3.4 The Complete Solution to $AX = B$

Note Title 5/1/2007

1. (Recommended) Execute the six steps of Worked Example **3.4 A** to describe the column space and nullspace of A and the complete solution to $A\boldsymbol{x} = \boldsymbol{b}$:

$$A = \begin{bmatrix} 2 & 4 & 6 & 4 \\ 2 & 5 & 7 & 6 \\ 2 & 3 & 5 & 2 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$$

1. Reduce $[A\ b]$ to $[U\ c]$

$$\begin{bmatrix} 2 & 4 & 6 & 4 & b_1 \\ 2 & 5 & 7 & 6 & b_2 \\ 2 & 3 & 5 & 2 & b_3 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 4 & 6 & 4 & b_1 \\ 0 & 1 & 1 & 2 & b_2 - b_1 \\ 0 & -1 & -1 & -2 & b_3 - b_1 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} 2 & 4 & 6 & 4 & b_1 \\ 0 & 1 & 1 & 2 & b_2 - b_1 \\ 0 & 0 & 0 & 0 & b_2 + b_3 - 2b_1 \end{bmatrix}$$

2. $b_2 + b_3 - 2b_1 = 0$ for $AX = b$ to be solvable.

3. Column space of A is all vectors of form

$$c_1 \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix}$$ which is the plane described by $b_2 + b_3 - 2b_1 = 0$

4. Free variables are x_3, x_4. $\therefore$ Special solutions are $(-1, -1, 1, 0)$ and $(2, -2, 0, 1)$

$\therefore$ Nullspace is of form: $c_1\begin{bmatrix}-1\\-1\\1\\0\end{bmatrix}+c_2\begin{bmatrix}2\\-2\\0\\1\end{bmatrix}$

5. For $b=(0,6,-6)$, it satifies $b_2+b_3-2b_1=0$.
Set free variables to 0, so $x_3=x_4=0$.

$\therefore$ from U, $x_2=6$, $x_1=-12$.

$\therefore x_p=(-12,6,0,0)$

$\therefore$ Complete solution to $Ax=(0,6,-6)$
is $x_p+c_1s_1+c_2s_2$, $s_1=\begin{bmatrix}-1\\-1\\1\\0\end{bmatrix}$, $s_2=\begin{bmatrix}2\\-2\\0\\1\end{bmatrix}$

6. $U\to R$:

$$\begin{bmatrix}2&4&6&4&b_1\\0&1&1&2&b_2-b_1\\0&0&0&0&b_2+b_3-2b_1\end{bmatrix}\to\begin{bmatrix}1&2&3&2&\frac{b_1}{2}\\0&1&1&2&b_2-b_1\\0&0&0&0&b_2+b_3-2b_1\end{bmatrix}$$

$$\to\begin{bmatrix}1&0&1&-2&\frac{b_1}{2}-2b_2+2b_1\\0&1&1&2&b_2-b_1\\0&0&0&0&b_2+b_3-2b_1\end{bmatrix}$$

Write the complete solution as x_p plus any multiple of s in the nullspace:

3.
$$x + 3y + 3z = 1$$
$$2x + 6y + 9z = 5$$
$$-x - 3y + 3z = 5.$$

$$\begin{bmatrix} 1 & 3 & 3 & 1 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 3 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 6 & 6 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 3 & 3 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ y is free, special solution: $(-3, 1, 0)$

Particular solution x_p: $(-2, 0, 1)$

$\therefore$ $(-2, 0, 1) + s(-3, 1, 0)$

13. Explain why these are all false:

(a) The complete solution is any linear combination of x_p and x_n.

(b) A system $Ax = b$ has at most one particular solution.

(c) The solution x_p with all free variables zero is the shortest solution (minimum length $\|x\|$). Find a 2 by 2 counterexample.

(d) If A is invertible there is no solution x_n in the nullspace.

(a) Complete solution is of form $x_p + s x_n$, not $v x_p + w x_n$

(b) If $Ax = b$ has infinite solutions, any one may serve as a particular solution.

(c) As in (b), any one may serve as x_p.

Consider $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ x_p would be $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$

But $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ is longer than $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(d) $x_n = 0$ is in the null space.

30. Reduce to $Ux = c$ (Gaussian elimination) and then $Rx = d$ (Gauss-Jordan):

$$Ax = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 1 & 3 & 2 & 0 \\ 2 & 0 & 4 & 9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 10 \end{bmatrix} = b.$$

Find a particular solution x_p and all homogeneous solutions x_n.

$$\begin{bmatrix} 1 & 0 & 2 & 3 & 2 \\ 1 & 3 & 2 & 0 & 5 \\ 2 & 0 & 4 & 9 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 & 3 & 2 \\ 0 & 3 & 0 & -3 & 3 \\ 0 & 0 & 0 & 3 & 6 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} 1 & 0 & 2 & 3 & 2 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 2 & 0 & -4 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} = [R \; d]$$

$$\therefore \; x_p = \begin{bmatrix} -4 \\ 3 \\ 0 \\ 2 \end{bmatrix}, \quad x_n = x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

32. Find the LU factorization of A and the complete solution to $A\boldsymbol{x} = \boldsymbol{b}$:

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 1 & 5 \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 3 \\ 6 \\ 5 \end{bmatrix} \quad \text{and then} \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 3 & 1 \\ 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 1 & 5 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} A = E_1 A$$

$$= \begin{bmatrix} 1 & 3 & 1 \\ 0 & -1 & 2 \\ 0 & -2 & 4 \\ 0 & -2 & 4 \end{bmatrix} = A_1$$

$$\therefore \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{bmatrix} A_1 = E_2 A_1 = \begin{bmatrix} 1 & 3 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = U$$

$\therefore E_2 E_1 A = U$, so $A = E_1^{-1} E_2^{-1} U$

$$E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix}, \quad E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\therefore L = E_1^{-1} E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}$$

$$\therefore A = LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Note: $L^{-1} = E_2 E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 1 & -2 & 0 & 1 \end{bmatrix}$$

$\therefore Ax = b = \begin{bmatrix} 1 \\ 3 \\ 6 \\ 5 \end{bmatrix} \Rightarrow LUx = b$

$\Rightarrow Ux = L^{-1}b = c$

$$L^{-1}b = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 1 & -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 6 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} = c$$

$$\therefore \begin{bmatrix} 1 & 3 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore x_3$ free, so $x_n = x_3 \begin{bmatrix} -7 \\ 2 \\ 1 \end{bmatrix}$

$$x_p = \begin{bmatrix} 7 \\ -2 \\ 0 \end{bmatrix}$$

$$\therefore x = \begin{bmatrix} 7 \\ -2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -7 \\ 2 \\ 1 \end{bmatrix}$$