (F_y)^2 = (-f_y)^2, \quad (F_z)^2 = 1$$

$$\therefore \therefore \nabla F \cdot \nabla F = f_x^2 + f_y^2 + 1$$

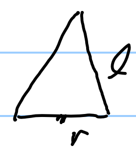
$$\therefore A(s) = \iint_D \sqrt{f_x^2 + f_y^2 + 1}\, dA$$

$$= \iint_D \sqrt{\nabla F \cdot \nabla F}\, dA$$

$$= \underline{\iint_D \|\nabla F\|\, dA}$$

23.

(a) Surface area of the slanted side of a cone (i.e., ignoring base area) is $\pi r l$, where r = radius of circular base,



l = length of slant height

$\therefore$ For this frustrum, $\pi r_2 (l+s) - \pi r_1 l$ =

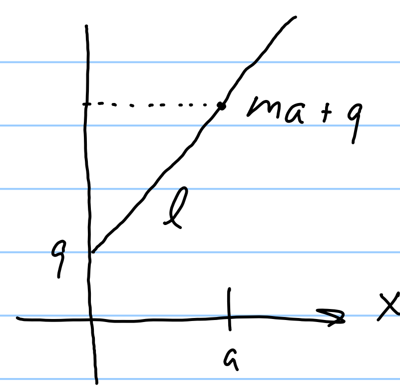
area cone with base r_2, slant height $l+s$
- area cone with base r_1, slant height l

$= \pi l (r_2 - r_1) + \pi r_2 s = \pi l (b-a) + \pi b s$

Here, $s^2 = (b-a)^2 + [(mb+q)-(ma+q)]^2$

$= (b-a)^2 + m^2(b-a)^2 = (b-a)^2(1+m^2)$

$\therefore s = (b-a)\sqrt{m^2+1}$



$l^2 = [(ma+q)-q]^2 + a^2$

$= m^2a^2 + a^2 = a^2(m^2+1)$

$\therefore l = a\sqrt{m^2+1}$

$\therefore$ Area of frustrum $= \pi(a\sqrt{m^2+1})(b-a) + \pi b(b-a)\sqrt{m^2+1}$

$= \underline{\underline{\pi(b-a)\sqrt{m^2+1}\,(a+b)}} = \underline{\underline{\pi(b^2-a^2)\sqrt{m^2+1}}}$

(b) This is a surface of revolution about y-axis.

$$\therefore \; A = 2\pi \int_a^b |x| \sqrt{1 + f'(x)^2}\, dx$$

$|x| = x$ since $0 \le a < b$, $f(x) = mx + q \Rightarrow f'(x) = m$

$$\therefore \; A = 2\pi \int_a^b x\sqrt{1+m^2}\, dx = 2\pi\sqrt{1+m^2}\left[\frac{x^2}{2}\right]_{x=a}^{x=b}$$

$$= \underline{\pi\sqrt{1+m^2}\,(b^2 - a^2)} = \underline{\pi\sqrt{1+m^2}\,(b+a)(b-a)}$$

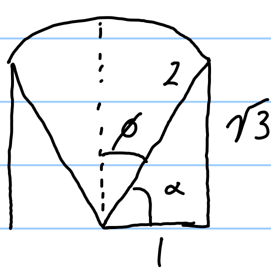
24.

(a) Volume

Volume of sphere $= \frac{4}{3}\pi(2)^3 = \frac{32}{3}\pi$

Volume of bored cylinder:

$\cos\alpha = \frac{1}{2} \Rightarrow \alpha = \frac{\pi}{3}$, $\phi = \pi/6$



Use spherical coordinates, find volume of upper half of bored cylinder.

Volume of "ice cream cone" part is:

$$\int_0^{2\pi}\int_0^{\pi/6}\int_0^2 \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta =$$

$$\int_0^{2\pi} d\theta \int_0^{\frac{\pi}{6}} \sin\phi \, d\phi \int_0^2 \rho^2 \, d\rho = 2\pi\left[-\cos\phi\right]_0^{\frac{\pi}{6}} \left[\frac{\rho^3}{3}\right]_0^2$$

$$= 2\pi\left[-\frac{\sqrt{3}}{2} - (-1)\right]\left[\frac{8}{3}\right] = \underline{\frac{8}{3}\pi(2-\sqrt{3})}$$

Volume of side part:

Note $\rho\sin\phi = 1$ for $\frac{\pi}{6} \leq \phi \leq \frac{\pi}{2}$

$$\therefore \int_0^{2\pi} \int_{\pi/6}^{\pi/2} \int_0^{\frac{1}{\sin\phi}} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= 2\pi \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin\phi \left.\frac{\rho^3}{3}\right|_{\rho=0}^{\rho=\frac{1}{\sin\phi}} d\phi = \frac{2\pi}{3} \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{d\phi}{\sin^2\phi}$$

$$= \frac{2\pi}{3}\left[-\cot\phi\right]_{\pi/6}^{\pi/2} = \frac{2\pi}{3}\left[0 - (-\sqrt{3})\right] = \underline{\frac{2\sqrt{3}}{3}\pi}$$

$\therefore$ Volume of top half of bore =

$$\frac{8}{3}\pi(2-\sqrt{3}) + \frac{2\sqrt{3}}{3}\pi = \frac{\pi}{3}(16 - 8\sqrt{3} + 2\sqrt{3})$$

$$= \frac{\pi}{3}(16 - 6\sqrt{3}) = \frac{2\pi}{3}(8 - 3\sqrt{3})$$

$\therefore$ Volume of bored cylinder $= \frac{4}{3}\pi(8-3\sqrt{3})$

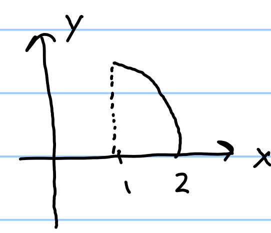
$\therefore$ Volume of coupler $= \frac{32}{3}\pi - \frac{4}{3}\pi(8-3\sqrt{3})$

$$= \frac{32}{3}\pi - \frac{32}{3}\pi + 4\sqrt{3}\,\pi = \underline{4\sqrt{3}\,\pi}$$

Cylindrical coordinates may have been easier

(b) Surface area of outer coupler

View surface as a circular arc rotated about y-axis. $x^2+y^2=4 \Rightarrow y=\sqrt{4-x^2}$, $1 \le x \le 2$



Just compute top half of coupler.

$$y' = \tfrac{1}{2}(4-x^2)^{-\frac{1}{2}}(-2x) = \frac{x}{\sqrt{4-x^2}}$$

$$\therefore A(S) = 2\pi \int_1^2 x\sqrt{1+\left(\frac{x}{\sqrt{4-x^2}}\right)^2}\,dx$$

$$= 2\pi \int_1^2 x\sqrt{\frac{4-x^2+x^2}{4-x^2}}\,dx = 4\pi \int_1^2 \frac{x}{\sqrt{4-x^2}}\,dx$$

$$= 4\pi\left[-(4-x^2)^{\frac{1}{2}}\right]_{x=1}^{x=2} = 4\pi\left[0-(-\sqrt{3})\right]$$

$$= 4\sqrt{3}\,\pi$$

$\therefore$ Coupler surface area $= \underline{8\sqrt{3}\,\pi}$

25.

$f_x = x^{\frac{1}{2}} \qquad f_y = y^{\frac{1}{2}}$

$$A(S) = \iint_D \sqrt{f_x^2 + f_y^2 + 1}\, dA = \int_0^1 \int_0^1 \sqrt{x+y+1}\, dx\, dy$$

$$= \int_0^1 \frac{2}{3}(x+y+1)^{3/2} \Big|_{x=0}^{x=1} dy$$

$$= \int_0^1 \frac{2}{3}(y+2)^{3/2} - \frac{2}{3}(y+1)^{\frac{3}{2}}\, dy$$

$$= \left(\frac{2}{3}\right)\left(\frac{2}{5}\right)(y+2)^{5/2} \Big|_{y=0}^{y=1} - \left(\frac{2}{3}\right)\left(\frac{2}{5}\right)(y+1)^{5/2} \Big|_{y=0}^{y=1}$$

$$= \frac{4}{15}\left[3^{5/2} - 2^{5/2}\right] - \frac{4}{15}\left[2^{5/2} - 1\right]$$

$$= \frac{4}{15}\left[9\sqrt{3} - 4\sqrt{2} - 4\sqrt{2} + 1\right]$$

$$= \underline{\frac{4}{15}\left(9\sqrt{3} - 8\sqrt{2} + 1\right)}$$

26.

$$A(S) = \iint_D \sqrt{f_x^2 + f_y^2 + 1}\, dA$$

(a) $f_x = 2(x+2y)$ $\quad f_y = 4(x+2y)$

$f_x^2 = 4(x+2y)^2$ $\quad f_y^2 = 16(x+2y)^2$

$$\therefore A(S) = \int_{-1}^{2}\int_{0}^{2} \sqrt{20(x+2y)^2}\, dy\, dx$$

$$= 2\sqrt{5} \int_{-1}^{2}\int_{0}^{2} \sqrt{(x+2y)^2}\, dy\, dx$$

(b) $f_x = y + \frac{1}{y+1} = \frac{y^2+y+1}{y+1}$ $\quad f_y = x - \frac{x}{(y+1)^2} = \frac{x(y^2+2y)}{(y+1)^2}$

$f_x^2 = \frac{(y^2+y+1)^2}{(y+1)^2}$ $\quad f_y^2 = \frac{x^2(y^2+2y)^2}{(y+1)^4}$

$$f_x^2 + f_y^2 + 1 = \frac{(y^2+y+1)^2(y+1)^2 + x^2(y^2+2y)^2 + (y+1)^4}{(y+1)^4}$$

$$\therefore A(S) = \int_{1}^{4}\int_{1}^{2} \frac{\sqrt{(y^2+y+1)^2(y+1)^2 + x^2(y^2+2y)^2 + (y+1)^4}}{(y+1)^2}\, dy\, dx$$

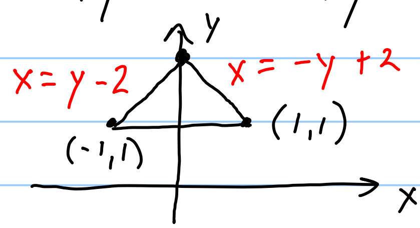
(c) $f_x = y^3 e^{x^2y^2} + 2x^2y^5 e^{x^2y^2} = (y^3+2x^2y^5)e^{x^2y^2}$

$f_y = 3xy^2 e^{x^2y^2} + 2x^3y^4 e^{x^2y^2} = (3xy^2+2x^3y^4)e^{x^2y^2}$

$$\therefore A(S) = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1+[(y^3+2x^2y^5)^2+(3xy^2+2x^3y^4)]e^{2x^2y^2}}\,dy\,dx$$

(d) $f_x = -2y^3\cos x\sin x \qquad f_y = 3y^2\cos^2 x$

$1+f_x^2+f_y^2 = 1+4y^6\cos^2 x\sin^2 x+9y^4\cos^4 x$

$D =$ 

an x-simple region

$$\therefore A(S) = \int_{1}^{2}\int_{y-2}^{-y+2} \sqrt{1+4y^6\cos^2 x\sin^2 x+9y^4\cos^4 x}\,dx\,dy$$

27.

$z_x = -x(R^2-x^2-y^2)^{-\frac{1}{2}} \qquad z_y = -y(R^2-x^2-y^2)^{-\frac{1}{2}}$

$$1+z_x^2+z_y^2 = 1+\frac{x^2}{R^2-x^2-y^2}+\frac{y^2}{R^2-x^2-y^2} = \frac{R^2}{R^2-x^2-y^2}$$

$D =$ circle in xy-plane of radius R, center $(0,0)$.

$$\therefore A(s) = \int_{-1}^{1} \int_{-\sqrt{R^2-x^2}}^{\sqrt{R-x^2}} \frac{R}{\sqrt{R^2-x^2-y^2}} \, dy\, dx$$

Switch to polar coordinates: $x = r\cos\theta$, $y = r\sin\theta$

Jacobian = $r\,dr\,d\theta$, $0 \le r \le R$, $0 \le \theta \le 2\pi$

$$R^2 - x^2 - y^2 = R^2 - r^2\cos^2\theta - r^2\sin^2\theta = R^2 - r^2$$

$$\therefore A(s) = R \int_0^{2\pi} \int_0^{R} \frac{r}{\sqrt{R^2-r^2}} \, dr\, d\theta$$

$$= 2\pi R \lim_{x \to R^-} \int_0^{x} \frac{r}{\sqrt{R^2-r^2}} \, dr$$

$$= 2\pi R \lim_{x \to R^-} \left[-(R^2-r^2)^{\frac{1}{2}} \right]_{r=0}^{r=x}$$

$$= 2\pi R \lim_{x \to R^-} \left[-\sqrt{R^2-x^2} - (-R) \right]$$

$$= 2\pi R \left[0 + R \right] = \underline{2\pi R^2}$$

7.5 Integrals of Scalar Functions Over Surfaces

Note Title 3/15/2017

1.

$$\phi_u = (2\cos v, 2\sin v, 1) \qquad \phi_v = (-2u\sin v, 2u\cos v, 0)$$

$$\phi_u \times \phi_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2\cos v & 2\sin v & 1 \\ -2u\sin v & 2u\cos v & 0 \end{vmatrix}$$

$$= (-2u\cos v, -2u\sin v, 4u\cos^2 v + 4u\sin^2 v)$$

$$= (-2u\cos v, -2u\sin v, 4u)$$

$$\|\phi_u \times \phi_v\| = \sqrt{4u^2 + 16u^2} = 2\sqrt{5}\,u$$

$$f(x(u,v), y(u,v), z(u,v)) = 2u\cos v + 2u\sin v$$

$$\therefore \iint_S f\,dS = \int_0^4 \int_0^\pi (2u\cos v + 2u\sin v)\,2\sqrt{5}\,u\,dv\,du$$

$$= \int_0^4 4\sqrt{5}\,u^2\,du \int_0^\pi (\cos v + \sin v)\,dv$$

$$= \left(4\sqrt{5}\,\frac{u^3}{3}\Big|_0^4\right)\left(\sin v - \cos v\Big|_0^\pi\right)$$

$$= \left(\frac{256\sqrt{5}}{3}\right)(2) = \frac{512\sqrt{5}}{3}$$

2.

$$f(x(u,v), y(u,v), z(u,v)) = v + 6$$

$$\phi_u \times \phi_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ 0 & 1/3 & 1 \end{vmatrix} = (0, -1, 1/3)$$

$$\therefore \|\phi_u \times \phi_v\| = \sqrt{1 + 1/9} = \sqrt{10}/3$$

$$\therefore \iint_S f\,dS = \int_0^2 \int_0^3 \frac{\sqrt{10}}{3}(v+6)\,dv\,du$$

$$= \frac{2\sqrt{10}}{3}\left(\frac{v^2}{2} + 6v\right)\Big|_{v=0}^{v=3} = \frac{2\sqrt{10}}{3}\left(\frac{9}{2} + 18\right)$$

$$= 15\sqrt{10}$$

3.

Normal to the plane $= (2,3,1)$. Unit $\vec{n} = \frac{1}{\sqrt{14}}(2,3,1)$.

$\therefore \vec{n} \cdot \vec{k} = \cos\theta = \frac{1}{\sqrt{14}}$. The plane is $z = g(x,y) = 6 - 2x - 3y$

$$\therefore \iint_S f\,ds = \iint_D \frac{f(x,y,g(x,y))}{\cos\theta}\,dx\,dy$$

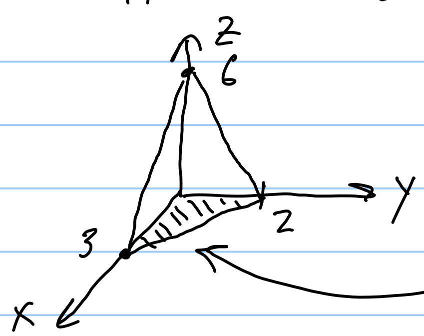
$$= \iint_D \sqrt{14}\left[3x - 2y + (6-2x-3y)\right] dx\,dy$$

$$= \iint_D \sqrt{14}\,(x - 5y + 6)\,dx\,dy$$

For first octant: $2x + 3(0) + (0) = 6 \Rightarrow x = 3$

$2(0) + 3y + (0) = 6 \Rightarrow y = 2$

$2(0) + 3(0) + z = 6 \Rightarrow z = 6$



$y = -\frac{2}{3}x + 2$

$\therefore D = \{(x,y) : 0 \le x \le 3,\ 0 \le y \le -\frac{2}{3}x + 2\}$

$$\therefore \sqrt{14}\int_0^3 \int_0^{-\frac{2}{3}x+2} (x - 5y + 6)\,dy\,dx = \sqrt{14}\int_0^3 xy - \frac{5}{2}y^2 + 6y \Big|_0^{-\frac{2}{3}x+2} dx$$

$$= \sqrt{14}\int_0^3 -\frac{2}{3}x^2 + 2x - \frac{5}{2}\left(-\frac{2}{3}x+2\right)^2 + 6\left(-\frac{2}{3}x+2\right) dx$$

$-4x + 12$

$$= \sqrt{14} \int_0^3 -\frac{2}{3}x^2 - 2x + 12 - \frac{5}{2}\left(-\frac{2}{3}x+2\right)^2 dx$$

$$= \sqrt{14}\left[-\frac{2}{9}x^3 - x^2 + 12x - \frac{5}{2}\left(\frac{1}{3}\right)\left(-\frac{3}{2}\right)\left(-\frac{2}{3}x+2\right)^3\right]_{x=0}^{x=3}$$

$$= \sqrt{14}\left[-6 - 9 + 36 - 0 - \left(0 - 0 + 0 + \frac{5}{4}(8)\right)\right]$$

$$= \sqrt{14}\left[21 - 10\right] = \underline{11\sqrt{14}}$$

4.

Let $\phi(x,\theta) = (x, 2\cos\theta, 2\sin\theta)$ be a parametrization of the cylinder, $0 \le x \le 5$, $0 \le \theta \le 2\pi$

$$\phi_x \times \phi_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ 0 & -2\sin\theta & 2\cos\theta \end{vmatrix} = (0, -2\cos\theta, -2\sin\theta)$$

$$\therefore \|\phi_x \times \phi_\theta\| = 2$$

$$\therefore \iint_S (x+z)\,dS = \int_0^5 \int_0^{2\pi} (x + 2\sin\theta)(2)\,d\theta\,dx$$

$$= \int_0^5 2x\theta - 4\cos\theta \Big|_0^{2\pi} dx$$

$$= \int_0^5 (4\pi x - 4) - (0-4)\, dx = \int_0^5 4\pi x\, dx = 2\pi x^2 \Big|_0^5$$

$$= \underline{50\pi}$$

5.

(a) $x = u+v,\ y = u-v,\ x^2 = u^2 + 2uv + v^2,\ y^2 = u^2 - 2uv + v^2$

$\therefore x^2 - y^2 = 4uv = 4z$

$\therefore$ every point of $\phi(u,v)$ is a point on $4z = x^2 - y^2$

Also, given (x,y,z), let $u = \frac{x+y}{2}$, $v = \frac{x-y}{2}$

$\therefore u+v = x$, $u-v = y$, and $uv = z$. $\therefore$ Every point of the graph is in the image of ϕ.

(b) $\phi_u \times \phi_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & v \\ 1 & -1 & u \end{vmatrix} = (u+v,\ v-u,\ -2)$

$\therefore \|\phi_u \times \phi_v\| = \sqrt{2u^2 + 2v^2 + 4}$

For $x^2 + y^2 \leq 1$, $(u+v)^2 + (u-v)^2 \leq 1$, or $u^2 + v^2 \leq \frac{1}{2}$

$\therefore$ D is a circular disc of radius $\frac{1}{\sqrt{2}}$

$$\iint_S x \, dS = \iint_D (u+v)\sqrt{2u^2+2v^2+4} \, dA$$

Make a polar substition : $0 \leq r \leq \frac{1}{\sqrt{2}}$, $0 \leq \theta \leq 2\pi$

$u = r\cos\theta$, $v = r\sin\theta$

$\therefore \sqrt{2u^2+2v^2+4} = \sqrt{2r^2+4}$

$$\therefore \iint_S x \, dS = \int_0^{2\pi}\int_0^{\frac{1}{\sqrt{2}}} (r\cos\theta + r\sin\theta)\sqrt{2r^2+4} \, r \, dr \, d\theta$$

$$= \sqrt{2}\int_0^{2\pi}\int_0^{\frac{1}{\sqrt{2}}} (\cos\theta + \sin\theta) r^2\sqrt{r^2+2} \, dr \, d\theta$$

$$= \sqrt{2}\int_0^{2\pi} (\cos\theta + \sin\theta) \, d\theta \int_0^{\frac{1}{\sqrt{2}}} r^2\sqrt{r^2+2} \, dr$$

$$= \sqrt{2} \left(\sin\theta - \cos\theta\right)\Big|_0^{2\pi} \int_0^{\frac{1}{\sqrt{2}}} r^2\sqrt{r^2+2} \, dr$$

$$= \sqrt{2}\left[0-1-(0-1)\right]\int_0^{\frac{1}{\sqrt{2}}} r^2\sqrt{r^2+2}\,dr$$

$$= 0\int_0^{\frac{1}{\sqrt{2}}} r^2\sqrt{r^2+2}\,dr = \underline{\underline{0}}$$

6.

Normal to plane is $(1,1,-1)$, unit $\vec{n} = \frac{1}{\sqrt{3}}(-1,-1,1)$

$\therefore \cos\theta = \vec{n}\cdot\hat{k} = \frac{1}{\sqrt{3}}$

$$\therefore \iint_S z(x^2+y^2)\,dS = \sqrt{3}\iint_D z(x^2+y^2)\,dA$$

Now make a polar conversion: $x = r\cos\theta$, $y = r\sin\theta$,

$0 \le r \le 2$, $0 \le \theta \le 2\pi$, $z = 4 + r\cos\theta + r\sin\theta$

$$\therefore \sqrt{3}\int_0^{2\pi}\int_0^2 (4 + r\cos\theta + r\sin\theta)(r^2)\,r\,dr\,d\theta$$

$$= \sqrt{3}\int_0^{2\pi} 4r^3 + (\cos\theta + \sin\theta)r^4 \Big|_0^2 \,d\theta$$

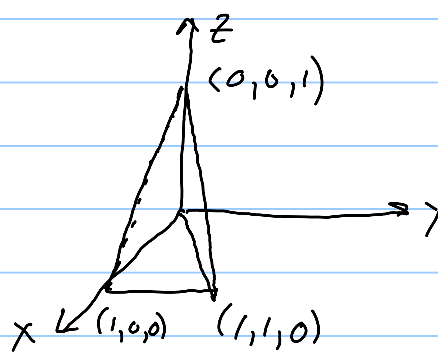
$$= \sqrt{3} \int_0^{2\pi} 32 + 16(\cos\theta + \sin\theta)\, d\theta$$

$$= \sqrt{3} \left[32\theta + 16(\sin\theta - \cos\theta) \right]_0^{2\pi}$$

$$= 64\sqrt{3}\pi + 16\sqrt{3}\left[0 - 1 - (0-1)\right]$$

$$= \underline{64\sqrt{3}\pi}$$

7.



Since 4 sides, this is the sum of 4 surface integrals:

(1) xy-plane (3) $x+z=1$ plane

(2) xz-plane (4) $x=y$ plane

Note $f(x,y,z) = xy$

(1) xy-plane Here, $\vec{n} = \hat{k}$. $\therefore \cos\theta = \vec{n}\cdot\hat{k} = 1$

$$\therefore \iint_S f(x,y,z)\,dS = \iint_D \frac{f(x,y,g(x,y))}{\cos\theta}\,dx\,dy$$

$$= \iint_D f(x,y,0)\,dx\,dy = \iint_D xy\,dx\,dy$$

$D = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le x\}$

$$\therefore \int_0^1 \int_0^x xy\, dy\, dx = \int_0^1 x \frac{y^2}{2} \Big|_{y=0}^{y=x} dx = \int_0^1 \frac{x^3}{2} dx$$

$$= \frac{x^4}{8} \Big|_0^1 = \underline{\frac{1}{8}}$$

(2) xz-plane Here $f(x,y,z) = 0$ as $y=0$, so $xy = 0$

$$\therefore \iint_S f(x,y,z)\, dS = \underline{0}$$

(3) $x+z=1$ plane Here, $\vec{n} = \frac{1}{\sqrt{2}}(1,0,1)$.

$$\therefore \vec{n} \cdot \hat{k} = \frac{1}{\sqrt{2}} = \cos\theta$$

$$\therefore \iint_S f(x,y,z)\, dS = \iint_D \frac{f(x,y,g(x,y))}{\cos\theta}\, dx\, dy$$

$$= \sqrt{2} \iint_D xy\, dx\, dy$$

D is the projection of this plane onto the xy-plane, and so is $D = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le x\}$

$$\therefore \sqrt{2}\int_0^1\int_0^x xy\,dy\,dx = \underline{\sqrt{2}\left(\tfrac{1}{8}\right)} \quad \text{from } (1)$$

(4) $x = y$ plane. A parametrization of this side is $\phi(u,v) = (u,u,v)$, $0 \le u \le 1$, $0 \le v \le 1-u$ as the upper boundary of the z component comes from the $x+z=1$ plane, or $z = 1-x$.

$$\therefore \phi_u \times \phi_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = (1,-1,0)$$

$$\therefore \|\phi_u \times \phi_v\| = \sqrt{2} \qquad f(x(u,v), y(u,v), z(u,v)) = u^2$$

$$\therefore \iint_S f\,dS = \int_0^1\int_0^{1-u} u^2(\sqrt{2})\,dv\,du$$

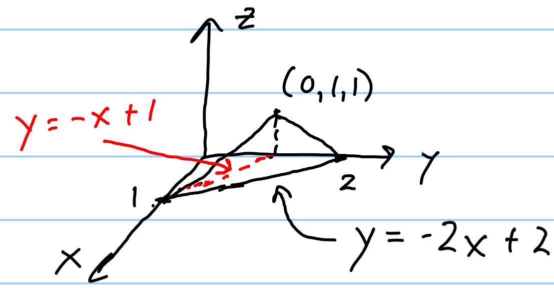
$$= \int_0^1 \sqrt{2}u^2 v\Big|_{v=0}^{v=1-u} du = \int_0^1 \sqrt{2}u^2(1-u)\,du$$

$$= \sqrt{2}\left[\frac{u^3}{3} - \frac{u^4}{4}\right]_0^1 = \sqrt{2}\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{\sqrt{2}}{12}$$

$$\therefore (1)+(2)+(3)+(4) = \frac{1}{8} + 0 + \frac{\sqrt{2}}{8} + \frac{\sqrt{2}}{12} = \frac{3}{24} + \frac{3\sqrt{2}}{24} + \frac{2\sqrt{2}}{24}$$

$$= \frac{5\sqrt{2}+3}{24}$$

8.



Projection of triangle onto the xy-plane is bounded below by $y=-x+1$ and above by $y=-2x+2$, $0 \le x \le 1$.

$\therefore D = \{(x,y) : 0 \le x \le 1,\ -x+1 \le y \le -2x+2\}$

A normal to the triangular surface is:

$$[(0,1,1)-(1,0,0)] \times [(0,1,1)-(0,2,0)]$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & -1 \\ 0 & 1 & -1 \end{vmatrix} = (2,1,1)$$

$\therefore$ plane of triangle is $2x+y+z=K$, so choose a point of triangle (in plane),

$\therefore 2(1)+(0)+(0)=K,\ K=2.$

$\therefore 2x+y+z=2$, so $z=2-2x-y$

$\therefore$ unit normal $= \frac{1}{\sqrt{6}}(2,1,1) = \vec{n}$

$\therefore \vec{n}\cdot\hat{k} = \frac{1}{\sqrt{6}} = \cos\theta$

$$\therefore \iint_S xyz\, dS = \iint_D \frac{xyz}{\cos\theta}\, dx\,dy = \sqrt{6} \iint_D xyz\, dx\,dy$$

$$= \sqrt{6} \int_0^1 \int_{-x+1}^{-2x+2} xy\,(2-2x-y)\, dy\, dx$$

$$= \sqrt{6} \int_0^1 \int_{-x+1}^{-2x+2} 2xy - 2x^2y - xy^2\, dy\,dx$$

$$= \sqrt{6} \int_0^1 (x-x^2)y^2 - \frac{xy^3}{3} \Bigg|_{y=-x+1}^{y=-2x+2} dx$$

$$= \sqrt{6} \int_0^1 (x-x^2)\Big[\overbrace{4x^2-8x+4-(x^2-2x+1)}^{3x^2-6x+3}\Big]$$

$$- x\Big[\overbrace{-8x^3+24x^2-24x+8-(-x^3+3x^2-3x+1)}^{-7x^3+21x^2-21x+7}\Big]\, dx$$

$$= \sqrt{6} \int_0^1 \Big[3x^3-6x^2+3x-3x^4+6x^3-3x^2+(7x^4-21x^3+21x^2-7x)\Big]\, dx$$

$$= \sqrt{6} \int_0^1 4x^4-12x^3+12x^2-4x\, dx$$

$$= \sqrt{6} \left[\frac{4}{5}x^5 - 3x^4 + 4x^3 - 2x^2\right]_0^1$$

$$= \sqrt{6}\left(\frac{4}{5} - 3 + 4 - 2\right) = -\frac{\sqrt{6}}{5}$$

9.

Use spherical coordinates: $x = a\cos\theta\sin\phi$, $y = a\sin\theta\sin\phi$,
$z = a\cos\phi$

$0 \le \theta \le 2\pi$, $0 \le \phi \le \pi/2$

$\therefore \vec{T}(\theta,\phi) = (a\cos\theta\sin\phi, a\sin\theta\sin\phi, a\cos\phi)$

describes the hemisphere

$$\vec{T}_\theta \times \vec{T}_\phi = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a\sin\theta\sin\phi & a\cos\theta\sin\phi & 0 \\ a\cos\theta\cos\phi & a\sin\theta\cos\phi & -a\sin\phi \end{vmatrix}$$

$$= (-a^2\cos\theta\sin^2\phi, -a^2\sin\theta\sin^2\phi, -a^2\sin^2\theta\sin\phi\cos\phi - a^2\cos^2\theta\sin\phi\cos\phi)$$

$$= (-a^2\cos\theta\sin^2\phi, -a^2\sin\theta\sin^2\phi, -a^2\sin\phi\cos\phi)$$

$$\therefore \|\vec{T}_\theta \times \vec{T}_\phi\| = \sqrt{a^4\cos^2\theta\sin^4\phi + a^4\sin^2\theta\sin^4\phi + a^4\sin^2\phi\cos^2\phi}$$

$$= a^2\sqrt{\sin^4\phi + \sin^2\phi\cos^2\phi} = a^2\sqrt{\sin^2\phi}$$

$$= a^2\sin\phi$$

$$\therefore \iint_S z\,dS = \int_0^{2\pi}\int_0^{\pi/2} (a\cos\phi)a^2\sin\phi\,d\phi\,d\theta$$

$$= 2\pi a^3 \int_0^{\frac{\pi}{2}} \cos\phi \sin\phi \, d\phi = 2\pi a^3 \left[\frac{\sin^2\phi}{2} \right]_{\phi=0}^{\phi=\pi/2}$$

$$= \underline{\underline{\pi a^3}}$$

10.

