1.1 Vectors and Linear Combinations

Note Title 9/4/2006

6. Every combination of $v = (1,-2,1)$ and $w = (0,1,-1)$ has components that add to ____. Find c and d s.t. $cv + dw = (4,2,-6)$.

$$cv + dw = c\begin{bmatrix}1\\-2\\1\end{bmatrix} + d\begin{bmatrix}0\\1\\-1\end{bmatrix} = \begin{bmatrix}c\\-2c+d\\c-d\end{bmatrix}$$

(a) Adding the components: $c + (-2c+d) + (c-d)$
$= \underline{\underline{0}}$

(b) $\begin{bmatrix}c\\-2c+d\\c-d\end{bmatrix} = \begin{bmatrix}4\\2\\-6\end{bmatrix}$, $c = 4,\ d = 10$

9. If three corners of a parallelogram are $(1,1)$, $(4,2)$, and $(1,3)$, what are all the possible fourth corners?

Let $u = (1,1)$, $v = (4,2)$, $w = (1,3)$

Consider each point as the "origin" of an axis system. Use vectors from new "origin", and then add sides. Then add back origin

The possible sides are: $v-u \quad u-v \quad u-w$
$w-u \quad w-v \quad v-w$

$\therefore$ The possible 4th sides are: $(v-u)+(w-u) = v+w-2u$

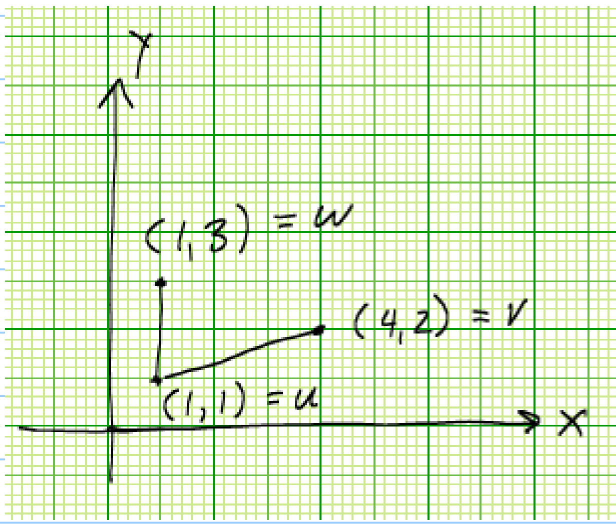
$(u-v)+(w-v) = u+w-2v$

$(u-w)+(v-w) = u+v-2w$

Adding back the origins: $v+w-u = (4,4)$

$u+w-v = (-2,2)$

$u+v-w = (4,0)$



Basicly, considering, alternately, two closed edges and finding the last point related to the open edge.

12. How many corners does a cube have in 4 dimensions? How many faces? How many edges? A typical corner is $(0,0,1,0)$.

(a) Corners: For each component, just 2 choices (0 or 1). There are four components. $\therefore 2^4$ possible expressions for (a,b,c,d). $\therefore$ $\underline{2^4 \text{ corners}}$.

(c) Edges: These are segments between corners. For any corner (or possible point

of an n-tuple), There are n "directions" to take. These are the segments between the chosen corner and adjacent corner. For example, let $f(x)$ be the binary operator s.t. $f(0)=1$ and $f(1)=0$. Given a corner (a, b, c, d), the adjacent corners are $(f(a), b, c, d)$, $(a, f(b), c, d)$, $(a, b, f(c), d)$ and $(a, b, c, f(d))$.
$\therefore$ for each point in n-space, n segments arise. $\therefore$ $n \cdot 2^n$ segments, but each gets counted twice, so $n \cdot 2^{n-1}$ edges. For $n=4$, $4 \cdot 2^3 = \underline{32 \text{ edges}}$

(6) Faces: What is a face? In R^3, it is a fixed 2-dimensional structure; i.e., given a corner, such as $(0,0,0)$, one such face is $(0, a_1, a_2)$, $0 \le a_i \le 1$. Another is $(1, a_1, a_2)$, $0 \le a_i \le 1$.
$\therefore$ 2 2-dimensional faces for each component. So for a cube (R^3), there are $2 \cdot 3 = 6$ "faces".
For R^n, there are $2n$ $n-1$ dimensional "faces". For $n=4$, there are $\underline{8}$ possible 3-dimensional faces (cubes).

