6.1 Introduction to Eigenvalues

Note Title 12/17/2007

1. The example at the start of the chapter has

$$A = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \quad \text{and} \quad A^2 = \begin{bmatrix} .70 & .45 \\ .30 & .55 \end{bmatrix} \quad \text{and} \quad A^\infty = \begin{bmatrix} .6 & .6 \\ .4 & .4 \end{bmatrix}.$$

The matrix A^2 is halfway between A and A^∞. Explain why $A^2 = \frac{1}{2}(A + A^\infty)$ from the eigenvalues and eigenvectors of these three matrices.

(a) Show from A how a row exchange can produce different eigenvalues.

(b) Why is a zero eigenvalue *not* changed by the steps of elimination?

A has eigenvalues of 1 and $\frac{1}{2}$

A^2 has eigenvalues of 1^2 and $(\frac{1}{2})^2 = \frac{1}{4}$

A^∞ has eigenvalues of 1^∞ and $(\frac{1}{2})^\infty = 0$

$\therefore \frac{1}{2}(1+1) = 1, \quad \frac{1}{2}(\frac{1}{2} + 0) = \frac{1}{4}$

(a) Let $A' = \begin{bmatrix} .2 & .7 \\ .8 & .3 \end{bmatrix}$

$\therefore \det \begin{bmatrix} .2-\lambda & .7 \\ .8 & .3-\lambda \end{bmatrix} = (.2-\lambda)(.3-\lambda) - 0.56 = 0$

$\lambda^2 - 0.5\lambda - 0.5 = 0$

$(\lambda - 1)(\lambda + 0.5) = 0$

$\therefore \lambda = 1, -0.5$

(b) A $\lambda = 0$ means eigenvector is in nullspace of matrix A. Steps of elimination don't change nullspace.

2\. Find the eigenvalues and the eigenvectors of these two matrices:

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad A + I = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix}.$$

$A + I$ has the _____ eigenvectors as A. Its eigenvalues are _____ by 1.

$$\begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = \lambda^2 - 4\lambda - 5 = 0$$

$$(\lambda - 5)(\lambda + 1) = 0$$

$$\underline{\lambda = 5, -1}$$

If $Ax = \lambda x$, then $(A+I)x = Ax + Ix = \lambda x + x$
$= (\lambda + 1)x$

$\therefore$ for $A+I$, $\lambda = 6, 0$

$A+I$ has <u>same</u> eigenvectors as A.

Its eigenvalues are <u>shifted</u> up by 1.

7\. Elimination produces $A = LU$. The eigenvalues of U are on its diagonal; they are the _____. The eigenvalues of L are on its diagonal; they are all _____. The eigenvalues of A are not the same as _____.

Let $U = \begin{bmatrix} x_1 & x & x & x & x \\ & x_2 & x & x & x \\ & & \ddots & & \vdots \\ & & & & x_n \end{bmatrix}$ det $(U - \lambda I)$ is unaffected by row operations.

$$\therefore \det(U - \lambda I) = \begin{vmatrix} x_1-\lambda & 0 & 0 & \cdots & 0 \\ & x_2-\lambda & 0 & \cdots & 0 \\ & & \ddots & & \vdots \\ & & & & x_n-\lambda \end{vmatrix}$$

$$= (x_1-\lambda)(x_2-\lambda)\cdots(x_n-\lambda) = 0$$

$$\Rightarrow \lambda = x_1, x_2, \ldots, x_n$$

$\therefore$ Eigenvalues of U are the <u>pivots</u>

Similary, for $L = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ x & 1 & & 0 \\ x & & \ddots & \vdots \\ x & x & x \cdots & 1 \end{bmatrix}$, the eigenvalues are the "1's", or <u>1</u>.

Eigenvalues of A are not the same as the pivots.

11. Here is a strange fact about 2 by 2 matrices with eigenvalues $\lambda_1 \neq \lambda_2$: The columns of $A - \lambda_1 I$ are multiples of the eigenvector x_2. Any idea why this should be?

Let x_1, x_2 be the eigenvectors.

$\lambda_1 \neq \lambda_2 \Rightarrow x_1 \neq x_2$, for if $x_1 = x_2$, then $Ax = \lambda_1 x = \lambda_2 x$. But $\lambda_1 x = \lambda_2 x \Rightarrow \lambda_1 = \lambda_2$.

$\therefore x_1, x_2$ are independent.

