3.5 Independence, Basis and Dimension

Note Title 5/30/2007

1. Show that $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ are independent but $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4$ are dependent:

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \boldsymbol{v}_4 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}.$$

Solve either $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + c_3\boldsymbol{v}_3 = \boldsymbol{0}$ or $A\boldsymbol{x} = \boldsymbol{0}$. The $\boldsymbol{v}$'s go in the columns of A.

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$ so $Vx = 0 \Rightarrow x = 0$ $\Rightarrow V_1, V_2, V_3$ are independent.

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

$\therefore$ $Vx = 0$ is solved by $x = (1, 1, -4, 1)$

$\therefore$ non-zero solution $\Rightarrow V_1, V_2, V_3, V_4$ are dependent.

12. The vector $\boldsymbol{b}$ is in the subspace spanned by the columns of A when there is a solution to _____. The vector $\boldsymbol{c}$ is in the row space of A when there is a solution to _____.

True or false: If the zero vector is in the row space, the rows are dependent.

(a) $Ax = b$. (b) $c^T = x^T A$, or $A^T x = c$

(c) False. Zero vector is in every row space.

13. Find the dimensions of these 4 spaces. Which two of the spaces are the same? (a) column space of A, (b) column space of U, (c) row space of A, (d) row space of U:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 3 & 1 & -1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

(a) Since $EA = U$, and $U \to \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} = R$

$\therefore$ 2 pivot cols in R, so 2 pivot cols in A,

$\therefore$ dim(col space of A) = 2

(b) dim(col space of U) same as A = 2

(c) row space (A) = row space(U) = row space(R)

$\therefore$ dim (row space (A)) = 2

(d) as in (c), dim = 2.

21. Find a basis for the plane $x - 2y + 3z = 0$ in $\mathbf{R}^3$. Then find a basis for the intersection of that plane with the xy plane. Then find a basis for all vectors perpendicular to the plane.

(a) For the plane $x - 2y + 3z = 0$, look at the nullspace of $\begin{bmatrix} 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$

This has 2 free variables, so 2 special solutions: $(-3, 0, 1)$ and $(2, 1, 0)$

Dim(nullspace) $= 2$, so these are a basis as they are independent.

(b) (x, y, z) is in the xy plane if $z = 0$. The intersection is a line, so $(2, 1, 0)$ from (a) satisfies requirement.

Another way to look at this is to look at nullspace of $\begin{bmatrix} 1 & -2 & 3 \\ 0 & 0 & 1 \end{bmatrix}$

This has 1 free variable, and the special solution is: $(2, 1, 0)$.

(c) All vectors perpendicular to $x - 2y + 3 = 0$ must be perpendicular to the basis vectors in (a). $\therefore$ Look at their nullspace: $\begin{bmatrix} -3 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 3 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix} \to$

$$\begin{bmatrix} 3 & 0 & -1 \\ 0 & 1 & 2/3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1/3 \\ 0 & 1 & 2/3 \end{bmatrix}$$

$\therefore \left(\frac{1}{3}, -\frac{2}{3}, 1\right)$, or $\underline{(1, -2, 3)}$

37. **37** Find a basis for the space of polynomials $p(x)$ of degree ≤ 3. Find a basis for the subspace with $p(1) = 0$.

$$p(x) = a + bx + cx^2 + dx^3 = \begin{bmatrix} 1 & x & x^2 & x^3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

A basis would be $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ x \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ x^2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ x^3 \end{bmatrix}$

For $p(1) = 0$, look at nullspace of $\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 0$

$\therefore$ Special solutions: $\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

or $x^3 - 1, x^2 - 1, x - 1$

38. **38** Find a basis for the space $\mathbf{S}$ of vectors (a, b, c, d) with $a + c + d = 0$ and also for the space $\mathbf{T}$ with $a + b = 0$ and $c = 2d$. What is the dimension of the intersection $\mathbf{S} \cap \mathbf{T}$?

For S, consider nullspace for $\begin{bmatrix} 1 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 0$

Special solutions are: $\begin{bmatrix}-1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}-1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}$

These are independent and form $\therefore$ form a basis.

For T: nullspace for $\begin{bmatrix}1&1&0&0\\0&0&1&-2\end{bmatrix}\begin{bmatrix}a\\b\\c\\d\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

Already row-reduced.

$\therefore$ Special solutions: $\begin{bmatrix}0\\0\\2\\1\end{bmatrix}, \begin{bmatrix}-1\\1\\0\\0\end{bmatrix}$

For $S \cap T$, look at nullspace for:

$$\begin{bmatrix}1&0&1&1\\1&1&0&0\\0&0&1&-2\end{bmatrix}\begin{bmatrix}a\\b\\c\\d\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$\therefore$ Row reduce:

$$\begin{bmatrix}1&0&1&1\\1&1&0&0\\0&0&1&-2\end{bmatrix} \to \begin{bmatrix}1&0&1&1\\0&1&-1&-1\\0&0&1&-2\end{bmatrix} \to \begin{bmatrix}1&0&0&3\\0&1&0&-3\\0&0&1&-2\end{bmatrix}$$

$\therefore$ Special solutions: $\underline{\begin{bmatrix} -3 \\ 3 \\ 2 \\ 1 \end{bmatrix}}$

$\dim(S \cap T) = \underline{1}$