

9.2 The Sigma Notation

Note Title

12/23/2014

12. 12. Prove 9.2.2. (Hint: Since $i^3 - (i-1)^3 = 3i^2 - 3i + 1$, $\sum_{i=1}^n [i^3 - (i-1)^3] = \sum_{i=1}^n (3i^2 - 3i + 1)$; use Exercise 10 on the left member of the latter equation, and Exercises 7, 8 and 9, and 9.2.1 on the right member.)

$$\text{Use } i^3 - (i-1)^3 = 3i^2 - 3i + 1$$

$$\text{Since } \sum_{i=1}^n (a_i - a_{i-1}) = a_n - a_0,$$

$$\text{Then } \sum_{i=1}^n [i^3 - (i-1)^3] = n^3 - 0 = n^3$$

$$\begin{aligned} \text{Now } \sum_{i=1}^n (3i^2 - 3i + 1) &= 3 \sum_{i=1}^n i^2 - 3 \sum_{i=1}^n i + n \\ &= 3 \sum_{i=1}^n i^2 - 3 \frac{n(n+1)}{2} + n \end{aligned}$$

$$\therefore 3 \sum_{i=1}^n i^2 = n^3 + \frac{3n(n+1)}{2} - n$$

$$= \frac{2n^3 + 3n^2 + 3n - 2n}{2}$$

$$\therefore \sum_{i=1}^n i^2 = \frac{2n^3 + 3n^2 + n}{6} = \frac{n(2n^2 + 3n + 1)}{6}$$

$$= \frac{n(2n+1)(n+1)}{6}$$

$$17. \sum_{i=1}^n [(i+2)(3i-5)].$$

$$(i+2)(3i-5) = 3i^2 + i - 10$$

$$\therefore 3 \sum_{i=1}^n i^2 + \sum_{i=1}^n i - 10n$$

$$= 3 \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} - 10n$$

$$= \frac{n(n+1)(2n+1) + n(n+1) - 20n}{2}$$

$$= \frac{n[(n+1)(2n+2) - 20]}{2}$$

$$= n[(n+1)^2 - 10]$$

$$= n(n^2 + 2n - 9)$$

||

$$7. \sum_{i=1}^n c_i = c_1 + c_2 + \dots + c_n = \underbrace{c + c + \dots + c}_{n \text{ terms}} = nc$$

if $c_i = c$

$$\therefore \sum_{i=1}^n 10 = 10n$$

9.3 the Definite Integral

4. In Exercises 1-4, write out the Riemann sum $R(f, [a, b], P, \{\xi_i\})$ for the given data and find its value. Graph the given function and show the Riemann sum as the sum of areas of rectangles, as in Fig. 138.

4. $f(x) = x^4$, $[a, b] = [-1, 2]$, $P = \{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1, 1\frac{1}{2}, 2\}$, $\{\xi_i\} = \{-\frac{1}{2}, 0, \frac{1}{2}, 1, 1\frac{1}{2}, 2\}$.

$$\text{For, } \sum_{i=1}^6 f(\xi_i) \Delta x_i, \quad \Delta x_1 = \frac{1}{2} \quad \Delta x_4 = \frac{1}{2}$$

$$\Delta x_2 = \frac{1}{2} \quad \Delta x_5 = \frac{1}{2}$$

$$\Delta x_3 = \frac{1}{2} \quad \Delta x_6 = \frac{1}{2}$$

$$f(\xi_1) = \left(-\frac{1}{2}\right)^4 = \frac{1}{16} \quad f(\xi_4) = 1^4 = 1$$

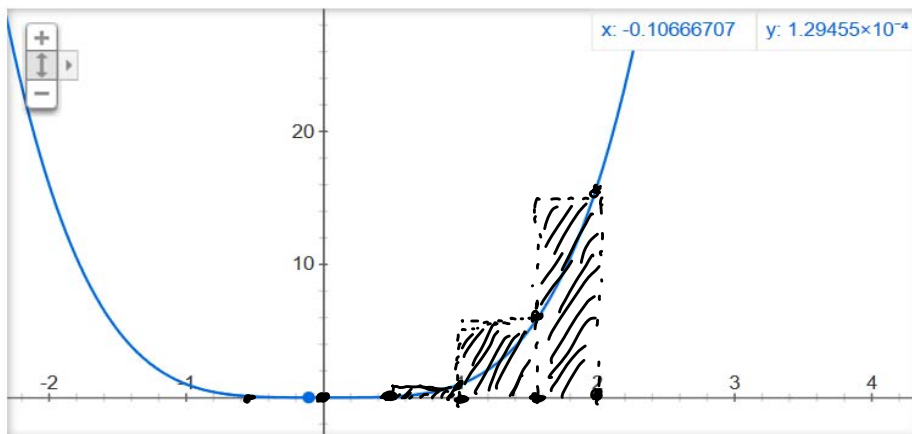
$$f(\xi_2) = 0^4 = 0 \quad f(\xi_5) = \left(\frac{3}{2}\right)^4 = \frac{81}{16}$$

