12. **12.** Prove 9.2.2. (Hint: Since $i^3 - (i-1)^3 = 3i^2 - 3i + 1$, $\sum_{i=1}^{n} [i^3 - (i-1)^3] = \sum_{i=1}^{n} (3i^2 - 3i + 1)$; use Exercise 10 on the left member of the latter equation, and Exercises 7, 8 and 9, and 9.2.1 on the right member.)

Use $i^3 - (i-1)^3 = 3i^2 - 3i + 1$

Since $\sum_{i=1}^{n} (a_i - a_{i-1}) = a_n - a_0$,

Then $\sum_{i=1}^{n} [i^3 - (i-1)^3] = n^3 - 0 = n^3$

Now $\sum_{i=1}^{n} (3i^2 - 3i + 1) = 3\sum_{i=1}^{n} i^2 - 3\sum_{i=1}^{n} i + n$

$$= 3\sum_{i=1}^{n} i^2 - 3\frac{n(n+1)}{2} + n$$

$$\therefore 3\sum_{i=1}^{n} i^2 = n^3 + \frac{3n(n+1)}{2} - n$$

$$= \frac{2n^3 + 3n^2 + 3n - 2n}{2}$$

$$\therefore \sum_{i=1}^{n} i^2 = \frac{2n^3 + 3n^2 + n}{6} = \frac{n(2n^2 + 3n + 1)}{6}$$

$$= \frac{n(2n+1)(n+1)}{6}$$

17. **17.** $\sum_{i=1}^{n} [(i+2)(3i-5)]$.

$$(i+2)(3i-5) = 3i^2 + i - 10$$

$$\therefore 3\sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i - 10n$$

$$= 3\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} - 10n$$

$$= \frac{n(n+1)(2n+1) + n(n+1) - 20n}{2}$$

$$= \frac{n[(n+1)(2n+2) - 20]}{2}$$

$$= n[(n+1)^2 - 10]$$

$$= n(n^2 + 2n - 9)$$

---

7. $$\sum_{i=1}^{n} c_i = c_1 + c_2 + \dots + c_n = \underbrace{c + c + \dots + c}_{n \text{ terms}} = nc$$

if $c_i = c$

$$\therefore \sum_{i=1}^{n} 10 = 10n$$

9.3 the Definite Integral

Note Title 12/26/2014

4.

In Exercises 1–4, write out the Riemann sum $R(f, [a, b], P, \{\xi_i\})$ for the given data and find its value. Graph the given function and show the Riemann sum as the sum of areas of rectangles, as in Fig. 138.

4. $f(x) = x^4$, $[a, b] = [-1, 2]$, $P = \{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1, 1\frac{1}{2}, 2\}$, $\{\xi_i\} = \{-\frac{1}{2}, 0, \frac{1}{2}, 1, 1\frac{1}{2}, 2\}$.

For, $\sum_{i=1}^{6} f(\xi_i)\Delta x_i$, $\Delta x_1 = \frac{1}{2}$ $\Delta x_4 = \frac{1}{2}$

$\Delta x_2 = \frac{1}{2}$ $\Delta x_5 = \frac{1}{2}$

$\Delta x_3 = \frac{1}{2}$ $\Delta x_6 = \frac{1}{2}$

$f(\xi_1) = (-\frac{1}{2})^4 = \frac{1}{16}$ $f(\xi_4) = 1^4 = 1$

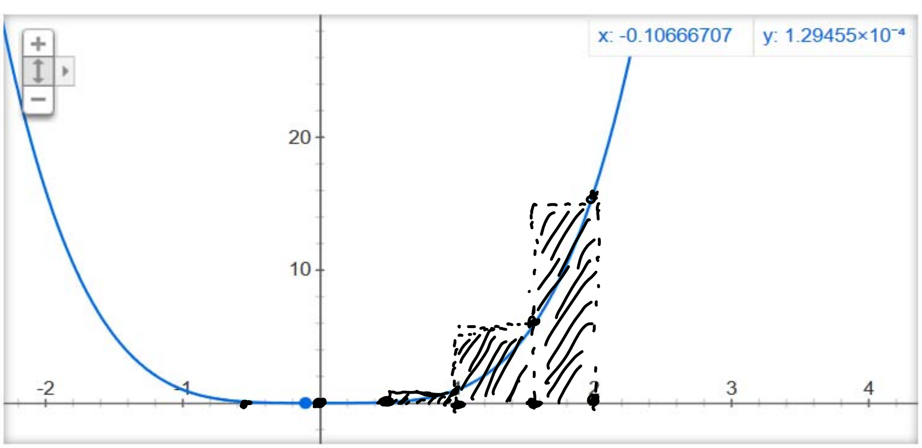
$f(\xi_2) = 0^4 = 0$ $f(\xi_5) = (\frac{3}{2})^4 = \frac{81}{16}$

