Chapter 11 - Rolling Motion and Angular Momentum

Note Title 5/23/2005

19. For the 3 kg mass: $L = (0.5\,m)(3.00\,kg)(5.00\,m/sec)$
$= 7.5\ kg\cdot m^2/sec$

For the 4 kg mass: $L = (0.5\,m)(4.00\,kg)(5.00\,m/sec)$
$= 10.0\ kg\cdot m^2/sec$

The rod is "light", $\therefore m = 0$, $\therefore L = 0$.

$\therefore 10 + 7.5 = 17.5\ kg\cdot m^2/sec$

21. $\vec{r}(t) = (6.00,\ 5.00\,t)$

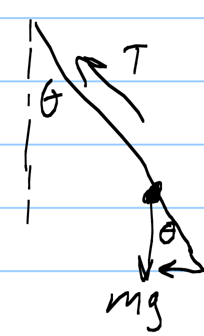
$\therefore \vec{v}(t) = (0,\ 5.00)$

$$\vec{L} = \vec{r} \times m\vec{v} = m \cdot \begin{vmatrix} i & j & k \\ 6.0 & 5.00t & 0 \\ 0 & 5.00 & 0 \end{vmatrix}$$

$$= (0\hat{i} - 0\hat{j} + 30.0\hat{k})(2.00\,kg)$$

$$= 60.0\,\hat{k}\ kg\cdot m^2/sec$$

22. Radius of circle $= l \sin\theta$



Centripetal force $= mg\tan\theta$

$$\therefore \text{acceleration} = \frac{v^2}{R} = \frac{mg\tan\theta}{m} = g\tan\theta$$

$$\therefore v^2 = Rg\tan\theta = g(l\sin\theta)(\sin\theta/\cos\theta)$$

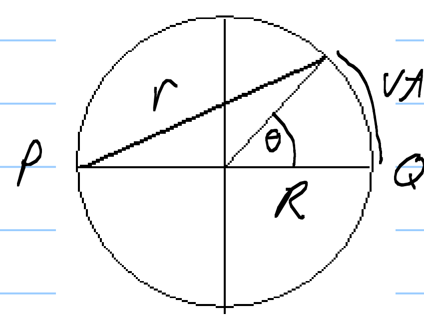
$$= gl\frac{\sin^2\theta}{\cos\theta}$$

$$L = mvR = m(l\sin\theta)\left(gl\frac{\sin^2\theta}{\cos\theta}\right)^{\frac{1}{2}}$$

$$= (m^2l^2\sin^2\theta)^{\frac{1}{2}}\left(gl\frac{\sin^2\theta}{\cos\theta}\right)^{\frac{1}{2}}$$

$$= \left(m^2l^3g\frac{\sin^2\theta}{\cos\theta}\right)^{\frac{1}{2}}$$

23. For time t, particle has traveled a length vt, and subtended an angle of $\frac{vt}{R} = \theta$, in radians



Let P be origin

$$\therefore \vec{r}(t) = (R + R\cos\theta, \sin\theta)$$

$$= \left(R + R\cos\left(\frac{vt}{R}\right), R\sin\left(\frac{vt}{R}\right)\right)$$

$$\therefore \vec{r}'(t) = v\left(-\sin\left(\frac{vt}{R}\right), \cos\left(\frac{vt}{R}\right)\right)$$

$\therefore \vec{L}(t) = \vec{r}(t) \times \vec{p}(t)$, $\vec{p}(t) = m\vec{r}'(t)$

$$\therefore \vec{L}(t) = m \begin{vmatrix} i & j & k \\ R + R\cos\left(\frac{vt}{R}\right) & R\sin\left(\frac{vt}{R}\right) & 0 \\ -v\sin\left(\frac{vt}{R}\right) & v\cos\left(\frac{vt}{R}\right) & 0 \end{vmatrix}$$

$$= m\left[Rv\cos\theta + Rv\cos^2\theta + Rv\sin^2\theta\right]\hat{k}$$

$$= mvR\left[\cos\theta + 1\right]\hat{k}$$

$$= mvR\left[\cos\left(\frac{vt}{R}\right) + 1\right]\hat{k}$$

25. (a) Since $\vec{r} = 0$, $\vec{r} \times \vec{p} = 0$

(b) At height, $h = \dfrac{v_i^2 \sin^2\theta_i}{2g}$, and $x = \frac{1}{2}$ max Range $= \frac{1}{2}\dfrac{v_i^2 \sin 2\theta_i}{g}$

$$\therefore \vec{r} = \left(\frac{v_i^2 \sin 2\theta_i}{2g}, \frac{v_i^2 \sin^2\theta_i}{2g}\right)$$

