2.5 Inverse Matrices

Note Title 12/6/2006

1. Find the inverses of A, B, C:

$$A = \begin{bmatrix} 0 & 3 \\ 4 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 0 \\ 4 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix}$$

$$A^{-1} = \frac{1}{-12}\begin{bmatrix} 0 & -3 \\ -4 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{4} \\ \frac{1}{3} & 0 \end{bmatrix}$$

$$B^{-1} = \frac{1}{4}\begin{bmatrix} 2 & 0 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ -1 & \frac{1}{2} \end{bmatrix}$$

$$C^{-1} = \frac{1}{1}\begin{bmatrix} 7 & -4 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} 7 & -4 \\ -5 & 3 \end{bmatrix}$$

2. For these "permutation matrices", find P^{-1}:

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

(a) For $P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, row 3 → row 1, row 2 → row 2, row 1 → row 3

∴ Flip rows 1 & 3 again.

$$\therefore P^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

(b) For $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, row 2 $\rightarrow$ row 1
row 3 $\rightarrow$ row 2
row 1 $\rightarrow$ row 3

$\therefore$ want row 1 $\rightarrow$ row 2
row 2 $\rightarrow$ row 3
row 3 $\rightarrow$ row 1

$\therefore P^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

7. If A has row 1 + row 2 = row 3, show That A is not invertible:

(a) Explain why $Ax = (1,0,0)$ cannot have a solution

$$\left[\begin{array}{ccc|c} a & a' & a'' & 1 \\ b & b' & b'' & 0 \\ a+b & a'+b' & a''+b'' & 0 \end{array}\right] \xrightarrow[\text{row 2 from 3}]{\text{subtract}} \left[\begin{array}{ccc|c} a & a' & a'' & 1 \\ b & b' & b'' & 0 \\ a & a' & a'' & 0 \end{array}\right]$$

$$\therefore \left.\begin{array}{c} a x_1 + a' x_2 + a'' x_3 = 1 \\ \text{and} \\ a x_1 + a' x_2 + a'' x_3 = 0 \end{array}\right\} \text{ or } 1 = 0$$

(b) Which right sides (b_1, b_2, b_3) might allow a solution to $Ax = b$.

$b_3 = b_1 + b_2$, just like $\text{row}_1 + \text{row}_2 = \text{row}_3$.

(c) What happens to row 3 in elimination?

From (a), row 3 will become all zeros, so there will be no third pivot.

8. If A has column 1 + column 2 = column 3, show that A is not invertible:

(a) Find a nonzero solution x to $Ax = 0$. The matrix is 3×3.

$x = (1, 1, -1)$ will yield $Ax = 0$ since

$$\begin{bmatrix} A_1 \end{bmatrix} \cdot 1 + \begin{bmatrix} A_2 \end{bmatrix} \cdot 1 + \begin{bmatrix} A_3 \end{bmatrix} \cdot (-1) =$$

$$\begin{bmatrix} A_1 \end{bmatrix} + \begin{bmatrix} A_2 \end{bmatrix} + \left(\begin{bmatrix} A_1 \end{bmatrix} + \begin{bmatrix} A_2 \end{bmatrix} \right)(-1) = 0$$

(b) Elimination keeps $col_1 + col_2 = col_3$. Explain why there is no third pivot.

For the last row, if there were a third pivot, then $0_{col_1} + 0_{col_2} = 3rd\ pivot_{col_3}$, but

$0_{col_1} + 0_{col_2} = 0_{col_3} \Rightarrow 0 = 3\text{rd pivot}.$

Note: elimination keeps $col_1 + col_2 = col_3$

consider $\begin{bmatrix} a & b & c \\ x & y & z \end{bmatrix}$ $\quad a + b = c$, $\quad x + y = z$

consider $r(row_i) + row_j$

$\therefore \begin{bmatrix} a & b & c \\ x+ra & y+rb & z+rc \end{bmatrix}$

Does $x + ra + y + rb \stackrel{?}{=} z + rc$
$= x + y + r(a+b)$
$= x + ra + y + rb$,
so yes.

9. Suppose A is invertible and you exchange its first two rows to reach B. Is the new matrix B invertible and how do you find B^{-1} from A^{-1}?

Let $B = P_{12} A$. P_{12}^{-1} exists, so $B^{-1} = (P_{12} A)^{-1}$

$= A^{-1} P_{12}^{-1}$

$$P_{12} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ & & \vdots & & \end{bmatrix}$$

Since $P_{12} \cdot P_{12} = I$, then $P_{12}^{-1} = P_{12}$

$\therefore$ $B^{-1} = A^{-1} P_{12}^{-1} = A^{-1} P_{12}$

This effectively swaps col_1 & col_2 of A^{-1}.