$f(\xi_3) = (\frac{1}{2})^4 = \frac{1}{16}$ $f(\xi_6) = 2^4 = 16$

$\therefore \sum_{i=1}^{6} f(\xi_i)\Delta x_i = \frac{1}{16}(\frac{1}{2}) + 0(\frac{1}{2}) + \frac{1}{16}(\frac{1}{2}) + 1(\frac{1}{2}) + \frac{81}{16}(\frac{1}{2}) + \frac{1}{2}(\frac{1}{2})$

$= \frac{1}{2}(\frac{1}{16} + 0 + \frac{1}{16} + 1 + \frac{81}{16} + \frac{1}{2})$

$= \frac{1}{2}(\frac{91}{16}) = \frac{91}{32}$



5.

In each of Exercises 5–10, proceed as in the Examples in 9.2 and 9.3 to find the exact value of the given definite integral.

5. $\int_1^4 (x^2 + 1)\, dx.$ $\left[\text{Hint: } \xi_i = 1 + i\left(\frac{3}{n}\right)\cdot\right]$

Let $\xi_i = 1 + i\left(\frac{3}{n}\right)$, $a = 1$, $b = 4$, $\Delta x_i = \frac{3}{n}$

$$\therefore f(\xi_i) = \left[1 + i\left(\frac{3}{n}\right)\right]^2 + 1$$

$$= i^2\left(\frac{9}{n^2}\right) + 2i\left(\frac{3}{n}\right) + 2$$

$$\therefore \int_1^4 = \lim_{n\to\infty} \sum_{i=1}^{n} f(\xi_i)\Delta x_i$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[i^2\left(\frac{9}{n^2}\right) + 2i\left(\frac{3}{n}\right) + 2\right]\frac{3}{n}$$

$$= \lim_{n\to\infty} \frac{27}{n^3}\sum_{i=1}^{n} i^2 + \frac{18}{n^2}\sum_{i=1}^{n} i + \frac{3}{n}\sum_{i=1}^{n} 2$$

$$= \lim_{n\to\infty} \frac{27}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] + \frac{18}{n^2}\cdot\frac{n(n+1)}{2} + \frac{3}{n}\cdot 2n$$

$$= \lim_{n\to\infty} \frac{27}{6}\left[1\cdot\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\right] + \frac{18}{2}\left(1+\frac{1}{n}\right) + 6$$

$$= \frac{27}{6}(1\cdot 1\cdot 2) + 9\cdot 1 + 6 = 9 + 9 + 6$$

$$= 24$$

In each of Exercises 1-12, find an approximate value of the given definite integral by the trapezoidal rule, using the given value of n. Keep four decimal places in your calculations and round off your answer to three decimal places.

5. **5.** $\int_1^2 \frac{dx}{1+x}$, $n = 5$.

$x_0 = 1 \quad f(x_0) = 0.5000 \qquad n=5 \quad \Delta x_i = \frac{2-1}{5} = 0.2$

$x_1 = 1.2 \quad f(x_1) = 0.4545$

$x_2 = 1.4 \quad f(x_2) = 0.4167$

$x_3 = 1.6 \quad f(x_3) = 0.3846$

$x_4 = 1.8 \quad f(x_4) = 0.3571$

$x_5 = 2.0 \quad f(x_5) = 0.3333$

$$\int_1^2 f(x)\,dx \approx \frac{\Delta x}{2}\left[0.5 + 2(0.4545) + 2(0.4167) + 2(0.3846) + 2(0.3571) + 0.3333\right]$$

$$= \frac{0.2}{2}(4.0593) = 0.406$$

15. **15.** The work done by a horizontal force $F(x)$ in moving an object along the x-axis from $x = a$ to $x = b$ is defined to be $W = \int_a^b F(x)\,dx$. The horizontal force, in pounds, is measured at two-foot intervals as it moves an object from $x = 1$ to $x = 11$, and is recorded in the following table:

x	1	3	5	7	9	11
$F(x)$	14	13	10	8	5	1

Use the trapezoidal rule to find approximately the work done.

$$\frac{2}{2}\left[14 + 2(13) + 2(10) + 2(8) + 2(5) + 1\right]$$

$= 87$ ft-lbs. $\quad \left(n = 5,\ \Delta x_i = \frac{11-1}{n} = 2\right)$

9.5 Properties of Definite Integrals

Note Title 1/5/2015

1\. **1.** Prove 9.5.2. (Hint: By Exercise 9 of 9.2, $\sum_{i=1}^{n} kf(\xi_i)\Delta x_i = k\sum_{i=1}^{n} f(\xi_i)\Delta x_i$ for any Riemann sum for f on $[a, b]$.)

