7\. In Exercises 7 and 8, the equation of a curve and the coordinates of a point $P_1:(x_1, y_1)$ on the curve are given. Find the equation of the tangent to the curve at P_1 and make a sketch.

7. $y = \frac{1}{x}$, $P_1:(2, \frac{1}{2})$. (Hint: $\dfrac{\frac{1}{2+\Delta x} - \frac{1}{2}}{\Delta x} = \dfrac{-1}{2(2+\Delta x)}$ for all $\Delta x \neq 0$).

$$\frac{\Delta f(2)}{\Delta x} = \frac{\frac{1}{2+\Delta x} - \frac{1}{2}}{\Delta x} = \frac{2-(2+\Delta x)}{2(2+\Delta x)\Delta x}$$

$$= \frac{-1}{4+2\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{\Delta f(2)}{\Delta x} = -\frac{1}{4}$$

$\therefore\ y - \frac{1}{2} = -\frac{1}{4}(x-2)$, or $4y - 2 = 2 - x$

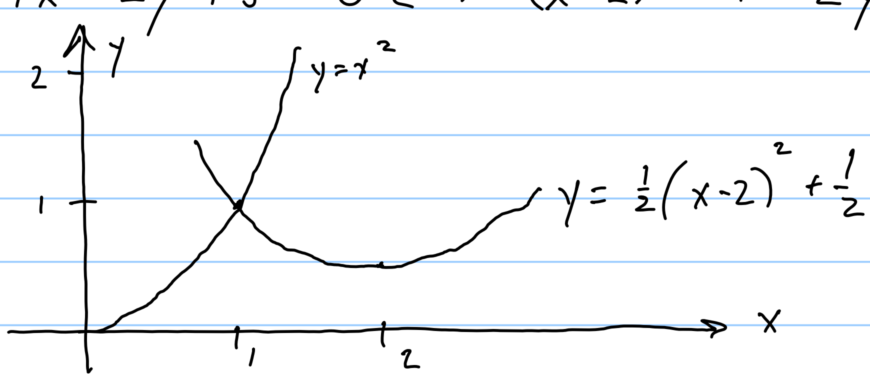
or $x + 4y - 4 = 0$

9\. In each of Exercises 9–12, the equations of two curves are given, and the coordinates of a point of intersection, Q, of the two curves are given. Find $\tan \varphi$, where φ is the angle between the two tangents to the given curves at Q, and make a sketch. (Hint: Find the slopes of the two tangents and use 2.6.2).

9. $x^2 - y = 0$, $x^2 - 4x - 2y + 5 = 0$, $Q:(1, 1)$.

$x^2 - 4x - 2y + 5 = 0 \Leftrightarrow (x-2)^2 + 1 = 2y \Leftrightarrow y = \frac{1}{2}(x-2)^2 + \frac{1}{2}$

$\therefore$



For $y = x^2$, $m = 2$ at $(1,1)$

for $y = -\frac{1}{2}(x-2)^2 + \frac{1}{2}$, $m = -1$

$$\therefore \tan\varphi = \frac{m_2 - m_1}{1 + m_2 m_1} = \frac{(-1) - 2}{1 + (-1)(2)} = \frac{-3}{-1}$$

$$= 3$$

In Exercises 8–12, a particle moves along a coordinate line, and s, its directed distance from the origin at the end of t seconds, is given in feet. Find the instantaneous velocity of the particle at the end of a seconds.

8. $s = 2t - 1, \quad a = 7.$

9. $s = t^2 + 11, \quad a = 1\frac{1}{2}.$

10. $s = \sqrt{t}, \quad a = 9.$

11. $s = \sqrt{t-2}, \quad a = 6.$

12. $s = 1/(3t), \quad a = \frac{1}{3}.$

10. $$\lim_{\Delta t \to 0} \frac{\sqrt{9+\Delta t} - \sqrt{9}}{\Delta t} =$$

$$\lim_{\Delta t \to 0} \frac{\sqrt{9+\Delta t} - 3}{\Delta t} \cdot \frac{\sqrt{9+\Delta t} + 3}{\sqrt{9+\Delta t} + 3}$$

$$= \lim_{\Delta t \to 0} \frac{\Delta t}{\Delta t\left(\sqrt{9+\Delta t} + 3\right)}$$

$$= \lim_{\Delta t \to 0} \frac{1}{\sqrt{9+\Delta t} + 3} = \frac{1}{3+3} = \frac{1}{6}$$

since $\lim_{\Delta t \to 0} \sqrt{9+\Delta t} = \sqrt{9} = 3$

24. **24.** $G(x) = \dfrac{2}{\sqrt{3-x}}$.

