Chapter 13 - Oscillatory Motion

Note Title 10/16/2005

6\. **6.** The initial position, velocity, and acceleration of an object moving in simple harmonic motion are x_i, v_i, and a_i; the angular frequency of oscillation is ω. (a) Show that the position and velocity of the object for all time can be written as

$$x(t) = x_i \cos \omega t + \left(\frac{v_i}{\omega}\right) \sin \omega t$$

$$v(t) = -x_i \omega \sin \omega t + v_i \cos \omega t$$

(b) If the amplitude of the motion is A, show that

$$v^2 - ax = v_i^2 - a_i x_i = \omega^2 A^2$$

(a) $x(t) = A\cos(\omega t + \phi)$

$= A\cos(\omega t)\cos\phi - A\sin(\omega t)\sin\phi$

At $t=0$, $x_i = A\cos\phi$

$\therefore x(t) = x_i \cos(\omega t) - A\sin(\omega t)\sin\phi$

Also at $t=0$, $v_i = -A\omega\sin\phi$, $\frac{v_i}{\omega} = -A\sin\phi$

$\therefore x(t) = x_i\cos(\omega t) + \frac{v_i}{\omega}\sin(\omega t)$

$\therefore x'(t) = v(t) = -x_i\omega\sin(\omega t) + v_i\cos(\omega t)$

(b) From (a), $v^2(t) = x_i^2 w^2 \sin^2(wt) + V_i^2 \cos^2(wt)$

$-2x_i v_i w \sin(wt)\cos(wt)$ [1]

$v'(t) = a(t) = -x_i w^2 \cos(wt) - V_i w \sin(wt)$

$\therefore a(t)\,x(t) = -x_i^2 w^2 \cos^2(wt) - V_i^2 \sin^2(wt)$

$-x_i V_i w \sin(wt)\cos(wt) - x_i V_i w \sin(wt)\cos(wt)$

$= -x_i^2 w^2 \cos^2(wt) - V_i^2 \sin^2(wt)$

$-2x_i V_i w \sin(wt)\cos(wt)$ [2]

$\therefore v^2(t) - a(t)\,x(t) = [1] - [2]$

$= x_i^2 w^2 \sin^2(wt) + V_i^2 \cos^2(wt) +$

$x_i^2 w^2 \cos^2(wt) + V_i^2 \sin^2(wt)$

$= x_i^2 w^2 + V_i^2$ [3]

Note from $x(t)$ and $a(t)$, $a(t) = -w^2 x(t)$

$\therefore a_i = a(0) = -w^2 x(0) = -w^2 x_i$, or $-w^2 = a_i / x_i$

$\therefore$ From [3], $v^2 - ax = x_i^2 \omega^2 + V_i^2 = x_i^2\left(-\frac{a_i}{x_i}\right) + V_i^2$

$= V_i^2 - a_i x_i \quad [4]$

Also, from (a), $x_i = A\cos\phi$, $V_i = -A\omega\sin\phi$,

$$V^2 - ax = x_i^2 \omega^2 + V_i^2$$

$$= (A\cos\phi)^2 \omega^2 + (-A\omega\sin\phi)^2$$

$$= A^2\omega^2\cos^2\phi + A^2\omega^2\sin^2\phi$$

$$= A^2\omega^2$$

23. $V = \pm\omega\sqrt{A^2 - x^2}$, $V_{max} = \omega A$

$\therefore \frac{1}{2}\omega A = \pm\omega\sqrt{A^2 - x^2}$, $\frac{A^2}{4} = A^2 - x^2$

$x^2 = \frac{3}{4}A^2$, $x = \pm A\frac{\sqrt{3}}{2} = \pm 3\frac{\sqrt{3}}{2}$ cm

24. (a) $U(t) = \frac{1}{2}Kx^2(t)$, $\therefore U'(t) = Kx(t)\cdot x'(t)$

$x(t) = 0.05\cos(3.6t)$

$x'(t) = -0.18\sin(3.6t)$

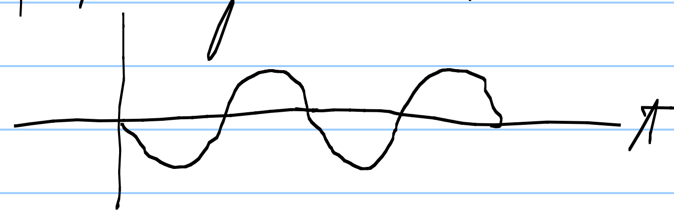
$\therefore U'(t) = -(3.24)(0.009)\sin(3.6t)\cos(3.6t)$

$\sin 2\theta = 2\sin\theta\cos\theta$, $\therefore U'(t) = -0.0146\sin(7.2t)$

Since $\omega = 7.2$, $T = 2\pi/7.2 = 0.872$ sec.