Should be 0: Octant I $(x,y,z>0)$ cancels octant where $x,y,z<0$. Similarly for other octants.

To check: use spherical coordinates.

$$T(\theta,\phi) = (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi), \quad 0 \le \theta \le 2\pi, \quad 0 \le \phi \le \pi$$

From #9 above, $\|\vec{T}_\theta \times \vec{T}_\phi\| = \sin\phi$

$$\therefore \iint_S (x+y+z)\, dS = \int_0^{2\pi} \int_0^{\pi} (\cos\theta \sin\phi + \sin\theta \sin\phi + \cos\phi) \sin\phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi} (\cos\theta + \sin\theta) \sin^2\phi + \cos\phi \sin\phi \; d\phi \, d\theta$$

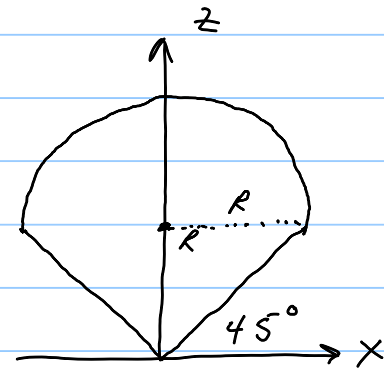
$$= \int_0^{2\pi} (\cos\theta + \sin\theta)\, d\theta \int_0^{\pi} \sin^2\phi \, d\phi + \int_0^{2\pi} \int_0^{\pi} \cos\phi \sin\phi \, d\phi \, d\theta$$

$$= (\sin\theta - \cos\theta)\Big|_0^{2\pi} \int_0^{\pi} \sin^2\phi \, d\phi + 2\pi \left[\frac{\sin^2\phi}{2} \right]_{\phi=0}^{\phi=\pi}$$

$$= \left[0 - 1 - (0-1)\right] \int_0^{\pi} \sin^2\phi \, d\phi + 2\pi(0)$$

$$= 0 + 0 = \underline{\underline{0}}$$

11.



(a) $x^2 + y^2 + z^2 - 2Rz = x^2 + y^2 + (z-R)^2 = R^2$

$\therefore$ The sphere has center $(0,0,R)$ and radius R.

Choose polar coordinates. The cone can be described as $\phi(r,\theta) = (r\cos\theta, r\sin\theta, r)$

since the cone makes a 45° angle with z-axis

$0 \le r \le R$, $0 \le \theta \le 2\pi$. Cone + sphere meet at (x, y, R)

$$\phi_r \times \phi_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \left(-r\cos\theta, -r\sin\theta, r(\cos^2\theta + \sin\theta)\right)$$

$$= (-r\cos\theta, -r\sin\theta, r)$$

$\therefore \|\phi_r \times \phi_\theta\| = \sqrt{r^2 + r^2} = \sqrt{2}\, r$

$$\therefore A(S) = \int_0^{2\pi}\int_0^R \sqrt{2}\, r\, dr\, d\theta = 2\sqrt{2}\pi \left[\frac{r^2}{2}\right]_0^R$$

$$= \underline{\underline{\sqrt{2}\pi R^2}}$$

(b) It's just the top half of a sphere of radius R

$$= \frac{4\pi R^2}{2} = \underline{\underline{2\pi R^2}}$$

12.

As in problem #9, sphere of radius R can be described as $T(\theta, \phi) = (R\cos\theta\sin\phi,\ R\sin\theta\sin\phi,\ R\cos\phi)$ using spherical coordinates.

$$\therefore \vec{T}_\theta \times \vec{T}_\phi = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -R\sin\theta\sin\phi & R\cos\theta\sin\phi & 0 \\ R\cos\theta\cos\phi & R\sin\theta\cos\phi & -R\sin\phi \end{vmatrix}$$

$$= \left(-R^2\cos\theta\sin^2\phi,\ -R^2\sin\theta\sin^2\phi,\ -R^2\sin^2\theta\sin\phi\cos\phi - R^2\cos^2\theta\sin\phi\cos\phi\right)$$

$$= (-R^2\cos\theta\sin^2\phi, -R^2\sin\theta\sin^2\phi, -R^2\sin\phi\cos\phi)$$

$$\therefore \|\vec{T}_\theta \times \vec{T}_\phi\| = \sqrt{R^4\cos^2\theta\sin^4\phi + R^4\sin^2\theta\sin^4\phi + R^4\sin^2\phi\cos^2\phi}$$

$$= R^2\sqrt{\sin^4\phi + \sin^2\phi\cos^2\phi} = R^2\sqrt{\sin^2\phi}$$

$$= R^2\sin\phi$$

$$\therefore \|\vec{T}_\theta \times \vec{T}_\phi\|\, d\phi\, d\theta = \underline{R^2\sin\phi\, d\phi d\theta}$$

13.

$z = x^2+y^2$ is a paraboloid, $x^2+y^2 \le 1$ means within unit disc in xy-plane.

Use polar coordinates. A parametrization of the paraboloid is $\phi(r,\theta) = (r\cos\theta, r\sin\theta, r^2)$

since $x^2+y^2 = (r\cos\theta)^2 + (r\sin\theta)^2 = r^2 = z$

$0 \le \theta \le 2\pi$, $0 \le r \le 1$ as $r^2 = x^2+y^2 \le 1 \Rightarrow r \le 1$.

$$\therefore \phi_r \times \phi_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & 2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = (-2r^2\cos\theta, -2r^2\sin\theta, r)$$

as $r\cos^2\theta + r\sin^2\theta = r$

$$\therefore \|\phi_r \times \phi_\theta\| = \sqrt{4r^4\cos^2\theta + 4r^4\sin^2\theta + r^2}$$

$$= \sqrt{4r^4 + r^2} = r\sqrt{4r^2+1}$$

$$\therefore \iint_S z\,dS = \int_0^{2\pi}\int_0^1 (r^2)\, r\sqrt{4r^2+1}\; dr\,d\theta$$

$$= 2\pi \int_0^1 r^3\sqrt{4r^2+1}\;dr$$

Let $s = 2r$, $s^2 = 4r^2$, $r^3 = \frac{s^3}{8}$
$r=0 \Rightarrow s=0$, $r=1 \Rightarrow s=2$
$ds = 2dr$

$$= 2\pi\left(\frac{1}{16}\right)\int_0^2 s^3\sqrt{s^2+1}\;ds$$

Now use table of integrals #69 from text

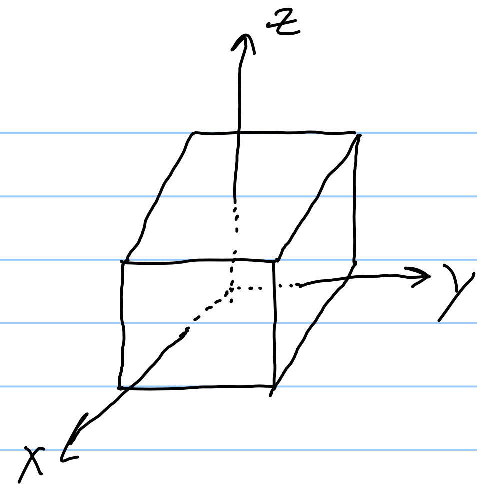
$$= \frac{\pi}{8}\left[\left(\frac{s^2}{5} - \frac{2}{15}\right)\sqrt{(1+s^2)^3}\,\right|_{s=0}^{s=2}$$

$$= \frac{\pi}{8}\left[\left(\frac{4}{5} - \frac{2}{15}\right)\sqrt{125} - \left(-\frac{2}{15}\right)\sqrt{1}\,\right]$$

$$= \frac{\pi}{8}\left[\frac{4}{5}\,5\sqrt{5} - \frac{2}{15}\,5\sqrt{5} + \frac{2}{15}\right] = \frac{\pi}{8}\left[4\sqrt{5} - \frac{2}{3}\sqrt{5} + \frac{2}{15}\right]$$

$$= \frac{\pi}{4}\left[2\sqrt{5} - \frac{\sqrt{5}}{3} + \frac{1}{15}\right] = \frac{\pi}{4}\left[\frac{25\sqrt{5}+1}{15}\right]$$

14.



By symmetry $\iint_S z^2 dS$ is the same over the top surface as for the bottom surface.

Each side is also the same and z^2 is the same over each side.

$\therefore$ Only need to compute $\iint_S z^2 dS$ for the top surface and one side surface.

Answer = 4 (side surface) + 2 (top surface).

For top surface, $\phi(x,y) = (x,y,1)$, $-1 \le x \le 1$, $-1 \le y \le 1$.

$$\therefore \phi_x \times \phi_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = (0,0,1).$$

$\therefore \|\phi_x \times \phi_y\| = 1$. Note $z^2 = 1$.

$$\therefore \text{Top Surface} = \iint_S z^2 \|\phi_x \times \phi_y\| \, dx\,dy = \int_{-1}^{1}\int_{-1}^{1} dx\,dy = \underline{4}$$

For a side surface, use $\phi(y,z) = (1,y,z)$,

$-1 \le y \le 1$, $-1 \le z \le 1$.

$$\therefore \phi_y \times \phi_z = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = (1,0,0).$$

$$\therefore \|\phi_y \times \phi_z\| = 1.$$

$$\therefore \text{Side surface} = \iint_S z^2 \|\phi_y \times \phi_z\| \, dy \, dz$$

$$= \int_{-1}^{1}\int_{-1}^{1} z^2 \, dy \, dz = 2\int_{-1}^{1} z^2 \, dz = 2\left[\frac{z^3}{3}\right]_{z=-1}^{z=1}$$

$$= 2\left[\frac{1}{3} - \left(-\frac{1}{3}\right)\right] = \underline{\frac{4}{3}}$$

$$\therefore 2(4) + 4\left(\frac{4}{3}\right) = 8 + \frac{16}{3} = \underline{13\tfrac{1}{3}} = \frac{40}{3}$$

15.

Using symmetry, assume $(x_0, y_0, z_0) = (0,0,R)$,

$\therefore m(x,y,z) = \sqrt{x^2 + y^2 + (z-R)^2}$, since $(x_0,y_0,z_0) \in S$.

Use spherical coordinates for S: $\vec{T}(\theta, \phi) =$

$(R\cos\theta\sin\phi,\ R\sin\theta\sin\phi,\ R\cos\phi),\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi$

$\therefore$ From problem # 12, $\|\vec{T}_\theta \times \vec{T}_\phi\| = R^2\sin\phi$

$m(R\cos\theta\sin\phi,\ R\sin\theta\sin\phi,\ R\cos\phi) =$

$$\sqrt{R^2(\cos^2\theta + \sin^2\theta)\sin^2\phi + (R\cos\phi - R)^2}$$

$$= \sqrt{R^2\sin^2\phi + R^2\cos^2\phi - 2R^2\cos\phi + R^2}$$

$$= \sqrt{2R^2 - 2R^2\cos\phi} = \sqrt{2}\,R\sqrt{1-\cos\phi}$$

$$\therefore\ M = \iint_S m(x,y,z)\,dS = \iint_D m(\theta,\phi)\|\vec{T}_\theta \times \vec{T}_\phi\|\,d\phi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \sqrt{2}\,R\sqrt{1-\cos\phi}\,(R^2\sin\phi)\,d\phi\,d\theta$$

$$= \sqrt{2}\,R^3(2\pi)\left(\tfrac{2}{3}\right)(1-\cos\phi)^{3/2}\Big|_{\phi=0}^{\phi=\pi}$$

$$= \sqrt{2}\,\tfrac{4}{3}\pi R^3\left[(1+1)^{3/2} - (1-1)^{3/2}\right]$$

$$= \sqrt{2}\,\tfrac{4}{3}\pi R^3(2\sqrt{2}) = \underline{\tfrac{16}{3}\pi R^3}$$

16.

Using spherical coordinates, $0 \le \theta \le 2\pi$, $0 \le \phi \le \frac{\pi}{2}$,

$\vec{T}(\theta, \phi) = [R\cos\theta\sin\phi, R\sin\theta\sin\phi, R\cos\phi]$.

$\|\vec{T}_\theta \times \vec{T}_\phi\| = R^2 \sin\phi$.

$$m(x,y,z) \Rightarrow m(\theta,\phi) = (R\cos\theta\sin\phi)^2 + (R\sin\theta\sin\phi)^2$$

$$= R^2\sin^2\phi$$

$$\therefore M = \iint_S m(x,y,z)\, dS = \iint_D m(\theta,\phi) \|\vec{T}_\theta \times \vec{T}_\phi\|\, d\theta\, d\phi$$

$$= \int_0^{2\pi} \int_0^{\pi/2} (R^2\sin^2\phi)(R^2\sin\phi)\, d\phi\, d\theta$$

$$= 2\pi R^4 \int_0^{\pi/2} \sin^3\phi\, d\phi = 2\pi R^4 \int_0^{\frac{\pi}{2}} (1-\cos^2\phi)\sin\phi\, d\phi$$

$$= 2\pi R^4 \int_0^{\pi/2} \sin\phi\, d\phi + 2\pi R^4 \int_0^{\pi/2} \cos^2\phi\,(-\sin\phi)\, d\phi$$

$$= 2\pi R^4 \left[-\cos\phi\right]_0^{\pi/2} + 2\pi R^4 \left[\frac{\cos^3\phi}{3}\right]_0^{\pi/2}$$

$$= 2\pi R^4 \left[0-(-1)\right] + 2\pi R^4 \left[0 - \tfrac{1}{3}\right]$$

$$= 2\pi R^4 \left[1 - \tfrac{1}{3}\right] = \underline{\tfrac{4}{3}\pi R^4}$$

17.

(a) The summation of a function solely along an axis is the same irrespective of the chosen axis because of the symmetry of the sphere relative to any axis.

(b)

$$x^2 + y^2 + z^2 = R^2$$

$$\therefore \iint_S (x^2+y^2+z^2)\,dS = \iint_S R^2\,ds = R^2 \iint_S ds$$

$$= R^2 (4\pi R^2) = 4\pi R^4$$

$$\iint_S (x^2+y^2+z^2)\,ds = \iint_S x^2\,ds + \iint_S y^2\,ds + \iint_S z^2\,ds$$

$$= 3\iint_S x^2\,ds$$

$$\therefore\ 4\pi R^4 = \iint_S (x^2+y^2+z^2)\,ds = 3\iint_S x^2\,ds$$

$$\therefore \iint_S x^2\,ds = \underline{\tfrac{4}{3}\pi R^4}$$

(c)

In #16, integrating only over half a sphere.
If integrating over entire sphere,

$$\text{Mass} = \iint_S (x^2+y^2)\,ds = \iint_S x^2\,ds + \iint_S y^2\,ds$$

$$= 2\iint_S x^2\,ds = 2\left(\tfrac{4}{3}\pi R^4\right)$$

From symmetry, mass of whole sphere = twice mass of hemisphere.

$$\therefore \text{Hemisphere mass} = \tfrac{1}{2}(2)\left(\tfrac{4}{3}\pi R^4\right) = \underline{\tfrac{4}{3}\pi R^4}$$

So, yes, this does help in #16.

18.

If $\phi(u,v)$ is a paramtrization of S, $\phi: D \to S$, and R_{ij} is a small rectangle in D, $\phi(R_{ij}) = S_{ij}$ a portion of S corresponding to R_{ij} through ϕ,
Then the total sum of $f(x,y,z)$ over S is

$$S_n = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} f(\phi(u_i, v_j)) A(S_{ij}), \text{ where}$$

$A(S_{ij})$ = area of S_{ij}. The approximation of the total sum, S_n, becomes better as $n \to \infty$.

If $A(S)$ = total surface area, an approximation of the average of $f()$ over S is: $\frac{1}{A(S)}[S_n]$.

$\therefore$ As $n \to \infty$, the average of $f()$ over S is

$$\frac{1}{A(S)} \iint_S f(x,y,z)\,dS \quad \text{since} \quad \lim_{n\to\infty} S_n = \iint_S f(x,y,z)\,dS.$$

(b)

From Example 3, p. 395 of text, $\iint_S z^2\,dS = \frac{4\pi}{3}$ where S is a unit sphere.

But $A(S) = 4\pi(1)^2 = 4\pi$.

$\therefore$ Ave value $= \frac{4\pi}{3} \Big/ A(S) = \frac{4\pi}{3} \Big/ 4\pi = \frac{1}{3}$

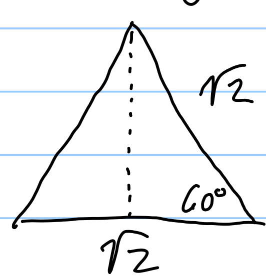
(c)

i.e., $\bar{x} = \dfrac{\iint_S x\,dS}{A(S)}$

From Example 4, p. 397 of text, $\iint_S x\,dS = \frac{\sqrt{3}}{6}$

The triangle is an equilateral triangle with side length $= \sqrt{(1-0)^2 + (0-1)^2 + (0-0)^2} = \sqrt{2}$

$\therefore$



Height $= \sqrt{2}\sin 60° = \sqrt{2}\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{6}}{2}$

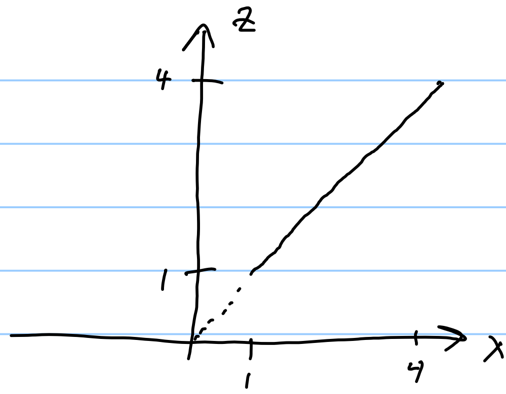
$\therefore$ Area $= \frac{1}{2}(\sqrt{2})\left(\frac{\sqrt{6}}{2}\right) = \frac{\sqrt{12}}{4}$

$\therefore\ \bar{x} = \frac{\sqrt{3}}{6} \Big/ \frac{\sqrt{12}}{4} = \frac{4}{6}\frac{\sqrt{3}}{\sqrt{12}} = \frac{2}{3}\sqrt{\frac{1}{4}} = \frac{2}{3}\cdot\frac{1}{2} = \frac{1}{3}$

By symmetry, $\bar{y} = \frac{1}{3}$, $\bar{z} = \frac{1}{3}$. $\therefore (\bar{x}, \bar{y}, \bar{z}) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$

19.

$$\text{Ave} = \frac{\iint_S (x + z^2)\, dS}{A(S)}$$



(a) From geometry, or using problem #23 of section 7.4, p. 392 of text, $A(S) = \pi(b^2 - a^2)\sqrt{m^2+1}$

$= \pi(16-1)\sqrt{1+1} = \underline{15\sqrt{2}\,\pi}$, as $b = 4$, $a = 1$, $m = 1$.

(b) Find parametrization of S using cylindrical coordinates.

$x = r\cos\theta$, $y = r\sin\theta$, $z = z$, $0 \le \theta \le 2\pi$, $1 \le z \le 4$,

$x^2 + y^2 = r^2 = z^2$, $\therefore r = z$.

$\therefore \phi(\theta, z) = (z\cos\theta, z\sin\theta, z)$

$$\therefore \phi_\theta \times \phi_z = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -z\sin\theta & z\cos\theta & 0 \\ \cos\theta & \sin\theta & 1 \end{vmatrix} = (z\cos\theta,\ z\sin\theta,\ -z\sin^2\theta - z\cos^2\theta)$$

$$= (z\cos\theta, z\sin\theta, -z)$$

$$\therefore \|\phi_\theta \times \phi_z\| = \sqrt{z^2 + z^2} = \sqrt{2}\, z$$

$$\therefore \iint_S (x+z^2)\,dS = \int_0^{2\pi}\int_1^4 (z\cos\theta + z^2)(\sqrt{2}\,z)\,dz\,d\theta$$

$$= \sqrt{2}\int_0^{2\pi}\int_1^4 (z^2\cos\theta + z^3)\,dz\,d\theta = \sqrt{2}\int_0^{2\pi}\left[\cos\theta\,\frac{z^3}{3} + \frac{z^4}{4}\right]_{z=1}^{4} d\theta$$

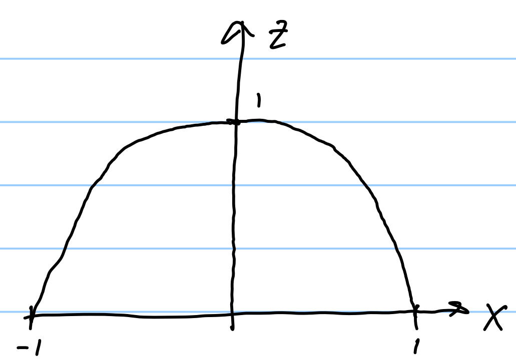
$$= \sqrt{2}\int_0^{2\pi}\left[\frac{64}{3}\cos\theta + \frac{256}{4} - \left(\frac{\cos\theta}{3} + \frac{1}{4}\right)\right) d\theta$$

$$= \sqrt{2}\int_0^{2\pi}\left(21\cos\theta + \frac{255}{4}\right)d\theta = \sqrt{2}\left[21\sin\theta + \frac{255}{4}\theta\right]_0^{2\pi}$$

$$= \sqrt{2}\left[0 + \frac{255}{2}\pi - (0+0)\right] = \underline{\frac{255\sqrt{2}}{2}\pi}$$

(c) $\text{Avc} = \dfrac{\frac{255\sqrt{2}}{2}\pi}{15\sqrt{2}\,\pi} = \underline{\dfrac{17}{2}}$

20.



A paraboloid. Use cylindrical coordinates

Let $x = r\cos\theta$, $y = r\sin\theta$, $z = 1-(x^2+y^2) = 1-r^2$

$0 \leq \theta \leq 2\pi,\ 0 \leq r \leq 1.$

Let $\phi(r,\theta) = (r\cos\theta, r\sin\theta, 1-r^2)$

$$\therefore\ \phi_r \times \phi_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & -2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = (2r^2\cos\theta,\ 2r^2\sin\theta,\ r\cos^2\theta + r\sin^2\theta)$$

$$= (2r^2\cos\theta,\ 2r^2\sin\theta,\ r)$$

$$\|\phi_r \times \phi_\theta\| = \sqrt{4r^4 + r^2} = r\sqrt{4r^2+1}$$

$$\therefore \iint_S (1-z)\,dS = \int_0^{2\pi}\int_0^1 (1-r^2)\left(r\sqrt{4r^2+1}\right) dr\,d\theta$$

$$= 2\pi \int_0^1 r\sqrt{4r^2+1}\,dr - 2\pi\int_0^1 r^3\sqrt{4r^2+1}\,dr$$

$$= \frac{2\pi}{8}\int_0^1 8r\sqrt{4r^2+1}\,dr - \frac{\pi}{4}\left[\frac{25\sqrt{5}+1}{15}\right]$$

See answer to #13 above

$$= \frac{\pi}{4}\left(\frac{2}{3}\right)(4r^2+1)^{3/2}\Big|_{r=0}^{r=1} - \frac{\pi}{4}\left[\frac{25\sqrt{5}+1}{15}\right]$$

$$= \frac{\pi}{4}\left(\frac{2}{3}\right)\left[5\sqrt{5}-1\right] - \frac{\pi}{4}\left[\frac{5\sqrt{5}}{3} + \frac{1}{15}\right]$$

$$= \frac{\pi}{4}\left[\frac{10\sqrt{5}}{3} - \frac{2}{3} - \frac{5\sqrt{5}}{3} - \frac{1}{15}\right]$$

$$= \frac{\pi}{4}\left[\frac{5\sqrt{5}}{3} - \frac{11}{15}\right]$$

21.

Interpret "solid sphere" to be the shell, not the ball.

$\therefore\ \bar{x} = \dfrac{\iint_S x\,dS}{A(S)}$, $\bar{y} = \dfrac{\iint_S y\,dS}{A(S)}$, $\bar{z} = \dfrac{\iint_S z\,dS}{A(S)}$,

using #18(c) above. By symmetry, $\bar{x} = \bar{y} = \bar{z}$.

$A(S) = 4\pi R^2/8 = \underline{\dfrac{\pi R^2}{2}}$, as there are 8 octands to a sphere.

For the octant $x \ge 0, y \ge 0, z \ge 0$, let

$T(\theta, \phi) = (R\cos\theta\sin\phi, R\sin\theta\sin\phi, R\cos\phi)$,

$0 \le \theta \le \frac{\pi}{2}$, $0 \le \phi \le \frac{\pi}{2}$.

$\|\vec{T}_\theta \times \vec{T}_\phi\| = R^2\sin\phi$ as seen from Example 3 and pages 395-396 of text.

$$\therefore \iint_S x\,dS = \int_0^{\pi/2}\int_0^{\pi/2} (R\cos\theta\sin\phi)(R^2\sin\phi)\,d\phi\,d\theta$$

$$= R^3 \int_0^{\frac{\pi}{2}} \cos\theta\,d\theta \int_0^{\frac{\pi}{2}} \sin^2\phi\,d\phi$$

use $\sin^2\theta = \frac{1-\cos 2\theta}{2}$

$$= R^3 \Big[\sin\theta\Big]_0^{\pi/2} \left[\tfrac{1}{2}\theta - \frac{\sin 2\theta}{4}\right]_0^{\pi/2}$$

$$= R^3 \{1-0\} \left[\frac{\pi}{4} - 0 - (0-0)\right] = \underline{\frac{\pi R^3}{4}}$$

$$\therefore \bar{x} = \frac{\frac{\pi R^3}{4}}{\frac{\pi R^2}{2}} = \underline{\frac{R}{2}}$$

$$\therefore (\bar{x}, \bar{y}, \bar{z}) = \underline{\left(\frac{R}{2}, \frac{R}{2}, \frac{R}{2}\right)}$$

22.

From #21 above, $\bar{z}$ for each octant with $z \le 0$ is $-\frac{r}{2}$. $\therefore$ The entire hemisphere with $z \le 0$ will have $\bar{z} = -\frac{r}{2}$.

From symmetry, $|\bar{x}| = |\bar{y}| = |\bar{z}|$ for the 4 octants below the xy-plane. If you add each $(\bar{x}, \bar{y}, \bar{z})$ and divide by 4, you will get

$$\left[\left(\frac{r}{2}, \frac{r}{2}, -\frac{r}{2}\right) + \left(-\frac{r}{2}, \frac{r}{2}, -\frac{r}{2}\right) + \left(\frac{r}{2}, -\frac{r}{2}, -\frac{r}{2}\right) + \left(-\frac{r}{2}, -\frac{r}{2}, -\frac{r}{2}\right)\right] / 4$$

$$= \left(0, 0, -\frac{r}{2}\right).$$

Alternatively, $\bar{z} = \iint_S z\,dS / A(S)$ from #18 above, and $\iint_S z\,dS = -\pi r^3$ from #9.

$$A(S) = \frac{4\pi r^2}{2} = 2\pi r^2.$$

$$\therefore \bar{z} = -\pi r^3 / 2\pi r^2 = -\frac{r}{2}$$

23.

