5.2 Permutations and Cofactors

Note Title 11/13/2007

1. Compute the determinants of A, B, C from six terms. Are their rows independent?

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 3 & 2 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 4 & 4 \\ 5 & 6 & 7 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

$$A = \begin{vmatrix} 1 & & \\ & 1 & \\ & & 1 \end{vmatrix} + \begin{vmatrix} & 2 & \\ 3 & & \\ & & 1 \end{vmatrix} + \begin{vmatrix} & & 3 \\ 3 & & \\ & 2 & \end{vmatrix}$$

$$+ \begin{vmatrix} 1 & & \\ & & 2 \\ & 2 & \end{vmatrix} + \begin{vmatrix} & 2 & \\ & & 2 \\ 3 & & \end{vmatrix} + \begin{vmatrix} & & 3 \\ & 1 & \\ 3 & & \end{vmatrix}$$

$$= (1-4) + (-6+12) + (18-9)$$

$$= -3 + 6 + 9$$

$= \underline{12}$, independent rows.

$$B = \begin{vmatrix} 1 & & \\ & 4 & \\ & & 7 \end{vmatrix} + \begin{vmatrix} & 2 & \\ 4 & & \\ & & 7 \end{vmatrix} + \begin{vmatrix} & & 3 \\ 4 & & \\ & 6 & \end{vmatrix}$$

$$+ \begin{vmatrix} 1 & & \\ & & 4 \\ & 6 & \end{vmatrix} + \begin{vmatrix} & 2 & \\ & & 4 \\ 5 & & \end{vmatrix} + \begin{vmatrix} & & 3 \\ & 4 & \\ 5 & & \end{vmatrix}$$

$$= (28-24) + (-56+40) + (72-60)$$

$= 4 + -16 + 12 = \underline{0}$, dependent rows.

$$C = \begin{vmatrix} 1 & & \\ & 1 & \\ & & 0 \end{vmatrix} + \begin{vmatrix} & 1 & \\ 1 & & \\ & & 0 \end{vmatrix} + \begin{vmatrix} & & 1 \\ & 1 & \\ 0 & & \end{vmatrix}$$

$$+ \begin{vmatrix} 1 & & \\ & & 0 \\ & 0 & \end{vmatrix} + \begin{vmatrix} & 1 & \\ & & 0 \\ 1 & & \end{vmatrix} + \begin{vmatrix} & & 1 \\ & 1 & \\ 1 & & \end{vmatrix}$$

$= 0 + 0 + 0 + 0 + 0 - 1 = \underline{-1}$, independent rows.

13. Find the cofactor matrix C and multiply A times C^{T}. Compare AC^{T} with A^{-1}:

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \qquad A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$

(a) $m_{11} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \therefore C_{11} = 3$ $\qquad m_{21} = \begin{bmatrix} -1 & 0 \\ -1 & 2 \end{bmatrix} \therefore C_{21} = 2$

$m_{12} = \begin{bmatrix} -1 & -1 \\ 0 & 2 \end{bmatrix} \therefore C_{12} = 2$ $\qquad m_{22} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ $C_{22} = 4$

$m_{13} = \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix} \therefore C_{13} = 1$ $\qquad m_{23} = \begin{bmatrix} 2 & -1 \\ 0 & -1 \end{bmatrix}$ $C_{23} = 2$

$m_{31} = \begin{bmatrix} -1 & 0 \\ 2 & -1 \end{bmatrix}$ $C_{31} = 1$ $\qquad m_{33} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ $C_{33} = 3$

$m_{32} = \begin{bmatrix} 2 & 0 \\ -1 & -1 \end{bmatrix}$ $C_{32} = 2$

$\therefore\ C = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ note $C = C^T$

(b) $AC^T = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

$\therefore\ AC^T = 4I$

(c) $A^{-1} = \frac{1}{4} C^T$