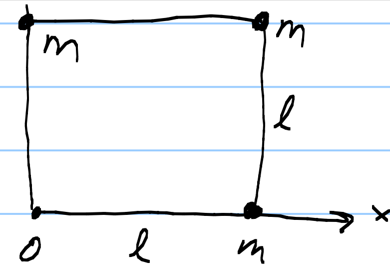
Chapter 14 - The Law of Gravity

Note Title 12/12/2005



3. $\vec{g} = -\frac{Gm}{r^2}\hat{r}$

3 masses at $(0,l), (l,l), (l,0)$

$\therefore \vec{g}$ at $(l,0)$: $\frac{Gm}{l^2}\hat{i}$

$\vec{g}$ at $(0,l)$: $\frac{Gm}{l^2}\hat{j}$

At (l,l), $\vec{g} = \frac{Gm}{2l^2}$, directed toward (l,l)

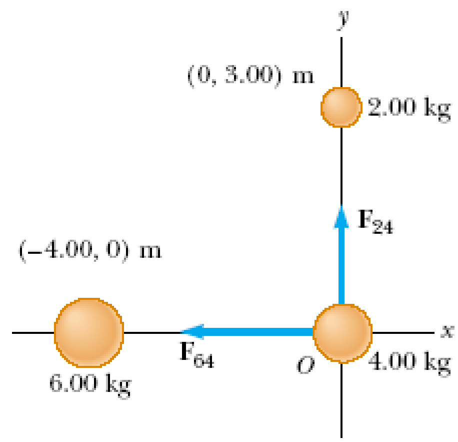
$\cos 45^\circ = \sin 45^\circ = \frac{\sqrt{2}}{2}$

$\therefore$ horiz and vert. component $= \frac{\sqrt{2}}{4}\frac{Gm}{l^2}$

$\therefore \vec{g}$ at (l,l): $\frac{\sqrt{2}}{4}\frac{Gm}{l^2}(\hat{i}+\hat{j})$

$\therefore \vec{g}$ at $(0,0)$: $\left(1+\frac{\sqrt{2}}{4}\right)\frac{Gm}{l^2}(\hat{i}+\hat{j})$

5. Three uniform spheres of mass 2.00 kg, 4.00 kg, and 6.00 kg are placed at the corners of a right triangle as in Figure P13.5. Calculate the resultant gravitational force on the 4.00-kg object, assuming the spheres are isolated from the rest of the Universe.



5. $\vec{F}_{64} = -\frac{G(6)(4)}{4^2}\hat{i}$

$= -G\left(\frac{3}{2}\right)\hat{i}$

$\vec{F}_{24} = -G\frac{(2)(4)}{3^2}(-\hat{j})$

$= G\left(\frac{8}{9}\right)\hat{j}$

$G = 6.67 \times 10^{-11}\ \frac{N\text{-}m^2}{kg^2}$

$\therefore \vec{F}_{64} + \vec{F}_{24} = -1.0\times10^{-10}\hat{i} + 5.92\times10^{-11}\hat{j}\ N$

$= -100\times10^{-12}\hat{i} + 59.2\times10^{-12}\hat{j}\ N$

$= (-100\hat{i} + 59.2\hat{j})\ pN \quad (p = 10^{-12})$

6. $g_m = \frac{GM_m}{R_m^2}$, $g_e = \frac{GM_e}{R_e^2}$, $\frac{g_m}{g_e} = \frac{1}{6}$

$R_m = 0.250 R_e$, and $\rho = m/v$

$$\therefore \frac{G M_m}{R_m^2} \bigg/ \frac{G M_e}{R_e^2} = \frac{M_m / R_m^2}{M_e / R_e^2} = \frac{1}{6}$$

$$V = \frac{4}{3}\pi r^3 . \quad \therefore \frac{\rho_m}{\rho_e} = \frac{M_m / R_m^2}{M_e / R_e^2} \cdot \frac{\frac{1}{R_m}}{\frac{1}{R_e}}$$

$$= \frac{1}{6}\left(\frac{R_e}{R_m}\right) = \frac{1}{6}\left(\frac{R_e}{0.250 R_e}\right)$$