(a)

$$\vec{T_u} \times \vec{T_v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} = (y_u z_v - y_v z_u,\ x_v z_u - x_u z_v,\ x_u y_v - x_v y_u)$$

$$\therefore \|\vec{T}_u \times \vec{T}_v\|^2 = (x_u y_v - x_v y_u)^2 + (x_v z_u - x_u z_v)^2 + (y_u z_v - y_v z_u)^2$$

$$E = x_u^2 + y_u^2 + z_u^2 \qquad G = x_v^2 + y_v^2 + z_v^2$$

$$F = x_u x_v + y_u y_v + z_u z_v$$

One could go through the algebraic manipulations, but note from Exercise # 24, Section 1.3, p. 50 of the text that $(\vec{u}\times\vec{v})\cdot(\vec{u}'\times\vec{v}') = \begin{vmatrix} \vec{u}\cdot\vec{u}' & \vec{u}\cdot\vec{v}' \\ \vec{u}'\cdot\vec{v} & \vec{v}\cdot\vec{v}' \end{vmatrix}$

so that $\|\vec{T}_u \times \vec{T}_v\|^2 = (\vec{T}_u \times \vec{T}_v)\cdot(\vec{T}_u \times \vec{T}_v)$

$$= \begin{vmatrix} \vec{T}_u\cdot\vec{T}_u & \vec{T}_u\cdot\vec{T}_v \\ \vec{T}_u\cdot\vec{T}_v & \vec{T}_v\cdot\vec{T}_v \end{vmatrix}$$

$$= \|\vec{T}_u\|^2 \|\vec{T}_v\|^2 - (\vec{T}_u\cdot\vec{T}_v)(\vec{T}_u\cdot\vec{T}_v)$$

$$= (E)(G) - (F)(F)$$

$$\therefore \|\vec{T}_u \times \vec{T}_v\|^2 = EG - F^2 \Rightarrow \underline{\|\vec{T}_u \times \vec{T}_v\| = \sqrt{EG - F^2}}$$

$$\therefore \underline{A(S)} = \iint_D \underline{\|\vec{T}_u \times \vec{T}_v\| \, du\,dv} = \iint_D \underline{\sqrt{EG-F^2}\, du\,dv}$$

$$\therefore \iint_S f\,dS = \iint_D f(x(u,v), y(u,v), z(u,v)) \sqrt{EG-F^2}\, du\,dv$$

$$= \iint_D f(\phi(u,v)) \sqrt{EG-F^2}\, du\,dv$$

(b)

$$\vec{T_u} \perp \vec{T_v} \Rightarrow \vec{T_u} \cdot \vec{T_v} = 0 \Rightarrow F = 0$$

$$\therefore A(S) = \iint_D \sqrt{EG}\, du\,dv$$

(c)

Let $T(\theta, \phi) = (a\cos\theta\sin\phi, a\sin\theta\sin\phi, a\cos\phi)$.

$$\vec{T_\theta} = (-a\sin\theta\sin\phi, a\cos\theta\sin\phi, 0)$$

$$\vec{T_\phi} = (a\cos\theta\cos\phi, a\sin\theta\cos\phi, -a\sin\phi)$$

Note $\vec{T_\theta} \cdot \vec{T_\phi} = -a^2\sin\theta\cos\theta\sin\phi\cos\phi$
$\quad + a^2\sin\theta\cos\theta\sin\phi\cos\phi$
$\quad + 0$
$\quad = 0$

$$E = a^2\sin^2\theta\sin^2\phi + a^2\cos^2\theta\sin^2\phi + 0$$

$= a^2\sin^2\phi(\sin^2\theta + \cos^2\theta) = a^2\sin^2\phi$

$G = a^2\cos^2\theta\cos^2\phi + a^2\sin^2\theta\cos^2\phi + a^2\sin^2\phi$

$= a^2\cos^2\phi(\cos^2\theta + \sin^2\theta) + a^2\sin^2\phi$

$= a^2\cos^2\phi + a^2\sin^2\phi = a^2$

$\therefore EG = (a^2\sin^2\phi)(a^2) = a^4\sin^2\phi$

$\therefore$

$A(S) = \iint_D \sqrt{EG}\, d\theta\, d\phi = \iint_D a^2\sin\phi\, d\theta\, d\phi$

$= a^2\int_0^{2\pi}\int_0^{\pi}\sin\phi\, d\phi\, d\theta = 2\pi a^2\int_0^{\pi}\sin\phi\, d\phi$

$= 2\pi a^2\left[-\cos\phi\right]_0^{\pi} = 2\pi a^2\left[1-(-1)\right]$

$= \underline{4\pi a^2}$

24.

$$A(\phi) = \iint_D \sqrt{EG - F^2}\, du\, dv \quad \text{and}$$

$$J(\phi) = \frac{1}{2}\iint_D (E + G)\, du\, dv$$

dot product

where $E = \|\phi_u\|^2$, $G = \|\phi_v\|^2$, $F = \phi_u \cdot \phi_v$

(1) If $\phi_u \cdot \phi_v = 0$, then $F = 0$ and if $\|\phi_u\|^2 = \|\phi_v\|^2$, then $E = G$, so

$$\sqrt{EG - F^2} = \sqrt{E^2} = \sqrt{G^2} = E = G$$

$$\therefore A(\phi) = \iint_D E\, du\, dv = \iint_D G\, du\, dv \quad \text{and}$$

$$J(\phi) = \frac{1}{2}\iint_D (E + E)\, du\, dv = \iint_D E\, du\, dv$$

$$\therefore A(\phi) = J(\phi)$$

(2) Since $E \geq 0$, $G \geq 0$, $F^2 \geq 0$, then

$$EG - F^2 \leq EG \Rightarrow \sqrt{EG - F^2} \leq \sqrt{EG}$$

$$\sqrt{EG} \leq \frac{1}{2}(E + G) \Leftrightarrow 2\sqrt{EG} \leq (E + G)$$

$$\Leftrightarrow 4EG \leq (E + G)^2 = E^2 + 2EG + G^2$$

$$\iff 0 \le E^2 - 2EG + G^2$$

$$\iff 0 \le (E-G)^2$$

Since $(E-G)^2 \ge 0$, then $\sqrt{EG} \le \frac{1}{2}(E+G)$

$\therefore \sqrt{EG - F^2} \le \frac{1}{2}(E+G)$

$\therefore$ Given any partition of D, for each R_{ij} of D, $\sqrt{EG-F^2} \le \frac{1}{2}(E+G)$.

$$\therefore A_n = \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \sqrt{EG-F^2}\,\Delta u \Delta v \le \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \frac{1}{2}(E+G)\,\Delta u \Delta v = J_n$$

where E, G, and F are evaluated over each R_{ij}.

$$\therefore A(\phi) = \lim_{n\to\infty} A_n \le \lim_{n\to\infty} J_n = J(\phi)$$

25. Error: (a) and (b) of Exercise 24

Exercise 24 was Exercise 16 in the 5th edition.

For ease of visualization, let $\phi_u = (x_u, y_u) = (a, b)$

$\phi_v = (x_v, y_v) = (c, d)$.

Condition (a) $\Rightarrow \|\phi_u\|^2 = \|\phi_v\|^2 \Rightarrow a^2 + b^2 = c^2 + d^2$ [1]

Condition (b) $\Rightarrow \phi_u \cdot \phi_v = 0 \Rightarrow (a, b) \cdot (c, d) = 0 \Rightarrow$

$ac + bd = 0$ [2]

$\det \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix} > 0 \Rightarrow ad - bc > 0$ [3]

Need to prove: $a = d$ and $b = -c$

(1) If $a = 0$, then [2] $\Rightarrow bd = 0 \Rightarrow b = 0$ or $d = 0$

If $b = 0$, then $ad - bc = 0$, contradicting [3]

$\therefore d = 0$, and [1] $\Rightarrow b^2 = c^2 \Rightarrow b = c$ or $b = -c$

If $b = c$, then [3] becomes $-b^2 > 0$, a contradiction

$\therefore b = -c$, no contradiction in [1], [2], [3]

$\therefore a = 0 \Rightarrow d = 0 \Rightarrow b = -c$, so

$\underline{a = d}$ and $\underline{b = -c}$ and Theorem true.

(2) Assume $a \neq 0$ $\therefore$ [2] $\Rightarrow c = -\dfrac{bd}{a}$

$\therefore$ [1] $\Rightarrow a^2+b^2=\left(-\frac{bd}{a}\right)^2+d^2$, or

$$a^4+a^2b^2=b^2d^2+a^2d^2, \text{ or}$$

$$a^2(a^2-d^2)=b^2(d^2-a^2) \qquad [4]$$

If $a \neq d$, then $a^2-d^2 \neq 0$, so [4] $\Rightarrow$

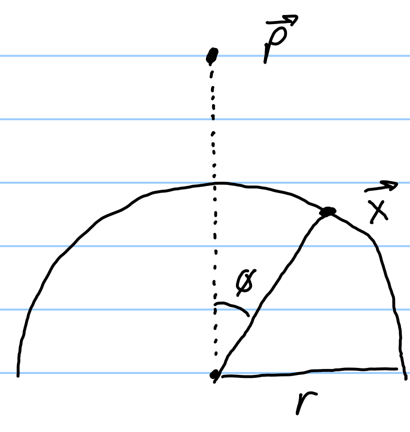
$a^2=-b^2$. But $a^2>0$ and it can't be true that $-b^2>0$.

$\therefore a=d$ and $\therefore d \neq 0$

$\therefore$ [2] $\Rightarrow ac+ba=0 \Rightarrow c+b=0 \Rightarrow b=-c$

$\therefore a \neq 0 \Rightarrow \underline{a=d}$ and $\underline{b=-c}$ and The Theorem is true.

26.



Using symmetry, can assume $\vec{p}$ is on any axis.

So, assume it is on the z-axis for ease of calculations.

$\vec{x}$ is on S.

Using spherical coordinates, $\vec{x}$ on S can be described as $(r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi)$.

$\vec{p}$ can be described as $(0,0,d)$

$$\therefore \|\vec{x}-\vec{p}\| = \sqrt{(r^2\cos^2\theta\sin^2\phi) + (r^2\sin^2\theta\sin^2\phi) + (r\cos\phi - d)^2}$$

$$= \sqrt{r^2\sin^2\phi + r^2\cos^2\phi - 2rd\cos\phi + d^2}$$

$$= \sqrt{r^2 - 2rd\cos\phi + d^2}$$

This expression holds for $\vec{p}$ inside or outside the sphere. If $\vec{p}$ were on the sphere, $r=d$. If $\vec{p}$ is at the center, $d=0$, so $\|\vec{x}-\vec{p}\| = r$.

For a sphere using spherical coordinates as a parametrization, $\|\vec{T}_\theta \times \vec{T}_\phi\| = r^2\sin\phi$

$$\therefore \iint_S \frac{1}{\|\vec{x}-\vec{p}\|}\,dS = \int_0^{2\pi}\int_0^{\pi} \frac{r^2\sin\phi}{\sqrt{r^2+d^2-2rd\cos\phi}}\,d\phi\,d\theta$$

$$= 2\pi r^2 \int_0^{\pi} \frac{\sin\phi}{\sqrt{r^2+d^2-2rd\cos\phi}}\,d\phi \qquad [1]$$

If $d=0$, this becomes $2\pi r^2 \int_0^\pi \frac{\sin\phi}{\sqrt{r^2}}\,d\phi$

$$= 2\pi r \int_0^\pi \sin\phi\, d\phi = 2\pi r\Big[-\cos\phi\Big]_0^\pi = 2\pi r\,[1-(-1)]$$

$$= \underline{4\pi r}$$

For $d \neq 0$, [1] becomes $\frac{2\pi r^2}{2rd} \int_0^\pi \frac{2rd\sin\phi}{\sqrt{r^2+d^2-2rd\cos\phi}}\,d\phi$

$$= \frac{\pi r}{d}\Big[(2)\,(r^2+d^2-2rd\cos\phi)^{\frac{1}{2}}\Big]_{\phi=0}^{\phi=\pi}$$

$$= \frac{2\pi r}{d}\Big[(r^2+d^2+2rd)^{\frac{1}{2}} - (r^2+d^2-2rd)^{\frac{1}{2}}\Big]$$

$$= \frac{2\pi r}{d}\Big[\sqrt{(r+d)^2} - \sqrt{(r-d)^2}\Big] \qquad [2]$$

<u>Inside S</u>, $d<r$ and $\sqrt{(r-d)^2} = r-d$

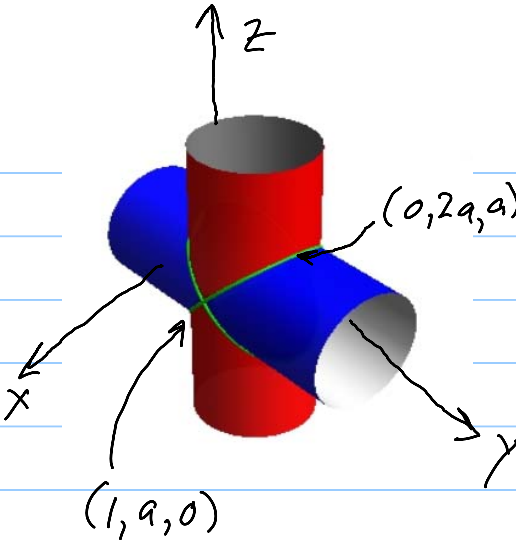
$\therefore$ [2] becomes $\frac{2\pi r}{d}\Big[(r+d)-(r-d)\Big] = \underline{4\pi r}$

$4\pi r$ is also the result for $d=0$.

<u>Outside S</u>, $d>r$, so $\sqrt{(r-d)^2} = \sqrt{(d-r)^2} = d-r$

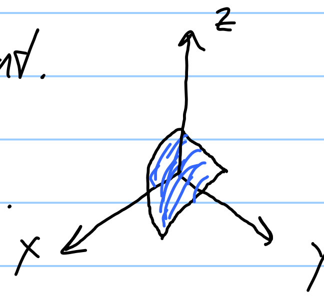
$\therefore$ [2] becomes $\frac{2\pi r}{d}\Big[(r+d)-(d-r)\Big] = \underline{\frac{4\pi r^2}{d}}$

27.



$x^2+y^2=2ay \Leftrightarrow x^2+(y-a)^2=a^2$, a cylinder perpendicular to the xy-plane, centered at $(0,a,z)$

From The image, it would be the top portion of the blue pipe in The positive octant.

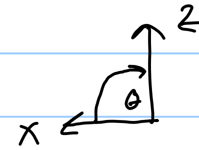


Cylinders suggest cylindrical coordinates.

$T(\theta,y)=(a\cos\theta,\ y,\ a\sin\theta)=(x,y,z)$.

where T represents The "blue" cylinder.

Here, $0\le\theta\le\frac{\pi}{2}$:



From $x^2+(y-a)^2=a^2$, $(y-a)^2=a^2-x^2$,

$y-a=\sqrt{a^2-x^2}$ since in positive octant.

$\therefore y=a+\sqrt{a^2-x^2}=a+\sqrt{a^2-(a\cos\theta)^2}=a+a\sin\theta$

Note when $\theta=0$, $y=a$, so $(x,y,z)=(1,a,0)$

when $\theta=\frac{\pi}{2}$, $y=2a$, so $(x,y,z)=(0,2a,a)$

$\therefore$ For $0 \leq \theta \leq \frac{\pi}{2}$, $y(\theta) = a + a\sin\theta$, so $a \leq y \leq 2y$

$\therefore$ Will be getting surface area of "outer" half of blue cylinder. $\therefore$ Multiply by 2.

$$\therefore A(S) = 2\int_0^{\frac{\pi}{2}}\int_a^{a+a\sin\theta} \|\vec{T}_\theta \times \vec{T}_y\|\, dy\, d\theta$$

$$\vec{T}_\theta \times \vec{T}_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a\sin\theta & 0 & a\cos\theta \\ 0 & 1 & 0 \end{vmatrix} = (-a\cos\theta, 0, -a\sin\theta)$$

$$\therefore \|\vec{T}_\theta \times \vec{T}_y\| = \sqrt{a^2\cos^2\theta + 0 + a^2\sin^2\theta} = a$$

$$\therefore A(S) = 2\int_0^{\frac{\pi}{2}}\int_a^{a+a\sin\theta} a\, dy\, d\theta$$

$$= 2a\int_0^{\frac{\pi}{2}} \left[y\right]_a^{a+a\sin\theta} d\theta$$

$$= 2a\int_0^{\frac{\pi}{2}} a\sin\theta = 2a^2\left[-\cos\theta\right]_{\theta=0}^{\frac{\pi}{2}}$$

$$= 2a^2\left[0-(-1)\right] = \underline{2a^2}$$

28.

For $A(S) = \iint_S f(x,y,z)\,dS$, here $f(x,y,z) = \left|\frac{\partial F}{\partial z}\right|$

Since S is over D in xy-plane, it is assumed there is a function $z = g(x,y)$.

$\therefore$ as in p. 395 of text,

$$\iint_S f(x,y,z)\,dS = \iint_D \left|\frac{\partial F}{\partial z}\right| \sqrt{1 + g_x^2 + g_y^2}\,dx\,dy \quad [1]$$

Since $F(x,y,z) = 0$, by the chain rule,

$$\frac{\partial F}{\partial x} = 0 = \frac{\partial F}{\partial x}\cdot\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\cdot\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial x}$$

$$\therefore \quad 0 = \frac{\partial F}{\partial x}\cdot 1 + 0 + \frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial x}$$

$$\text{or, } \frac{\partial z}{\partial x} = -\frac{\partial F}{\partial x} \Big/ \frac{\partial F}{\partial z}$$

Similarly, $\dfrac{\partial z}{\partial y} = -\dfrac{\partial F}{\partial y} \Big/ \dfrac{\partial F}{\partial z}$

But $\dfrac{\partial z}{\partial x} = g_x$, $\dfrac{\partial z}{\partial y} = g_y$

$$\therefore \sqrt{1 + g_x^2 + g_y^2} = \sqrt{1 + (\partial z/\partial x)^2 + (\partial z/\partial y)^2}$$

$$= \sqrt{1 + \left(-\frac{\partial F}{\partial x} \Big/ \frac{\partial F}{\partial z}\right)^2 + \left(-\frac{\partial F}{\partial y} \Big/ \frac{\partial F}{\partial z}\right)^2}$$

$$= \sqrt{\frac{\left(\frac{\partial F}{\partial z}\right)^2 + \left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2}{\left(\frac{\partial F}{\partial z}\right)^2}}$$

$$= \frac{\sqrt{(\partial F/\partial x)^2 + (\partial F/\partial y)^2 + (\partial F/\partial z)^2}}{|\partial F/\partial z|}$$

$\therefore$ [1] becomes

$$\iint_D \left|\frac{\partial F}{\partial z}\right| \sqrt{F_x^2 + F_y^2 + F_z^2} \Big/ |F_z| \, dx\, dy$$

$$= \iint_D \sqrt{(\partial F/\partial x)^2 + (\partial F/\partial y)^2 + (\partial F/\partial z)^2} \, dx\, dy$$

7.6 Surface Integrals of Vector Fields

Note Title 4/12/2017

1.

Use cylindrical coordinates : $0 \le \theta \le 2\pi$, $0 \le r \le 1$

$\therefore$ $x = r\cos\theta$, $y = r\sin\theta$, $z = 1 - x^2 - y^2 = 1 - r^2$

(a) $T(r,\theta) = (x, y, 1-x^2-y^2) = (r\cos\theta, r\sin\theta, 1-r^2)$

$$\vec{T}_r \times \vec{T}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & -2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = (2r^2\cos\theta, 2r^2\sin\theta, r\cos^2\theta + r\sin\theta)$$

$\therefore$ $\vec{N} = (2r^2\cos\theta, 2r^2\sin\theta, r)$ for paraboloid

Note: $\hat{k}$ component positive, so points up/out.

(b) $$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot \vec{N}\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^1 (2r\cos\theta, 2r\sin\theta, 1-r^2) \cdot (2r^2\cos\theta, 2r^2\sin\theta, r)\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 4r^3\cos^2\theta + 4r^3\sin^2\theta + r - r^3 \, dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 3r^3 + r \, dr\,d\theta = 2\pi\int_0^1 3r^3 + r\, dr$$

$$= 2\pi\left[\frac{3}{4}r^4 + \frac{r^2}{2}\right]_0^1 = 2\pi\left[\frac{3}{4}+\frac{1}{2}\right] = \underline{\frac{5}{2}\pi}$$

(c) Since over "closed" surface, must compute

$\iint_S \vec{F}\cdot d\vec{S}$ where S is unit disk.

Again use cylindrical coordinates, $0 \le \theta \le 2\pi, 0 \le r \le 1$.

$T(r,\theta) = (r\cos\theta, r\sin\theta, 0)$

$$\vec{T}_r \times \vec{T}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = (0,0,r)$$

To make $\vec{N}$ point "out", choose $\vec{N} = (0,0,-r)$

$$\therefore \iint_S \vec{F}\cdot d\vec{S} = \int_0^{2\pi}\int_0^1 \vec{F}\cdot\vec{N}\, dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 (2r\cos\theta, 2r\sin\theta, 0)\cdot(0,0,-r)\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 0\,dr\,d\theta = \underline{0}$$

(d) $\therefore \frac{5}{2}\pi + 0 = \underline{\underline{\frac{5}{2}\pi}}$

2.

$$\phi_u \times \phi_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2\cos u & -3\sin u & 0 \\ 0 & 0 & 1 \end{vmatrix} = (-3\sin u, -2\cos u, 0)$$

$\vec{F}$, at the surface, is $(2\sin u, 3\cos u, v^2)$

$$\therefore \vec{F}\cdot(\phi_u \times \phi_v) = -6\sin^2 u - 6\cos^2 u + 0 = -6$$

$$\therefore \iint_S \vec{F}\cdot d\vec{S} = \iint_D (-6)\,du\,dv = -6\int_0^{2\pi}\int_0^1 dv\,du$$

$$= \underline{\underline{-12\pi}}$$

3.

For the sphere, let $T(\phi,\theta)=(R\cos\theta\sin\phi, R\sin\theta\sin\phi, R\cos\phi)$,
here, $R=3$

$$\therefore \vec{T_\phi}\times\vec{T_\theta} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ R\cos\theta\cos\phi & R\sin\theta\cos\phi & -R\sin\phi \\ -R\sin\theta\sin\phi & R\cos\theta\sin\phi & 0 \end{vmatrix}$$

$$=(R^2\cos\theta\sin^2\phi, R^2\sin\theta\sin^2\phi, R^2\cos\phi\sin\phi)$$

$$= R\sin\phi\,(R\cos\theta\sin\phi, R\sin\theta\sin\phi, R\cos\phi)$$

$$= R\sin\phi\,(x,y,z) = R\sin\phi\,\vec{r} = R^2\sin\phi\,\vec{n},$$

where $\vec{n}=(\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)$

$$\therefore \|\vec{T_\phi}\times\vec{T_\theta}\| = R^2\sin\phi$$

Also, $\vec{F}(x,y,z)=(R\cos\theta\sin\phi, R\sin\theta\sin\phi, R\cos\phi) = R\vec{n}$

(a) $$\iint_S \vec{F}\cdot dS = \iint_D \vec{F}\cdot(\vec{T_\phi}\times\vec{T_\theta})\,d\phi\,d\theta$$

$$= \iint_D (R\vec{n}) \cdot (R^2 \sin\phi\, \vec{n})\, d\phi\, d\theta \quad , \text{ note } \vec{n}\cdot\vec{n} = 1$$

$$= R^3 \iint_D (\vec{n}\cdot\vec{n}) \sin\phi\, d\phi\, d\theta = R^3 \int_0^{2\pi}\int_0^{\pi/2} \sin\phi\, d\phi\, d\theta$$

$$= 2\pi R^3 \left[-\cos\phi\right]_0^{\pi/2} = 2\pi R^3 [0 - (-1)] = 2\pi R^3$$

As $R = 3$, $2\pi R^3 = \underline{54\pi}$

(b) $R^3 \int_0^{2\pi}\int_0^{\pi} \sin\phi\, d\phi\, d\theta = 4\pi R^3 = \underline{108\pi}$

4.

Using cylindrical coordinates, the surface S is

$T(\theta, z) = (2\cos\theta, 2\sin\theta, z)$, $0 \le \theta \le 2\pi$, $0 \le z \le 1$.

$\therefore \vec{F}(T(\theta, z)) = (4\cos\theta, -4\sin\theta, z^2)$

$$\vec{T}_\theta \times \vec{T}_z = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2\sin\theta & 2\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= (2\cos\theta, 2\sin\theta, 0)$$

$$\therefore \iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot (\vec{T}_\theta \times \vec{T}_z)\, d\theta\, dz$$

$$= \int_0^{2\pi} \int_0^1 (4\cos\theta, -4\sin\theta, z^2) \cdot (2\cos\theta, 2\sin\theta, 0)\, dz\, d\theta$$

$$= \int_0^{2\pi} \int_0^1 8\cos^2\theta - 8\sin^2\theta \; dz\, d\theta \qquad \cos^2\theta - \sin^2\theta = \cos 2\theta$$

$$= 8 \int_0^{2\pi} \cos 2\theta\, d\theta = 4 \sin 2\theta \Big|_0^{2\pi} = \underline{0}$$

5.

From p. 407 of text, $\vec{F} = -K \nabla T = -(6x, 0, 6z)$

Use cylindrical coordinates for S:

$$T(\theta, y) = (\sqrt{2}\cos\theta, y, \sqrt{2}\sin\theta), \quad 0 \le \theta \le 2\pi, \quad 0 \le y \le 2$$

$$\therefore \vec{F}(T(\theta, y)) = (-6\sqrt{2}\cos\theta, y, -6\sqrt{2}\sin\theta)$$

$$\vec{T}_\theta \times \vec{T}_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sqrt{2}\sin\theta & 0 & \sqrt{2}\cos\theta \\ 0 & 1 & 0 \end{vmatrix}$$

$$= (-\sqrt{2}\cos\theta,\ 0,\ -\sqrt{2}\sin\theta)$$

$$\therefore \vec{F}(T(\theta,y)) \cdot (\vec{T_\theta} \times \vec{T_y}) = 12\cos^2\theta + 0 + 12\sin^2\theta = 12$$

$$\therefore \iint_S \vec{F} \cdot d\vec{S} = \iint_D 12\, d\theta\, dy = \int_0^{2\pi}\int_0^2 12\, dy\, d\theta$$

$$= \underline{\underline{48\pi}}$$

6.

$$\vec{F} = -K\nabla T = -K(1,0,0)$$

For a unit sphere, $T(\phi,\theta) = (\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)$

$$\vec{T_\phi} \times \vec{T_\theta} = (\sin^2\phi\cos\theta, \sin^2\phi\sin\theta, \sin\phi\cos\phi)$$

$$\therefore \vec{F}(T(\phi,\theta)) \cdot (\vec{T_\phi} \times \vec{T_\theta}) = -K\sin^2\phi\cos\theta$$

$$\therefore \iint_S \vec{F} \cdot d\vec{S} = -K\int_0^{2\pi}\int_0^{\pi} \sin^2\phi\cos\theta\, d\phi\, d\theta$$

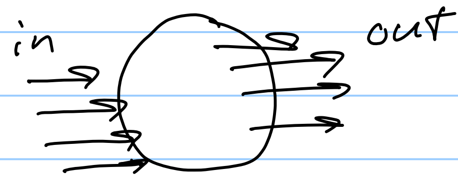
$$= -K\int_0^{2\pi}\cos\theta\, d\theta \int_0^{\pi}\sin^2\phi\, d\phi$$

$$= -k\left[\sin\theta\right]_0^{2\pi} \int_0^{\pi} \sin^2\phi \, d\phi = -k[0-0]\int_0^{\pi} \sin^2\phi \, d\phi$$

$$= \underline{\underline{0}}$$

As $T(x,y,z)=x$ only depends on x-coordinate, temp. gradient is one-way. As sphere is symmetrical, flow in on one side equals flow out on other side.

Net is 0.



7.

(a) Upper hemisphere: $T(\phi,\theta) = (\cos\theta\sin\phi, \sin\theta\sin\theta, \cos\phi)$

$0 \le \theta \le 2\pi, \quad 0 \le \phi \le \pi/2$

$$\vec{E}(T(\phi,\theta)) = (2\cos\theta\sin\phi, 2\sin\theta\sin\phi, 2\cos\phi)$$

$$\vec{T}_\phi \times \vec{T}_\theta = (\sin^2\phi\cos\theta, \sin^2\phi\sin\theta, \sin\phi\cos\phi)$$

$$\therefore \vec{E}\cdot d\vec{S} = 2\sin^3\phi\cos^2\theta + 2\sin^3\phi\sin^2\theta + 2\sin\phi\cos^2\phi$$

$$= 2\sin^3\phi + 2\sin\phi\cos^2\phi$$

$$= 2\sin\phi$$

$$\therefore \iint_S \vec{E} \cdot d\vec{S} = \iint_D 2\sin\phi \, d\phi \, d\theta = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} 2\sin\phi \, d\phi \, d\theta$$

$$= 4\pi \left[-\cos\phi \right]_0^{\pi/2} = 4\pi [0 - (-1)] = \underline{4\pi}$$

(b) Bottom base: $T(\theta, r) = (r\cos\theta, r\sin\theta, 0)$

$0 \le \theta \le 2\pi, \; 0 \le r \le 1$

$$\vec{E}(T(\theta, r)) = (2r\cos\theta, 2r\sin\theta, 0)$$

$$\vec{T}_\theta \times \vec{T}_r = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -r\sin\theta & r\cos\theta & 0 \\ \cos\theta & \sin\theta & 0 \end{vmatrix} = (0, 0, -r)$$

Note $\vec{T}_\theta \times \vec{T}_r$ points down or "out".

$$\therefore \vec{E} \cdot (\vec{T}_\theta \times \vec{T}_r) = 0$$

$\therefore \iint_S \vec{E} \cdot dS = 0$, which makes sense since

$\vec{E}(x, y, 0)$ is parallel to xy-plane and has

no component (z-coord) crossing bottom base.

(c) (a) + (b) = $4\pi + 0 = \underline{4\pi}$

8.

Cylinder can be parametrized as:

$$\phi(y,\theta) = (\cos\theta, y, \sin\theta), \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}, \quad 0 \le y \le 1$$

$$\therefore \vec{F}(\phi(y,\theta)) = (\sqrt{y}, 0, 0)$$

$$\phi_y \times \phi_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{vmatrix} = (\cos\theta, 0, \sin\theta)$$

$\therefore \phi_y \times \phi_\theta$ points "out"

$$\therefore \vec{F} \cdot (\phi_y \times \phi_\theta) = \sqrt{y}\cos\theta + 0 + 0$$

$$\therefore \iint_S \vec{F} \cdot d\vec{S} = \int_{-\pi/2}^{\pi/2} \int_0^1 \sqrt{y}\cos\theta \, dy \, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \cos\theta \, d\theta \int_0^1 \sqrt{y} \, dy = \sin\theta \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \; \frac{2}{3} y^{3/2} \Big|_0^1$$

$$= [1 - (-1)] [2/3 - 0] = \underline{\frac{4}{3}}$$

$\therefore \frac{4}{3}$ meters^3/sec flows out of the half cylinder

9.

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & z x^3 y^2 \end{vmatrix} = (2x^3yz - 0, \; 0 - 3x^2y^2z, \; -1-1)$$

$$= (2x^3yz, -3x^2y^2z, -2)$$

For S: $z^2 = \frac{1-x^2-y^2}{3}$, $z \le 0$, so $z = -\sqrt{\frac{1-x^2-y^2}{3}}$

Let $T(x,y) = \left(x, y, -\left(\frac{1-x^2-y^2}{3}\right)^{\frac{1}{2}}\right)$, $-1 \le x \le 1$, $-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$

$$\vec{T}_x \times \vec{T}_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & \frac{x}{3}\left(\frac{1-x^2-y^2}{3}\right)^{-\frac{1}{2}} \\ 0 & 1 & \frac{y}{3}\left(\frac{1-x^2-y^2}{3}\right)^{-\frac{1}{2}} \end{vmatrix}$$

$$= \left[-\frac{x}{3}\left(\frac{1-x^2-y^2}{3}\right)^{-\frac{1}{2}}, -\frac{y}{3}\left(\frac{1-x^2-y^2}{3}\right)^{-\frac{1}{2}}, 1\right]$$

Note $\vec{T}_x \times \vec{T}_y$ points "up" as $\hat{k}$ component $= 1$.