$A - \lambda_1 I$, as a matrix, has eigenvalues of
$\lambda_1 - \lambda_1 = 0$, and $\lambda_2 - \lambda_1 \neq 0$.
It has The same eigenvectors as A (x_1, x_2)
But $(A - \lambda_1 I)x_1 = 0$, so x_1 is in The
nullspace of $A - \lambda_1 I$.
And $(A - \lambda_1 I)x_2 = (\lambda_2 - \lambda_1)x_2 \neq 0$, so x_2 is
not in nullspace of $A - \lambda_1 I$.
$\therefore$ x_2 must be in col space of $A - \lambda_1 I$
since dim (col space of $A - \lambda_1 I$) = 1.

Another proof:

Lemma: If A_1 has eigenvalues of λ_1 and 0
with corresponding eigenvectors x_1, x_2
and A_2 has eigenvalues of 0 and λ_2
with The same corresponding vectors x_1, x_2,
Then, $A_1 A_2$ is The 0 matrix, assuming
x_1 and x_2 are independent, and
A_1, A_2 are 2×2.

Pf: Let $x = a x_1 + b x_2$

$$\therefore A_1 A_2 x = A_1 (A_2 x) = A_1 (0 + b \lambda_2 x_2)$$

$$= A_1 (b \lambda_2 x_2) = b \lambda_2 A_1 x_2 = 0$$

Now consider, for #11, $(A-\lambda_2 I)(A-\lambda_1 I)$.
By the above Lemma, this is the zero matrix
$\therefore$ Columns of $A-\lambda_1 I$ are in the nullspace of $A-\lambda_2 I$. Since $(A-\lambda_2 I)x_2 = 0$, then columns of $A-\lambda_1 I$ are multiples of x_2.

16. Prove that the determinant of A equals the product $\lambda_1\lambda_2\cdots\lambda_n$. Start with the polynomial $\det(A-\lambda I)$ separated into its n factors. Then set $\lambda =$ ____:

$$\det(A-\lambda I) = (\lambda_1-\lambda)(\lambda_2-\lambda)\cdots(\lambda_n-\lambda) \quad \text{so} \quad \det A = \text{\_\_\_\_\_}.$$

Pf: $\det(A-\lambda I) = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & \cdots & \cdots & a_{nn}-\lambda \end{vmatrix}$

$= (-1)^n \begin{vmatrix} \lambda - a_{11} & \cdots & -a_{1n} \\ \vdots & & \\ -a_{n1} & \cdots & \lambda - a_{nn} \end{vmatrix} = (-1)^n \det(\lambda I - A) = (-1)^n p(\lambda)$

where $p(\lambda) =$ a polynomial of degree n.

$p(\lambda) = (\lambda-\lambda_1)\cdots(\lambda-\lambda_n)$, where λ_i are zeros of $p(\lambda)$ and so are also the eigenvalues which make

$\det(A-\lambda_i I) = 0 = (-1)^n p(\lambda_i)$. $\therefore \det(A-\lambda I) = (-1)^n p(\lambda)$

Now set $\lambda = 0$ for $\det(A-\lambda I) = (-1)^n p(\lambda)$

$\therefore \det(A) = (-1)^n p(0) = (-1)^n (-\lambda_1)(-\lambda_2)\cdots(-\lambda_n)$

$= (-1)^n (-1)^n \lambda_1 \lambda_2 \cdots \lambda_n$

$= \lambda_1 \lambda_2 \cdots \lambda_n$ as $(-1)^n(-1)^n = 1$.

$\therefore \det(A) = \lambda_1 \lambda_2 \cdots \lambda_n$

17. The sum of the diagonal entries (the *trace*) equals the sum of the eigenvalues:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{has} \quad \det(A-\lambda I) = \lambda^2 - (a+d)\lambda + ad - bc = 0.$$

If A has $\lambda_1 = 3$ and $\lambda_2 = 4$ then $\det(A-\lambda I) =$ _____. The quadratic formula gives the eigenvalues $\lambda = (a+d+\sqrt{})/2$ and $\lambda =$ _____. Their sum is _____.

(a) If $\lambda_1 = 3$, $\lambda_2 = 4$, Then $\det(A-\lambda I) =$

$(3-\lambda)(4-\lambda) = 12 - 7\lambda + \lambda^2$

(b) $\dfrac{-b \pm \sqrt{b^2-4ac}}{2a} \Rightarrow \dfrac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$

$(a+d)^2 - 4(ad-bc) = a^2 + 2ad + d^2 - 4ad + 4bc$

$= a^2 - 2ad + d^2 + 4bc$

$= (a-d)^2 + 4bc$

$\therefore \lambda = \dfrac{(a+d) \pm \sqrt{(a-d)^2 + 4bc}}{2}$

Their sum is $a+d$.

19\. A 3 by 3 matrix B is known to have eigenvalues $0, 1, 2$. This information is enough to find three of these:

(a) the rank of B

(b) the determinant of $B^{\mathrm{T}}B$

(c) the eigenvalues of $B^{\mathrm{T}}B$

(d) the eigenvalues of $(B+I)^{-1}$.