10. Find the inverses of

$$A = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 3 & 0 \\ 0 & 4 & 0 & 0 \\ 5 & 0 & 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & 2 & 0 & 0 \\ 4 & 3 & 0 & 0 \\ 0 & 0 & 6 & 5 \\ 0 & 0 & 7 & 6 \end{bmatrix}$$

Note $\begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{5} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$

$\therefore$ From #9 above, flip rows 1 & 4 $\Rightarrow$ flip cols 1 & 4

$$\therefore A^{-1} = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{5} \\ \\ \frac{1}{2} & 0 & 0 & 0 \end{bmatrix}$$ and flip rows 2 & 3 $\Rightarrow$ flip cols 2 & 3.

$$\therefore A^{-1} = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{5} \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \end{bmatrix}$$

For B^{-1}, invert blocks $\begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix}$

and $\begin{bmatrix} 6 & 5 \\ 7 & 6 \end{bmatrix}^{-1} = \begin{bmatrix} 6 & -5 \\ -7 & 6 \end{bmatrix}$

$$\therefore B^{-1} = \begin{bmatrix} 3 & -2 & 0 & 0 \\ -4 & 3 & 0 & 0 \\ 0 & 0 & 6 & -5 \\ 0 & 0 & -7 & 6 \end{bmatrix}$$

---

since if $B = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}$, then

$$\begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \begin{bmatrix} B_1^{-1} & 0 \\ 0 & B_2^{-1} \end{bmatrix} = \begin{bmatrix} B_1 B_1^{-1} & 0 \\ 0 & B_2 B_2^{-1} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

18. If B is the inverse of A^2, show AB is the inverse of A.

Pf: $A^2B = I \quad \therefore A(AB) = I$

Reasoning on p. 76 showed that if a right inverse exists ($AC = I$), then A must have n pivots, and $\therefore$ A has a right and left inverse which are the same.

$\therefore (AB)A = I$, so $A^{-1} = AB$

21. There are sixteen 2 by 2 matrices whose entries are 1's and 0's. How many of them are invertible?

A 2x2 is invertible when its determinant is nonzero: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow ad - bc \neq 0$.

First consider $ad = bc = 0$.

When a is zero, 3 choices for b, c: $b=0$, $c=0$, and $b=c=0$.

When d is zero, same 3 choices will give zero.

When $a=d=0$, same 3 choices for b, c will give zero.

$\therefore$ so far, 9 possibilities $\Rightarrow ad-bc=0$.

Now consider $ad = bc = 1$. All entries must be 1.

$\therefore$ Only ten possibilities $\Rightarrow ad-bc=0$.

$\therefore$ $16-10=6$ possibilities for A^{-1}.

23. Follow the 3 by 3 text example but with plus signs in A. Eliminate above and below the pivots to reduce $[\,A \quad I\,]$ to $[\,I \quad A^{-1}\,]$:

$$[\,A \quad I\,] = \begin{bmatrix} 2 & 1 & 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{bmatrix}.$$

$$\begin{bmatrix} 2 & 1 & 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & 1 & -\frac{1}{2} & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & 1 & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 4/3 & \frac{1}{3} & -\frac{2}{3} & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & 0 & -\frac{3}{4} & \frac{3}{2} & -\frac{3}{4} \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & -\frac{2}{3} & 1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 2 & 0 & 0 & \frac{3}{2} & -1 & \frac{1}{2} \\ 0 & \frac{3}{2} & 0 & -\frac{3}{4} & \frac{3}{2} & -\frac{3}{4} \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & -\frac{2}{3} & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & \frac{3}{4} & -\frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{4} & -\frac{1}{2} & \frac{3}{4} \end{bmatrix}$$

For which three numbers c is this matrix not invertible, and why not?

30.

$$A = \begin{bmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{bmatrix}.$$

$c = 0$ gives a zero row,
$c = 2$ gives two identical rows ($\therefore$ won't get 3 pivots)
$c = 7$ gives two identical columns ($\therefore$ a nonzero vector, $x = (0, 1, -1)$, gives $Ax = 0$).

36. If an invertible matrix A commutes with C (this means $AC = CA$) show that A^{-1} commutes with C. If also B commutes with C, show that AB commutes with C. Translation: If $AC = CA$ and $BC = CB$ then $(AB)C = C(AB)$.

(a) $AC = CA \Rightarrow (AC)A^{-1} = (CA)A^{-1}$

$\Rightarrow A(CA^{-1}) = C(AA^{-1}) = C$

$\therefore A^{-1}[A(CA^{-1})] = A^{-1}C$

$\therefore [A^{-1}A](CA^{-1}) = A^{-1}C$,

or $I(CA^{-1}) = CA^{-1} = A^{-1}C$

(b) $BC = CB$, $\therefore A(BC) = A(CB)$, or

$$(AB)C = A(CB) \Rightarrow (AB)C = (AC)B$$
$$\Rightarrow (AB)C = (CA)B$$
$$\Rightarrow (AB)C = C(AB)$$

42 If $AC = I$ and $AC^* = I$ (all square matrices) use **2I** to prove that $C = C^*$.

If $AC = I$, then A has n pivots, so that A^{-1} exists. $\therefore$ $A^{-1}(AC) = A^{-1}I$, or $(A^{-1}A)C = IC = C = A^{-1}$.

Similarly, $C^* = A^{-1}$.

$\therefore$ $C^* = C$.