$\therefore (\nabla \times \vec{F}) \cdot (\vec{T}_x \times \vec{T}_y) =$

$$-\frac{2x^4yz}{3}\left(\frac{1-x^2-y^2}{3}\right)^{-\frac{1}{2}} + x^2y^3z\left(\frac{1-x^2-y^2}{3}\right)^{-\frac{1}{2}} - 2$$

$$= z\left(x^2y^3 - \frac{2x^4y}{3}\right)\left(\frac{1-x^2-y^2}{3}\right)^{-\frac{1}{2}} - 2$$

Note: $z = -\left(\frac{1-x^2-y^2}{3}\right)^{\frac{1}{2}}$

$$= \frac{2x^4y}{3} - x^2y^3 - 2$$

$$\therefore \iint_S (\nabla \times \vec{F}) \cdot dS = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left(\frac{2x^4y}{3} - x^2y^3 - 2 \right) dy\, dx$$

$$= \int_{-1}^{1} \left. \frac{x^4y^2}{3} - \frac{x^2y^4}{4} - 2y \right|_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} dx$$

$$= \int_{-1}^{1} \frac{x^4(1-x^2)}{3} - \frac{x^2(1-x^2)^2}{4} - 2\sqrt{1-x^2} - \left[\frac{x^4(1-x^2)}{3} - \frac{x^2(1-x^2)^2}{4} + 2\sqrt{1-x^2} \right] dx$$

Text Table of Integrals, # 38

$$= -4 \int_{-1}^{1} \sqrt{1-x^2}\, dx = -4 \left[\frac{x}{2}\sqrt{1-x^2} + \frac{x^2}{2} \operatorname{Arcsin} x \right]_{-1}^{1}$$

$$= -4 \left[\tfrac{1}{2} \operatorname{Arcsin}(1) - \tfrac{1}{2} \operatorname{Arcsin}(-1) \right]$$

$$= -4 \left[\frac{\pi}{4} + \frac{\pi}{4} \right] = \underline{-2\pi}$$

10.

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y-4 & 3xy & 2xz+z^2 \end{vmatrix} = (0-0,\ 0-2z,\ 3y-1)$$

$$= \underline{(0, -2z, 3y-1)}$$

Let $T(x,y) = (x, y, \sqrt{16-x^2-y^2})$, $-4 \le x \le 4$, $-\sqrt{16-x^2} \le y \le \sqrt{16-x^2}$

$$\vec{T_x} \times \vec{T_y} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -x\sqrt{16-x^2-y^2} \\ 0 & 1 & -y\sqrt{16-x^2-y^2} \end{vmatrix}$$

$$= \underline{\left(x\sqrt{16-x^2-y^2},\ y\sqrt{16-x^2-y^2},\ 1\right)}$$

Note $\vec{T_x} \times \vec{T_y}$ points "up" as $\hat{k}$ component is 1.

$$\therefore (\nabla \times \vec{F}) \cdot (\vec{T_x} \times \vec{T_y}) = 0 + \left(-2zy\sqrt{16-x^2-y^2}\right) + 3y - 1$$

$$= 3y - 2y\left(\sqrt{16-x^2-y^2}\right)\left(\sqrt{16-x^2-y^2}\right) - 1$$

$$= 3y - 2y(16-x^2-y^2) - 1 = -29y + 2x^2y + 2y^3 - 1$$

$$\therefore \iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \int_{-4}^{4} \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} (-29y + 2x^2y + 2y^3 - 1)\, dy\, dx$$

$$= \int_{-4}^{4} -\frac{29}{2}y^2 + x^2y^2 + \frac{y^4}{2} - y \Big|_{y=-\sqrt{16-x^2}}^{y=\sqrt{16-x^2}} dx$$

Using #38 text table of integrals

$$= \int_{-4}^{4} -2\sqrt{16-x^2}\, dx = -2\left[\frac{x}{2}\sqrt{16-x^2} + 8 \operatorname{Arcsin}\frac{x}{4}\right]_{x=-4}^{x=4}$$

$$= -2\left[8\,\mathrm{Arcsin}(1) - 8\,\mathrm{Arcsin}(-1)\right]$$

$$= -2\left[8\left(\tfrac{\pi}{2}\right) - 8\left(-\tfrac{\pi}{2}\right)\right] = \underline{\underline{-16\pi}}$$

11.

(a) Top surface: Use spherical coordinates

$$T(\phi,\theta) = (\cos\theta\sin\phi,\ \sin\theta\sin\phi,\ \cos\phi),\quad 0 \le \theta \le 2\pi,\ 0 \le \phi \le \tfrac{\pi}{2}$$

$$T_\phi \times T_\theta = (\sin^2\phi\cos\theta,\ \sin^2\phi\sin\theta,\ \sin\phi\cos\phi)$$ $\sin\phi\cos\phi\,\hat{k}$ points "out"

$$\vec{F}(\phi,\theta) = \left[\cos\theta\sin\phi + 3(\sin\theta\sin\phi)^5,\ \sin\theta\sin\phi + 10\cos\theta\sin\phi\cos\phi,\ \cos\phi - \cos\theta\sin\theta\sin^2\phi\right]$$

$$\therefore\ \vec{F}(\phi,\theta)\cdot(T_\phi \times T_\theta) =$$

$$\boxed{\sin^3\phi\cos^2\theta + \sin^3\phi\sin^2\theta} + 3\sin^5\theta\cos\theta\sin^7\phi + 10\sin^3\phi\cos\phi\sin\theta\cos\theta$$

$$+ \sin\phi\cos^2\phi - \sin\theta\cos\theta\sin^3\phi\cos\phi$$

$$= \sin^3\phi + 3\sin^5\theta\cos\theta\sin^7\phi + 10\sin^3\phi\cos\phi\sin\theta\cos\theta$$

$$+ \sin\phi\cos^2\phi - \sin\theta\cos\theta\sin^3\phi\cos\phi$$

Integrating each term of $\vec{F}\cdot(T_\phi \times T_\theta)$ yields many cancellations.

$$\int_0^{2\pi}\int_0^{\pi/2} 3\sin^5\theta\cos\theta\sin^7\phi\, d\phi\, d\theta = \frac{3\sin^6\theta}{6}\Bigg|_{\theta=0}^{\theta=2\pi}\int_0^{\frac{\pi}{2}}\sin^7\phi\, d\phi$$

$$= (0)\int_0^{\frac{\pi}{2}}\sin^7\phi\, d\phi = 0$$ $\sin(2\pi) = \sin(0) = 0$

$$\int_0^{2\pi}\int_0^{\frac{\pi}{2}} 10\sin^3\phi\cos\phi\sin\theta\cos\theta\, d\phi\, d\theta = \frac{\sin^2\theta}{2}\Bigg|_0^{2\pi}\int_0^{\pi/2} 10\sin^3\phi\cos\phi\, d\phi$$

$$= (0)\int_0^{\pi/2} 10\sin^3\phi\cos\phi\, d\phi = 0$$ $\sin(2\pi) = \sin(0) = 0$

Similarly, $\int_0^{2\pi}\int_0^{\frac{\pi}{2}} \sin\theta\cos\theta\sin^3\phi\cos\phi\, d\phi\, d\theta = 0$

$$\therefore \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \vec{F}\cdot(T_\phi \times T_\theta) = \int_0^{2\pi}\int_0^{\pi/2}(\sin^3\phi + \sin\phi\cos^2\phi)\, d\phi\, d\theta$$

$$= 2\pi\int_0^{\pi/2}(\sin^3\phi + \sin\phi\cos^2\phi)\, d\phi$$

$$= 2\pi\int_0^{\pi/2}(1-\cos^2\phi)\sin\phi + \sin\phi\cos^2\phi\; d\phi$$

$$= 2\pi \int_0^{\pi/2} \sin\phi \, d\phi = 2\pi \left[-\cos\phi \right]_0^{\pi/2} = \underline{2\pi}$$

$\therefore$ Top Surface $\iint_S \vec{F} \cdot d\vec{S} = \underline{\underline{2\pi}}$

(b) Bottom Surface

$$T(\theta, r) = (r\cos\theta, r\sin\theta, 0), \quad 0 \le r \le 1 \quad 0 \le \theta \le 2\pi$$

$$T_\theta = (-r\sin\theta, r\cos\theta, 0) \qquad T_r = (\cos\theta, \sin\theta, 0)$$

$$T_\theta \times T_r = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -r\sin\theta & r\cos\theta & 0 \\ \cos\theta & \sin\theta & 0 \end{vmatrix} = (0, 0, -r)$$

$-r\hat{k}$ points "out"

$$\vec{F}(\theta, r) = (r\cos\theta + 3r^5\sin^5\theta, r\sin\theta, -r^2\sin\theta\cos\theta)$$

$$\therefore \vec{F} \cdot (T_\theta \times T_r) = r^3 \sin\theta\cos\theta$$

$$\therefore \int_0^{2\pi}\int_0^1 r^3 \sin\theta\cos\theta \, dr \, d\theta = \frac{\sin^2\theta}{2}\Big|_0^{2\pi} \frac{r^4}{4}\Big|_0^1 = \underline{0}$$

as $\sin(2\pi) = \sin(0) = 0$

$$\therefore \iint_S \vec{F} \cdot d\vec{S} = 2\pi + 0 = \underline{\underline{2\pi}}$$

12.

The region of interest is completely below the top disk, so use the disk for parametrizations - for surface area & volume.

(a) The disk admits a polar parametrization:

$x = R\cos\theta$, $y = R + R\sin\theta$, $0 \le \theta \le 2\pi$, where θ is measured from line parallel to x-axis.

$\therefore$ A parametrization of the portion of cylinder of interest is: $T(\theta, z) = (R\cos\theta, R + R\sin\theta, z)$.

Need to find a range for z, the vertical height.

The cylinder "height" depends on θ: i.e., where x and y are located. A line of "height" height goes from the paraboloid to the to disk.

For the paraboloid, $z = 4R^2 - (x^2 + y^2)$, and from the cylinder, $x^2 + y^2 - 2yR + R^2 = R^2$, so $x^2 + y^2 = 2yR$

$\therefore$ At intersection of cylinder and paraboloid,

$$z = 4R^2 - (2yR) = 4R^2 - 2(R + R\sin\theta)R$$

$$= 2R^2 - 2R^2\sin\theta$$

$$\therefore\ 2R^2 - 2R^2\sin\theta \le z \le 4R^2$$

So, for

$\theta =$	0	$\pi/2$	π	$\frac{3}{2}\pi$
$z =$	$2R^2$	0	$2R^2$	$4R^2$

The above table gives the lower boundary of the cylinder portion in question.

$$A(S) = \iint_D \|\vec{T}_\theta \times \vec{T}_z\|\, d\theta\, dz$$

$$\vec{T_\theta} \times \vec{T_z} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -R\sin\theta & R\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = (R\cos\theta, R\sin\theta, 0)$$

$$\therefore \|\vec{T_\theta} \times \vec{T_z}\| = R$$

$$\therefore A(S) = \int_0^{2\pi} \int_{2R^2-2R^2\sin\theta}^{4R^2} R\,dz\,d\theta$$

$$= R\int_0^{2\pi} 2R^2 + 2R^2\sin\theta\,d\theta = 4\pi R^3 + 2R^3\int_0^{2\pi} \sin\theta\,d\theta$$

$$= 4\pi R^3 + \left[-\cos\theta\right]_0^{2\pi} = 4\pi R^2 + 0$$

$\therefore$ Surface area of glass wall $= \underline{\underline{4\pi R^3}}$

(b)

Modifying parametrization in (a), the volume is described as $T(r,\theta,z) = (r\cos\theta, R + r\sin\theta, z)$,

$$0 \leq r \leq R, \quad 0 \leq \theta \leq 2\pi$$

For z: $z = 4R^2 - (x^2+y^2) = 4R^2 - (r\cos\theta)^2 - (R + r\sin\theta)^2$

$$= 4R^2 - r^2\cos^2\theta - R^2 - 2Rr\sin\theta - r^2\sin^2\theta$$

$$= 3R^2 - r^2 - 2Rr\sin\theta$$

$$\therefore\ 3R^2 - r^2 - 2Rr\sin\theta \le z \le 4R^2$$

Jacobian $\dfrac{\partial(x,y,z)}{\partial(r,\theta,z)}$ from $T(\,) = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$

$$\therefore\ V = \iiint_W dx\,dy\,dz = \iiint_{W^*} r\,dr\,d\theta\,dz$$

$$= \int_0^{2\pi}\int_0^R \int_{3R^2 - r^2 - 2Rr\sin\theta}^{4R^2} r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^R r(R^2 + r^2 + 2Rr\sin\theta)\,dr\,d\theta$$

$$= \int_0^{2\pi} R^2\frac{r^2}{2} + \frac{r^4}{4} + Rr^2\sin\theta \Big|_{r=0}^{r=R} d\theta$$

$$= \int_0^{2\pi} \left(\frac{3}{4}R^4 + R^4\sin\theta\right) d\theta = \frac{3}{4}R^4(2\pi) + 0$$

$$= \frac{3}{2}\pi R^4$$

$\therefore$ Volume $= \frac{3}{2}\pi R^4 > \frac{\pi}{2}R^4$, so any $R > 0$ works.

(c)

$$\nabla T = (6x, 2y-2R, 32z)$$

$$\therefore \vec{V} = -K(6x, 2y-2R, 32z)$$

(1) Top surface: parametrization is,

$$T(r,\theta) = (r\cos\theta, R + r\sin\theta, 4R^2)$$

$$0 \le \theta \le 2\pi, \quad 0 \le r \le R$$

$$\therefore \vec{T_r} \times \vec{T_\theta} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{K} \\ \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = (0,0,r)$$ which points "out"

$$\therefore \vec{V} \cdot (\vec{T_r} \times \vec{T_\theta}) = -32Kzr = -32K(4R^2)r$$

$$= -128KR^2 r$$

$$\therefore \iint_T \vec{V} \cdot (\vec{T_r} \times \vec{T_\theta})\, dr\, d\theta = \int_0^{2\pi}\int_0^R -128KR^2 r \, dr\, d\theta$$

$$= -256\pi K R^2 \left[\frac{r^2}{2}\right]_0^R = \underline{-128\pi K R^4}$$

(2) Side surface: parametrization is, as in (a),

$$T(\theta, z) = (R\cos\theta, R + R\sin\theta, z), \quad 0 \le \theta \le 2\pi,$$

$$2R^2 - 2R^2\sin\theta \le z \le 4R^2$$

$$\therefore \vec{T}_\theta \times \vec{T}_z = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -R\sin\theta & R\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = (R\cos\theta, R\sin\theta, 0)$$

Note this points "out".

$$\therefore \vec{V} \cdot (\vec{T}_\theta \times \vec{T}_z) = -K(6x, 2y - 2R, 32z) \cdot (R\cos\theta, R\sin\theta, 0)$$

$$= -K(6xR\cos\theta + 2yR\sin\theta - 2R^2\sin\theta)$$

$$= -K[6(R\cos\theta)R\cos\theta + 2(R + R\sin\theta)R\sin\theta - 2R^2\sin\theta]$$

$$= -K[6R^2\cos^2\theta + 2R^2\sin^2\theta]$$

$$= -K[4R^2\cos^2\theta + 2R^2]$$

$$\therefore \iint_T \vec{V} \cdot (\vec{T}_\theta \times \vec{T}_z)\, d\theta\, dz = \int_0^{2\pi} \int_{2R^2 - 2R^2\sin\theta}^{4R^2} -K[4R^2\cos^2\theta + 2R^2]\, dz\, d\theta$$

$$= -K \int_0^{2\pi} (4R^2\cos^2\theta + 2R^2)(2R^2 + 2R^2\sin\theta)\, d\theta$$

$$(2)(2)(2R^2\cos^2\theta + R^2)(R^2\sin\theta + R^2)$$

$$= -4K\int_0^{2\pi} 2R^4\cos^2\theta\sin\theta + R^4\sin\theta + 2R^4\cos^2\theta + R^4\, d\theta$$

$\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$

$$= -4KR^4\left[-\frac{2\cos^3\theta}{3} - \cos\theta + \theta + \frac{\sin 2\theta}{2} + \theta\right]_0^{2\pi}$$

$$= -4KR^4\left[-\frac{2}{3} - 1 + 0 + 4\pi - \left(-\frac{2}{3} - 1 + 0\right)\right]$$

$$= \underline{-16\pi K R^4}$$

(3) Paraboloid surface: a preliminary parametrization is:

$T(r,\theta) = (r\cos\theta,\ R + r\sin\theta,\ z)$, $0 \le \theta \le 2\pi$, $0 \le r \le R$

This time, $z=0$ is the bottom, and the paraboloid is the top.

Here, $z = 4R^2 - (x^2+y^2) = 4R^2 - (r\cos\theta)^2 - (R + r\sin\theta)^2$

$$= 4R^2 - r^2\cos^2\theta - R^2 - 2Rr\sin\theta - r^2\sin^2\theta$$

$$= 3R^2 - r^2 - 2Rr\sin\theta$$

$\therefore T(r,\theta) = (r\cos\theta,\ R + r\sin\theta,\ 3R^2 - r^2 - 2Rr\sin\theta)$

$$\vec{T}_\theta \times \vec{T}_r = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -r\sin\theta & r\cos\theta & -2Rr\cos\theta \\ \cos\theta & \sin\theta & -2r - 2R\sin\theta \end{vmatrix}$$

$= (-2r^2\cos\theta - 2Rr\sin\theta\cos\theta + 2Rr\sin\theta\cos\theta,$
$-2Rr\cos^2\theta - 2r^2\sin\theta - 2Rr\sin^2\theta,$
$-r\sin^2\theta - r\cos^2\theta)$

$= (-2r^2\cos\theta, -2Rr - 2r^2\sin\theta, -r)$

Notice this points down $(-r)$ and in, and so points "out", consistent with the normals to the other two sides.

$\therefore \vec{V}\cdot(\vec{T}_\theta \times \vec{T}_r) =$

$-K(6x, 2y-2R, 32z)\cdot(-2r^2\cos\theta, -2Rr - 2r^2\sin\theta, -r)$

$= -K(6r\cos\theta, 2r\sin\theta, 96R^2 - 32r^2 - 64Rr\sin\theta)\cdot$
$(-2r^2\cos\theta, -2Rr - 2r^2\sin\theta, -r)$

$= K[12r^3\cos^2\theta + 4Rr^2\sin\theta + 4r^3\sin^2\theta$
$+ 96R^2 r - 32r^3 - 64Rr^2\sin\theta]$

$= K[8r^3\cos^2\theta - 28r^3 - 60Rr^2\sin\theta + 96R^2 r]$

$\therefore \iint_T \vec{V}\cdot(\vec{T}_r \times \vec{T}_\theta)\,dr\,d\theta =$

$$4k\int_0^{2\pi}\int_0^R (2r^3\cos^2\theta - 7r^3 - 15Rr^2\sin\theta + 24R^2 r)\,dr\,d\theta$$

$$= 4k\int_0^{2\pi} \frac{r^4}{2}\cos^2\theta - \frac{7}{4}r^4 - 5Rr^3\sin\theta + 12R^2r^2\Big|_0^R \,d\theta$$

$$= 4k\int_0^{2\pi} \frac{R^4}{2}\cos^2\theta - \frac{7R^4}{4} - 5R^4\sin\theta + 12R^4\,d\theta$$

$\cos^2\theta = \frac{1+\cos 2\theta}{2}$

$$= 4k\left[\frac{R^4}{4}\theta + \frac{R^4}{2}\frac{\sin 2\theta}{4} - \frac{7R^4}{4}\theta + 5R^4\cos\theta + 12R^4\theta\right]_0^{2\pi}$$

$$= 4k\left[\frac{R^4\pi}{2} + 0 - \frac{7R^4\pi}{2} + 5R^4 + 24R^4\pi - (0 + 0 - 0 + 5R^4 + 0)\right]$$

$$= 4k\left[\frac{R^4\pi}{2} - \frac{7R^4\pi}{2} + 24R^4\pi\right]$$

$$= 2kR^4\pi - 14R^4\pi + 96kR^4\pi = \underline{84\pi kR^4}$$

$$\therefore (1) + (2) + (3) = -128\pi kR^4 - 16\pi kR^4 + 84\pi kR^4$$

$$= \underline{60\pi kR^4}$$

13.

Using spherical coordinates, unit sphere is described by

$T(\theta,\phi) = (\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi) = (x,y,z)$

As seen in Example 1, p. 401 of text,

$\vec{T}_\phi \times \vec{T}_\theta = -\vec{T}_\theta \times \vec{T}_\phi = (\sin^2\phi\cos\theta, \sin^2\phi\sin\theta, \sin\phi\cos\phi).$

$= \sin\phi\,(x,y,z)$

$\therefore\ \vec{V}\cdot(\vec{T}_\phi \times \vec{T}_\theta) = \sin\phi\,(3xy^2, 3x^2y, z^3)\cdot(x,y,z)$

$= \sin\phi\,[3x^2y^2 + 3x^2y^2 + z^4]$

$= \sin\phi\,(6x^2y^2 + z^4)$

$= \sin\phi\,[6(\cos^2\theta\sin^2\phi)(\sin^2\theta\sin^2\phi) + \cos^4\phi]$

$= \sin\phi\,[6\cos^2\theta\sin^2\theta\sin^4\phi + \cos^4\phi]$

$= 6\cos^2\theta\sin^2\theta\sin^5\phi + \cos^4\phi\sin\phi$

$\therefore \iint_T \vec{V}\cdot(\vec{T}_\phi\times\vec{T}_\theta)\,d\theta\,d\phi =$

$$\int_0^{2\pi}\int_0^{\pi}\left[6\cos^2\theta\sin^2\theta\sin^5\phi + \cos^4\phi\sin\phi\right]d\phi\,d\theta$$

$$= \int_0^{2\pi} 6\cos^2\theta \sin^2\theta (1-\cos^2\phi)^2 \sin\phi \, d\phi \, d\theta + 2\pi \left[-\frac{\cos^5\phi}{5} \right]_{\phi=0}^{\phi=\pi}$$

$$= \int_0^{2\pi} \int_0^{\pi} 6\cos^2\theta \sin^2\theta \left(1 - 2\cos^2\phi + \cos^4\phi\right) \sin\phi \, d\phi \, d\theta + 2\pi \left[\frac{1}{5} - \left(-\frac{1}{5}\right) \right]$$

$$= \int_0^{2\pi} 6\cos^2\theta \sin^2\theta \left[-\cos\phi + \frac{2}{3}\cos^3\phi - \frac{\cos^5\phi}{5} \right]_{\phi=0}^{\phi=\pi} d\theta + \frac{4}{5}\pi$$

$$= \int_0^{2\pi} 6\cos^2\theta \sin^2\theta \left[1 - \frac{2}{3} + \frac{1}{5} - \left(-1 + \frac{2}{3} - \frac{1}{5} \right) \right] d\theta + \frac{4}{5}\pi$$

$$= \int_0^{2\pi} 6\left(\frac{16}{15}\right) \cos^2\theta \sin^2\theta \, d\theta + \frac{4}{5}\pi$$

$$= \int_0^{2\pi} \frac{8}{5} 4\cos^2\theta \sin^2\theta \, d\theta + \frac{4}{5}\pi = \frac{8}{5} \int_0^{2\pi} \sin^2 2\theta \, d\theta + \frac{4}{5}\pi$$

$$\frac{1-\cos 2\theta}{2} = \sin^2\theta$$

$$= \frac{8}{5} \int_0^{2\pi} \frac{1-\cos 4\theta}{2} \, d\theta + \frac{4}{5}\pi = \frac{4}{5}\pi + \frac{8}{5} \left[\frac{\theta}{2} - \frac{\sin 4\theta}{8} \right]_0^{2\pi}$$

$$= \frac{4}{5}\pi + \frac{8}{5} \left[\pi - 0 - (0-0) \right] = \frac{4}{5}\pi + \frac{8}{5}\pi$$

$$= \underline{\frac{12}{5}\pi}$$

14.

The cylinder has a circular side, a top and bottom.

(a) Side: $T(\theta, z) = (\cos\theta, \sin\theta, z)$, $0 \leq \theta \leq 2\pi$, $0 \leq z \leq 1$.

$$\therefore \vec{T}_\theta \times \vec{T}_z = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = (\cos\theta, \sin\theta, 0) = \vec{n},$$

and $\|\vec{n}\| = 1$

$$\therefore \vec{F} \cdot \vec{n} = (1, 1, z(x^2+y^2)^2) \cdot (x, y, 0) = x + y + 0 = x + y$$

$$\therefore \vec{F} \cdot \vec{n} = \cos\theta + \sin\theta$$

$$\therefore \iint_S \vec{F} \cdot \vec{n}\, dA = \int_0^{2\pi} \int_0^1 (\cos\theta + \sin\theta)\, dz\, d\theta$$

$$= \int_0^{2\pi} (\cos\theta + \sin\theta)\, d\theta = \Big[\sin\theta - \cos\theta\Big]_0^{2\pi}$$

$$= [0 - 1 - (0 - 1)] = \underline{0}$$

(b) Top: $T(r, \theta) = (r\cos\theta, r\sin\theta, 1)$, $0 \leq r \leq 1$, $0 \leq \theta \leq 2\pi$

$$\vec{T}_r \times \vec{T}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = (0, 0, r) = r\hat{k}$$

Note $x^2 + y^2 = r^2$

$$\therefore \vec{n} = \hat{k}$$

$$\therefore \vec{F}\cdot\vec{n} = (1,1,z(x^2+y^2)^2)\cdot(0,0,1) = z(x^2+y^2)^2$$

$$= r^4$$

$$\therefore \iint_S \vec{F}\cdot\vec{n}\,dA = \int_0^{2\pi}\int_0^1 r^4\,dr\,d\theta = 2\pi\left[\frac{r^5}{5}\right]_0^1$$

$$= \underline{\frac{2}{5}\pi}$$

(c) Bottom: $T(r,\theta) = (r\cos\theta, r\sin\theta, 0)$, $0 \le r \le 1, 0 \le \theta \le 2\pi$

Keeping with orientation of "out", $\vec{n} = -\hat{k}$

$$\therefore \vec{F}\cdot\vec{n} = (1,1,z(x^2+y^2)^2)\cdot(0,0,-1) = -z(x^2+y^2)^2$$

$$= 0 \text{ since } z = 0$$

$$\therefore \iint_S \vec{F}\cdot\vec{n}\,dA = 0$$

$$\therefore (a) + (b) + (c) = \underline{\frac{2}{5}\pi}$$

15.

Using spherical coordinates, a parametrization for S is: $T(\phi,\theta) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi) = (x,y,z) = \vec{r}$

$0 \le \phi \le \pi,\ 0 \le \theta \le 2\pi$

As in Example 1, p. 401 of text,

$$\vec{T}_\phi \times \vec{T}_\theta = \sin\phi(\cos\theta, \sin\theta, \cos\phi) = \sin\phi\,\vec{r},$$

where $\vec{r}$ (i.e., the normal) points "out".

$$\therefore\ \vec{F}\cdot(\vec{T}_\phi \times \vec{T}_\theta) = \vec{F}\cdot(\sin\phi\,\vec{r}) = \sin\phi(\vec{F}\cdot\vec{r})$$

$$= \sin\phi\, F_r,$$

where F_r = component of $\vec{F}$ along $\vec{r}$, by geometric interpretation of dot product.

$$\therefore \iint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}\cdot(\vec{T}_\phi \times \vec{T}_\theta)\, d\phi\, d\theta$$

$$= \iint_D F_r \sin\phi\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi} F_r \sin\phi\, d\phi\, d\theta$$

For real valued functions, $\|\vec{T}_\phi \times \vec{T}_\theta\| = \|\sin\phi\,\vec{r}\| = \sin\phi$ since $\|\vec{r}\| = 1$

$$\therefore \iint_S F\, dS = \iint_D F(T(\phi,\theta))\,\|\vec{T}_\phi \times \vec{T}_\theta\|\, d\phi\, d\theta$$

$$= \iint_D F(T(\phi,\theta)) \sin\phi \, d\phi d\theta$$

16.

Assume S is a C^1 surface, so that the parametrization $\phi(u,v): R^2 \to R^3$ is class C^1.

$\therefore$ $\phi_u \times \phi_v$ is continuos, and so is $\|\phi_u \times \phi_v\|$, and $\therefore$

$\vec{F} \cdot \left(\frac{\phi_u \times \phi_v}{\|\phi_u \times \phi_v\|}\right) = \vec{F} \cdot \vec{n}$ is continuous on D as $\vec{F}$ is continuous. Let $G(u,v) = \vec{F}(\phi(u,v)) \cdot \vec{n}(u,v)$

and assume D is closed. $\therefore$ G contains a maximum and minimum on D.

Let $Q_{min} \in D$ s.t. $G(Q_{min})$ = the minimum.

$Q_{max} \in D$ s.t. $G(Q_{max})$ = The maximum.

$\therefore$ $G(Q_{min}) \leq G(u,v) \leq G(Q_{max})$, $(u,v) \in D$.

$\therefore$ Given a partition of D,

$$\sum_u \sum_v G(Q_{min}) \leq \sum_u \sum_v G(u,v) \leq \sum_u \sum_v G(Q_{max}),$$

or $$\iint_D G(Q_{min})\,dS \leq \iint_S G\,dS \leq \iint_S G(Q_{max})\,dS$$

or $$G(Q_{min})\iint_S dS \leq \iint_S G\,dS \leq G(Q_{max})\iint dS$$

or $$G(Q_{min})\,A(S) \leq \iint_S G\,dS \leq G(Q_{max})\,A(S)$$

$$\therefore \quad G(Q_{min}) \leq \frac{\iint_S G\,dS}{A(S)} \leq G(Q_{max})$$

By Intermediate Value Theorem for continuous functions on a closed domain, there exists an element $Q \in D$, s.t.

$$G(Q) = \frac{\iint_D G\,dS}{A(S)}$$

But $G(Q) = \vec{F}(\phi(Q)) \cdot \vec{n}(Q)$

$\therefore$ There exists a $Q \in D$ s.t.

$$\left[\vec{F}(\phi(Q)) \cdot \vec{n}(Q)\right] A(s) = \iint_S \vec{F} \cdot \vec{n}\, dS$$

17.

A parametrization for a unit cylinder perpendicular to the xy-plane is: $T(\theta, z) = (\cos\theta, \sin\theta, z)$,
$0 \leq \theta \leq 2\pi$, $a \leq z \leq b$
where the cylinder has a height from a to b.

$\therefore$ Normal to cylinder is:

$$\vec{T}_\theta \times \vec{T}_z = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = (\cos\theta, \sin\theta, 0)$$

and this normal points "out" from origin.

Let $\vec{n} = (\cos\theta, \sin\theta, 0)$.

Let $\vec{F} = (F_x, F_y, F_z)$. $\therefore \vec{F} \cdot \vec{n} = F_x \cos\theta + F_y \sin\theta$

But $\vec{F} \cdot \vec{n}$ is the radial component of $\vec{F}$: i.e., the component in the xy-plane parallel to $\vec{n}$.

$\therefore$ Let $F_r = F_x \cos\theta + F_y \sin\theta$.

$$\therefore \iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot \vec{n}\, dA = \int_0^{2\pi} \int_a^b F_r\, dz\, d\theta$$

18.