Since f is integrable on $[a,b]$, then given any $\epsilon > 0$, $\exists\ \delta > 0$ s.t. if $|P| < \delta$, P a partition of $[a,b]$, then

$$\left| \sum_{i=1}^{n} f(\xi_i)\Delta x_i - \int_a^b f(x)\,dx \right| < \epsilon$$

Let $\epsilon' = k\epsilon$

$$\therefore \left| k\sum_{i=1}^{n} f(\xi_i)\Delta x_i - k\int_a^b f(x)\,dx \right| < k\epsilon = \epsilon'$$

$$\therefore \left| \sum_{i=1}^{n} kf(\xi_i)\Delta x_i - k\int_a^b f(x)\,dx \right| < \epsilon'$$

$\therefore$ Given any $\epsilon' > 0$, we choose $\delta > 0$ found from choosing $\epsilon = \frac{\epsilon'}{k}$ for $\int_a^b f(x)\,dx$

$$\int_a^b kf(x)\,dx = \lim_{|P|\to 0} \sum_{i=1}^{n} kf(\xi_i)\Delta x_i = k\int_a^b f(x)\,dx$$

2\. **2.** Prove 9.5.3. (Hint: Assume $\int_a^b f(x)\,dx = L < 0$, and show that this leads to a contradiction.)

Suppose $\int_a^b f(x)\,dx = L < 0$.

Since $\int_a^b f(x)\,dx = \lim_{|P|\to 0} \sum_{i=1}^{n} f(\xi_i)\Delta x_i = L$

There is an $\delta > 0$ s.t. if $|P| < \delta$

$$\left| \sum_{i=1}^{n} f(\xi_i)\Delta x_i - L \right| < \frac{L}{2}, \text{ or}$$

$$\frac{L}{2} < \sum_{i=1}^{n} f(\xi_i)\Delta x_i < \frac{3L}{2} < 0, \text{ since } L < 0.$$

But $f(\xi_i) \geq 0$ for $a \leq x \leq b$, and $\Delta x_i > 0$

$\therefore f(\xi_i)\Delta x_i \geq 0$ for all i, and

$\therefore \sum_{i=1}^{n} f(\xi_i)\Delta x_i \geq 0$, a contradiction.

$\therefore$ if $f(\xi_i) \geq 0$ for $a \leq x \leq b$, $\int_a^b f(x)\,dx \geq 0$

5. **5.** Prove 9.5.5 for the case where $a < b < c$. (Hint: Use Exercise 4, and 9.5.1, 9.3.1, and the properties of Riemann sums.)

Let $a < b < c$. The assumption is that

f is integrable on $[a,b]$, $[b,c]$, and $[a,b]$.

Then $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

Let $\epsilon > 0$

Given any $\frac{\epsilon}{2} > 0$, $\exists\, \delta_1, \delta_2 > 0$ s.t. if $|P_1| < \delta_1$ and $|P_2| < \delta_2$, P_1 a partition for $[a,b]$, P_2 a partition for $[b,c]$, Then

$$\left| \sum_{i=1}^{n} f(\xi_i')\Delta x_i' - \int_a^b f \right| < \frac{\epsilon}{2} \text{ and } \left| \sum_{i=1}^{m} f(\xi_i'')\Delta x_i'' - \int_b^c f \right| < \frac{\epsilon}{2}$$

if $\delta_1 \leq \delta_2$, let P_2' be a refinement of P_2 s.t. $|P_2'| \leq |P_1|$.