$$f(\xi_3) = \left(\frac{1}{2}\right)^4 = \frac{1}{16} \quad f(\xi_6) = 2^4 = 16$$

$$\therefore \sum_{i=1}^6 f(\xi_i) \Delta x_i = \frac{1}{16} \left(\frac{1}{2}\right) + 0 \left(\frac{1}{2}\right) + \frac{1}{16} \left(\frac{1}{2}\right) + 1 \left(\frac{1}{2}\right) +$$

$$= \frac{1}{2} \left(\frac{1}{16} + 0 + \frac{1}{16} + 1 + \frac{81}{16} + \frac{1}{2} \right) \quad \frac{81}{16} \left(\frac{1}{2}\right) + \frac{1}{2} \left(\frac{1}{2}\right)$$

$$= \frac{1}{2} \left(\frac{91}{16} \right) = \frac{91}{32}$$



5.

In each of Exercises 5–10, proceed as in the Examples in 9.2 and 9.3 to find the exact value of the given definite integral.

$$5. \int_1^4 (x^2 + 1) dx. \quad \left[\text{Hint: } \xi_i = 1 + i\left(\frac{3}{n}\right). \right]$$

$$\text{Let } \xi_i = 1 + i\left(\frac{3}{n}\right), \quad a = 1, \quad b = 4, \quad \Delta x_i = \frac{3}{n}$$

$$\therefore f(\xi_i) = \left[1 + i\left(\frac{3}{n}\right) \right]^2 + 1$$

$$= i^2 \left(\frac{9}{n^2}\right) + 2i\left(\frac{3}{n}\right) + 2$$

$$\therefore \int_1^4 = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(\xi_i) \Delta x_i$$

$$= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[i^2 \left(\frac{9}{n^2}\right) + 2i\left(\frac{3}{n}\right) + 2 \right] \frac{3}{n}$$

$$= \lim_{n \rightarrow \infty} \frac{27}{n^3} \sum_{i=1}^n i^2 + \frac{18}{n^2} \sum_{i=1}^n i + \frac{3}{n} \sum_{i=1}^n 2$$

$$= \lim_{n \rightarrow \infty} \frac{27}{n^3} \left[\frac{n(n+1)(2n+1)}{6} \right] + \frac{18}{n^2} \cdot \frac{n(n+1)}{2} + \frac{3}{n} \cdot 2n$$

$$= \lim_{n \rightarrow \infty} \frac{27}{6} \left[1 \cdot \left(1 + \frac{1}{n}\right) \left(2 + \frac{1}{n}\right) \right] + \frac{18}{2} \left(1 + \frac{1}{n}\right) + 6$$

$$= \frac{27}{6} (1 \cdot 1 \cdot 2) + 9 \cdot 1 + 6 = 9 + 9 + 6$$

$$= 24$$

9.4 Approximate Integration by the Trapezoidal Rule

Note Title

12/28/2014

In each of Exercises 1-12, find an approximate value of the given definite integral by the trapezoidal rule, using the given value of n . Keep four decimal places in your calculations and round off your answer to three decimal places.

5. $\int_1^2 \frac{dx}{1+x}, n=5.$

$$\begin{array}{lll} x_0 = 1 & f(x_0) = 0.5000 & n=5 \quad \Delta x_i = \frac{2-1}{5} = 0.2 \\ x_1 = 1.2 & f(x_1) = 0.4545 & \\ x_2 = 1.4 & f(x_2) = 0.4167 & \\ x_3 = 1.6 & f(x_3) = 0.3846 & \\ x_4 = 1.8 & f(x_4) = 0.3571 & \\ x_5 = 2.0 & f(x_5) = 0.3333 & \end{array}$$

$$\begin{aligned} \int_1^2 f(x) dx &\approx \frac{\Delta x}{2} \left[0.5 + 2(0.4545) + 2(0.4167) + \right. \\ &\quad \left. 2(0.3846) + 2(0.3571) + 0.3333 \right] \\ &= \frac{0.2}{2} (4.0593) = 0.406 \end{aligned}$$