(a) A parametrization for S is $T(x,y) = (x,y,0)$.

$$\vec{T}_x \times \vec{T}_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = (0,0,1) = \hat{k}$$

$$\therefore \|\vec{T}_x \times \vec{T}_y\| = 1.$$

By definition,

$$\iint_S f(x,y,z)\,dS = \iint_D f(T(x,y))\,\|\vec{T}_x \times \vec{T}_y\|\,dx\,dy$$

$$= \iint_D f(x,y,0)\,dx\,dy$$

(b) The surface integral of $\vec{F} = (F_x, F_y, F_z)$ is:

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot (\vec{T}_x \times \vec{T}_y)\,dx\,dy = \iint_D F_z\,dx\,dy$$

19.

Using spherical coordinates, a parametrization of the surface is: $T(\phi, \theta) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi) = (x,y,z)$

$0 \leq \phi \leq \frac{\pi}{2}$, $0 \leq \theta \leq 2\pi$.

As in Example 1, p. 401 of the text,

$$\vec{T}_\phi \times \vec{T}_\theta = \sin\phi(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$$

$$= \sin\phi\,(x,y,z).$$

$$\therefore \vec{F}\cdot(\vec{T}_\phi \times \vec{T}_\theta) = (1, x, z)\cdot[\sin\phi\,(x,y,z)]$$

$$= \sin\phi\,(x + xy + z^2)$$

$$= \sin\phi[\sin\phi\cos\theta + (\sin\phi\cos\theta)(\sin\phi\sin\theta) + \cos^2\phi]$$

$$= \sin^2\phi\cos\theta + \sin^3\phi\cos\theta\sin\theta + \sin\phi\cos^2\phi$$

$$\therefore \iint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}\cdot(\vec{T}_\phi \times \vec{T}_\theta)\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \sin^2\phi\cos\theta\,d\phi\,d\theta$$

$$+ \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \sin^3\phi\cos\theta\sin\theta\,d\phi\,d\theta + \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \sin\phi\cos^2\phi\,d\phi\,d\theta$$

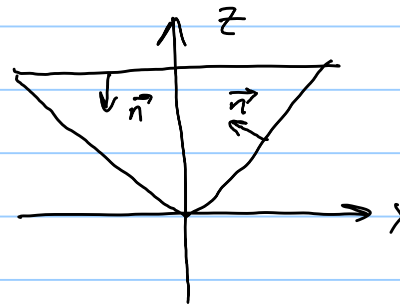
$$= \int_0^{2\pi} \cos\theta\, d\theta \int_0^{\frac{\pi}{2}} \sin^2\phi\, d\phi + \int_0^{2\pi} \cos\theta \sin\theta\, d\theta \int_0^{\frac{\pi}{2}} \sin^3\phi\, d\phi + 2\pi \int_0^{\frac{\pi}{2}} \sin\phi \cos^2\phi\, d\phi$$

$$= \sin\theta \Big|_0^{2\pi} \int_0^{\frac{\pi}{2}} \sin^2\phi\, d\phi + \frac{\sin^2\theta}{2}\Big|_0^{2\pi} \int_0^{\frac{\pi}{2}} \sin^3\phi\, d\phi + 2\pi \left[-\frac{\cos^3\phi}{3}\right]_0^{\frac{\pi}{2}}$$

$$= 0 \cdot \int_0^{\pi/2} \sin^2\phi\, d\phi + 0 \cdot \int_0^{\pi/2} \sin^3\phi\, d\phi + 2\pi \left[0 - \left(-\frac{1}{3}\right)\right]$$

$$= 0 + 0 + \frac{2}{3}\pi = \underline{\frac{2\pi}{3}}$$

20.



(a) Choose normals (orientation) of surfaces that point inward to the cone, and ∴ computing flux "into" the cone.

(1) Top surface, a disk, with parametrization

$$T(r,\theta) = (r\cos\theta, r\sin\theta, 1), \quad 0 \le r \le 1,\ 0 \le \theta \le 2\pi$$

$$\therefore T_r \times T_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = (0, 0, r)$$

We want $\vec{n}$ to point down, $\therefore$ take

$$\vec{T}_\theta \times \vec{T}_r = -\vec{T}_r \times \vec{T}_\theta = (0,0,-r)$$

$$\therefore \vec{F} \cdot (\vec{T}_\theta \times \vec{T}_r) = (0,0,-1) \cdot (0,0,-r) = r$$

$$\therefore \iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot (\vec{T}_\theta \times \vec{T}_r)\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^1 r\, dr\, d\theta = 2\pi \left[\frac{r^2}{2} \right]_0^1 = \underline{\pi}$$

(2) Side of cone: parametrization is:

$$T(r,\theta) = (r\cos\theta, r\sin\theta, z), \text{ where}$$

$$z = (x^2+y^2)^{\frac{1}{2}} = (r^2\cos^2\theta + r^2\sin^2\theta)^{\frac{1}{2}} = r$$

$$\therefore T(r,\theta) = (r\cos\theta, r\sin\theta, r), \quad 0 \le r \le 1, \ 0 \le \theta \le 2\pi$$

$$\vec{T}_r \times \vec{T}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = (-r\cos\theta, -r\sin\theta, r)$$

This normal points up ($r\hat{k}$ component) and in ($-r\cos\theta\,\hat{i} - r\sin\theta\,\hat{j}$), which is toward inside of cone.

$$\therefore \vec{F}\cdot(\vec{T}_r \times \vec{T}_\theta) = (0,0,-1)\cdot(-r\cos\theta, -r\sin\theta, r)$$

$$= -r.$$

$$\therefore \iint_S \vec{F}\cdot d\vec{S} = \iint_D (-r)\,dr\,d\theta = \int_0^{2\pi}\int_0^1 (-r)\,dr\,d\theta$$

$$= \underline{-\pi}$$

$$(1)+(2) = \pi - \pi = \underline{0}$$

(5)

(1) Top surface

From (a) $\vec{T}_\theta \times \vec{T}_r = (0,0,-r)$

$$\therefore \vec{F}\cdot(\vec{T}_\theta \times \vec{T}_r) = \left(-\frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2}\right)\cdot(0,0,-r) = \frac{\sqrt{2}}{2}r$$

$$\therefore \int_0^{2\pi}\int_0^1 \frac{\sqrt{2}}{2} r\,dr\,d\theta = \underline{\frac{\sqrt{2}}{2}\pi}$$

(2) Side of cone

From (a), $\vec{T}_r \times \vec{T}_\theta = (-r\cos\theta, -r\sin\theta, r)$

$$\therefore \vec{F}\cdot(\vec{T}_r \times \vec{T}_\theta) = \left(-\frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2}\right)\cdot(-r\cos\theta, -r\sin\theta, r)$$

$$= \frac{\sqrt{2}}{2} r\cos\theta - \frac{\sqrt{2}}{2} r$$

$$\therefore \int_0^{2\pi}\int_0^1 \left(\frac{\sqrt{2}}{2} r\cos\theta - \frac{\sqrt{2}}{2} r\right) dr\,d\theta$$

$$= \int_0^{2\pi} \cos\theta\, d\theta \int_0^1 \frac{\sqrt{2}}{2} r\,dr - \int_0^{2\pi}\int_0^1 \frac{\sqrt{2}}{2} r\,dr\,d\theta$$

$$= 0\cdot\frac{\sqrt{2}}{4} - 2\pi\left(\frac{\sqrt{2}}{4}\right) = \underline{-\frac{\sqrt{2}}{2}\pi}$$

$$(1) + (2) = \frac{\sqrt{2}}{2}\pi - \frac{\sqrt{2}}{2}\pi = \underline{0}$$

21.

Let $T(r,\theta) = (ar\cos\theta, br\sin\theta, z)$, $0 \le r \le 1$, $0 \le \theta \le 2\pi$

where $z = c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} = c\sqrt{1 - \frac{a^2r^2\cos^2\theta}{a^2} - \frac{b^2r^2\sin^2\theta}{b^2}}$

$$= c\sqrt{1 - r^2}$$

$\therefore T(r,\theta) = \left(ar\cos\theta,\ br\sin\theta,\ c\sqrt{1-r^2}\right)$

$$\therefore \vec{T}_r \times \vec{T}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a\cos\theta & b\sin\theta & -cr(1-r^2)^{-\frac{1}{2}} \\ -ar\sin\theta & br\cos\theta & 0 \end{vmatrix}$$

$$= \left(bcr^2(1-r^2)^{-\frac{1}{2}}\cos\theta,\ acr^2(1-r^2)^{-\frac{1}{2}},\ abr\right)$$

Since $abr \geq 0$, $abr\hat{k}$ points up.

$$\therefore \vec{F}\cdot(\vec{T}_r \times \vec{T}_\theta) = x^3\left[bcr^2(1-r^2)^{-\frac{1}{2}}\cos\theta\right]$$

$$= (a^3r^3\cos^3\theta)(bcr^2(1-r^2)^{-\frac{1}{2}}\cos\theta)$$

$$= \frac{a^3bcr^5\cos^4\theta}{\sqrt{1-r^2}}$$

$$\therefore \iint_S \vec{F}\cdot d\vec{S} = a^3bc\int_0^{2\pi}\int_0^1 \frac{\cos^4\theta\, r^5}{\sqrt{1-r^2}}\,dr\,d\theta$$

$$= a^3bc\int_0^{2\pi}\cos^4\theta\,d\theta \int_0^1 \frac{r^5}{\sqrt{1-r^2}}\,dr$$

let $r = \sin\phi$

$\therefore dr = \cos\phi\, d\phi$

$\sqrt{1-r^2} = \cos\phi$

$$= a^3bc\int_0^{2\pi}\cos^2\theta(1-\sin^2\theta)\,d\theta \int_0^{\frac{\pi}{2}} \sin^5\phi\, d\phi$$

$\sin^5\phi = \sin^4\phi\sin\phi$

$= (1-\cos^2\phi)^2\sin\phi$

$$= a^3 bc \int_0^{2\pi} \cos^2\theta - (\cos\theta \sin\theta)^2 \, d\theta \int_0^{\frac{\pi}{2}} (1 - 2\cos^2\phi + \cos^4\phi)\sin\phi \, d\phi$$

$\cos^2\theta = \frac{1+\cos 2\theta}{2}$, $\sin 2\theta = 2\sin\theta\cos\theta$

$$= a^3 bc \int_0^{2\pi} \left[\frac{1+\cos 2\theta}{2} - \frac{\sin^2 2\theta}{4}\right] d\theta \left[-\cos\phi + \frac{2}{3}\cos^3\phi - \frac{\cos^5\phi}{5}\right]_0^{\frac{\pi}{2}}$$

$\sin^2 x = \frac{1-\cos 2x}{2}$

$$= a^3 bc \int_0^{2\pi} \frac{1}{2} + \frac{\cos 2\theta}{2} - \left(\frac{1-\cos 4\theta}{8}\right) d\theta \left[0 - \left(-1 + \frac{2}{3} - \frac{1}{5}\right)\right]$$

$$= a^3 bc \left[\frac{\theta}{2} + \frac{\sin 2\theta}{4} - \frac{\theta}{8} + \frac{\sin 4\theta}{32}\right]_0^{2\pi} \left[\frac{8}{15}\right]$$

$$= a^3 bc \left[\pi + 0 - \frac{\pi}{4} + 0\right]\left[\frac{8}{15}\right] = a^3 bc \left(\frac{3}{4}\pi\right)\left(\frac{8}{15}\right)$$

$$= \underline{\underline{\frac{2}{5} a^3 bc \pi}}$$

22.

Let $T(\phi, \theta) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$, $0 \le \phi \le \frac{\pi}{2}$, $0 \le \theta \le 2\pi$

$\therefore \vec{T}_\phi \times \vec{T}_\theta = (\sin^2\phi\cos\theta, \sin^2\phi\sin\theta, \sin\phi\cos\phi)$, as in

Example 1, p. 401 of the text. This normal points "out".

(a)

$\vec{F} = (\sin\phi\cos\theta, \sin\phi\sin\theta, 0).$

$\therefore \vec{F}\cdot(\vec{T}_\phi \times \vec{T}_\theta) = \sin^3\phi\cos^2\theta + \sin^3\phi\sin^2\theta = \sin^3\phi$

$\therefore \iint_S \vec{F}\cdot d\vec{S} = \int_0^{2\pi}\int_0^{\pi} \sin^3\phi \, d\phi \, d\theta = 2\pi\int_0^{\pi}\sin^3\phi \, d\phi$

$= 2\pi\int_0^{\pi}(1-\cos^2\phi)\sin\phi \, d\phi = 2\pi\left[-\cos\phi + \frac{\cos^3\phi}{3}\right]_0^{\pi}$

$= 2\pi\left[1 - \frac{1}{3} - \left(-1 + \frac{1}{3}\right)\right] = 2\pi\left(\frac{4}{3}\right) = \underline{\frac{8}{3}\pi}$

(b)

$\vec{F} = (\sin\phi\sin\theta, \sin\phi\cos\theta, 0)$

$\therefore \vec{F}\cdot(\vec{T}_\phi \times \vec{T}_\theta) = \sin^3\phi\sin\theta\cos\theta + \sin^3\phi\sin\theta\cos\theta$

$= \sin^3\phi(2\sin\theta\cos\theta) = \sin^3\phi\sin 2\theta$

$\therefore \iint_S \vec{F}\cdot d\vec{S} = \int_0^{2\pi}\int_0^{\pi}\sin^3\phi\sin 2\theta \, d\phi \, d\theta$

$= \int_0^{2\pi}\sin 2\theta \, d\theta \int_0^{\pi}\sin^3\phi \, d\phi$

$$= \left[-\frac{\cos 2\theta}{2} \Big|_0^{2\pi} \right] \left[\frac{4}{3} \right]$$ from (a), $\int_0^{\pi} \sin^3\phi \, d\phi = \frac{4}{3}$

$$= \left[-\frac{1}{2} - \left(-\frac{1}{2}\right) \right] \left[\frac{4}{3} \right] = 0 \cdot \frac{4}{3} = \underline{\underline{0}}$$

(c)

(1) $\vec{F} = (x, y, 0)$

$$\therefore \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & 0 \end{vmatrix} = (0, 0, 0)$$

$$\therefore (\nabla \times \vec{F}) \cdot d\vec{S} = 0$$

$$\therefore \iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \underline{0}$$

(2) $\vec{F} = (y, x, 0)$ $\quad \therefore \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & x & 0 \end{vmatrix} = (0, 0, 1-1) = (0, 0, 0)$

$$\therefore (\nabla \times \vec{F}) \cdot d\vec{S} = 0 \quad \therefore \iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \underline{\underline{0}}$$

(3) Let $\vec{c}(t) = (\cos t, \sin t, 0)$, $0 \le t \le 2\pi$

$\therefore \vec{c}\,'(t) = (-\sin t, \cos t, 0)$.

$\vec{F}(\vec{c}(t)) \cdot \vec{c}\,'(t) = (\cos t, \sin t, 0) \cdot (-\sin t, \cos t, 0)$

$= -\sin t \cos t + \sin t \cos t = 0$

$\therefore \iint_C \vec{F} \cdot d\vec{s} = \int_0^{2\pi} 0 \, dt = \underline{0}$

(4) Use $\vec{c}(t)$ as in (3)

$\vec{F}(\vec{c}(t)) = (\sin t, \cos t, 0)$

$\therefore \vec{F}(\vec{c}(t)) \cdot \vec{c}\,'(t) = (\sin t, \cos t, 0) \cdot (-\sin t, \cos t, 0)$

$= \cos^2 t - \sin^2 t = \cos 2t$

$\therefore \iint_C \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \cos 2t \, dt = \frac{\sin 2t}{2} \Big|_0^{2\pi} = \underline{0}$

7.7 Applications to Differential Geometry, Physics, and Forms of Life

Note Title 4/24/2017

1.

$\phi_u = (\cos v, \sin v, 0) \quad \phi_v = (-u\sin v, u\cos v, 6)$

$\phi_{uu} = (0,0,0) \quad \phi_{vv} = (-u\cos v, -u\sin v, 0)$

$\phi_{uv} = (-\sin v, \cos v, 0)$

$$\phi_u \times \phi_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 6 \end{vmatrix} = (6\sin v, -6\cos v, u)$$

$\|\phi_u \times \phi_v\| = \sqrt{6^2+u^2} \quad \therefore W = 6^2+u^2$

$\therefore l(u,v) = \vec{N}(u,v) \cdot \phi_{uu} = 0$

$$m(u,v) = \vec{N}(u,v)\cdot\phi_{uv} = \frac{(6\sin v, -6\cos v, u)}{\sqrt{6^2+u^2}} \cdot (-\sin v, \cos v, 0)$$

$$= \frac{-6}{\sqrt{6^2+u^2}}$$

$$n(u,v) = \vec{N}(u,v)\cdot\phi_{vv} = \frac{(6\sin v, -6\cos v, u)}{\sqrt{6^2+u^2}} \cdot (-u\cos v, -u\sin v, 0)$$

$$= \frac{-ub\sin v\cos v + ub\cos v\sin v + 0}{\sqrt{b^2+u^2}} = 0$$

$$\therefore K = \frac{\ell n - m^2}{W} = \frac{(0)\left(\frac{-2bu\cos v\sin v}{\sqrt{b^2+u^2}}\right) - \left(\frac{-b}{\sqrt{b^2+u^2}}\right)^2}{b^2+u^2}$$

$$= \underline{\underline{-\frac{b^2}{(b^2+u^2)^2}}}$$

$$H = \frac{G\ell + En - 2Fm}{2W}$$

$$= \frac{\|\phi_v\|^2 \overset{\ell=0}{(0)} + \|\phi_u\|^2 \overset{n=0}{(0)} - 2\phi_u\cdot\phi_v\left(\frac{-b}{\sqrt{b^2+u^2}}\right)}{2(b^2+u^2)}$$

$$= \frac{-2(\cos v, \sin v, 0)\cdot(-u\sin v, u\cos v, b)\left(\frac{-b}{\sqrt{b^2+u^2}}\right)}{2(b^2+u^2)}$$

$$= \frac{-ub\cos v\sin v + ub\sin v\cos v + 0}{(b^2+u^2)(\sqrt{b^2+u^2})} = 0$$

$$\therefore \underline{H = 0}$$

2.

Let $\phi(x,y) = (x, y, xy)$

$\therefore \phi_x = (1,0,y) \quad \phi_y = (0,1,x)$

$\phi_{xx} = (0,0,0) \quad \phi_{yy} = (0,0,0) \quad \phi_{xy} = (0,0,1)$

$E = \|\phi_x\|^2 = 1+y^2 \quad G = \|\phi_y\|^2 = 1+x^2$

$F = \phi_x \cdot \phi_y = xy$

$$\phi_x \times \phi_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & y \\ 0 & 1 & x \end{vmatrix} = (-y, -x, 1)$$

$W = \|\phi_x \times \phi_y\|^2 = 1 + x^2 + y^2$

$$\vec{N} = \frac{\phi_x \times \phi_y}{\|\phi_x \times \phi_y\|} = \frac{1}{\sqrt{1+x^2+y^2}}(-y,-x,1)$$

$\therefore l = \vec{N} \cdot \phi_{xx} = \vec{N} \cdot (0,0,0) = 0$

$$m = \vec{N} \cdot \phi_{xy} = \frac{(-y,-x,1)}{\sqrt{1+x^2+y^2}} \cdot (0,0,1) = \frac{1}{\sqrt{1+x^2+y^2}}$$

$n = \vec{N} \cdot \phi_{yy} = \vec{N} \cdot (0,0,0) = 0$

$$\therefore K = \frac{\ell n - m^2}{W} = \frac{(0)(0) - \frac{1}{1+x^2+y^2}}{1+x^2+y^2} = \underline{-\frac{1}{(1+x^2+y^2)^2}}$$

$$H = \frac{G\ell + En - 2Fm}{2W} = \frac{G(0) + E(0) - 2(xy)\left(\frac{1}{\sqrt{1+x^2+y^2}}\right)}{2(1+x^2+y^2)}$$

$$= \underline{\frac{-xy}{(1+x^2+y^2)^{3/2}}}$$

3.

$$\phi_u = (1, 0, \tan u) \qquad \phi_v = (0, 1, -\tan v)$$

$$\phi_{uu} = (0, 0, \sec^2 u) \qquad \phi_{vv} = (0, 0, -\sec^2 v)$$

$$\phi_{uv} = (0, 0, 0)$$

$$\phi_u \times \phi_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & \tan u \\ 0 & 1 & \tan v \end{vmatrix} = (-\tan u, -\tan v, 1)$$

$$\|\phi_u \times \phi_v\| = \sqrt{1 + \tan^2 u + \tan^2 v}, \quad W = 1 + \tan^2 u + \tan^2 v$$

$$E = \|\phi_u\|^2 = 1 + \tan^2 u \qquad G = \|\phi_v\|^2 = 1 + \tan^2 v$$

$$F = \phi_u \cdot \phi_v = -\tan u \tan v$$

$$\vec{N} = \frac{(-\tan u, -\tan v, 1)}{\sqrt{1+\tan^2 u + \tan^2 v}}$$

$$l = \vec{N} \cdot \phi_{uu} = \frac{(-\tan u, -\tan v, 1)}{\sqrt{1+\tan^2 u + \tan^2 v}} \cdot (0, 0, \sec^2 u)$$

$$= \frac{\sec^2 u}{\sqrt{1+\tan^2 u + \tan^2 v}} = \frac{\sec^2 u}{\sqrt{W}}$$

$$m = \vec{N} \cdot \phi_{uv} = \vec{N} \cdot (0,0,0) = 0$$

$$n = \vec{N} \cdot \phi_{vv} = \frac{(-\tan u, -\tan v, 1)}{\sqrt{1+\tan^2 u + \tan^2 v}} \cdot (0, 0, -\sec^2 v)$$

$$= \frac{-\sec^2 v}{\sqrt{1+\tan^2 u + \tan^2 v}} = \frac{-\sec^2 v}{\sqrt{W}}$$

$$\therefore H = \frac{Gl + En - 2Fm}{2W} \quad m=0$$

$$= \frac{\frac{(1+\tan^2 v)(\sec^2 u)}{\sqrt{W}} + \frac{(1+\tan^2 u)(-\sec^2 v)}{\sqrt{W}}}{2W}$$

$$\sec^2 x = 1 + \tan^2 x$$

$$= \frac{(1+\tan^2 v)(1+\tan^2 u) - (1+\tan^2 u)(1+\tan^2 v)}{2W^2}$$

$$= \underline{0}$$

4.

Let $\phi(x,y) = \left(x, y, \frac{x^2}{a^2} + \frac{y^2}{b^2}\right)$

$\therefore \phi_x = \left(1, 0, \frac{2x}{a^2}\right) \quad \phi_y = \left(0, 1, \frac{2y}{b^2}\right)$

$\phi_{xx} = \left(0, 0, \frac{2}{a^2}\right) \quad \phi_{yy} = \left(0, 0, \frac{2}{b^2}\right)$

$$\phi_{xy} = (0,0,0)$$

$$\phi_x \times \phi_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 2x/a^2 \\ 0 & 1 & 2y/b^2 \end{vmatrix} = \left(-2x/a^2, -2y/b^2, 1\right)$$

$$\therefore \vec{N} = \frac{\left(-\frac{2x}{a^2}, -\frac{2y}{b^2}, 1\right)}{\sqrt{1 + \frac{4x^2}{a^4} + \frac{4y^2}{b^2}}} = \frac{\left(-\frac{2x}{a^2}, \frac{-2y}{b^2}, 1\right)}{\sqrt{W}}$$

$$l = \vec{N} \cdot \phi_{xx} = \frac{\left(-\frac{2x}{a^2}, -\frac{2y}{b^2}, 1\right)}{\sqrt{W}} \cdot \left(0, 0, \frac{2}{a^2}\right) = \frac{2}{a^2\sqrt{W}}$$

$$m = \vec{N} \cdot \phi_{xy} = 0$$

$$n = \vec{N} \cdot \phi_{yy} = \frac{\left(-\frac{2x}{a^2}, -\frac{2y}{b^2}, 1\right)}{\sqrt{W}} \cdot \left(0, 0, \frac{2}{b^2}\right) = \frac{2}{b^2\sqrt{W}}$$

$$\therefore K = \frac{ln - m^2}{W} = \frac{\left(\frac{2}{a^2\sqrt{W}}\right)\left(\frac{2}{b^2\sqrt{W}}\right) - 0}{W}$$

$$= \frac{4}{a^2b^2W^2} = \frac{4}{a^2b^2\left(1 + \frac{4x^2}{a^4} + \frac{4y^2}{b^4}\right)^2}$$

$$= \frac{4}{a^2b^2\left(\frac{a^4b^4 + 4b^4x^2 + 4a^4y^2}{a^4b^4}\right)^2}$$

$$= \underline{\frac{4a^6b^6}{(a^4b^4 + 4b^4x^2 + 4a^4y^2)^2}}$$

5.

Let $\phi(x,y) = \left(x, y, \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)$

$\phi_x = (1, 0, 2x/a^2) \qquad \phi_y = (0, 1, -2y/b^2)$

$\phi_{xx} = \left(0, 0, \frac{2}{a^2}\right) \qquad \phi_{yy} = \left(0, 0, -\frac{2}{b^2}\right)$

$$\phi_{xy} = (0,0,0)$$

$$\phi_x \times \phi_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 2x/a^2 \\ 0 & 1 & -2y/b^2 \end{vmatrix} = \left(-2x/a^2, 2y/b^2, 1\right)$$

$$W = 1 + \frac{4x^2}{a^4} + \frac{4y^2}{b^4} = \frac{a^4b^4 + 4b^4x^2 + 4a^4y^2}{a^4b^4}$$

$$\vec{N} = \frac{\left(-\frac{2x}{a^2}, \frac{2y}{b^2}, 1\right)}{\sqrt{W}}$$

$$l = \vec{N} \cdot \phi_{xx} = \frac{\left(-\frac{2x}{a^2}, \frac{2y}{b^2}, 1\right)}{\sqrt{w}} \cdot \left(0, 0, \frac{2}{a^2}\right) = \frac{2}{a^2\sqrt{W}}$$

$$m = \vec{N} \cdot \phi_{xy} = 0$$

$$n = \vec{N} \cdot \phi_{yy} = \frac{\left(-\frac{2x}{a^2}, \frac{2y}{b^2}, 1\right)}{\sqrt{W}} \cdot \left(0, 0, -\frac{2}{b^2}\right) = -\frac{2}{b^2\sqrt{W}}$$

$$\therefore K = \frac{ln - m^2}{W} = \frac{\left(\frac{2}{a^2\sqrt{W}}\right)\left(-\frac{2}{b^2\sqrt{W}}\right) - 0}{W}$$

$$= -\frac{4}{a^2b^2W^2} = \frac{-4}{a^2b^2\left(\frac{a^4b^4 + 4b^4x^2 + 4a^4y^2}{a^4b^4}\right)^2}$$

$$= \underline{\frac{-4a^6b^6}{(a^4b^4 + 4b^4x^2 + 4a^4y^2)^2}}$$

6.

Let $x = a\cos\theta\sin\phi$, $y = a\sin\theta\sin\phi$, $z = c\cos\phi$

$$\therefore \frac{x^2}{a^2} + \frac{y^2}{a^2} = \frac{a^2\sin^2\phi(\cos^2\theta + \sin^2\theta)}{a^2} = \sin^2\phi$$

$$\therefore \frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{c^2} = \sin^2\phi + \frac{c^2\cos^2\phi}{c^2} = 1.$$

$\therefore T(\theta, \phi) = (a\cos\theta\sin\phi, a\sin\theta\sin\phi, c\cos\phi)$,

$0 \le \theta \le 2\pi$, $0 \le \phi \le \pi$.

$T_\theta = (-a\sin\theta\sin\phi, a\cos\theta\sin\phi, 0)$ $\quad a^2\sin^2\phi$

$T_\phi = (a\cos\theta\cos\phi, a\sin\theta\cos\phi, -c\sin\phi)$ $\quad a^2\cos^2\phi + c^2\sin^2\phi$

$T_{\theta\theta} = (-a\cos\theta\sin\phi, -a\sin\theta\sin\phi, 0) = (-x, -y, 0)$

$T_{\phi\phi} = (-a\cos\theta\sin\phi, -a\sin\theta\sin\phi, -c\cos\phi) = (-x, -y, -z)$

$T_{\theta\phi} = (-a\sin\theta\cos\phi, a\cos\theta\cos\phi, 0)$

$$T_\theta \times T_\phi = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a\sin\theta\sin\phi & a\cos\theta\sin\phi & 0 \\ a\cos\theta\cos\phi & a\sin\theta\cos\phi & -c\sin\phi \end{vmatrix}$$

$$= (-ac\cos\theta\sin^2\phi, -ac\sin\theta\sin^2\phi, -a^2\sin\phi\cos\phi)$$

$$\| T_\theta \times T_\phi \|^2 = a^2 c^2 \sin^4\phi + a^4 \sin^2\phi \cos^2\phi = W$$

$$\therefore \vec{N} = \frac{(-ac\cos\theta \sin^2\phi, -ac\sin\theta\sin^2\phi, -a^2\sin\phi\cos\phi)}{\sqrt{W}}$$

$$= -\frac{c\sin\phi}{\sqrt{W}} \left(x, y, \frac{a^2}{c^2} z \right)$$

$$\therefore \ell = \vec{N} \cdot T_{\theta\theta} = -\frac{c\sin\phi}{\sqrt{W}} \left(x, y, \frac{a^2}{c^2} z\right) \cdot (-x, -y, 0)$$

$$= \frac{c\sin\phi}{\sqrt{W}} (x^2 + y^2)$$

$$= \frac{c\sin\phi}{\sqrt{W}} (a^2 \sin^2\phi) = \underline{\frac{a^2 c \sin^3\phi}{\sqrt{W}}}$$

$$m = \vec{N} \cdot T_{\theta\phi} = \frac{(-ac\cos\theta\sin^2\phi, -ac\sin\theta\sin^2\phi, -a^2\sin\phi\cos\phi)}{\sqrt{W}} \cdot (-a\sin\theta\cos\phi, a\cos\theta\cos\phi, 0)$$

$$= \frac{a^2 c \sin\theta\cos\theta\sin^2\phi\cos\phi - a^2 c\sin\theta\cos\theta\sin^2\phi\cos\phi + 0}{\sqrt{W}} = \underline{0}$$

$$n = \vec{N} \cdot T_{\phi\phi} = -\frac{c\sin\phi}{\sqrt{W}} \left(x, y, \frac{a^2}{c^2} z\right) \cdot (-x, -y, -z)$$

$$= \frac{c\sin\phi}{\sqrt{W}} \left(x^2 + y^2 + \frac{a^2}{c^2} z^2 \right)$$

$$= \frac{c\sin\phi}{\sqrt{W}} \left[x^2 + y^2 + a^2 \left(1 - \frac{x^2}{a^2} - \frac{y^2}{a^2} \right) \right] = \underline{\frac{a^2 c \sin\phi}{\sqrt{W}}}$$

$$\therefore K = \frac{ln - m^2}{W} = \frac{\left(\frac{a^2 c \sin^3\phi}{\sqrt{W}}\right)\left(\frac{a^2 c \sin\phi}{\sqrt{W}}\right) - 0}{W}$$

$$= \frac{a^4 c^2 \sin^4\phi}{W^2} = \frac{a^4 c^2 \sin^4\phi}{(a^2 c^2 \sin^4\phi + a^4 \sin^2\phi \cos^2\phi)^2}$$

$$= \frac{a^4 c^2 \sin^4\phi}{[(a^2 \sin^2\phi)(c^2 \sin^2\phi + a^2 \cos^2\phi)]^2}$$

$$= \frac{c^2}{(c^2 \sin^2\phi + a^2 \cos^2\phi)^2}$$

7.

