In each of Exercises 6–10, describe the graph of the given set of points and sketch it.

9. \{P: (x, y, z) \mid x = y\}.

A plane that is perpendicular to the xy plane, and includes the z-axis, making a 45° angle between it and the x-axis and y-axis.
1. Is the line through $P:(4, 2, -1)$ and $Q:(4, 2, 5)$ parallel to one of the coordinate axes? Find $PQ$.

Yes, the z-axis. The x and y coordinates are the same. $PQ = 5 - (-1) = 6$.

3. Find the directed distance from the point whose coordinates are $(1, 4, 2)$ to the point whose coordinates are $(1, 4, 5)$.

The x and y coordinates are equal. 

Directed distance $= \frac{5 - 2}{1 - 1}$.

9. Express by an equation the statement that the undirected distance between the point $P$ whose coordinates are $(x, y, z)$ and the point $(2, 3, 1)$ is always equal to 5. Name the graph of the set of all such points $P$.

$(x - 2)^2 + (y - 3)^2 + (z - 1)^2 = 5^2$, a sphere with center $(2, 3, 1)$ and radius 5.

11. What is the graph of the equation $(x - 1)^2 + (y + 4)^2 + (z - 2)^2 = 9$?

A sphere with center $(1, -4, 2)$ and radius 3.
1. Find the direction cosines of the line determined by each of the following pairs of points:
   
   (a) \((3, 4, 1), (-1, 8, 3)\);
   (b) \((0, -1, 7), (-6, 2, 5)\);
   (c) \((-1, 2, 3), (3, 5, 3)\);
   (d) \((3, -1, 0), (6, -1, 4)\).

\[\text{(a)}\] \(\text{Distance}^2 = (3 - (-1))^2 + (4 - 8)^2 + (1 - 3)^2 = 16 + 16 + 4 = 36\)

\[\therefore \text{Distance} = 6\]

\[\therefore \cos \alpha = \frac{3 - (-1)}{6} = \frac{2}{3}\]
\[\cos \beta = \frac{4 - 8}{6} = \frac{-2}{3}\]
\[\cos \gamma = \frac{1 - 3}{6} = -\frac{1}{3}\]

\[\text{(b)}\] \(\text{Distance}^2 = (-6 - 0)^2 + [2 - (-1)]^2 + [5 - 7]^2 = 36 + 9 + 4 = 49\)

\[\therefore \text{Distance} = 7\]

\[\therefore \cos \alpha = \frac{-6}{7} \quad \cos \beta = \frac{3}{7} \quad \cos \gamma = \frac{-2}{7}\]
\[\therefore \left(\frac{6}{7}, \frac{3}{7}, \frac{2}{7}\right)\]

\[\text{(c)}\] \(\text{Distance}^2 = (-1 - 3)^2 + (2 - 5)^2 + (3 - 3)^2 = 16 + 9 + 0 = 25\)

\[\therefore \text{Distance} = 5\]

\[\therefore \cos \alpha = \frac{-4}{5} \quad \cos \beta = \frac{-3}{5} \quad \cos \gamma = 0\]
\[\therefore \left(\frac{4}{5}, \frac{3}{5}, 0\right)\]
\( \textbf{6. Distance:} \quad [3 - 6]^2 + [-1 - (-1)]^2 + [0 - 4]^2 = 9 + 0 + 16 = 25 \)

\[ \therefore \text{Distance} = 5 \]

\[ \therefore \cos \alpha = -\frac{3}{5}, \quad \cos \beta = 0, \quad \cos \gamma = -\frac{4}{5} \]

\[ \text{or} \quad \left( \frac{3}{5}, 0, \frac{4}{5} \right) \]

4. What are the direction angles of each of the coordinate axes? The direction cosines?

\[ \text{x-axis:} \quad (0, 90^\circ, 90^\circ) \quad \therefore \quad (1, 0, 0) \]

\[ \text{y-axis:} \quad (90^\circ, 0, 90^\circ) \quad (0, 1, 0) \]

\[ \text{z-axis:} \quad (90^\circ, 90^\circ, 0) \quad (0, 0, 1) \]

5. Given that \( \cos \alpha = \frac{3}{5} \) and \( \cos \beta = -\frac{6}{5} \). Find \( \cos \gamma \).

\[ \left( \frac{3}{5} \right)^2 + \left( -\frac{6}{5} \right)^2 + (\cos \gamma)^2 = 1 \]

\[ \frac{9}{25} + \frac{36}{25} + x^2 = \frac{49}{25} \quad x^2 = \frac{4}{5} \quad x = \pm \frac{2}{\sqrt{5}} \]

7. Given that \( \alpha = 120^\circ \) and \( \gamma = 45^\circ \). Find \( \beta \). (Two solutions.)

\[ \cos \alpha = -\frac{1}{2}, \quad \cos \gamma = \frac{\sqrt{2}}{2} \]

\[ \cos^2 \beta = 1 - \left( -\frac{1}{2} \right)^2 + \left( \frac{\sqrt{2}}{2} \right)^2 = 1 - \frac{1}{4} - \frac{2}{4} = \frac{1}{4} \]

\[ \therefore \cos \beta = \frac{1}{2} \quad \therefore \beta = 60^\circ \text{ or } 120^\circ \]
9. Find the direction angles of the line through the points \((-3, 0, 5)\) and \((-7, 4\sqrt{2}, 9)\). Make a sketch.

\[
\begin{align*}
\sqrt{P_1P_2}^2 &= \left[ -7 - (-3) \right]^2 + \left[ 4\sqrt{2} - 0 \right]^2 + \left[ 9 - 5 \right]^2 \\
&= 16 + 32 + 16 \\
&= 64 \\
\therefore \sqrt{P_1P_2} &= 8
\end{align*}
\]

\[
\begin{align*}
\cos \alpha &= \frac{-4}{8} = -\frac{1}{2} \quad \therefore \alpha = 120^\circ \\
\cos \beta &= \frac{4\sqrt{2}}{8} = \frac{\sqrt{2}}{2} \quad \beta = 45^\circ \\
\cos \gamma &= \frac{4}{8} = \frac{1}{2} \quad \gamma = 60^\circ
\end{align*}
\]

\[
\alpha, \beta, \gamma \quad \left( \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{3} \right) = \left( 60^\circ, 45^\circ, 120^\circ \right)
\]
17.4 Direction Numbers

1. Find direction numbers of the lines determined by the following pairs of points:
   
   (a) \((2, 3, 5)\) and \((4, 5, 6)\);
   (b) \((-2, 0, -4)\) and \((-3, 5, 0)\);
   (c) \((5, -3, 3)\) and \((8, -7, 3)\);
   (d) \((-4, 2, -4)\) and \((0, 6, -2)\);
   (e) \((7, 2, -3)\) and \((6, 7, 1)\);
   (f) \((0, 0, 6)\) and \((1, 4, 14)\);
   (g) \((3, -1, 2)\) and \((7, 2, 2)\);
   (h) \((-2, 1, 5)\) and \((-5, 3, 7)\).

   Using the convention that for \((a, b, c)\), \(c \geq 0\),
   
   \(b \geq 0\) if \(c = 0\), then \(a \geq 0\) if \(b = 0\).

   1. \((a)\) \((4, 5, 6) - (2, 3, 5) = (2, 2, 1)\)
   
   \((b)\) \((-3, 5, 0) - (-2, 0, 4) = (-1, 5, 4)\)
   
   \((c)\) \((5, -3, 3) - (8, -7, 3) = (-3, 4, 0)\)
   
   \((d)\) \((0, 6, -2) - (-4, 2, -4) = (4, 4, 2)\)
   
   \((e)\) \((6, 7, 1) - (7, 2, -3) = (-1, 5, 4)\)
   
   \((f)\) \((1, 4, 14) - (0, 0, 6) = (1, 4, 8)\)
   
   \((g)\) \((7, 2, 2) - (3, -1, 2) = (4, 3, 0)\)
   
   \((h)\) \((-5, 3, 7) - (-2, 1, 5) = (-3, 2, 2)\)

3. Which lines in Exercise 1 are parallel to each other?
   
   \((a)\) \& \((d)\) \([d = 2(a)]\)
   
   \((b)\) \& \((c)\)

5. Which lines in Exercise 1 are perpendicular to each other?
\[(a) \cdot (b) = 0 \quad \therefore (a) \perp (b) \quad \therefore (a) \perp (h)\]

\[(c) \cdot (g) = 0 \quad \therefore (c) \perp (g)\]