For $0 < t < 1.75$, two periods are covered

Graph looks like:



$\therefore$ for $t = \frac{3}{4}T$, or $\frac{3}{4}T + T$, $U'(t)$ will be a max. $\therefore$ $\frac{3}{4}(0.872)$ and $\frac{7}{4}(0.872)$, or

$t = 0.654$ sec, or

$t = 1.53$ sec.

(d) Max is amplitude of $U'(t) = 0.0146$ J/sec.

52.

52. An object of mass $m_1 = 9.00$ kg is in equilibrium while connected to a light spring of constant $k = 100$ N/m that is fastened to a wall as shown in Figure P15.52a. A second object, $m_2 = 7.00$ kg, is slowly pushed up against m_1, compressing the spring by the amount $A = 0.200$ m, (see Figure P15.52b). The system is then released, and both objects start moving to the right on the frictionless surface. (a) When m_1 reaches the equilibrium point, m_2 loses contact with m_1 (see Fig. P15.5c) and moves to the right with speed v. Determine the value of v. (b) How far apart are the objects when the spring is fully stretched for the first time (D in Fig. P15.52d)? (*Suggestion:* First determine the period of oscillation and the amplitude of the m_1–spring system after m_2 loses contact with m_1.)

(a) $v = v_{max}$ = speed when m_1 is at $x = 0$

$\therefore V = \omega A = \omega (0.2\ m)$

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{100\ n/m}{m_1 + m_2}} = \sqrt{\frac{100}{16}} = \frac{10}{4}\ sec^{-1}$$

$$\therefore V = \frac{(0.2)(10)}{4} = 5\ m/sec$$

(b) Time to max stretch is $T/4$

$$\omega T = 2\pi, \therefore T/4 = \frac{2\pi}{4\omega} = \frac{\pi}{2\omega}$$

For just m_1, $\omega = \sqrt{\frac{k}{m_1}} = \sqrt{\frac{100}{9}} = \frac{10}{3}\ sec^{-1}$

$$\therefore T/4 = \frac{\pi}{2\left(\frac{10}{3}\right)} = 0.471\ sec$$

From $V_{max} = \omega A$, $A = \dfrac{5\ m/sec}{\frac{10}{3}\ sec^{-1}} = 1.5\ m$

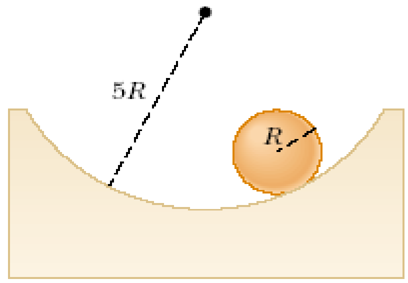
$\therefore$ at $T/4 = 0.471$ sec, m_1 will be at $x = 1.5\ m$

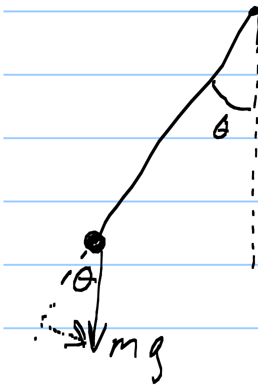
and m_2 will be at $5(0.471) = 2.36\ m$

$\therefore$ Separation $= 2.36 - 1.50 = \underline{0.86\ m}$

56

56. A solid sphere (radius = R) rolls without slipping in a cylindrical trough (radius = $5R$) as shown in Figure P15.56. Show that, for small displacements from equilibrium perpendicular to the length of the trough, the sphere executes simple harmonic motion with a period $T = 2\pi\sqrt{28R/5g}$.





Strategy: Find equation for acceleration, show it is proportional to negative of displacement from equilibrium.

