2.6 Elimination = Factorization : A = LU

Note Title 1/22/2007

5. What matrix E puts A into triangular form $EA = U$? Multiply by $E^{-1} = L$ to factor A into LU:

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 6 & 3 & 5 \end{bmatrix}.$$

$l_{21} = 0, \quad l_{31} = 3, \quad l_{32} = 0$

$\therefore E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \quad \therefore EA = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{bmatrix}$

$\therefore E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \quad \therefore A = E^{-1}U$, so

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{bmatrix}$$

6. What two elimination matrices E_{21} and E_{32} put A into upper triangular form $E_{32}E_{21}A = U$? Multiply by E_{32}^{-1} and E_{21}^{-1} to factor A into $LU = E_{21}^{-1}E_{32}^{-1}U$:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 5 \\ 0 & 4 & 0 \end{bmatrix}.$$

$l_{21} = 2, \quad \therefore E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \therefore l_{32} = 2, \quad E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

$\therefore U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & -6 \end{bmatrix} \quad \therefore A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & -6 \end{bmatrix}$

8.

Suppose A is already lower triangular with 1's on the diagonal. Then $U = I$!

$$A = L = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}.$$

The elimination matrices E_{21}, E_{31}, E_{32} contain $-a$ then $-b$ then $-c$.

(a) Multiply $E_{32}E_{31}E_{21}$ to find the single matrix E that produces $EA = I$.

(b) Multiply $E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}$ to bring back L (nicer than E).

(a) $$E = E_{32}E_{31}E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -c & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -b & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -c & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ -b & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ ac-b & -c & 1 \end{bmatrix}$$

(b) $$E_{21}^{-1}E_{31}^{-1}E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ b & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ b & c & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}$$

11. What are L and D for this matrix A? What is U in $A = LU$ and what is the new U in $A = LDU$?

$$A = \begin{bmatrix} 2 & 4 & 8 \\ 0 & 3 & 9 \\ 0 & 0 & 7 \end{bmatrix}.$$

A is already upper triangular $= U$

$$\therefore DD^{-1}A = \begin{bmatrix} 2 & & \\ & 3 & \\ & & 7 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & & \\ & \frac{1}{3} & \\ & & \frac{1}{7} \end{bmatrix} A$$

$$= \begin{bmatrix} 2 & & \\ & 3 & \\ & & 7 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$L = I$, so $A = I \begin{bmatrix} 2 & & \\ & 3 & \\ & & 7 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$

13. (Recommended) Compute L and U for the symmetric matrix

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}.$$

Find four conditions on a, b, c, d to get $A = LU$ with four pivots.

$$E_{41}E_{31}E_{21}A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} A = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & b-a & c-a & c-a \\ 0 & b-a & c-a & d-a \end{bmatrix}$$

Let $E_1 = E_{41}E_{31}E_{21}$

$$\therefore E_{42}E_{32}(E_1)A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & b-a & c-a & c-a \\ 0 & b-a & c-a & d-a \end{bmatrix}$$

$$= \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & c-b & d-b \end{bmatrix}$$

Let $E_2 = E_{42}E_{32}E_1$

$$\therefore E_{43}(E_2)(A) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & c-b & d-b \end{bmatrix}$$

$$= \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix} = \underline{\underline{U}}$$

$$L = (E_{43}E_{42}E_{32}E_{41}E_{31}E_{21})^{-1} = E_{21}^{-1}E_{31}^{-1}E_{41}^{-1}E_{32}^{-1}E_{42}^{-1}E_{43}^{-1}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}}}$$

The conditions, from U, must be $a \neq 0$, $b \neq a$, $c \neq b$, $d \neq c$

15 Solve the triangular system $Lc = b$ to find c. Then solve $Ux = c$ to find x:

$$L = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 2 \\ 11 \end{bmatrix}.$$

For safety find $A = LU$ and solve $Ax = b$ as usual. Circle c when you see it.

$$\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 11 \end{bmatrix} \Rightarrow c_1 = 2, \; c_2 = 3$$

$$\begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \Rightarrow x_2 = 3, \; x_1 = -5$$

$$A = LU = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 8 & 17 \end{bmatrix}$$

$$\therefore Ax = b : \quad \begin{bmatrix} 2 & 4 \\ 8 & 17 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 11 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 \\ 3 \end{bmatrix}}_{c} \Rightarrow x_2 = 3, \; x_1 = -5$$

17. (a) When you apply the usual elimination steps to L, what matrix do you reach?

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}.$$

(b) When you apply the same steps to I, what matrix do you get?

(c) When you apply the same steps to LU, what matrix do you get?

(a) For this L, $E_{21}L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$,

$E_{31}E_{21}L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & l_{32} & 1 \end{bmatrix}$, $E_{32}E_{21}E_{21}L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \underline{I}$

(b) Since $LL^{-1} = I$, then

$(E_{32}E_{31}E_{21}L)L^{-1} = E_{32}E_{31}E_{21}I$

$\therefore IL^{-1} = \underline{L^{-1}} = E_{32}E_{31}E_{21}I$

(c) Since $E_{32}E_{31}E_{21}L = I$, then $E_{32}E_{31}E_{21}LU = \underline{U}$

18. If $A = LDU$ and also $A = L_1D_1U_1$ with all factors invertible, then $L = L_1$ and $D = D_1$ and $U = U_1$. *"The factors are unique."*

Derive the equation $L_1^{-1}LD = D_1U_1U^{-1}$. Are the two sides triangular or diagonal? Deduce $L = L_1$ and $U = U_1$ (they all have diagonal 1's). Then $D = D_1$.

(a) $LDU = L_1D_1U_1$

$\therefore L_1^{-1}LDU = L_1^{-1}L_1D_1U_1 = D_1U_1$

$\therefore L_1^{-1} L D U U^{-1} = D_1 U_1 U^{-1}$

$\therefore L_1^{-1} L D = D_1 U_1 U^{-1}$

(b) The inverse of a lower triangular matrix is lower triangular (see #17).
$\therefore L_1^{-1}$ is lower triangular.
A lower triangular x a lower triangular = a lower triangular matrix.
$\therefore L_1^{-1} L$ is lower triangular
A lower triangular x D is a lower triangular

$\therefore L_1^{-1} L D$ is lower triangular

Inverse of an upper triangular must be upper triangular: consider columns of U^{-1} when calculating $U^{-1}U = I$, or $U^{-1}[u_1\ u_2 \dots u_n] = [e_1\ e_2 \dots e_n]$
Also, an upper triangular times an upper triangular = upper triangular

$\therefore U_1 U^{-1}$ is upper triangular.

And, (D)(upper triangular) = upper

triangular.

$\therefore D_1 U_1 U^{-1}$ is upper triangular.

$\therefore L_1^{-1} L D = D_1 U_1 U^{-1}$ means upper = lower, so both sides must be diagonal.

(c) L, U, L_1, U_1 all have "1's" on their diagonal. So does L^{-1} (consider elimination matrices to get L^{-1}). So does U^{-1} (consider $U[u_1 \dots u_n] = [e_1 \dots e_n]$).

$\therefore L_1^{-1} L$ has "1's" on its diagonal and so does $U_1 U^{-1}$.

$\therefore$ Since $L_1^{-1} L D$ is diagonal, then $L_1^{-1} L D = D$.

Similarly, $D_1 U_1 U^{-1} = D_1$

$\therefore D = D_1$

Also, $(L_1^{-1} L) D = D$, so $L_1^{-1} L = I$, so $L = L_1$

And since $D_1 (U_1 U^{-1}) = D_1$, then $U_1 U^{-1} = I$, so $U_1 = U$.