$\vec{v} = \vec{v}_x = (v_i\cos\theta, 0)$, so $\vec{p} = (mv_i\cos\theta, 0)$

$$\therefore \vec{r} \times \vec{p} = \begin{vmatrix} i & j & K \\ \frac{v_i^2 \sin 2\theta}{2g} & \frac{v_i^2 \sin^2\theta}{2g} & 0 \\ m v_i \cos\theta & 0 & 0 \end{vmatrix}$$

$$= -\frac{m v_i^3 \sin^2\theta \cos\theta}{2g} \hat{K}$$

(c) $\vec{r} = \left(\frac{v_i^2 \sin 2\theta}{g}, 0 \right)$ $\vec{p} = m \left(v_i \cos\theta, -v_i \sin\theta \right)$

$$\therefore \vec{r} \times \vec{p} = \begin{vmatrix} i & j & K \\ \frac{v_i^2 \sin 2\theta}{g} & 0 & 0 \\ v_i \cos\theta & -v_i \sin\theta & 0 \end{vmatrix}$$

$$= -\frac{v_i^3 \sin 2\theta \cdot \sin\theta}{g} \hat{K}$$

(d) gravity provides the torque

41. (a) Angular momentum conserved.

$L_i = m v_i d \qquad L_f = I\omega. \quad I_{cylinder} = \frac{1}{2} M R^2$

$I_{clay\ when\ stuck} = m R^2$

$\therefore L_f = \left(m R^2 + \frac{1}{2} M R^2\right)\omega = \left(\frac{2m + M}{2}\right) R^2 \omega$

$L_i = L_f = m v_i d = \left(\frac{2m + M}{2}\right) R^2 \omega$

$$\therefore \omega = \frac{2 m v_i d}{(2m + M) R^2}$$

(b) No, some mechanical energy goes into deforming clay and sticking to cylinder.

45. Initial height of sphere: $R - (R-r)\cos\theta$

At bottom, height is: r

$\therefore \Delta h = R - r - (R-r)\cos\theta = (R-r)(1-\cos\theta)$

$\therefore \Delta P.E = mg(R-r)(1-\cos\theta)$

At bottom, all rotation about bottom point

$I_{bottom\ point} = I_{cm} + m r^2$

$$= \frac{2}{5}mr^2 + mr^2 = \frac{7}{5}mr^2$$

$$\therefore \frac{1}{2}I\omega^2 = \frac{1}{2}\left(\frac{7}{5}mr^2\right)\omega^2 = \frac{7}{10}mr^2\omega^2$$

$$= mg(R-r)(1-\cos\theta)$$

$$\therefore \omega^2 = \frac{10g(R-r)(1-\cos\theta)}{7r^2}$$

$$\therefore \omega = \sqrt{\frac{10g(R-r)(1-\cos\theta)}{7r^2}}$$

51.(a) To complete loop, sphere must have a translational velocity such that the centrypetal acceleration is at least g.

$$\therefore \frac{V_{cm}^2}{R} = g$$

Total energy at top of loop is P.E + K.E
PE = $2mgR$
KE = rotational + translational
translational = $\frac{1}{2}mV_{cm}^2 = \frac{1}{2}mgR$

$$\text{rotational} = \frac{1}{2} I_{cm}\,\omega^2 = \frac{1}{2} I_{cm}\left(\frac{v_{cm}}{r}\right)^2$$

$$= \frac{1}{2}\left(\frac{2}{5} m r^2\right)\left(\frac{v_{cm}^2}{r^2}\right)$$

$$= \frac{mgR}{5}$$

$\therefore$ Total KE $= \frac{1}{2} mgR + \frac{1}{5} mgR = \frac{7}{10} mgR$

$\therefore$ Total energy at top $= 2 mgR + \frac{7}{10} mgR$

$= 2.7\, mgR$

Total Energy at top must = energy at bottom

Energy at bottom was obtained from the starting P.E. = mgh.