From #6 above, $K = \dfrac{c^2}{(c^2 \sin^2\phi + a^2 \cos^2\phi)^2}$

$$\frac{1}{2\pi} \iint_S f\, dS = \frac{1}{2\pi} \iint_S K\, dA = \frac{1}{2\pi} \iint_S K \|T_\theta \times T_\phi\|\, d\theta\, d\phi$$

From #6 above, $\|T_\theta \times T_\phi\|^2 = a^2 c^2 \sin^4\phi + a^4 \sin^2\phi \cos^2\phi$,

so $\|T_\theta \times T_\phi\| = a \sin\phi (c^2 \sin^2\phi + a^2 \cos^2\phi)^{\frac{1}{2}}$

$$\therefore K \|T_\theta \times T_\phi\| = \frac{a c^2 \sin\phi (c^2 \sin^2\phi + a^2 \cos^2\phi)^{\frac{1}{2}}}{(c^2 \sin^2\phi + a^2 \cos^2\phi)^2}$$

$$= \frac{ac^2 \sin\phi}{(c^2\sin^2\phi + a^2\cos^2\phi)^{3/2}}$$

$$\therefore \frac{1}{2\pi}\iint_S K\,dA = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\pi} \frac{ac^2\sin\phi}{(c^2\sin^2\phi + a^2\cos^2\phi)^{3/2}}\,d\phi\,d\theta$$

$$= ac^2 \int_0^{\pi} \frac{\sin\phi}{(c^2\sin^2\phi + a^2\cos^2\phi)^{3/2}}\,d\phi \qquad [1]$$

Note $\sin\phi = -d(\cos\phi)$. From text table of integrals,

#42: $\displaystyle\int \frac{dx}{(a^2-x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2-x^2}}$

and from other tables, $\displaystyle\int \frac{dx}{(a^2+x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2+x^2}}$

$$\therefore \frac{\sin\phi}{(c^2\sin^2\phi + a^2\cos^2\phi)^{3/2}} = \frac{\sin\phi}{[c^2(1-\cos^2\phi) + a^2\cos^2\phi]^{3/2}}$$

$$= \frac{\sin\phi}{[c^2 - (c^2-a^2)\cos^2\phi]^{3/2}} = \frac{\sin\phi}{(c^2-a^2)^{3/2}\left(\frac{c^2}{c^2-a^2} - \cos^2\phi\right)^{3/2}} \qquad c > a$$

or
$$= \frac{\sin\phi}{[c^2 + (a^2-c^2)\cos^2\phi]^{3/2}} = \frac{\sin\phi}{(a^2-c^2)^{3/2}\left(\frac{c^2}{a^2-c^2} + \cos^2\phi\right)^{3/2}} \qquad a > c$$

$\therefore$ Let $x = -\cos\phi$ $\therefore dx = \sin\phi\, d\phi$

$\phi = 0 \Rightarrow x = -1$, $\phi = \pi \Rightarrow x = 1$

(a) Assume $c > a$

$\therefore$ [1] becomes $\dfrac{ac^2}{(c^2-a^2)^{3/2}} \displaystyle\int_{-1}^{1} \frac{dx}{\left(\frac{c^2}{c^2-a^2} - x^2\right)^{3/2}} =$

$$\frac{ac^2}{(c^2-a^2)^{3/2}} \cdot \frac{x}{\left(\frac{c^2}{c^2-a^2}\right)\sqrt{\frac{c^2}{c^2-a^2} - x^2}} \Bigg|_{x=-1}^{x=1}$$

using $\displaystyle\int \frac{du}{(a^2-u^2)^{3/2}}$

$$= \frac{2ac^2}{(c^2-a^2)^{3/2}\left(\frac{c^2}{c^2-a^2}\right)\sqrt{\frac{c^2}{c^2-a^2}-1}}$$

$$= \frac{2a}{(c^2-a^2)(c^2-a^2)^{1/2}\left(\frac{1}{c^2-a^2}\right)\sqrt{\frac{a^2}{c^2-a^2}}}$$

$$= \frac{2a}{(c^2-a^2)^{1/2}\,\frac{1}{\sqrt{c^2-a^2}}\cdot a} = \underline{\underline{2}}$$

$$\therefore \frac{1}{2\pi}\iint_S K\,dA = \underline{2}$$, assuming $c > a$.

(b) Now assume $a < c$

$\therefore$ [1] becomes $\dfrac{ac^2}{(a^2-c^2)^{3/2}} \displaystyle\int_{-1}^{1} \frac{dx}{\left(\frac{c^2}{a^2-c^2} + x^2\right)^{3/2}}$

$$= \frac{ac^2}{(a^2-c^2)^{3/2}} \cdot \frac{x}{\left(\frac{c^2}{a^2-c^2}\right)\sqrt{\frac{c^2}{a^2-c^2}+x^2}}\Bigg|_{x=-1}^{x=1}$$

using $\int \frac{du}{(a^2+u^2)^{3/2}}$

$$= \frac{2ac^2}{(a^2-c^2)^{3/2}\left(\frac{c^2}{a^2-c^2}\right)\sqrt{\frac{c^2}{a^2-c^2}+1}}$$

$$= \frac{2a}{(a^2-c^2)(a^2-c^2)^{1/2}\left(\frac{1}{a^2-c^2}\right)\sqrt{\frac{a^2}{a^2-c^2}}}$$

$$= \frac{2a}{(a^2-c^2)^{1/2}\frac{1}{\sqrt{a^2-c^2}}\cdot a} = \underline{2}$$

$$\therefore \frac{1}{2\pi}\iint_S K\,dA = 2, \quad \text{assuming } a<c$$

(c) For $a=c$, the ellipsoid is a sphere of radius a.

Example 2 on p. 415 of text showed $K = \frac{1}{a^2}$.

$$\therefore \frac{1}{2\pi}\iint_S K\,dA = \frac{1}{2\pi a^2}\iint_S dA = \frac{1}{2\pi a^2}(4\pi a^2) = 2$$

$$\therefore \underline{\frac{1}{2\pi}\iint_S K\,dA = 2} \quad \text{for } a<c,\ a>c,\ a=c.$$

8.

(a) $\phi_u = (0,0,1)$ $\phi_v = (-2\sin v, 2\cos v, 0)$

$\phi_{uu} = (0,0,0)$ $\phi_{vv} = (-2\cos v, -2\sin v, 0)$ $\phi_{uv} = (0,0,0)$

$$\phi_u \times \phi_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 1 \\ -2\sin v & 2\cos v & 0 \end{vmatrix} = (-2\cos v, -2\sin v, 0)$$

$$\|\phi_u \times \phi_v\|^2 = 4\cos^2 v + 4\sin^2 v = 4 = W$$

$$\vec{N} = \frac{\phi_u \times \phi_v}{\sqrt{W}} = (-\cos v, -\sin v, 0)$$

$$l = \vec{N} \cdot \phi_{uu} = \vec{N} \cdot (0,0,0) = 0$$

$$m = \vec{N} \cdot \phi_{uv} = \vec{N} \cdot (0,0,0) = 0$$

$$n = \vec{N} \cdot \phi_{vv} = (-\cos v, -\sin v, 0) \cdot (-2\cos v, -2\sin v, 0) = 2$$

$$\therefore K = \frac{ln - m^2}{W} = \frac{0 \cdot 2 - 0}{4} = \underline{0}$$

(b) $\phi_u = (1, 0, 2u)$ $\phi_v = (0,1,0)$

$\phi_{uu} = (0,0,2)$ $\phi_{vv} = (0,0,0)$ $\phi_{uv} = (0,0,0)$

$$\phi_u \times \phi_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 2u \\ 0 & 1 & 0 \end{vmatrix} = (-2u, 0, 1)$$

$$\|\phi_u \times \phi_v\|^2 = 4u^2 + 1 = W$$

$$\vec{N} = \frac{\phi_u \times \phi_v}{\sqrt{W}} = \frac{1}{\sqrt{4u^2+1}}(-2u, 0, 1)$$

$$l = \vec{N} \cdot \phi_{uu} = \frac{1}{\sqrt{4u^2+1}}(-2u, 0, 1) \cdot (0, 0, 2) = \frac{2}{4u^2+1}$$

$$n = \vec{N} \cdot \phi_{uv} = \vec{N} \cdot (0,0,0) = 0$$

$$m = \vec{N} \cdot \phi_{vv} = \vec{N} \cdot (0,0,0) = 0$$

$$K = \frac{ln - m^2}{W} = \frac{l(0) - (0)^2}{W} = \underline{0}$$

9.

$$\phi_u = (1 - u^2 + v^2, 2uv, 2u) \quad \phi_v = (2uv, 1 - v^2 + u^2, -2v)$$

$$\phi_{uu} = (-2u, 2v, 2) \quad \phi_{vv} = (2u, -2v, -2) \quad \phi_{uv} = (2v, 2u, 0)$$

$$\begin{aligned} E &= (1 - u^2 + v^2)^2 + 4u^2v^2 + 4u^2 \\ &= 1 - 2u^2 + 2v^2 - 2u^2v^2 + u^4 + v^4 + 4u^2v^2 + 4u^2 \\ &= 1 + 2u^2 + 2v^2 + 2u^2v^2 + u^4 + v^4 \end{aligned}$$

$$F = (2uv - 2u^3v + 2uv^3) + (2uv - 2uv^3 + 2u^3v) - 4uv$$
$$= 0$$

$$G = 4u^2v^2 + (1 - v^2 + u^2)^2 + 4v^2$$
$$= 4u^2v^2 + (1 + 2u^2 - 2v^2 - 2u^2v^2 + u^4 + v^4) + 4v^2$$
$$= 1 + 2u^2 + 2v^2 + 2u^2v^2 + u^4 + v^4$$

$\therefore E = G$ Also note $\phi_{uu} = -\phi_{vv}$

$$\phi_u \times \phi_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 - u^2 + v^2 & 2uv & 2u \\ 2uv & 1 - v^2 + u^2 & -2v \end{vmatrix}$$

$$= (-2u - 2uv^2 - 2u^3,\ 2v + 2u^2v + 2v^3,\ 1 - 2u^2v^2 - u^4 - v^4)$$

$$\|\phi_u \times \phi_v\|^2 = EG - F^2 = E^2 - 0 = E^2 = W$$

$$\vec{N} = \frac{\phi_u \times \phi_v}{\sqrt{W}}$$

$$l = \vec{N} \cdot \phi_{uu} = \vec{N} \cdot (-\phi_{vv}) = -\vec{N} \cdot \phi_{vv} = -n$$

$$m = \vec{N} \cdot \phi_{uv}$$

$$H = \frac{Gl + En - 2Fm}{2W} = \frac{Gl + (G)(-l) - 2(0)m}{2W}$$

$$= \frac{0 - 0}{2W} = \underline{0}$$

10.

(a) From #4 of Section 7.4,

$$T(\theta,\phi) = [(R+\cos\phi)\cos\theta, (R+\cos\phi)\sin\theta, \sin\phi]$$

$0 \le \theta \le 2\pi,\ 0 \le \phi \le 2\pi,\ R>1$

And, from that problem,

$$\|T_\theta \times T_\phi\|^2 = \left[\frac{\partial(x,y)}{\partial(\theta,\phi)}\right]^2 + \left[\frac{\partial(y,z)}{\partial(\theta,\phi)}\right]^2 + \left[\frac{\partial(x,z)}{\partial(\theta,\phi)}\right]^2$$

$$= (R+\cos\phi)^2 = W$$

$$T_\theta = [-(R+\cos\phi)\sin\theta, (R+\cos\phi)\cos\theta, 0]$$

$$T_\phi = [-\cos\theta\sin\phi, -\sin\theta\sin\phi, \cos\phi]$$

$$T_{\theta\theta} = [-(R+\cos\phi)\cos\theta, -(R+\cos\phi)\sin\theta, 0]$$

$$T_{\phi\phi} = [-\cos\theta\cos\phi, -\sin\theta\cos\phi, -\sin\phi]$$

$$T_{\theta\phi} = [\sin\theta\sin\phi, -\cos\theta\sin\phi, 0]$$

$$T_\theta \times T_\phi = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -(R+\cos\phi)\sin\theta & (R+\cos\phi)\cos\theta & 0 \\ -\cos\theta\sin\phi & -\sin\theta\sin\phi & \cos\phi \end{vmatrix}$$

$$[(R+\cos\phi)\cos\theta\cos\phi, (R+\cos\phi)\sin\theta\cos\phi, -(R+\cos\phi)\sin\phi]$$

$$\vec{N} = \frac{T_\theta \times T_\phi}{\sqrt{W}} = \frac{T_\theta \times T_\phi}{R+\cos\phi} \quad (\text{Note } R>1 \Rightarrow R+\cos\phi>0)$$

$$= (\cos\theta\cos\phi, \sin\theta\cos\phi, -\sin\phi)$$

$$l = \vec{N}\cdot T_{\theta\theta} = (\cos\theta\cos\phi, \sin\theta\cos\phi, -\sin\phi)\cdot -(R+\cos\phi)[\cos\theta, \sin\theta, 0]$$
$$= -(R+\cos\phi)\cos\phi$$

$$n = \vec{N}\cdot T_{\phi\phi} = (\cos\theta\cos\phi, \sin\theta\cos\phi, -\sin\phi)\cdot [-\cos\theta\cos\phi, -\sin\theta\cos\phi, -\sin\phi]$$

$$= -\cos^2\theta\cos^2\phi - \sin^2\theta\cos^2\phi - \sin^2\phi$$
$$= -\cos^2\phi - \sin^2\phi$$
$$= -1$$

$$m = \vec{N}\cdot T_{\theta\phi} = (\cos\theta\cos\phi, \sin\theta\cos\phi, -\sin\phi)\cdot (\sin\theta\sin\phi, -\cos\theta\sin\phi, 0) = 0$$

$$\therefore K = \frac{ln - m^2}{W} = \frac{-(R+\cos\phi)\cos\phi(-1) - 0}{(R+\cos\phi)^2}$$

$$= \underline{\underline{\frac{\cos\phi}{(R+\cos\phi)}}}$$

(6) $$\frac{1}{2\pi}\iint_S K\,dA = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} K\,\|T_\theta \times T_\phi\|\,d\theta\,d\phi$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} \frac{\cos\phi}{(R+\cos\phi)} (R+\cos\phi)\, d\theta\, d\phi$$

$$= \frac{2\pi}{2\pi} \int_0^{2\pi} \cos\phi\, d\phi = \sin\phi \Big|_0^{2\pi} = \underline{\underline{0}}$$

11.

Given x-coordinate u, get $h(u)$, and $h(u)$ revolves around x-axis.

$\phi_u = (1, h'\cos v, h'\sin v) \quad \phi_v = (0, -h\sin v, h\cos v)$

$\phi_{uu} = (0, h''\cos v, h''\sin v)$

$\phi_{vv} = (0, -h\cos v, -h\sin v)$

$\phi_{uv} = (0, -h'\sin v, h'\cos v)$

$$\phi_u \times \phi_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & h'\cos v & h'\sin v \\ 0 & -h\sin v & h\cos v \end{vmatrix}$$

$$= (hh', -h\cos v, -h\sin v)$$

$$\|\phi_u \times \phi_v\|^2 = (hh')^2 + h^2 = h^2[1+(h')^2] = W$$

$$\vec{N} = \frac{\phi_u \times \phi_v}{\sqrt{W}} = \frac{(h', -\cos v, -\sin v)}{\sqrt{1+(h')^2}}, \text{ as } h>0.$$

$$l = \vec{N}\cdot\phi_{uu} = \frac{1}{\sqrt{1+(h')^2}}(h', -\cos v, -\sin v)\cdot(0, h''\cos v, h''\sin v)$$

$$= \frac{-h''}{\sqrt{1+(h')^2}}$$

$$n = \vec{N}\cdot\phi_{vv} = \frac{1}{\sqrt{1+(h')^2}}(h', -\cos v, -\sin v)\cdot(0, -h\cos v, -h\sin v)$$

$$= \frac{h}{\sqrt{1+(h')^2}}$$

$$m = \vec{N}\cdot\phi_{uv} = \frac{1}{\sqrt{1+(h')^2}}(h', -\cos v, -\sin v)\cdot(0, -h'\sin v, h'\cos v)$$

$$= \frac{0 + h'\cos v\sin v - h'\sin v\cos v}{\sqrt{1+(h')^2}} = 0$$

$$\therefore K = \frac{ln - m^2}{W} = \frac{\left(\frac{-h''}{\sqrt{1+(h')^2}}\right)\left(\frac{h}{\sqrt{1+(h')^2}}\right) - 0}{h^2\left[1+(h')^2\right]}$$

$$= \underline{\underline{\frac{-h''}{h\left[1+(h')^2\right]^2}}}$$

12.

$$K = 0 \Rightarrow \frac{\ell n - m^2}{W} = 0 \Rightarrow \ell n = m^2$$

$$H = 0,\ E = G,\ F = 0 \Rightarrow \frac{G\ell + En - 2(0)m}{2W} = 0$$

$$\Rightarrow G(\ell + n) = 0$$

(1) If $E = G = 0$, then $\phi_u = 0$ and $\phi_v = 0$

$\phi_u = 0 \Rightarrow \phi(u,v) = (f(v), g(v), h(v))$

$\phi_v = 0 \Rightarrow \phi(u,v) = (a, b, c)$, a, b, c constants.

So S is a point

(2) If $E \neq 0$, $G \neq 0$, then $G(\ell + n) = 0 \Rightarrow \ell = -n$.

$\therefore \ell n = m^2 \Rightarrow -n^2 = m^2$, so $\ell = n = m = 0$.

$\therefore \vec{N} \cdot \phi_{uu} = 0,\ \vec{N} \cdot \phi_{vv} = 0,\ \vec{N} \cdot \phi_{uv} = 0$

$\therefore \vec{N} \perp \phi_{uu},\ \vec{N} \perp \phi_{vv},\ \vec{N} \perp \phi_{uv}$ [0]

Since $F = 0$, $\phi_u \cdot \phi_v = 0$, and assuming

$\phi_u \neq 0$ and $\phi_v \neq 0$ (S is regular or smooth), then ϕ_u and ϕ_v form a basis for the plane determined by ϕ_u and ϕ_v ($\phi_u \times \phi_v \neq 0$ for a regular surface).

Since $\phi_u \perp \vec{N}$, $\phi_v \perp \vec{N}$, then $\phi_{uu}, \phi_{vv}, \phi_{uv}$ are in the same plane of ϕ_u and ϕ_v from [0]

$$\therefore \quad \phi_{uu} = \alpha_1 \phi_u + \beta_1 \phi_v \qquad [1]$$

$$\phi_{vv} = \alpha_2 \phi_u + \beta_2 \phi_v \qquad [2]$$

$$\phi_{uv} = \alpha_3 \phi_u + \beta_3 \phi_v \qquad [3]$$

where α_i, β_i are constants.

To show S is in a plane (or is planar), it suffices to show $\vec{N}$ is constant (i.e., does not depend on u or v from $S = \phi(u,v)$).

$\therefore$ Need to show $\frac{\partial}{\partial u}\vec{N} = \vec{0}$ and $\frac{\partial}{\partial v}\vec{N} = \vec{0}$

$\vec{N} = \dfrac{\phi_u \times \phi_v}{\|\phi_u \times \phi_v\|}$, by definition and $\|\phi_u \times \phi_v\| = \sqrt{EG - F^2}$ (p. 414 of text)

$$= \sqrt{E^2} = E = G$$

and $E = \|\phi_u\|^2 = \phi_u \cdot \phi_u = G = \|\phi_v\|^2 = \phi_v \cdot \phi_v$

$$\therefore \vec{N} = \frac{\phi_u \times \phi_v}{\phi_u \cdot \phi_u} = \frac{\phi_u \times \phi_v}{\phi_v \cdot \phi_v}$$

$\therefore$ Look at $\dfrac{\partial \vec{N}}{\partial u} = \dfrac{\partial}{\partial u} \dfrac{\phi_u \times \phi_v}{\phi_u \cdot \phi_u}$

$$= \frac{(\phi_u \cdot \phi_u)\left[\phi_{uu} \times \phi_v + \phi_u \times \phi_{uv}\right] - (\phi_u \times \phi_v)(2\phi_{uu} \cdot \phi_u)}{[\phi_u \cdot \phi_u]^2} \quad [4]$$

From [1], $\phi_{uu} \times \phi_v = (\alpha_1 \phi_u + \beta_1 \phi_v) \times \phi_v$ $\quad \phi_v \times \phi_v = 0$

$$= \alpha_1 \phi_u \times \phi_v \quad [4.1]$$

From [3], $\phi_u \times \phi_{uv} = \phi_u \times (\alpha_3 \phi_u + \beta_3 \phi_v)$ $\quad \phi_u \times \phi_u = 0$

$$= \beta_3 \phi_u \times \phi_v \quad [4.2]$$

and $\phi_{uu} \cdot \phi_u = (\alpha_1 \phi_u + \beta_1 \phi_v) \cdot \phi_u$ $\quad F = 0 = \phi_u \cdot \phi_v$

$$= \alpha_1 \phi_u \cdot \phi_u \quad [4.3]$$

$\therefore$ [4] becomes $\dfrac{(\phi_u \cdot \phi_u)(\alpha_1 + \beta_3)\phi_u \times \phi_v - 2\alpha_1(\phi_u \cdot \phi_u)\phi_u \times \phi_v}{[\phi_u \cdot \phi_u]^2}$

$$\therefore \frac{\partial \vec{N}}{\partial u} = (\beta_3 - \alpha_1)\frac{\phi_u \times \phi_v}{\phi_u \cdot \phi_u} \qquad [5]$$

Now using $E = G \Rightarrow \phi_u \cdot \phi_u = \phi_v \cdot \phi_v$,

$$\frac{\partial \vec{N}}{\partial u} = \frac{\partial}{\partial u}\frac{\phi_u \times \phi_v}{\phi_v \cdot \phi_v}$$

$$= \frac{(\phi_v \cdot \phi_v)[\phi_{uu} \times \phi_v + \phi_u \times \phi_{uv}] - (\phi_u \times \phi_v)(2\phi_{uv} \cdot \phi_v)}{[\phi_v \cdot \phi_v]^2}$$

using [4.1] and [4.2]

$$= \frac{(\phi_v \cdot \phi_v)(\alpha_1 + \beta_3)\phi_u \times \phi_v - (2\phi_{uv} \cdot \phi_v)\phi_u \times \phi_v}{[\phi_v \cdot \phi_v]^2} \qquad [5.1]$$

From [3], $\phi_{uv} = \alpha_3 \phi_u + \beta_3 \phi_v$, so

$$\phi_{uv} \cdot \phi_v = (\alpha_3 \phi_u + \beta_3 \phi_v) \cdot \phi_v \qquad F = 0 = \phi_u \cdot \phi_v$$

$$= \beta_3(\phi_v \cdot \phi_v)$$

$\therefore$ [5.1] becomes $\dfrac{(\phi_v \cdot \phi_v)(\alpha_1 + \beta_3)\phi_u \times \phi_v - 2\beta_3(\phi_v \cdot \phi_v)\phi_u \times \phi_v}{[\phi_v \cdot \phi_v]^2}$

$$\therefore \frac{\partial \vec{N}}{\partial u} = (\alpha_1 - \beta_3)\frac{\phi_u \times \phi_v}{\phi_v \cdot \phi_v} = (\alpha_1 - \beta_3)\frac{\phi_u \times \phi_v}{\phi_u \cdot \phi_u} \qquad [6]$$

$E = G \Rightarrow \phi_u \cdot \phi_u = \phi_v \cdot \phi_v$

$\therefore$ [5], [6] $\Rightarrow$ $\dfrac{\partial \vec{N}}{\partial u} = (\beta_3 - \alpha_1)\dfrac{\phi_u \times \phi_v}{\phi_u \cdot \phi_u} = (\alpha_1 - \beta_3)\dfrac{\phi_u \times \phi_v}{\phi_u \cdot \phi_u}$

so $\beta_3 - \alpha_1 = \alpha_1 - \beta_3 \Rightarrow \alpha_1 = \beta_3$

so $\dfrac{\partial \vec{N}}{\partial u} = \vec{0}$ [7]

Now look at $\dfrac{\partial \vec{N}}{\partial v} = \dfrac{\partial}{\partial v}\dfrac{\phi_u \times \phi_v}{\phi_u \cdot \phi_u}$

$$= \frac{(\phi_u \cdot \phi_u)\left[\phi_{uv} \times \phi_v + \phi_u \times \phi_{vv}\right] - \phi_u \times \phi_v (2\phi_{uv} \cdot \phi_u)}{[\phi_u \cdot \phi_u]^2} \quad [8]$$

From [3], $\phi_{uv} \times \phi_v = (\alpha_3 \phi_u + \beta_3 \phi_v) \times \phi_v$ $\quad \phi_v \times \phi_v = 0$

$= \alpha_3(\phi_u \times \phi_v)$ [8.1]

and $\phi_{uv} \cdot \phi_u = (\alpha_3 \phi_u + \beta_3 \phi_v) \cdot \phi_u$ $\quad F = 0 = \phi_u \cdot \phi_v$

$= \alpha_3(\phi_u \cdot \phi_u)$ [8.2]

From [2], $\phi_u \times \phi_{vv} = \phi_u \times (\alpha_2 \phi_u + \beta_2 \phi_v)$ $\quad \phi_u \times \phi_u = 0$

$= \beta_2(\phi_u \times \phi_v)$ [8.3]

From [8.1], [8.2], [8.3], [8] becomes

$$\frac{(\phi_u \cdot \phi_u)(\alpha_3+\beta_2)\,\phi_u \times \phi_v - (2\alpha_3)(\phi_u \cdot \phi_u)\,\phi_u \times \phi_v}{[\phi_u \cdot \phi_u]^2}$$

$$= (\beta_2 - \alpha_3)\,\frac{\phi_u \times \phi_v}{\phi_u \cdot \phi_u}$$

$$\therefore \frac{\partial \vec{N}}{\partial v} = (\beta_2 - \alpha_3)\,\frac{\phi_u \times \phi_v}{\phi_u \cdot \phi_u} \qquad [9]$$

Now using $E = G = \phi_u \cdot \phi_u = \phi_v \cdot \phi_v$,

$$\frac{\partial \vec{N}}{\partial v} = \frac{\partial}{\partial v}\,\frac{\phi_u \times \phi_v}{\phi_v \cdot \phi_v}$$

$$= \frac{(\phi_v \cdot \phi_v)\left[\phi_{uv} \times \phi_v + \phi_u \times \phi_{vv}\right] - (\phi_u \times \phi_v)(2\,\phi_{vv} \cdot \phi_v)}{[\phi_v \cdot \phi_v]^2} \qquad [10]$$

From [8.1], [8.3],

$$\phi_{uv} \times \phi_v + \phi_u \times \phi_{vv} = (\alpha_3 + \beta_2)\,\phi_u \times \phi_v \qquad [10.1]$$

From [2] $\phi_{vv} \cdot \phi_v = (\alpha_2 \phi_u + \beta_2 \phi_v) \cdot \phi_v$ $\quad \phi_u \cdot \phi_v = 0 = F$

$$= \beta_2 (\phi_v \cdot \phi_v) \qquad [10.2]$$

From [10.1], [10.2], [10] becomes

$$\frac{(\phi_v \cdot \phi_v)(\alpha_3 + \beta_2)\, \phi_u \times \phi_v \;-\; (2\beta_2)(\phi_v \cdot \phi_v)\, \phi_u \times \phi_v}{[\phi_v \cdot \phi_v]^2}$$

$$= (\alpha_3 - \beta_2)\, \frac{\phi_u \times \phi_v}{\phi_v \cdot \phi_v}$$

$$\therefore \frac{\partial \vec{N}}{\partial v} = (\alpha_3 - \beta_2)\, \frac{\phi_u \times \phi_v}{\phi_v \cdot \phi_v} \qquad [11]$$

$\therefore$ [9], [11] mean $\dfrac{\partial \vec{N}}{\partial v} = (\beta_2 - \alpha_3)\dfrac{\phi_u \times \phi_v}{\phi_u \cdot \phi_u} = (\alpha_3 - \beta_2)\dfrac{\phi_u \times \phi_v}{\phi_v \cdot \phi_v}$

and since $\phi_u \cdot \phi_u = \phi_v \cdot \phi_v$ as $E = G$,

$$\beta_2 - \alpha_3 = \alpha_3 - \beta_2 \Rightarrow \alpha_3 = \beta_2 \Rightarrow \frac{\partial \vec{N}}{\partial v} = \vec{0} \qquad [12]$$

$\therefore$ [7], [12] $\dfrac{\partial \vec{N}}{\partial u} = \vec{0}$, $\dfrac{\partial \vec{N}}{\partial v} = \vec{0}$

$\therefore$ $\vec{N}$, the normal vector to S, is a constant vector for all u, v.

$\therefore$ S is part of a plane (S is planar).

Review Exercises for Chapter 7

Note Title 5/1/2017

1.