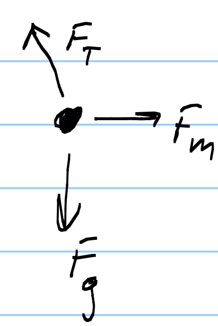
$$= \frac{1}{1.5} = 2/3$$

$$\therefore \underline{\frac{\rho_m}{\rho_e} = \frac{2}{3}}$$

11. Assume d between masses $\ll 1.0$ m

$$\therefore F_m = \frac{G(10^2 \text{kg})(10^2 \text{kg})}{(1.0 \text{ m})^2} = 6.67 \times 10^{-7} \text{ N}$$

$$F_g = (10^2 \text{kg})(9.8) = 980 \text{ N}$$



$$\vec{F}_T = -(\vec{F}_m + \vec{F}_g)$$

$= (-6.67\times10^{-7}\,\hat{i} - 980\,\hat{j})\,N$

$\therefore \theta =$ angle of deflection

$= \arctan\left(\dfrac{6.67\times10^{-7}}{9.80\times10^{2}}\right)$

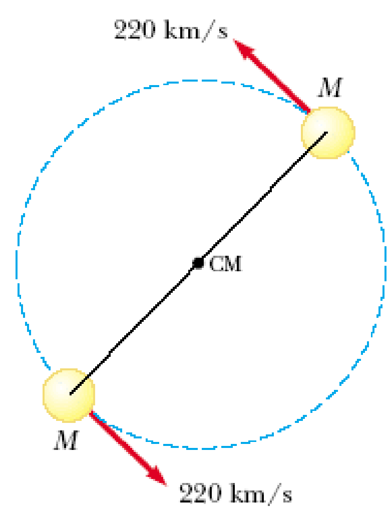
At such a small angle, $\sin\theta = \tan\theta = \theta$

$\therefore d = (45.00\,m)\sin\theta \approx (45.00)(\tan\theta)$

$= 45\left(\dfrac{6.67\times10^{-7}}{9.8\times10^{2}}\right) = 30.6 \times10^{-9}\ m = 30.6\,nm$

$\therefore 1.000\,m - 2(d) = 1.00\,m - 61.2\,nm$

15. Plaskett's binary system consists of two stars that revolve in a circular orbit about a center of mass midway between them. This means that the masses of the two stars are equal (Fig. P13.13). Assume the orbital speed of each star is 220 km/s and the orbital period of each is 14.4 days. Find the mass M of each star. (For comparison, the mass of our Sun is 1.99×10^{30} kg.)



Only force on one mass is gravity, so,

$F = \dfrac{Gm^2}{(2r)^2} = \dfrac{Gm^2}{4r^2}$

$$F = ma = \frac{mv^2}{r} = \frac{Gm^2}{4r^2}, \quad m = \frac{4rv^2}{G}$$

$$\text{But } v = \frac{2\pi r}{T}, \text{ so } r = \frac{vT}{2\pi}$$

$$\therefore m = \frac{4v^3 T}{2\pi G} = \frac{2v^3 T}{\pi G}$$

$$= \frac{2(2.2\times10^5)^3 (14.4d)(24 hr/d)(60 \ min/hr)(60 \ sec/min)}{\pi(6.67\times10^{-11})}$$

$$= 1.26\times10^6 \times 10^{15} \times 10^{11} = 1.26\times10^{32} \text{ Kg}$$

16. As in #15, $m = \frac{2v^3 T}{\pi G}$

17. The *Explorer VIII* satellite, placed into orbit November 3, 1960, to investigate the ionosphere, had the following orbit parameters: perigee, 459 km; apogee, 2 289 km (both distances above the Earth's surface); period, 112.7 min. Find the ratio v_p/v_a of the speed at perigee to that at apogee.

From conservation of ang. momentum, $mv_a r_a = mv_p r_p$,

$$\frac{v_p}{v_a} = \frac{r_a}{r_p}$$

$$r_{earth} = 6.37\times10^6 \text{ m}$$

$$\therefore \frac{V_p}{V_a} = \frac{2.289\times10^6 + 6.37\times10^6}{0.459\times10^6 + 6.37\times10^6} = \underline{1.27}$$

18. Comet Halley (Figure P13.17) approaches the Sun to within 0.570 AU, and its orbital period is 75.6 years. (AU is the symbol for astronomical unit, where 1 AU = 1.50×10^{11} m is the mean Earth–Sun distance.) How far from the Sun will Halley's comet travel before it starts its return journey?

From Kepler's law, $T^2 = K_s R^3$, R = semi-major axis.

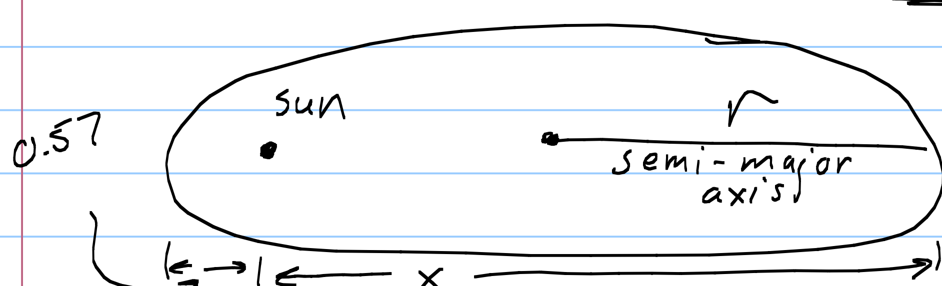
$K_s = 2.97\times10^{-19}\ s^2/m^3$

$$\therefore R^3 = \frac{T^2}{K_s} = \frac{\left[75.6(365)(24)(60)(60)\right]^2}{2.97\times10^{-19}}$$

$$= \frac{5.68\times10^{18}}{2.97\times10^{-19}} = 1.91\times10^{37} = 19.1\times10^{36}$$

