4.4 Orthogonal Bases and Gram-Schmidt

Note Title 9/19/2007

2. The vectors (2, 2, −1) and (−1, 2, 2) are orthogonal. Divide them by their lengths to find orthonormal vectors $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$. Put those into the columns of Q and multiply $Q^{\mathrm{T}}Q$ and QQ^{T}.

$$\begin{bmatrix} 2/3 & -1/3 \\ 2/3 & 2/3 \\ -1/3 & 2/3 \end{bmatrix} = Q \qquad Q^TQ = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$Q^T = \begin{bmatrix} 2/3 & 2/3 & -1/3 \\ -1/3 & 2/3 & 2/3 \end{bmatrix} \quad \therefore QQ^T = \begin{bmatrix} 5/9 & 2/9 & -4/9 \\ 2/9 & 8/9 & 2/9 \\ -4/9 & 2/9 & 5/9 \end{bmatrix}$$

6. If Q_1 and Q_2 are orthogonal matrices, show that their product Q_1Q_2 is also an orthogonal matrix. (Use $Q^{\mathrm{T}}Q = I$.)

Must show $(Q_1Q_2)^TQ_1Q_2 = I$

$$\begin{aligned}(Q_1Q_2)^TQ_1Q_2 &= (Q_2^TQ_1^T)Q_1Q_2 \\ &= Q_2^T(Q_1^TQ_1)Q_2 \\ &= Q_2^T(I)Q_2 \\ &= Q_2^TQ_2 = I\end{aligned}$$

7. If Q has orthonormal columns, what is the least squares solution $\widehat{x}$ to $Qx = b$?

Assume no solution to $Qx = b$.

$\therefore$ Find closest solution $\hat{x}$ s.t. $Q\hat{x}$ is closest to b: $b \perp Q\hat{x}$

$\therefore Q^T \cdot (b - Q\hat{x}) = 0$, or $Q^T Q \hat{x} = Q^T b$

But $Q^T Q = I$, so $\hat{x} = Q^T b$.

9. (a) Compute $P = QQ^{\mathrm{T}}$ when $q_1 = (.8, .6, 0)$ and $q_2 = (-.6, .8, 0)$. Verify that $P^2 = P$.

(b) Prove that always $(QQ^{\mathrm{T}})(QQ^{\mathrm{T}}) = QQ^{\mathrm{T}}$ by using $Q^{\mathrm{T}}Q = I$. Then $P = QQ^{\mathrm{T}}$ is the projection matrix onto the column space of Q.

(a) $$\begin{bmatrix} .8 & -.6 \\ .6 & .8 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} .8 & .6 & 0 \\ -.6 & .8 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = P$$

$$\therefore P^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = P$$

(b) $(QQ^T)(QQ^T) = Q(Q^T Q)Q^T = Q(I)Q^T$

$= QQ^T$

The projection matrix onto the

column space of A is: $A(A^TA)^{-1}A^T$

$\therefore$ Projection matrix onto column space of Q is: $Q(Q^TQ)^{-1}Q^T = Q(I)Q^T$
$= QQ^T$.

11. (a) Find orthonormal vectors $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ in the plane of $\boldsymbol{a} = (1, 3, 4, 5, 7)$ and $\boldsymbol{b} = (-6, 6, 8, 0, 8)$.

(b) Which vector in this plane is closest to $(1, 0, 0, 0, 0)$?

(a) Let $A = a$, $B = b - \frac{a^Tb}{a^Ta}a$

$\therefore B = (-6, 6, 8, 0, 8) - \frac{100}{100}(1, 3, 4, 5, 7)$

$= (-7, 3, 4, -5, 1)$

$\therefore q_1 = \frac{A}{\|A\|} = \frac{1}{10}(1, 3, 4, 5, 7)$

$q_2 = \frac{B}{\|B\|} = \frac{(-7, 3, 4, -5, 1)}{\sqrt{49+9+16+25+1}} = \frac{1}{10}(-7, 3, 4, -5, 1)$

(b) Let $b = (1, 0, 0, 0, 0)$

$\therefore$ Want projection $p = (q_1^T \cdot b)q_1 + (q_2^T \cdot b)q_2$

$= \frac{1}{10}q_1 + -\frac{7}{10}q_2$

$= (.01, .03, .04, .05, .07) + (.49, -.21, -.28, .35, -.07)$

$= (.5, -.18, -.24, .4, 0)$

15. (a) Find orthonormal vectors q_1, q_2, q_3 such that q_1, q_2 span the column space of

$$A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix}.$$

(b) Which of the four fundamental subspaces contains q_3?