(a) $\vec{c}\,'(t) = (e^t\cos t - e^t\sin t,\ e^t\sin t + e^t\cos t,\ 0)$

$\|\vec{c}\,'(t)\| = [e^{2t}\cos^2 t - 2e^{2t}\cos t\sin t + e^{2t}\sin^2 t + e^{2t}\cos^2 t + 2e^{2t}\cos t\sin t + e^{2t}\sin^2 t + 0]^{\frac{1}{2}}$

$= [e^{2t} + e^{2t}]^{\frac{1}{2}} = \sqrt{2}\, e^t$

$f(x,y,z) = (e^t\cos t)(e^t\sin t)(3) = 3e^{2t}\cos t\sin t$

$\therefore f(x(t),y(t),z(t))\|\vec{c}\,'(t)\| = 3\sqrt{2}\, e^{3t}\cos t\sin t$

$= \frac{\sqrt{2}}{2}\sin 2t\,(3e^{3t})$

$\therefore \int_{\vec{c}} f\,ds = \frac{\sqrt{2}}{2}\int_0^{2\pi} \sin 2t\,(3e^{3t})\,dt$ [1]

$u = \sin 2t \quad v = e^{3t}$

$dv = 3e^{3t}dt$

$= \frac{\sqrt{2}}{2} e^{3t}\sin 2t\Big|_0^{2\pi} - \frac{\sqrt{2}}{2}\int_0^{2\pi} e^{3t}\, 2\cos 2t\,dt$

$\int u\,dv = uv - \int v\,du$

$= 0 - \frac{\sqrt{2}}{3}\int_0^{2\pi} 3e^{3t}\cos 2t\,dt$

$$= -\frac{\sqrt{2}}{3} e^{3t}\cos 2t\Big|_0^{2\pi} + \frac{\sqrt{2}}{3}\int_0^{2\pi} e^{3t}(-2\sin 2t)\,dt$$

$$= \left[-\frac{\sqrt{2}}{3}e^{6\pi} - \left(-\frac{\sqrt{2}}{3}\right)\right] - \frac{2\sqrt{2}}{3}\int_0^{2\pi} \left(\frac{1}{3}\right)3e^{3t}\sin 2t\,dt$$

$$= \frac{\sqrt{2}}{3}\left(1-e^{6\pi}\right) - \frac{2\sqrt{2}}{9}\int_0^{2\pi} 3e^{3t}\sin 2t\,dt \qquad [2]$$

Combining [1] and [2],

$$\therefore \left(\frac{\sqrt{2}}{2} + \frac{2\sqrt{2}}{9}\right)\int_0^{2\pi} 3e^{3t}\sin 2t\,dt = \frac{\sqrt{2}}{3}\left(1-e^{6\pi}\right)$$

$$\therefore \left(1+\frac{4}{9}\right)\frac{\sqrt{2}}{2}\int_0^{2\pi} 3e^{3t}\sin 2t\,dt = \frac{\sqrt{2}}{3}\left(1-e^{6\pi}\right)$$

$$\therefore \int_{\vec{c}} f\,ds = \frac{\sqrt{2}}{2}\int_0^{2\pi} 3e^{3t}\sin 2t\,dt = \frac{9}{13}\left(\frac{\sqrt{2}}{3}\right)\left(1-e^{6\pi}\right)$$

$$= \underline{\underline{\frac{3\sqrt{2}}{13}\left(1-e^{6\pi}\right)}}$$

(b)

$$\vec{c}\,'(t) = (-\sin t, \cos t, 1). \quad \therefore \|\vec{c}\,'(t)\| = \sqrt{2}$$

$$f(x,y,z) = (\cos t)(\sin t)(t)$$

$$\therefore \int_{\vec{C}} f\,ds = \int_0^{2\pi} \sqrt{2}\, t \cos t \sin t \, dt$$

$\sin 2\theta = 2\sin\theta\cos\theta$

$$\frac{\sqrt{2}}{2}\int_0^{2\pi} t \sin(2t)\,dt = -\frac{\sqrt{2}}{4}\int_0^{2\pi} t\, d\left(\cos(2t)\right)$$

$$= -\frac{\sqrt{2}}{4}\Big[t\cos(2t)\Big]_0^{2\pi} + \frac{\sqrt{2}}{4}\int_0^{2\pi} \cos(2t)\,dt$$

$\int u\,dv = uv - \int v\,du$

$$= -\frac{\sqrt{2}}{4}\left[2\pi - 0\right] + \frac{\sqrt{2}}{4}\,\frac{\sin 2t}{2}\Big|_0^{2\pi}$$

$$= -\frac{\sqrt{2}}{2}\pi$$

(c)

$$\vec{C}'(t) = (3t, 4t, 1). \quad \|\vec{C}'(t)\| = \sqrt{9t^2+16t^2+1} = \sqrt{1+25t^2}$$

$$f(x,y,z) = \left(\tfrac{3}{2}t^2\right)(2t^2)(t) = 3t^5$$

$$\therefore \int_{\vec{C}} f\,ds = \int_0^1 3t^5\sqrt{1+25t^2}\,dt$$

From Wolfram: $\int x^5 \sqrt{1+25x^2}\, dx = \frac{(25x^2+1)^{3/2}(9375x^4 - 300x^2 + 8)}{1640625}$ + constant

$$\therefore \int_{\vec{c}} f\, ds = 3\frac{(25t^2+1)^{3/2}(9375t^4 - 300t^2 + 8)}{1,640,625}\Bigg|_0^1$$

$$= \frac{3(26)^{3/2}(9375 - 300 + 8) - 3(1)(8)}{1,640,625}$$

$1,640,625 = 3(35)(25^3)$

$$= \frac{3(26)\sqrt{26}(9083) - 24}{3(35)(25)^3}$$

$(26)(9083) = 236,158$

$$= \frac{(236,158)\sqrt{26} - 8}{35(25)^3}$$

(d)

$$\vec{c}\,'(t) = (1, \sqrt{2}t, t^2)$$

$$\therefore \|\vec{c}(t)\| = \sqrt{1 + 2t^2 + t^4} = \sqrt{(1+t^2)^2} = 1 + t^2$$

$$f(x,y,z) = f(\vec{c}(t)) = (t)\left(\frac{t^2}{\sqrt{2}}\right)\left(\frac{1}{3}t^3\right) = \frac{t^6}{3\sqrt{2}}$$

$$\therefore \int_{\vec{c}} f\, ds = \int_0^1 \left(\frac{t^6}{3\sqrt{2}}\right)(1+t^2)\, dt$$

$$= \frac{1}{3\sqrt{2}} \int_0^1 t^6 + t^8 \, dt = \frac{1}{3\sqrt{2}} \left[\frac{t^7}{7} + \frac{t^9}{9} \right]_0^1$$

$$= \frac{1}{3\sqrt{2}} \left(\frac{1}{7} + \frac{1}{9} \right) = \frac{1}{3\sqrt{2}} \left(\frac{16}{63} \right) = \underline{\frac{8\sqrt{2}}{189}}$$

2.

(a) $\vec{c}\,'(t) = (\cos t, -\sin t, 1) \quad \|\vec{c}\,'(t)\| = \sqrt{2}$

$f(\vec{c}(t)) = \sin t + \cos t + t\cos t$

$$\therefore \int_{\vec{c}} f \, ds = \sqrt{2} \int_0^{2\pi} \sin t + \cos t + t\cos t \, dt$$

$$= 0 + 0 + \sqrt{2} \int_0^{2\pi} t\cos t \, dt = \sqrt{2} \int_0^{2\pi} t \, d(\sin t)$$

$$= \sqrt{2}\, t \sin t \Big|_0^{2\pi} - \sqrt{2} \int_0^{2\pi} \sin t \, dt$$

$$= 0 - 0 = \underline{0}$$

(b)

$\vec{c}'(t) = (\cos t, -\sin t, 1) \quad \|\vec{c}'(t)\| = \sqrt{2}$

$f(\vec{c}(t)) = \sin t + \cos^2 t$

$\therefore \int_{\vec{c}} f\,ds = \sqrt{2}\int_0^{2\pi} \sin t + \cos^2 t \; dt = \sqrt{2}\int_0^{2\pi} \cos^2 t\, dt$

$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1$

$= \frac{\sqrt{2}}{2}\int_0^{2\pi} (1 + \cos 2t)\, dt = \frac{\sqrt{2}}{2}\left[t + \frac{\sin 2t}{2}\right]_0^{2\pi}$

$= \frac{\sqrt{2}}{2}(2\pi) = \underline{\pi\sqrt{2}}$

(c)

$\vec{c}'(t) = (1, 2t, 2t^2)$

$\|\vec{c}'(t)\| = \sqrt{1 + 4t^2 + 4t^4} = \sqrt{(1+2t^2)^2} = 1 + 2t^2$

$f(\vec{c}(t)) = t + (t^2) + (\frac{2}{3}t^3)$

$\therefore \int_{\vec{c}} f\,ds = \int_0^1 f(\vec{c}(t))\,\|\vec{c}'(t)\|\,dt$

$$= \int_0^1 [t + t^2 + \frac{2}{3}t^3](1+2t^2)\,dt$$

$$= \int_0^1 t + t^2 + \frac{8}{3}t^3 + 2t^4 + \frac{4}{3}t^5\,dt$$

$$= \frac{t^2}{2} + \frac{t^3}{3} + \frac{2}{3}t^4 + \frac{2}{5}t^5 + \frac{2}{9}t^6 \Big|_0^1$$

$$= \frac{1}{2} + \frac{1}{3} + \frac{2}{3} + \frac{2}{5} + \frac{2}{9} = \frac{3}{2} + \frac{2}{5} + \frac{2}{9} = \frac{135 + 36 + 20}{90}$$

$$= \frac{191}{90}$$

3.

(a) $C_1 : (1,0,0)$ to $(0,1,0)$: $(1,0,0) + t[(0,1,0) - (1,0,0)]$, $0 \le t \le 1$

$\therefore \vec{c}_1(t) = (1,0,0) + t(-1,1,0) = (1-t, t, 0) = [x(t), y(t), z(t)]$

$\vec{c}_1{}'(t) = (-1, 1, 0) = (x', y', z')$

$\therefore (\sin \pi x)\,dy - (\cos \pi y)\,dz =$

$$[\sin \pi(1-t)](1) - [\cos \pi t](0) = \sin \pi(1-t)$$

$$\therefore \int_{C_1} \sin \pi x \, dy - \cos \pi y \, dz = \int_0^1 \sin \pi(1-t)\, dt$$

$$= \frac{\cos \pi(1-t)}{\pi}\Big|_0^1 = \frac{1}{\pi}\left[\cos(0) - \cos(\pi)\right]$$

$$= \underline{\frac{2}{\pi}}$$

$C_2 : (0,1,0)$ to $(0,0,1) : (0,1,0) + t[(0,0,1)-(0,1,0)],\ 0 \le t \le 1$

$$\therefore \vec{c}_2(t) = (0,1,0) + t(0,-1,1) = (0,1-t,t) = (x,y,z)$$

$$\vec{c}\,'_2(t) = (0,-1,1) = (x',y',z')$$

$$\therefore (\sin \pi x)\, dy - (\cos \pi y)\, dz =$$

$$(\sin \pi(0))(-1) - [\cos \pi(1-t)](1) = -\cos \pi(1-t)$$

$$\therefore \int_{C_2} \sin \pi x \, dy - \cos \pi y \, dz = \int_0^1 -\cos(\pi - \pi t)\, dt$$

$$= \frac{\sin(\pi - \pi t)}{\pi}\Big|_0^1 = \frac{1}{\pi}\left[\sin(0) - \sin(\pi)\right] = \underline{0}$$

$C_3 : (0,0,1)$ to $(1,0,0)$: $(0,0,1) + t[(1,0,0) - (0,0,1)]$, $0 \le t \le 1$

$\therefore \vec{c}_3(t) = (0,0,1) + t(1,0,-1) = (t, 0, 1-t) = (x,y,z)$

$\vec{c}_3{}'(t) = (1,0,-1) = (x', y', z')$

$\therefore (\sin \pi x)\,dy - (\cos \pi y)\,dz =$

$(\sin \pi t)(0) - (\cos \pi(0))(-1) = 1$

$$\therefore \int_{C_3} \sin \pi x \, dy - \cos \pi y \, dz = \int_0^1 1 \, dt = \underline{1}$$

$$\therefore C_1 + C_2 + C_3 = \frac{2}{\pi} + 0 + 1 = \underline{1 + \frac{2}{\pi}}$$

(b)

$\vec{c}(\theta) = (\cos^3\theta, \sin^3\theta, \theta) = (x,y,z)$

$\vec{c}\,'(\theta) = (-3\cos^2\theta \sin\theta, 3\sin^2\theta\cos\theta, 1) = (x', y', z')$

$\therefore \sin z \, dx = (\sin\theta)(-3\cos^2\sin\theta) = -3\cos^2\theta \sin^2\theta$

$\cos z \, dy = (\cos\theta)(3\sin^2\theta\cos\theta) = 3\cos^2\theta\sin^2\theta$

$(xy)^{1/3} = (\cos^3\theta \sin^3\theta)^{1/3} = \cos\theta\sin\theta$

$$\therefore \int_C \sin z\,dx + \cos z\,dy - (xy)^{\frac{1}{3}} = \int_0^{\frac{7\pi}{2}} -\cos\theta \sin\theta \, d\theta$$

$$= -\frac{1}{2}\int_0^{7\pi/2} \sin 2\theta \, d\theta = \frac{1}{4}\cos 2\theta \Big|_0^{\frac{7\pi}{2}}$$

$$= \frac{1}{4}\left[\cos(7\pi) - \cos(0)\right] = \frac{1}{4}[-1-1] = \underline{-\frac{1}{2}}$$

4.

$$\int_{\vec{C}} \vec{F} \cdot d\vec{s} = \int_{\vec{C}} \vec{F}(\vec{C}(t)) \cdot \vec{C}'(t) = \int_{\vec{C}} (0)\,dt = \underline{0}$$

5.

Let $C = C_1 + C_2 + C_3 + C_4$

C_1 : $(0,0)$ to $(a,0)$: $(0,0) + t\left[(a,0) - (0,0)\right]$, $0 \le t \le 1$

$\therefore \vec{C}_1(t) = (at, 0) = (x, y)$

$\vec{C}_1'(t) = (a, 0) = (x', y')$

$\vec{F}(\vec{C}_1(t)) = (a^2t^2, 0)$

$$\therefore \vec{F}(\vec{c}_1(t)) \cdot \vec{c}_1{}'(t) = a^3 t^2$$

$$\therefore \int_{C_1} \vec{F} \cdot d\vec{s} = \int_0^1 a^3 t^2\, dt = \frac{a^3}{3} t^3 \Big|_0^1 = \underline{\frac{a^3}{3}}$$

$C_2 : (a,0)$ to (a,a) : $(a,0) + t[(a,a) - (a,0)] =$

$(a,0) + t\,(0,a),\ 0 \le t \le 1$

$$\therefore \vec{c}_2(t) = (a, at) = (x,y)$$

$$\vec{c}_2{}'(t) = (0, a) = (x', y')$$

$$\therefore \vec{F}(\vec{c}_2(t)) = (a^2 - a^2 t^2, 2a^2 t)$$

$$\therefore \vec{F}(\vec{c}_2(t)) \cdot \vec{c}_2{}'(t) = 2a^3 t$$

$$\therefore \int_{C_2} \vec{F} \cdot d\vec{s} = \int_0^1 2a^3 t = a^3 t^2 \Big|_0^1 = a^3$$

$C_3 : (a,a)$ to $(0,a)$: $(a,a) + t[(0,a) - (a,a)] =$

$(a,a) + t\,(-a,0),\ 0 \le t \le 1$

$$\therefore \vec{c}_3(t) = (a - at, a) = (x, y)$$

$$\vec{c}_3{}'(t) = (-a, 0) = (x', y')$$

$$\vec{F}(\vec{c}_3(t)) = [(a-at)^2 - a^2, 2a(a-at)]$$

$$= (a^2t^2 - 2a^2t, 2a^2 - 2a^2t)$$

$$\therefore \vec{F}(\vec{c_3}(t)) \cdot \vec{c_3}'(t) = 2a^3t - a^3t^2$$

$$\int_{c_3} \vec{F} \cdot d\vec{s} = \int_0^1 2a^3t - a^3t^2 = a^3t^2 - \frac{a^3}{3}t^3 \Big|_0^1$$

$$= \underline{a^3 - \frac{a^3}{3}}$$

$$C_4 : (0,a) \text{ to } (0,0) : (0,a) + t[(0,0)-(0,a)] =$$
$$(0,a) + t(0,-a), \quad 0 \le t \le 1$$

$$\therefore \vec{c_4}(t) = (0, a-at) = (x, y)$$

$$\vec{c_4}'(t) = (0, -a) = (x', y')$$

$$\vec{F}(\vec{c_4}(t)) = [0 - (a-at)^2, 2(0)(a-at)]$$

$$= [-a^2 + 2a^2t - a^2t^2, 0]$$

$$\therefore \vec{F}(\vec{c_4}(t)) \cdot \vec{c_4}'(t) = 0$$

$$\int_{c_4} \vec{F} \cdot d\vec{s} = \int_0^1 0 \, dt = \underline{0}$$

$$\therefore C_1 + C_2 + C_3 + C_4 = \frac{a^3}{3} + a^3 + \left(a^3 - \frac{a^3}{3}\right) + 0 = \underline{\underline{2a^3}}$$

6.

Using symmetry, only need to calculate for Quadrant I, then multiply by 4.

$\therefore F(x,y) = x+y. \quad \vec{c}(\theta) = (a\cos\theta, a\sin\theta) = (x,y)$

$0 \le \theta \le \pi/2$

$\therefore \vec{c}\,'(\theta) = (-a\sin\theta, a\cos\theta) \quad \|\vec{c}\,'(\theta)\| = a$

$$\therefore \int_C F\,ds = \int_0^{\pi/2} F(\vec{c}(\theta))\|\vec{c}\,'(\theta)\|\,d\theta$$

$$= \int_0^{\pi/2} (a\cos\theta + a\sin\theta)\,a\,d\theta$$

$$= a^2\int_0^{\pi/2} (\cos\theta + \sin\theta)\,d\theta = a^2\Big[\sin\theta - \cos\theta\Big]_0^{\pi/2}$$

$$= a^2\left[1 - 0 - (0-1)\right] = 2a^2$$

$\therefore$ Mass $= \underline{8a^2}$

7.

(a) $(x^2-4x+4)+(y^2-6y+9)+z^2=12+4+9$

$\therefore (x-2)^2+(y-3)^2+z^2=5^2$, a sphere, or

$\|(x,y,z)-(2,3,0)\|=5$

Use spherical coordinates

$x'=\rho\sin\phi\cos\theta,\ y'=\rho\sin\phi\sin\theta,\ z'=\rho\cos\phi$

Here, $\rho=5$, and (x',y',z') centered at $(0,0,0)$.

$\therefore (x,y,z)=(x',y',z')+(2,3,0)$

$\therefore x=2+5\sin\phi\cos\theta \quad y=3+5\sin\phi\sin\theta \quad z=5\cos\phi$

Or, $\underline{\underline{T(\theta,\phi)}}=(2+5\sin\phi\cos\theta,\ 3+5\sin\phi\sin\theta,\ 5\cos\phi)$,

$0\le\phi\le\pi,\ 0\le\theta\le 2\pi$

(b) $2x^2-8x=2(x^2-4x)=2(x^2-4x+4)-8=2(x-2)^2-8$

$=\dfrac{(x-2)^2}{\frac{1}{2}}-8$

$\therefore \dfrac{(x-2)^2}{\frac{1}{2}}+y^2+z^2=9$, or

$$\frac{(x-2)^2}{\left(\frac{3}{\sqrt{2}}\right)^2} + \frac{y^2}{3^2} + \frac{z^2}{3^2} = 1 \qquad [1]$$

$\therefore$ an ellipsoid centred at $(2,0,0)$, with minor axes of $\frac{3}{\sqrt{2}}$ for x-axis, 3 for y,z axes.

$\therefore$ Use spherical coordinates as a framework.

From $\frac{(x-2)^2}{\left(\frac{\sqrt{3}}{2}\right)^2}$, let $x = 2 + \frac{\sqrt{3}}{2} f(\theta,\phi)$

$\therefore \frac{(x-2)^2}{\left(\frac{\sqrt{3}}{2}\right)^2}$ becomes $f^2(\theta,\phi)$

If $y = 3g(\theta,\phi)$, $z = 3h(\theta,\phi)$, then [1]

becomes $f^2 + g^2 + h^2 = 1$.

Using the sphere as an analogy,

$f(\theta,\phi) = \cos\theta\sin\phi$, $g(\theta,\phi) = \sin\theta\sin\phi$,

$h(\theta,\phi) = \cos\phi$

$\therefore x = 2 + \frac{\sqrt{3}}{2}\cos\theta\sin\phi$, $y = 3\sin\theta\sin\phi$, $z = 3\cos\phi$

$\therefore$ The "ρ" coefficient is the length of the relevant minor axis.

$$\therefore T(\theta,\phi) = \left[2 + \frac{\sqrt{3}}{2}\cos\theta\sin\phi,\ 3\sin\theta\sin\phi,\ 3\cos\phi\right],$$

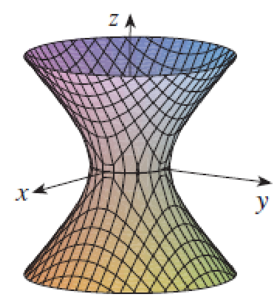
$$0 \le \theta \le 2\pi,\quad 0 \le \phi \le \pi$$

(c)

$$\frac{4}{8}x^2 + \frac{9}{8}y^2 - \frac{2}{8}z^2 = 1, \quad \frac{x^2}{(\sqrt{2})^2} + \frac{y^2}{\left(\frac{2\sqrt{2}}{3}\right)^2} - \frac{z^2}{2^2} = 1$$

Analogous to an hyperboloid of one sheet, each horizontal section an ellipse

Hyperboloid of One Sheet



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

Horizontal traces are ellipses.

Vertical traces are hyperbolas.

The axis of symmetry corresponds to the variable whose coefficient is negative.

Use cylindrical coordinates

from Stewart - Calculus, 7th ed.

$$4x^2 + 9y^2 = 8 + 2z^2, \quad \frac{x^2}{\frac{8+2z^2}{4}} + \frac{y^2}{\frac{8+2z^2}{9}} = 1$$

Using $\cos^2\theta + \sin^2\theta = 1$, $-\infty < z < \infty$

$$\therefore \frac{x^2}{\left(\frac{\sqrt{8+2z^2}}{2}\right)^2} + \frac{y^2}{\left(\frac{\sqrt{8+2z^2}}{3}\right)^2} = 1$$

$$\therefore \text{Let } x = \left(\frac{\sqrt{8+2z^2}}{2}\right)\cos\theta, \quad y = \left(\frac{\sqrt{8+2z^2}}{3}\right)\sin\theta$$

$$\therefore T(\theta, z) = \left[\frac{\sqrt{8+2z^2}}{2} \cos\theta, \frac{\sqrt{8+2z^2}}{3} \sin\theta, z \right],$$

$$0 \leq \theta \leq 2\pi, \; -\infty < z < \infty$$

8.

$$\phi(u,v) = (u+v, u, v)$$

$$\phi_u = (1,1,0) \quad \phi_v = (1,0,1)$$

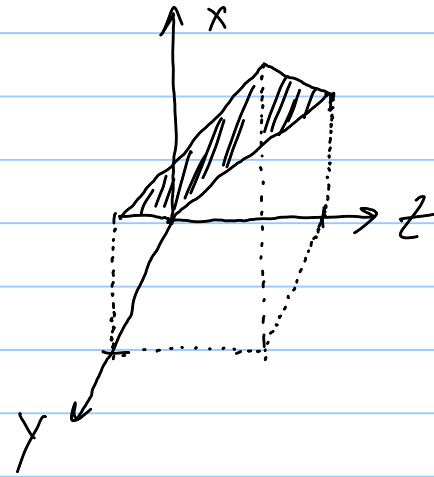
$$\phi_u \times \phi_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = (1,-1,-1)$$

$$\therefore \|\phi_u \times \phi_v\| = \sqrt{3}$$

$$\therefore A(S) = \iint_D \|\phi_u \times \phi_v\| \, du\,dv = \int_0^1 \int_0^1 \sqrt{3} \, du\,dv$$

$$= \underline{\sqrt{3}}$$

Much like the graph of $z = g(x,y)$, this is the graph of $x = g(y,z)$, so using yz plane as domain D, and surface above/below that.



$x = y + z$
a plane

9.

$\phi(r,\theta) = (r\cos\theta, 2r\sin\theta, r)$

$\phi_r = (\cos\theta, 2\sin\theta, 1) \qquad \phi_\theta = (-r\sin\theta, 2r\cos\theta, 0)$

$$\phi_r \times \phi_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & 2\sin\theta & 1 \\ -r\sin\theta & 2r\cos\theta & 0 \end{vmatrix} = (-2r\cos\theta, -r\sin\theta, 2r)$$

$$\therefore \|\phi_r \times \phi_\theta\| = \sqrt{4r^2\cos^2\theta + r^2\sin^2\theta + 4r^2}$$

$$= \sqrt{3r^2\cos^2\theta + 5r^2} = r\sqrt{3\cos^2\theta + 5}$$

$$\therefore A(S) = \int_0^{2\pi}\int_0^1 r\sqrt{3\cos^2\theta + 5}\, dr\, d\theta$$

$$= \int_0^{2\pi} \sqrt{3\cos^2\theta + 5}\, d\theta \left[\frac{r^2}{2} \right]_{r=0}^{r=1}$$

$$= \frac{1}{2} \int_0^{2\pi} \sqrt{3\cos^2\theta + 5}\, d\theta$$

Note: $2x = 2r\cos\theta$ $\quad \therefore (2x)^2 + y^2 = 4r^2 = 4z^2$

$\therefore 4x^2 + y^2 = 4z^2$, $z^2 = x^2 + \frac{1}{4}y^2$

$f(x,y) = x^2 + \frac{1}{4}y^2$ is a series of ellipsis.

Note $z \geq 0$ since $0 \leq r \leq 1$.

$\therefore z^2 = x^2 + \frac{1}{4}y^2$ is the upper portion of a cone in which each horizontal section is an ellipse of form $x^2 + \frac{1}{4}y^2 = K$, a constant.

10.

Given the parametrization $\phi(x,y) = (x, y, f(x,y))$,

$\| \phi_x \times \phi_y \| = \sqrt{1 + f_x^2 + f_y^2} = \sqrt{1 + c}$

$$\therefore A(S) = \iint_D \sqrt{1+c}\, dx\,dy = \sqrt{1+c} \iint_D dx\,dy$$

$$= \underline{\sqrt{1+c}\, A(D)}$$

11.

From #8, $\|\phi_u \times \phi_v\| = \sqrt{3}$, $\phi(u,v) = (u+v, u, v)$

$$\therefore f(\phi(u,v)) = (u+v)^2 + u^2 + v^2 = 2u^2 + 2v^2 + 2uv$$

$$\therefore \iint_S f\, dS = \int_0^1 \int_0^1 \sqrt{3}\,(2u^2 + 2v^2 + 2uv)\, du\,dv$$

$$= 2\sqrt{3} \int_0^1 \int_0^1 u^2 + v^2 + uv\, du\,dv$$

$$= 2\sqrt{3} \int_0^1 \frac{u^3}{3} + uv^2 + \frac{u^2 v}{2} \Bigg|_{u=0}^{u=1} dv$$

$$= 2\sqrt{3} \int_0^1 \left(\frac{1}{3} + v^2 + \frac{v}{2}\right) dv = 2\sqrt{3} \left[\frac{v}{3} + \frac{v^3}{3} + \frac{v^2}{4}\right]_0^1$$

$$= 2\sqrt{3}\left(\frac{1}{3} + \frac{1}{3} + \frac{1}{4}\right) = \frac{22}{12}\sqrt{3} = \underline{\frac{11\sqrt{3}}{6}}$$

12.

(a) $z = 1 - x - y \quad \therefore \phi(x,y) = (x, y, 1-x-y)$

The plane has axis intercepts: $(1,0,0), (0,1,0), (0,0,1)$.

$\therefore \; 0 \leq x \leq 1, \; 0 \leq y \leq 1.$

$\phi_x = (1, 0, -1) \quad \phi_y = (0, 1, -1)$

$$\phi_x \times \phi_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = (1, 1, 1)$$

$\|\phi_x \times \phi_y\| = \sqrt{3}$

$$\therefore \iint_S f \, dS = \int_0^1 \int_0^1 (x)(\sqrt{3}) \, dx \, dy$$

$$= \int_0^1 \frac{\sqrt{3}}{2} \, dy = \underline{\frac{\sqrt{3}}{2}}$$

(b)

For S, $\phi(r, \theta) = (r\cos\theta, r\sin\theta, r\cos\theta)$

$0 \leq r \leq 1, \; 0 \leq \theta \leq 2\pi$

$\therefore f(\phi(r,\theta)) = r^2 \cos^2\theta$

$$\phi_r = (\cos\theta, \sin\theta, \cos\theta) \quad \phi_\theta = (-r\sin\theta, r\cos\theta, -r\sin\theta)$$

$$\phi_r \times \phi_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & \cos\theta \\ -r\sin\theta & r\cos\theta & -r\sin\theta \end{vmatrix} = (-r, 0, r)$$

$$\|\phi_r \times \phi_\theta\| = \sqrt{2}\, r$$

$$\therefore \iint_S f\, dS = \int_0^{2\pi}\int_0^1 (r^2\cos^2\theta)(\sqrt{2}\, r)\, dr\, d\theta$$

$$= \sqrt{2}\int_0^{2\pi} \cos^2\theta\, d\theta \int_0^1 r^3\, dr = \sqrt{2}\int_0^{2\pi} \cos^2\theta\, d\theta \left[\frac{r^4}{4}\right]_0^1$$

$\cos^2\theta - \sin^2\theta = \cos 2\theta = 2\cos^2\theta - 1$

$$= \frac{\sqrt{2}}{4}\int_0^{2\pi} \cos^2\theta\, d\theta = \frac{\sqrt{2}}{4}\int_0^{2\pi} \frac{1+\cos 2\theta}{2}\, d\theta$$

$$= \frac{\sqrt{2}}{4}\left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{2\pi} = \frac{\sqrt{2}}{4}\left[\pi + 0 - (0)\right]$$

$$= \underline{\frac{\sqrt{2}}{4}\pi}$$

(c)

$x^2 + y^2 = 2x \Leftrightarrow (x-1)^2 + y^2 = 1$, cylinder with center at $(1,0,0)$, radius 1.

$z=\sqrt{x^2+y^2}$ is upper half of cone.