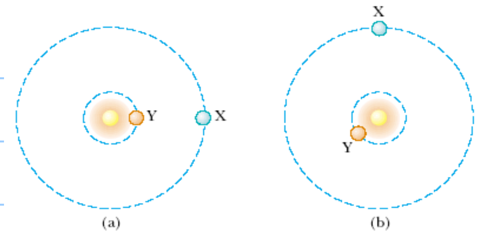
$$\therefore R = 2.67\times10^{12}\ m = \frac{2.67\times10^{12}}{1.5\times10^{11}} = 17.8\ AU$$

$$\therefore 2R - 0.57 = 35.6 - 0.57 = \underline{35.0}\ AU$$



$x = 2r - 0.57$

20. Two planets X and Y travel counterclockwise in circular orbits about a star as in Figure P13.18. The radii of their orbits are in the ratio 3:1. At some time, they are aligned as in Figure P13.18a, making a straight line with the star. During the next five years, the angular displacement of planet X is 90.0°, as in Figure P13.18b. Where is planet Y at this time?



$T^2 = KR^3$, $\therefore \dfrac{T_y^2}{T_x^2} = \dfrac{R_y^3}{R_x^3} = \left(\dfrac{1}{3}\right)^3 = \dfrac{1}{27}$

$\therefore T_x^2 = 27 T_y^2$, $T_x = \sqrt{27}\, T_y$

$\omega T = 2\pi$, so $T = \dfrac{2\pi}{\omega}$, so $\dfrac{2\pi}{\omega_x} = \sqrt{27}\, \dfrac{2\pi}{\omega_y}$

$\therefore \omega_y = \sqrt{27}\, \omega_x$

$\therefore 90^\circ = \frac{1}{4}$ of a period, let T = time for X to go 90°.

$\omega_x T = \frac{\pi}{2}$ radians, $\therefore \omega_y T = \sqrt{27}\, \omega_x T$, so

Y will have gone $\sqrt{27}\, \frac{\pi}{2}$ radians $= \sqrt{27}\, \frac{\pi}{2}\left(\frac{180}{\pi}\right)$ degrees

$= 90\sqrt{27}^{\,0} = 467.7^\circ = 467.7 - 360 = 107.7^\circ$

So, angular displacement of Y is $\underline{108^\circ}$

23. The Solar and Heliospheric Observatory (SOHO) spacecraft has a special orbit, chosen so that its view of the Sun is never eclipsed and it is always close enough to the Earth to transmit data easily. It moves in a near-circle around the Sun that is smaller than the Earth's circular orbit. Its period, however, is just equal to 1 yr. It is always located between the Earth and the Sun along the line joining them. Both objects exert gravitational forces on the observatory. Show that its distance from the Earth must be between 1.47×10^9 m and 1.48×10^9 m. In 1772 Joseph Louis Lagrange determined theoretically the special location allowing this orbit. The SOHO spacecraft took this position on February 14, 1996. *Suggestion:* Use data that are precise to four digits. The mass of the Earth is 5.983×10^{24} kg.

Want $F_{sun} - F_{earth} = \frac{mv^2}{r_s}$, r_s = distance to sun

$\therefore \frac{GM_s m}{r_s^2} - \frac{GM_e m}{r_e^2} = \frac{mv^2}{r_s}$, r_e = distance to earth

Note: $r_s + r_e = R_{earth\text{-}to\text{-}sun} = 1.496 \times 10^{11}$ m
(R_{e-s})

$\therefore r_s = R_{e-s} - r_e$

$$\therefore \frac{GM_s}{(R_{e-s} - r_e)^2} - \frac{GM_e}{r_e^2} = \frac{v^2}{r_s}$$

But $v = \frac{2\pi r_s}{T}$, $v^2 = \frac{4\pi^2 r_s^2}{T^2}$

$$\therefore \frac{GM_s}{(R_{e-s} - r_e)^2} - \frac{GM_e}{r_e^2} = \frac{4\pi^2 (R_{e-s} - r_e)}{T^2}, \text{ or}$$

$$4\pi^2 r_e^2 (R_{e-s} - r_e)^3 - GM_s T^2 r_e^2 + GM_e T^2 (R_{e-s} - r_e)^2 = 0$$

This is a rather complex equation in r_e (5th order), but is a continuous function of variable r_e, so if it can be shown that $F(x_1) > 0$ and $F(x_2) < 0$, then $\frac{x_1 + x_2}{2}$ would be a good approximation of one of the roots.