7. Find the direction cosines of each of the lines in Exercise 1.

\[(a) \ \text{Length} = \sqrt{4 + 4 + 1} = 3 \quad \therefore \left( \frac{2}{3}, \frac{2}{3}, \frac{1}{3} \right)\]

\[(b) \ \text{Length} = \sqrt{1 + 25 + 16} = \sqrt{42} \quad \therefore \left( \frac{1}{\sqrt{42}}, \frac{5}{\sqrt{42}}, \frac{4}{\sqrt{42}} \right)\]

\[(c) \ \text{Length} = \sqrt{9 + 16 + 0} = 5 \quad \therefore \left( \frac{3}{5}, \frac{4}{5}, 0 \right)\]

\[(d) \ \text{Length} = \sqrt{16 + 16 + 4} = 6 \quad \therefore \left( \frac{4}{6}, \frac{4}{6}, \frac{2}{6} \right) = \left( \frac{2}{3}, \frac{2}{3}, \frac{1}{3} \right)\]

\[(e) \ \text{Length} = \sqrt{1 + 25 + 16} = \sqrt{42} \quad \therefore \left( \frac{1}{\sqrt{42}}, \frac{5}{\sqrt{42}}, \frac{4}{\sqrt{42}} \right)\]

\[(f) \ \text{Length} = \sqrt{1 + 16 + 14} = 9 \quad \therefore \left( \frac{1}{9}, \frac{4}{9}, \frac{2}{9} \right)\]

\[(g) \ \text{Length} = \sqrt{16 + 9 + 0} = 5 \quad \therefore \left( \frac{4}{5}, \frac{3}{5}, 0 \right)\]

\[(h) \ \text{Length} = \sqrt{9 + 4 + 4} = \sqrt{17} \quad \therefore \left( \frac{3}{\sqrt{17}}, \frac{2}{\sqrt{17}}, \frac{2}{\sqrt{17}} \right)\]
17.7 Normal Equation of a Plane

1. Write the equation of the plane through the point \((-5, 7, -2)\)
   (a) parallel to the \(zx\)-plane;
   (b) perpendicular to the \(x\)-axis;
   (c) parallel to both the \(x\)- and \(y\)-axes.

   \[
   (a) \quad \gamma = 7 \quad (b) \quad x = -5 \quad (c) \quad z = -2
   \]

3. Find the equation of a plane if a normal to it has direction cosines \([\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}]\), and if the directed distance from the origin to the plane is 6.

   \[
   \frac{2}{3} x - \frac{1}{3} y + \frac{2}{3} z - d = 0, \quad \text{here} \quad d = 6
   \]

   \[
   \therefore \frac{2}{3} x - \frac{1}{3} y + \frac{2}{3} z - 6 = 0, \quad \text{or} \quad 2x - y + 2z - 18 = 0
   \]

5. Direction numbers of a normal to a certain plane are \([-2, 1, 3]\), and the directed distance from the origin to the plane is \(-5\). Find the equation of the plane.

   \[
   \text{Length of normal}^2 = (-2)^2 + 1^2 + 3^2 = 14
   \]

   \[
   \therefore \quad \text{Direction cosines of normal:} \quad \left(\frac{-2}{\sqrt{14}}, \frac{1}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right)
   \]

   \[
   \therefore \quad \text{Equation of plane:} \quad \frac{-2}{\sqrt{14}} x + \frac{1}{\sqrt{14}} y + \frac{3}{\sqrt{14}} z - (-5) = 0
   \]

   \[
   \text{or} \quad -2x + y + 3z + 5\sqrt{14} = 0
   \]

7. The line through the origin perpendicular to a plane intersects the plane in the point \((1, -1, -2)\). Find the equation of the plane.

   \[
   \lambda x - \lambda y - 2\lambda + k = 0 \quad \therefore \quad 1(1) - 1(-1) - 2(-2) + k = 0
   \]

   \[
   1 + 1 + 4 + k = 0, \quad k = -6
   \]

   \[
   \therefore \quad x - y - 2z - 6 = 0
   \]
9. Direction cosines of a normal to a certain plane are \([-\frac{3}{7}, \frac{6}{7}, \frac{2}{7}\]). If the plane goes through the point \((5, 2, -1)\), find the normal equation of the plane.

\[-\frac{3}{7}x + \frac{6}{7}y + \frac{2}{7}z - d = 0\] 
\[-\frac{3}{7} \cdot 5 + \frac{6}{7} \cdot 2 + \frac{2}{7} \cdot (-1) + k = 0\]
\[-\frac{15}{7} + \frac{12}{7} - \frac{2}{7} + k = 0\]
\[k = 5\]
\[-\frac{3}{7}x + \frac{6}{7}y + \frac{2}{7}z + 5 = 0\]
\[-\frac{3}{7}x + \frac{6}{7}y + \frac{2}{7}z + \frac{35}{7} = 0\]

11. Find the equation of the plane through the point \((4, 1, -6)\) perpendicular to the line joining the points \((-1, 6, 2)\) and \((-8, 10, -2)\).

Direction numbers of normal: \((-1, -8, 10, 2, -2)\)

\[= (7, -4, 4)\]
\[7x - 4y + 4z + k = 0\]
\[7(4) - 4(1) + 4(-6) + k = 0\]
\[28 - 4 - 24 + k = 0\]
\[k = 0\]
\[7x - 4y + 4z = 0\]

13. A line through the point \((4, -2, 3)\) perpendicular to a certain plane intersects that plane in \((2, 3, 6)\). Find the equation of the plane.

Direction number of normal to plane:

\[\langle 4, -2, 3 \rangle - \langle 2, 3, 6 \rangle = \langle 2, -5, -3 \rangle\]
\[2x - 5y - 3z + k = 0\]
(2, 3, 6) is in plane, \[ 2(2) - 5(3) - 3(6) + k = 0 \]
\[ 4 - 15 - 18 + k = 0, \quad k = 29 \]
\[ 2x - 5y - 3z + 29 = 0 \]
1. Find the intercepts of the plane

\[ 3x - 4y + 2z - 12 = 0. \]

- \( x \)-intercept: \((4,0,0)\)
- \( y \)-intercept: \((0,-3,0)\)
- \( z \)-intercept: \((0,0,6)\)

3. Find the normal equations of the planes \(2x + 2y - z - 27 = 0\) and \(2x - 3y + 6z + 8 = 0\).

\[
\sqrt{2^2 + 2^2 + (-1)^2} = 3. \quad \text{Coefficient of } z = -1. \quad \therefore \text{Use } -3.
\]

\[
- \frac{2}{3}x - \frac{2}{3}y + \frac{1}{3}z + 9 = 0
\]

\[
\sqrt{2^2 + (-3)^2 + 6^2} = 7. \quad \therefore \frac{2}{7}x - \frac{3}{7}y + \frac{6}{7}z + \frac{8}{7} = 0
\]

5. Find the directed distance from the origin to the plane \(5x - 12y - 26 = 0\).

\[
\sqrt{5^2 + (-12)^2 + 0^2} = 13.
\]

- Coefficient of \( y \): negative. \( \therefore \text{Use } -13. \)

\[
- \frac{5}{13}x + \frac{12}{13}y + 2 = 0.
\]

- Directed distance from origin: \(-\frac{2}{13}\)
In each of the Exercises 7–10, find the direction cosines of a normal to the plane whose equation is given.

7. \(2x - 4y - 4z + 7 = 0\). \(\sqrt{2^2 + (-4)^2 + (-4)^2} = c\)

Since \(-4^2\), use \(-6\). \(\left(\frac{-2}{6}, \frac{4}{6}, \frac{4}{6}\right)\) or \(\left(\frac{-1}{3}, \frac{2}{3}, \frac{2}{3}\right)\)

11. Find the equation of the plane which passes through the point \((-3, 8, -1)\) and is perpendicular to a line with direction numbers \([-1, 0, 11]\).

From direction numbers, \((-1)x + (0)y + (11)z + k = 0\)

\((-1)(-3) + 11(-1) + k = 0\), \(k = 8\)

\(-x + 11z + 8 = 0\)

13. Find the equation of the plane which passes through the point \((2, -4, -5)\) and is perpendicular to the line joining the points \((-1, 5, -7)\) and \((4, 1, 1)\).

Direction numbers of a normal: \((4, 1, 1) - (-1, 5, -7) = (5, -4, 8)\)

\(5x - 4y + 8z + k = 0\)

\(5(2) - 4(-4) + 8(-5) + k = 0\)

\(10 + 16 - 40 + k = 0\), \(k = 14\)

\(5x - 4y + 8z + 14 = 0\)

15. A line through the origin, perpendicular to a certain plane, intersects that plane in the point \((-1, 5, 4)\). Find the equation of the plane.

