Chapter 24 - Gauss's Law

Note Title 4/14/2008

51. A sphere of radius R surrounds a point charge Q, located at its center. (a) Show that the electric flux through a circular cap of half-angle θ (Fig. P24.53) is

$$\Phi_E = \frac{Q}{2\epsilon_0}(1 - \cos\theta)$$

What is the flux for (b) $\theta = 90°$ and (c) $\theta = 180°$?

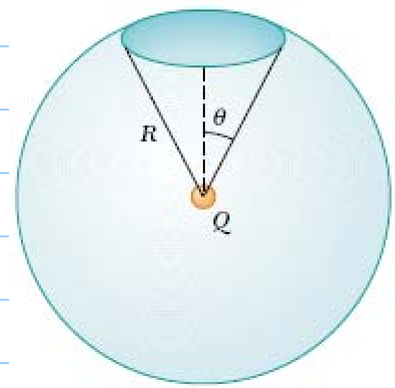
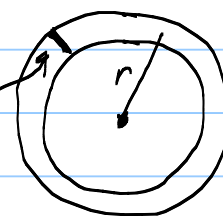


Figure P24.53

(a) To find the area of the cap, consider a thin circular rim of thickness $R\,d\theta$



The area of this strip is: $(R\,d\theta)(2\pi r)$

But $r = R\sin\theta$.

$\therefore$ Area of strip $= 2\pi R^2 \sin\theta\, d\theta$

$\therefore$ Area of cap $= \int_0^\theta 2\pi R^2 \sin\theta\, d\theta$

$= -2\pi R^2 \cos\theta \Big|_0^\theta$

$2\pi R^2(1 - \cos\theta)$

$$\therefore \text{Flux} = EA = \frac{k_e Q}{R^2} \cdot 2\pi R^2 (1-\cos\theta)$$

$$= 2\pi k_e Q (1-\cos\theta)$$

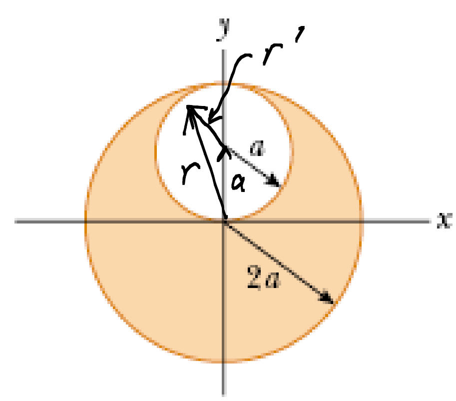
$$= 2\pi \left(\frac{1}{4\pi\epsilon_0}\right) Q (1-\cos\theta)$$

$$= \frac{Q}{2\epsilon_0}(1-\cos\theta)$$

(b) for $\theta = 90°$, $\text{Flux} = \frac{Q}{2\epsilon_0}$

(c) For $\theta = 180°$, $\text{Flux} = \frac{Q}{\epsilon_0}$

60. A sphere of radius $2a$ is made of a nonconducting material that has a uniform volume charge density ρ. (Assume that the material does not affect the electric field.) A spherical cavity of radius a is now removed from the sphere, as shown in Figure P24.64. Show that the electric field within the cavity is uniform and is given by $E_x = 0$ and $E_y = \rho a/3\epsilon_0$. (*Suggestion:* The field within the cavity is the superposition of the field due to the original uncut sphere, plus the field due to a sphere the size of the cavity with a uniform negative charge density $-\rho$.)



In The figure, let $\vec{r}\,'$ be The radial vector from the center of The cavity, and $\vec{r}$ from The center of The large sphere to a point inside the cavity. Let $\vec{a}$ be from the center of the large sphere to the center of the cavity.

$$\therefore \vec{r} = \vec{a} + \vec{r}\,'$$

From the large sphere:

$$\vec{E}^{+}(4\pi r^2) = \frac{4\pi r^3}{3\epsilon_0}\rho\,\hat{r}$$, $\hat{r}$ a unit vector parallel to $\vec{r}$

From the cavity:

$$\vec{E}^{-}[4\pi(r')^2] = -\frac{4\pi(r')^3}{3\epsilon_0}\rho\,\hat{r}\,'$$, $\hat{r}\,'$ a unit vector parallel to $\vec{r}\,'$

$$\vec{E} = \vec{E}^{+} + \vec{E}^{-}$$

$$= \frac{r\rho}{3\epsilon_0}\hat{r} - \frac{r'\rho}{3\epsilon_0}\hat{r}\,'$$

$$= \frac{\rho}{3\epsilon_0}\vec{r} - \frac{\rho}{3\epsilon_0}\vec{r}\,' = \frac{\rho}{3\epsilon_0}(\vec{r} - \vec{r}\,')$$

$$= \frac{\rho}{3\epsilon_0}\vec{a} = 0\hat{i} + \frac{\rho a}{3\epsilon_0}\hat{j}$$

$$\therefore \vec{E_x} = 0\hat{i}, \text{ and } \vec{E_y} = \frac{\rho a}{3\epsilon_0}\hat{j}$$

So, field uniform within cavity, directed along y-axis.

69. (a) Using the mathematical similarity between Coulomb's law and Newton's law of universal gravitation, show that Gauss's law for gravitation can be written as

$$\oint \mathbf{g} \cdot d\mathbf{A} = -4\pi G m_{\text{in}}$$

where m_{in} is the net mass inside the gaussian surface and $\mathbf{g} = \mathbf{F}_g/m$ represents the gravitational field at any point on the gaussian surface. (b) Determine the gravitational field at a distance r from the center of the Earth where $r < R_E$, assuming that the Earth's mass density is uniform.

(a) $\vec{g} = -\frac{Gm}{r^2}\hat{r}$

$$\therefore \oint \vec{g} \cdot d\vec{A} = -\frac{Gm}{r^2}(4\pi r^2) = -4\pi G m,$$

where m = mass inside a sphere of radius r.

(b) Let ρ = uniform density of earth.

$$\therefore \rho = \frac{M_e}{\frac{4}{3}\pi R_e^3}$$

$\therefore$ At a distance $r < R_e$ from center,

$$g(4\pi r^2) = -4\pi G m$$

$$= -4\pi G\left(\rho \tfrac{4}{3}\pi r^3\right)$$

$$\therefore g = -4\pi G\left(\frac{\rho}{3} r\right)$$

$$= -\frac{4}{3}\pi G r\left(\frac{M_e}{\frac{4}{3}\pi R_e^3}\right)$$

$$\therefore \underline{\underline{g = -\frac{G M_e}{R_e^3} r}}$$ (minus means directed to center of earth).