2.4 Rules for Matrix Operations

Note Title 11/20/2006

5. Given $A = \begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}$,
compute A^2 and A^3, B^2 and B^3.
Make a prediction for A^n and B^n

(a) $A^2 = \begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 26 \\ 0 & 1 \end{bmatrix}$

$A^3 = \begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 26 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 36 \\ 0 & 1 \end{bmatrix}$

$B^2 = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 0 & 0 \end{bmatrix}$

$B^3 = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 4 & 4 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 8 & 8 \\ 0 & 0 \end{bmatrix}$

(b) $A^n = \begin{bmatrix} 1 & n6 \\ 0 & 1 \end{bmatrix}$ $\quad B^n = \begin{bmatrix} 2^n & 2^n \\ 0 & 0 \end{bmatrix}$

7. True or False:

(a) If columns 1 and 3 of B are the same, so are colums 1 and 3 of AB.

<u>True</u>. Column j of AB is A(column j of B).

(b) If rows 1 and 3 of B are the same, so are rows 1 and 3 of AB.

<u>False</u>. (Row j of A) B = row j of AB

(c) If rows 1 and 3 of A are the same, so are rows 1 and 3 of ABC.

<u>True</u>. $ABC = A(BC)$. (Row j of A) BC = row j of $A(BC)$.

(d) $(AB)^2 = ABAB \neq A^2B^2$ since $AB \neq BA$
$\therefore$ <u>False</u>

16. Prove $A(BC) = (AB)C$

Pf: Let A be $m \times n$
B be $n \times p$
C be $p \times q$

$\therefore$ BC is a $n \times q$ matrix, so $A(BC)$ is $m \times q$

AB is $m \times p$ matrix, so $(AB)C$ is $m \times q$

$\therefore$ show each element is the same; i.e.,

$[a(bc)]_{mq} = [(ab)c]_{mq}$

Consider $[a(bc)]_{mq}$. This is the result of $A_{row\,m} \cdot (BC)_{col\,q}$, or

$$\begin{bmatrix} \vdots & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \\ \vdots & & & \\ \vdots & & & \end{bmatrix} \begin{bmatrix} \vdots & \vdots & (bc)_{1q} & \vdots & \vdots \\ \vdots & \vdots & (bc)_{2q} & \vdots & \vdots \\ & & \vdots & & \\ & & (bc)_{nq} & & \end{bmatrix}$$

$\therefore [a(bc)]_{mq} = a_{m1}(bc)_{1q} + a_{m2}(bc)_{2q} + \dots + a_{mn}(bc)_{nq}$

Now, $(bc)_{ij} = B_{row\,i} \cdot C_{col\,j}$, so

$(bc)_{1q} = b_{11}c_{1q} + b_{12}c_{2q} + \dots + b_{1p}c_{pq}$

$(bc)_{2q} = b_{21}c_{1q} + b_{22}c_{2q} + \dots + b_{2p}c_{pq}$

$\vdots$

$(bc)_{nq} = b_{n1}c_{1q} + b_{n2}c_{2q} + \dots + b_{np}c_{pq}$

$$\therefore a_{m1}(bc)_{1q} = a_{m1}(b_{11}c_{1q} + b_{12}c_{2q} + \dots + b_{1p}c_{pq})$$

$$a_{m2}(bc)_{2q} = a_{m2}(b_{21}c_{1q} + b_{22}c_{2q} + \dots + b_{2p}c_{pq})$$

$$\vdots$$

$$a_{mn}(bc)_{nq} = a_{mn}(b_{n1}c_{1q} + b_{n2}c_{2q} + \dots + b_{np}c_{pq})$$

Now use assoc. law for addition and multiplication to rearrange terms (look at colums of c_{iq}).

$$\therefore [a(bc)]_{mq} = (a_{m1}b_{11} + a_{m2}b_{21} + \dots + a_{mn}b_{n1})c_{1q}$$

$$+ (a_{m1}b_{12} + a_{m2}b_{22} + \dots + a_{mn}b_{n2})c_{2q}$$

$$\vdots$$

$$+ (a_{m1}b_{1p} + a_{m2}b_{2p} + \dots + a_{mn}b_{np})c_{pq}$$

$$= (A_{\text{row } m} \cdot B_{\text{col } 1})c_{1q}$$

$$+ (A_{\text{row } m} \cdot B_{\text{col } 2})c_{2q}$$

$$\vdots$$

$$+ (A_{\text{row } m} \cdot B_{\text{col } p})c_{pq}$$

$$= (AB)_{\text{row } m} \cdot C_{\text{col } q} = [(ab)c]_{mq}$$

20. The entries of A are a_{ij}. Assuming that zeros don't appear, what is

(a) the first pivot? $\underline{a_{11}}$

(b) The multiplier ℓ_{31} of row 1 to be subtracted from row 3? $\underline{\dfrac{a_{31}}{a_{11}}}$

(c) the new entry that replaces a_{32} after that subtraction? $\underline{a_{32} - \ell_{31} a_{12}}$

(d) The second pivot? $\underline{a_{22} - \dfrac{a_{21}}{a_{11}} \cdot a_{12}}$

26. Multiply AB using columns times rows.

$$AB = \begin{bmatrix} 1 & 0 \\ 2 & 4 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 3 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \begin{bmatrix} 3 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 0 \\ 6 & 6 & 0 \\ 6 & 6 & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 4 & 8 & 4 \\ 1 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 3 & 0 \\ 10 & 14 & 4 \\ 7 & 8 & 1 \end{bmatrix}$$