(a) Let $\lambda_1 = 0, \lambda_2 = 1, \lambda_3 = 2$, x_i The corresponding eigenvectors

$\lambda_1 = 0 \Rightarrow Ax_1 = 0$ has non-zero $x_1 \Rightarrow$
A is not invertible. $\therefore$ Rank is < 3

Since $\lambda_2 \neq \lambda_3$, x_2 and x_3 are independent.

Pf: Consider $c_2 x_2 + c_3 x_3 = 0$ [1]

$\therefore B(c_2 x_2 + c_3 x_3) = c_2 \lambda_2 x_2 + c_3 \lambda_3 x_3 = 0$ [2]

and from [1], $c_2 \lambda_2 x_2 + c_3 \lambda_2 x_3 = 0$ [3]

Subtracting [2] - [3],

$c_3(\lambda_3 - \lambda_2) x_3 = 0 \Rightarrow c_3 x_3 = 0$
since $\lambda_3 - \lambda_2 \neq 0$

But $c_3 x_3 = 0 \Rightarrow c_3 = 0$ as $x_3 \neq 0$

Similarly, multiplying [1] by λ_3 and subtracting from [2], $c_2 = 0$.

$\therefore c_2 x_2 + c_3 x_3 = 0 \Rightarrow c_2 = c_3 = 0$, so x_2 and x_3 are independent.

$\therefore$ Rank of $B \geq 2$, and so $\underline{\text{Rank } B = 2}$

(b) $\det(B^T B) = \det(B^T)\det(B)$
$= \{\det(B)\}^2$
$= (0 \cdot 1 \cdot 2)^2 = 0$

Also, $\lambda = 0 \Rightarrow B$ not invertible $\Rightarrow \det B = 0$.

$\therefore \underline{\det(B^T B) = 0}$

(c) Can't determine eigenvalues of $B^T B$

For example, Let $B = \begin{bmatrix} 0 & & \\ & 1 & \\ & & 2 \end{bmatrix}$, which has eigenvalues 0, 1, 2. $B^T = \begin{bmatrix} 0 & & \\ & 1 & \\ & & 2 \end{bmatrix}$, so

$B^T B = \begin{bmatrix} 0 & & \\ & 1 & \\ & & 4 \end{bmatrix}$ with eigenvalues of

$\lambda = 0, 1, 4$

Now suppose $B = \begin{bmatrix} 0 & 1 & \\ & 1 & \\ & & 2 \end{bmatrix}$ so $B^T = \begin{bmatrix} 0 & & \\ 1 & 1 & \\ & & 2 \end{bmatrix}$

$\therefore B^T B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ with eigenvalues

$\lambda = 0, 2, 4$

These have different eigenvalues.

$\therefore$ Insufficient information to determine.

(d) eigenvalues $B + I = 0+1, 1+1, 2+1 = 1, 2, 3$

$\therefore$ eigenvalues of $(B+I)^{-1} = \frac{1}{1}, \frac{1}{2}, \frac{1}{3} = 1, \frac{1}{2}, \frac{1}{3}$

20. Choose the second row of $A = \begin{bmatrix} 0 & 1 \\ * & * \end{bmatrix}$ so that A has eigenvalues 4 and 7.

Let $A = \begin{bmatrix} 0 & 1 \\ c & d \end{bmatrix}$ $\therefore$ $0 + d = 4 + 7 = 11$, $d = 11$

$\det A = 4 \cdot 7 = 28 = 0 \cdot d - c \cdot 1$

$\therefore c = -28$

$\therefore A = \begin{bmatrix} 0 & 1 \\ -28 & 11 \end{bmatrix}$

33. Suppose A has eigenvalues $0, 3, 5$ with independent eigenvectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$.

(a) Give a basis for the nullspace and a basis for the column space.

(b) Find a particular solution to $A\boldsymbol{x} = \boldsymbol{v} + \boldsymbol{w}$. Find all solutions.

(c) Show that $A\boldsymbol{x} = \boldsymbol{u}$ has no solution. (If it did then ____ would be in the column space.)

(a) Assume A is 3×3. As shown in #19, u is a basis for nullspace. Dim col. space $= 2$, so v, w is a basis for col. space.

(b) Since $Av = 3v$, $Aw = 5w$, Then $A(\frac{1}{3}v + \frac{1}{5}w) = v + w$.

$\therefore x = \frac{1}{3}v + \frac{1}{5}w$. All solutions are $x + cu$

(c) If $Ax = u$ had a solution, then u would be in col. space of A, which is impossible since u is in nullspace, and $u \neq 0$.