Direction numbers of normal: \((-1, 5, 4)\)
\[
\therefore -x + 5y + 42 + k = 0. \text{ Since point is in plane,}
\]
\[
\therefore -(1) + 5(5) + 4(4) + k = 0
\]
\[
1 + 25 + 16 + k = 0, \quad k = -42.
\]
\[
\therefore -x + 5y + 42 - 42 = 0.
\]
17.9 Parallel and Perpendicular Planes

1. Which of the following planes are identical, which are parallel to each other, and which are perpendicular to each other?

(a) $3x - 5y + z + 12 = 0$;  
(b) $7x + y - 16z + 11 = 0$;  
(c) $6x - 10y + 2z - 1 = 0$;  
(d) $x - 7y - 7 = 0$;  
(e) $2x + 13 = 0$;  
(f) $6x - 10y + 2z + 24 = 0$.

Identical: $(f) = 2(a) \therefore (a) \neq (f)$

Parallel: $(6, -10, 2) = 2(3, -5, 1)$

$\therefore (a), (f) \parallel (c)$

Perpendicular: $(0, 0, 2) \cdot (1, -7, 0) = 0$

$\therefore (c) \perp (d)$

$(3, -5, 1) \cdot (7, 1, -16) = 0$

$\therefore (a), (c), (f) \perp (b)$

$(7, 1, -16) \cdot (1, -7, 0) = 0$

$\therefore (b) \perp (d)$

3. Find the equation of the plane which passes through the origin and is parallel to the plane $4x + 2y - 7z + 10 = 0$. Make a sketch.

Has normal: $(4, 2, -7)$.

$\therefore 4x + 2y - 7z + k = 0$.

$\therefore 4(0) + 2(0) - 7(0) + k = 0$, $k = 0$

$\therefore 4x + 2y - 7z = 0$. 
5. Find the equation of the plane through the point \((0, 0, 1)\), parallel to the plane \(x - 2y + z + 4 = 0\). Make a sketch.

Has form \(x - 2y + z + k = 0\)

\[
\therefore 0 - 2(0) + 1 + k = 0, \quad k = -1
\]

\[
\therefore x - 2y + z - 1 = 0
\]

7. Find the value of \(B\) if the plane \(2x + By - z + 8 = 0\) is perpendicular to the plane \(3x - 2y + 10z + 1 = 0\).

\[
(2, B, -1) \cdot (3, -2, 10) = 0
\]

\[
\therefore 6 - 2B - 10 = 0, \quad 2B = -4, \quad B = -2
\]
17.10 Conditions That Determine a Plane

1. Find the equation of the plane through the three points:
   (a) \((0, 5, -2), (3, 4, 4), (2, -6, -1)\);
   (b) \((2, 3, -1), (-1, 5, 2), (-4, -2, 2)\).

\[
\begin{align*}
(a) & \quad 5B - 2C + D = 0 \quad (1) \\
& \quad 3A + 4B + 4C + D = 0 \quad (2) \\
& \quad 2A - 6B - C + D = 0 \quad (3)
\end{align*}
\]

\[
\begin{align*}
(2) - (1) & : \quad 3A - B + 6C = 0 \quad (4) \\
(3) - (1) & : \quad 2A - 11B + C = 0 \quad (5)
\end{align*}
\]

\[
\begin{align*}
2(4) & : \quad 6A - 2B + 12C = 0 \quad (6) \\
3(5) & : \quad 6A - 33B + 3C = 0 \quad (7)
\end{align*}
\]

\[
\begin{align*}
(6) - (7) & : \quad 3B + 9C = 0, \quad 3B = -9C \\
& \implies \frac{B}{C} = \frac{-9}{31}
\end{align*}
\]

\[
\begin{align*}
\therefore \quad B = -9, \quad C = 31 \\
\therefore \quad \text{From (5)} \quad 2A - 11(-9) + 31 = 0 \\
& \quad 2A + 99 + 31 = 0 \\
& \quad 2A = -130, \quad A = -65
\end{align*}
\]

\[
\begin{align*}
& \implies -65x - 9y + 31z + D = 0 \\
& \therefore \text{Using } \quad (0, 5, -2), \quad -65(0) - 9(5) + 31(-2) + D = 0 \\
& \quad -45 - 62 + D = 0
\end{align*}
\]
$A = 107$

\[
\therefore -65x - 9y + 31z + 107 = 0
\]

(5) \quad 2A + 3B - C + D = 0 \quad (1)

\[-A + 5B + 2C + D = 0 \quad (2)
\]

\[-4A - 2B + 2C + 0 = 0 \quad (3)
\]

(1)-(2) \quad 3A - 2B - 3C = 0 \quad (4)

(1)-(3) \quad 6A + 5B - 3C = 0 \quad (5)

(5) - 2(4) \quad 9B + 3C = 0, \quad \frac{B}{C} = -\frac{3}{9} = -\frac{1}{3}

\therefore B = -1, \quad C = 3

\therefore From (4), \quad 3A - 2(-1) - 3(3) = 0

\[3A + 2 - 9 = 0, \quad A = \frac{7}{3}
\]

\[\frac{7}{3}x - y + 3z + A = 0
\]

Using (2, 3, -1), \quad \frac{7}{3}(2) - (3) + 3(-1) + D = 0

\[\frac{14}{3} - 3 - 3 + D = 0
\]

\[D = 6 - \frac{14}{3} = \frac{4}{3}
\]

\[\therefore \frac{7}{3}x - y + 3z + \frac{4}{3} = 0, \quad \text{or} \quad 7x - 3y + 9z + 4 = 0
\]
3. Find the equation of the plane which passes through the point \((2, 3, -1)\) and is perpendicular to the line joining the points \((-1, 4, 5)\) and \((6, 3, 2)\).

Normal: \((6, 3, 2) - (-1, 4, 5) = (7, -1, -3)\)

\[
\therefore 7x - y - 3z + k = 0
\]

Using \((2, 3, -1)\), 
\[
7(2) - (3) - 3(-1) + k = 0 \\
14 - 3 + 3 + k = 0, \quad k = -14
\]

\[
\therefore 7x - y - 3z - 14 = 0
\]

5. Find the equation of the plane through the point \((-2, 1, 3)\), perpendicular to both of the planes \(x + 5y - z + 2 = 0\) and \(2x - 4y - 5z + 3 = 0\).

Let \((a, b, c)\) be a normal to the plane.

\[
\therefore (a, b, c) \cdot (1, 5, -1) = 0, \quad a + 5b - c = 0 \quad (1)
\]

\[
(a, b, c) \cdot (2, -4, -5) = 0, \quad 2a - 4b - 5c = 0 \quad (2)
\]

\[
\therefore 2(1) - (2): \quad 14b + 3c = 0, \quad b = -\frac{3}{14}c
\]

Let \(b = -3, \quad c = 14\)

\[
\therefore \text{In (1), } a + 5(-3) - (14) = 0 \\
a - 15 - 14 = 0, \quad a = 29
\]

\[
\therefore 29x - 3y + 14z + k = 0
\]
Using \((-2, 1, 3)\), \(29(-2) - 3(1) + 14(3) + k = 0\)
\[-58 -3 + 42 + k = 0\]
\[k = 19\]

\[\therefore 29x - 3y + 14z + 19 = 0\]

6. Find the equation of the plane through the point \((2, -1, 6)\), perpendicular to the line of intersection of the planes \(5x + 4y - z - 11 = 0\) and \(2x - y + 7z + 2 = 0\).

Let \((a, b, c)\) be direction numbers of the line of intersection. Since it is in each plane, it is perpendicular to the normal of each plane.

\[\therefore (a, b, c) \cdot (5, 4, -1) = 0, \quad 5a + 4b - c = 0 \quad (1)\]
\[\quad (a, b, c) \cdot (2, -1, 7) = 0, \quad 2a - b + 7c = 0 \quad (2)\]

\[\text{Subtracting } (2) \times 4 \text{ from } (1): \quad 13a + 27c = 0, \quad \text{or} \quad \frac{a}{c} = \frac{-27}{13}\]

\[\therefore a = -27, \quad c = 13\]

\[\therefore \text{From } (1), \quad 5(-27) + 4b - (13) = 0\]
\[-135 + 4b - 13 = 0\]
\[4b = 148, \quad b = 37\]

\[\therefore \text{Direction numbers of intersection line:}\]
\[-(27, 37, 13)\]
Plane perpendicular to intersection line:
\[-27x + 37y + 13z + k = 0\]

This plane contains \((2, -1, 6)\).
\[-27(2) + 37(-1) + 13(6) + k = 0\]
\[-54 - 37 + 78 + k = 0\]
\[k = 13\]
\[-27x + 37y + 13z + 13 = 0\]

7. Find the equation of the plane which intersects the coordinate axes in the points \((3, 0, 0)\), \((0, 7, 0)\), and \((0, 0, -4)\).

