# 4-2 Projections

Note Title 8/15/2007

1. Project the vector $\boldsymbol{b}$ onto the line through $\boldsymbol{a}$. Check that $\boldsymbol{e}$ is perpendicular to $\boldsymbol{a}$:

(a) $\boldsymbol{b} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ and $\boldsymbol{a} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ (b) $\boldsymbol{b} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$ and $\boldsymbol{a} = \begin{bmatrix} -1 \\ -3 \\ -1 \end{bmatrix}$.

(a) $p = \dfrac{a^T b}{a^T a} a = \dfrac{(1,1,1)\cdot(1,2,2)}{(1,1,1)\cdot(1,1,1)}(1,1,1) = \dfrac{5}{3}(1,1,1)$

$= \underline{\left(\frac{5}{3}, \frac{5}{3}, \frac{5}{3}\right)}$

$e = b - p = \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$

$e \cdot a = -\frac{2}{3} + \frac{1}{3} + \frac{1}{3} = 0$

(b) $\dfrac{a^T b}{a^T a} = \dfrac{(-1,-3,-1)\cdot(1,3,1)}{(-1,-3,-1)\cdot(-1,-3,-1)} = -\dfrac{11}{11} = -1$

$\therefore\ p = (-1)a = (1,3,1)$

$e = b - p = (0,0,0)$. $\therefore\ e \cdot a = 0$

11. Project $\boldsymbol{b}$ onto the column space of $A$ by solving $A^{\mathrm{T}} A \widehat{\boldsymbol{x}} = A^{\mathrm{T}} \boldsymbol{b}$ and $\boldsymbol{p} = A \widehat{\boldsymbol{x}}$:

(a) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix}$.

Find $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p}$. It should be perpendicular to the columns of $A$.

(a) $A^TA = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

$\therefore (A^TA)^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

$A^Tb = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$

$\therefore \hat{x} = (A^TA)^{-1}A^Tb = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

$\therefore p = A\hat{x} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \underline{\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}}$

$e = b - p = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} = \underline{\begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}}$

(b) $A^TA = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix}$

$\therefore (A^TA)^{-1} = \frac{1}{2} \begin{bmatrix} 3 & -2 \\ -2 & 2 \end{bmatrix}$

$A^Tb = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 8 \\ 14 \end{bmatrix}$

$\therefore \hat{x} = (A^TA)^{-1}A^Tb = \frac{1}{2} \begin{bmatrix} 3 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 8 \\ 14 \end{bmatrix}$

$$= \frac{1}{2}\begin{bmatrix} -4 \\ 12 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$$

$$\therefore p = A\hat{x} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix}$$

$$e = b - p = \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix} - \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

13. (Quick and Recommended) Suppose $A$ is the 4 by 4 identity matrix with its last column removed. $A$ is 4 by 3. Project $\boldsymbol{b} = (1, 2, 3, 4)$ onto the column space of $A$. What shape is the projection matrix $P$ and what is $P$?

$$A^TA = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$\therefore P = A(A^TA)^{-1}A^T = A(I)A^T = AA^T$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\therefore p = Pb = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}$$

20. To find the projection matrix $P$ onto the same plane $x-y-2z=0$, write down a vector $\boldsymbol{e}$ that is perpendicular to that plane. Compute the projection $Q=\boldsymbol{e}\boldsymbol{e}^{\mathrm{T}}/\boldsymbol{e}^{\mathrm{T}}\boldsymbol{e}$ and then $P=I-Q$.

$$x-y-2z=0 \iff (1,-1,-2)\cdot(x,y,z)=0$$

$$\therefore e=(1,-1,-2)$$

$$\therefore Q=\frac{e\cdot e^T}{e^T\cdot e}=\frac{1}{6}\begin{bmatrix}1\\-1\\-2\end{bmatrix}\begin{bmatrix}1 & -1 & -2\end{bmatrix}=\frac{1}{6}\begin{bmatrix}1 & -1 & -2\\-1 & 1 & 2\\-2 & 2 & 4\end{bmatrix}$$

$$=\begin{bmatrix}1/6 & -1/6 & -\frac{1}{3}\\-\frac{1}{6} & \frac{1}{6} & \frac{1}{3}\\-\frac{1}{3} & \frac{1}{3} & \frac{2}{3}\end{bmatrix}$$

$$\therefore P=I-Q=\begin{bmatrix}\frac{5}{6} & \frac{1}{6} & \frac{1}{3}\\\frac{1}{6} & \frac{5}{6} & -\frac{1}{3}\\\frac{1}{3} & -\frac{1}{3} & \frac{1}{3}\end{bmatrix}$$

21. Multiply the matrix $P=A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$ by itself. Cancel to prove that $P^2=P$. Explain why $P(P\boldsymbol{b})$ always equals $P\boldsymbol{b}$: The vector $P\boldsymbol{b}$ is in the column space so its projection is ______.

