2.7 Transposes and Permutations

1. Find A^{T} and A^{-1} and $(A^{-1})^{\mathrm{T}}$ and $(A^{\mathrm{T}})^{-1}$ for

$$A = \begin{bmatrix} 1 & 0 \\ 9 & 3 \end{bmatrix} \quad \text{and also} \quad A = \begin{bmatrix} 1 & c \\ c & 0 \end{bmatrix}.$$

(a) $A^T = \underline{\begin{bmatrix} 1 & 9 \\ 0 & 3 \end{bmatrix}}$

For A^{-1}, $E_{21} = \begin{bmatrix} 1 & 0 \\ -9 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 \\ 0 & 1/3 \end{bmatrix}$

$\therefore A^{-1} = D E_{21} = \underline{\begin{bmatrix} 1 & 0 \\ -3 & 1/3 \end{bmatrix}}$

$\therefore (A^{-1})^T = \begin{bmatrix} 1 & -3 \\ 0 & 1/3 \end{bmatrix}$

For $(A^T)^{-1}$, $D^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1/3 \end{bmatrix}$, $E_{12} = \begin{bmatrix} 1 & -9 \\ 0 & 1 \end{bmatrix}$

$\therefore (A^T)^{-1} = E_{12} D^{-1} = \begin{bmatrix} 1 & -3 \\ 0 & 1/3 \end{bmatrix}$ $\therefore (A^{-1})^T = (A^T)^{-1}$

(b) $A^T = \underline{\begin{bmatrix} 1 & c \\ c & 0 \end{bmatrix}}$

For A^{-1}, $E_{21} = \begin{bmatrix} 1 & 0 \\ -c & 1 \end{bmatrix} \Rightarrow E_{21} A = \begin{bmatrix} 1 & c \\ 0 & -c^2 \end{bmatrix}$

$\therefore E_{12} = \begin{bmatrix} 1 & 1/c \\ 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & -c^2 \end{bmatrix}$

$\therefore D = \begin{bmatrix} 1 & 0 \\ 0 & -1/c^2 \end{bmatrix}$

$\therefore D E_{12} E_{21} = \underline{\begin{bmatrix} 0 & 1/c \\ 1/c & -1/c^2 \end{bmatrix}} = A^{-1}$

$$\therefore (A^{-1})^T = \begin{bmatrix} 0 & 1/c \\ 1/c & -1/c^2 \end{bmatrix}$$

For $(A^T)^{-1}$, $E_{21} = \begin{bmatrix} 1 & 0 \\ -c & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & c \\ 0 & -c^2 \end{bmatrix}$

$E_{12} = \begin{bmatrix} 1 & 1/c \\ 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & -c^2 \end{bmatrix} \quad \therefore D = \begin{bmatrix} 1 & 0 \\ 0 & -1/c^2 \end{bmatrix}$

$\therefore D\, E_{12} E_{21} = D \begin{bmatrix} 0 & 1/c \\ -c & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1/c \\ 1/c & -1/c^2 \end{bmatrix}$

$\therefore (A^{-1})^T = (A^T)^{-1}$

3. (a) The matrix $((AB)^{-1})^{\mathrm{T}}$ comes from $(A^{-1})^{\mathrm{T}}$ and $(B^{-1})^{\mathrm{T}}$. *In what order?*

(b) If U is upper triangular then $(U^{-1})^{\mathrm{T}}$ is _____ triangular.

(a) $((AB)^{-1})^T = (B^{-1}A^{-1})^T = (A^{-1})^T (B^{-1})^T$

(b) $(U^{-1})^T = (U^T)^{-1}$. U^T is lower triangular.

$EU^T = I$, and E is lower triangular.

$E = (U^T)^{-1}$, so $(U^{-1})^T$ is lower triangular.

7. True or false:

(a) The block matrix $\begin{bmatrix} \mathbf{0} & \mathbf{A} \\ \mathbf{A} & \mathbf{0} \end{bmatrix}$ is automatically symmetric.

(b) If A and B are symmetric then their product AB is symmetric.

(c) If A is not symmetric then A^{-1} is not symmetric.

(d) When A, B, C are symmetric, the transpose of ABC is CBA.

(a) False, unless $A = A^T$

(b) $A = A^T$, $B = B^T$. Does $AB \stackrel{?}{=} (AB)^T$

$(AB)^T = B^T A^T = BA$. There's no guarantee $AB = BA$, so False.

(c) If A^{-1} is symmetric, then $(A^{-1})^T = A^{-1}$.

But $(A^{-1})^T = (A^T)^{-1}$. $\therefore$ $(A^T)^{-1} = A^{-1}$,

so $[(A^T)^{-1}]^{-1} = (A^{-1})^{-1} \Rightarrow A^T = A$.

$\therefore$ A^{-1} is not symmetric: True

(d) $(ABC)^T = C^T B^T A^T = CBA$. $\therefore$ True

14. If you take powers of a permutation matrix, why is some P^k eventually equal to I?

There are a finite number of permutation matrices for an $n \times n$ array: $n!$

Let P be one such permutation matrix.

If $P^r = P^s$, $r \neq s$, $r, s \leq n!$, then if $s<r$, $P^{r-s} \cdot P^s = P^r$, so $P^{r-s} \cdot P^s = P^s$, and $\therefore$ $P^{r-s} = I$

If $P^1, P^2, \ldots, P^{n!}$ are all unique, then $P^{n!+1} = P^j$, some $1 \leq j \leq n!$

$\therefore$ $1 \leq n!+1-j \leq n!$, and $P^{n!+1-j} = I$.

16. If $A = A^{\mathrm{T}}$ and $B = B^{\mathrm{T}}$, which of these matrices are certainly symmetric?

(a) $A^2 - B^2$

(b) $(A+B)(A-B)$

(c) ABA

(d) $ABAB$.

(a) $(A^2 - B^2)^T = (A^2)^T - (B^2)^T = (A \cdot A)^T - (B \cdot B)^T$

$= A^T \cdot A^T - B^T \cdot B^T = A^2 - B^2$

$\therefore$ symmetric

(b) $[(A+B)(A-B)]^T = [A^2 + BA - AB - B^2]^T$

$= (A^2)^T + (BA)^T - (AB)^T - (B^2)^T$

$= A^2 + A^T B^T - B^T A^T - B^2$

$= A^2 + AB - BA - B^2$

$= A(A+B) - B(A+B)$

$= (A-B)(A+B)$

$\therefore$ not symmetric

(c) $(ABA)^T = A^T B^T A^T = ABA$. $\therefore$ symmetric

(d) $(ABAB)^T = B^T A^T B^T A^T = BABA$

$\therefore$ not symmetric

20. Factor these symmetric matrices into $A = LDL^{\mathrm{T}}$. The pivot matrix D is diagonal:

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & b \\ b & c \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

(a) $\begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & -7 \end{bmatrix}$

$\rightarrow \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -7 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & b \\ b & c \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}\begin{bmatrix} 1 & b \\ 0 & c-b^2 \end{bmatrix}$

$\rightarrow \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & c-b^2 \end{bmatrix}\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 & 0 \\ 0 & 3/2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

$\rightarrow \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & -2/3 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 & 0 \\ 0 & 3/2 & -1 \\ 0 & 0 & 4/3 \end{bmatrix}$

$\rightarrow \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & -2/3 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3/2 & 0 \\ 0 & 0 & 4/3 \end{bmatrix}\begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -2/3 \\ 0 & 0 & 1 \end{bmatrix}$