15. The work done by a horizontal force $F(x)$ in moving an object along the x -axis from $x = a$ to $x = b$ is defined to be $W = \int_a^b F(x) dx$. The horizontal force, in pounds, is measured at two-foot intervals as it moves an object from $x = 1$ to $x = 11$, and is recorded in the following table:

x	1	3	5	7	9	11
$F(x)$	14	13	10	8	5	1

Use the trapezoidal rule to find approximately the work done.

$$\begin{aligned} &\frac{2}{2} \left[14 + 2(13) + 2(10) + 2(8) + 2(5) + 1 \right] \\ &= 87 \text{ ft-lbs.} \quad \left(n=5, \Delta x_i = \frac{11-1}{5} = 2 \right) \end{aligned}$$

9.5 Properties of Definite Integrals

Note Title

1/5/2015

1. Prove 9.5.2. (Hint: By Exercise 9 of 9.2, $\sum_{i=1}^n kf(\xi_i)\Delta x_i = k \sum_{i=1}^n f(\xi_i)\Delta x_i$ for any Riemann sum for f on $[a, b]$.)

Since f is integrable on $[a, b]$, then given

any $\epsilon > 0$, $\exists \delta > 0$ s.t. if $|P| < \delta$, P a partition of $[a, b]$, then

$$\left| \sum_{i=1}^n f(\xi_i) \Delta x_i - \int_a^b f(x) dx \right| < \epsilon$$

$$\text{Let } \epsilon' = k\epsilon$$

$$\therefore \left| k \sum_{i=1}^n f(\xi_i) \Delta x_i - k \int_a^b f(x) dx \right| < k\epsilon = \epsilon'$$

$$\therefore \left| \sum_{i=1}^n kf(\xi_i) \Delta x_i - k \int_a^b f(x) dx \right| < \epsilon'$$

\therefore Given any $\epsilon' > 0$, we choose $\delta > 0$ found from choosing $\epsilon = \frac{\epsilon'}{k}$ for $\int_a^b f(x) dx$

$$\int_a^b kf(x) dx = \lim_{|P| \rightarrow 0} \sum_{i=1}^n kf(\xi_i) \Delta x_i = k \int_a^b f(x) dx$$

2. Prove 9.5.3. (Hint: Assume $\int_a^b f(x) dx = L < 0$, and show that this leads to a contradiction.)

$$\text{Suppose } \int_a^b f(x) dx = L < 0.$$

$$\text{Since } \int_a^b f(x) dx = \lim_{|P| \rightarrow 0} \sum_{i=1}^n f(\xi_i) \Delta x_i = L$$

There is an $\delta > 0$ s.t. if $|P| < \delta$

$$\left| \sum_{i=1}^n f(\xi_i) \Delta x_i - L \right| < -\frac{L}{2} \quad \left(-\frac{L}{2} > 0\right), \text{ or}$$

$$\frac{3}{2}L < \sum_{i=1}^n f(\xi_i) \Delta x_i < \frac{L}{2} < 0.$$

But $f(\xi_i) \geq 0$ for $a \leq x \leq b$, and $\Delta x_i > 0$

$\therefore f(\xi_i) \Delta x_i \geq 0$ for all i , and

$\therefore \sum_{i=1}^n f(\xi_i) \Delta x_i \geq 0$, a contradiction.

\therefore if $f(\xi_i) \geq 0$ for $a \leq x \leq b$, $\int_a^b f(x) dx \geq 0$

5. Prove 9.5.5 for the case where $a < b < c$. (Hint: Use Exercise 4, and 9.5.1, 9.3.1, and the properties of Riemann sums.)

Let $a < b < c$. The assumption is that

f is integrable on $[a, b]$, $[b, c]$, and $[a, c]$.

$$\text{Then } \int_a^b f(x) dx + \int_b^c f(x) dx = \int_a^c f(x) dx$$

Let $\epsilon > 0$

Given any $\frac{\epsilon}{2} > 0$, $\exists \delta_1, \delta_2 > 0$ s.t. if $|P_1| < \delta_1$
and $|P_2| < \delta_2$, P_1 a partition for $[a, b]$,

P_2 a partition for $[b, c]$, Then

$$\left| \sum_{i=1}^n f(\xi_i) \Delta x_i - \int_a^b f \right| < \frac{\epsilon}{2} \quad \text{and} \quad \left| \sum_{i=1}^m f(\xi_i) \Delta x_i - \int_b^c f \right| < \frac{\epsilon}{2}$$

if $\delta_1 \leq \delta_2$, let P_2' be a refinement of P_2 s.t.
 $|P_2'| \leq |P_1|$.