$$\frac{\dfrac{2}{\sqrt{3-(x+\Delta x)}} - \dfrac{2}{\sqrt{3-x}}}{\Delta x} = \frac{2\left(\sqrt{3-x} - \sqrt{3-(x+\Delta x)}\right)}{\Delta x\sqrt{3-(x+\Delta x)}\sqrt{3-x}}$$

$$= \frac{2\left(\sqrt{3-x} - \sqrt{3-(x+\Delta x)}\right)\left(\sqrt{3-x} + \sqrt{3-(x+\Delta x)}\right)}{\Delta x\sqrt{3-(x+\Delta x)}\sqrt{3-x}\left(\sqrt{3-x} + \sqrt{3-(x+\Delta x)}\right)}$$

$$= \frac{2\left(3-x-(3-(x+\Delta x))\right)}{\Delta x\sqrt{3-(x+\Delta x)}\sqrt{3-x}\left(\sqrt{3-x} + \sqrt{3-(x+\Delta x)}\right)}$$

$$= \frac{2\Delta x}{\Delta x\sqrt{3-(x+\Delta x)}\sqrt{3-x}\left(\sqrt{3-x} + \sqrt{3-(x+\Delta x)}\right)}$$

$$= \frac{2}{\sqrt{3-(x+\Delta x)}\sqrt{3-x}\left(\sqrt{3-x} + \sqrt{3-(x+\Delta x)}\right)}$$

$$\therefore \lim_{\Delta x \to 0} \frac{G(x+\Delta x) - G(x)}{\Delta x} =$$

$$\lim_{\Delta x \to 0} \frac{2}{\sqrt{3-(x+\Delta x)}\sqrt{3-x}\left(\sqrt{3-x} + \sqrt{3-(x+\Delta x)}\right)} =$$

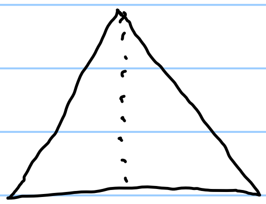
$$\frac{2}{\sqrt{3-x}\,\sqrt{3-x}\,\left(\sqrt{3-x}+\sqrt{3-x}\right)}$$

$$= \frac{2}{(3-x)\,2\sqrt{3-x}}$$

$$= \frac{1}{(3-x)\sqrt{3-x}} = (3-x)^{-\frac{3}{2}}$$

3.

3. Find the instantaneous rate of change of the area of an equilateral triangle with respect to its perimeter.



Let x = length of a side.

$\therefore$ P = perimeter $= 3x$

Height of triangle $= \sqrt{x^2 - \left(\frac{1}{2}x\right)^2} = \frac{\sqrt{3}}{2}x$

$\therefore$ Area $= \frac{1}{2}$ base $\cdot$ height $= \frac{1}{2}(x)\left(\frac{\sqrt{3}}{2}x\right)$

But $x = \frac{P}{3}$. $\therefore$ Area $= \frac{\sqrt{3}}{4}x^2 = \frac{\sqrt{3}}{36}P^2$

$\therefore$ $\frac{dA}{dP} = \frac{\sqrt{3}}{18}P$

6.

6. Show that the rate of change of the area of a circle with respect to its radius is equal to the circumference.

$A = \pi r^2$ $\quad \frac{dA}{dr} = 2\pi r$ = circumference

7.

7. Find, and justify, a theorem for spheres which is analogous to the preceding exercise.

$V = \frac{4}{3}\pi r^3$ $\quad \frac{dV}{dr} = 4\pi r^2$ = surface area

16.

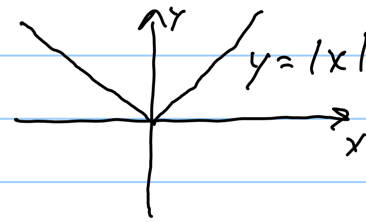
16. Water is pouring into a cylindrical tank, whose radius is 4 feet, at the rate of 20 cubic feet a minute. How fast is the water level rising? (Hint: Express the depth of the water, h, as a function of the time, t, and find $D_t h$.)

$$\text{Volume} = \pi r^2 h = 16\pi h, \quad h = \frac{V}{16\pi}$$

$$V = 20t \qquad \therefore h = \frac{20t}{16\pi} = \frac{5t}{4\pi}$$

$$\therefore \frac{dh}{dt} = \frac{5}{4\pi}$$

6. **6.** Sketch the graph of $y = |x|$ and prove that the absolute value function is continuous at 0. (Hint: Show that $\lim_{x\to 0^+} |x| = \lim_{x\to 0^-} |x|$.)



$\lim_{x\to 0} x = 0$, so $\lim_{x\to 0^+} x = 0$

$\lim_{x\to 0}(-x) = 0$, so $\lim_{x\to 0^-}(-x) = 0$

if $x \geq 0$, $|x| = x$.

$\therefore \lim_{x\to 0^+} |x| = \lim_{x\to 0^+} x = 0$

if $x < 0$, $|x| = -x$

$\therefore \lim_{x\to 0^-} |x| = \lim_{x\to 0^-}(-x) = 0$

$\therefore \lim_{x\to 0} |x| = 0$

Alternatively, let $\epsilon > 0$

$||x| - 0| = ||x|| = |x| < \epsilon \iff |x - 0| < \epsilon$

Choose $\delta = \epsilon$. $\therefore$ if $0 < |x - 0| < \delta$, then $||x| - 0| < \epsilon$

7.

