1.2 Lengths and Dot Products

Note Title 9/10/2006

12. With $\vec{v}=(1,1)$ and $\vec{w}=(1,5)$, choose a number c s.t. $\vec{w}-c\vec{v}$ is perpendicular to $\vec{v}$. Then find the formula that gives this number c for any nonzero $\vec{v}$ and $\vec{w}$.

(a) $\vec{w}-c\vec{v}=(1-c,\ 5-c)$

$$\therefore (\vec{w}-c\vec{v})\cdot\vec{v}=(1-c,5-c)\cdot(1,1)=1-c+5-c$$
$$=6-2c=0,\quad c=3.$$

(b) Let $\vec{v}=(v_1,v_2)$, $\vec{w}=(w_1,w_2)$

$$\therefore (\vec{w}-c\vec{v})\cdot\vec{v}=(w_1-cv_1,\ w_2-cv_2)\cdot(v_1,v_2)$$
$$=v_1w_1-cv_1^2+v_2w_2-cv_2^2=0$$
$$\therefore v_1w_1+v_2w_2=c(v_1^2+v_2^2)$$
$$\therefore c=\frac{v_1w_1+v_2w_2}{v_1^2+v_2^2}=\frac{\vec{v}\cdot\vec{w}}{\vec{v}\cdot\vec{v}}$$

Cauchy-Schwarz Inequality by algebra

$$v_1 w_1 + \cdots + v_n w_n \le \sqrt{v_1^2 + \cdots + v_2^2}\sqrt{w_1^2 + \cdots + w_n^2}$$

Pf: Let $v = \sqrt{v_1^2 + \cdots v_n^2}$, $w = \sqrt{w_1^2 + \cdots w_n^2}$

Since, for all a, $0 \le a^2$, Then

$$0 \le \left(\frac{v_i}{v} - \frac{w_i}{w}\right)^2 \quad [v, w \ne 0]$$

$$\therefore\ 0 \le \left(\frac{v_1}{v} - \frac{w_1}{w}\right)\left(\frac{v_1}{v} - \frac{w_1}{w}\right) + \cdots + \left(\frac{v_n}{v} - \frac{w_n}{w}\right)\left(\frac{v_n}{v} - \frac{w_n}{w}\right)$$

$$\therefore 0 \le \frac{v_1^2}{v^2} + \frac{w_1^2}{w^2} - \frac{2 v_1 w_1}{vw} + \cdots + \frac{v_n^2}{v^2} + \frac{w_n^2}{w^2} - \frac{2 v_n w_n}{vw}$$

$$= \frac{v_1^2 + \cdots + v_n^2}{v^2} + \frac{w_1^2 + \cdots + w_n^2}{w^2} - \frac{2(v_1 w_1 + \cdots + v_n w_n)}{v\,w}$$

$$= 1 + 1 - \frac{2(v_1 w_1 + \cdots + v_n w_n)}{vw}$$

$$\therefore\ 2(v_1 w_1 + \cdots + v_n w_n) \le 2vw$$

$$\therefore\ v_1 w_1 + \cdots + v_n w_n \le \sqrt{v_1^2 + \cdots + v_n^2}\sqrt{w_1^2 + \cdots + w_n^2}$$