6.2 Derivative of a Product or Quotient of Functions

Note Title 11/13/2014

16. **16.** $f(v) = \frac{14v^2}{(3v^2 - 2v)^2}$.

$$D_v (3v^2-2v)^2 = (3v^2-2v)\cdot(6v-2) + (6v-2)(3v^2-2v)$$

$$f'(v) = \frac{(3v^2-2v)^2 \cdot 28v - 14v^2 \cdot 2(6v-2)(3v^2-2v)}{(3v^2-2v)^4}$$

$$= \frac{(3v^2-2v)^3 \left[28v - 28v^2(6v-2)\right]}{(3v^2-2v)^4}$$

$$= \frac{-168v^3 + 56v^2 + 28v}{(3v^2-2v)}$$

17. **17.** $f(p) = \frac{k}{p}$, (k a constant).

$$f'(p) = \frac{p \cdot D_p k - k \cdot D_p p}{p^2} = \frac{0 - k}{p^2} = \frac{-k}{p^2}$$

23. **23.** By mathematical induction, prove 6.1.6.

6.1.6 Corollary. *The derivative of the sum of any finite number of differentiable functions is equal to the sum of their derivatives.*

(1) $n = 2$: Let $f_1(x)$, $f_2(x)$ be differentiable.

$$D_x (f_1(x) + f_2(x)) = D_x f_1(x) + D_x f_2(x)$$

by Theorem 6.1.5

(2) Consider $k > 2$, and assume

$$D_x(f_1(x) + f_2(x) + \dots + f_k(x))$$
$$= D_x f_1(x) + D_x f_2(x) + \dots + D_x f_k(x)$$

$\therefore$ $D_x(f_1 + f_2 + \dots + f_k + f_{k+1})$

$$= D_x(f_1 + \dots + f_k) + D f_{k+1} \quad \text{by (1)}$$
$$= D_x f_1 + D_x f_2 + \dots + D_x f_k + D_x f_{k+1}$$

$\therefore$ When true for k, also true for $k+1$.

$\therefore$ (1) & (2) $\Rightarrow$ true for all $n \geq 2$.

27. **27.** Use mathematical induction to prove that if f is a differentiable function and n is a positive integer, then $D_x[f(x)]^n = n[f(x)]^{n-1} \cdot f'(x)$.

The crux of the argument is:

$$D_x[f(x)]^{n+1} = D_x([f(x)]^n \cdot f(x))$$
$$= (D_x[f(x)]^n) \cdot f(x) + [f(x)]^n \cdot D_x f(x)$$
$$= (n \cdot [f(x)]^{n-1} \cdot D_x f(x)) \cdot f(x) + [f(x)]^n \cdot D_x f(x)$$
$$= n \cdot [f(x)]^n \cdot D_x f(x) + [f(x)]^n \cdot D_x f(x)$$
$$= (n[f(x)]^n + [f(x)]^n) D_x f(x)$$

$= (n+1) \cdot [f(x)]^n \cdot D_x f(x)$

$\therefore$ When true for n, also true for $n+1$.

30. **30.** Find $D_x(13x^4 - 6x + 6)^{-5}$. (Hint: Use 6.2.3 and Exercise 27.)

$D_x(13x^4 - 6x + 6)^{-5}$

$= -5(13x^4 - 6x + 6)^{-6} \cdot (52x^3 - 6)$

$$= \frac{-260x^3 + 30}{(13x^4 - 6x + 6)^6}$$

7.

7. $y = \dfrac{1}{(3x^3 - 4x^2 + 16)^5}$. (Hint: Let $u = (3x^3 - 4x^2 + 16)^5$; use 6.2.3 and the chain rule.)