$\therefore$ minimum P.E. $= mgh = 2.7\, mgR$

$\therefore$ minimum $h = 2.7\, R$

(b) Vertical force component at P is $-mg$

If start at $h = 3R$, energy at bottom is $3mgR$

$\therefore$ at P, lose mgR in P.E., so

total $KE_P = 2mgR$

$$= \frac{1}{2} I_{cm} \omega^2 + \frac{1}{2} m V_{cm}^2$$

$$\therefore 2mgR = \frac{1}{2}\left(\frac{2}{5}mr^2\right)\left(\frac{V_{cm}}{r}\right)^2 + \frac{1}{2} m V_{cm}^2,$$

or

$$4gR = \frac{2}{5} V_{cm}^2 + V_{cm}^2 = \frac{7}{5} V_{cm}^2,$$

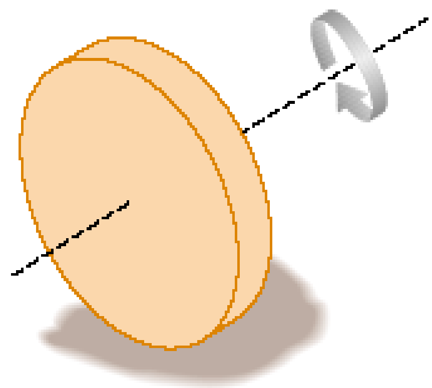
or

$$\frac{20}{7} gR = V_{cm}^2 \qquad [*]$$

Horizontal centripetal acceleration is $-\frac{V_{cm}^2}{R}$, which from [*] is (to the left is negative) $-\frac{20}{7} g$

$\therefore$ Horizontal force component $= ma = -\frac{20}{7} mg$

A uniform solid disk is set into rotation with an angular speed ω_i about an axis through its center. While still rotating at this speed, the disk is placed into contact with a horizontal surface and released as in Figure P11.56. (a) What is the angular speed of the disk once pure rolling takes place? (b) Find the fractional loss in kinetic energy from the time the disk is released until pure rolling occurs. (*Hint:* Consider torques about the center of mass.)



64. (a). $\Delta L = \tau \Delta t$, $\Delta L = I\omega_f - I\omega_i$, $\tau = -(F_{friction})(R)$

R = disk radius

Also, $V_{cm_f} = V_{cm_i} + a\Delta t$, $V_{cm_i} = 0$

$\therefore V_{cm_f} = R\omega_f$ (when rolling), so $R\omega_f = a\Delta t$

Also, $F_{friction} = Ma$, so $a = \frac{F_f}{M}$

$\therefore R\omega_f = \frac{F_f}{M}\Delta t$, $\Delta t = \frac{MR\omega_f}{F_{friction}}$

$\therefore -(F_f)(R)(\Delta t) = -MR^2\omega_f = \Delta L = I(\omega_f - \omega_i)$

But $I = \frac{1}{2}MR^2$. $\therefore -MR^2\omega_f = \frac{1}{2}MR^2(\omega_f - \omega_i)$

$\therefore -2\omega_f = \omega_f - \omega_i$, $\omega_i = 3\omega_f$, or $\underline{\omega_f = \frac{1}{3}\omega_i}$

(b) K.E. at start: $\frac{1}{2}I\omega_i^2$, and $I = \frac{1}{2}mR^2$

K.E. at pure rolling is $\frac{1}{2}I\omega_f^2 + \frac{1}{2}mv_{cm}^2$

From (a), $\omega_f = \frac{1}{3}\omega_i$.