(c) Solve $Ax = (1, 2, 7)$ by least squares.

(a) Let $A_1 = (1, 2, -2)$, $A_2 = (1, -1, 4)$

$$\underline{q_1} = \frac{A_1}{\|A_1\|} = \underline{\tfrac{1}{3}(1, 2, -2)}$$

$$\text{Let } B = A_2 - \frac{A_1^T \cdot A_2}{A_1^T \cdot A_1} A_1 = (1, -1, 4) - \frac{-9}{9}(1, 2, -2)$$

$(2, 1, 2)$

$$\therefore \underline{q_2} = \frac{B}{\|B\|} = \underline{\tfrac{1}{3}(2, 1, 2)}$$

Looking at pattern for q_1 & q_2, choose

$\underline{q_3 = \tfrac{1}{3}(-2, 2, 1)}$. $q_3 \perp q_2$, and $q_3 \perp q_1$.

(b) The left nullspace of A is orthogonal to A, or the nullspace of A^T

(c) $\begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$

Since $\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 2 \\ -2 & 4 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & 0 \\ 0 & 6 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & 0 \\ 0 & 0 & 9 \end{bmatrix}$

Then there is no solution to $Ax = b$.
Closest solution is projection of b onto column space of A

$\therefore$ Using $A^T(b - A\hat{x}) = 0 \Rightarrow A^Tb = A^TA\hat{x}$
Using $A = QR$, using Q from (a), we get $R = Q^TA$, so $A^Tb = A^TA\hat{x}$ becomes $R^TQ^Tb = R^TQ^TQR\hat{x} = R^TR\hat{x}$, and since R^T is lower triangular, $R^TQ^Tb = R^TR\hat{x}$ becomes $Q^Tb = R\hat{x}$

$\therefore Q^Tb = \frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -9 \\ 18 \end{bmatrix}$

$$R = Q^T A = \frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 9 & -9 \\ 0 & 9 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 9 & -9 \\ 0 & 9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -9 \\ 18 \end{bmatrix}, \quad x_2 = 2, \quad x_1 = 1$$

$$\therefore \hat{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

17. Find the projection of $\boldsymbol{b}$ onto the line through $\boldsymbol{a}$:

$$\boldsymbol{a} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \quad \text{and} \quad \boldsymbol{p} = \,? \quad \text{and} \quad \boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p} = \,?$$

Compute the orthonormal vectors $\boldsymbol{q}_1 = \boldsymbol{a}/\|\boldsymbol{a}\|$ and $\boldsymbol{q}_2 = \boldsymbol{e}/\|\boldsymbol{e}\|$.

$$p = \left(\frac{a^T b}{a^T a}\right) a = \frac{\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}}{\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} a = \frac{9}{3} a = 3a = \underline{(3,3,3)}$$

$$e = b - p = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} - \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = \underline{\begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}}$$

$$q_1 = \frac{a}{\|a\|} = \frac{1}{\sqrt{3}}(1,1,1) = \left(\sqrt{3}/3, \sqrt{3}/3, \sqrt{3}/3\right)$$

$$q_2 = \frac{e}{\|e\|} = \frac{1}{2\sqrt{2}}(-2,0,2) = \left(-\sqrt{2}/2, 0, \sqrt{2}/2\right)$$

18. (Recommended) Find orthogonal vectors $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ by Gram-Schmidt from $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$:

$$\boldsymbol{a} = (1, -1, 0, 0) \qquad \boldsymbol{b} = (0, 1, -1, 0) \qquad \boldsymbol{c} = (0, 0, 1, -1).$$

$\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ and $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ are bases for the vectors perpendicular to $\boldsymbol{d} = (1, 1, 1, 1)$.