Let $x=1+\cos\theta$, $y=\sin\theta$, $0\le\theta\le 2\pi$

$\therefore (x-1)^2+y^2=1.$

For z, $z=\sqrt{(1+\cos\theta)^2+\sin^2\theta}=\sqrt{1+2\cos\theta+1}$

$$=\sqrt{2(1+\cos\theta)}=\sqrt{4\left(\frac{1+\cos\theta}{2}\right)}=\sqrt{4\cos^2\frac{\theta}{2}}$$

$$=2\left|\cos\frac{\theta}{2}\right| \text{ for } 0\le\theta\le 2\pi$$

$$\therefore 0\le z\le 2\cos\frac{\theta}{2} \text{ for } -\pi\le\theta\le\pi$$

$\therefore T(\theta,z)=(1+\cos\theta, \sin\theta, z)$, $-\pi\le\theta\le\pi$, $0\le z\le 2\cos\frac{\theta}{2}$

$f(T(\theta,z))=1+\cos\theta$

$T_\theta=(-\sin\theta, \cos\theta, 0)$ $T_z=(0,0,1)$

$$T_\theta\times T_z=\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}=(\cos\theta,\sin\theta,0)$$

$\|T_\theta\times T_z\|=1$

$$\therefore \iint_S f\,dS=\int_{-\pi}^{\pi}\int_0^{2\cos\frac{\theta}{2}}(1+\cos\theta)(1)\,dz\,d\theta$$

$$= \int_{-\pi}^{\pi} (1+\cos\theta)\left(2\cos\frac{\theta}{2}\right) d\theta$$

$\cos\theta = \cos\left(\frac{\theta}{2}+\frac{\theta}{2}\right) = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}$

$$= \int_{-\pi}^{\pi} 2\cos\frac{\theta}{2} + 2\cos^3\frac{\theta}{2} - 2\sin^2\frac{\theta}{2}\cos\frac{\theta}{2}\, d\theta$$

$$= \int_{-\pi}^{\pi} 2\cos\frac{\theta}{2} + 2\left(1-\sin^2\frac{\theta}{2}\right)\cos\frac{\theta}{2} - 2\sin^2\frac{\theta}{2}\cos\frac{\theta}{2}\, d\theta$$

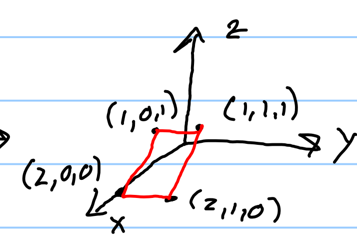
$$= \int_{-\pi}^{\pi} 4\cos\frac{\theta}{2} - 4\sin^2\frac{\theta}{2}\cos\frac{\theta}{2}\, d\theta$$

$$= 8\sin\frac{\theta}{2}\Big|_{-\pi}^{\pi} - \frac{8}{3}\sin^3\frac{\theta}{2}\Big|_{-\pi}^{\pi}$$

$$= 8[1-(-1)] - \frac{8}{3}[1-(-1)] = 16 - \frac{16}{3} = \underline{\frac{32}{3}}$$

13.

Note the order $\rightarrow$



Assume this is a surface integral over entire rectangular area, not the path integral.

Find normal to plane: $(2,1,0)-(2,0,0)=(0,1,0)$

$(1,0,1)-(2,0,0)=(-1,0,1)$

$$\therefore \vec{N} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = (1,0,1) \qquad \|\vec{N}\| = \sqrt{2}$$

$\therefore$ Plane: $(1,0,1)\cdot[(x,y,z)-(2,0,0)]$

$= (1,0,1)\cdot(x-2,y,z) = 0$

Or, $x-2+z=0$, $z=2-x$

$\therefore \phi(x,y) = (x,y,2-x)$, $1 \le x \le 2$, $0 \le y \le 1$

$\|\phi_x \times \phi_y\| = \|\vec{N}\| = \sqrt{2}$

$f(\phi(x,y)) = (x)(y)(2-x) = 2xy - x^2y$

$$\therefore \iint_S f\,ds = \int_1^2 \int_0^1 (2xy - x^2y)(\sqrt{2})\,dy\,dx$$

$$= \sqrt{2}\int_1^2 xy^2 - \frac{x^2y^2}{2}\Big|_{y=0}^{y=1} dx = \sqrt{2}\int_1^2 x - \frac{x^2}{2}\,dx$$

$$= \sqrt{2}\left[\frac{x^2}{2} - \frac{x^3}{6}\right]\Big|_1^2 = \sqrt{2}\left[2 - \frac{4}{3} - \left(\frac{1}{2} - \frac{1}{6}\right)\right]$$

$$= \sqrt{2}\left(\frac{2}{3} - \frac{1}{3}\right) = \sqrt{2}/3$$

14.

$T(\theta,\phi) = (\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi), \; 0\le\theta\le 2\pi, 0\le\phi\le\pi$

$f(T(\theta,\phi)) = \cos\theta\sin\phi + \sin\theta\sin\phi$

$T_\theta = (-\sin\theta\sin\phi, \cos\theta\sin\phi, 0)$

$T_\phi = (\cos\theta\cos\phi, \sin\theta\cos\phi, -\sin\phi)$

As shown in Example 1, p. 401 of text,

$T_\theta \times T_\phi = (-\sin^2\phi\cos\theta, -\sin^2\phi\sin\theta, -\sin\phi\cos\phi)$

$\therefore \|T_\theta \times T_\phi\| = \sqrt{\sin^4\phi(\cos^2\theta+\sin^2\theta) + \sin^2\phi\cos^2\phi}$

$= \sin\phi$, and $\sin\phi \ge 0$ for $0\le\phi\le\pi$

$$\therefore \iint_S f\,dS = \int_0^{2\pi}\int_0^{\pi} (\cos\theta\sin\phi + \sin\theta\sin\phi)(\sin\phi)\,d\phi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \sin^2\phi\,(\cos\theta+\sin\theta)\,d\phi\,d\theta$$

$$= \int_0^{2\pi}(\cos\theta+\sin\theta)\,d\theta \int_0^{\pi}\sin^2\phi\,d\phi$$

$$= (0)\int_0^{\pi} \sin^2\phi \, d\phi = \underline{0}$$

15.

Assume surface integral over area of triangle.

Normal to plane of triangle: $(2,1,1)-(1,1,1) = (1,0,0)$
$(2,0,3)-(1,1,1) = (1,-1,2)$

$$\therefore \vec{N} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ 1 & -1 & 2 \end{vmatrix} = (0,-2,-1)$$

Choose $\vec{N}$ s.t. coefficient to $\hat{k}$ is positive

$\therefore \vec{N} = (0,2,1)$

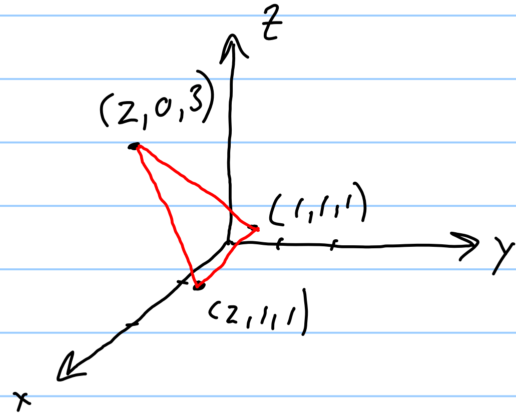
$\therefore \vec{n} =$ unit normal $= \frac{1}{\sqrt{5}}(0,2,1)$

$\cos\theta =$ angle of plane of triangle with $\hat{k}$

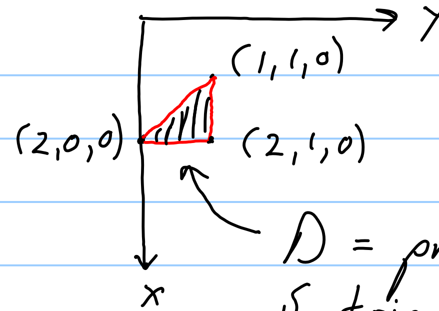
$$= \vec{n}\cdot\hat{k} = \frac{1}{\sqrt{5}}$$

$$\therefore \iint_S f\,ds = \iint_D \frac{f(x,y,g(x,y))}{\cos\theta}\,dx\,dy$$

$$= \sqrt{5}\iint_D x\,dx\,dy$$



D in the xy-plane is described as: $1 \le x \le 2$



D = projection of triangle onto xy plane

For y: slanted line is
$y = -x + 2$
$\therefore \; -x+2 \le y \le 1$

$$\therefore \iint_S f\,dS = \sqrt{5} \int_1^2 \int_{-x+2}^{1} x \, dy \, dx$$

$$= \sqrt{5} \int_1^2 xy \Big|_{y=-x+2}^{y=1} dx = \sqrt{5} \int_1^2 x - (-x^2 + 2x) \, dx$$

$$= \sqrt{5} \int_1^2 x^2 - x \, dx = \sqrt{5} \left[\frac{x^3}{3} - \frac{x^2}{2} \right]_1^2$$

$$= \sqrt{5} \left[\frac{8}{3} - 2 - \left(\frac{1}{3} - \frac{1}{2} \right) \right] = \sqrt{5} \left[\frac{7}{3} - \frac{3}{2} \right]$$

$$= \frac{5}{6}\sqrt{5}$$

16.

(a) $z = u^2 = (u\cos v)^2 + (u\sin v)^2 = x^2 + y^2$

$\therefore z = x^2 + y^2$, $-2 \le x \le 2$, $-\sqrt{4-x^2} \le y \le \sqrt{4-x^2}$

(b)

v represents the radian angle around a circle of radius u

(c)

$\phi_u = (\cos v, \sin v, 2u)$ $\phi_v = (-u\sin v, u\cos v, 0)$

$$\phi_u \times \phi_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos v & \sin v & 2u \\ -u\sin v & u\cos v & 0 \end{vmatrix} = (-2u^2\cos v, -2u^2\sin v, u)$$

$$\|\phi_u \times \phi_v\| = \sqrt{4u^4\cos^2 v + 4u^4\sin^2 v + u^2}$$

$$= u\sqrt{4u^2+1}$$

$$\therefore \vec{n}(u,v) = \frac{1}{\sqrt{4u^2+1}}(-2u\cos v, -2u\sin v, 1)$$

(d)

$$\phi(u_0, v_0) = (1,1,2) = (u_0\cos v_0, u_0\sin v_0, u_0^2)$$

$$\therefore u_0^2 = 2, \ u_0 = \sqrt{2} \text{ as } 0 \le u \le 2$$

$\therefore \sqrt{2}\cos v_0 = 1,\ \sqrt{2}\sin v_0 = 1,\ \therefore v_0 = \frac{\pi}{4}$

$\therefore (u_0, v_0) = (\sqrt{2}, \frac{\pi}{4}).$

$\phi_u(\sqrt{2}, \frac{\pi}{4}) = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 2\sqrt{2})$

$\phi_v(\sqrt{2}, \frac{\pi}{4}) = (-\sqrt{2}\frac{\sqrt{2}}{2}, \sqrt{2}\frac{\sqrt{2}}{2}, 0) = (-1, 1, 0)$

(i) Plane $= (1,1,2) + u(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 2\sqrt{2}) + v(-1,1,0)$

(ii) $\vec{n}(\sqrt{2}, \frac{\pi}{4}) = \frac{1}{3}(-2\sqrt{2}\cdot\frac{\sqrt{2}}{2}, -2\sqrt{2}\cdot\frac{\sqrt{2}}{2}, 1)$

$= \frac{1}{3}(-2, -2, 1),$

so just use $\vec{N} = (-2,-2,1)$

$\therefore (-2,-2,1)\cdot(x-1, y-1, z-2) = 0$

or $-2(x-1) - 2(y-1) + z - 2 = 0,$

or $z = 2 + 2x - 2 + 2y - 2,$

or $\underline{z = 2x + 2y - 2}$

(e)

$$A(S) = \iint_D \|\phi_u \times \phi_v\|\, du\, dv = \int_0^{2\pi}\int_0^1 u\sqrt{4u^2+1}\, du\, dv$$

$$= 2\pi \int_0^1 u\sqrt{4u^2+1}\, du = \frac{\pi}{4}\int_0^1 8u(4u^2+1)^{1/2}\, du$$

$$= \frac{\pi}{4}\left[\frac{2}{3}(4u^2+1)^{3/2}\right]_0^1 = \frac{\pi}{6}\left[5^{3/2}-1\right]$$

$$= \underline{\frac{\pi}{6}\left(5\sqrt{5}-1\right)}$$

17.

(a) $\nabla f = (f_x, f_y, f_z) = (e^y\cos\pi z,\ xe^y\cos\pi z,\ -\pi x e^y \sin\pi z)$

(b) $$\int_C \vec{F}\cdot d\vec{s} = \int_C \nabla f\cdot d\vec{s} = \int_0^{\pi} \nabla f(\vec{c}(t))\cdot \vec{c}\,'(t)\, dt$$

$$= \int_0^{\pi} (f\circ\vec{c})'(t)\, dt = (f\circ\vec{c})(t)\Big|_{t=0}^{t=\pi}$$

$$= f(\vec{c}(\pi)) - f(\vec{c}(0)) = f(3,0,0) - f(3,0,0) = \underline{0}$$

18.

Let $T(\phi,\theta) = (\cos\theta\sin\phi,\ \sin\theta\sin\phi,\ \cos\phi)$,

$0 \le \theta \le 2\pi,\ 0 \le \phi \le \frac{\pi}{2}$

$\therefore$ as in Example 1, p. 401 of the text,

$T_\phi \times T_\theta = (\sin^2\phi\cos\theta, \sin^2\phi\sin\theta, \sin\phi\cos\phi)$

and $\vec{F} \cdot (T_\phi \times T_\theta) = \sin\phi$

$$\therefore \iint_S \vec{F} \cdot d\vec{S} = \int_0^{2\pi}\int_0^{\pi/2} \sin\phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \left[-\cos\phi\right]_{\phi=0}^{\phi=\frac{\pi}{2}} d\theta = \int_0^{2\pi} [1-(0)] \, d\theta = \underline{\underline{2\pi}}$$

19.

$\vec{c}\,'(t) = (e^t, 1, 2t)$

$\vec{F} \cdot \vec{c}\,'(t) = (e^t, t, t^2) \cdot (e^t, 1, 2t) = e^{2t} + t + 2t^3$

$$\therefore \int_{\vec{c}} \vec{F} \cdot d\vec{s} = \int_0^1 (e^{2t} + t + 2t^3) \, dt = \frac{e^{2t}}{2} + \frac{t^2}{2} + \frac{t^4}{2}\Big|_0^1$$

$$= \frac{e^2}{2} + \frac{1}{2} + \frac{1}{2} - \left(\frac{1}{2} + 0 + 0\right) = \frac{e^2}{2} + \frac{1}{2} = \underline{\frac{1}{2}(e^2+1)}$$

20.

$$\int_{\vec{c}} \vec{F} \cdot d\vec{s} = \int_a^b \nabla f \cdot \vec{c}\,'(t)\,dt = \int_a^b (f \circ \vec{c})'(t)\,dt$$

$$= (f \circ \vec{c})(t) \Big|_{t=a}^{t=b} = f(\vec{c}(b)) - f(\vec{c}(a)) = \underline{0}$$

21.

(a) $\phi_u = (2u\cos v, 2u\sin v, 1)$ $\phi_v = (-u^2\sin v, u^2\cos v, 0)$

$\phi_u(1,0) = (2,0,1)$ $\phi_v(1,0) = (0,1,0)$

$$\phi_u \times \phi_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = (-1,0,2) = \vec{N}$$

$\therefore \|\vec{N}\| = \sqrt{5}$

$\therefore$ unit normal $\vec{n}(1,0) = \underline{\underline{\frac{1}{\sqrt{5}}(-1,0,2)}}$

(b) $\phi(1,0) = (1,0,1)$

$\therefore \vec{N} \cdot [(x,y,z) - (1,0,1)] = 0$

$(-1,0,2) \cdot (x-1, y, z-1) = 0$

or, $1-x+2z-2=0$, $\underline{\underline{2z-x=1}}$

22.

Let $x=r\cos\theta$, $y=r\sin\theta$, $\therefore x^2+y^2=r^2=z^2 \Rightarrow z=r$

$\therefore T(r,\theta)=(r\cos\theta, r\sin\theta, r)$, $0\le\theta\le 2\pi$, $1\le r\le 2$

$T_r=(\cos\theta, \sin\theta, 1)$ $\quad T_\theta=(-r\sin\theta, r\cos\theta, 0)$

$$\therefore T_r\times T_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = (-r\cos\theta, -r\sin\theta, r)$$

$T_r\times T_\theta$ points "inward" and "upward" as $r>0$.

$\therefore$ Choose $\vec{N}=T_\theta\times T_r=(r\cos\theta, r\sin\theta, -r)$

$\vec{F}=(r^2\cos^2\theta, r^2\sin^2\theta, r^2)$

$\therefore \vec{F}\cdot\vec{N}=r^3\cos^3\theta+r^3\sin^3\theta-r^3=r^3(\cos^3\theta+\sin^3\theta-1)$

$$\therefore \iint_S \vec{F}\cdot d\vec{S}=\int_0^{2\pi}\int_1^2 r^3(\cos^3\theta+\sin^3\theta-1)\,dr\,d\theta$$

$$=\int_0^{2\pi}(\cos^3\theta+\sin^3\theta-1)\left.\frac{r^4}{4}\right|_{r=1}^{r=2}d\theta$$

$$= \frac{15}{4}\int_0^{2\pi} (1-\sin^2\theta)\cos\theta + (1-\cos^2\theta)\sin\theta - 1 \, d\theta$$

$$= \frac{15}{4}\int_0^{2\pi} \cos\theta + \sin\theta - 1 - \sin^2\theta\cos\theta - \cos^2\theta\sin\theta \, d\theta$$

$$= \frac{15}{4}\left[0 + 0 - 2\pi\right] - \frac{15}{4}\int_0^{2\pi} \left[\sin^2\theta\cos\theta + \cos^2\theta\sin\theta\right] d\theta$$

$$= -\frac{30}{4}\pi - \frac{15}{4}\left[\frac{\sin^3\theta}{3} - \frac{\cos^3\theta}{3}\right]_0^{2\pi}$$

$$= -\frac{15}{2}\pi - \frac{15}{4}\left[0 - \frac{1}{3} - \left(0 - \frac{1}{3}\right)\right] = \underline{-\frac{15}{2}\pi}$$

23.

The square in the xy-plane can be described as

$$\phi(x,y) = (x,y,0), \quad 0 \leq x \leq 1, \; 0 \leq y \leq 1$$

$$\phi_x = (1,0,0) = \hat{i} \quad \phi_y = (0,1,0) = \hat{j}$$

$$\phi_x \times \phi_y = \hat{i} \times \hat{j} = \hat{k} = (0,0,1)$$

$$\vec{F}(\phi(x,y)) = (x, x^2, 0) \quad \therefore \vec{F}\cdot(\phi_x \times \phi_y) = 0$$

$$\therefore \int_{\phi} \vec{F} \cdot d\vec{S} = \underline{0}$$

which makes sense as $\vec{F}$ flows parallel to xy-plane at $z = 0$.

24.

A parametrization of the upper hemisphere is

$$\phi(x,y) = (x, y, \sqrt{1-x^2-y^2}), \ -a \le x \le a, \ -a \le y \le a, \ 2a^2 < 1.$$

$$\phi_x = [1, 0, -x(1-x^2-y^2)^{-\frac{1}{2}}]$$

$$\phi_y = [0, 1, -y(1-x^2-y^2)^{-\frac{1}{2}}]$$

Let $w = \sqrt{1-x^2-y^2}$

$$\therefore \phi_x \times \phi_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -x/w \\ 0 & 1 & -y/w \end{vmatrix} = \left(\frac{x}{w}, \frac{y}{w}, 1\right)$$

$$\therefore \|\phi_x \times \phi_y\| = \frac{\sqrt{x^2+y^2+w^2}}{w} = \frac{1}{w} = \frac{1}{\sqrt{1-x^2-y^2}}$$

$$\therefore A(S) = \iint_D \|\phi_x \times \phi_y\| \, dx\, dy$$

$$= \int_{-a}^{a} \int_{-a}^{a} \frac{1}{\sqrt{1-x^2-y^2}}\, dy\, dx = \int_{-a}^{a}\int_{-a}^{a} \frac{1}{\sqrt{c^2-y^2}}\, dy\, dx$$

letting $c^2 = 1-x^2$

$$= \int_{-a}^{a} \operatorname{Arcsin}\frac{y}{c}\Big|_{y=-a}^{y=a}\, dx$$

$$= \int_{-a}^{a} \operatorname{Arcsin}\left(\frac{a}{c}\right) - \operatorname{Arcsin}\left(\frac{-a}{c}\right) dx$$

$\operatorname{Arcsin}(-x) = -\operatorname{Arcsin}(x)$

$$= 2\int_{-a}^{a} \operatorname{Arcsin}\left(\frac{a}{c}\right) dx = 2\int_{-a}^{a} \operatorname{Arcsin}\left(\frac{a}{\sqrt{1-x^2}}\right) dx$$

25.

If $\vec{F}$ is a gradient of some function f, $\nabla f = \vec{F}$,
then $\nabla \times (\nabla f) = \vec{0}$ (see Theorem 1, p. 252 of text).

$$\therefore \nabla \times \vec{F} = \vec{0} \Rightarrow \iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = 0$$

But by #20 above, $\int_c \vec{F} \cdot d\vec{s} = 0$.

$$\therefore \iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{s}$$

26.

Let $\phi(r, \theta) = (\cos\theta, \sin\theta, r)$, $0 \le \theta \le 2\pi$, $0 \le r \le 1$

$\therefore \phi_r = (0, 0, 1) \quad \phi_\theta = (-\sin\theta, \cos\theta, 0)$

$$\phi_r \times \phi_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 1 \\ -\sin\theta & \cos\theta & 0 \end{vmatrix} = (-\cos\theta, -\sin\theta, 0)$$

$\phi_r \times \phi_\theta$ points "in", so choose $\phi_\theta \times \phi_r = (\cos\theta, \sin\theta, 0)$

$\vec{F}(\phi(r,\theta)) = (\cos\theta, \sin\theta, -\sin\theta)$

$\therefore \vec{F} \cdot (\phi_\theta \times \phi_r) = \cos^2\theta + \sin^2\theta + 0 = 1$

$$\therefore \iint_S \vec{F} \cdot d\vec{S} = \int_0^{2\pi} \int_0^1 (1)\, dr\, d\theta = \underline{\underline{2\pi}}$$

27.

Define S by $\phi(\theta, z) = (2\cos\theta, 2\sin\theta, z)$

$0 \le \theta \le 2\pi$, $0 \le z \le 2\cos\theta + 3$

$\therefore \phi_\theta = (-2\sin\theta, 2\cos\theta, 0) \quad \phi_z = (0,0,1)$

$$\therefore \phi_\theta \times \phi_z = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2\sin\theta & 2\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = (2\cos\theta, 2\sin\theta, 0)$$

$$\|\phi_\theta \times \phi_z\| = 2$$

(a) $$\iint_S x^2 \, dS = \int_0^{2\pi} \int_0^{2\cos\theta + 3} 4\cos^2\theta \, (2) \, dz \, d\theta$$

$$= 8 \int_0^{2\pi} \cos^2\theta \, (2\cos\theta + 3) \, d\theta = 8 \int_0^{2\pi} 2\cos^3\theta + 3\cos^2\theta \, d\theta$$

$\cos 2\theta = 2\cos^2\theta - 1$

$$= 16 \int_0^{2\pi} (1 - \sin^2\theta)\cos\theta \, d\theta + 24 \int_0^{2\pi} \frac{1 + \cos 2\theta}{2} \, d\theta$$

$$= 16 \left[\sin\theta - \frac{\sin^3\theta}{3} \right]_0^{2\pi} + 24 \left[\frac{\theta}{2} + \frac{\sin 2\theta}{4} \right]_0^{2\pi}$$

$$= 16(0) + 24 \left[\pi + 0 - (0 + 0) \right] = \underline{24\pi}$$

(b) $$\iint_S y^2 \, dS = \int_0^{2\pi} \int_0^{2\cos\theta + 3} (4\sin^2\theta)(2) \, dz \, d\theta$$

$$= 8\int_0^{2\pi} \sin^2\theta\,(2\cos\theta + 3)\,d\theta = 8\int_0^{2\pi} (2\sin^2\theta\cos\theta + 3\sin^2\theta)\,d\theta$$

$\cos 2\theta = 1 - 2\sin^2\theta$

$$= 16\int_0^{2\pi} \sin^2\theta\cos\theta\,d\theta + 24\int_0^{2\pi} \frac{1-\cos 2\theta}{2}\,d\theta$$

$$= 16\left[\frac{\sin^3\theta}{3}\right]_0^{2\pi} + 24\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{2\pi}$$

$$= 16(0) + 24\left[\pi - 0 - (0-0)\right] = \underline{\underline{24\pi}}$$

(c) $$\iint_S z^2\,dS = \int_0^{2\pi}\int_0^{2\cos\theta+3} (z^2)(2)\,dz\,d\theta$$

$$= 2\int_0^{2\pi} \left.\frac{z^3}{3}\right|_{z=0}^{2\cos\theta+3} d\theta$$

$$= \frac{2}{3}\int_0^{2\pi} 8\cos^3\theta + 36\cos^2\theta + 54\cos\theta + 27\,d\theta \qquad [1]$$

From (a), $\int_0^{2\pi} \cos^3\theta\,d\theta = 0$, $\int_0^{2\pi} \cos^2\theta\,d\theta = \pi$

and $\int_0^{2\pi} \cos\theta\,d\theta = 0$

$\therefore$ [1] becomes $\frac{2}{3}\left[8(0) + 36(\pi) + 54(0) + 27(2\pi)\right]$

$= \frac{2}{3}\left[36\pi + 54\pi\right] = \underline{\underline{60\pi}}$

28.

Let $\Gamma(\theta) = (\cos\theta, \sin\theta, a\cos\theta + b\sin\theta)$, $0 \le \theta \le 2\pi$

$dx = -\sin\theta \, d\theta \quad dy = \cos\theta \, d\theta \quad dz = (-a\sin\theta + b\cos\theta)\, d\theta$

$\therefore \; y\,dx = -\sin^2\theta \, d\theta$

$(z - x)\,dy = (a\cos\theta + b\sin\theta - \cos\theta)\cos\theta \, d\theta$

$= (a\cos^2\theta + b\sin\theta\cos\theta - \cos^2\theta)\, d\theta$

$-y\,dz = -\sin\theta(-a\sin\theta + b\cos\theta)\, d\theta$

$= (a\sin^2\theta - b\sin\theta\cos\theta)\, d\theta$

$\therefore \; y\,dx + (z-x)\,dy - y\,dz = a\cos^2\theta + a\sin^2\theta - \sin^2\theta - \cos^2\theta$

$= a - 1$

$\int_\Gamma y\,dx + (z-x)\,dy - y\,dz = \int_\Gamma a - 1 = \int_0^{2\pi} (a-1)\, d\theta$

$= 2\pi(a-1) = 0 \Rightarrow a = 1$

$\therefore a^2 + b^2 = 1 \Rightarrow b = 0$

$\therefore \underline{a = 1, \ b = 0}$

29.

(a) Let $\phi(\theta) = (R\cos\theta, R\sin\theta, \rho\theta), \quad \theta \geq 0$

$\phi'(\theta) = (-R\sin\theta, R\cos\theta, \rho)$

$\|\phi'(\theta)\| = \sqrt{R^2\sin^2\theta + R^2\cos^2\theta + \rho^2} = \sqrt{R^2+\rho^2}$

$\therefore$ Arc length $= \displaystyle\int_{z_1/\rho}^{z_0/\rho} \sqrt{R^2+\rho^2}\, d\theta = \underline{\frac{\sqrt{R^2+\rho^2}}{\rho}(z_0 - z_1)}$

Note $z_0 = \rho\theta_0$, $z_1 = \rho\theta_1$, $\therefore \theta_0 = z_0/\rho$, $\theta_1 = z_1/\rho$

(b) $\dfrac{\Delta \text{ length (at point } z)}{\Delta \text{ time (at point } z)} \approx$ speed, instantaneous (at z)

$\therefore \dfrac{\Delta \text{ Arc length}}{\text{speed (instantaneous)}} \approx \Delta$ time at point z

$\therefore \dfrac{\Delta \text{ Arc length}/\Delta z}{\text{speed (instantaneous)}} \approx \Delta \text{time}/\Delta z$

$\therefore \displaystyle\int_0^{z_0} \left(\frac{\Delta \text{Time}}{\Delta z}\right) dz = T_0$, as quantities are functions of z.

From (a), $\text{Arc length}(z) = \dfrac{\sqrt{R^2+\rho^2}}{\rho}(z_0 - z)$

$\therefore \dfrac{\Delta \text{ length}}{\Delta z} = -\dfrac{d}{dz}\left[\dfrac{\sqrt{R^2+\rho^2}}{\rho}(z_0 - z)\right]$,

the minus sign appearing as the length gets shorter with increasing z toward z_0, the maximum value of z.

$\therefore -\dfrac{d}{dz}\text{Arc length} = -\dfrac{d}{dz}\left[\dfrac{\sqrt{R^2+\rho^2}}{\rho}(z_0 - z)\right] = \dfrac{\sqrt{R^2+\rho^2}}{\rho}$

$\therefore \dfrac{\Delta \text{ Arc length}/\Delta z}{\text{speed (instantaneous)}} \approx \dfrac{\dfrac{\sqrt{R^2+\rho^2}}{\rho}}{\sqrt{2g(z_0 - z)}}$

$$= \frac{1}{\rho}\sqrt{\frac{R^2+\rho^2}{2g}} \cdot \frac{1}{\sqrt{z_0 - z}}$$

$$\therefore T_0 = \frac{1}{\rho}\sqrt{\frac{R^2+\rho^2}{2g}} \int_0^{z_0} \frac{1}{\sqrt{z_0 - z}}\, dz$$

$$= -\frac{1}{\rho}\sqrt{\frac{R^2+\rho^2}{2g}} \left[2\sqrt{z_0 - z} \right] \Bigg|_{z=0}^{z=z_0}$$

$$= \frac{1}{\rho}\sqrt{\frac{R^2+\rho^2}{2g}} \cdot 2\sqrt{z_0} = \sqrt{\frac{4z_0(R^2+\rho^2)}{2g\rho^2}}$$

$$= \sqrt{\frac{2z_0(R^2+\rho^2)}{g\rho^2}}$$