Using
$$M_e = 5.983 \times 10^{24}\ \text{Kg}$$
$$M_s = 1.991 \times 10^{30}\ \text{Kg}$$
$$G = 6.672 \times 10^{-11}\ \text{N}\cdot\text{m}^2/\text{Kg}^2$$
$$T = 1\,\text{yr} = 3.156 \times 10^7\ \text{sec.}$$
$$R_{e-s} = 1.496 \times 10^{11}\ \text{m}$$

$\therefore$ $GM_s = 1.328 \times 10^{20}$, $GM_e = 3.992 \times 10^{14}$
$T^2 = 9.960 \times 10^{14}$

Try $r_e = 1.470 \times 10^9\,\text{m}$: $(R_{e-s} - r_e) = (1.496 - 0.0147) \times 10^{11} = 1.481 \times 10^{11}$

$$\frac{1.328 \times 10^{20}}{(1.481 \times 10^{11})^2} - \frac{3.992 \times 10^{14}}{(1.470 \times 10^9)^2} - \frac{4\pi^2 (1.481 \times 10^{11})}{9.96 \times 10^{14}} =$$

$$6.054\times10^{-3} - 0.1847\times10^{-3} - 5.870\times10^{-3} = -0.0003$$

Try $r_e = 1.48\times10^9\,m$: $(R_{e\text{-}s} - r_e) = (1.496 - 0.0148)\times10^{11} = 1.481\times10^{11}$

$$\frac{1.328\times10^{20}}{(1.481\times10^{11})^2} - \frac{3.992\times10^{14}}{(1.480\times10^9)^2} - \frac{4\pi^2(1.481\times10^{11})}{9.96\times10^{14}} =$$

$$6.054\times10^{-3} - 0.1822\times10^{-3} - 5.87\times10^{-3} = +0.002$$

$\therefore \exists$ a value $1.47\times10^9 < r_e < 1.48\times10^9$
s.t. computation$(r_e) = 0$.

Could set $r_e = 1.475\times10^9\,m$:

$$\therefore R_{e\text{-}s} - r_e = (1.496 - 0.01475) = 1.481\times10^{11}$$

$$\therefore 6.054\times10^{-3} - 0.1835\times10^{-3} - 5.870\times10^{-3} = +0.0009$$

30. At the Earth's surface a projectile is launched straight up at a speed of 10.0 km/s. To what height will it rise? Ignore air resistance and the rotation of the Earth.

Height may be such that g is not 9.8 at apogee.

$$\therefore \frac{1}{2}mv^2 - \frac{GM_em}{R_e} = 0 - \frac{GM_em}{R_e+h}$$

$$\therefore \frac{R_e v^2 - 2GM_e}{2R_e} = -\frac{GM_e}{R_e + h}$$

$$\therefore h = \frac{2GM_e R_e}{2GM_e - R_e v^2} - R_e$$

$$= \frac{2(6.67\times10^{-11})(5.98\times10^{24})(6.37\times10^{6})}{2(6.67\times10^{-11})(5.98\times10^{24}) - (6.37\times10^{6})(10^{4})^2} - 6.37\times10^{6}$$

$$= 3.16\times10^{7} - 0.637\times10^{7} = \underline{2.52\times10^{7}\,m}$$

To confirm assumption, g at 2.52×10^{7} is:

$$\frac{g_{surface}}{g_h} = \frac{\frac{1}{r_e^2}}{\frac{1}{(r_e+h)^2}} = \left(\frac{6.38\times10^{6} + 25.2\times10^{6}}{6.38\times10^{6}}\right)^2$$

$$g_h = 0.40\ m/sec^2$$

34. (a) What is the minimum speed, relative to the Sun, necessary for a spacecraft to escape the solar system if it starts at the Earth's orbit? (b) *Voyager 1* achieved a maximum speed of 125 000 km/h on its way to photograph Jupiter. Beyond what distance from the Sun is this speed sufficient to escape the solar system?

$F_{g-sun} >>> F_{g-earth}$

so, can ignore effect of earth.

(a) $$V_{esc} = \sqrt{\frac{2GM_{sun}}{r_{sun-earth}}} = \sqrt{\frac{2(6.67\times10^{-11})(1.99\times10^{30})}{1.50\times10^{11}}}$$

$$= \sqrt{17.7 \times 10^8} = 4.21 \times 10^4 \text{ m/sec.}$$

(b) $125{,}000$ Km/hr $= 34.7$ Km/sec $= 3.47 \times 10^4$ m/sec.

$$\therefore v^2 = \frac{2GM_s}{r}, \quad r = \frac{2GM_s}{v^2} = \frac{2(6.67\times10^{-11})(1.99\times10^{30})}{(3.47\times10^4)^2}$$

$= 2.20 \times 10^{11}$ meters, which is pretty close to orbit of Mars.