$\therefore \frac{1}{2}I\omega_f^2 = \frac{1}{2}I\left(\frac{1}{3}\omega_i\right)^2 = \frac{1}{18}\left(\frac{1}{2}mR^2\right)\omega_i^2 = \frac{1}{36}mR^2\omega_i^2$

At rolling, $V_{cm} = R\omega_f = R\left(\frac{1}{3}\omega_i\right)$

$\therefore \frac{1}{2}mv_{cm}^2 = \frac{1}{2}m\left(\frac{R\omega_i}{3}\right)^2 = \frac{1}{18}mR^2\omega_i^2$

$$\therefore \frac{KE_i - KE_f}{K.E._i} = \frac{\frac{1}{4}mR^2\omega_i^2 - \left(\frac{1}{36}mR^2\omega_i^2 + \frac{1}{18}mR^2\omega_i^2\right)}{\frac{1}{4}mR^2\omega_i^2}$$

$$= \frac{\frac{1}{4} - \frac{3}{36}}{\frac{1}{4}} = \frac{\frac{9}{36} - \frac{3}{36}}{\frac{9}{36}} = \frac{2}{3}$$ $\therefore$ Loose $\frac{2}{3}$ initial K.E.

67. (a). Let f_T = friction between plank and cylinders

f_b = Friction between cylinders and table

a_p = acceleration of plank

a_{cm} = acceleration of cylinders (center-of-mass)

67. A plank with a mass $M = 6.00$ kg rides on top of two identical solid cylindrical rollers that have $R = 5.00$ cm and $m = 2.00$ kg (Fig. P11.67). The plank is pulled by a constant horizontal force of magnitude $F = 6.00$ N applied to the end of the plank and perpendicular to the axes of the cylinders (which are parallel). The cylinders roll without slipping on a flat surface. Also, no slipping occurs between the cylinders and the plank. (a) Find the acceleration of the plank and that of the rollers. (b) What frictional forces are acting?

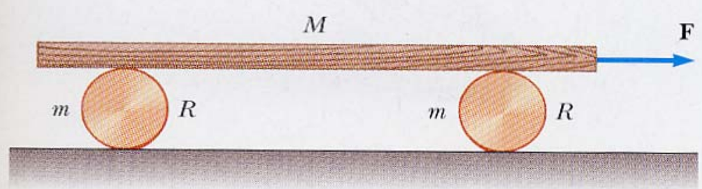


Figure P11.67

v_{cm} = velocity of cylinders at center-of-mass

Note that velocity of plank will be $2v_{cm}$ = speed at top rim of cylinders.

Note that for any distance at time t, if distance plank travels is $x(t)$, then distance cylinders travel is $\frac{1}{2}x(t)$. $\therefore v_{cm} = \frac{1}{2}x'(t)$,

$a_p = x''(t)$, so a_{cm} of cylinders $= \frac{1}{2}x''(t) = v'_{cm}(t)$

$\therefore a_{cm}$ cylinders $= \frac{1}{2}a_p$

For the plank, $F - 2f_T = Ma_p$ [1]

For any one cylinder, $(f_T - f_b)R = I\alpha$

Since $R\alpha = a_{cm}$,

$$\alpha = \frac{a_{cm}}{R} = \frac{\frac{1}{2}a_p}{R}. \quad \therefore f_T - f_b = \frac{I\,a_p}{2R^2}$$

For a cylinder, $I = \frac{1}{2}mR^2$. $\therefore f_T - f_b = \frac{ma_p}{4}$ [2]

Now consider energy, and work done by F for a distance x.

$\therefore$ Work = K.E.$_{plank}$ + 2 (Energy$_{clinders}$)

$$\therefore Fx = \frac{1}{2} M (2v_{cm})^2 + 2\left[\frac{1}{2} I \omega^2 + \frac{1}{2} m v_{cm}^2\right]$$

But $\omega = \frac{v_{cm}}{R}$

$$\therefore Fx = 2 M v_{cm}^2 + 2\left[\frac{1}{2}\left(\frac{1}{2} m R^2\right)\left(\frac{v_{cm}}{R}\right)^2 + \frac{1}{2} m v_{cm}^2\right]$$

$$= 2 M v_{cm}^2 + \frac{1}{2} m v_{cm}^2 + m v_{cm}^2$$

$$= v_{cm}^2 \left(2M + \frac{1}{2} m + m\right)$$

Using $v_f^2 - v_i^2 = 2ad$, $v_i = 0$, $a = a_{cm}$
and $d = \frac{1}{2}x$

$$\therefore v_{cm}^2 = 2a_{cm}\left(\frac{1}{2}x\right) = a_{cm} x$$

$$\therefore Fx = a_{cm} x \left(2M + \frac{1}{2} m + m\right)$$

$$a_{cm} = \frac{F}{\left(2M + \frac{1}{2} m + m\right)} = \frac{6\ N}{\left(2(6) + \frac{1}{2}(2) + 2\right) kg} = \frac{6}{15}$$

$$= \underline{0.4\ m/sec^2} \text{ (cylinders)}$$

$$\therefore a = 2a_{cm} = \underline{0.8\ m/sec^2} \text{ (plank)}$$

(3) From [1] above, $F - 2f_T = M a_p$

$\therefore\ 6 - 2f_T = 6(0.8)$, $\underline{f_T = 0.6\ N}$ (top)