Let P be the combined partition of P_1 and P_2'

$\therefore |P| \leq |P_1|$ and $|P| \leq |P_2'| \leq |P_2|$

$\therefore |P| < \delta_1$ and $|P| < \delta_2$

Similarly if $\delta_2 \leq \delta_1$

Let $\delta = \min\{\delta_1, \delta_2\}$

$\therefore |P| < \delta$ since $|P| < \delta_1$ and $|P| < \delta_2$

$\therefore$ Given $\epsilon > 0$, $\exists\, \delta$ s.t. $|P| < \delta$ and

$$\therefore \left| \sum_{i=1}^{n} f(\xi_i')\Delta x_i' - \int_a^b f \right| + \left| \sum_{i=1}^{m} f(\xi_i'')\Delta x_i'' - \int_b^c f \right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

$\therefore$ By triangle inequality,

$$\left| \sum_{i=1}^{n} f(\xi_i') \Delta x_i' + \sum_{i=1}^{m} f(\xi_i'') \Delta x_i'' - \left(\int_a^b f + \int_b^c f \right) \right| < \epsilon$$

But $\sum_{i=1}^{n} f(\xi_i') \Delta x_i' + \sum_{i=1}^{m} f(\xi_i'') \Delta x_i''$ is, from

the combined partition P over $[a,b]$,

$$\sum_{i=1}^{m+n} f(\xi_i) \Delta x_i, \quad \text{where } f(\xi_{n+i}) = f(\xi_i'')$$

$\therefore$ Given any $\epsilon > 0$, $\exists\, \delta > 0$ s.t. if $|P| < \delta$, then

$$\left| \sum_{i=1}^{N} f(\xi_i) \Delta x_i - \left(\int_a^b f + \int_b^c f \right) \right| < \epsilon$$

$$\therefore \int_a^c f(x)\,dx = \lim_{|P| \to 0} \sum_{i=1}^{N} f(\xi_i) \Delta x_i$$

$$= \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$

8. **8.** Prove 9.5.6.

9.5.6 Theorem. *If k is a constant, then*

$$\int_a^b k\,dx = k(b-a).$$

First prove $\int_a^b dx = (b-a)$, Then use 9.5.2,

where $f(x) = 1$.

Pf: $f(x) = 1$ is continuous on $[a,b]$, and
$\therefore$ is integrable over $[a,b]$.

$\therefore$ limit exists: $\lim\limits_{|P| \to 0} \sum_{i=1}^{n} \Delta x_i = \int_a^b dx$

But $\sum_{i=1}^{n} \Delta x_i = (b-a)$

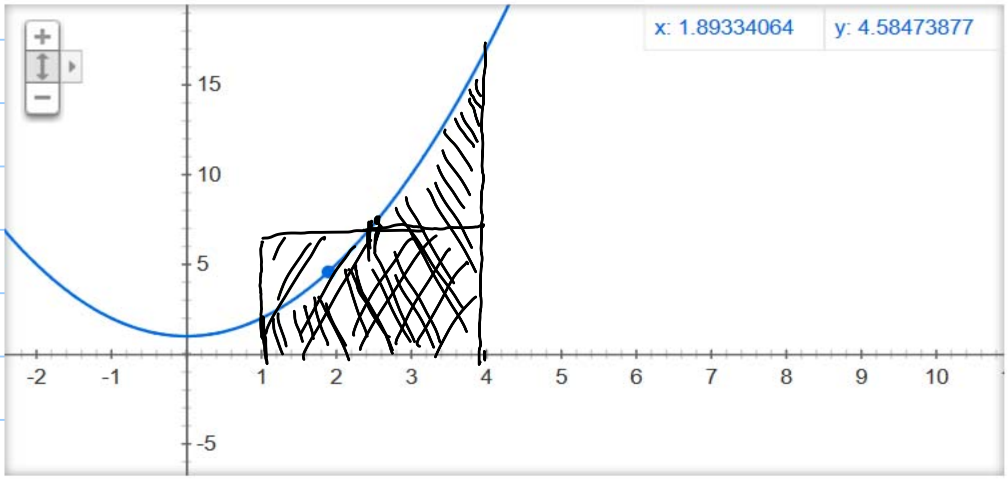
$\therefore \lim\limits_{|P| \to 0} (b-a) = (b-a) = \int_a^b dx$

1. **1.** If $\int_a^b f(x)\,dx = \int_1^4 (x^2+1)\,dx$, use the result of Exercise 5 of 9.3 to find the number μ whose existence is guaranteed by the mean value theorem for integrals, 9.6.2. Illustrate this application of the theorem by a sketch (see Fig. 140).

