4.1 Orthogonality of the Four Subspaces

Note Title 7/27/2007

3. Construct a matrix with the required property or say why that is impossible:

(a) Column space contains $\begin{bmatrix}1\\2\\-3\end{bmatrix}$ and $\begin{bmatrix}2\\-3\\5\end{bmatrix}$, nullspace contains $\begin{bmatrix}1\\1\\1\end{bmatrix}$

(b) Row space contains $\begin{bmatrix}1\\2\\-3\end{bmatrix}$ and $\begin{bmatrix}2\\-3\\5\end{bmatrix}$, nullspace contains $\begin{bmatrix}1\\1\\1\end{bmatrix}$

(c) $Ax = \begin{bmatrix}1\\1\\1\end{bmatrix}$ has a solution and $A^{\mathrm{T}}\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

(d) Every row is orthogonal to every column (A is not the zero matrix)

(e) Columns add up to a column of zeros, rows add to a row of 1's.

(a) $1\cdot\begin{bmatrix}1\\2\\-3\end{bmatrix} + 1\cdot\begin{bmatrix}2\\-3\\5\end{bmatrix} + 1\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$, so $\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}-3\\1\\-2\end{bmatrix}$

$\therefore \begin{bmatrix}1 & 2 & -3\\2 & -3 & 1\\-3 & 5 & 2\end{bmatrix}$

(b) $\begin{bmatrix}1\\2\\-3\end{bmatrix} \perp \begin{bmatrix}1\\1\\1\end{bmatrix}$, but $\begin{bmatrix}2\\-3\\5\end{bmatrix} \not\perp \begin{bmatrix}1\\1\\1\end{bmatrix}$

$\therefore$ impossible

(c) $Ax = \begin{bmatrix}1\\1\\1\end{bmatrix} \Rightarrow \begin{bmatrix}1\\1\\1\end{bmatrix}$ is in col. sp. of A.

$\therefore \begin{bmatrix}1\\1\\1\end{bmatrix}$ must be $\perp$ to left nullspace of A.

$A^T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow [1\ 0\ 0]\, A = [0\ 0\ 0]$, so $[1\ 0\ 0]$ must be in left nullspace of A, so $[1\ 0\ 0] \perp \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, but they aren't $\perp$, $\therefore$ impossible.

(d). $\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$

(e) Columns add up to a col. of zeros $\Rightarrow$ $\begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$ is in nullspace

Rows add up to a row of 1's $\Rightarrow [1\ 1 \dots 1]$ is in row space.

$\therefore$ The $\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$ and $[1\ 1 \dots 1]$ must be $\perp$, which is impossible because dot product equals $1 + 1 + 1 \dots + 1 \neq 0$.

5. (a) If $Ax = b$ has a solution and $A^T y = 0$, then y is perpendicular to ______.

(b) If $A^T y = c$ has a solution and $Ax = 0$, then x is perpendicular to ______.

(a) $A^T y = 0 \Rightarrow y^T A = 0$, $\therefore y^T(Ax) = (y^T A)x$

$= (0)x = 0. \therefore 0 = y^T(Ax) = y^T(b)$, so $y \perp b$.

(b) $A^T y = c \Rightarrow y^T A = c^T \Rightarrow (y^T A)x = c^T x$

$\therefore y^T(Ax) = y^T 0 = 0$, so $c^T x = 0$,

$\therefore x \perp c$.

6. This is a system of equations $Ax = b$ with *no solution*:

$$\begin{aligned} x + 2y + 2z &= 5 \\ 2x + 2y + 3z &= 5 \\ 3x + 4y + 5z &= 9 \end{aligned}$$

Find numbers y_1, y_2, y_3 to multiply the equations so they add to $0 = 1$. You have found a vector y in which subspace? Its dot product $y^T b$ is 1.

$[y_1\ y_2\ y_3]\, A = 0 \Rightarrow [y_1\ y_2\ y_3]$ in left nullspace of A. By inspection, $[y_1\ y_2\ y_3] = [1\ 1\ {-1}]$

And $[1\ 1\ {-1}]\begin{bmatrix} 5 \\ 5 \\ 9 \end{bmatrix} = 1 = y^T b$.

9. If Ax is in the nullspace of A^T then $Ax = \mathbf{0}$. Reason: Ax is also in the _____ of A and the spaces are _____. *Conclusion*: $A^T A$ *has the same nullspace as* A.

Ax is in the column space of A. Since Col. sp. of A is orthogonal to left nullspace of A,

Then Ax is perpendicular to any element in left nullspace of A.

$\therefore$ If $Ax \in$ nullspace of A^T, then $Ax \perp Ax$.
$\therefore$ $Ax = 0$.

Let $x \in N(A)$. $\therefore$ $Ax = 0$, so $A^T A x = 0 \Rightarrow$
$x \in N(A^T A)$.
Now suppose $x \in N(A^T A)$. $\therefore$ $A^T A x = 0$, so
$Ax \in N(A^T)$, and from above $Ax = 0 \Rightarrow$
$x \in N(A)$.
$\therefore$ $N(A) = N(A^T A)$

16. Suppose A is a symmetric matrix ($A^{\mathrm{T}} = A$).

(a) Why is its column space perpendicular to its nullspace?

(b) If $A\boldsymbol{x} = \mathbf{0}$ and $A\boldsymbol{z} = 5\boldsymbol{z}$, which subspaces contain these "eigenvectors" $\boldsymbol{x}$ and $\boldsymbol{z}$? Symmetric matrices have perpendicular eigenvectors.

(a) For $A^T = A$, column space = row space.
Since $N(A)$ is $\perp$ row space, it is $\therefore$
$\perp$ col. space.

(b) $Ax = 0 \Rightarrow x \in N(A)$, and since $N(A) \perp$ row

space, and row space = col. space, Then $N(A) \perp C(A)$! But $Az = 5z \Rightarrow 5z$, and
$\therefore z$, is $\in C(A)$.
$\therefore x \perp z$, so $x^T \cdot z = 0$

23. If a subspace S is contained in a subspace V, prove that $S^\perp$ contains $V^\perp$.

Let $v^T \in V^T$, $v \in V$. $\therefore v^T \cdot v = 0$
If $s \in S$, then $s \in V$, since $S \subset V$.
$\therefore v^T \cdot s = 0 \Rightarrow$ every element of V^T is perpendicular to every element of S.
$\therefore v^T \in V^T \Rightarrow v^T \in S^T$!
$\therefore V^T \subset S^T$.

25. Find A^TA if the columns of A are unit vectors, all mutually perpendicular.

Let a_i be any column of A. Then $a_i \cdot a_i = 1$, since the cols. are unit vectors.
Also, $a_i \cdot a_j = 0$, when $i \neq j$.

$$\therefore \begin{bmatrix} \leftarrow a_1 \rightarrow \\ \leftarrow a_2 \rightarrow \\ \vdots \\ \leftarrow a_n \rightarrow \end{bmatrix} \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ a_1 & a_2 & \cdots & a_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdot & 1 \end{bmatrix} = I$$

30. Suppose A is 3 by 4 and B is 4 by 5 and $AB = 0$. Prove $\text{rank}(A)+\text{rank}(B) \leq 4$.

$\dim N(A) = 4 - \text{rank}(A)$.

Since $AB = 0$, columns of B are in $N(A)$.

$\therefore \text{rank}(B) \leq \dim N(A)$

$\therefore \text{rank}(B) \leq 4 - \text{rank}(A) \Rightarrow$

$\text{rank}(A) + \text{rank}(B) \leq 4$