3.6 Dimensions of The Four Subspaces

Note Title 6/18/2007

1. (a) If a 7 by 9 matrix has rank 5, what are the dimensions of the four subspaces? What is the sum of all four dimensions?

(b) If a 3 by 4 matrix has rank 3, what are its column space and left nullspace?

(a) Dim (col. space) = dim (row space) = 5
Dim (nullspace) = 9-5 = 4
Dim (left nullspace) = 7-5 = 2

(b) Column space has dim = 3, so space is R^3
Left nullspace dim = 3-3 = 0, so space is (0,0,0).

2. Find bases for the four subspaces associated with A and B:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 5 & 8 \end{bmatrix}.$$

(a) A clearly has rank 1

$A \rightarrow \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 0 \end{bmatrix}$ ∴ pivot col. is 1.

∴ Base col. space : $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Base row space : (1, 2, 4)

Special solutions : $\begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$, which are bases for nullspace.

For left nullspace, note $EA = R$, where $E = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$

Since dim (left nullspace) $= 2 - 1 = 1$, and row $[-2 \ 1]$ gives the zero row, $[-2 \ 1]$ is a base for left nullspace.

(b) $\begin{bmatrix} 1 & 2 & 4 \\ 2 & 5 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \end{bmatrix}$, so rank is 2

pivot columns of B are cols. #1 & 2, so base for col. space is: $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix}$

base for row space: $(1,0,4), (0,1,0)$

base for null space: $(-4, 0, 1)$

left null space is $(0,0)$, which has no basis since, e.g., $3 \cdot (0,0) = 0$, so the zero vector is not independent.

3. Find a basis for each of the four subspaces associated with

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Here, $A = EU$. Row space (A) = Row Sp. (U),

(a) $\therefore$ Row space basis: $(0,1,2,3,4)$ and $(0,0,0,1,2)$

(b) Cols. 2 and 4 of U are pivot cols., $\therefore$ cols 2 & 4 for A are a basis for Col. Sp. (A)

$\therefore$ Col. Sp. (A) basis: $\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}3\\4\\1\end{bmatrix}$

(c) From U, special solutions of $UX=0$ (and $\therefore$ for $AX=0$) are from setting each "free" variable to 1, while setting the others to zero. Free variables in cols. 1, 3, 5.

$\therefore$ $(1,0,0,0,0)$, $(0,-2,1,0,0)$, $(0,2,0,-2,1)$

These form the basis for nullspace of A.

(d) From $A = EU$, $E^{-1}A = U$. Last row of E^{-1} is a solution to $XA = 0$ since last row of U is zero. $\therefore$ Find E^{-1}.

$$EI = \begin{bmatrix}1&0&0\\1&1&0\\0&1&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \rightarrow \begin{bmatrix}1&0&0\\0&1&0\\0&1&1\end{bmatrix}\begin{bmatrix}1&0&0\\-1&1&0\\0&0&1\end{bmatrix}$$

$$\rightarrow \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\-1&1&0\\1&-1&1\end{bmatrix} \quad \therefore E^{-1} = \begin{bmatrix}1&0&0\\-1&1&0\\1&-1&1\end{bmatrix}$$

Last row of E^{-1} is $(1,-1,1)$

$\therefore$ Basis for left nullspace is: $(1,-1,1)$

23. Without multiplying matrices, find bases for the row and column spaces of A:

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} 3 & 0 & 3 \\ 1 & 1 & 2 \end{bmatrix}.$$

How do you know from these shapes that A is not invertible?

A is a $(3\times 2)(2\times 3) = 3\times 3$ matrix, but since A is a linear combination of the rows $[3\ 0\ 3]$ and $[1\ 1\ 2]$, it has rank 2. Invertible would have rank 3.

24. $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{d}$ is solvable when the right side $\boldsymbol{d}$ is in which of the four subspaces? The solution is unique when the _____ contains only the zero vector.

Solvable when d is in col. sp. (A^T) = row space of A.

Solution unique when nullspace of A^T is zero, or left nullspace of A is zero.
(Think of partic. solutions + special solutions).