7. Show that the derivative of $|x|$ exists for all $x \neq 0$, and that the derivative fails to exist at 0.

For $x > 0$, $|x| = x$

$$\therefore \lim_{x \to a} \frac{|x| - |a|}{x - a} = \lim_{x \to a} \frac{x - a}{x - a} = 1$$

For $x < 0$, $|x| = -x$

$$\therefore \lim_{x \to a} \frac{|x| - |a|}{x - a} = \lim_{x \to a} \frac{-x + a}{x - a} = -1$$

$\therefore D_x |x|$ exists for $x \neq 0$

For $x = 0$, suppose $D_x |x|$ exists.

$$\therefore \lim_{x \to 0} \frac{|x| - |0|}{x - 0} = \lim_{x \to 0} \frac{|x|}{x} = L$$

$\therefore$ Let $\epsilon > 0$. $\therefore$ A $\delta > 0$ must exist s.t.

if $0 < |x| < \delta$, then $\left| \frac{|x|}{x} - L \right| < \epsilon$

$$\therefore L - \epsilon < \frac{|x|}{x} < L + \epsilon$$

Choose $\epsilon = L$. $\therefore$ There must be a $\delta > 0$
s.t. $0 < \frac{|x|}{x} < 2L$

This is impossible if $x < 0$, for then

$0 < -1 < 2L$

$\therefore D_x |x|$ does not exist.

Alternatively, $\lim\limits_{x \to 0^+} \frac{|x|}{x} \neq \lim\limits_{x \to 0^-} \frac{|x|}{x}$

$\lim\limits_{x \to 0^+} \frac{|x|}{x} = \lim\limits_{x \to 0^+} \frac{x}{x} = 1$

$\lim\limits_{x \to 0^-} \frac{|x|}{x} = \lim\limits_{x \to 0^-} -\frac{x}{x} = -1$

$\therefore \lim\limits_{x \to 0} \frac{|x|}{x}$ doesn't exist, so $D_x |x|$ doesn't exist at $x = 0$.

11. **11.** Prove that 5.5.3 and 4.4.1 are equivalent; that is, prove that 5.5.3 $\Rightarrow$ 4.4.1 and that 4.4.1 $\Rightarrow$ 5.5.3. (Hint: See the above proof that 5.5.1 $\Leftrightarrow$ 5.3.1.)

(a) Assume 5.5.3:

5.5.3 Definition (Alternate form). *Let f be a function which is defined throughout an interval containing x_1 and $x_1 + \Delta x$ as interior points or endpoints; then the function f is* **continuous** *at x_1 if*

$$\lim_{\Delta x \to 0} f(x_1 + \Delta x) = f(x_1).$$

Note, $f(x_1)$ exists by definition, satisfying (b) of 4.4.1.

Let $\epsilon > 0$. By 5.5.3, $\exists\, \delta > 0$ s.t. if $0 < |\Delta x| < \delta$

Then $|f(x_1+\Delta x) - f(x_1)| < \epsilon$.

Let $x = x_1 + \Delta x$. $\therefore \Delta x = x - x_1$

$\therefore$ if $0 < |x - x_1| < \delta$ then $|f(x_1 + x - x_1) - f(x_1)| < \epsilon$

i.e., $|f(x) - f(x_1)| < \epsilon$.

$\therefore$ By def. of limit, $\lim_{x \to x_1} f(x) = f(x_1)$.

$\therefore$ conditions (a) and (c) of 4.4.1 are satisfied.
$\therefore$ 5.53 $\Rightarrow$ 4.4.1

(b) Assume 4.4.1:

4.4.1 Definition. *A function f is* **continuous** *at a number a if*

$$\lim_{x \to a} f(x) = f(a).$$

Let $g(\Delta x) = x_1 + \Delta x$, where x_1 is some number in the interval under consideration.
$\therefore (f \circ g)(\Delta x) = f(g(\Delta x)) = f(x_1 + \Delta x)$.

But $g(\Delta x)$ is continuous at $\Delta x = 0$, since

$$\lim_{\Delta x \to 0} g(\Delta x) = x_1 = g(0)$$

$\therefore$ By 4.4.4, $f \circ g$ is continuous at $\Delta x = 0$,

since, by assumption, f is continuous at

$x_1 = g(0)$. $\left[\text{i.e., } \lim_{x \to x_1} f(x) = f(x_1)\right]$

$f \circ g$ continuous at $\Delta x = 0$ means:

$$\lim_{\Delta x \to 0} f(x_1 + \Delta x) = \lim_{\Delta x \to 0} (f \circ g)(\Delta x) = f(g(0)) = f(x_1)$$

$\therefore \lim_{\Delta x \to 0} f(x_1 + \Delta x) = f(x_1)$, so $4.4.1 \Rightarrow 5.5.3$