5.3 Cramer's Rule, Inverses, and Volumes

Note Title 11/28/2007

1. Solve these linear equations by Cramer's Rule $x_j = \det B_j / \det A$:

(a) $\begin{aligned} 2x_1 + 5x_2 &= 1 \\ x_1 + 4x_2 &= 2 \end{aligned}$ (b) $\begin{aligned} 2x_1 + x_2 \quad\quad &= 1 \\ x_1 + 2x_2 + x_3 &= 0 \\ x_2 + 2x_3 &= 0. \end{aligned}$

(a)

$$\det A = \begin{vmatrix} 2 & 5 \\ 1 & 4 \end{vmatrix} = 3$$

$$\det B_1 = \begin{vmatrix} 1 & 5 \\ 2 & 4 \end{vmatrix} = -6 \qquad \det B_2 = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$$

$$\therefore x_1 = \frac{-6}{3} = -2 \qquad x_2 = \frac{3}{3} = 1$$

(b)

$$\det A = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 2(4-1) - 1(2-0) + 0 = 6 - 2 = \underline{\underline{4}}$$

$$\det B_1 = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 3$$

$$\det B_2 = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 2 \end{vmatrix} = -2$$

$$\det B_3 = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 1$$

$$\therefore\ x_1 = 3/4 \qquad x_2 = -\frac{1}{2} \qquad x_3 = 1/4$$

6. Find A^{-1} from the cofactor formula $C^{\mathrm{T}}/\det A$. Use symmetry in part (b).

(a) $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 4 & 1 \end{bmatrix}$ (b) $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$.

(a) $\det A = 3 - 2(0) = 3$

$C_{11} = 3,\ C_{12} = 0,\ C_{13} = 0$ $\quad \therefore\ C = \begin{bmatrix} 3 & 0 & 0 \\ -2 & 1 & -4 \\ 0 & 0 & 3 \end{bmatrix}$

$C_{21} = -2,\ C_{22} = 1,\ C_{23} = -4$

$C_{31} = 0,\ C_{32} = 0,\ C_{33} = 3$ $\quad C^T = \begin{bmatrix} 3 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 3 \end{bmatrix}$

$$\therefore\ A^{-1} = \frac{1}{3}\begin{bmatrix} 3 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 3 \end{bmatrix} = \underline{\begin{bmatrix} 1 & -2/3 & 0 \\ 0 & 1/3 & 0 \\ 0 & -4/3 & 1 \end{bmatrix}}$$

(b) $\det A = 2(4-1) + 1(-2-0) = 4$

$C_{11} = 3,\ C_{12} = 2,\ C_{13} = -1$ $\quad \therefore$

$C_{22} = 4,\ C_{23} = 2$

$C_{33} = 3$

$$C = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 4 & 2 \\ -1 & 2 & 3 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 2 & -1 \\ 2 & 4 & 2 \\ -1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 3/4 & 1/2 & -\frac{1}{4} \\ 1/2 & 1 & 1/2 \\ -1/4 & 1/2 & 3/4 \end{bmatrix}$$

Note: if $A = A^T$, then $A^{-1} = (A^T)^{-1} = (A^{-1})^T$,
so A^{-1} is symmetric

9. Suppose $\det A = 1$ and you know all the cofactors. How can you find A?

$AC^T = (\det A)\, I = I \quad \therefore \quad A^{-1} = C^T$

Now $A \cdot A^{-1} = I \Rightarrow \det A \cdot \det A^{-1} = 1.$
Since $\det A = 1$, $\det A^{-1} = 1.$

But $A^{-1} = C^T \Rightarrow \det(A^{-1}) = 1 = \det(C^T)$

$\therefore C^T \cdot (\text{cofactor matrix of } C^T)^T = (\det C^T)\, I = I$

$\therefore A^{-1} \cdot (\text{cofactor matrix of } C^T)^T = I$,

$\therefore A = (\text{cofactor matrix of } C^T)^T$

10. From the formula $AC^{\mathrm{T}} = (\det A)I$ show that $\det C = (\det A)^{n-1}$.

$(AC^T)^T = (\det A)\, I$, so $CA^T = (\det A)\, I$

$\therefore (\det C)(\det A^T) = \det[(\det A)I]$
$= (\det A)^n$

But $\det A^T = \det A$

$\therefore \det C = (\det A)^{n-1}$

11. (for professors only) If you know all 16 cofactors of a 4 by 4 invertible matrix A, how would you find A?

From #10 above, $\det A = (\det C)^{1/3}$, so $\det A$ can be computed, since C is known.

$\therefore$ Can determine A^{-1} from $\dfrac{C^T}{\det A}$

$\therefore$ From A^{-1}, can compute A by solving $A^{-1}X = I$

12. If all entries of A are integers, and $\det A = 1$ or -1, prove that all entries of A^{-1} are integers. Give a 2 by 2 example.

(a) If all entries of A are integers, then all cofactors are integers, so that the cofactor matrix has all integer entries.
Since $A^{-1} = \dfrac{C^T}{\det A}$, then A^{-1} has

all integer entries.