Let \(Ax + By + Cz + D = 0\) be the plane.

\[(3, 0, 0): 3A + D = 0\]
\[(0, 7, 0): 7B + D = 0\]
\[(0, 0, -4): -4C + D = 0\]
\[\therefore \frac{D}{C} = 4\]
\[\therefore D = 4, \quad C = 1\]

\[\therefore 7B + 4 = 0 \Rightarrow B = -\frac{4}{7}\]
\[\therefore 3A + 4 = 0 \Rightarrow A = -\frac{4}{3}\]
\[\therefore -\frac{4}{3}x - \frac{4}{7}y + z + 4 = 0, \quad \text{or}\]
\[-28x - 12y + 21z + 84 = 0\]
8. Find the equation of the plane which passes through the point \((-1, 7, 2)\) and has the same trace in the xy-plane as the plane \(5x + y - 2z + 4 = 0\).

Same face \iff same intercepts.

\[\therefore \text{x-intercept: } 5x + (0) - 2(0) + 4 = 0, \quad x = -\frac{4}{5}\]

\[\text{y-intercept: } 5(0) + y - 2(0) + 4 = 0, \quad y = -4\]

\[\therefore \text{Plane contains } (-\frac{4}{5}, 0, 0), (0, -4, 0), (-1, 7, 2)\]

Let \(Ax + By + Cz + D = 0\) be the plane.

\[\therefore -\frac{4}{5}A + D = 0 \quad (1)\]

\[-4B + D = 0 \quad (2)\]

\[-A + 7B + 2C + D = 0 \quad (3)\]

From (2), \(-4B + D = 0\), \(\therefore D = 4B, \quad \frac{D}{B} = 4\)

\[\therefore \text{Let } A = 4, \quad B = 1\]

From (1), \(-\frac{4}{5}A + 4 = 0\), \(A = 5\)

From (3) \(-5 + 7(1) + 2C + 4 = 0\), \(C = -3\)

\[\therefore 5x + y - 3z + 4 = 0\]

10. Find the equation of the plane through the origin, perpendicular to the trace of the plane \(7x + 4y - 11z - 5 = 0\) in the xy-plane.
Trace of \(7x + 4y - 11z - 5 = 0:\)

**x-intercept:** \(7x - 5 = 0, \ x = \frac{5}{7}\)

**y-intercept:** \(4y - 5 = 0, \ y = \frac{5}{4}\)

Direction numbers of line between \((\frac{5}{7}, 0, 0)\) and \((0, \frac{5}{4}, 0)\): \((\frac{5}{7}, -\frac{5}{4}, 0)\)

This line is a normal for the plane in question.

\[
\frac{5}{7}x - \frac{5}{4}y + \lambda = 0
\]

Since \((0, 0, 0)\) in plane, \(\lambda = 0\).

\[
\frac{5}{7}x - \frac{5}{4}y = 0, \ 20x - 35y = 0
\]

\[4x - 7y = 0\]
17.11 General Equations of a Line in Space

Find two piercing points for each of the following lines. Then draw the line and find direction numbers for it.

1. \[\begin{cases} 
  x - 2y + 4z - 14 = 0, \\
  x + 20y - 18z + 30 = 0.
\end{cases}\]

For \(x = 0\):
\[\begin{align*}
  -2y + 4z &= 14 \quad (1) \\
  20y - 18z &= -30 \quad (2)
\end{align*}\]

From (1):
\[-20y + 40z = 140\]
\[20y - 18z = -30\]

\[\therefore 22z = 110, \quad z = 5\]

\[\therefore -2y + 4(5) = 14, \quad y = 3\]

\[\therefore \text{One set: } (0, 3, 5)\]

For \(y = 0\):
\[\begin{align*}
  x + 4z &= 14 \\
  x - 18z &= -30
\end{align*}\]

\[\therefore 22z = 44, \quad z = 2, \quad \therefore x = 6\]

\[\therefore (6, 0, 2)\]
17.12 Projecting Planes of a Line

In Exercises 1–6, find the projecting planes of the lines whose general equations are given.

1. \[ \begin{cases} 2x - 3y + 4z + 5 = 0, \\ x + 5y - z - 2 = 0. \end{cases} \]

\[ \begin{align*}
\text{yz: } & \quad 2x - 3y + 4z + 5 = 0 \quad -13y + 6z + 9 = 0 \\
\text{zx: } & \quad 2x + 10y - 2z - 4 = 0 \\
\text{xy: } & \quad 5x - 15y + 20z + 25 = 0 \quad 18x + 17z + 19 = 0
\end{align*} \]

2. \[ \begin{cases} 3x + 15y - 3z - 6 = 0. \end{cases} \]

\[ \begin{align*}
\text{xy: } & \quad 2x - 3y + 4z + 5 = 0 \quad 6x + 17y - 3 = 0 \\
\text{yz: } & \quad 4x + 20y - 4z - 8 = 0
\end{align*} \]

7. Find projecting equations of the line joining the points \((-3, 1, 7)\) and \((4, 6, -5)\). Make a sketch. (Hint: The equation of a plane perpendicular to the \(xy\)-plane is of the form \(Ax + By + D = 0\). Determine values for \(A\), \(B\), and \(D\) so that the given points lie on this plane.)

\[ \begin{align*}
\text{xy: } & \quad Ax + By + D = 0 \\
\therefore & \quad A(-3) + B(1) + D = 0 \quad -3A + B + D = 0 \quad (1) \\
& \quad A(4) + B(6) + D = 0 \quad 4A + 6B + D = 0 \quad (2)
\end{align*} \]

\[ \begin{align*}
\text{C(1): } & \quad -18A + 6B + 6D = 0 \quad -22A + 5D = 0 \\
& \quad 4A + 6B + D = 0 \quad A = \frac{5}{22}
\end{align*} \]

\[ \begin{align*}
\therefore & \quad \text{if } A = \frac{5}{22}, D = 22 \\
& \quad \text{From (1), } -5(5) + B + 22 = 0, \quad B = -7
\end{align*} \]
\[ 5x - 7y + 22 = 0 \]

\[ y_2 : \beta y + \gamma z + \lambda = 0 \]

\[ \beta (1) + \gamma (7) + \lambda = 0 \quad \beta + 7\gamma + \lambda = 0 \quad (1) \]

\[ \beta (6) + \gamma (-5) + \lambda = 0 \quad 6\beta - 5\gamma + \lambda = 0 \quad (2) \]

\[ 6(1) \quad 6\beta + 42\gamma + 6\lambda = 0 \quad 47\gamma + 5\lambda = 0 \]

\[ 6\beta - 5\gamma + \lambda = 0 \quad \frac{\beta}{\lambda} = \frac{-5}{47} \]

\[ \therefore \lambda = \frac{-5}{47} \]

\[ \therefore \text{From (1), } \beta + 7(-5) + 47 = 0, \quad \beta = -12 \]

\[ -12y - 5z + 47 = 0, \text{ or } 12y + 5z - 47 = 0 \]

\[ x_2 : \Delta x + \gamma z + \lambda = 0 \]

\[ 4(1) \quad -12\Delta + 28\gamma + 4\lambda = 0 \quad 13\gamma + 7\lambda = 0 \]

\[ 3(2) \quad 12\Delta - 15\gamma + 3\lambda = 0 \quad \frac{\gamma}{\lambda} = \frac{-7}{13} \]

\[ \therefore \gamma = -7, \quad \lambda = 13 \]

\[ \therefore \text{From (1), } -3\Delta + 7(-7) + 13 = 0, \quad \Delta = -12 \]

\[ -12x - 7z + 13 = 0, \text{ or } 12x + 7z - 13 = 0 \]
\[ \begin{align*}
5x - 7y + 22 &= 0 \\
12y + 5z - 47 &= 0 \\
12x + 7z - 13 &= 0
\end{align*} \]

\text{Take any } z \text{ to make projecting equations.}
17.13 Symmetric Equations of a Line

In Exercises 1–6, find symmetric equations of the line which passes through the given point and has the given direction numbers.

1. \((-4, 2, 1), [3, 5, -8]\).

\[(x, y, z) - (-4, 2, 1) = k (3, 5, -8)\]

\[\therefore \quad \frac{x + 4}{3} = \frac{y - 2}{5} = \frac{z - 1}{-8}\]

5. \((1, 2, 4), [0, -2, 3]\).

\[(x, y, z) - (1, 2, 4) = k (0, -2, 3)\]

\[\therefore \quad \frac{x - 1}{0} = \frac{y - 2}{-2} = \frac{z - 4}{3}\]

7. Find symmetric equations of the line through the point \((-2, 1, 5)\) perpendicular to the plane \(3x + 7y - 6z + 19 = 0\).

