3.2 The Nullspace of A: Solving AX=0

Note Title 4/6/2007

1. Reduce these matrices to their ordinary echelon forms U:

(a) $A = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 6 & 9 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix}$ (b) $B = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 8 & 8 \end{bmatrix}$.

Which are the free variables and which are the pivot variables?

(a) $\begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 6 & 9 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Free variables: x_2, x_4, x_5
Pivot variables: x_1, x_3

(b) $\begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 8 & 8 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \end{bmatrix}$

Free: x_3 Pivot: x_1, x_2

2. For the matrices in Problem 1, find a special solution for each free variable. (Set the free variable to 1. Set the other free variables to zero.)

(a) $x_2 = 1$, $x_4 = 0$, $x_5 = 0$ $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ $x_2 = 0$, $x_4 = 1$, $x_5 = 0$ $\begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}$ $x_2 = 0$, $x_4 = 0$, $x_5 = 1$ $\begin{bmatrix} 0 \\ 0 \\ -3 \\ 0 \\ 1 \end{bmatrix}$

(b) $x_3 = 1$ $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

3. By combining the special solutions in Problem 2, describe every solution to $Ax = \mathbf{0}$ and $Bx = \mathbf{0}$. The nullspace contains only $\boldsymbol{x} = \mathbf{0}$ when there are no _____.

$Ax = 0$:

$$x_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ -3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2x_1 \\ x_1 \\ -2x_2 - 3x_3 \\ x_2 \\ x_3 \end{bmatrix}$$

$Bx = 0$:

$$x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} x_3 \\ -x_3 \\ x_3 \end{bmatrix}$$

The nullspace contains only $x = 0$ when there are no free variables.

4. By further row operations on each U in Problem 1, find the reduced echelon form R. *True or false*: The nullspace of R equals the nullspace of U.

(a) $$\begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = R$$

$$\begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 0 & -2 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = R$$

(b) True.

5. By row operations reduce each matrix to its echelon form U. Write down a 2 by 2 lower triangular L such that $B = LU$.

(a) $A = \begin{bmatrix} -1 & 3 & 5 \\ -2 & 6 & 10 \end{bmatrix}$ (b) $B = \begin{bmatrix} -1 & 3 & 5 \\ -2 & 6 & 7 \end{bmatrix}$.

$$\begin{bmatrix} -1 & 3 & 5 \\ -2 & 6 & 10 \end{bmatrix} \longrightarrow \begin{bmatrix} -1 & 3 & 5 \\ 0 & 0 & 0 \end{bmatrix} = U$$

$$\begin{bmatrix} -1 & 3 & 5 \\ -2 & 6 & 7 \end{bmatrix} \longrightarrow \begin{bmatrix} -1 & 3 & 5 \\ 0 & 0 & -3 \end{bmatrix} = U$$

2 was the factor for each reduction, so

$$E_1 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$\therefore \quad A = E_1 U, \quad B = E_1 U$

6. Find the special solutions to $Ax = \mathbf{0}$ and $Bx = \mathbf{0}$. For an m by n matrix, the number of pivot variables plus the number of free variables is ______.

$$\begin{bmatrix} -1 & 3 & 5 \\ 0 & 0 & 0 \end{bmatrix}; \quad \text{free}: x_2, x_3 \quad \therefore \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 3 & 5 \\ 0 & 0 & -3 \end{bmatrix} \quad \text{free}: x_2 \quad \therefore \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

For $m \times n$, pivot + free = $\underline{n}$

7. In Problem 5, describe the nullspaces of A and B in two ways. Give the equations for the plane or the line, and give all vectors x that satisfy those equations as combinations of the special solutions.

$$A: \begin{bmatrix} -1 & 3 & 5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = -x_1 + 3x_2 + 5x_3 = 0$$

= equation of plane

$$x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3x_2 + 5x_3 \\ x_2 \\ x_3 \end{bmatrix}$$ all vectors of this form

$$B: \begin{bmatrix} -1 & 3 & 5 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \rightarrow \begin{aligned} -x_1 + 3x_2 + 5x_3 &= 0 \\ -3x_3 &= 0 \end{aligned}$$

$\therefore$ line equation: $-x_1 + 3x_2 = 0$

$$x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3x_2 \\ x_2 \\ 0 \end{bmatrix}$$ all vectors of this form.