Let θ = angle of radius of cylindrical trough with the vertical.
m = mass of sphere.
f = friction force of trough surface

$\therefore$ Force of gravity tangential to surface of trough $= -mg\sin\theta$ (right is positive)

$\therefore -mg\sin\theta - f = m\,a_{sphere}$

torque on sphere from friction $= fR$
$= I_{cm}\alpha$, α = rotational acceleration of sphere

For sphere, $I_{cm} = \frac{2}{5} m R^2$, and

$a_{sphere} = R\alpha$.

$$\therefore fR = \frac{2}{5} m R^2 \left(\frac{a_{sphere}}{R} \right)$$

$$\therefore f = \frac{2}{5} m a_{sphere}$$

$$\therefore -mg\sin\theta - \frac{2}{5} m a_{sphere} = m a_{sphere}$$

$$\therefore -g\sin\theta = \frac{7}{5} a_{sphere}$$

For small θ, $-g\theta = \frac{7}{5} a_{sphere}$

$\theta = \frac{x}{4R}$, x = displacement of center of mass perpendicular to vertical.

$$\therefore \frac{d^2\theta}{dt^2} = \frac{1}{4R} \frac{d^2x}{dt^2}, \text{ and } \frac{d^2x}{dt^2} = a_{sphere}$$

$$\therefore 4R \frac{d^2\theta}{dt^2} = a_{sphere}$$

$$\therefore -g\theta = \frac{7}{5}\left(4R \frac{d^2\theta}{dt^2} \right)$$

Or, $-\frac{5g}{28R}\theta = \frac{d^2\theta}{dt^2}$

$\therefore$ simple harmonic motion.

Let $\omega^2 = \frac{5g}{28R}$, $\omega T = 2\pi$, $\omega^2 T^2 = 4\pi^2$

$\therefore \frac{5g}{28R} T^2 = 4\pi^2$, $T^2 = 4\pi^2\left(\frac{28R}{5g}\right)$

$T = 2\pi\sqrt{\frac{28R}{5g}}$

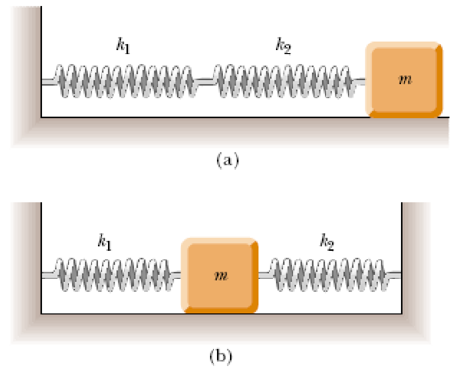
71. **71.** A block of mass m is connected to two springs of force constants k_1 and k_2 as shown in Figures P15.71a and P15.71b. In each case, the block moves on a frictionless table after it is displaced from equilibrium and released. Show that in the two cases the block exhibits simple harmonic motion with periods

(a) $$T = 2\pi\sqrt{\frac{m(k_1 + k_2)}{k_1 k_2}}$$

(b) $$T = 2\pi\sqrt{\frac{m}{k_1 + k_2}}$$

(a). Let x_1 = displacement from K_1

x_2 = displacement from K_2



$x = x_1 + x_2$ = total displacement.

Note That $K_1 x_1 = K_2 x_2$ by Newton's 3rd Law.

$$\therefore -K_1 x_1 = -K_2 x_2 = m x'' = -Kx = -K(x_1 + x_2)$$

$$x_1 = \frac{K_2}{K_1} x_2, \quad \therefore -K_2 x_2 = -K\left(\frac{K_2}{K_1} x_2 + x_2\right)$$

$$\therefore K_2 = K\left(\frac{K_2}{K_1} + 1\right)$$

$$\frac{K_1 K_2}{K_1 + K_2} = K$$

$$\therefore m x'' = -\frac{K_1 K_2}{K_1 + K_2} x, \quad x'' = -\frac{K_1 K_2}{m(K_1 + K_2)} x$$

$$\therefore \text{Let } \omega^2 = \frac{K_1 K_2}{m(K_1 + K_2)}, \quad \omega = \sqrt{\frac{K_1 K_2}{m(K_1 + K_2)}}$$

$$\therefore T = 2\pi/\omega = 2\pi \sqrt{\frac{m(K_1 + K_2)}{K_1 K_2}}$$

(b) $F_m = -K_1 x - K_2 x = m x''$

$\therefore -\dfrac{(K_1 + K_2)}{m} x = x''$

$\therefore$ let $\omega^2 = \dfrac{K_1 + K_2}{m}$, $\omega = \sqrt{\dfrac{K_1 + K_2}{m}}$

$T = 2\pi/\omega = 2\pi \sqrt{\dfrac{m}{K_1 + K_2}}$