Normal to the plane has direction numbers \([3, 7, -6]\).

\[\therefore \quad (x, y, z) - (-2, 1, 5) = k (3, 7, -6)\]

\[\therefore \quad \frac{x + 2}{3} = \frac{y - 1}{7} = \frac{z - 5}{-6}\]
11. Find symmetric equations for the line in Exercise 6, Section 17.12.

\[ \begin{align*}
6x + y + 2z + 3 &= 0, \\
2x - y + z - 4 &= 0.
\end{align*} \]  

Eliminate \( y \): (1) + (2): \[ 8x + 3z - 1 = 0 \]  

Eliminate \( z \): (1) - 2(2): \[ 6x + y + 2z + 3 = 0 \] \[ 4x - 2y + 2z - 8 = 0 \]

\[ \therefore 2x + 3y + 11 = 0 \]  

From (3), \( x = \frac{-3z + 1}{8} \)  

From (4), \( x = \frac{-3y - 11}{2} \)  

\[ \therefore \frac{x}{-3} = \frac{z - \frac{1}{3}}{8} = \frac{y + \frac{11}{3}}{2} \]

\[ \therefore \frac{x}{-3} = \frac{y}{2} + \frac{11}{6}, \quad \therefore \frac{y}{2} = \frac{x}{-3} - \frac{11}{6} = \frac{x + \frac{11}{2}}{-3} \]

\[ \therefore \frac{y}{2} = \frac{x + \frac{11}{2}}{-3} \]

\[ \text{Also}, \quad \frac{z - \frac{1}{3}}{8} = \frac{y + \frac{11}{3}}{2} = \frac{y}{2} + \frac{11}{6} \]

\[ \therefore \frac{y}{2} = \frac{z}{8} - \frac{11}{6} = \frac{z}{8} - \frac{1}{24} - \frac{44}{24} = \frac{z}{8} - \frac{45}{24} = \frac{z}{8} - \frac{15}{8} = \frac{z - 15}{8} \]
Another form is: \( \frac{y}{2} = \frac{x + \frac{11}{2}}{-3} = \frac{z - 15}{8} \)

(as given in back-of-book answers).

13. Find symmetric equations of the line which passes through the point \((2, 4, 5)\) and intersects the \(x\)-axis at right angles. Make the sketch first.

The line contains the point: \((2, 0, 0)\).

\[ \text{Direction numbers: } (2, 4, 5) - (2, 0, 0) = (0, 4, 5) \]

\[ (x, y, z) - (2, 0, 0) = k (0, 4, 5) \]

\[ \frac{y}{4} = \frac{z}{5}, \quad x = 2 \]

15. Show that the two lines

\[ \frac{x + 2}{3} = \frac{y - 4}{2} = \frac{z + 1}{6} \quad \text{and} \quad \frac{x + 1}{-1} = \frac{y + 1}{5} = z + 2, \]

lie in the same plane.

Two lines lie in the same plane if they share a common point (i.e., they intersect).

\[ \text{Look at } \frac{x + 2}{3} = \frac{y - 4}{2} \iff 2x + 4 = 3y - 12, \quad \text{or} \quad 2x - 3y = -16 \]

\[ \frac{x + 1}{-1} = \frac{y + 1}{5} \iff 5x + 5 = -y - 1, \quad \text{or} \quad 5x + y = -6 \]
\[ 2x - 3y = -16 \iff 2x - 3y = -16 \]
\[ 5x + y = -6 \quad 15x + 3y = -18 \]

\[ 17x = -34, \quad x = -2 \]

\[ 5(-2) + y = -6, \quad y = 4 \]

\[ \therefore \text{ Is there a common } z \text{ s.t. } (-2, 4, z) \text{ is on both lines?} \]

First set: \[ \frac{y - 4}{2} = \frac{z + 1}{6}, \quad \text{or} \quad \frac{4 - y}{2} = \frac{z + 1}{6}, \quad z = -1 \]

Second set: \[ \frac{y + 1}{5} = z + 2, \quad \frac{4 + 1}{5} = z + 2, \quad z = -1 \]

\[ \therefore (-2, 4, -1) \text{ is on both lines.} \]

Note also, the direction numbers: \[ [3, 2, 6] \text{ and } [-1, 5, 1], \text{ indicate lines are not parallel.} \]

\[ \therefore \text{ Lines intersect at exactly 1 point, and} \]
\[ \therefore \text{ line in a plane determined by the two lines.} \]
2. Find parametric equations of the line through the point \((-7, -6, 2),\) with direction angles \(\beta = \frac{3}{4}\pi, \gamma = \frac{1}{4}\pi.\) Then sketch the line. (Two solutions.)

\[
\cos \beta = -\frac{1}{2}, \quad \cos \gamma = \frac{\sqrt{2}}{2},
\]

\[
\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1, \quad \cos^2 \alpha = 1 - \frac{1}{4} - \frac{1}{4} = \frac{1}{4}
\]

\[
\therefore \cos \alpha = \pm \frac{1}{2}
\]

\[
\therefore (x, y, z) = (-7 - 6, 2) + k \left(\pm \frac{1}{2}, -\frac{1}{2}, \frac{\sqrt{2}}{2}\right)
\]

One solution: \(x = -7 + \frac{1}{2}k,\)
\[y = -6 - \frac{1}{2}k,\]
\[z = 2 + \frac{\sqrt{2}}{2}k\]

2nd solution: \(x = -7 - \frac{1}{2}k,\)
\[y = -6 + \frac{1}{2}k,\]
\[z = 2 + \frac{\sqrt{2}}{2}k\]

3. Find the direction cosines of the line with parametric equations
\(x = 2 + 3k, \quad y = -3 + 4k, \quad z = 5 - k.\)

Direction numbers: \([3, 4, -1]\)

\[
\text{Length} = \sqrt{3^2 + 4^2 + (-1)^2} = \sqrt{26}
\]

\[
\therefore \text{Use } -\frac{3}{\sqrt{26}}, \frac{4}{\sqrt{26}}, \frac{1}{\sqrt{26}} \text{ since the component is } < 0.
\]

\[
\left[ \frac{-3}{\sqrt{26}}, \frac{4}{\sqrt{26}}, \frac{1}{\sqrt{26}} \right]
\]
In Exercises 5–10, find parametric equations of the lines whose general equations are given. (See the Exercises, Section 17.11.)

7. \begin{align*}
2x + 5y + 4z - 4 &= 0, \\
28x - 4y + 19z + 92 &= 0.
\end{align*} \hfill (2)

Find 2 piercing points.

\( (1) \) \( z = 0 \):
\begin{align*}
2x + 5y - 4 &= 0 \quad (1) \\
28x - 4y + 92 &= 0 \quad (2)
\end{align*}

\( 14(1): \) \( 28x + 70y - 56 = 0 \quad (3) \\
\( 2): \) \( 28x - 4y + 92 = 0 \quad (4) \\
\( 3) - (4): \) \quad 74y - 148 = 0, \quad y = 2

From \( 1) \) \( 2x + 5(2) - 4 = 0, \quad x = -3 \)
\[ \therefore (-3, 2, 0) \]

\( (2) \) \( y = 0 \):
\begin{align*}
2x + 4z - 4 &= 0 \quad (1) \\
28x + 19z + 92 &= 0 \quad (2)
\end{align*}

\( 14(1): \) \( 28x + 56z - 56 = 0 \quad (3) \\
\( 2): \) \( 28x + 19z + 92 = 0 \quad (4) \\
\( 3) - (4): \) \quad 37z - 148 = 0, \quad z = 4

From \( 1) \) \( 2x + 4(4) - 4 = 0, \quad x = -6 \)
\[ \therefore (-6, 0, 4) \]
\[ \text{Direction numbers: } (-6, 0, 4) - (-3, 2, 0) = [-3, -2, 4] \]

\[ \therefore (x, y, z) = (-3, 2, 0) + k(-3, -2, 4), \]

or
\[ \begin{align*}
x &= -3 - 3k \\
y &= 2 - 2k \\
z &= 4k 
\end{align*} \]

Note: For \( k = -1 \), the point \((0, 4, -4)\) is on the line.

\[ \therefore \text{Another form is } (x, y, z) = (0, 4, -4) + k(-3, -2, 4) \]

or
\[ \begin{align*}
x &= -3k' \\
y &= 4 - 2k' \\
z &= -4 + 4k' 
\end{align*} \]

\( \text{The form given in back of book.} \)

13. Find parametric equations of the line which goes through the origin and is perpendicular to the plane \(7x - 5y + z + 11 = 0\). Then sketch the line.