From Exercise 5 of 9.3, $\int_1^4 (x^2+1)\,dx = 24$

$\therefore \exists \mu$ s.t. $1 \leq \mu \leq 4$ and $f(\mu)(4-1) = 24$

$\therefore (\mu)^2 + 1 = 8$, $\mu^2 = 7$, $\mu = \sqrt{7} \approx 2.65$



5. In each of Exercises 5–10, a function F is defined on an interval by means of a definite integral with the variable upper limit x. Express $F(x)$ as an algebraic expression without an integral sign by using 9.7.1 to obtain $F'(x)$ and then finding $F(x) + C$ by antidifferentiation (8.2 and 8.3); finally, determine the value of C from the fact that $F(a) = \int_a^a f(t)\,dt = 0$.

5. $F(x) = \int_{-3}^{x} (4t+1)\,dt, \quad -3 \leq x \leq 20.$

$F'(x) = 4x+1$, $\therefore F(x) = 2x^2 + x + C$

$F(-3) = 0 = 2(-3)^2 + (-3) + C = 15 + C$, $C = -15$

$\therefore F(x) = 2x^2 + x - 15$

9.9 Finding the Exact Value of a Definite Integral

Note Title 1/8/2015

6. **6.** $\displaystyle\int_{-4}^{2} \frac{dy}{(y+5)^2}$. $D_x^{-1}(y+5)^{-2} = -(y+5)^{-1}$

$$\therefore \int_{-4}^{2} \frac{dy}{(y+5)^2} = -\frac{1}{y+5}\Bigg|_{-4}^{2} = -\frac{1}{7} + 1 = \frac{6}{7}$$

19. **19.** $\displaystyle\int_{0}^{16} (a^{1/2} - x^{1/2})^2\, dx$.

$$\int_0^{16} (a^{1/2} - x^{1/2})^2\, dx = \int_0^{16} (a - 2a^{1/2}x^{1/2} + x)\, dx$$

$$= ax - 2\left(\tfrac{2}{3}\right)a^{1/2}x^{3/2} + \frac{x^2}{2}\Bigg|_0^{16}$$

$$= 16a - \tfrac{4}{3}a^{1/2}(64) + \frac{(256)}{2} - 0$$

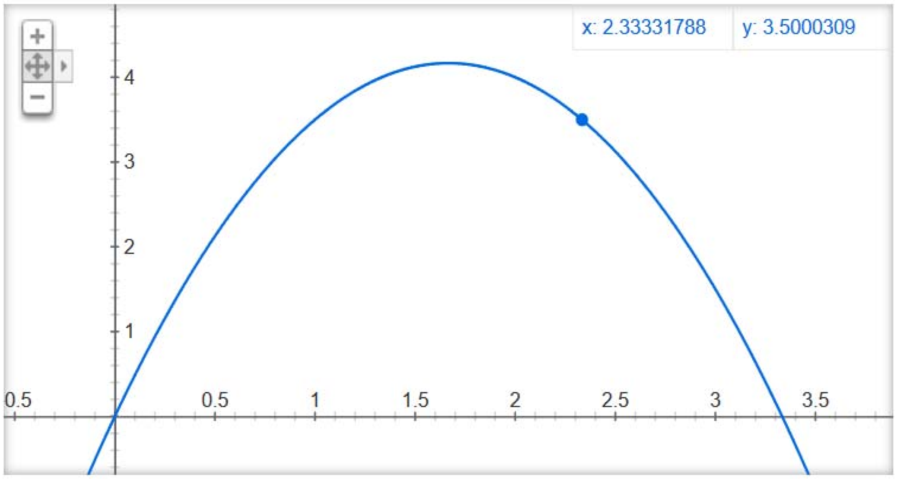
$$= 16a - \frac{256}{3}a^{1/2} + 128, \text{ or } \tfrac{4}{3}(12a - 64\sqrt{a} + 96)$$

21. **21.** Find the area of the region bounded by the curve $2y = 10x - 3x^2$ and the x-axis. Make a sketch.

$$y = 5x - \tfrac{3}{2}x^2$$

$$= x\left(5 - \tfrac{3}{2}x\right)$$

$$\therefore y = 0 \text{ at } x = 0,\; x = 10/3$$



$$\therefore \int_0^{10/3} \left(5x - \frac{3}{2}x^2\right) dx = \frac{5}{2}x^2 - \frac{x^3}{2} \Big|_0^{10/3}$$

$$= \frac{5}{2}\left(\frac{10}{3}\right)^2 - \frac{1}{2}\left(\frac{10}{3}\right)^3$$

$$= \frac{1}{2}\,\frac{10^2}{3^2}\left(5 - \frac{10}{3}\right) = \frac{50}{9}\left(\frac{5}{3}\right) = \frac{250}{27}$$

$$= 9\frac{7}{27} \text{ sq. units}$$