Let P be the combined partition of P_1 and P_2'

$$\therefore |P| \leq |P_1| \quad \text{and} \quad |P| \leq |P_2'| \leq |P_2|$$

$$\therefore |P| < \delta_1 \quad \text{and} \quad |P| < \delta_2$$

Similarly if $\delta_2 \leq \delta_1$

$$\text{Let } \delta = \min \{ \delta_1, \delta_2 \}$$

$$\therefore |P| < \delta \quad \text{since } |P| < \delta_1 \quad \text{and} \quad |P| < \delta_2$$

\therefore Given $\epsilon > 0$, $\exists \delta$ s.t. $|P| < \delta$ and

$$\therefore \left| \sum_{i=1}^n f(\xi_i) \Delta x_i - \int_a^b f \right| + \left| \sum_{i=1}^m f(\xi_i) \Delta x_i - \int_b^c f \right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

∴ By triangle inequality,

$$\left| \sum_{i=1}^n f(\xi_i') \Delta x_i' + \sum_{i=1}^m f(\xi_i'') \Delta x_i'' - \left(\int_a^b f + \int_b^c f \right) \right| < \epsilon$$

But $\sum_{i=1}^n f(\xi_i') \Delta x_i' + \sum_{i=1}^m f(\xi_i'') \Delta x_i''$ is, from

The combined partition P over $[a, c]$,

$$\sum_{i=1}^{m+n} f(\xi_i) \Delta x_i, \text{ where } f(\xi_{n+i}) = f(\xi_i'')$$

∴ Given any $\epsilon > 0$, $\exists \delta > 0$ s.t. if $|P| < \delta$, then

$$\left| \sum_{i=1}^N f(\xi_i) \Delta x_i - \left(\int_a^b f + \int_b^c f \right) \right| < \epsilon$$

$$\therefore \int_a^c f(x) dx = \lim_{|P| \rightarrow 0} \sum_{i=1}^N f(\xi_i) \Delta x_i$$

$$= \int_a^b f(x) dx + \int_b^c f(x) dx$$

8. Prove 9.5.6. **9.5.6 Theorem.** If k is a constant, then

$$\int_a^b k dx = k(b - a).$$

First prove $\int_a^b dx = (b-a)$, Then use 9.5.2,

where $f(x) = 1$.

Pf: $f(x) = 1$ is continuous on $[a, b]$, and
 \therefore is integrable over $[a, b]$.

$$\therefore \text{limit exists: } \lim_{|P| \rightarrow 0} \sum_{i=1}^n \Delta x_i = \int_a^b dx$$

$$\text{But } \sum_{i=1}^n \Delta x_i = (b-a)$$

$$\therefore \lim_{|P| \rightarrow 0} (b-a) = (b-a) = \int_a^b dx$$

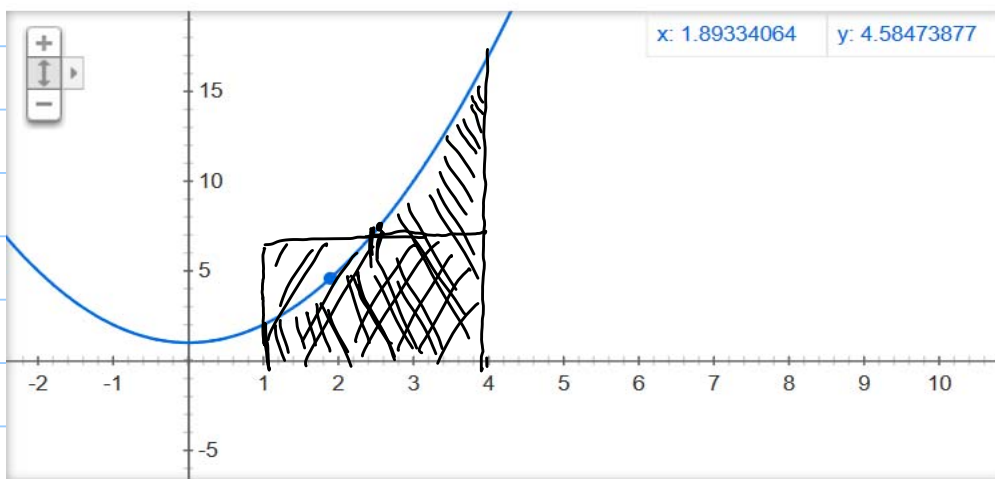
9.7 Integrals with Variable Upper Limits

1. If $\int_a^b f(x) dx = \int_1^4 (x^2 + 1) dx$, use the result of Exercise 5 of 9.3 to find the number μ whose existence is guaranteed by the mean value theorem for integrals, 9.6.2. Illustrate this application of the theorem by a sketch (see Fig. 140).