(b) Let $A = \begin{bmatrix} 5 & 1 \\ 4 & 1 \end{bmatrix}$, Then $\det A = 1$

$C = \begin{bmatrix} 1 & -4 \\ -1 & 5 \end{bmatrix}$, $C^T = \begin{bmatrix} 1 & -1 \\ -4 & 5 \end{bmatrix}$

$\therefore A^{-1} = \begin{bmatrix} 1 & -1 \\ -4 & 5 \end{bmatrix}$

15. L is lower triangular and S is symmetric. Assume they are invertible:

$$L = \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix} \qquad S = \begin{bmatrix} a & b & d \\ b & c & e \\ d & e & f \end{bmatrix}.$$

(a) Which three cofactors of L are zero? Then L^{-1} is lower triangular.

(b) Which three pairs of cofactors of S are equal? Then S^{-1} is symmetric.

(a) By crossing out rows & columns from b, d, and e, you always have zeroes so that $(\det M_{ij}) = 0$.

$\therefore C = \begin{bmatrix} & & \\ 0 & & \\ 0 & 0 & \end{bmatrix} \Rightarrow C^T = \begin{bmatrix} x & 0 & 0 \\ x & x & 0 \\ x & x & x \end{bmatrix}$

so C^T is lower triangular $\Rightarrow$
$L^{-1} = \dfrac{C^T}{\det L}$ is lower triangular.

(b) Since $M_{ij} = M_{ji}$, Then $C_{ij} = C_{ji}$

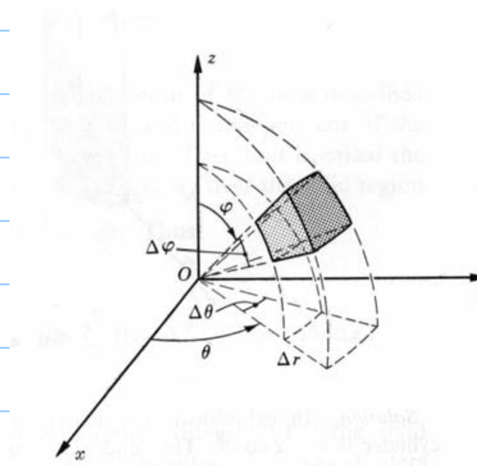
$\therefore$ C is symmetric $\Rightarrow$ C^T is symmetric

$\therefore$ $S^{-1} = \dfrac{C^T}{\det S}$ is symmetric

29. Spherical coordinates ρ, ϕ, θ satisfy $x = \rho \sin\phi \cos\theta$ and $y = \rho \sin\phi \sin\theta$ and $z = \rho\cos\phi$. Find the 3 by 3 matrix of partial derivatives: $\partial x/\partial\rho$, $\partial x/\partial\phi$, $\partial x/\partial\theta$ in row 1. Simplify its determinant to $J = \rho^2 \sin\phi$. Then dV in a sphere is $\rho^2 \sin\phi \, d\rho \, d\phi d\theta$.

Projection of ρ onto xy plane is: $\rho \sin\phi$

Projection onto z-axis is $\rho\cos\phi$



$\therefore$ $x = (\rho\sin\phi)\cos\theta$

$y = (\rho\sin\phi)\sin\theta$

$z = \rho\cos\phi$

Volume edges: $(\rho\sin\phi)\Delta\theta$, $\rho\Delta\phi$, and $\Delta\rho$

$\therefore dV = \rho^2 \sin\phi \, d\theta \, d\phi \, d\rho$

$\therefore \dfrac{\partial x}{\partial \rho} = \sin\phi\cos\theta$

$\dfrac{\partial x}{\partial \phi} = \rho\cos\phi\cos\theta$ $\qquad \dfrac{\partial x}{\partial \theta} = -\rho\sin\phi\sin\theta$

$$\frac{\partial y}{\partial \rho} = \sin\phi \sin\theta$$

$$\frac{\partial y}{\partial \phi} = \rho\cos\phi\sin\theta \qquad \frac{\partial y}{\partial \theta} = \rho\sin\phi\cos\theta$$

$$\frac{\partial z}{\partial \rho} = \cos\phi \quad \frac{\partial z}{\partial \phi} = -\rho\sin\phi \quad \frac{\partial z}{\partial \theta} = 0$$

$$\therefore J = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{bmatrix}$$

$$= \begin{bmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{bmatrix}$$

$\therefore \ |J| =$

$$\cos\phi\left(\rho^2\cos\phi\sin\phi\cos^2\theta + \rho^2\cos\phi\sin\phi\sin^2\theta\right)$$

$$+\ \rho\sin\phi\left(\rho\sin^2\phi\cos^2\theta + \rho\sin^2\phi\sin^2\theta\right)$$

$$+\ 0\left(\quad\right)$$

$$= \cos\phi\left(\rho^2\cos\phi\sin\phi\right)\left(\cos^2\theta + \sin^2\theta\right)$$

$$+\ \rho\sin\phi\left(\rho\sin^2\phi\right)\left(\cos^2\theta + \sin^2\theta\right)$$

$$= \rho^2\cos^2\phi\sin\phi + \rho^2\sin\phi\sin^2\phi$$

$$= \rho^2\sin\phi\left(\cos^2\phi + \sin^2\phi\right)$$

$$= \rho^2\sin\phi$$