Let $g(x) = 3x^3 - 4x^2 + 16$,

$h(g) = g^5 \quad \therefore h(g(x)) = g(x)^5 = (3x^3 - 4x^2 + 16)^5$

$f(h) = \dfrac{1}{h} \quad \therefore f(h(g)) = \dfrac{1}{g^5}$

$\therefore f(h(g(x))) = \dfrac{1}{(3x^3 - 4x^2 + 16)^5}$

$\therefore D_h f = -\dfrac{1}{h^2}$

$D_g f(h(g)) = [D_h f][D_g h]$

$= -\dfrac{1}{h^2} \cdot 5g^4 = -\dfrac{5g^4}{(g^5)^2}$

$= -\dfrac{5}{g^6}$

$\therefore D_x f(h(g(x))) = [D_h f] \cdot [D_g h] \cdot [D_x g]$

$= -\dfrac{5}{g^6} \cdot (9x^2 - 8x) = -\dfrac{5(9x^2 - 8x)}{(3x^3 - 4x^2 + 16)^6}$

11. **11.** $f(t) = \left(\dfrac{3t-5}{t+4}\right)^2$.

Let $u(t) = \dfrac{3t-5}{t+4}$, $g(u) = u^2$

$\therefore f(t) = g(u(t))$

$\therefore f'(t) = g'(u) \cdot u'(t) = 2u \cdot u'(t)$

$$u'(t) = \frac{(t+4)(3) - (3t-5)(1)}{(t+4)^2}$$

$$= \frac{3t+12-3t+5}{(t+4)^2} = \frac{17}{(t+4)^2}$$

$$\therefore f'(t) = 2\left(\frac{3t-5}{t+4}\right) \cdot \frac{17}{(t+4)^2}$$

$$= \frac{34(3t-5)}{(t+4)^3}$$

6.4 Derivative of Any Rational Power of a Function

Note Title 11/19/2014

23. **23.** $u = \sqrt{\dfrac{3v - 1}{v^2 + 3}}$.

$$D_v \left(\frac{3v-1}{v^2+3}\right)^{\frac{1}{2}} = \frac{1}{2}\left(\frac{3v-1}{v^2+3}\right)^{-\frac{1}{2}} \cdot D_v\left(\frac{3v-1}{v^2+3}\right)$$

$$= \frac{1}{2}\left(\frac{3v-1}{v^2+3}\right)^{-\frac{1}{2}}\left(\frac{(v^2+3)\cdot 3 - (3v-1)2v}{(v^2+3)^2}\right)$$

$$= \frac{1}{2}\left(\frac{3v-1}{v^2+3}\right)^{-\frac{1}{2}}\left(\frac{3v^2+9-6v^2+2v}{(v^2+3)^2}\right)$$

$$= \frac{1}{2}\left(\frac{3v-1}{v^2+3}\right)^{-\frac{1}{2}}\left(\frac{-3v^2+2v+9}{(v^2+3)^2}\right)$$

$$= \frac{1}{2}\frac{(3v-1)^{-\frac{1}{2}}}{(v^2+3)^{-\frac{1}{2}}}\left(\frac{-3v^2+2v+9}{(v^2+3)^2}\right)$$

$$= -\frac{1}{2}(3v-1)^{-\frac{1}{2}}(v^2+3)^{-3/2}(3v^2-2v-9)$$

29. **29.** $y = \sqrt{1 - \sqrt{1 + x}}$.

$$y = \left(1-(1+x)^{\frac{1}{2}}\right)^{\frac{1}{2}}$$

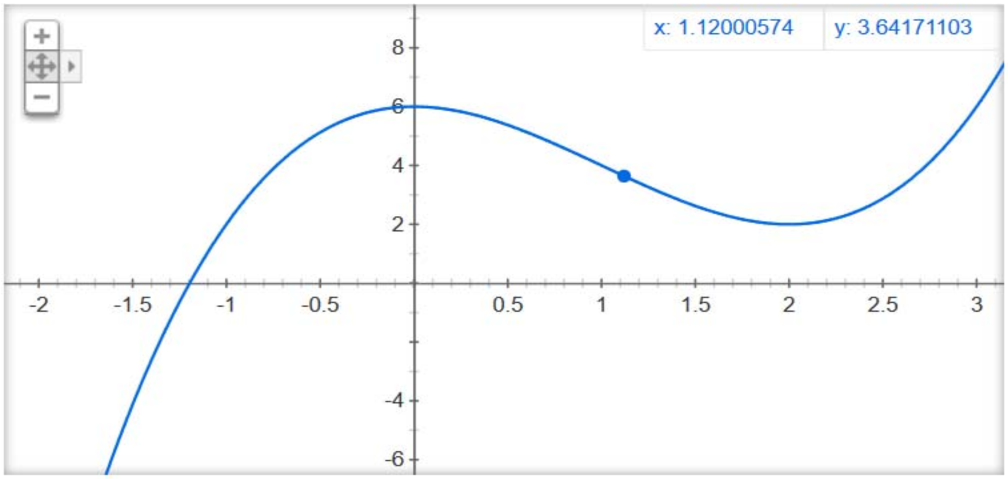
$$\therefore \frac{dy}{dx} = \frac{1}{2}\left(1-(1+x)^{\frac{1}{2}}\right)^{-\frac{1}{2}} \cdot D_x\left(-(1+x)^{\frac{1}{2}}\right)$$