27. The product of upper triangular matrices is always upper ~~t~~triangular:

$$AB = \begin{bmatrix} x & x & x \\ 0 & x & x \\ 0 & 0 & x \end{bmatrix} \begin{bmatrix} x & x & x \\ 0 & x & x \\ 0 & 0 & x \end{bmatrix} = \begin{bmatrix} & & \\ 0 & & \\ 0 & 0 & \end{bmatrix}$$

Row times column is dot product:

(Row 2 of A) · (colum 1 of B) = 0
(Row 3 of A) · (colum 1 of B) = 0
(Row 3 of A) · (colum 2 of B) = 0

Column times row is full matrix:

$$\begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} x & x & x \end{bmatrix} = \begin{bmatrix} x & x & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ x \\ 0 \end{bmatrix} \begin{bmatrix} 0 & x & x \end{bmatrix} = \begin{bmatrix} 0 & x & x \\ 0 & x & x \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ x \\ x \end{bmatrix} \begin{bmatrix} 0 & 0 & x \end{bmatrix} = \begin{bmatrix} 0 & 0 & x \\ 0 & 0 & x \\ 0 & 0 & x \end{bmatrix}$$

28. Draw the cuts in A (2×3) and B (3×4) and AB to show how each of the four multiplication rules is really a block multiplication:

(1) Matrix A times columns of B

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \left[\begin{array}{c|c|c|c} m & n & o & p \\ q & r & s & t \\ u & v & w & x \end{array}\right]$$

$$= \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} m' & n' & o' & p' \end{bmatrix}$$

(2) Rows of A times matrix B

$$\left[\begin{array}{ccc} a & b & c \\ \hline d & e & f \end{array}\right] \begin{bmatrix} m & n & o & p \\ q & r & s & t \\ u & v & w & x \end{bmatrix}$$

$$= \begin{bmatrix} a' \\ d' \end{bmatrix} \begin{bmatrix} m & n & o & p \\ q & r & s & t \\ u & v & w & x \end{bmatrix}$$

(3) Rows A times colums of B

$$\left[\begin{array}{ccc} a & b & c \\ \hline d & e & f \end{array}\right] \left[\begin{array}{c|c|c|c} m & n & o & p \\ q & r & s & t \\ u & v & w & x \end{array}\right]$$

$$= \begin{bmatrix} a' \\ d' \end{bmatrix} \begin{bmatrix} m' & n' & o' & p' \end{bmatrix}$$

(4) Columns of A times rows of B

$$\left[\begin{array}{c|c|c} a & b & c \\ d & e & f \end{array}\right] \left[\begin{array}{cccc} m & n & o & p \\ \hline q & r & s & t \\ \hline u & v & w & x \end{array}\right]$$

$$= \begin{bmatrix} a' & b' & c' \end{bmatrix} \begin{bmatrix} m' \\ q' \\ u' \end{bmatrix}$$

29. Draw cuts in A and x to multiply Ax as a column at a time: x_1 (column 1) + x_2 (column 2) + ...

$$Ax = \left[\begin{array}{c|c|c|c} & & & \cdots \end{array}\right] \left[\begin{array}{c} x_1 \\ \hline x_2 \\ \hline \vdots \end{array}\right] = x_1(Col_1) + x_2(Col_2) + \dots$$

34. If the three solutions in # 33 are $x_1 = (1,1,1)$, $x_2 = (0,1,1)$, and $x_3 = (0,0,1)$, then solve

$Ax = b$ when $b = (3, 5, 8)$.

From # 33, $A \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Then $a_{31} \cdot 0 + a_{32} \cdot 0 + a_{33} \cdot 1 = 1$, $\therefore a_{33} = 1$

$a_{31} \cdot 0 + a_{32} \cdot 1 + a_{33} \cdot 1 = 0$

or $a_{32} + 1 = 0$, $\therefore a_{32} = -1$

$a_{31} \cdot 1 + a_{32} \cdot 1 + a_{33} \cdot 1 = 0$,

or $a_{31} - 1 + 1 = 0$, so $a_{31} = 0$

$(A_{row2})(x_3) = a_{23} \cdot 1 = 0$, $a_{23} = 0$

$(A_{row2})(x_2) = a_{22} \cdot 1 + a_{23} \cdot 1 = a_{22} = 1$

$(A_{row3})(x_1) = a_{21} \cdot 1 + 1 \cdot 1 + 0 \cdot 1 = 0$, $a_{21} = -1$

$(A_{row1})(x_3) = a_{11} \cdot 0 + a_{12} \cdot 0 + a_{13} \cdot 1 = 0$, $a_{13} = 0$

$(A_{row1})(x_2) = a_{11} \cdot 0 + a_{12} \cdot 1 + a_{13} \cdot 1 = 0$, $a_{12} = 0$

$(A_{row1})(x_1) = a_{11} + a_{12} + a_{13} = 1$, $a_{11} = 1$

$$\therefore A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\therefore Ax = \begin{bmatrix} 3 \\ 5 \\ 8 \end{bmatrix}, \text{ or } \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 8 \end{bmatrix}$$

$\therefore x_1 = 3$

$-x_1 + x_2 = 5, \quad x_2 = 8$

$-x_2 + x_3 = 8, \quad x_3 = 16$

$$\therefore x = \begin{bmatrix} 3 \\ 8 \\ 16 \end{bmatrix}$$