Normal to plane has direction numbers: \([7, -5, 1] \)

\[ \therefore (x, y, z) = (0, 0, 0) + k(7, -5, 1), \text{ or} \]

\[ \begin{align*}
x &= 7k \\
y &= -5k \\
z &= k 
\end{align*} \]

15. Find parametric equations of the line which goes through the point \((-4, 2, 6)\) and is parallel to the z-axis. Draw the line.
Line parallel to $z$-axis has direction angles of $(90^\circ, 90^\circ, 0)$, \( t \) = direction cosines \( (0, 0, 1) \).

\[
\therefore (x, y, z) = (-4, 2, 0) + k(0, 0, 1)
\]

\[
\therefore x = -4 \\
y = 2 \\
z = 6 + k
\]

17. Find parametric equations of the line through the point $(4, 0, -3)$, parallel to the line whose general equations are

\[
\begin{align*}
2x + y - 5z + 1 &= 0, \\
3x - 3y + z - 10 &= 0.
\end{align*}
\]

Then sketch the line.

The line parallel to the general equations will have the same direction numbers.

\[
\therefore \text{Find piercing points to get direction numbers.}
\]

\[
X = 0 : \quad y - 5z + 1 = 0 \quad (1) \\
-3y + 2 - 10 = 0 \quad (2)
\]

\[
(1) : \quad y - 5z + 1 = 0 \quad (3) \\
5(2) : -15y + 5z - 50 = 0 \quad (4)
\]

\[
(3) + (4) : \quad -14y - 49 = 0, \quad y = -\frac{7}{2}
\]

From (1): \( -\frac{7}{2} - 5z + 1 = 0 \), \( z = -\frac{1}{2} \)

\[
\therefore (0, -\frac{7}{2}, -\frac{1}{2}) \text{ one point.}
\]
$y = 0; \quad 2x - 5z + 1 = 0 \quad (1)$

$\quad 3x + 2 = -10 = 0 \quad (2)$

$3(1): \quad 6x - 15z + 3 = 0 \quad (3)$

$2(2): \quad 6x + 2z - 20 = 0 \quad (4)$

$(3) - (4): \quad -17z + 23 = 0, \quad z = \frac{23}{17}$

From $(1): \quad 2x - 5\left(\frac{23}{17}\right) + 1 = 0$

$2x = \frac{115}{17} - \frac{17}{17} = \frac{98}{17}, \quad x = \frac{49}{17}$

$\therefore \left(\frac{49}{17}, 0, \frac{23}{17}\right)$ another point

$\therefore$ Direction numbers: \left(\frac{49}{17}, 0, \frac{23}{17}\right) - \left(0, -\frac{2}{2}, -\frac{1}{2}\right) = \left(\frac{49}{17}, \frac{7}{2}, \frac{23}{34}\right) \quad \text{To simplify},$

$\frac{34}{7} \left(\frac{49}{17}, \frac{7}{2}, \frac{23}{34}\right) = (14, 17, 9)$

$\therefore (x, y, z) = (4, 0, -3) + k(14, 17, 9), \quad \text{or}$

$x = 4 + 14k \quad \therefore \text{Another point is, } k = 1,$

$y = 17k$

$z = -3 + 9k \quad (18, 17, 6)$
17.15 The Sphere

In Exercises 3–8, find the center and radius of the sphere whose equation is given.

3. \( x^2 + y^2 + z^2 + 2x - 6y + 22z + 122 = 0 \).

\[
(x^2 + 2x + 1) + (y^2 - 6y + 9) + (z^2 + 22z + 121) + 122 - 1 - 9 - 121 = 0
\]

\[
(x + 1)^2 + (y - 3)^2 + (z + 11)^2 = 9 = 3^2
\]

\[ \text{Center is: (-1, 3, -11), radius = 3} \]

9. Find the equation of the sphere whose center is on the z-axis and which passes through the points (3, 4, 3) and (-2, -1, 1).

Let \((0, 0, c)\) be coordinates of the center.

\[
(3 - 0)^2 + (4 - 0)^2 + (3 - c)^2 = (-2 - 0)^2 + (-1 - 0)^2 + (1 - c)^2
\]

\[
9 + 16 + c^2 - 6c + 9 = 4 + 1 + c^2 - 2c + 1
\]

\[
34 - 6c = 6 - 2c
\]

\[
28 = 4c
\]

\[
c = 7
\]

\[ \text{Center = (0, 0, 7)} \]

Radius: \( \sqrt{(3 - 0)^2 + (4 - 0)^2 + (3 - 7)^2} = \sqrt{41} \)

\[ x^2 + y^2 + (z - 7)^2 = 41, \text{ or} \]

\[ x^2 + y^2 + z^2 - 14z + 8 = 0 \]
11. Find the equation of the sphere which is tangent to the plane
\[ x - 8y + 4z + 7 = 0 \]
and which has the same center as \( x^2 + y^2 + z^2 - 12x - 4y - 6z + 33 = 0 \).

(a) Find center: 
\[
(x^2 + y^2 + z^2 - 12x - 4y - 6z + 33 = 0),
\]
\[
(x - 6)^2 + (y - 2)^2 + (z - 3)^2 + r = 0,
\]
where \( r \) = number from completing squares.

\[ \therefore \text{center} = (6, 2, 3). \]

(b) Find line perpendicular to plane containing center.

Normal to plane \( x - 8y + 4z + 7 = 0 \) has direction numbers \([1, -8, 4]\).

\[ \therefore \text{line equation}: \]
\[
(x, y, z) = (6, 2, 3) + k(1, -8, 4), \text{ or}
\]
\[
x = 6 + k, \\
y = 2 - 8k, \\
z = 3 + 4k.
\]

(c) Find point on line containing sphere center which intersects plane, using plane equation.

\[
(6 + k) - 8(2 - 8k) + 4(3 + 4k) + 7 = 0
\]
\[
6 + k - 16 + 64k + 12 + 16k + 7 = 0
\]
\[
81k + 9 = 0, \text{ or } k = -\frac{1}{9}
\]

\[ \therefore \text{intersection point} : \left(6 - \frac{1}{9}, 2 + \frac{8}{9}, 3 - \frac{4}{9}\right). \]
(a) Find \( r^2 \) (center - intersection point)^2

\[
\begin{align*}
\therefore \ [6 - (6 - \frac{1}{9})]^2 + [2 - (2 + \frac{6}{9})]^2 + [3 - (3 - \frac{4}{9})]^2 &= (\frac{1}{9})^2 + (\frac{6}{9})^2 + (\frac{4}{9})^2 = \frac{1}{81} + \frac{36}{81} + \frac{16}{81} = \frac{81}{81} = 1 \\
\therefore \text{ radius } &= 1 \\
\therefore (x-6)^2 + (y-2)^2 + (z-3)^2 &= 1
\end{align*}
\]

\[
\begin{align*}
\therefore x^2 - 12x + 36 + y^2 - 4y + 4 + z^2 - 6z + 9 &= 1 \\
\therefore x^2 + y^2 + z^2 - 12x - 2y - 6z + 48 &= 0
\end{align*}
\]

13. Find the equation of the sphere which has its center on the x-axis and passes through the points \((0, 5, 0)\) and \((-2, 1, 0)\).

**Center**: \((x, 0, 0)\)

\[
\begin{align*}
\therefore \ [ (x,0,0) - (0,5,0) ]^2 &= [ (x,0,0) - (-2,1,0) ]^2 = \text{ radius}^2 \\
\therefore x^2 + 25 + 0 = (x+2)^2 + 1 + 0 \\
\therefore x^2 + 4x + 4 + 1 &= x^2 + 25 + 0 \\
\therefore 25 &= 4x + 25 \\
\therefore x &= 5 \\
\therefore \text{ radius}^2 = [ (5,0,0) - (0,5,0) ]^2 &= 50 \\
\text{ Center } &= (5,0,0) \\
\therefore (x-5)^2 + y^2 + z^2 &= 50
\end{align*}
\]
15. Find the equation of the sphere which has its center in the $xz$-plane and which is tangent to the plane $2x - y + z - 4 = 0$ at the point $(1, 5, 7)$.