$$= -\frac{1}{2}\left(1-(1+x)^{\frac{1}{2}}\right)^{-\frac{1}{2}} \cdot \left(\frac{1}{2}(1+x)^{-\frac{1}{2}}\right)$$

$$= -\frac{1}{4}\left(1-(1+x)^{\frac{1}{2}}\right)^{-\frac{1}{2}}(1+x)^{-\frac{1}{2}}$$

$$= \frac{-1}{4\sqrt{1-\sqrt{1+x}}\ \sqrt{1+x}}$$

17. **17.** Graph $f(x) = x^3 - 3x^2 + 6$. Find the points on the curve at which $f'(x) = 0$, and the point where $f''(x) = 0$. Draw the tangents to the curve at these points.



$f(x) = x^3 - 3x^2 + 6$

$f'(x) = 3x^2 - 6x = 3x(x-2) = 0$ at $x = 0, 2$

$f''(x) = 6x - 6 = 0$ at $x = 1$

19. **19.** $y = \dfrac{1}{x}$.

$D_x y = -\dfrac{1}{x^2} = -x^{-2}$

$D_x^2 y = D_x(-x^{-2}) = 2x^{-3}$

$D_x^3 y = 2(-3)x^{-4}$

$D_x^4 y = 2(-3)(-4)x^{-5}$

$\therefore D_x^n y = (-1)^n n!\, x^{-(n+1)}$

True for $n = 1, 2, 3, 4$

Assume true for K: $D_x^k y = (-1)^k k! \, x^{-(k+1)}$

$\therefore D_x^{k+1} y = D_x (D_x^k y) = D_x \left((-1)^k k! \, x^{-(k+1)} \right)$

$= (-1)^k k! \left(-(k+1)\right) x^{-(k+1)-1}$

$= (-1)(-1)^k k! (k+1) \, x^{-k-2}$

$= (-1)^{k+1} (k+1)! \, x^{-[(k+1)+1]}$

$\therefore$ True for $K+1$

$\therefore$ True for all integers $n > 0$

6.6 Implicit Differentiation

Note Title 11/19/2014

9. **9.** $y^2/x^3 - 1 = y^{3/2}$.

$$\frac{y^2}{x^3} - 1 = y^{3/2}$$

$$D_x\left(\frac{y^2}{x^3} - 1\right) = \frac{x^3 \cdot 2y \cdot D_x y - y^2 \cdot 3x^2}{x^6}$$

$$D_x\left(y^{3/2}\right) = \frac{3}{2} y^{\frac{1}{2}} \cdot D_x y$$

$$\therefore x^3 2y\, D_x y - y^2 \cdot 3x^2 = \frac{3}{2} y^{\frac{1}{2}} x^6\, D_x y$$

$$D_x y\left(2x^3 y - \frac{3}{2} x^6 y^{\frac{1}{2}}\right) = 3x^2 y^2$$

$$\therefore D_x y = \frac{6x^2 y^2}{4x^3 y - 3x^6 y^{\frac{1}{2}}}$$

$$= \frac{6y^2}{4xy - 3x^4 y^{\frac{1}{2}}}$$

10. **10.** $x^{2/3} + y^{2/3} = a^{2/3}$.