(a) $P\cdot P=\left[A(A^TA)^{-1}A^T\right]\left[A(A^TA)^{-1}A^T\right]$

$$=A(A^TA)^{-1}(A^TA)(A^TA)^{-1}A^T$$

$$= A I (A^T A)^{-1} A^T = A (A^T A)^{-1} A^T = P$$

(b) $P(Pb) = (P \cdot P)b = Pb$

(c) Since $Pb$ is in column space of $A$, and since the projection of every column space vector is itself, then $P(Pb) = Pb$.

22. Prove that $P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$ is symmetric by computing $P^{\mathrm{T}}$. Remember that the inverse of a symmetric matrix is symmetric.

$$P^T = [A (A^T A)^{-1} A^T]^T = (A^T)^T [(A^T A)^{-1}]^T A^T$$

$$= A [(A^T A)^T]^{-1} A^T$$

$$= A [A^T A]^{-1} A^T = P$$

23. If $A$ is square and invertible, the warning against splitting $(A^{\mathrm{T}}A)^{-1}$ does not apply. It is true that $AA^{-1}(A^{\mathrm{T}})^{-1}A^{\mathrm{T}} = I$. *When $A$ is invertible, why is $P = I$? **What is the error $e$?***

When $A$ is invertible, its independent columns span (assuming column vectors are elements of $R^n$) all of $R^n$. $\therefore$ You are not projecting onto a subspace separate from $R^n$. The projection is itself, and the error $e = 0$.

26. If an $m$ by $m$ matrix has $A^2 = A$ and its rank is $m$, prove that $A = I$.

$m \times m$ matrix of rank $m \Rightarrow$ matrix is invertible.

$\therefore$ $A^{-1}$ exists

$$\therefore \quad A^2 = A \Rightarrow A^{-1}(A^2) = A^{-1}(A)$$

$$\Rightarrow (A^{-1}A)A = I$$

$$\Rightarrow IA = I \Rightarrow A = I$$

28. Use $P^{\mathrm{T}} = P$ and $P^2 = P$ to prove that the length squared of column 2 always equals the diagonal entry $p_{22}$. This number is $\frac{2}{6} = \frac{4}{36} + \frac{4}{36} + \frac{4}{36}$ for

$$P = \frac{1}{6}\begin{bmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{bmatrix}.$$

$P^T = P$ and $P^2 = P \Rightarrow P^T P = P$

$(P^T P)_{22} = (\text{row } 2 \text{ of } P^T) \cdot (\text{column } 2 \text{ of } P)$

$= (\text{column } 2 \text{ of } P) \cdot (\text{column } 2 \text{ of } P)$ since $P^T = P$.

$= (\text{column } 2 \text{ of } P)^2$

$\therefore P_{22} = (P^T P)_{22} = (\text{column } 2 \text{ of } P)^2$

29. If $B$ has rank $m$ (full row rank, independent rows) show that $BB^T$ is invertible.

Pf: (1) $B$ and $BB^T$ have same left nullspace

(a) if $x^TB = 0$, Then $x^TBB^T = 0 \cdot B^T = 0$.

$\therefore$ left nullspace of $B \subset$ left nullspace of $B^T$

(b) if $x^TBB^T = 0$, Then $x^TBB^Tx = 0 \Rightarrow$

$x^TB(x^TB)^T = 0 \Rightarrow \|x^TB\|^2 = 0 \Rightarrow$

$x^TB = 0$

$\therefore$ left nullspace of $B^T <$ left nullspace of $B$.

$\therefore$ (a) + (b) $\Rightarrow$ $B$ and $BB^T$ have same left nullspace.

(2) Rows of $B$ are independent $\Leftrightarrow$ rows of $BB^T$ are independent.

This directly follows from (1) above.

(3) $BB^T$ is square, since if $B$ is $m \times n$, Then $B^T$ is $n \times m$, so $BB^T$ is $m \times m$.

(4) If $B$ has full row rank, from (2), $BB^T$ has row rank of $m$ and $\therefore$ also col. rank of $m$ since dim (row space) = dim (col. space).

(5) $m$ independent rows $\Rightarrow$ they span $\mathbb{R}^m$

$\Rightarrow X(BB^T) = I$ is solvable, so left inverse exists.

$m$ independent cols $\Rightarrow$ they span $\mathbb{R}^m$

$\Rightarrow (BB^T)X' = I$ is solvable, so right inverse exists.

(6) Left + right inverse identical since

$$[X(BB^T)]X' = IX' = X'$$
$$X[(BB^T)X'] = XI = X$$

$\therefore$ $BB^T$ is invertible.