Center: $(x, 0, z)$

\[
(x-1)^2 + (0-5)^2 + (z-7)^2 = r^2, \quad r = \text{radius.}
\]

Normal to plane has direction numbers: $(2, -1, 1)$. Line from $(1, 5, 7)$ to center has form

\[
(x, 0, z) = (1, 5, 7) + k[2, -1, 1]
\]

\[
\begin{align*}
\begin{cases}
  x = 1 + 2k \\
  0 = 5 - k & \Rightarrow k = 5 \\
  z = 7 + k
\end{cases}
\end{align*}
\]

\[
\begin{align*}
  x = 1 + 2(5) &= 11 \\
  0 = 5 - k &= 5 \\
  z = 7 + 5 &= 12
\end{align*}
\]

Center of sphere: $(11, 0, 12) = (x, 0, z)$

\[
r^2 = (11-1)^2 + (0-5)^2 + (12-7)^2
\]

\[
= 100 + 25 + 25
\]

\[
= 150
\]

\[
(x-11)^2 + (y-0)^2 + (z-12)^2 = 150
\]

\[
x^2 + y^2 + z^2 - 22x - 24z + 121 + 144 = 150
\]

\[
\therefore \quad x^2 + y^2 + z^2 - 22x - 24z + 115 = 0
\]
1. In three-dimensional geometry, write equations of a parabola in standard position in the \( yz \)-plane. What are the coordinates of its focus?

\[
x = 0, \quad z^2 = 4\rho y, \quad \text{focus} = (0, \rho, 0)
\]

3. Write three-dimensional equations of an ellipse in standard position in the \( zx \)-plane. Find the coordinates of its foci.

\[
y = 0, \quad \frac{x^2}{a^2} + \frac{z^2}{b^2} = 1
\]

where \( b^2 = a^2(1 - \varepsilon^2) \)

\[
\varepsilon = \text{concentricity}.
\]

\[
\therefore \quad b^2 = a^2 - a^2 \varepsilon^2,
\]

\[
a^2 \varepsilon^2 = a^2 - b^2
\]

\[
a \varepsilon = \pm \sqrt{a^2 - b^2}
\]

\[
a \varepsilon = \text{coords. of focus}
\]

5. Write three-dimensional equations of a hyperbola in standard position in the \( xy \)-plane. Find equations for each of its asymptotes.

\[
z = 0, \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1
\]

\[
b^2 = a^2(\varepsilon^2 - 1) = a^2\varepsilon^2 - a^2
\]

\[
\therefore \quad \pm \sqrt{a^2 + b^2} = a \varepsilon = \text{focus}
\]

Asymptotes: \( x/a = \pm \frac{b}{a} y, \) or \( y = \pm \frac{b}{a} x, z = 0 \)
6. Find the equations of the hyperbola whose vertices are \((0, \pm 3, 0)\) and which passes through the point \((0, 3\sqrt{2}, 2)\).

**Hyperbola is in \(yz\) plane, given point \((0, 3\sqrt{2}, 2)\).**

Since vertices on \(y\)-axis, form of the hyperbola is

\[
\frac{y^2}{a^2} - \frac{z^2}{b^2} = 1
\]

\[
\therefore \frac{(3\sqrt{2})^2}{a^2} - \frac{(0)^2}{b^2} = 1 \quad \therefore a^2 = 9
\]

\[
\therefore \frac{y^2}{9} - \frac{z^2}{b^2} = 1
\]

Using \((0, 3\sqrt{2}, 2)\):

\[
\frac{(3\sqrt{2})^2}{9} - \frac{(2)^2}{b^2} = 1
\]

\[
\therefore \frac{18}{9} - \frac{4}{b^2} = 1 \quad \frac{4}{b^2} = 1 \quad b^2 = 4
\]

\[
\therefore \frac{y^2}{9} - \frac{z^2}{4} = 1 \quad x = 0
\]

7. Name and describe the graph of the set

\[
\{ P; (x, y, z) \mid x^2 + y^2 + z^2 = 16 \text{ and } z = -3 \}.
\]

**Plane of \(z = -3\) intersecting sphere = circle.**

\[
x^2 + y^2 + (-3)^2 = 16, \quad x^2 + y^2 = 7
\]

\[
\therefore \text{Circle with radius } \sqrt{7}, \text{ center } (0,0,-3).
\]
9. What is the graph of the set
\( \{ P(x, y, z) \mid x^2 + y^2 + z^2 = 4 \text{ and } x = -3 \} \)?

With \( x = -3 \),

\[ x^2 + y^2 + z^2 = (-3)^2 + y^2 + z^2 = 9 + y^2 + z^2 \]

But \( y^2 \geq 0 \), \( z^2 \geq 0 \), \( \therefore 9 + y^2 + z^2 \geq 9 \geq 4 \)

\( \therefore \) There is no \( y, z \) that satisfies \( 9 + y^2 + z^2 = 4 \)

\( \therefore \) Empty set.
2. Name the graph of each of the following equations:

(a) \(25x^2 + 16y^2 = 400\);  
(b) \(y^2 + z^2 = 9\);  
(c) \(2x + 5z - 12 = 0\);  
(d) \(z^2 = 6y\);  
(e) \(x^2 + y^2 - 8x + 2y + 13 = 0\);  
(f) \(z^2 = y^3\).

\[
(a) \quad \frac{25x^2}{400} + \frac{16y^2}{400} = 1 \quad \therefore \frac{x^2}{16} + \frac{y^2}{25} = 1
\]

\text{elliptic cylinder}

\[
(b) \quad y^2 + z^2 = 9 \quad \text{circular cylinder.}
\]

\[
(c) \quad 2x + 5z - 12 = 0 \quad \text{plane (parallel to y-axis)}
\]

\[
(d) \quad z^2 = 6y \quad \text{parabolic cylinder (parallel to x-axis)}
\]

\[
(e) \quad x^2 - 8x + 16 + y^2 + 2y + 1 + 13 - 16 - 1 = 0 \quad (x - 4)^2 + (y + 1)^2 = 4 \quad \text{sphere of radius 2}
\]

\[
(f) \quad z^2 = y^3 \quad \therefore \text{Negative values of } y \text{ not allowed}
\]

\[\therefore y = \pm z^{2/3} \quad \text{An open cylinder, parallel to x-axis, something like a parabola.}\]

3. Write the equation of a parabolic cylinder whose generators are parallel to the y-axis.
5. Write the equation of the circular cylinder whose generators are parallel to the \( x \)-axis, if its directrix has its center at \((0, -2, 5)\) and its radius is equal to 3.

\[ \text{No } x \text{-term, } \therefore [y - (-2)]^2 + (z - 5)^2 = 3^2, \] or
\[ (y + 2)^2 + (z - 5)^2 = 9, \quad y^2 + 2y + 4 + z^2 - 10z + 25 = 9 \]
\[ \therefore y^2 + z^2 + 2y - 10z + 20 = 0 \]

7. Find a projecting cylinder of the curve whose equations are

\[ x^2 + 2y^2 - z - 50 = 0, \quad y = 5. \]

\[ x^2 + 2(5)^2 - z - 50 = 0, \quad x^2 + 50 - z - 50 = 0 \]
\[ \therefore \quad x^2 - z = 0 \]

9. As in the Example, above, find simpler equations for the ellipse

\[ \frac{x^2}{27} + \frac{y^2}{9} + \frac{z^2}{3} = 1, \quad x = 3. \]

\[ \frac{3^2}{27} + \frac{y^2}{9} + \frac{z^2}{3} = 1 \quad \therefore \frac{1}{3} + \frac{y^2}{9} + \frac{z^2}{3} = 1 \]
\[ \therefore 1 + \frac{y^2}{3} + \frac{z^2}{3} = 3, \quad \frac{y^2}{3} + \frac{z^2}{3} = 2 \]
\[ \therefore \frac{y^2}{6} + \frac{z^2}{2} = 1, \quad x = 3 \]
11. Find the equations of two projecting cylinders of the space curve

\[ y^2 + z^2 = x, \quad 2x^2 + z^2 = 3y, \]

and use them to write new equations of the curve. No sketch is required.

(1) Substitute \( y^2 + z^2 = x \) into \( 2x^2 + z^2 = 3y \)

\[ \Rightarrow 2 (y^2 + z^2)^2 + z^2 = 3y \]

\[ \Rightarrow 2 (y^4 + 2y^2z^2 + z^4) + z^2 = 3y \]

\[ \Rightarrow 2y^4 + 2z^4 + 4y^2z^2 + z^2 - 3y = 0 \]

(2) Substitute \( 2x^2 + z^2 = 3y \) into \( y^2 + z^2 = x \)

\[ \Rightarrow y^2 + (3y - 2x^2) = x \]

\[ y^2 - 2x^2 + 3y - x = 0 \]
17.18 Surfaces of Revolution

In Exercises 1–10, find the equation of the surface of revolution generated by revolving the given plane curve about the indicated axis. Make a sketch.