36. A satellite of mass m, originally on the surface of the Earth, is placed into Earth orbit at an altitude h. (a) With a circular orbit, how long does the satellite take to complete one orbit? (b) What is the satellite's speed? (c) What is the minimum energy input necessary to place this satellite in orbit? Ignore air resistance but include the effect of the planet's daily rotation. At what location on the Earth's surface and in what direction should the satellite be launched to minimize the required energy investment? Represent the mass and radius of the Earth as M_E and R_E.

(a) $$\frac{GM_e m_s}{(R_e+h)^2} = \frac{m_s V^2}{(R_e+h)}$$

$$\therefore \sqrt{\frac{GM_e}{R_e+h}} = V, \quad \frac{2\pi(R_e+h)}{T} = V$$

$$\therefore T = \frac{1}{2\pi(R_e+h)} \cdot \sqrt{\frac{R_e+h}{GM_e}}$$

(b) as in (a), $V_{orbit} = \sqrt{\frac{GM_e}{R_e+h}}$

(c) Total $E_{earth\ surface}$ = Total E_{orbit} + Input E

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r}$$

V at earth surface $= \frac{2\pi R_e(\cos\theta)}{(60)(60)(24)} = \frac{2\pi R_e(\cos\theta)}{86400}$

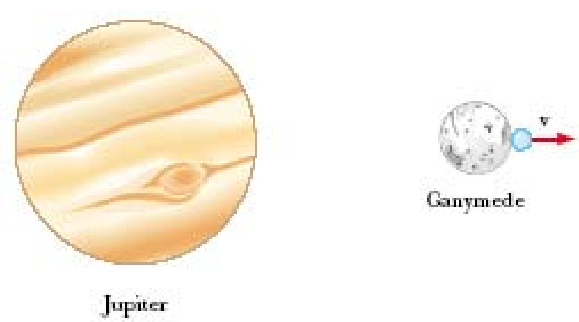
θ = latitude.

V is max at equator ($\theta = 0°$), and earth rotates toward east. ∴ Easiest to launch at equator toward east.

$$\therefore \frac{1}{2}m\left(\frac{2\pi R_e\cos\theta}{86400}\right)^2 - \frac{GM_em}{R_e} - \left[\frac{1}{2}m\left(\frac{GM_e}{R_e+h}\right) - \frac{GM_em}{R_e+h}\right] =$$

$$\frac{1}{2}m\left(\frac{2\pi R_e\cos\theta}{86400}\right)^2 - \frac{GM_em}{R_e} + \frac{GM_em}{2(R_e+h)} = \text{Input } E$$

41. Determine the escape speed for a rocket on the far side of Ganymede, the largest of Jupiter's moons (Figure P13.41). The radius of Ganymede is 2.64×10^6 m, and its mass is 1.495×10^{23} kg. The mass of Jupiter is 1.90×10^{27} kg, and the distance between Jupiter and Ganymede is 1.071×10^9 m. Be sure to include the gravitational effect due to Jupiter, but you may ignore the motion of Jupiter and Ganymede as they revolve about their center of mass.



$\frac{1}{2}mv^2 - U_G - U_J = 0$, for $v =$ escape speed.

$\therefore \frac{1}{2}mv^2 = \frac{GM_G m}{R_G} + \frac{GM_J m}{R_{J-G}}$

$\therefore v^2 = 2(6.67\times10^{-11})\left[\frac{1.495\times10^{23}}{2.64\times10^{6}} + \frac{1.90\times10^{27}}{1.071\times10^{9}}\right]$

$= 2(6.67\times10^{-11})\left[5.66\times10^{16} + 1.77\times10^{18}\right]$

$= 24.4\times10^{7}$

$\therefore v = 1.56\times10^{4}$ m/sec.

71. The acceleration of an object moving in the gravitational field of the Earth is

$$\mathbf{a} = -\frac{GM_E\mathbf{r}}{r^3}$$

where $\mathbf{r}$ is the position vector directed from the center of the Earth toward the object. Choosing the origin at the center of the Earth and assuming that the small object is moving in the xy plane, we find that the rectangular (Cartesian) components of its acceleration are

$$a_x = -\frac{GM_E x}{(x^2+y^2)^{3/2}} \qquad a_y = -\frac{GM_E y}{(x^2+y^2)^{3/2}}$$

Use a computer to set up and carry out a numerical prediction of the motion of the object, according to Euler's method. Assume the initial position of the object is $x = 0$ and $y = 2R_E$, where R_E is the radius of the Earth. Give the object an initial velocity of 5 000 m/s in the x direction. The time increment should be made as small as practical. Try 5 s. Plot the x and y coordinates of the object as time goes on. Does the object hit the Earth? Vary the initial velocity until you find a circular orbit.