$$\frac{2}{3} x^{-\frac{1}{3}} + \frac{2}{3} y^{-\frac{1}{3}} D_x y = 0$$

$$\therefore D_x y = -\frac{x^{-\frac{1}{3}}}{y^{-\frac{1}{3}}} = -\frac{y^{\frac{1}{3}}}{x^{1/3}}$$

$$= -\sqrt[3]{\frac{y}{x}}$$

13. **13.** $xy - 2x + y - 6 = 0;\quad P:(1, 4).$

$$xy' + y - 2 + y' = 0$$

$$y' = \frac{2-y}{x+1} = \frac{2-4}{1+1} = -1$$

$$\therefore x + y + c = 0,\quad 1 + 4 + c = 0,\quad c = -5$$

$$\therefore x + y - 5 = 0$$

19. **19.** $3x^2 - 2xy + y^2 = 0.$

$$D_x(3x^2 - 2xy + y^2)$$

$$= 6x - 2y - 2x D_x y + 2y D_x y = 0$$

$$\therefore 6x - 2y - D_x y\,(2x - 2y) = 0$$

$$D_x y = \frac{6x - 2y}{2x - 2y} = \frac{3x - y}{x - y}$$

$$D_x\left(6x-2y-(2x-2y)D_x y\right)=D_x 0$$

$$6-2D_x y-(2x-2y)D_x^2 y-D_x y\left(2-2D_x y\right)=0$$

$$D_x^2 y\,(2x-2y)=6-D_x y\left(2+2-2D_x y\right)$$

$$=6-D_x y\left(4-2D_x y\right)$$

$$\therefore D_x^2 y\,(x-y)=3-D_x y\left(2-D_x y\right)$$

$$=3-2D_x y+(D_x y)^2$$

$$=\left(\frac{3x-y}{x-y}\right)^2-2\left(\frac{3x-y}{x-y}\right)+3$$

$$\therefore D_x^2 y=\frac{(3x-y)^2}{(x-y)^3}-\frac{2(3x-y)}{(x-y)^2}+\frac{3}{(x-y)}$$

$$=\frac{9x^2-6xy+y^2-2(3x^2-4xy+y^2)+3(x^2-2xy+y^2)}{(x-y)^3}$$

$$=\frac{9x^2-6xy+y^2-6x^2+8xy-2y^2+3x^2-6xy+3y^2}{(x-y)^3}$$

$$=\frac{6x^2-4xy+2y^2}{(x-y)^3}=\frac{2(3x^2-2xy+y^2)}{(x-y)^3}$$

1. **1.** Let $f(x) = x^2 - 1$. Find the value of df when: (*a*) $x = 2$, $dx = 0.75$; (*b*) $x = 0$, $dx = 3$; (*c*) $x = -1$, $dx = 0.5$; (*d*) $x = -2$, $dx = -1$. Make a careful drawing of the graph of f, and the tangents to the curve at $x = 2, 0, -1$ and -2; on this drawing show the dx and df for each of the given sets of data in (*a*), (*b*), (*c*) and (*d*).