1. $z^2 = 6x, y = 0$, about the $x$-axis.

$$\left(\pm \sqrt{y^2 + z^2}\right)^2 = 6x,$$

$$y^2 + z^2 = 6x$$

2. The same curve as in Exercise 1, about the $z$-axis.

$$z^2 = C\left(\pm \sqrt{x^2 + y^2}\right),$$

$$z^4 = 3C\left(x^2 + y^2\right)$$

3. $9x^2 + 16y^2 = 144, z = 0$, about the $x$-axis.

$$9x^2 + 16\left(\pm \sqrt{y^2 + z^2}\right)^2 = 144,$$

$$9x^2 + 16(y^2 + z^2) = 144$$
5. \(25x^2 - 4z^2 = 100, y = 0,\) about the \(z\)-axis.

\[
25\left( \pm \sqrt{x^2 + y^2} \right)^2 - 4z^2 = 100
\]

\[\therefore 25 \left( x^2 + y^2 \right) - 4z^2 = 100\]

7. \((x - 7)^2 + z^2 = 4, y = 0,\) about the \(z\)-axis.

\[
x^2 - 14x + 49 + z^2 = 4
\]

\[\text{or } x^2 + z^2 + 45 = 14x\]

\[\therefore \left( \pm \sqrt{x^2 + y^2} \right)^2 + z^2 + 45 = 14 \pm \sqrt{x^2 + y^2}\]

\[\therefore x^2 + y^2 + z^2 + 45 = 14 \left( \pm \sqrt{x^2 + y^2} \right)\]

\[\therefore (x^2 + y^2 + z^2 + 45)^2 = 196 \left( x^2 + y^2 \right)\]

\[x^4 + x^2y^2 + x^2z^2 + 45x^2 + y^4 + y^2z^2 + 45y^2 + x^2z^2 + y^2z^2 + 2y^2z^2 + 45x^2 + 45y^2 + 45z^2 + 2025 = 196x^2 + 196y^2\]

\[\therefore x^4 + y^4 + z^4 + 2x^2y^2 + 2x^2z^2 + 2y^2z^2 - 106x^2 - 106y^2 + 90z^2 + 2025 = 0\]

9. \(z = \sin y, x = 0,\) about the \(y\)-axis.

\[\pm \sqrt{x^2 + z^2} = \sin y\]

\[\therefore x^2 + z^2 = \sin^2 y\]
11. Some of the following equations represent surfaces of revolution. Find the axis and the generating curve for each such surface of revolution.

(a) \( x^2 + y^2 + z^2 = 25 \); 
(b) \( 2x^2 + 3y^2 - 4z^2 = 24 \);
(c) \( x^2 + z^2 - 2y = 0 \).

(a) One possibility: \( y^2 + z^2 = 25 \), about \( z \)-axis
\( x = 0 \)

Other: \( x^2 + z^2 = 25 \), about \( y \)-axis
\( y = 0 \)

(b) \( 2x^2 + 3y^2 - 4z^2 = 24 \)

The coefficients: 2, 3, 4 do not permit a combination of \((\pm \sqrt{1})\) to come out evenly. \( \therefore \) The surface is not a revolution in a circular motion.

i.e., The cross-section of the surface with \( x = c \), or \( y = c \), or \( z = c \), \( c \) a constant, is not a circle.

(c) One possibility: \( z^2 - 2y = 0 \), about \( y \)-axis
\( x = 0 \)

Another: \( x^2 - 2y = 0 \), about \( y \)-axis
\( z = 0 \)

Here, when \( y = c \), a constant, \( x^2 + z^2 - 2y = 0 \) becomes \( x^2 + z^2 = 2c \), a circle.
17.19 Symmetry, Traces, and Plane Sections of a Surface

1. \(x^2 + y^2 - z^2 - 1 = 0\).

   For a given \(z\), trace through \(z = k\) is a circle, expanding as \(z\) increases: 
   \[x^2 + y^2 = 1 + k^2\]

   For \(y = 0\), hyperbola \(x^2 - z^2 = 1\)

   For \(x = 0\), hyperbola \(y^2 - z^2 = 1\) "one-sheeted hyperboloid"

3. \(3x^2 + 4y^2 + 9z^2 - 24 = 0\).

   Symmetric with respect to each coordinate plane
   Trace on each coord. plane is an ellipse:
   e.g. \(x = 0\), \(4y^2 + 9z^2 = 24\)
   or \(\frac{y^2}{6} + \frac{z^2}{24} = 1\)

   \(\therefore\) Ellipsoid.

7. \(x^2 + y^2 - z^2 = 0\).

   Same as \#1, but here origin \((0, 0, 0)\) is included. \(\therefore\) an
   infinite cone.
8. \(400x^2 - 144y^2 + 225z^2 - 3600 = 0\).

Same as #1, but this time rotating around \(y\)-axis (rotation occurs around axis with negative coefficient, if just one term has a negative coefficient).

11. \(x^2 + y^2 - 4x = 0\).

This is \((x-2)^2 + y^2 = 4\)

- Circular cylinder, center at \((2, 0, 0)\), parallel to \(z\)-axis.

13. \(x^2 + 4y^2 + 4z^2 - 4 = 0\).

\(\frac{x^2}{4} + \frac{y^2}{1} + \frac{z^2}{1} = 1\)

Ellipsoid

Endpoints: 
- \((\pm 2, 0, 0)\)
- \((0, \pm 1, 0)\)
- \((0, 0, \pm 1)\)
15. \( x^2 + z^2 - y = 0. \)

For \( x=0 \) or \( z=0 \),
The traces are parabolas.

\[ \therefore \text{a paraboloid around y-axis.} \]
17.21 Procedure for Sketching a Surface

In Exercises 3–14, name and sketch the quadric surfaces whose equations are given.

3. \[4x^2 + 36y^2 + 9z^2 - 1 = 0.\]

Same as \[\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\]

\[\therefore \text{ Ellipsoid}\]

Very tiny with \((\frac{1}{2}, 0, 0), (0, \frac{1}{6}, 0), (0, 0, \frac{1}{3})\) as endpoints.

5. \[4x^2 - y^2 + 4z^2 - 4 = 0.\]

For \(x = 0\), \[\frac{z^2}{1} - \frac{y^2}{4} = 1\]

For \(y = 0\), \[\frac{x^2}{4} - \frac{z^2}{1} = 1\]

\[\therefore \text{ hyperboloid of 1 sheet}\]

\((z \neq 0, x \neq 0, \therefore \text{ rotating around y-axis})\)

For \(y = k\), traces are ellipses.
7. \[ 144x^2 + 16y^2 - 9z^2 - 144 = 0. \]

\[ \frac{x^2}{9} + \frac{y^2}{16} - \frac{z^2}{16} = 1 \]

\[ \therefore \text{Hyperboloid of 1-sheet, around z-axis.} \]

For \( z = k \), trace is ellipse.

9. \[ 36x^2 + 4y^2 + 9z = 0. \]

\[ \frac{x^2}{4} + \frac{y^2}{9} = -\frac{z}{9} \]

For \( x = 0 \), or \( y = 0 \), a parabola opening down toward negative \( z \).

For \( z = k \), a negative constant, traces are ellipses.

11. \[ 9x^2 + 36y^2 - 4z^2 + 36 = 0. \]

\[ -\frac{x^2}{4} - \frac{y^2}{9} + \frac{z^2}{2} = 1 \]

\[ \therefore \text{Hyperboloid of 2 sheets, rotating around z-axis.} \]

For \( x = 0 \), \( \frac{z^2}{2} - \frac{y^2}{9} = 1 \), for \( y = 0 \), \( \frac{z^2}{2} - \frac{x^2}{4} = 1 \).
15. \( y - \ln z = 0 \).

A \textit{1-sheeted surface}, parallel to \textit{x-axis}.

17. \( y = e^{-x^2} \).

A \textit{1-sheeted surface}, parallel to \textit{z-axis}.

19. \( -\sqrt{x^2 + z^2} = \cos y \).

\begin{align*}
\text{Same as } x^2 + z^2 &= \cos^2 y \\
\text{For } z=0, x^2 &= \cos^2 y \\
\text{Looks like:}
\end{align*}

\begin{align*}
\text{Same for } x=0, z^2 &= \cos^2 y \\
\text{For a given } y = k, x^2 + z^2 &= \cos^2 k, \\
\text{a circle.}
\end{align*}

\begin{itemize}
\item A surface of revolution about the \textit{y-axis}, where the radius oscillates between 0 and 1.
\end{itemize}