14. Suppose the first and last columns of a 3 by 5 matrix are the same (not zero). Then _____ is a free variable. Find the special solution for this variable.

$$\begin{bmatrix} x_1 & & x_1 \\ y_1 & & y_1 \\ z_1 & & z_1 \end{bmatrix} \xrightarrow{\text{elimination}} \begin{bmatrix} x_1 & & x_1 \\ 0 & & 0 \\ 0 & & 0 \end{bmatrix}$$

$\therefore$ x_5 $(=x_1)$ is a free variable.

$\therefore$ set $x_5 = 1$, other free to zero, get $x_1 = -1$.

$\therefore$ $(-1, 0, 0, 0, 1)$

Another way to see this is to look at the columns:

$$x_1 \begin{bmatrix} 1 \end{bmatrix} + 0 \begin{bmatrix} 2 \end{bmatrix} + 0 \begin{bmatrix} 3 \end{bmatrix} + 0 \begin{bmatrix} 4 \end{bmatrix} + x_5 \begin{bmatrix} 5 \end{bmatrix} = 0$$

since $\begin{bmatrix} 1 \end{bmatrix} = \begin{bmatrix} 5 \end{bmatrix}$, then $x_1 = -x_5$.

15. Suppose an m by n matrix has r pivots. The number of special solutions is _____. The nullspace contains only $\boldsymbol{x} = \mathbf{0}$ when $r =$ _____. The column space is all of $\mathbf{R}^m$ when $r =$ _____.

Special solutions: $n-r$

When $r=n$, then $m=n$, matrix is invertible, so $x=0$ is only solution.

When $r=m$, consider $\begin{bmatrix} 1 & 0 & 0 & \cdots & \\ & 1 & 0 & \cdots & \\ & & 1 & 0 & \cdots \\ & & & 1 & x\ x\ x \end{bmatrix} x = b$

Let b be anything in R^m

$\therefore$ Set $x_1 = b_1$, $x_2 = b_2$, ..., $x_m = b_m$

For $i > m$, $x_i = 0$. $\therefore$ $x = \begin{bmatrix} b_1 \\ \vdots \\ b_m \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ is a solution

$\therefore$ since b was any element of R^m, column space $= R^m$.

18. (Recommended) The plane $x - 3y - z = 12$ is parallel to the plane $x - 3y - z = 0$ in Problem 17. One particular point on this plane is $(12, 0, 0)$. All points on the plane have the form (fill in the first components)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} \\ 0 \\ 1 \end{bmatrix}.$$

$$\begin{bmatrix}1 & -3 & -1\end{bmatrix}\left(\begin{bmatrix} \\ 0 \\ 0\end{bmatrix} + y\begin{bmatrix}1 \\ \\ 0\end{bmatrix} + z\begin{bmatrix}0 \\ \\ 1\end{bmatrix}\right) = \begin{bmatrix}12\end{bmatrix}$$

Or, $\begin{bmatrix}1 & -3 & -1\end{bmatrix}\begin{bmatrix}(\quad) \\ y \\ z\end{bmatrix} = \begin{bmatrix}12\end{bmatrix}$

$\therefore$ ( ) must be $(12 + 3y + z)$

$\therefore \begin{bmatrix}12 \\ 0 \\ 0\end{bmatrix}, \begin{bmatrix}3 \\ 1 \\ 0\end{bmatrix}, \begin{bmatrix}1 \\ 0 \\ 1\end{bmatrix}$

19. Prove that U and $A = LU$ have the same nullspace when L is invertible:

If $U\boldsymbol{x} = \mathbf{0}$ then $LU\boldsymbol{x} = \mathbf{0}$. If $LU\boldsymbol{x} = \mathbf{0}$, how do you know $U\boldsymbol{x} = \mathbf{0}$?

(a) If $Ux = 0$, Then clearly $LUx = L0 = 0$.

(b) If $LUx = 0$, Then $L^{-1}LUx = L^{-1}0$, or $Ux = 0$.

$\therefore$ (a), (b) $\Rightarrow$ $Ux = 0 \Leftrightarrow LUx = 0$.