Spreadsheet shown below for $\Delta t = 10$ sec.

Usid $a = \frac{GM_e}{r^2}$, $r = \sqrt{x^2+y^2}$

$$a_x = a\cos\theta = a\left(\frac{x}{r}\right), \quad a_y = a\sin\theta = a\left(\frac{y}{r}\right)$$

$$V_x = V_{prev.} + a_x \Delta t, \quad V_y = V_{prev} + a_y \Delta t$$

$$x = x_{prev} + (V_{x\,prev})\Delta t + \frac{1}{2}(a_{x\,prev})\Delta t^2$$

$$y = y_{prev} + (V_{y\,prev})\Delta t + \frac{1}{2}(a_{y\,prev})\Delta t^2$$

	A	B	C	D	E	F	G	H	I
1	Mass of Planet:		5.98E+24						
2	Radius of Planet:		6.37E+06						
3	Initial X:		0.00	Planet radii		Acc = GM/r^2			
4	Initial Y:		2.00	Planet radii		Acc-x = Acc * (cosine angle = x/r)			
5	Initial Vx		5,000.00			Acc-y = Acc * (sine angle = y/r)			
6	Initial Vy:		0.00						
7	G:		6.67E-11						
8	Time interval:		10.00						
9									
10	t (sec)	X (m)	Y (m)	r object (m)	Vx	Vy	Acc	Acc-x	Acc-y
11									
12	0.00	0.00	12,740,000.00	12,740,000.00	5,000.00	0.00	2.4584	0.0000	-2.4584
13	10.00	50,000.00	12,739,877.08	12,739,975.20	5,000.00	-24.58	2.4584	-0.0096	-2.4584
14	20.00	99,999.52	12,739,508.32	12,739,900.79	4,999.90	-49.17	2.4585	-0.0193	-2.4584
15	30.00	149,997.59	12,738,893.71	12,739,776.78	4,999.71	-73.75	2.4585	-0.0289	-2.4583
16	40.00	199,993.25	12,738,033.27	12,739,603.17	4,999.42	-98.34	2.4586	-0.0386	-2.4583
17	50.00	249,985.53	12,736,927.00	12,739,379.98	4,999.04	-122.92	2.4587	-0.0482	-2.4582
18	60.00	299,973.47	12,735,574.91	12,739,107.20	4,998.55	-147.50	2.4588	-0.0579	-2.4581

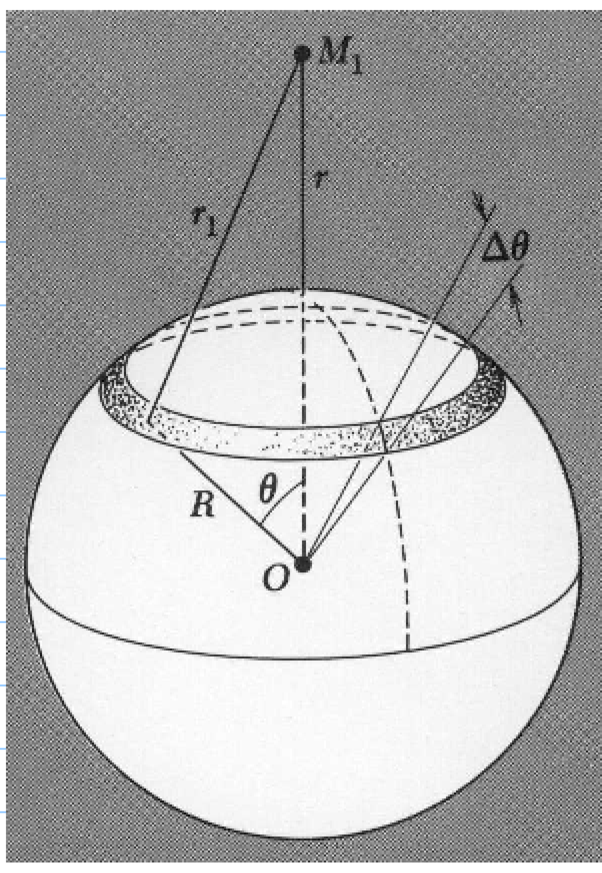
10	t (sec)	X (m)	Y (m)	r object (m)	Vx	Vy	Acc	Acc-x	Acc-y
1126	11,140.00	-274,859.10	13,321,325.12	13,324,160.41	4,869.81	124.13	2.2476	0.0464	-2.2471
1127	11,150.00	-226,158.64	13,322,454.03	13,324,373.49	4,870.28	101.65	2.2475	0.0381	-2.2472
1128	11,160.00	-177,453.95	13,323,358.22	13,324,539.92	4,870.66	79.18	2.2475	0.0299	-2.2473
1129	11,170.00	-128,745.86	13,324,037.68	13,324,659.68	4,870.96	56.71	2.2474	0.0217	-2.2473
1130	11,180.00	-80,035.19	13,324,492.42	13,324,732.79	4,871.18	34.24	2.2474	0.0135	-2.2474
1131	11,190.00	-31,322.76	13,324,722.43	13,324,759.24	4,871.31	11.76	2.2474	0.0053	-2.2474
1132	11,200.00	17,390.61	13,324,727.70	13,324,739.05	4,871.36	-10.71	2.2474	-0.0029	-2.2474
1133	11,210.00	66,104.10	13,324,508.23	13,324,672.20	4,871.33	-33.18	2.2474	-0.0111	-2.2474
1134	11,220.00	114,816.89	13,324,064.02	13,324,558.72	4,871.22	-55.66	2.2474	-0.0194	-2.2474
1135	11,230.00	163,528.15	13,323,395.08	13,324,398.59	4,871.03	-78.13	2.2475	-0.0276	-2.2473
1136	11,240.00	212,237.06	13,322,501.40	13,324,191.84	4,870.75	-100.60	2.2476	-0.0358	-2.2473