From [2] above, $f_T - f_b = \frac{m a_p}{4}$

$\therefore\ f_b = 0.6 - \frac{(2)(0.8)}{4} = \underline{0.2\ N}$ (bottom)

70.

70. In a demonstration that employs a ballistics cart, a ball is projected vertically upward from a cart moving with constant velocity along the horizontal direction. The ball lands in the catching cup of the cart because both the cart and the ball have the same horizontal component of velocity. Now consider a ballistics cart on an incline making an angle θ with the horizontal, as shown in Figure P11.70. The cart (including its wheels) has a mass M, and the moment of inertia of each of the two wheels is $mR^2/2$. (a) Using conservation of energy considerations (assuming that there is no friction between the cart and the axles) and assuming pure rolling motion (that is, no slipping), show that the acceleration of the cart along the incline is

$$a_x = \left(\frac{M}{M + 2m}\right) g \sin \theta$$

(b) Note that the x component of acceleration of the ball released by the cart is $g \sin \theta$. Thus, the x component of the cart's acceleration is *smaller* than that of the ball by the factor $M/(M + 2m)$. Use this fact and kinematic equations to show that the ball overshoots the cart by an amount Δx, where

$$\Delta x = \left(\frac{4m}{M + 2m}\right)\left(\frac{\sin \theta}{\cos^2 \theta}\right) \frac{v_{yi}^2}{g}$$

and v_{yi} is the initial speed of the ball imparted to it by the spring in the cart. (c) Show that the distance d that the ball travels measured along the incline is

$$d = \frac{2v_{yi}^2}{g} \frac{\sin \theta}{\cos^2 \theta}$$

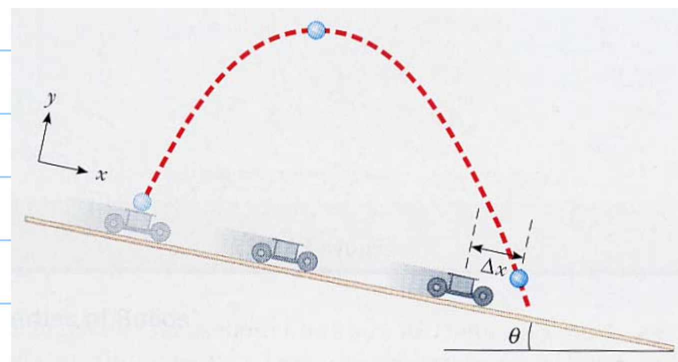
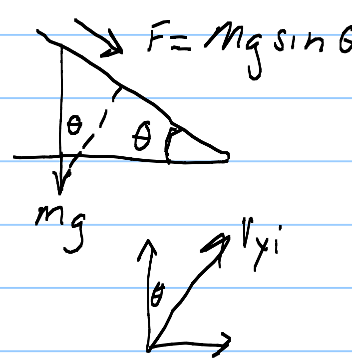


Figure P11.70



(a) Force along plank is $Mg\sin\theta$

Suppose gravity works over a distance x.

$\therefore Fx$ = work on cart

= translational K.E. + rotational K.E.

$$= \frac{1}{2}Mv^2 + 4\left(\frac{1}{2}I\omega^2\right) \qquad [4 \text{ wheels}]$$

$$= \frac{1}{2}Mv^2 + 2\left(\frac{1}{2}mR^2\right)\left(\frac{v}{R}\right)^2$$

$$= \frac{1}{2}Mv^2 + mv^2$$

Note that v = velocity of cart and center-of-mass of wheels.

