2.1 Vectors and Linear Equations

Note Title 9/18/2006

14. (a) A matrix with m rows and n columns multiplies a vector with $\underline{n}$ components to produce a vector with $\underline{m}$ components

(b) The planes from the m equations $Ax = b$ are in n-dimensional space. The combination of the columns of A is in $\underline{m}$-dimensional space.

16. (a) What is the 2 by 2 identity matrix?

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

(b) What is the 2 by 2 exchange matrix?

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = x\begin{bmatrix} 0 \\ 1 \end{bmatrix} + y\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$$

17. (a) What 2 by 2 matrix R rotates every vector by $90°$? $R\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ -x \end{bmatrix}$

$$R = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

(b) What 2 by 2 matrix rotates every vector by $180°$?

$$R \cdot R = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$$

18. Find the matrix P that multiplies (x, y, z) to give (y, z, x). Find the matrix Q that multiplies (y, z, x) to bring back (x, y, z).

(a) $$x \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + y \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} y \\ z \\ x \end{bmatrix}$$

$$\therefore\ P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

(b) $$y \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\therefore\ Q = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

22. What 2 by 2 matrix R rotates every vector through 45°? The vector $(1, 0)$ goes to $(\sqrt{2}/2, \sqrt{2}/2)$. The vector $(0, 1)$ goes to $(-\sqrt{2}/2, \sqrt{2}/2)$.

$$\begin{bmatrix} & \\ & \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix} \Rightarrow 1 \begin{bmatrix} \\ \end{bmatrix} + 0 \begin{bmatrix} \\ \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$$

$$\therefore \begin{bmatrix} & \\ & \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 & * \\ \sqrt{2}/2 & * \end{bmatrix}$$

$$\begin{bmatrix} \sqrt{2}/2 & * \\ \sqrt{2}/2 & * \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix} \Rightarrow 0 \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix} + 1 \begin{bmatrix} * \\ * \end{bmatrix} = \begin{bmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$$

$$\therefore \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$$

30. Start with the vector $u_0 = (1, 0)$. Multiply again and again by the same "Markov matrix" A below. The next three vectors are u_1, u_2, u_3:

$$u_1 = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} .8 \\ .2 \end{bmatrix}$$

$$u_2 = Au_1 = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \begin{bmatrix} .8 \\ .2 \end{bmatrix} = \begin{bmatrix} .7 \\ .3 \end{bmatrix}$$

$$u_3 = Au_2 = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \begin{bmatrix} .7 \\ .3 \end{bmatrix} = \begin{bmatrix} .65 \\ .35 \end{bmatrix}$$

(b). What properties do you notice for all four vectors u_0, u_1, u_2, u_3?

Their components all add up to 1.