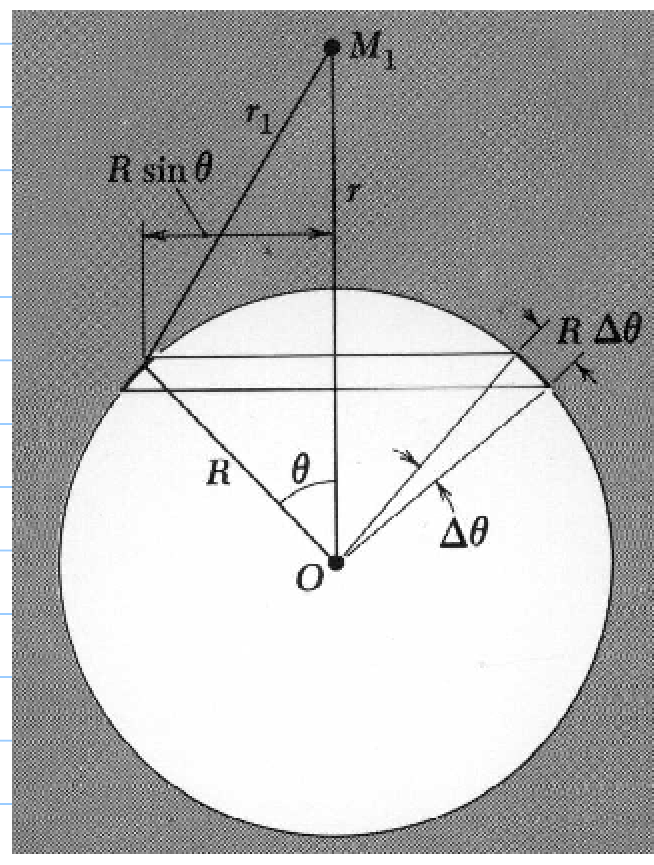
Around $t = 11,200$ sec, The x-coord becomes positive again, so period $\approx 1.12 \times 10^4$ sec.

The object does <u>not</u> hit The earth.

With a velocity of about 5500 ($= v_x$), The orbit is roughly circular.

Force between a Point Mass and Spherical Shell





In the figures above, let M_1 = point mass a distance r from the center of a shell of mass M. Let σ be the density of the shell (assume uniform density), R = radius of the shell.

$$\therefore \sigma = \frac{4\pi R^2}{M}$$

Consider a thin ring of the shell at position θ in the shell, and so of angular thickness $\Delta\theta$, and so of linear thickness $R\Delta\theta$. The thin ring has a radius of $R\sin\theta$, circumference $2\pi R\sin\theta$, and area of:

$$\text{Area of ring} \simeq (2\pi R\sin\theta)(R\Delta\theta)$$

The smaller the $\Delta\theta$, the closer the approximation.

$$\therefore \text{Mass of ring} \simeq (2\pi R^2 \sin\theta\, \Delta\theta)\sigma$$

The potential energy of M_1 in the field of the ring is the sum of the P.E. of each Δm of the ring, and, due to symmetry, is (as each Δm is a distance r_1 from M_1):

$$U_{ring} \approx -\frac{G M_1 (2\pi R^2 \sin\theta \, \Delta\theta)\, \sigma}{r_1} \qquad [1]$$

The approximation becomes exact as $\Delta\theta \to 0$.