If just want a structure with only 2 degrees of freedom relative to a corner, then need to consider, for example in $\mathbb{R}^4$, (a_1, x, a_3, y) where $a_i = 0$ or 1, and $0 \le x \le 1$, $0 \le y \le 1$.
For $a_1 = 0, 1$, $a_3 = 0, 1$, there are 2 choices for each a_i, so $2^2 = 4$ choices.

$\therefore$ Consider

(a_1, a_2, x, y) 4 choices
(a_1, x, y, a_4) "
(x, y, a_3, a_4) "
(a_1, x, a_3, y) "
(x, a_2, y, a_4) "
(x, a_3, a_4, y) "

$\therefore$ $6 \times 4 = \underline{\underline{24}}$ 2-dimensional faces

For $\mathbb{R}^n$, there are 2^{n-2} choices for a fixed position of x, y. Multiply this by the number of possible ordered positions of x, y in an n-tuple.
In general, $n-1$ positions for x, y
$n-2$ positions for x, a_i, y
$\vdots$
$n-j$ positions for $x, \overbrace{\ldots\ldots}^{j-1 \text{ components}}, y$

up to $j-1+2=n$, or $j=n-1$

$$\therefore \sum_{j=1}^{n-1} n-j = (n-1)+(n-2)+\cdots+(1)$$

$$= \frac{n(n-1)}{2}$$

$\therefore$ Number of 2-dimensional faces for R^n is $\frac{n(n-1)}{2} \cdot 2^{n-2}$

16. Consider $c\vec{v}+d\vec{w}$ with $c+d=1$.

$\vec{v}+a(\vec{w}-\vec{v}) = (1-a)\vec{v}+a\vec{w}$, and $1-a+a=1$.

$\vec{v}+a(\vec{w}-\vec{v})$, $a\in R$, is clearly a line through $\vec{v}$ parallel to $\vec{w}-\vec{v}$.

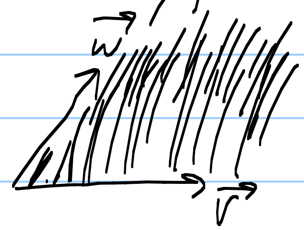
$\therefore$ with $c+d=1$, $d=1-c$, and $c=1-d$, so $c\vec{v}+d\vec{w}$, with $c+d=1$, is a line through $\vec{v}$ and $\vec{w}$ (i.e., the tips of the vectors).

18. Restricted by $0\le c\le 1$ and $0\le d\le 1$, shade in all combinations of $c\vec{v}+d\vec{w}$.

This is just like the perpendicular x-y axes, but with $\vec{v}$ and $\vec{w}$ at angles.

19. Restricted only by $c \geq 0$ and $d \geq 0$, draw the "cone" of all combinations $c\vec{v} + d\vec{w}$.

Basically, an angled x-y plane.



28. If (a, b) is a multiple of (c, d) with $abcd \neq 0$, show that (a, c) is a multiple of (b, d).

Pf: $abcd \neq 0$ means none of a, b, c, d is zero.

$$\therefore (a, b) = x(c, d)$$
$$a = xc$$
$$b = xd$$
$$\therefore \frac{a}{c} = \frac{b}{d} = x$$
$$\therefore a = \left(\frac{c}{d}\right) b$$

clearly $c = \left(\frac{c}{d}\right) d$

$\therefore (a, c) = \frac{c}{d}(b, d)$