3.3 The Rank and Row Reduced Form

Note Title 4/20/2007

17\. (a) Suppose column j of B is a combination of previous columns of B. Show that column j of AB is the same combination of previous columns of AB. Then $\mathbf{rank}(AB) \le \mathbf{rank}(B)$, because AB cannot have new pivot columns.

(b) Find A_1 and A_2 so that $\text{rank}(A_1B) = 1$ and $\text{rank}(A_2B) = 0$ for $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

(a) Let $B_j = r_1 B_1 + r_2 B_2 + \cdots + r_i B_i$,

where $0 \le r_i$ and $i < j$.

$\therefore$ Column j of $AB = A B_j$

$= A(r_1 B_1 + \cdots + r_i B_i)$

$= r_1 A B_1 + \cdots + r_i A B_i$

$= r_1$ (col 1 of AB) $+ \cdots + r_i$ (column i of AB)

$\therefore$ it's the same combination (the r_i's) of the same numbered columns of AB.

rank$(AB) \le$ rank(B) because AB is a linear combination of the <u>rows</u> of B, and so new pivots of B cannot be created.

Note that if $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \\ \vdots & & \cdots & a_{mn} \end{bmatrix}$

Then $A = \cdots\cdots\cdots \begin{bmatrix} 1 & a_{12} & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \cdot & \vdots & \vdots & \cdots & 1 \end{bmatrix} \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{bmatrix}$

These are like elimination matrices, call them $\cdots, E_{12}, D_{11}$, and they are invertable.

$\therefore AB = \cdots E_{12} D_{11} B$

$\therefore D_{11}^{-1} E_{12}^{-1} \cdots AB = B$

Let E' represent the elimination matrices on B to give R.

$\therefore [E'(D_{11} E_{12}^{-1} \cdots)] AB = R$. $\therefore$ steps can be taken on AB to yield R, which has the # of pivots for B, so AB can't increase the # of pivots.

(6) Since $\text{rank}(B) = 1$, then $\text{rank}(A_1B) \leq 1$,

so to ensure $\text{rank}(A_1B) = 1$, make sure A_1 doesn't make $A_1B = 0$.

The only A_1 to do this is $A_1 = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ or $A_1 = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$.

$\therefore A_1$ can be anything except $\pm r \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$, r any real number (including 0).

From above, $A_2 = r \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$

18. Problem 17 proved that $\text{rank}(AB) \le \text{rank}(B)$. Then the same reasoning gives $\text{rank}(B^T A^T) \le \text{rank}(A^T)$. How do you deduce that $\textbf{rank}(AB) \le \textbf{rank}\, A$?

$\text{rank}(A) = \text{rank}(A^T)$

Then $\text{rank}(B^T A^T) \le \text{rank}(A)$

Since $B^T A^T = (AB)^T$, Then

$\text{rank}(B^T A^T) = \text{rank}[(AB)^T] = \text{rank}(AB)$

$\therefore \text{rank}(AB) = \text{rank}(B^T A^T) \le \text{rank}(A)$

19. *(Important)* Suppose A and B are n by n matrices, and $AB = I$. Prove from $\text{rank}(AB) \le \text{rank}(A)$ that the rank of A is n. So A is invertible and B must be its two-sided inverse (Section 2.5). Therefore $BA = I$ *(which is not so obvious!)*.

$\text{rank}(I) = n$. $\therefore \text{rank}(AB) = \text{rank}(I) = n$

$\therefore n = \text{rank}(AB) \le \text{rank}(A)$.

$\therefore n \le \text{rank}(A)$, but since A is $n \times n$,

$\text{rank}(A) \le n$.

$\therefore \text{rank}(A) = n$.

20. If A is 2 by 3 and B is 3 by 2 and $AB = I$, show from its rank that $BA \ne I$. Give an example of A and B. For $m < n$, a right inverse is not a left inverse.

$\text{rank}(A) \le 2$ and $\text{rank}(B) \le 2$, since rank $\leq$ min(# rows, # cols).

If $BA = I$, a 3×3 matrix, so that rank$(I) = 3$,
Then rank$(BA) = 3$.
But rank$(BA) \leq$ rank$(B) \Rightarrow 3 \leq 2$.
$\therefore BA \neq I$.

21. Suppose A and B have the *same* reduced row echelon form R.

(a) Show that A and B have the same nullspace and the same row space.

(b) We know $E_1A = R$ and $E_2B = R$. So A equals an ______ matrix times B.

(a) Since $N(A) = R$ and $N(B) = R$,
Then $N(A) = N(B)$
Also, row space(A) = row space(R)
row space(B) = row space(R)
$\therefore$ row space(A) = row space(B)

(b) $E_1A = R$, $E_2B = R$, $\therefore E_1A = E_2B$, so

$A = E_1^{-1}E_2B$, and E_2 is invertible

$\therefore E_1^{-1}E_2$ is invertible, so

A = (invertible matrix) B