5.1 The Properties of Determinants

Note Title 11/5/2007

5. For $n = 5, 6, 7$, count the row exchanges to permute the reverse identity J_n to the identity matrix I_n. Propose a rule for every size n and predict whether J_{101} has determinant $+1$ or -1.

J_2 and $J_3 \Rightarrow 1$ row exchange
J_4 and $J_5 \Rightarrow 2$ row exchanges
J_6 and $J_7 \Rightarrow 3$ row exchanges

$\therefore$ Let $\mathrm{Int}(n)$ = integer part of $(n/2)$

$\therefore \det J_n = (-1)^{\mathrm{Int}(n)}$

$\therefore |J_{101}| = (-1)^{\mathrm{Int}(101)} = (-1)^{50} = 1$

6. Show how Rule 6 (determinant $= 0$ if a row is all zero) comes from Rule 3.

$$\begin{vmatrix} 0 \cdot a_{11} & 0 \cdot a_{12} & \cdots & 0 \cdot a_{1n} \\ a_{21} & \cdots & & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & \cdots & & a_{nn} \end{vmatrix} = 0 \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = 0$$

10. If the entries in every row of A add to zero, solve $A\boldsymbol{x} = \mathbf{0}$ to prove $\det A = 0$. If those entries add to one, show that $\det(A - I) = 0$. Does this mean $\det A = 1$?

(a) If A is invertible, only solution to $Ax = 0$ is $x = 0$. But $x = (1, 1, \ldots, 1)$ is a solution, so A is not invertible $\Rightarrow \det A = 0$.

(b) Entries in every row of $A - I$ add to zero. $\therefore$ By (a) $\det(A-I) = 0$.

13. Reduce A to U and find $\det A =$ product of the pivots:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 3 & 3 \end{bmatrix}.$$

(a) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ $\det A = 1$

(b) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 3 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -3 \\ 0 & -3 & -6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -3 \\ 0 & 0 & -\frac{3}{2} \end{bmatrix}$

$\therefore \det A = (1)(-2)(-\frac{3}{2}) = \underline{3}$