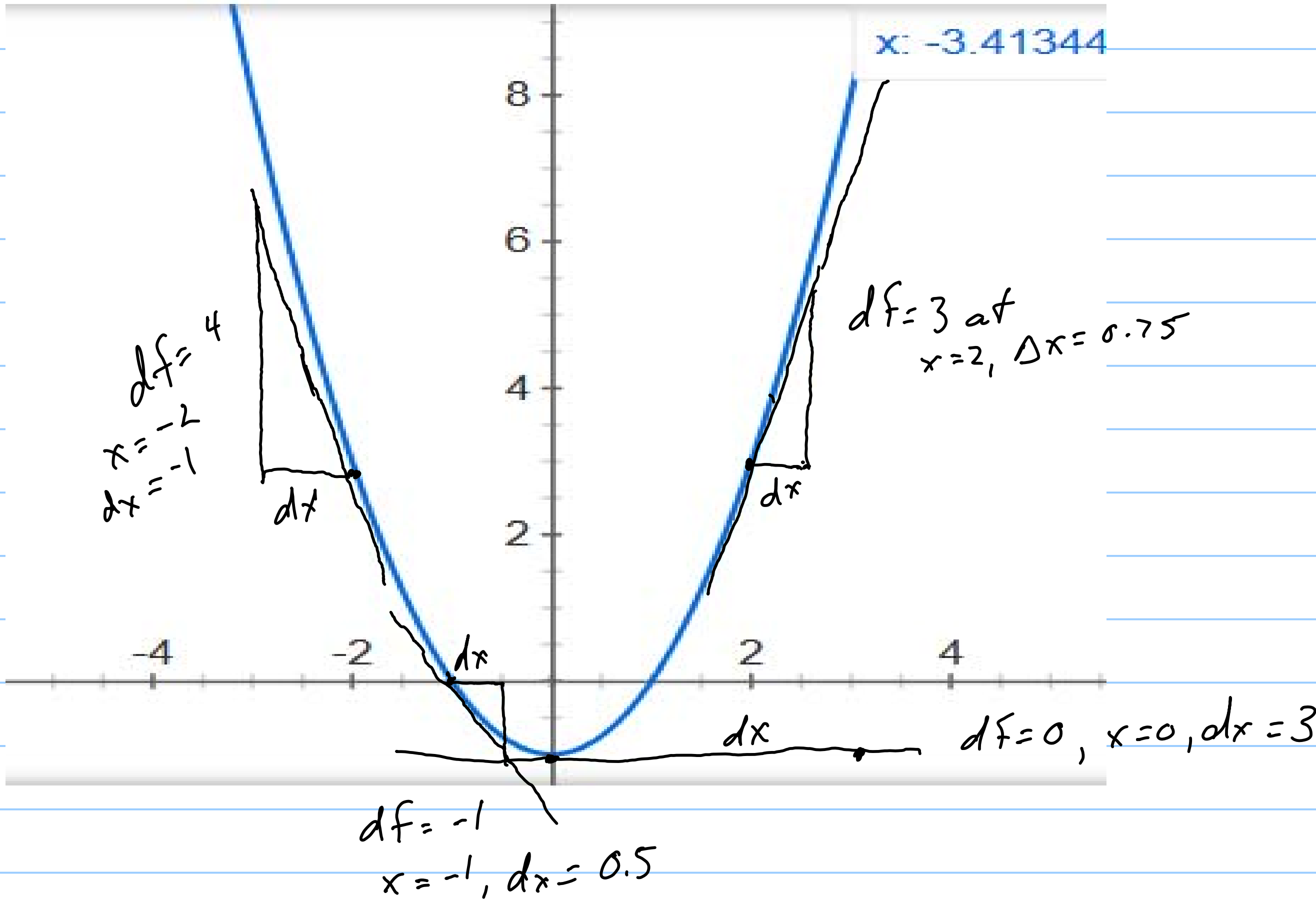
$df = 2x\,dx$

(a) $df = 2(2)(0.75) = 3.0$

(b) $df = 2(0)(3) = 0$

(c) $df = 2(-1)(0.5) = -1.0$

(d) $df = 2(-2)(-1) = 4$



8. **8.** If $s = \sqrt[5]{(t^2-3)^2}$, find ds.

$$S = (t^2-3)^{2/5}$$

$$\therefore ds = \frac{2}{5}(t^2-3)^{-3/5}(2t\,dt)$$

$$= \frac{4t\,dt}{5\sqrt[5]{(t^2-3)^3}}$$

15. **15.** $2xy^2 = 3x - 1$.

$$d(2xy^2) = 3dx$$

$$2x(2y)dy + 2dx(y^2) = 3dx$$

$$dy = \frac{3-2y^2}{4xy}dx$$

$$\frac{dy}{dx} = \frac{3-2y^2}{4xy} = y'$$

$$dy' = \frac{(4xy)(-4y\,dy) - (3-2y^2)(4x\,dy + 4y\,dx)}{16x^2y^2}$$

$$= \frac{-16xy^2dy - 12x\,dy + 8xy^2dy - 12y\,dx + 8y^3dx}{16x^2y^2}$$

$$= \frac{-8xy^2dy - 12x\,dy + (8y^3 - 12y)\,dx}{16x^2y^2}$$

$$\therefore \frac{dy'}{dx} = \frac{d^2y}{dx^2} = \frac{(-8xy^2 - 12x)\frac{dy}{dx} + (8y^3 - 12y)}{16x^2y^2}$$