From the law of cosines,

$$r_1^2 = R^2 + r^2 - 2Rr\cos\theta$$

As R and r are constant, and r_1 is a function of θ,

$$\frac{d}{d\theta} r_1^2(\theta) = \frac{d}{d\theta}\left(R^2 + r^2 - 2Rr\cos\theta\right)$$

$$2 r_1 \frac{dr_1}{d\theta} = 2Rr\sin\theta \text{, or}$$

$$\frac{dr_1}{d\theta} = \frac{Rr\sin\theta}{r_1}$$

$$\therefore \frac{\Delta r_1}{\Delta\theta} \approx \frac{Rr\sin\theta}{r_1}$$

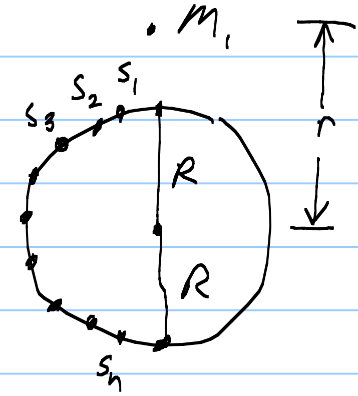
$$\therefore \Delta\theta \approx \frac{r_1 \Delta r_1}{Rr\sin\theta} \qquad [2]$$

Substituting $\Delta\theta$ from [2] into U_{ring} in [1],

$$U_{ring} \approx -\frac{GM_1 (2\pi R^2 \sin\theta)\sigma}{r_1} \cdot \frac{r_1 \Delta r_1}{Rr\sin\theta}$$

$$= -\frac{GM_1 (2\pi R \Delta r_1)\sigma}{r}$$

Thus, U_{ring} is a function of Δr_1, as all the other terms are constant.
Summing Δr_1 from $r-R$ ("north pole") to $r+R$ ("south pole"), which is shortest to longest distance, you get, where each s_i is a point on the semicircle,



$$\begin{aligned}\sum \Delta r_1 = &[s_1 - (r-R)] \\ &+ [s_2 - s_1] \\ &+ [s_3 - s_2] \\ &+ \quad \vdots \\ &+ [s_n - s_{n-1}] \\ &+ [(r+R) - s_n]\end{aligned}$$

$$= (r+R) - (r-R) = 2R$$

$$\therefore \sum U_{ring} \approx -\sum \frac{GM_1 (2\pi R\sigma)\Delta r_1}{r}$$

$$= -\frac{G m_1 2\pi R \sigma}{r} \sum \Delta r_1$$

$$= -\frac{G m_1 2\pi R \sigma (2R)}{r}$$

$$= -\frac{G m_1 4\pi R^2 \sigma}{r}$$

As mass of shell $m_s = 4\pi R^2 \sigma$,

$$\therefore \sum U_{ring} = -\frac{G m_1 m_s}{r}$$

As $\Delta r_1 \to 0$ ($n \to \infty$ in s_i), $\sum U_{ring} \to U_{shell}$

But $-\frac{G m_1 m_s}{r}$ contains all constants, so

$$U(r)_{shell} = -\frac{G m_1 m_s}{r}$$

Here, $r > R$ (from use of law of cosines).

Thus, P.E. of m_1 is same as if all mass were concentrated at the center of the shell.

And, $F_{gravity} = -\dfrac{dU}{dr} = \dfrac{GM_1 M_s}{r^2}$,

so force is same as if all mass concentrated at center of shell.

If $r < R$ (inside shell), analysis is the same, law of cosines is the same, but when sum U_{ring}, you sum from $R-r$ to $R+r$, so

$$\sum \Delta r_1 = [s_1 - (R-r)] + [s_2 - s_1] + \vdots + [(R+r) - s_n] = R + r - (R-r) = 2r$$

$$\therefore \sum U_{ring} \approx -\sum \frac{GM_1 (2\pi R\sigma)\Delta r_1}{r}$$

$$= -\frac{GM_1 (2\pi R\sigma)}{r} \sum \Delta r_1$$

$$= -G M_1 4\pi R\sigma$$

As $\Delta r_1 \to 0$, $U_{ring} \to U_{shell}$, so

$$U(r)_{shell} = -\frac{GM_1 4\pi R^2 \sigma}{R} = -\frac{GM_1 M_s}{R} \quad (r < R)$$

$\therefore$ $U(r)$ is constant anywhere within shell.

$$-\frac{dU(r)}{dr} = 0 = F_{gravity} \text{ within shell.}$$

For $r = R$, either analysis works, and summing Δr_1 from 0 to $2R$.

Summary:

$$U_{shell}(r) = \begin{cases} -\dfrac{G m_1 M_s}{r} & (r \geq R) \\[2ex] -\dfrac{G m_1 M_s}{R} & (r < R) \end{cases}$$

$$F(r) = -\frac{dU(r)}{dr} = \begin{cases} -\dfrac{G m_1 M_s}{r^2} & (r \geq R) \\[2ex] 0 & (r < R) \end{cases}$$

$U(r)$ is not differentiable at $r = R$, but the derivative exists as $r \to R^+$