But $v^2 = 2ax$ (from $v_f^2 - v_i^2 = 2ad$)

$$\therefore Fx = \left(\frac{1}{2}M + m\right)v^2 = \left(\frac{1}{2}M+m\right)2ax$$

$$= (M+2m)ax$$

$$\therefore F = Mg\sin\theta = (M+2m)a, \text{ or}$$

$$a = \frac{M}{(M+2m)}g\sin\theta$$

(b), (c) Let T = time ball takes to hit ramp on its downward flight.

V_{yi} = initial velocity in y-direction. (using y-axis as shown)

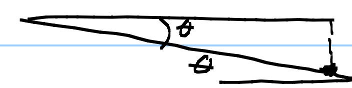
Note angle from horizontal (ground) is $90-\theta$.

$\therefore$ Horiz. velocity (ground) $= V_{yi}\cos(90-\theta)$
$= V_{yi}\sin\theta$

$\therefore$ Vert. velocity (to ground) $= V_{yi}\sin(90-\theta) = V_{yi}\cos\theta$

$\therefore$ Horiz. distance when ball hits plank $= (V_{yi}\sin\theta)T$

Vertical distance is $-(\tan\theta)(V_{yi}\sin\theta)T$



$$\therefore -(\tan\theta)(V_{yi}\sin\theta)T = (V_{yi}\cos\theta)T - \tfrac{1}{2}gT^2$$

$$\therefore (\tan\theta)(V_{yi}\sin\theta) + (V_{yi}\cos\theta) = \tfrac{1}{2}gT$$

$$\therefore T = \frac{2\tan\theta\,(V_{yi}\sin\theta) + 2(V_{yi}\cos\theta)}{g}$$

$\therefore$ Dist. along plank when ball hits $= \dfrac{\text{horiz. dist.}}{\cos\theta}$

$$= \frac{(V_{y_i} \sin\theta)\, T}{\cos\theta}$$

$$= \left(\frac{V_{y_i} \sin\theta}{\cos\theta}\right)\left[\frac{2\tan\theta\,(V_{y_i}\sin\theta) + 2(V_{y_i}\cos\theta)}{g}\right]$$

$$= \frac{2V_{y_i}^2}{g}\left[\frac{\sin\theta\tan\theta\sin\theta + \sin\theta\cos\theta}{\cos\theta}\right]$$

$$= \frac{2V_{y_i}^2}{g}\left[\frac{\sin^3\theta + \sin\theta\cos^2\theta}{\cos^2\theta}\right]$$

$$= \frac{2V_{y_i}^2}{g}\left[\frac{\sin\theta\,(\sin^2\theta + \cos^2\theta)}{\cos^2\theta}\right]$$

$$= \underline{\underline{\frac{2V_{y_i}^2}{g}\,\frac{\sin\theta}{\cos^2\theta}}} \qquad [1]$$

For the cart, $d = \frac{1}{2} a T^2$

$$= \frac{1}{2}\left(\frac{m}{M+2m}\right) g\sin\theta \left[\frac{2\tan\theta\,(V_{y_i}\sin\theta) + 2V_{y_i}\cos\theta}{g}\right]^2$$

$$= \frac{1}{2}\left(\frac{m}{m+2m}\right) g\sin\theta \left(\frac{4V_{y_i}^2}{g^2}\right)\left(\tan\theta\sin\theta + \cos\theta\right)^2$$

$$= \frac{2V_{y_i}^2}{g}\left(\frac{m}{M+2m}\right)\sin\theta\left[\frac{\sin^2\theta+\cos^2\theta}{\cos\theta}\right]^2$$

$$= \frac{2V_{y_i}^2}{g}\left(\frac{m}{M+2m}\right)\frac{\sin\theta}{\cos^2\theta} \qquad [2]$$

$\therefore$ Ball - cart distance $= \Delta x = [1]-[2]$

$$= \frac{2V_{y_i}^2}{g}\frac{\sin\theta}{\cos^2\theta}\left(1-\frac{m}{M+2m}\right)$$

$$= \frac{2V_{y_i}^2}{g}\frac{\sin\theta}{\cos^2\theta}\left(\frac{2m}{M+2m}\right)$$

$$= \underline{\underline{\frac{4m}{M+2m}\cdot\frac{V_{y_i}^2}{g}\cdot\frac{\sin\theta}{\cos^2\theta}}} = \Delta x$$