$$\underline{A = a}, \quad B = b - \frac{a^T b}{a^T a} a = b - \frac{(-1)}{2} a = b + \frac{1}{2} a$$

$$= (0, 1, -1, 0) + \left(\frac{1}{2}, -\frac{1}{2}, 0, 0\right)$$

$$= \left(\frac{1}{2}, \frac{1}{2}, -1, 0\right)$$

$$C = c - \frac{A^T c}{A^T A} A - \frac{B^T c}{B^T B} B = c - 0 \cdot A - \frac{(-1)}{3/2} B$$

$$= c + \frac{2}{3} B = (0, 0, 1, -1) + (1/3, 1/3, -2/3, 0)$$

$$= (1/3, 1/3, 1/3, 0)$$

19. If $A = QR$ then $A^T A = R^T R =$ _____ triangular times _____ triangular. *Gram-Schmidt on A corresponds to elimination on $A^T A$.* Compare the pivots for $A^T A$ with $\|\boldsymbol{a}\|^2 = 3$ and $\|\boldsymbol{e}\|^2 = 8$ in Problem 17:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 3 \\ 1 & 5 \end{bmatrix} \quad \text{and} \quad A^T A = \begin{bmatrix} 3 & 9 \\ 9 & 35 \end{bmatrix}.$$

$$A^T A = (QR)^T (QR) = R^T Q^T Q R = R^T R$$

$=$ lower triangular times upper triangular.

21. Find an orthonormal basis for the column space of A:

$$A = \begin{bmatrix} 1 & -2 \\ 1 & 0 \\ 1 & 1 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} -4 \\ -3 \\ 3 \\ 0 \end{bmatrix}.$$

Then compute the projection of $\boldsymbol{b}$ onto that column space.

(a) $A_1 \cdot A_2 = -2+1+3 = 2$, so columns not orthogonal and are independent.

$\therefore$ Let $A = A_1$, $B = A_2 - \dfrac{A_1^T \cdot A_2}{A_1^T \cdot A_1} A_1$

$$= \begin{bmatrix} -2 \\ 0 \\ 1 \\ 3 \end{bmatrix} - \frac{2}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5/2 \\ -1/2 \\ 1/2 \\ 5/2 \end{bmatrix}$$

$$\therefore \frac{A}{\|A\|} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \quad \frac{B}{\|B\|} = \frac{B}{\sqrt{\frac{52}{4}}} = \frac{B}{\sqrt{13}}$$

$$= \begin{bmatrix} -5/2\sqrt{13} \\ -1/2\sqrt{13} \\ 1/2\sqrt{13} \\ 5/2\sqrt{13} \end{bmatrix}$$

(b) In general, $p = A\hat{x} = A(A^TA)^{-1}A^Tb$

So, $p = Q(Q^TQ)^{-1}Q^Tb = QQ^Tb$

since $Q^TQ = I$, where $Q = \left[\frac{A}{\|A\|} \; \frac{B}{\|B\|}\right]$

$$QQ^Tb = \begin{bmatrix} 1/2 & -5/\sqrt{52} \\ 1/2 & -1/\sqrt{52} \\ 1/2 & 1/\sqrt{52} \\ 1/2 & 5/\sqrt{52} \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ -5/\sqrt{52} & -1/\sqrt{52} & 1/\sqrt{52} & 5/\sqrt{52} \end{bmatrix} \begin{bmatrix} -4 \\ -3 \\ 3 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{4}+\frac{25}{52} & \frac{1}{4}+\frac{5}{52} & \frac{1}{4}-\frac{5}{52} & \frac{1}{4}-\frac{25}{52} \\ \frac{1}{4}+\frac{5}{52} & \frac{1}{4}+\frac{1}{52} & \frac{1}{4}-\frac{1}{52} & \frac{1}{4}-\frac{5}{52} \\ \frac{1}{4}-\frac{5}{52} & \frac{1}{4}-\frac{1}{52} & \frac{1}{4}+\frac{1}{52} & \frac{1}{4}+\frac{5}{52} \\ \frac{1}{4}-\frac{25}{52} & \frac{1}{4}-\frac{5}{52} & \frac{1}{4}+\frac{5}{52} & \frac{1}{4}+\frac{25}{52} \end{bmatrix} \begin{bmatrix} -4 \\ -3 \\ 3 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1-\frac{100}{52} \\ -1-\frac{20}{52} \\ -1+\frac{20}{52} \\ -1+\frac{100}{52} \end{bmatrix} = \begin{bmatrix} -\frac{152}{52} \\ -\frac{72}{52} \\ -\frac{32}{52} \\ \frac{48}{52} \end{bmatrix}$$