$$= \frac{-4x(2y^2+3)\left(\frac{3-2y^2}{4xy}\right) + 8y^3 - 12y}{16x^2y^2}$$

$$= \frac{4x(4y^4 - 9) + 32xy^4 - 48xy^2}{64x^3y^3}$$

$$= \frac{4y^4 + 8y^4 - 12y^2 - 9}{16x^2y^3}$$

$$= \frac{12y^4 - 12y^2 - 9}{16x^2y^3} = \frac{3(4y^4 - 4y^2 - 3)}{16x^2y^3}$$

$$= \frac{3(2y^2+1)(2y^2-3)}{16x^2y^3}$$

4\. **4.** All six sides of a cubical metal box are 0.25 inch thick, and the volume of the interior of the box is 40 cubic inches. Use differentials to find the approximate volume of metal used to make the box.

$V = r^3 \quad \therefore \ dV = 3r^2 dr \qquad r = V^{1/3}, \ r^2 = V^{2/3}$

$\therefore \ dV = 3V^{2/3} dr$

$= 3(40)^{2/3}(0.25)$

$= 3(11.696)(0.25) = 8.77 \text{ in}^3$

8\. **8.** Use differentials to approximate $\sqrt{402}$. (Hint: Let $y = \sqrt{x}$. Find the value of dy when $x = 400$ and $dx = 2$. Then $\sqrt{402} \doteq 20 + dy$.

$\Delta f = f(x+\Delta x) - f(x) \qquad dy \approx \Delta f$

$\therefore \ dy \approx f(x+\Delta x) - f(x)$

$dy + f(x) \approx f(x+\Delta x)$

$\therefore \ dy + f(400) \approx f(400+2)$

Here, $f(x) = \sqrt{x}$, so $f(400) = 20$

$\therefore \sqrt{402} \approx dy + 20 \qquad y = \sqrt{x}, \ dy = \frac{1}{2}(x)^{-\frac{1}{2}} dx$

$\therefore \sqrt{402} \approx \frac{1}{2}(400)^{-\frac{1}{2}}(2) + 20 = \frac{1}{20} + 20 = 20.05$

15.

15. Assuming that the equator is a circle whose radius is 4000 miles, approximately how much longer than the equator would a concentric, coplanar circle be if each point on it were 1 foot above the equator?

$$2\pi(r+\Delta r) - 2\pi(r) = \Delta C$$

$$\Delta C \approx dC = 2\pi\, dr = 2\pi(1) = 6.28 \text{ feet}$$

16.

16. The period of a simple pendulum of length L feet is given by

$$T = 2\pi\sqrt{L/g} \text{ seconds.}$$

We assume that g, the constant of acceleration due to gravity, is 32 feet per second per second. If the pendulum is that of a clock that keeps good time when $L = 3$, how much will it lose in 24 hours if its length is increased to 3.02 feet?

$$24 \text{ hrs} = 24(60)(60) = 86{,}400 \text{ secs.}$$

$\therefore$ There are $N = \dfrac{86{,}400}{T}$ swings in 24 hrs.

$\therefore$ $N\,dT$ will be change in time over 1 day.

$$T = 2\pi\left(\frac{L}{g}\right)^{\frac{1}{2}} \quad \therefore dT = 2\pi\left(\frac{1}{2}\right)\left(\frac{L}{g}\right)^{-\frac{1}{2}}\left(\frac{dL}{g}\right)$$

$$= \frac{\pi L}{g}\left(\frac{L}{g}\right)^{-\frac{1}{2}} dL$$

$$\therefore N\,dT = \frac{86{,}400}{2\pi\sqrt{L/g}} \cdot \frac{\pi L}{g}\, \frac{1}{\sqrt{L/g}}\, dL$$

$$= \frac{(86400)\, L\, dL}{g\,(L/g)} = 86{,}400\, dL$$

$= 86{,}400\,(.02) = 1{,}728$ secs

$= 28.8$ mins.

Note that 0.02 feet = .24 inches, quite a length increase for a pendulum.