21. Construct a matrix whose nullspace consists of all combinations of $(2, 2, 1, 0)$ and $(3, 1, 0, 1)$.

Use special solutions with x_3, x_4 as free.

$\therefore$ From $x_3 = 0,\ x_4 = 1$
$(3, 1, 0, 1)$

$$\begin{bmatrix} 1 & 0 & x & -3 \\ 0 & 1 & x & -1 \end{bmatrix}$$

From $x_3 = 1,\ x_4 = 0$
$(2, 2, 1, 0)$

$$\begin{bmatrix} 1 & 0 & -2 & x \\ 0 & 1 & -2 & x \end{bmatrix}$$

$\therefore$ Combining:

$$\begin{bmatrix} 1 & 0 & -2 & -3 \\ 0 & 1 & -2 & -1 \end{bmatrix}$$

22. Construct a matrix whose nullspace consists of all multiples of (4, 3, 2, 1).

Use one free variable: x_4 $\therefore$
$(0, 0, 0, 1)$

$$\begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

23. Construct a matrix whose column space contains (1, 1, 5) and (0, 3, 1) and whose nullspace contains (1, 1, 2).

Consider $$1 \cdot \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ $1 + 2x_1 = 0,\ x_1 = -\frac{1}{2}$

$1 + 3 + 2x_2 = 0,\ x_2 = -2$

$5 + 1 + 2x_3 = 0,\ x_3 = -3$

$$\therefore \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 1 & 3 & -2 \\ 5 & 1 & -3 \end{bmatrix}$$

24. Construct a matrix whose column space contains (1, 1, 0) and (0, 1, 1) and whose nullspace contains (1, 0, 1) and (0, 0, 1).

If (0,0,1) in null space, then

$$0\begin{bmatrix} Col_1 \end{bmatrix} + 0\begin{bmatrix} Col_2 \end{bmatrix} + 1\begin{bmatrix} Col_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore Col_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

For (1,0,1),

$$1\begin{bmatrix} Col_1 \end{bmatrix} + 0\begin{bmatrix} Col_2 \end{bmatrix} + 1\cdot\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore Col_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 0 & x & 0 \\ 0 & x & 0 \\ 0 & x & 0 \end{bmatrix}$$

This conflicts with column space containing $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$\therefore$ impossible.

27. Why does no 3 by 3 matrix have a nullspace that equals its column space?

Nullspace = # free variable : $r < 3$
pivots + free = 3

$\therefore$ One is a plane (2 independent vectors), the other a line (1 vector).

Another way to view this: column space = # pivots. $3 - p$ = nullspace = p, so $2p = 3$, but p must be integral.

28. If $AB = 0$ then the column space of B is contained in the ______ of A. Give an example of A and B.

Consider $A[B_1] + A[B_2] + \dots A[B_n] = 0$

$\therefore$ Each column, B_i, is in nullspace of A.
$\therefore$ Column space of B is in $\underline{\underline{\text{nullspace}}}$ of A.

31. If the nullspace of A consists of all multiples of $\boldsymbol{x} = (2, 1, 0, 1)$, how many pivots appear in U? What is R?

A is a $m \times 4$ matrix, one free, $\therefore$ $\underline{\underline{3 \text{ pivots}}}$

$$R = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

32. If the special solutions to $Rx = \mathbf{0}$ are in the columns of these N, go backward to find the nonzero rows of the reduced matrices R:

$$N = \begin{bmatrix} 2 & 3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} \ \end{bmatrix} \text{ (empty 3 by 1).}$$

(a) For $N = \begin{bmatrix} 2 & 3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$, R must be $m \times 3$ with 2 free variables. $\therefore$ 1 pivot

From $\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & x & -3 \end{bmatrix}$

From $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & x \end{bmatrix}$

$\therefore$ combining: $\underline{R = \begin{bmatrix} 1 & -2 & -3 \end{bmatrix}}$

(b) For $N = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, R must be $m \times 3$, with 1 free, $\therefore$ 2 pivots.

$\therefore R = \begin{bmatrix} 1 & 0 & x \\ 0 & 1 & x \end{bmatrix}$, $\therefore \underline{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}$

(c) $N = \begin{bmatrix} \ \end{bmatrix}$ (ϕ), R must be I