From Exercise 5 of 9.3, $\int_1^4 (x^2 + 1) dx = 24$

$\therefore \exists \mu$ s.t. $1 \leq \mu \leq 4$ and $f(\mu)(4-1) = 24$

$$\therefore (\mu)^2 + 1 = 8, \mu^2 = 7, \mu = \sqrt{7} \approx 2.65$$



5. In each of Exercises 5–10, a function F is defined on an interval by means of a definite integral with the variable upper limit x . Express $F(x)$ as an algebraic expression without an integral sign by using 9.7.1 to obtain $F'(x)$ and then finding $F(x) + C$ by antidifferentiation (8.2 and 8.3); finally, determine the value of C from the fact that $F(a) = \int_a^a f(t) dt = 0$.

5. $F(x) = \int_{-3}^x (4t + 1) dt, \quad -3 \leq x \leq 20.$

$$F'(x) = 4x + 1, \quad \therefore F(x) = 2x^2 + x + C$$

$$F(-3) = 0 = 2(-3)^2 + (-3) + C = 15 + C, \quad C = -15$$

$$\therefore F(x) = 2x^2 + x - 15$$

9.9 Finding the Exact Value of a Definite Integral

Note Title

1/8/2015

$$6. \int_{-4}^2 \frac{dy}{(y+5)^2} \quad \text{Power Rule } (y+5)^{-2} = -(y+5)^{-1}$$

$$\therefore \int_{-4}^2 \frac{dy}{(y+5)^2} = -\frac{1}{y+5} \Big|_{-4}^2 = -\frac{1}{7} + 1 = \frac{6}{7}$$

$$19. \int_0^{16} (a^{1/2} - x^{1/2})^2 dx.$$

$$\int_0^{16} (a^{1/2} - x^{1/2})^2 dx = \int_0^{16} (a - 2a^{1/2}x^{1/2} + x) dx$$

$$= ax - 2\left(\frac{2}{3}\right)a^{1/2}x^{3/2} + \frac{x^2}{2} \Big|_0^{16}$$

$$= 16a - \frac{4}{3}a^{1/2}(64) + \frac{(256)}{2} - 0$$

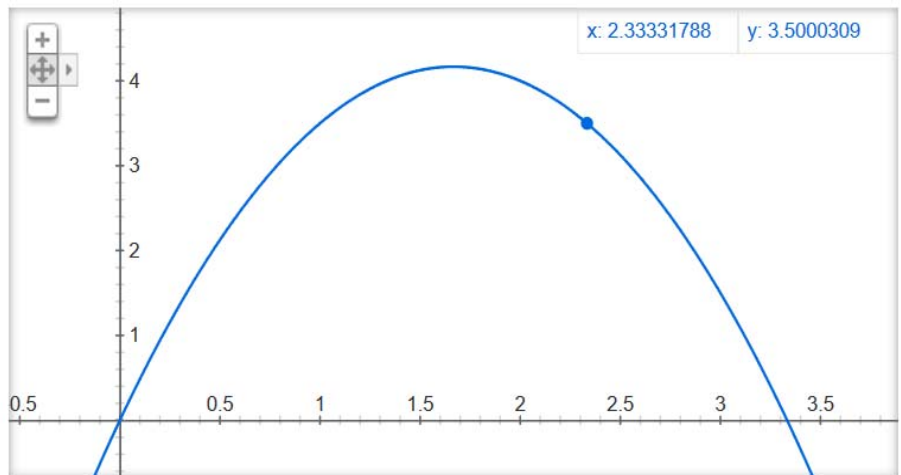
$$= 16a - \frac{256}{3}a^{1/2} + 128, \text{ or } \frac{4}{3}(12a - 64\sqrt{a} + 96)$$

21. Find the area of the region bounded by the curve $2y = 10x - 3x^2$ and the x -axis. Make a sketch.

$$y = 5x - \frac{3}{2}x^2$$

$$= x\left(5 - \frac{3}{2}x\right)$$

$$\therefore y = 0 \text{ at } x = 0, \\ x = \frac{10}{3}$$



$$\therefore \int_0^{10/3} (5x - \frac{3}{2}x^2) dx = \frac{5}{2}x^2 - \frac{x^3}{2} \Big|_0^{10/3}$$

$$= \frac{5}{2} \left(\frac{10}{3}\right)^2 - \frac{1}{2} \left(\frac{10}{3}\right)^3$$

$$= \frac{1}{2} \frac{10^2}{3^2} \left(5 - \frac{10}{3}\right) = \frac{50}{9} \left(\frac{5}{3}\right) = \frac{250}{27}$$

$$= 9 \frac{7}{27} \text{ sq. units}$$